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RESEARCH ARTICLE

NON DIFFERENTIABLE MULTI OBJECTIVE FRACTIONAL MINIMAX OPTIMIZATION WITH SUPPORT FUNCTION UNDER (V, ρ, σ)- CONVEXITY.

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Abstract

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In this paper, we derive some theorems and duality theorems on non differentiable Multi objectve Fractional Minimax Programming with support functions under V, ρ, σ convex functions.

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Introduction:-

There are several papers devoted to this topic by Mishra and Giorgio [11], Clarke [1] and Craven [2]. In the recent past, KUK et al. [4] have introduced the concept of (V, ρ)– Convexity, which is generalization of the V-convexity for vector valued functions and derived the generalized Karush – Kuhm – Tucker optimality conditions as well as weak and strong duality for non smooth multi objective fractional programs. Later, Kim et al. [3] extended their resulty in presence of support functions. Very recently, Kim et al [3] have introduced the assumption of (V, ρ) – convexity for (F.P) the following generalized non differentiable fractional programming problem (GFP);

$$\text{Minimize} \qquad \max \left\{ \frac{f_i(x) + s(x/C_i)}{g_i(x) - s(x/D_i)} \; / \, i = 1, \ldots. p \right\}$$

$$\text{Subject to } h_j(x) \le 0, \; j = 1, \ldots.. m$$

Where $f := (f_1, \ldots\ldots. f_p) : R^n \to R^p$, $g := (g_1, \ldots. g_p) : R^n \to R^p$ and

$h := (h_1, \ldots\ldots h_m) : R^n \to R^m$ are continuously differentiable and for each I = 1, …. P, C_i and D_i are compact convex sets of R^n

In this paper, we introduce (V, P, σ) - convex function to derive the Kurush – Kuhn – Tucker Sufficient optimality and Mond – weir type weak and strong duality theorems for a generalized non differentiable minimax multi objective fractional optimization problem (GEP), in which numerator and denominator of each term consists of support function, and a constraint set defined by differentiable functions.

Preliminaries and definitions:-

In this paper, we consider the following non differentiable multi objective fractional programming problem.

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(GFP) Minimize $\max \left\{ \frac{f_i(x)+s(x/C_i)}{g_i(x)-s(x/D_i)} \;/\; i=1,....p \right\}$

Subject to : $h_j(x) \le 0,\ j=1,.....m$

Where $f := (f_1,.......f_p) : R^n \to R^p,\ g := (g_1,....g_p) : R^n \to R^p$ and

$h := (h_1,......h_m) : R^n \to R^m$ are continuously differentiable.

We assume that $g_i(x)-s(x/D_i)>0$, I = 1,….p. For each i = 1,….p, C_i and D_i are compact convex sets of R^n and define the support functions with respect to C_i and D_i as follows.

$$s(x/C_i) = \max \left\{ < x_1\ y_i > / \ y_i \in C_i \right\}$$

And $$s(x/C_i) = \max \left\{ < x_1\ y_i > / \ y_i \in C_i \right\}$$

Further denote $I(x) = \left\{ j / hj(x) = 0 \right\}$ for any $x \in R^n$.

Let $k_i(x) = s(x/C_i)$ and $\tilde{k}_i(x) = s(x/D_i)$, i = 1,……P.

Hence $k_i(x)$ and $\tilde{k}_i(x)$ are convex functions.

Choose $w_i \in dk_i(x)$ and $\tilde{w}_i \in d\tilde{k}_i(x)$ such that

$$dk_i(x) = \left\{ w_i \in C_i / < w_i, x >= s(x/C_i) \right\}$$

And $$d\tilde{k}_i(x) = \left\{ \tilde{w}_i \in C_i / < \tilde{w}_i, x >= s(x/D_i) \right\}$$

Where dk_i and $d\tilde{k}_i$ are the sub differential of k_i and $\tilde{k}$ respectively. Further

Let $S = \left\{ x \in R^n / h_j(x) \le 0, \quad j=1,.....m \right\}$

Definition:-

A vector valued function $f : R^n \to R^p$ is said to be convex at $u \in R^n$ if for any $x \in R^n$ and for all i = 1, ….P one has

$$f_i(x) - f_i(u) \ge \nabla f_i(u)(x-u)^t$$

Definition:-

A vector valued function $f : R^n \to R^p$ is said to be (V,ρ)- Convex at $u \in R^n$ with respect to the function $\alpha_i : R^n \times R^n \to R_+ / \{0\}$ and $\rho_i \in R$, i =1,…p, such that for any $x \in R^n$ and for all i =1,…p it holds.

$$\alpha_i(x,u)\left[f_i(x) - f_i(u) \right] \ge \nabla f_i(u)(x-u)^t + \rho_i \left\| \theta_i(x,u) \right\|^2$$

The following Theorem from [ ] will be needed in the sequel.

Theorem:-

Assume that f and g are vector valued differentiable functions defined on R^n and $f(x) + < w, x > \ge 0,\ \ g(x) - < \tilde{w}, x > > 0$ for all $x \in R^n$. If $f(.) + < w\ ,. >$ and $-g(.) + < \tilde{w}, . >$ are (V,P) – Convex at $u \in R^n$ with respect to the function θ_i *and* α_i , i =1,…P, then $\frac{f(.) + < w,\ . >}{g(.) + < \tilde{w}, . >}$ is (V,ρ) convex at $u \in R^n$ with respect to the function $\bar{\theta}_i$ *and* $\bar{\alpha}_i$, i =1,….P where

$$\bar{\alpha}_i(x,u) = \frac{g_i(x) + <\tilde{w}_i, x>}{g_i(u) + <\tilde{w}_i, u>} \ \alpha_i(x,u)$$

And $\bar{\theta}_i(x,u) = \left(\frac{1}{g_i(u) + <\tilde{w}_i, u>} \right)^{\frac{1}{2}} \theta_i(x,u)$

That is for all i $\bar{\alpha}_i(x,u) \left[\frac{f_i(x) + <w_i, x>}{g_i(x) - <\tilde{w}_i, x>} - \frac{f_i(u) + <w_i, u>}{g_i(u) - <\tilde{w}_i, u>} \right]$

$$\geq \nabla \left[\frac{f_i(u) + <w_i, u>}{g_i(u) - <\tilde{w}_i, u>} \right] (x-u)^t + \rho_i \left\| \bar{\theta}_i(x,u) \right\|^2$$

Definition:-

Let $\left(\frac{f_i(x) + <w_i, x>}{g_i(x) - <\tilde{w}_i, x>} \right) = \theta_i \ (x)$, i=1,….P

$$<w_i, x> \ = \ s(x/C_i), \quad <\tilde{w}_i, x> \ = \ s(x/D_i)$$

The pair (ϕ_i, h_j) is called (V, ρ_i, σ_j) - convex at $u \in R^n$,

If there exist $\alpha_i : R^n \times R^n \to R_+ / \{0\}$

$$\bar{\alpha}_i(x,u) = \frac{g_i(x) + <\tilde{w}_i, x>}{g_i(x) - <\tilde{w}_i, x>} \ \alpha_i(x,u) > 0$$

$$\bar{\theta}_i(x,u) = \left(\frac{1}{g_i(u) + <\tilde{w}_i, u>} \right)^{\frac{1}{2}} \theta_i(x,u)$$

$$\beta_i : R^n \times R^n \to R_+ / \{0\}, \ \rho_i \in R, i = 1, \ldots.P, : \sigma_j \in R, j = 1, ..m$$

Such that $\theta_i(x) - \theta_i(u) \geq \bar{\alpha}_i(x,u) \nabla \theta_i(u)(x-u)^t + \rho_i \left\| \bar{\theta}_i(x,u) \right\|^2$ and

$$-h_j(u) \geq \beta_j(x,u) \nabla h_j(u)(x-u)^t + \sigma_j \left\| \bar{\theta}_i(x,u) \right\|^2$$

Optimality Conditions:-

The following Kurn –Tucker necessary optimality conditions for (GFP) from [ ] will be needed in the sequel.

Theorem :

(Kuhn – Tucker necessary optimality condition) if x_0 is a solution of the problem (GFP) and under the assumption that one has $0 \notin C_0 \ \{\nabla h_j(x_0) / j = 1, \ldots.m\}$ then there exist $\lambda_i \geq 0$.

$$i \in I(x_0) := \left\{ i / \max_j \frac{f_j(x_0) + s(x_0 / C_j)}{g_j(x_0) - s(x_0 / D_j)} = \frac{f_i(x_0) + s(x_0 / C_i)}{g_i(x_0) - s(x_0 / D_i)} \right\},$$

$$\sum_{i=1}^{p} \lambda_i = 1, \mu_j \geq 0, \text{ j=1,…. m and } w_i \in C_i, \tilde{w}_i \in D_i, i = 1, \ldots p,$$

Such that

$$\sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(x_0) + <w_i, x_o>}{g_i(x_o) - (\tilde{w}_i, x_o)} \right) + \sum_{j=1}^{m} u_j \sigma h_i(x_o) = 0$$

$$<w_i, x_o> = S(x_o / c_i), < \tilde{w}_i, x_o > = s(x_o / D_i)$$

$$\sum_{j=1}^{m} \mu_j h_j(x_o) = 0$$

Theorem: 3.2:
(KUHN-TUCKER type sufficient condition), support that there exist a feasible solution x_o for (GFP) and scalars $\lambda_i > 0, i = 1, \ldots, p, \sum_{i=1}^{p} \lambda_i = 1, \mu_j \geq 0, j = 1, \ldots, m$ and $w_i \in c_i, \tilde{w}_i \in D_i \; i = 1, \ldots, p$ such that

i) $$\sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) - < \tilde{w}_i, x_o >} \right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_o) =$$

$$< w_i, x_o > = s(x_o / c_i), < \tilde{w}_i, x_o > = S(x_o / D_i)$$

$$\sum_{j=1}^{m} \mu_i h_j(x_o) = 0$$

ii) (θ_i, h_i) is (V, ρ_i, σ_j) - convex at x_0.

Then x_0 is an efficient solution for (GFP)

Proof: Hypothesis (i) implies that

$$0 = \sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) - < \tilde{w}_i, x_o >} \right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_o) \qquad (3.1)$$

Since (ϕ_i, h_j) is (v, ρ_i, σ_j) convex at x_o, we have for all $x \in S$

$$\frac{f_i(x) + c w_{i,} x}{g_i(x) - < \tilde{w}_i x >} - \frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) - < \tilde{w}_i, x_o >}$$

$$\geq \bar{\alpha}_i(x, \; x_o) \nabla \left(\frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) - < \tilde{w}_i, x_o >} \right) (x - x_o)^t + \rho_i \left| \bar{\theta}_i (x_, x_o)^2 \right|$$

and

$$0 = -h_j(x_0) \geq \beta_j(x_, x_o) V h_j(x_o)(x - x_o)^t + \sigma_j \left\| \theta_j(x_, x_o) \right\|^2$$

By using $\bar{\alpha}_i(x_, x_o) > 0, i = 1, \ldots p$ and $\beta_j(x, x_o) > 0, j = 1, \ldots, m$ we get

$$\frac{1}{\bar{\alpha}_i(x_1 x_o)} \left(\frac{f_i(x) + < w_i, x >}{g_i(x) - < \tilde{w}_i, x >} \right) - \frac{1}{\bar{\alpha}_i(x_1 x_o)} \left(\frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) < \tilde{w}_i, x_o >} \right)$$

$$\geq \nabla \left(\frac{f_i(x_0) + < w_i, x_0 >}{g_i(x_0) - < \tilde{w}_i, x_0 >} \right) \; (x - x_0)^t + \frac{\rho_i \left\| \bar{\theta}_i (x_, x_0) \right\|^2}{\bar{\alpha}_i (x_, x_0)} \qquad (3.2)$$

And $$0 \geq \nabla h_j \; (x_0) \; (x - x_0)^t \; + \frac{\sigma_j \left\| \theta_j (x_, x_0) \right\|^2}{\beta_j (x_, x_0)} \qquad (3.3)$$

Adding (3.2) and (3.3), we get

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x_1x_0)}\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<v_i,x>}\right)-\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x_1x_0)}\left(\frac{f_i(x_0)+<w_i,x_0>}{g_i(x_0)-<v_i,x_0>}\right)$$

$$\geq\left[\sum_{i=1}^{p} \lambda_1\nabla\left(\frac{f_i(x_0)+<w_i,x_0>}{g_i(x_0)-<v_i,x_0>}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(x_o)\right](x-x_0)^t$$

$$+\sum_{i=1}^{p}\lambda_i\frac{\rho_i\left\|\bar{\theta}_i(x_1x_0)\right\|^2}{\bar{\alpha}_i(x_1x_0)}+\sum_{j=1}^{m}\mu_j\frac{\sigma_j\left\|\theta_j(x,x_0)\right\|^2}{\beta_j(x_,x_0)}$$

Using (3.1), we have

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x_1x_0)}\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<v_i,x>}\right)-\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x_1x_0)}\left(\frac{f_i(x_0)+<w_i,x_0>}{g_i(x_0)-<v_i,x_0>}\right)$$

$$\geq\sum_{i=1}^{p}\lambda_i\frac{\rho_i\left\|\bar{\theta}_i(x,x_0)\right\|^2}{\bar{\alpha}_i(x_,x_0)}+\sum_{j=1}^{m}\mu_j\frac{\sigma_j\left\|\theta_j(x_,x_0)\right\|^2}{\beta_j(x,x_0)}$$

As $\sum_{i=1}^{p}\lambda_i\frac{\rho_i\left\|\bar{\theta}_i(x,x_0)\right\|^2}{\bar{\alpha}_i(x_,x_0)} \geq 0$ and $\sum_{j=1}^{m}\mu_j\frac{\sigma_j\left\|\theta_j(x_,x_0)\right\|^2}{\beta_j(x,x_0)}\geq 0$ we get

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x_,x_0)}\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<v_i,x>}\right)-\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x_,x_0)}\left(\frac{f_i(x_0)+<w_i,x_0>}{g_i(x_0)-<v_i,x_0>}\right)\geq 0$$

Thus we have

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\ \frac{f_i(x)+<w_i,x>}{g_i(x)-<v_i,x>}\ \geq\ \sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x_,x_0)}\ \frac{f_i(x_0)+<w_i,x_0>}{g_i(x_0)-<v_i,x_0>}$$

Suppose that x_0 is not an efficient solution for (GEP), then there exist a feasible solution x for (GFP) and an index r such that $\theta_i(x)\leq\theta_i(x_0)$ for any $i\neq r$ and Φ, where

$$\theta_i(x)=\frac{f_i(x)+<w_i,x>}{g_i(x)-<v_i,x>}\qquad \text{for any i.}$$

Since $\lambda_i>0$ and $\bar{\alpha}_i(x_1x_0)>0$, i =1,…..P, we have

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\ \Phi_i(x)<\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x_,x_0)}\ \Phi_i(x_0)$$

It follows that one has

$$\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<v_i,x>}\right)<\sum_{i=1}^{p} \frac{\lambda_i}{\bar{\alpha}_i(x_,x_0)}\left(\frac{f_i(x_0)+<w_i,x_0>}{g_i(x_0)-<v_i,x_0>}\right)$$

Which contradicts the inequalities (34) and hence x_0 is an efficient solution for (GFP)

Mond – Weir type duality:-

We now consider the following Mond-Weir type dual for (GFP)

(DGFP) Maximize $\max\left\{\dfrac{f_i(u)+S(u/C_i)}{g_i(u)-S(u/D_i)} \,/\, i=1,....P\right\}$

Subject to $$\sum_{i=1}^{p} \lambda_i \nabla\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right) + \sum_{j=1}^{m}\mu_j \nabla h_j(u)=0 \qquad (4.1)$$

$$w_i \in C_i,\ <w_i,u>=S(u/C_i),\ \tilde{w}_i \in D_i,$$
$$<\tilde{w}_i,u>=S(u/D_i),\ i=1,...m\tilde{w}_i$$
$$\lambda_i>0, i=1,....P\ ,\sum_{i=1}^{p}\lambda_i=1,\ \mu_j \ge 0, j=1,....m, \sum_{j=1}^{m}\mu_j h_j(u)=0$$

Theorem: (4.1): (Weak Duality) Let x be a feasible solution for (GFP) and let $(u,\lambda,\mu,w,\tilde{w})$ be feasible for (DGFP) such that (Φ_i,h_j) is (Φ_i,h_i) is (V,ρ_i,σ_j) - convex at u. then the following cannot hold

$$\left(\frac{f_i(x)+S(x/C_i)}{g_i(x)-S(x/D_i)}\right) < \left(\frac{f_i(u)+S(u/C_i)}{g_i(u)-S(u/D_i)}\right) \qquad (4.2)$$

Proof: Suppose that (4.2) holds that is

$$\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>}\right) < \left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)$$

Using $\lambda_i>0, i=1,....P\ ,\sum_{i=1}^{p}\lambda_i=1,\ \mu_j \ge 0, j=1,....m$ we get

$$\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>}\right) < \sum_{i=1}^{p}\lambda_i\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)$$

That is

$$\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>}\right) - \sum_{i=1}^{p}\lambda_i\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right) < 0 \qquad (4.3)$$

and $-\sum_{j=1}^{m}\mu_j h_j(u)=0$ (4.4)

By (v,p,σ) convexity, we have

$$\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)(x-u)^t + \sum_{i=1}^{p}\lambda_i \frac{\rho_i\left\|\bar{\theta}_i(x_, x_0)\right\|^2}{\bar{\alpha}_i(x_1 u)}$$
$$\le \sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x_1 u)}\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>}\right) - \sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x_1 u)}\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right) < 0$$

And $$\sum_{j=1}^{m}\mu_j\nabla h_j(u)(x-u)^t + \sum_{j=1}^{m}\mu_j\frac{\sigma_j\left\|\theta_j(x_, u)\right\|^2}{\beta_j(x,u)} \le -\sum_{j=1}^{m}\frac{\mu_j}{\beta_j(x,u)}\nabla h_j(u)=0$$

That is, $\sum_{i=1}^{p}\lambda_i \nabla\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)(x-u)^t+\sum_{i=1}^{p}\lambda_i \frac{\rho_i\left\|\overline{\theta}_i(x,u)\right\|^2}{\overline{\alpha}_i(x,u)}<0$

And $\sum_{j=1}^{m}\mu_j \nabla h_j(u)\,(x-u)^t+\sum_{j=1}^{m}\mu_j \frac{\sigma_j\left\|\theta_j(x_1x_0)\right\|^2}{\beta_j(x,u)}\leq 0$

By adding the above inequalities, we get

$$\left[\sum_{i=1}^{p}\lambda_i \nabla\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)+\sum_{j=1}^{m}\mu_j \nabla h_j(u)\right](x-u)^t<0$$

Which contradicty the dual constraints (4.1). Hence (4.2) cannot hold.

Theorem 4.2:-

(Strong Duality) : Let $\overline{x}$ be a weakly efficient solution for (GFP). Then there exist $\overline{\lambda}\in R^P$, $\overline{\mu}\in R^m$ and $\overline{w}\in C$ such that $(\overline{x},\overline{\lambda},\overline{\mu},\overline{w},\overline{\tilde{w}})$ is feasible for (DGFP). Moreover, If the weak duality holds, then $(\overline{x},\overline{\lambda},\overline{\mu},\overline{w},\overline{\tilde{w}})$ is a weakly efficient solution for (DGFP)

Proof:-

Take $\overline{x}$ a wakly efficient solution for (GFP) and suppose that

$0\notin co\left\{\nabla h_j(\overline{x})/j=1,...m\right\}$.Then there exist $\overline{\lambda}\in R^P$, $\overline{\mu}\in R^m$ and

$\overline{w}_i\in C_i, \overline{\tilde{w}}_i\in D_i,\ i=1,....P$

Such that

$$\sum_{i=1}^{p}\lambda_i \nabla\left(\frac{f_i(\overline{x})+(w_i,\overline{x})}{g_i(\overline{x})-(\tilde{w}_i,\overline{x})}\right)+\sum_{j=1}^{m}\mu_j \nabla h_j(\overline{x})=0$$

$$(\overline{w}_i,\overline{x})\ =\ S(\overline{x}/C_i)\ ,\left(\overline{\tilde{w}}_i,\overline{x}\right)=S(\overline{x}/D_i)$$

$$\sum_{j=1}^{m}\overline{\mu}_j h_j(\overline{x})=0$$

$$\overline{\lambda}_i>0, i=1,....P,\ \sum_{i=1}^{p}\overline{\lambda}_i\ =1$$

Thus, $(\overline{x},\overline{\lambda},\overline{\mu},\overline{w},\overline{\tilde{w}})$ is a feasible solution for (DGFP). On the other hand, by weak duality (Theorem 4.1)

$$\max\left\{\frac{f_i(\overline{x})+S(\overline{x}/C_i)}{g_i(\overline{x})-S(\overline{x}/D_i)}/i=1,....P\right\}\geq \max\left\{\frac{f_i(u)+S(u/C_i)}{g_i(u)-S(u/D_i)}/i=1,....P\right\}$$

for any feasible solution $\left(x,\lambda,\mu,w,\tilde{w}\right)$ of (DGFP). Hence $(\overline{x},\overline{\lambda},\overline{\mu},\overline{w},\overline{\tilde{w}})$ is a weakly efficient solution for (DGFP).

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