Classification of motions of the 3-connected Harary graph on 7 vertices

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This paper is meant to be a supplement for a recently published paper by the same authors describing methods to find conditions on edge lengths for flexible instances of graphs. Using these methods, we fully classify the motions of the Harary graph $H_{3,7}$.

An edge labeling of a graph is called flexible if there are infinitely many non-congruent realizations of the graph in the plane such that the distance between adjacent vertices is equal to the corresponding label. The labeling is called proper if infinitely many of the realizations are injective.

In [5] a method is presented to find all proper flexible labelings of a given graph based on edge colorings [1, 4]. The graph which is of interest in this paper is one of the few interesting ones to investigate with seven vertices. It is named there by Q_1 but in general it is also known to be the Harary graph $H_{3,7}$ on 7 vertices being 3-connected with the minimal number of edges (see Figure 1).

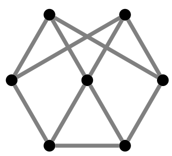

Figure 1: Harary graph $H_{3,7}$ also called Q_1 in this paper.

The goal of this paper is to classify all proper flexible labelings of the graph Q_1 using the tools developed in [5].

For notations, definitions and methods in use we refer to the main paper [5]. This paper describes general considerations on the motions of Q_1 in Section 1. The different motion families are then analyzed in detail in the following sections.

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1 Motion Families

At the beginning, we show that certain active NAC-colorings force the vertices of the unique triangle in Q_1 to be collinear, or some edge lengths must be equal.

Next, we determine consistent motion types and active NAC-colorings by the consistency check from [5, Section 4]. These are obtained by computer using our implementation of the method [2]. It appears that the triangle is actually degenerate for every consistent motion types. Then we obtain necessary algebraic conditions on λ from singleton NAC-colorings by the technique from [5, Section 2]. Using the computer algebra system `SageMath` [9], we identify six groups of motion types of 4-cycles, giving altogether eight motion families — irreducible algebraic sets of proper flexible labelings (see [8] for the computations and [3] for a detailed analysis). Animations of these motions can be found in [7].

In Section 8, we give a proof that the triangle is degenerate for every proper flexible labeling without using the consistent motion types, i.e., in computer-free manner. The proof was presented in [6].

Let $\Lambda \subset \mathbb{R}^{E_{Q_1}}$ be the set of all proper flexible labelings $\lambda : E_{Q_1} \to \mathbb{R}_+$ of Q_1. By *classification of motions*, or *proper flexible labelings* of Q_1 we mean the decomposition of the Zariski closure of Λ into irreducible algebraic sets $\Lambda_1, \ldots, \Lambda_k$, called motion families. We recall that the Zariski closure of Λ is the smallest algebraic set containing Λ.

Clearly, every proper flexible labeling is in some Λ_i, but not every $\lambda \in \Lambda_i$ is flexible — for instance, it is not guaranteed that it is realizable over $\mathbb{R}$, since this would require also inequalities. Notice also that a labeling in Λ_i does not have all edge lengths necessarily positive, but as long as they are not zero, the system

$$x_{\bar{u}} = y_{\bar{u}} = y_{\bar{v}} = 0\,,\; x_{\bar{v}} = \lambda_{\bar{u}\bar{v}}\,,\; (x_u - x_v)^2 + (y_u - y_v)^2 = \lambda_{uv}^2 \text{ for all } uv \in E_G \setminus \{\bar{u}\bar{v}\} \quad (1)$$

does not change due to taking squares, with the exception of the fixed edge — switching the sign of the fixed edge rotates the compatible realizations around the origin by 180°. There also might be a proper subvariety containing flexible labelings that are not proper.

Our goal is to provide equations defining the irreducible varieties and an instance for each of them that is proper flexible, namely, it has infinitely many non-congruent injective real realizations. The examples might have negative edge lengths of the degenerate triangle in order to handle all three possible arrangements of the collinear points using only one equation.

The NAC-colorings of Q_1, modulo conjugation, are shown in Figure 2. The figure also depicts the vertex labels we use in this section.

We start with the following lemma which states that the triangle $(5, 6, 7)$ is degenerate, or some edge lengths must be equal if a certain NAC-coloring is active. This is a direct corollary of [5, Proposition 2.2] but we give a direct proof here.

Lemma 1.1. *Let $\mathcal{C}$ be an algebraic motion of (Q_1, λ). If $\eta \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, resp. $\epsilon_{13} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, then the vertices of the triangle (5,6,7) are collinear, or $\lambda_{24} = \lambda_{23}$ and $\lambda_{14} = \lambda_{13}$, resp. $\lambda_{26} = \lambda_{67}$ and $\lambda_{24} = \lambda_{47}$.*

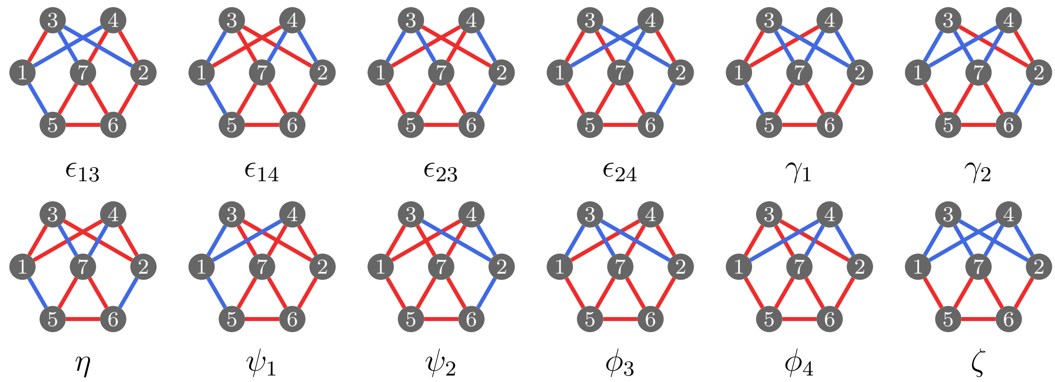


Figure 2: NAC-colorings of the graph Q_1.

Proof. The leading coefficient technique described in [5, Section 2] can be used, since the NAC-colorings η and ϵ_{13} are singletons. In case $\eta \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, the equation

$$\lambda_{57}^2 r^2 + \lambda_{67}^2 s^2 + \left(\lambda_{56}^2 - \lambda_{57}^2 - \lambda_{67}^2\right) rs = 0$$

is obtained, where $r = \lambda_{24}^2 - \lambda_{23}^2$ and $s = \lambda_{14}^2 - \lambda_{13}^2$. Considering the equation as a polynomial in r, the discriminant is

$$(\lambda_{56} + \lambda_{57} + \lambda_{67})(\lambda_{56} + \lambda_{57} - \lambda_{67})(\lambda_{56} - \lambda_{57} + \lambda_{67})(\lambda_{56} - \lambda_{57} - \lambda_{67}) s^2 \,.$$

But this is always negative or zero from the triangle inequality. Hence, the triangle must be degenerate, or $s = 0$. If $s = 0$, then also $r = 0$ and the statement follows. The proof for $\epsilon_{13} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$ is similar, since the obtained equation is

$$\lambda_{57}^2 (\underbrace{\lambda_{26}^2 - \lambda_{67}^2}_{r})^2 + \lambda_{67}^2 (\underbrace{\lambda_{24}^2 - \lambda_{47}^2}_{s})^2 + \left(\lambda_{56}^2 - \lambda_{57}^2 - \lambda_{67}^2\right)(\underbrace{\lambda_{26}^2 - \lambda_{67}^2}_{r})(\underbrace{\lambda_{24}^2 - \lambda_{47}^2}_{s}) \,.$$

□

Table 1 summarizes the output of the computation of consistent motion types using our implementation [2, 8]. The consistent motion types of 4-cycles and active NAC-colorings are given. The 4-cycles $H_1, \ldots, H_7$ are $(1,3,2,4)$, $(1,3,7,4)$, $(2,3,7,4)$, $(1,3,7,5)$, $(1,4,7,5)$, $(2,4,7,6)$, $(2,3,7,6)$ respectively. The number of isomorphic motion types is also indicated in the table. The column "Motion family" indicates to which family of proper flexible labelings of Q_1 a particular row belongs. The next column gives information about the dimension of this family. The last column provides the reference to the section where a particular motion family is elaborated.

2 Motion family I

We prove that Case (a) gives a 4-dimensional family of proper flexible labelings, with (b) and (c) being special cases. Although it might be seen rather easily that the triangle $(5,6,7)$ is degenerate, since the 4-cycles $(1,3,2,4)$, $(1,3,7,5)$ and $(2,4,7,6)$ are

	Motions types of $(H_1, \ldots, H_7)$	Active NAC-colorings	# isom.	Motion family	Dim.	Sec.
(a)	$\mathfrak{p}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{p}\ \mathfrak{g}\ \mathfrak{p}\ \mathfrak{g}$	$\{\epsilon_{13}, \epsilon_{24}, \eta\}$	2	I	4	
(b)	$\mathfrak{p}\ \mathfrak{o}\ \mathfrak{a}\ \mathfrak{p}\ \mathfrak{o}\ \mathfrak{p}\ \mathfrak{e}$	$\{\epsilon_{13}, \eta\}$	4	$\subset$ I, IV$_-$, V, VI	2	2
(c)	$\mathfrak{p}\ \mathfrak{e}\ \mathfrak{e}\ \mathfrak{p}\ \mathfrak{a}\ \mathfrak{p}\ \mathfrak{a}$	$\{\epsilon_{13}, \epsilon_{24}\}$	2	$\subset$ I, II, III	2	
(d)	$\mathfrak{o}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{g}$	$\{\epsilon_{ij}, \gamma_1, \gamma_2, \psi_1, \psi_2\}$	1	II$_-$ $\cup$ II$_+$	5	
(e)	$\mathfrak{p}\ \mathfrak{e}\ \mathfrak{e}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{g}$	$\{\epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24}\}$	1	$\subset$ II$_-$, II$_+$	4	3
(f)	$\mathfrak{o}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{p}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{a}$	$\{\epsilon_{13}, \epsilon_{24}, \gamma_1, \psi_2\}$	4	$\subset$ II$_-$	3	
(g)	$\mathfrak{o}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{e}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{e}$	$\{\epsilon_{13}, \epsilon_{23}, \gamma_1, \gamma_2\}$	2	$\subset$ II$_-$, deg.	2	
(h)	$\mathfrak{o}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{a}\ \mathfrak{g}\ \mathfrak{a}$	$\{\epsilon_{13}, \epsilon_{24}, \psi_1, \psi_2, \zeta\}$	2	III	3	4
(i)	$\mathfrak{g}\ \mathfrak{g}\ \mathfrak{a}\ \mathfrak{p}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{g}$	$\{\epsilon_{13}, \eta, \phi_4, \psi_2\}$	4	IV$_-$ $\cup$ IV$_+$	4	5
(j)	$\mathfrak{g}\ \mathfrak{g}\ \mathfrak{a}\ \mathfrak{e}\ \mathfrak{g}\ \mathfrak{p}\ \mathfrak{e}$	$\{\epsilon_{13}, \eta, \gamma_2, \phi_3\}$	4	V	3	6
(k)	$\mathfrak{p}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{e}\ \mathfrak{g}\ \mathfrak{g}\ \mathfrak{e}$	$\{\epsilon_{13}, \epsilon_{23}, \eta, \zeta\}$	2	VI	3	7

Table 1: The cases of consistent motion types and active NAC-colorings of Q_1.

parallelograms, and that the proper flexible labelings can be actually constructed by [1, Lemma 4.4] (see also Figure 4), we illustrate an approach that allows to deal also with the other cases.

In Case (a), the motion types enforce the following equalities of edge lengths:

$$\lambda_{13} = \lambda_{24} = \lambda_{57} = \lambda_{67}\,, \quad \lambda_{14} = \lambda_{23}\,, \quad \lambda_{15} = \lambda_{37}\,, \quad \lambda_{26} = \lambda_{47}\,.$$

They are also depicted by the same colors in Figure 3.

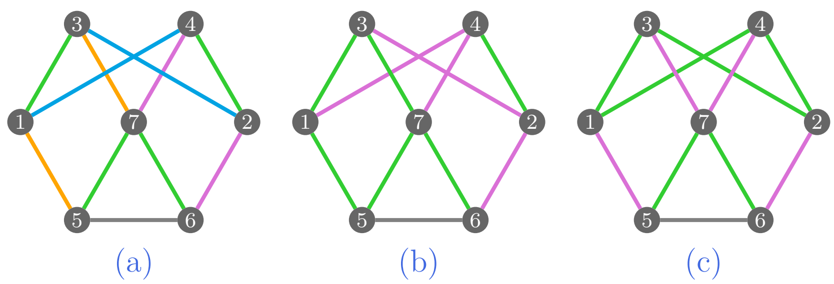


Figure 3: Edge lengths enforced by motion types (a)–(c): same color indicates equality of edge lengths.

Since the NAC-colorings $\epsilon_{13}, \epsilon_{24}$ and η are singletons, we can use the method of comparing leading coefficients described in [5, Section 2] to obtain the following equations

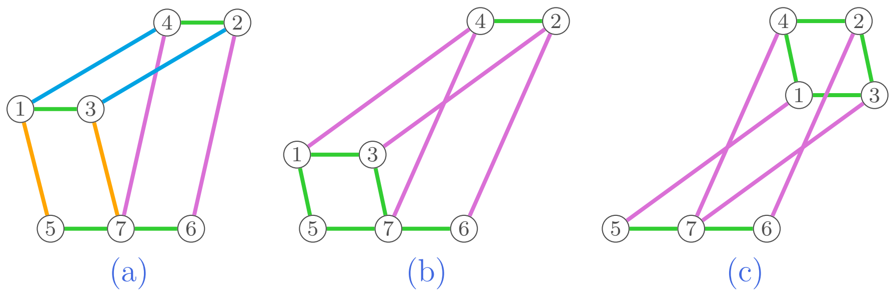


Figure 4: Realizations of Q_1 compatible with proper flexible labelings of family I.

for λ:

$$
\begin{aligned}
&(\lambda_{56}-2\lambda_{67})(\lambda_{56}+2\lambda_{67})(\lambda_{47}-\lambda_{67})(\lambda_{47}+\lambda_{67})=0 && (\epsilon_{13}\text{ active})\,,\\
&(\lambda_{56}-2\lambda_{67})(\lambda_{56}+2\lambda_{67})(\lambda_{37}-\lambda_{67})(\lambda_{37}+\lambda_{67})=0 && (\epsilon_{24}\text{ active})\,,\\
&(\lambda_{56}-2\lambda_{67})(\lambda_{56}+2\lambda_{67})(\lambda_{23}-\lambda_{67})(\lambda_{23}+\lambda_{67})=0 && (\eta\text{ active})\,.
\end{aligned} \tag{2}
$$

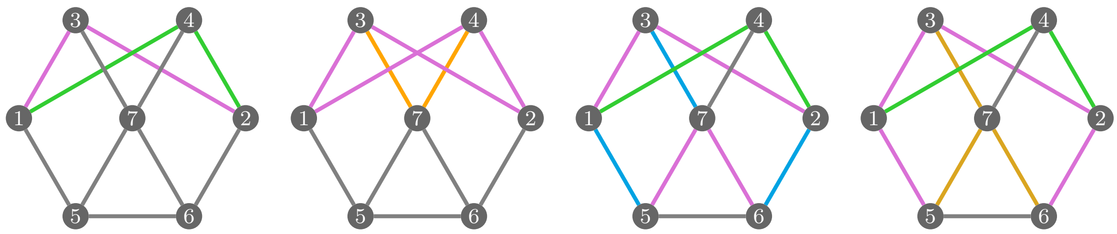
The second and fourth factor cannot vanish by the assumption that edge lengths are positive. If the first factor does not vanish, then $\lambda_{13}=\lambda_{24}=\lambda_{57}=\lambda_{67}=\lambda_{14}=\lambda_{23}=\lambda_{15}=\lambda_{37}=\lambda_{26}=\lambda_{47}$, which contradicts injective realizations (for instance the $K_{2,3}$ subgraph induced by $1,2,3,4$ and 7 has no injective realizations). Hence, we have that $\lambda_{56}=2\lambda_{67}=\lambda_{57}+\lambda_{67}$, i.e., the triangle $(5,6,7)$ is degenerate. We remark that instead of taking the factorizations, we could also argue by Lemma 8.1.

If we fix the vertices 5 and 6, then the system (1) has the following form

$$
\begin{aligned}
(x_1-x_3)^2+(y_1-y_3)^2 &= (x_2-x_4)^2+(y_2-y_4)^2=\lambda_{67}^2\\
(x_5-x_7)^2+(y_5-y_7)^2 &= (x_6-x_7)^2+(y_6-y_7)^2=\lambda_{67}^2\\
(x_1-x_4)^2+(y_1-y_4)^2 &= (x_2-x_3)^2+(y_2-y_3)^2=\lambda_{23}^2\\
(x_1-x_5)^2+(y_1-y_5)^2 &= (x_3-x_7)^2+(y_3-y_7)^2=\lambda_{37}^2\\
(x_2-x_6)^2+(y_2-y_6)^2 &= (x_4-x_7)^2+(y_4-y_7)^2=\lambda_{47}^2\\
x_5=y_5=y_6&=0\,, \qquad x_6=2\lambda_{67}\,.
\end{aligned}
$$

Considering $\lambda_{67},\lambda_{23},\lambda_{37}$ and λ_{47} also as variables, the zero set has dimension 5, as one can check by Gröbner basis computation. There are only 4 parameters, therefore if we fix $\lambda_{67},\lambda_{23},\lambda_{37}$ and λ_{47}, then there is a curve of solutions. Hence, the labeling is flexible whenever it is realizable. If the parameters are general enough, for instance pairwise distinct, then the labeling is proper flexible. A realization compatible with such an instance is shown in Figure 4.

In Case (b) and (c), the equations from comparing leading coefficients for corresponding active NAC-colorings always have the form of the first equation in (2). By the same reasons as before, the triangle $(5,6,7)$ is degenerate for every proper flexible labeling. Any flexible labeling enforced by motion types (b) or (c) and having the degenerate triangle satisfy also the equations of (a), i.e., (b) and (c) are subcases of (a). Figure 4 shows some examples.



(d) (e) (f) (g)

Figure 5: Edge lengths enforced by motion types (d)–(g): same color (except for gray) indicates equality of edge lengths.

3 Motion families II_- and II_+

If γ_1, resp. γ_2 is active, then the 4-cycle $H_3 = (2,3,7,4)$, resp. $H_2 = (1,3,7,4)$ has orthogonal diagonals by [5, Proposition 2.1]. Together with the motion types of H_1, H_2 and H_3, this implies that the vertices $1, 2$ ad 7 are collinear in all Cases (d)–(g).

The motion types in Cases (d)–(g) enforce $\lambda_{13} = \lambda_{23}$ and $\lambda_{14} = \lambda_{24}$ (see Figure 5). Assuming that the triangle $(5,6,7)$ is not degenerate, Lemma 8.1 for ϵ_{13} gives $\lambda_{24} = \lambda_{47}$. But this is a contradiction, since three distinct collinear points $1, 2$ and 7 cannot have the same distance to vertex 4. Hence, the triangle $(5,6,7)$ is degenerate. In order to avoid distinguishing the cases of which vertex is the middle one, we allow edge lengths in the triangle to be also negative. Then we can cover all three cases by the equation $\lambda_{56} = \lambda_{57} + \lambda_{67}$.

We focus now on Case (d). The active NAC-colorings $\epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24}, \gamma_1$ and γ_2 are singletons (ψ_1 and ψ_2 are not). The system of equations they provide together with the enforced edge lengths by the motion types has Gröbner basis:

$$\begin{aligned}
\left(\lambda_{26}^2\lambda_{57}^2 - \lambda_{15}^2\lambda_{67}^2\right) \cdot A(\lambda_{15}, \lambda_{24}, \lambda_{26}, \lambda_{47}, \lambda_{56}, \lambda_{57}, \lambda_{67}) &= 0\,,\\
\left(\lambda_{26}^2\lambda_{57}^2 - \lambda_{15}^2\lambda_{67}^2\right) \cdot B(\lambda_{15}, \lambda_{24}, \lambda_{26}, \lambda_{47}, \lambda_{57}, \lambda_{67}) &= 0\,,\\
D(\lambda_{15}, \lambda_{24}, \lambda_{47}, \lambda_{56}, \lambda_{57}, \lambda_{67}) &= 0\,,\\
E(\lambda_{24}, \lambda_{26}, \lambda_{47}, \lambda_{56}, \lambda_{57}, \lambda_{67}) &= 0\,,\\
\lambda_{13} = \lambda_{23}\,, \quad \lambda_{14} = \lambda_{24}\,, \quad \lambda_{24}^2 + \lambda_{37}^2 &= \lambda_{23}^2 + \lambda_{47}^2
\end{aligned}$$

where A, B, C, D, E are polynomials in λ_{ij}. Assume that the first factor vanishes. We introduce a new variable u, add the equations $\lambda_{56} = \lambda_{57} + \lambda_{67}$ and $(\lambda_{26}^2\lambda_{57}^2 - \lambda_{15}^2\lambda_{67}^2) \cdot u = 1$ and eliminate the variable u. Then the ideal is $\langle 1 \rangle$, namely it is not possible that $(\lambda_{26}^2\lambda_{57}^2 - \lambda_{15}^2\lambda_{67}^2)$ does not vanish. Therefore, we add the factor and the equation for the degenerate triangle to the Gröbner basis. After eliminating the case $\lambda_{67} = 0$, we obtain:

$$\begin{aligned}
-\lambda_{24}^4 + \lambda_{15}^2\lambda_{26}^2 + 2\lambda_{24}^2\lambda_{47}^2 - \lambda_{47}^4 - 2\lambda_{15}^2\lambda_{67}^2 + \lambda_{57}^2\lambda_{67}^2 &= 0\,,\\
\lambda_{24}^2\lambda_{57} - \lambda_{47}^2\lambda_{57} + \lambda_{15}^2\lambda_{67} - \lambda_{57}^2\lambda_{67} &= 0\,,\\
\lambda_{26}^2\lambda_{57} + \lambda_{24}^2\lambda_{67} - \lambda_{47}^2\lambda_{67} - \lambda_{57}\lambda_{67}^2 &= 0\,,\\
\lambda_{13} = \lambda_{23}\,, \quad \lambda_{14} = \lambda_{24}\,, \quad \lambda_{24}^2 + \lambda_{37}^2 = \lambda_{23}^2 + \lambda_{47}^2\,, \quad \lambda_{57} + \lambda_{67} &= \lambda_{56}\,.
\end{aligned}$$

The zero set of these equations is not irreducible. The two irreducible components are given by the following equations for $\alpha \in \{-1, 1\}$:

$$\begin{aligned}
-\lambda_{24}^2 + \alpha\lambda_{15}\lambda_{26} + \lambda_{47}^2 + \lambda_{57}\lambda_{67} &= 0\,,\\
\lambda_{26}\lambda_{57} + \alpha\lambda_{15}\lambda_{67} &= 0\,,\\
\lambda_{13} = \lambda_{23}\,, \quad \lambda_{14} = \lambda_{24}\,, \quad \lambda_{24}^2 + \lambda_{37}^2 = \lambda_{23}^2 + \lambda_{47}^2\,, \quad \lambda_{57} + \lambda_{67} &= \lambda_{56}\,.
\end{aligned}$$

To indicate α, we denote the components by II_- and II_+. We recall that the second equation is consistent also in II_+, since we allow negative edge lengths for the edges in the triangle $(5, 6, 7)$. The dimension of both varieties is 5. If we construct the system of equations for vertex coordinates, taking the λ_{ij} as variables, the dimension is 6. Therefore, a generic fiber of the projection from the whole zero set to II_-, resp. II_+, has positive dimension. Hence, a generic λ in $\mathrm{II}_- \cup \mathrm{II}_+$ that is realizable is flexible.

Before we provide an example of a proper flexible labeling for II_- and II_+, let us remark that using an analogous approach, one can show that Case (e) gives also two irreducible components, each of them being a subcase of II_- or II_+. Case (f) yields an irreducible component, which is a subvariety of II_-. This is the case also for (c). Labelings which satisfy equations given by motion types, active NAC-colorings and the degenerate triangle in Case (g) form a subvariety of II_-. Nevertheless, they are not proper. Assuming a non-degenerate motion, the deltoids and active NAC-colorings γ_1 and γ_2 imply that $1, 2, 7$ and $3, 4, 5, 6$ are collinear, respectively. These lines are perpendicular, but since there is also for instance an edge between 5 and 6, no motion is possible.

Case (g) shows that not all labelings in II_- are proper. The labelings which are not proper form a subset of an algebraic variety. Theoretically, the equations for this variety can be obtained by adding the equations for coordinates of coinciding vertices into the system (1) and eliminating the coordinate variables. Since, II_-, resp. II_+, are irreducible, giving an example of a proper flexible labeling implies that the subvariety of non-proper labelings is strict. In particular, a generic realizable labeling in II_-, resp. II_+, is proper. An example of a proper flexible labeling in II_+ is

$$\begin{aligned}
&\lambda_{13} = \lambda_{23} = 14\,, && \lambda_{15} = 9\,, && \lambda_{26} = 12\,, && \lambda_{37} = 10\,, && \lambda_{47} = 5\,,\\
&\lambda_{14} = \lambda_{24} = 11\,, && \lambda_{56} = 1\,, && \lambda_{57} = -3\,, && \lambda_{67} = 4\,.
\end{aligned}$$

The algebraic motion can be parametrized by

$$
\begin{aligned}
(x_1, y_1) &= \left(-\frac{9(t^2-1)}{t^2+1}, \frac{18t}{t^2+1}\right) ,\\
(x_2, y_2) &= \left(\frac{13t^2-44}{t^2+4}, -\frac{48t}{t^2+4}\right) ,\\
(x_3, y_3) &= \left(\frac{2t^4-3s(t)_1t-29t^2-4}{t^4+5t^2+4}, -\frac{15t^3+(t^2-2)s(t)_1-12t}{t^4+5t^2+4}\right) ,\\
(x_4, y_4) &= \left(\frac{2t^4-3s(t)_2t-29t^2-4}{t^4+5t^2+4}, -\frac{15t^3+(t^2-2)s(t)_2-12t}{t^4+5t^2+4}\right) ,\\
(x_5, y_5) &= (0,0) \ , \quad (x_6, y_6) = (1,0) \ , \quad (x_7, y_7) = (-3,0) \ ,
\end{aligned}
$$

where

$$s_1(t) = \pm\sqrt{3(5t^2+32)(5t^2+4)} \quad \text{and} \quad s_2(t) = \pm\sqrt{165t^2+84}\,.$$

There are two connected components, one is parametrized by taking $(+,+)$ and $(+,-)$ for s_1 and s_2, the other by taking $(-,-)$ and $(-,+)$. They are symmetric by reflection w.r.t. the line of the degenerate triangle. Figure 6 illustrates a part of the motion, with positive s_1 and s_2.

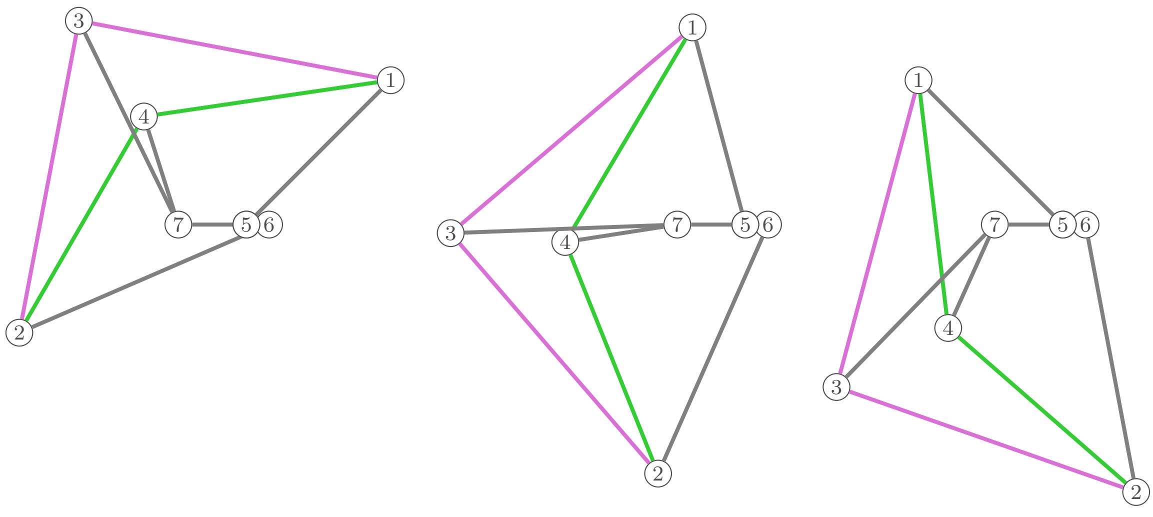


Figure 6: Example of a motion of family II_+.

An example of a proper flexible labeling in II_- is

$$
\begin{array}{lllll}
\lambda_{13} = \lambda_{23} = 10\,, & \lambda_{15} = 9\,, & \lambda_{26} = 12\,, & \lambda_{37} = 14\,, & \lambda_{47} = 11\,,\\
\lambda_{14} = \lambda_{24} = 5\,, & \lambda_{56} = 7\,, & \lambda_{57} = 3\,, & \lambda_{67} = 4\,. &
\end{array}
$$

A parametrization of the motion is

$$
\begin{aligned}
(x_1, y_1) &= \left(-\frac{9(t^2-1)}{t^2+1}, \frac{18t}{t^2+1}\right), \\
(x_2, y_2) &= \left(-\frac{20t^2-19}{4t^2+1}, \frac{48t}{4t^2+1}\right), \\
(x_3, y_3) &= \left(-\frac{28t^4+3s_1(t)t-13t^2-14}{4t^4+5t^2+1}, \frac{60t^3-(2t^2-1)s_1(t)+33t}{4t^4+5t^2+1}\right), \\
(x_4, y_4) &= \left(-\frac{28t^4+3s_2(t)t^2-13t^2-14}{4t^4+5t^2+1}, \frac{60t^3-(2t^3-t)s_2(t)+33t}{4t^4+5t^2+1}\right), \\
(x_5, y_5) &= (0,0)\,, \quad (x_6, y_6) = (7,0)\,, \quad (x_7, y_7) = (3,0)\,,
\end{aligned}
$$

where

$$
s_1(t) = \pm\sqrt{3\,(32t^2+5)\,(4t^2+5)} \quad \text{and} \quad s_2(t) = \pm\sqrt{84t^2+165}\,.
$$

There are again two connected components, which are symmetric to each other by reflection. Figure 7 illustrates a part of the motion, with positive s_1 and negative s_2.

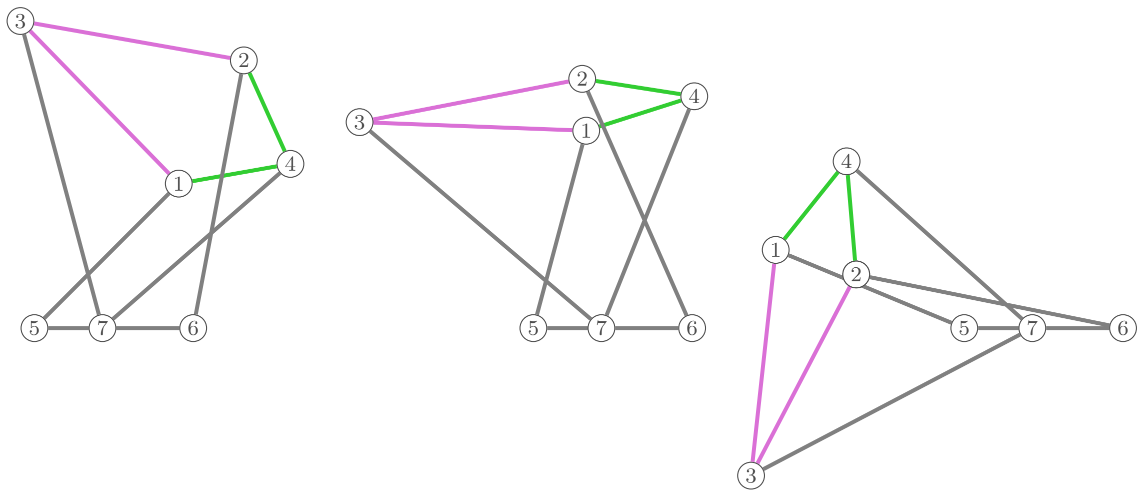


Figure 7: Example of a motion of family II_-.

4 Motion family III

The edge lengths enforced by the motion types in Case (h) are depicted in Figure 8 (h). The triangle $(5, 6, 7)$ is degenerate, otherwise the active NAC-coloring ϵ_{13} implies that $\lambda_{26} = \lambda_{67}$ by Lemma 1.1. But this contradicts that the 4-cycle $H_7 = (2, 3, 7, 6)$ is an antiparallelogram.

We consider the ideal generated by the edge lengths equalities, active NAC-colorings ϵ_{13}, ϵ_{24} and ζ, which are singletons w.r.t. all active NAC-colorings and the equation for

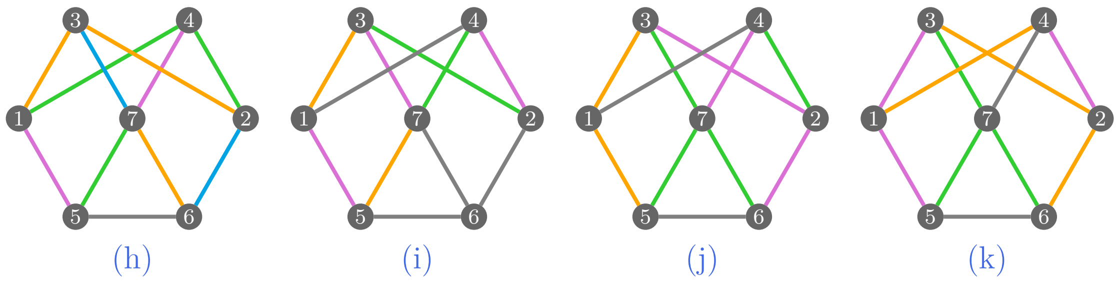


Figure 8: Edge lengths enforced by Cases (h)–(k): same color (except for gray) indicates equality of edge lengths.

the degenerate triangle. Notice that ζ is not singleton in general. The ideal has the following Gröbner basis:

$$(\lambda_{37}^2\lambda_{57} - \lambda_{47}^2\lambda_{67} + \lambda_{57}^2\lambda_{67} - \lambda_{57}\lambda_{67}^2)^2 = 0\,,$$
$$\lambda_{23} = \lambda_{13} = \lambda_{67}\,, \quad \lambda_{24} = \lambda_{14} = \lambda_{57}\,, \quad \lambda_{15} = \lambda_{47}\,, \quad \lambda_{26} = \lambda_{37}\,, \quad \lambda_{56} = \lambda_{57} + \lambda_{67}\,.$$

Clearly, we can skip taking the square in the first equation. Then the ideal is prime, i.e., the variety is irreducible. It has dimension 3, whereas the system for vertex coordinates enriched with the equations above has dimension 4. Hence, a generic realizable labeling is flexible. An example of a proper flexible labeling is

$$\begin{aligned} &\lambda_{23} = \lambda_{13} = \lambda_{67} = 10\,, && \lambda_{15} = \lambda_{47} = 2\,, && \lambda_{56} = 18\,,\\ &\lambda_{24} = \lambda_{14} = \lambda_{57} = 8\,, && \lambda_{26} = \lambda_{37} = 5\,. \end{aligned}$$

The motion can be parametrized by

$$\begin{aligned}
(x_1, y_1) &= \left(-\frac{2(t^2-1)}{t^2+1}, \frac{4t}{t^2+1}\right)\,,\\
(x_2, y_2) &= \left(\frac{68125t^4 - 25ts(t) + 39675t^2 + 4734}{8(625t^4+325t^2+36)}, -\frac{5(875t^3 - (25t^2+6)s(t) + 75t)}{8(625t^4+325t^2+36)}\right)\,,\\
(x_3, y_3) &= \left(\frac{675t^4 - 2s(t)t + 1208t^2 + 405}{4(25t^4+34t^2+9)}, \frac{50t^3 - (5t^2+3)s(t) - 78t}{4(25t^4+34t^2+9)}\right)\,,\\
(x_4, y_4) &= \left(\frac{30(5t^2+3)}{25t^2+9}, -\frac{60t}{25t^2+9}\right)\,,\\
(x_5, y_5) &= (0,0)\ , \quad (x_6, y_6) = (18,0)\ , \quad (x_7, y_7) = (8,0)\ ,
\end{aligned}$$

where $s(t) = \pm\sqrt{(125t^2+189)(75t^2+11)}$. Each sign determines one connected component of the motion. Figure 9 shows three realizations of the component for positive s.

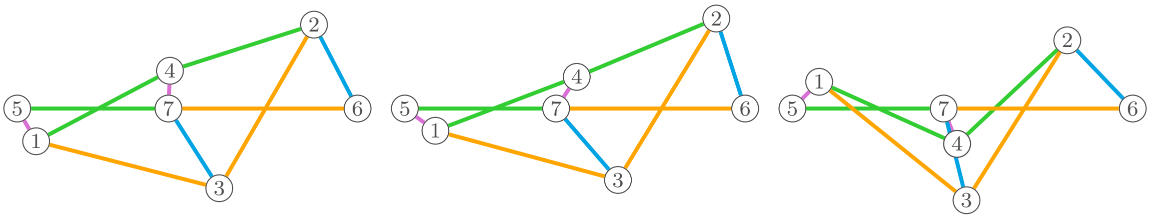


Figure 9: Example of a motion of family III.

5 Motion families IV_- and IV_+

The edge lengths enforced by the motion types in Case (i) are depicted in Figure 8. The triangle $(5,6,7)$ is degenerate, otherwise the active NAC-coloring ϵ_{13} implies that $\lambda_{24} = \lambda_{47}$ by Lemma 1.1. But this contradicts that the 4-cycle $H_3 = (2,3,7,4)$ is an antiparallelogram.

The irreducible components IV_- and IV_+ of the zero set of the ideal generated by the equalities of edge lengths, triangle equality and the equations implied by the active NAC-colorings $\epsilon_{13}, \eta, \phi_4$ and ψ_2, which are all singletons w.r.t. themselves, are described by

$$\begin{aligned} \lambda_{37}^2\lambda_{57} - \lambda_{47}^2\lambda_{57} + \lambda_{14}^2\lambda_{67} - \lambda_{57}^2\lambda_{67} &= 0\,, \\ \lambda_{47}^2 + \alpha\lambda_{14}\lambda_{26} - \lambda_{37}^2 + \lambda_{57}\lambda_{67} &= 0\,, \\ \lambda_{26}\lambda_{57} + \alpha\lambda_{14}\lambda_{67} &= 0\,, \\ \lambda_{13} = \lambda_{57}\,, \quad \lambda_{24} = \lambda_{15} = \lambda_{37}\,, \quad \lambda_{23} = \lambda_{47}\,, \quad \lambda_{56} = \lambda_{57} + \lambda_{67}\,, \end{aligned}$$

where $\alpha \in \{-1, 1\}$. The dimension of IV_- and IV_+ is 4, whereas the dimension of the systems for vertex coordinates is 5, which again proves that a generic realizable λ is flexible. Computation shows that Case (b) is a subvariety of IV_-. An example of a proper flexible labeling in IV_- is

$$\begin{aligned} &\lambda_{15} = \lambda_{24} = \lambda_{37} = 8\,, && \lambda_{14} = 14\,, && \lambda_{26} = 10\,, && \lambda_{56} = 12\,, \\ &\lambda_{13} = \lambda_{57} = 7\,, && \lambda_{23} = \lambda_{47} = 13\,, && \lambda_{67} = 5\,. \end{aligned}$$

A parametrization of the motion is

$$\begin{aligned} (x_1, y_1) &= \left(-\frac{8(t^2-1)}{t^2+1}, \frac{16t}{t^2+1}\right)\,, \\ (x_2, y_2) &= \left(\frac{2(689t^4 - 320t^2 s(t) + 1338t^2 + 9)}{169t^4 + 178t^2 + 9}, \frac{40(20t^3 - (13t^3 - 3t)s(t) - 12t)}{169t^4 + 178t^2 + 9}\right)\,, \\ (x_3, y_3) &= \left(-\frac{t^2-15}{t^2+1}, \frac{16t}{t^2+1}\right)\,, \\ (x_4, y_4) &= \left(\frac{2(45t^4 - 448t^2 s(t) + 938t^2 - 3)}{225t^4 + 226t^2 + 1}, \frac{8(198t^3 - 7(15t^3 - t)s(t) - 26t)}{225t^4 + 226t^2 + 1}\right)\,, \\ (x_5, y_5) &= (0,0)\,, \quad (x_6, y_6) = (12, 0)\,, \quad (x_7, y_7) = (7, 0)\,, \end{aligned}$$

where $s(t) = \pm\sqrt{9t^2 + 13}$. It has only one connected component. Figure 10 shows three realizations from the motion. An example of a proper flexible labeling in IV_+ is

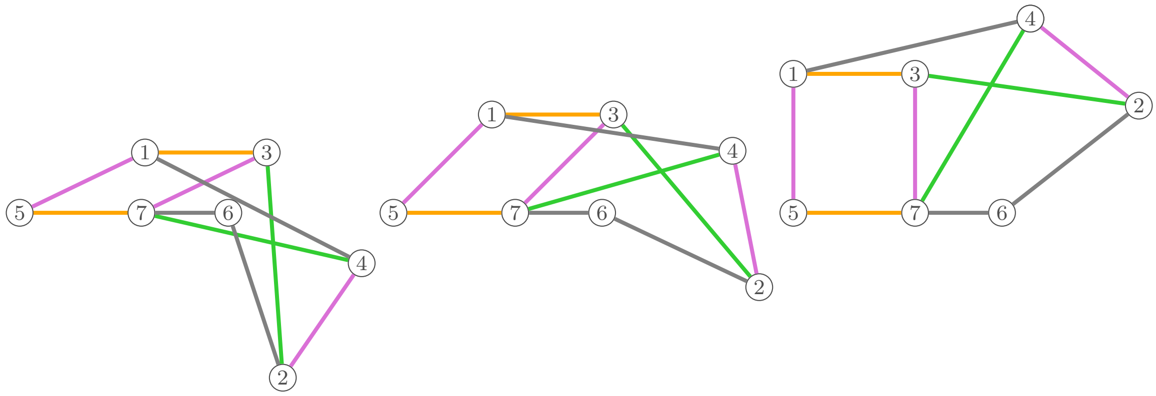


Figure 10: Example of a motion of family IV_-.

$$
\begin{aligned}
&\lambda_{15} = \lambda_{24} = \lambda_{37} = 13\,, && \lambda_{14} = 14\,, && \lambda_{26} = 10\,, && \lambda_{56} = 2\,,\\
&\lambda_{13} = \lambda_{57} = 7\,, && \lambda_{23} = \lambda_{47} = 8\,, && \lambda_{67} = -5\,.
\end{aligned}
$$

A parametrization of the motion is

$$
\begin{aligned}
(x_1, y_1) &= \left(-\frac{13(t^2-1)}{t^2+1}, \frac{26t}{t^2+1}\right)\,,\\
(x_2, y_2) &= \left(-\frac{68t^4 + 65s(t)t^2 - 59t^2 - 972}{16t^4 + 97t^2 + 81}, \frac{5(65t^3 - (4t^3 - 9t)s(t) + 234t)}{16t^4 + 97t^2 + 81}\right)\,,\\
(x_3, y_3) &= \left(-\frac{2(3t^2-10)}{t^2+1}, \frac{26t}{t^2+1}\right)\,,\\
(x_4, y_4) &= \left(\frac{30t^4 + 91s(t)t^2 + 1204t^2 - 9}{100t^4 + 109t^2 + 9}, \frac{871t^3 + 7(10t^3 - 3t)s(t) - 312t}{100t^4 + 109t^2 + 9}\right)\,,\\
(x_5, y_5) &= (0,0)\,, \quad (x_6, y_6) = (2,0)\,, \quad (x_7, y_7) = (7,0)\,,
\end{aligned}
$$

where $s(t) = \pm\sqrt{39t^2 + 208}$. Figure 11 shows three realizations from the motion.

6 Motion family V

Case (j) has the triangle degenerate by the same reason as (i). The obtained system of equations for edge lengths is

$$
\begin{gathered}
\lambda_{15}^2 + \lambda_{47}^2 = \lambda_{14}^2 + \lambda_{67}^2\,,\\
\lambda_{13} = \lambda_{15}\,, \quad \lambda_{23} = \lambda_{26} = \lambda_{47}\,, \quad \lambda_{24} = \lambda_{57} = \lambda_{37} = \lambda_{67}\,, \quad \lambda_{56} = 2\lambda_{67}\,.
\end{gathered}
$$

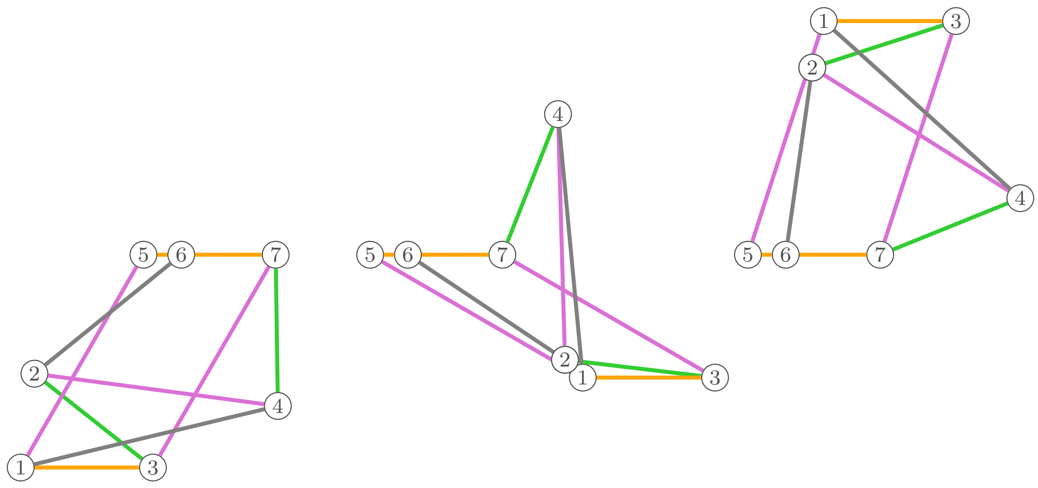


Figure 11: Example of a motion of family IV_+.

The dimension is 3. A generic realizable labeling is flexible, since the dimension of the system for coordinates is one higher. Case (b) is again a subcase. An example of a proper flexible labeling is

$$\lambda_{13} = \lambda_{15} = 6\,, \quad \lambda_{14} = 9\,, \quad \lambda_{23} = \lambda_{26} = \lambda_{47} = 7\,, \quad \lambda_{24} = \lambda_{37} = \lambda_{57} = \lambda_{67} = 2\,, \quad \lambda_{56} = 4\,.$$

The motion is parametrized by

$$\begin{aligned}
(x_1, y_1) &= \left(-\frac{6(t^2-1)}{t^2+1}, \frac{12t}{t^2+1}\right)\,,\\
(x_2, y_2) &= \left(\frac{8t^4 + 9s(t)t + 28t^2 + 2}{4t^4+5t^2+1}, \frac{3(4t^3 + (2t^2-1)s(t) - 2t)}{4t^4+5t^2+1}\right)\,,\\
(x_3, y_3) &= \left(\frac{36t^2}{4t^4+5t^2+1}, \frac{12(2t^3-t)}{4t^4+5t^2+1}\right)\,,\\
(x_4, y_4) &= \left(\frac{9(s(t)t+2t^2)}{4t^4+5t^2+1}, \frac{3(4t^3 + (2t^2-1)s(t) - 2t)}{4t^4+5t^2+1}\right)\,,\\
(x_5, y_5) &= (0, 0)\,, \quad (x_6, y_6) = (4, 0)\,, \quad (x_7, y_7) = (2, 0)\,,
\end{aligned}$$

where $s(t) = \pm\sqrt{(5t^2+1)(4t^2+5)}$. Figure 12 shows three realizations of one of the two connected components.

7 Motion family VI

The edge lengths enforced by the motion types in Case (k) are depicted in Figure 8. The triangle $(5, 6, 7)$ is degenerate, otherwise the active NAC-coloring ϵ_{13} implies that $\lambda_{26} = \lambda_{67}$ by Lemma 1.1. But this contradicts that the 4-cycle $H_7 = (2, 3, 7, 6)$ is an even deltoid.

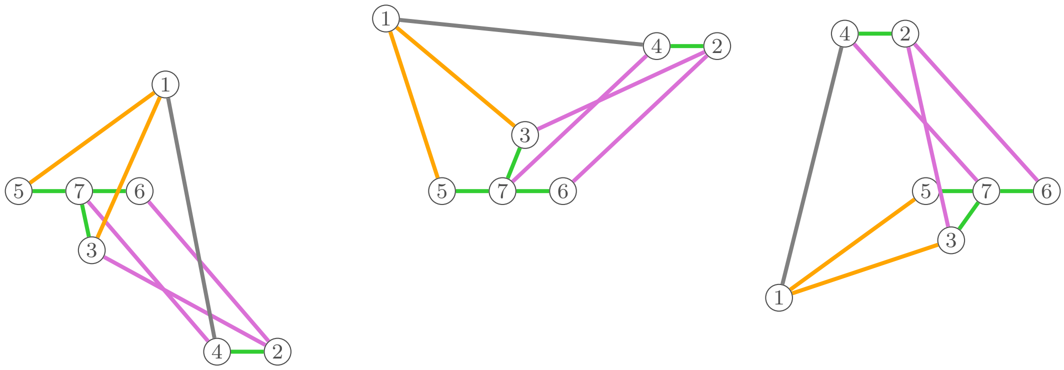


Figure 12: Example of a motion of family V.

In order to get the system of equations for λ, we again take the equalities of edge lengths, equations from the active NAC-colorings and triangle equality. The case $\lambda_{67} = 0$ must be excluded using elimination. By taking the radical ideal, we get a prime ideal generated by

$$-\lambda_{24}^2 - \lambda_{26}^2 + \lambda_{47}^2 + \lambda_{67}^2 = 0\,,$$
$$\lambda_{13} = \lambda_{15} = \lambda_{24}\,, \quad \lambda_{14} = \lambda_{23} = \lambda_{26}\,, \quad \lambda_{57} = \lambda_{37} = \lambda_{67}\,, \quad \lambda_{56} = 2\lambda_{67}\,.$$

The dimension is 3, a generic realizable labeling is flexible and Case (b) is a subcase. An instance of VI is

$$\lambda_{13} = \lambda_{15} = \lambda_{24} = 7\,, \quad \lambda_{14} = \lambda_{23} = \lambda_{26} = 6\,, \quad \lambda_{57} = \lambda_{37} = \lambda_{67} = 2\,, \quad \lambda_{47} = 9\,, \quad \lambda_{56} = 4\,.$$

The motion can be parametrized by

$$\begin{aligned}
(x_1, y_1) &= \left(-\frac{7(t^2-1)}{t^2+1}, \frac{14t}{t^2+1}\right)\,,\\
(x_2, y_2) &= \left(\frac{2(81t^4 + 28s(t)t + 302t^2 + 25)}{81t^4 + 106t^2 + 25}, \frac{4(63t^3 + (9t^2-5)s(t) - 35t)}{81t^4 + 106t^2 + 25}\right)\,,\\
(x_3, y_3) &= \left(\frac{784t^2}{81t^4 + 106t^2 + 25}, \frac{56(9t^3 - 5t)}{81t^4 + 106t^2 + 25}\right)\,,\\
(x_4, y_4) &= \left(-\frac{405t^4 - 56s(t)t - 212t^2 - 225}{81t^4 + 106t^2 + 25}, \frac{2(441t^3 + 2(9t^2-5)s(t) + 245t)}{81t^4 + 106t^2 + 25}\right)\,,\\
(x_5, y_5) &= (0, 0)\;, \quad (x_6, y_6) = (4, 0)\;, \quad (x_7, y_7) = (2, 0)\;,
\end{aligned}$$

where $s(t) = \pm\sqrt{(18t^2+25)(9t^2+2)}$. It has two connected components. One of them is illustrated in Figure 13.

To conclude the classification, we also have to check that the varieties $\mathrm{I}, \ldots, \mathrm{VI}$ are not subvarieties of each other. To see that none of them is a subvariety of $\mathrm{II}_- \cup \mathrm{II}_+$, we directly check ideal containment using the equations from the previous sections, since

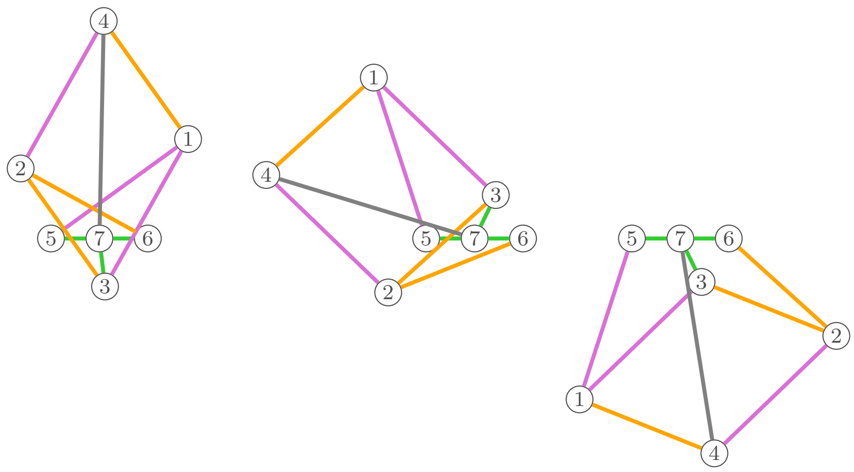


Figure 13: Example of a motion of family VI.

Case (d) has no isomorphic motion types (see [8]). Motion families III–VI are not special cases of I because they do not have three parallelograms. Clearly, III, V and VI are not contained in each other due to irreducibility and same dimension. To check that they are not subcases of $\mathrm{IV}_{-} \cup \mathrm{IV}_{+}$, we check ideal containment for all isomorphic cases of IV (see [8]).

8 Degenerate triangle $(5, 6, 7)$

Interestingly, we can prove that the triangle $(5, 6, 7)$ is degenerate for every proper flexible labeling of Q_1 even without the method deriving possible active NAC-colorings. The following approach was presented in [6].

Lemma 1.1 allows to give some conditions on active NAC-colorings of motions with injective realizations, under the assumption that the triangle $(5, 6, 7)$ is not degenerate. Then, we conclude that this assumption actually always yields a contradiction.

Lemma 8.1. *Let $\mathcal{C}$ be an algebraic motion of (Q_1, λ) such that λ is a proper flexible labeling and the vertices 5,6,7 are not collinear. If $\epsilon_{13} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, then*

(i) $\epsilon_{14}, \gamma_1, \gamma_2, \psi_1, \psi_2, \phi_4, \zeta \notin \mathrm{NAC}_{Q_1}(\mathcal{C})$,

(ii) either $\epsilon_{23} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$ *or* $\epsilon_{24} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$*, and*

(iii) $\eta \in \mathrm{NAC}_{Q_1}(\mathcal{C})$.

Proof. By the assumption and Lemma 1.1, if $\epsilon_{13} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, then $\lambda_{26} = \lambda_{67}$ and $\lambda_{24} = \lambda_{47}$. But then the 4-cycle $(2, 4, 7, 6)$ is an odd deltoid or rhombus. Hence, the restriction of any active NAC-coloring to $(2, 4, 7, 6)$ cannot be of type R (recall [5, Table 1]), i.e., $\epsilon_{14}, \gamma_1, \phi_4, \zeta \notin \mathrm{NAC}_{Q_1}(\mathcal{C})$ by Table 2. Since the 4-cycle $(2, 3, 7, 4)$ cannot be an antiparallelogram, there must be an active NAC-coloring whose restriction is of type O, namely, $\epsilon_{23} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$ or $\epsilon_{24} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$. Since ϵ_{13} excludes ϵ_{14} to be active, ϵ_{23} excludes ϵ_{24} by graph symmetry. This gives *(ii)*. By the symmetric approach to the fact that ϵ_{13} excludes γ_1, we also get that both ϵ_{23} and ϵ_{24} prohibit γ_2 to be active. Since the 4-cycle $(2,4,7,6)$, resp. $(2,3,7,6)$, is not an antiparallelogram, there must be an active NAC-coloring restricting to O, namely ϵ_{24} or η, resp. ϵ_{23} or η (γ_2 is already excluded). In the combination with the previous, we get *(iii)*. Therefore, $\lambda_{24} = \lambda_{23}$ and $\lambda_{14} = \lambda_{13}$ by Lemma 1.1. This shows that the 4-cycle $(1,3,2,4)$ is an even deltoid or rhombus which prohibits L. Thus, $\psi_1, \psi_2 \notin \mathrm{NAC}_{Q_1}(\mathcal{C})$, which finishes *(i)*. □

4-cycle	ϵ_{13}	ϵ_{14}	ϵ_{23}	ϵ_{24}	γ_1	γ_2	η	ψ_1	ψ_2	ϕ_3	ϕ_4	ζ
$H_1 = (1,3,2,4)$	O	O	O	O	L	L	S	L	L	R	R	S
$H_2 = (1,3,7,4)$	O	O	R	R	L	S	L	L	S	R	R	S
$H_3 = (2,3,7,4)$	R	R	O	O	S	L	L	S	L	R	R	S
$H_4 = (1,3,7,5)$	O	L	R	S	O	R	O	L	S	R	S	R
$H_5 = (1,4,7,5)$	L	O	S	R	O	R	O	L	S	S	R	R
$H_6 = (2,4,7,6)$	S	R	L	O	R	O	O	S	L	S	R	R
$H_7 = (2,3,7,6)$	R	S	O	L	R	O	O	S	L	R	S	R

Table 2: Types of NAC-colorings of Q_1 restricted to 4-cycles using the notation from [5, Figure 5] and S meaning all edges have the same color.

We conclude that the triangle $(5,6,7)$ in Q_1 is always degenerate.

Theorem 8.2. *If $\mathcal{C}$ is an algebraic motion of Q_1 with infinitely many injective realization, then the vertices $5,6$ and 7 are always collinear.*

Proof. If no ϵ_{ij} is active, then the 4-cycles $(1,3,2,4)$, $(1,3,7,4)$ and $(2,3,7,4)$ are all antiparallelograms. But this is not possible for injective realizations. Hence, by symmetry we can assume w.l.o.g. that ϵ_{13} is active. Suppose by contradiction that the triangle $(5,6,7)$ is not degenerate. By Lemma 8.1 (and its symmetric version for ϵ_{24}), the only possibilities for $\mathrm{NAC}_{Q_1}(\mathcal{C})$ are $\{\epsilon_{13}, \epsilon_{23}, \eta\}$, $\{\epsilon_{13}, \epsilon_{23}, \eta, \phi_3\}$ and $\{\epsilon_{13}, \epsilon_{24}, \eta\}$. In all cases, the motion types of 4-cycles given by the active NAC-colorings and Lemma 1.1 enforce the edge lengths to be the same for all edges in the $K_{2,3}$ subgraph induced by the vertices $1,2,3,4$ and 7. This contradicts a non-degenerate motion. □

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