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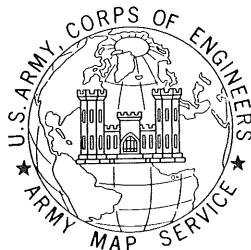
JORDAN'S HANDBOOK OF GEODESY

(JORDAN - EGGERT: HANDBUCH DER VERMESSUNGSKUNDE)

VOLUME III

SECOND HALF

SELECTED PORTIONS

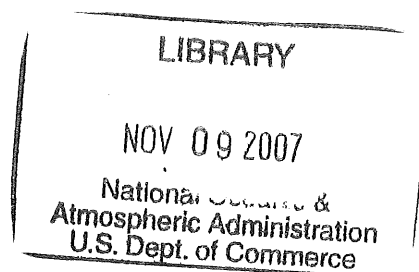


English Translation

by

Martha W. Carta

1962



CORPS OF ENGINEERS, U. S. ARMY

ARMY MAP SERVICE

Washington 25, D. C.

202973

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NOTE

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be disregarded.

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Chapter I

NORMAL SECTIONS AND GEODETIC LINE

Section 1. Counter-Normal Sections

In the first half-volume we have carried out all computations necessary for a triangulation on a sphere whose radius was assumed according to the mean geographic latitude of the territory in question. Now if we pass over to the computations on the terrestrial ellipsoid, then we begin in this chapter with a study of the special fundamental concepts valid for the ellipsoid of rotation.

The most essential difference in comparison to the sphere consists for the ellipsoid in the fact that between two points of its surface *two* normal sections exist in general.

Let in Fig. 1 A and B be two points of the ellipsoid of rotation, at different latitudes, and let AK_a be the normal to the surface at the point A , as well as BK_b the normal to the surface at the point B ; we notice here at first that the points at which the axis of rotation is hit by these normals, i.e. K_a and K_b , do *not* coincide in the case of an oblique section.

Just as we have in general *two* intersection points of axes K_a and K_b , there exist also *two* intersection planes of the normal which both pass through the *normals* of the points A and B .

More exactly, we have:

The intersection plane of the normal at the point A is that plane which passes through the normal AK_a and through the point B ; let this plane intersect the surface of the ellipsoid in an arc AaB . On the other hand, as the intersection plane of the normal at the point B we have that plane which passes through the normal BK_b and through the point A and intersects the surface of the ellipsoid in the arc BbA .

There are special cases in which the two normal sections between two points coincide:

First. For any two points which lie on the same meridian, this meridian is also the normal section in a twofold sense. There is included here also the special case that one of the two points considered lies at a pole of the earth; if a pole of the earth holds as first point A , and any other point of the earth as second point B , then the meridian of the point B is the normal section from B to A , as well as the normal section from A to B . However, since the poles of the earth are not accessible, this case has no practical significance for us.

Second. If two points lie at the same latitude φ , then the two intersection planes of the normal coincide, because the two axial sections K_a and K_b of the normal from A and from B become then identical on the terrestrial axis according to Fig. 1.

Because of the smallness of the flattening of our earth the recognized diverging of two counter-normal sections for measurable triangle sides is very small, and up to now we have implicitly neglected it when we treated the earth as a sphere. But in order to obtain a correct judgment, we will examine the deviation of the two normal sections more closely now.

In this connection, the conditions are shown in detail in Fig. 2, p. 2.

The two points A and B have the latitudes φ_1 and φ_2 , which are expressed in the case of K_a and K_b ; we have

$$\begin{aligned} \text{angle } AK_aT &= 90^\circ - \varphi_1 \\ BK_bT &= 90^\circ - \varphi_2. \end{aligned}$$

The symbols φ_1 and φ_2 inscribed in Fig. 2, p. 2, in the case of K_a and K_b correspond to this.

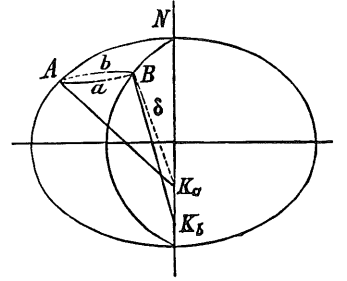


Fig. 1.

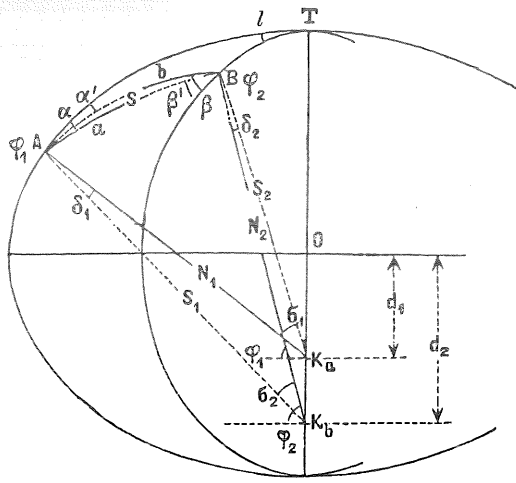


Fig. 2.

A theodolite set up at A has its axis directed to AK_a , and the sight to B occurs at AaB at the azimuth α . On the other hand, the sight from B to A lies at BbA with the azimuth β (counted from the south).

In Fig. 2, in addition, there are also indicated the angles α' and β' . We have to understand these angles in such a way that a tangent is laid at A to the arc AbB and at B to the arc BaA .

Determination of the angles δ_1 and δ_2 and of the distances S_1 and S_2

If we connect in Fig. 2 the point A with K_b and the point B with K_a , then these lines deviate from the normals to the ellipsoid by the small angles δ_1 and δ_2 , which we will determine now.

To do so, we need the distances of the sections K_a and K_b from the center point of the ellipsoid, entered in Fig. 2 with d_1 and d_2 .

By referring to the relations found in the first half-volume of this volume, sections 37 and 38, we have the ordinate y_1 for the point A , according to Fig. 1, p. 41, there, and we have

$$d_1 = N_1 \sin \varphi_1 - y_1. \quad (1)$$

Since according to Vol. III, first half-volume, p. 50, $N_1 = \frac{c}{V_1}$, and according to p. 49

$$y_1 = \frac{c \sin \varphi_1}{V_1 (1 + e'^2)} = \frac{c (1 - e^2) \sin \varphi_1}{V_1},$$

then we obtain

$$d_1 = e^2 \frac{c \sin \varphi_1}{V_1},$$

and accordingly

$$d_2 = e^2 \frac{c \sin \varphi_2}{V_2};$$

therefore

$$d_2 - d_1 = e^2 c \left(\frac{\sin \varphi_2}{V_2} - \frac{\sin \varphi_1}{V_1} \right). \quad (2)$$

The narrow triangle AK_aK_b of Fig. 2 yields after extension of AK_a :

$$\tan \delta_1 = \frac{(d_2 - d_1) \cos \varphi_1}{N_1 + (d_2 - d_1) \sin \varphi_1} = \frac{e^2 \left(\frac{V_1}{V_2} \sin \varphi_2 - \sin \varphi_1 \right) \cos \varphi_1}{1 + e^2 \left(\frac{V_1}{V_2} \sin \varphi_2 - \sin \varphi_1 \right) \sin \varphi_1}. \quad (3)$$

The corresponding formula for δ_2 is:

$$\tan \delta_2 = \frac{e^2 \left(\sin \varphi_2 - \sin \varphi_1 \frac{V_2}{V_1} \right) \cos \varphi_2}{1 - e^2 \left(\sin \varphi_2 - \sin \varphi_1 \frac{V_2}{V_1} \right) \sin \varphi_2}. \quad (4)$$

Similarly, we can also determine the distance S_1 according to Fig. 2:

$$S_1^2 = N_1^2 + (d_2 - d_1)^2 + 2 N_1 (d_2 - d_1) \sin \varphi_1.$$

Taking into account (1) and (2) this yields:

$$\left(\frac{S_1}{N_1} \right)^2 = 1 + 2 e^2 \left(\sin \varphi_2 \frac{V_1}{V_2} - \sin \varphi_1 \right) \sin \varphi_1 + e^4 \left(\sin \varphi_2 \frac{V_1}{V_2} - \sin \varphi_1 \right)^2 \quad (5)$$

and for the other distance S_2 :

$$\left(\frac{S_2}{N_2} \right)^2 = 1 - 2 e^2 \left(\sin \varphi_2 - \sin \varphi_1 \frac{V_2}{V_1} \right) \sin \varphi_2 + e^4 \left(\sin \varphi_2 - \sin \varphi_1 \frac{V_2}{V_1} \right)^2. \quad (6)$$

These formulae (3), (4), (5), (6) are rigorously correct, and can be used for any arbitrary latitudes φ_1 and φ_2 , where for $\log V_1^2$ and $\log V_2^2$ the auxiliary table on pp. [12] to [33] of the Appendix* to Vol. III, first half-volume, can be used.

For small differences of latitude, however, development in series is proper, which we will at first demonstrate for δ_2 .

If we set for the moment

$$\sin \varphi_2 - \sin \varphi_1 \frac{V_2}{V_1} = p, \quad (7)$$

then we have according to (4):

$$\tan \delta_2 = \frac{e^2 p \cos \varphi_2}{1 - e^2 p \sin \varphi_2}. \quad (8)$$

For the development in series we will set

$$\varphi_2 - \varphi_1 = \Delta \varphi,$$

and if we reduce all to the argument φ_1 , then we have according to Vol. III, first half-volume, section 40, p. 63, equation (n),

$$\frac{V_2}{V_1} = 1 - \eta_1^2 t_1 \frac{\Delta \varphi}{V_1^2} - \frac{1}{2} \eta_1^2 (1 - t_1^2 + \eta_1^2) \frac{\Delta \varphi^2}{V_1^4} + \dots, \quad (9)$$

* The Appendix containing auxiliary tables to the first half-volume is not included in the translation.

and according to section 34 of that place, p. 18,

$$\begin{aligned} \sin \varphi_2 = \sin \varphi_1 + \cos \varphi_1 \Delta \varphi - \frac{1}{2} \sin \varphi_1 \Delta \varphi^2 - \frac{1}{6} \cos \varphi_1 \Delta \varphi^3 \\ + \frac{1}{24} \sin \varphi_1 \Delta \varphi^4 + \dots \end{aligned} \quad (10)$$

With this, we obtain from (7) easily

$$\begin{aligned} p = \cos \varphi_1 (1 + \eta_1^2 t_1^2) \Delta \varphi - \frac{1}{2} \sin \varphi_1 (1 - \eta_1^2 + \eta_1^2 t_1^2) \Delta \varphi^2 - \frac{1}{6} \cos \varphi_1 \Delta \varphi^3 \\ + \frac{1}{24} \sin \varphi_1 \Delta \varphi^4. \end{aligned} \quad (11)$$

We take further from the first half-volume, section 37, p. 41, equation (5):

$$e^2 = e'^2 - e'^4 + \dots \quad \eta_1^2 = e'^2 \cos^2 \varphi_1, \quad (12)$$

and from section 34, p. 18:

$$\cos \varphi_2 = \cos \varphi_1 - \sin \varphi_1 \Delta \varphi - \frac{1}{2} \cos \varphi_1 \Delta \varphi^2 + \frac{1}{6} \sin \varphi_1 \Delta \varphi^3 + \frac{1}{24} \cos \varphi_1 \Delta \varphi^4, \quad (13)$$

and then we find:

$$\begin{aligned} e^2 p \cos \varphi_2 = \eta_1^2 (1 - \eta_1^2) \Delta \varphi - \frac{1}{2} \eta_1^2 t_1 (3 - 4 \eta_1^2) \Delta \varphi^2 - \frac{1}{6} \eta_1^2 (4 - 3 t_1^2) \Delta \varphi^3 \\ + \frac{5}{8} \eta_1^2 t_1 \Delta \varphi^4 + \dots \end{aligned} \quad (14)$$

If we take to this, in addition, with sufficient accuracy

$$\frac{1}{1 - e^2 p \sin \varphi_2} = 1 + \eta_1^2 t_1 \Delta \varphi + \dots,$$

then we will have

$$\begin{aligned} \tan \delta_2 = \eta_1^2 (1 - \eta_1^2) \Delta \varphi - \frac{3}{2} \eta_1^2 t_1 (1 - 2 \eta_1^2) \Delta \varphi^2 - \frac{1}{6} \eta_1^2 (4 - 3 t_1^2) \Delta \varphi^3 \\ + \frac{5}{8} \eta_1^2 t_1 \Delta \varphi^4. \end{aligned}$$

From $\tan \delta_2$ we also can pass over to δ_2 using the *arc tan* series of the first half-volume, section 34, p. 23. But since we have limited ourselves to the terms of sixth order in the case of the development of $\tan \delta_2$, we can set immediately δ_2 instead of $\tan \delta_2$. At the same time, we obtain, in addition, a small simplification, if we introduce

$$V_1^2 = 1 + \eta_1^2; \quad \text{therefore} \quad \frac{1}{V_1^2} = 1 - \eta_1^2 \quad \frac{1}{V_1^4} = 1 - 2 \eta_1^2.$$

Then we will have

$$\delta_2 = \eta_1^2 \frac{\Delta \varphi}{V_1^2} - \frac{3 \eta_1^2 t_1}{2} \frac{\Delta \varphi^2}{V_1^4} - \frac{\eta_1^2}{6} (4 - 3 t_1^2) \frac{\Delta \varphi^3}{V_1^6} + \frac{5 \eta_1^2 t_1}{8} \frac{\Delta \varphi^4}{V_1^8} + \dots \quad (15)$$

where in the last two terms V_1^6 and V_1^8 are added for the sake of symmetry.

In the same way, we also can develop δ_2 for the argument φ_2 , and for this reason we will limit ourselves to a few suggestions for this. According to (9) and (10) we have:

$$\frac{V_2}{V_1} = 1 - \eta_2^2 t_2 \frac{\Delta \varphi}{V_2^2} + \frac{1}{2} \eta_2^2 (1 - t_2^2 + \eta_2^2) \frac{\Delta \varphi^2}{V_2^4} + \dots \quad (16)$$

and

$$\sin \varphi_1 = \sin \varphi_2 - \cos \varphi_2 \Delta \varphi - \frac{1}{2} \sin \varphi_2 \Delta \varphi^2 + \frac{1}{6} \cos \varphi_2 \Delta \varphi^3 + \frac{1}{24} \sin \varphi_2 \Delta \varphi^4 - \dots$$

With these we will have:

$$p = \cos \varphi_2 (1 + \eta_2^2 + \eta_2^2 t_2^2) \frac{\Delta \varphi}{V_2^2} + \frac{1}{2} \cos \varphi_2 t_2 (1 - \eta_2^2 + \eta_2^2 t_2^2) \frac{\Delta \varphi^2}{V_2^4} - \frac{1}{6} \cos \varphi_2 \Delta \varphi^3 - \frac{1}{24} \sin \varphi_2 \Delta \varphi^4 + \dots$$

If we introduce the expressions (12) again, then we have:

$$\frac{1}{1 - e^2 p \sin \varphi_2} = 1 + \eta_2^2 t_2 \frac{\Delta \varphi}{V_2^2} + \dots,$$

and we find after simple conversion:

$$\tan \delta_2 = \eta_2^2 \frac{\Delta \varphi}{V_2^2} + \frac{\eta_2^2 t_2}{2} \frac{\Delta \varphi^2}{V_2^4} - \frac{\eta_2^2}{6} \Delta \varphi^3 - \frac{\eta_2^2 t_2}{24} \Delta \varphi^4$$

or else

$$\delta_2 = \eta_2^2 \frac{\Delta \varphi}{V_2^2} + \frac{\eta_2^2 t_2}{2} \frac{\Delta \varphi^2}{V_2^4} - \frac{\eta_2^2}{6} \frac{\Delta \varphi^3}{V_2^6} - \frac{\eta_2^2 t_2}{24} \frac{\Delta \varphi^4}{V_2^8} + \dots \quad (17)$$

For the later application, still a third formula for δ_2 is of use, which is based on the arithmetic mean

$\frac{\varphi_1 + \varphi_2}{2} = \varphi$ as argument. Then we have according to the Taylor series (first half-volume, section 34, p. 18):

$$\left. \begin{aligned} \sin \varphi_1 &= \sin \left(\varphi - \frac{\Delta \varphi}{2} \right) = \sin \varphi - \cos \varphi \frac{\Delta \varphi}{2} - \sin \varphi \frac{\Delta \varphi^2}{8} + \cos \varphi \frac{\Delta \varphi^3}{48} + \dots \\ \cos \varphi_1 &= \cos \left(\varphi - \frac{\Delta \varphi}{2} \right) = \cos \varphi + \sin \varphi \frac{\Delta \varphi}{2} - \cos \varphi \frac{\Delta \varphi^2}{8} - \sin \varphi \frac{\Delta \varphi^3}{48} + \dots \\ \sin \varphi_2 &= \sin \left(\varphi + \frac{\Delta \varphi}{2} \right) = \sin \varphi + \cos \varphi \frac{\Delta \varphi}{2} - \sin \varphi \frac{\Delta \varphi^2}{8} - \cos \varphi \frac{\Delta \varphi^3}{48} + \dots \\ \cos \varphi_2 &= \cos \left(\varphi + \frac{\Delta \varphi}{2} \right) = \cos \varphi - \sin \varphi \frac{\Delta \varphi}{2} - \cos \varphi \frac{\Delta \varphi^2}{8} + \sin \varphi \frac{\Delta \varphi^3}{48} + \dots \end{aligned} \right\} \quad (18)$$

Before introducing this into equations (7) and (8), we still want to transform the quotient $\frac{V_2}{V_1}$ occurring

in (7) to the mean latitude φ . According to the first half-volume, section 40, p. 63, equation (n), we have, if we limit ourselves to terms of fourth order,

$$V_2 = V - \frac{\eta^2 t}{V} \frac{\Delta \varphi}{2} - \frac{1}{8} \frac{\eta^2}{V^3} (1 - t^2) \Delta \varphi^2, \quad (19)$$

where there is set

$$t = \tan \varphi \quad \text{and} \quad \eta^2 = e'^2 \cos^2 \varphi$$

for abbreviation, as previously.

From (19) we obtain also immediately the value of V_1 , if we introduce $-\Delta \varphi$ instead of $\Delta \varphi$:

$$\left. \begin{aligned} V_2 &= V \left\{ 1 - \frac{1}{2} \frac{\eta^2 t}{V^2} \Delta \varphi - \frac{1}{8} \frac{\eta^2}{V^4} (1 - t^2) \Delta \varphi^2 \right\} \\ V_1 &= V \left\{ 1 + \frac{1}{2} \frac{\eta^2 t}{V^2} \Delta \varphi - \frac{1}{8} \frac{\eta^2}{V^4} (1 - t^2) \Delta \varphi^2 \right\} \end{aligned} \right\} \quad (20)$$

and with these we will have, omitting the terms of fifth order,

$$\frac{V_2}{V_1} = 1 - \frac{\eta^2 t}{V^2} \Delta \varphi. \quad (21)$$

After these preparations we can compute the auxiliary quantity p according to (7), namely with the help of (18) and (21) to an accuracy of fourth-order terms

$$p = \cos \varphi \Delta \varphi - \cos \varphi \frac{\Delta \varphi^3}{24} + \sin \varphi \frac{t \eta^2}{V^2} \Delta \varphi - \cos \varphi \frac{t \eta^2}{2} \Delta \varphi^2. \quad (22)$$

With this, we also can easily find the final expression for $\tan \delta_2$ according to (8), where the terms up to the sixth order are to be retained now. There follows

$$\begin{aligned} \tan \delta_2 &= e^2 \cos^2 \varphi \Delta \varphi - e^2 \sin \varphi \cos \varphi \frac{\Delta \varphi^2}{2} - e^2 \cos^2 \varphi \frac{\Delta \varphi^3}{6} + e^2 \sin \varphi \cos \varphi \frac{\Delta \varphi^4}{24} \\ &\quad + e^4 \sin \varphi \cos^3 \varphi \Delta \varphi^2 + e^2 \frac{t \eta^2}{V^2} \sin \varphi \cos \varphi \Delta \varphi - e^2 \frac{t \eta^2}{V^2} \frac{\Delta \varphi^2}{2}. \end{aligned}$$

This can be simplified still further by introducing again the auxiliary quantity e' instead of e , namely:

$$e^2 = \frac{e'^2}{1 + e'^2} = e'^2 (1 - e'^2) \quad \text{and} \quad e'^2 \cos^2 \varphi = \eta^2.$$

After simple conversions we will have then

$$\tan \delta_2 = \eta^2 (1 - \eta^2) \Delta \varphi - \eta^2 (1 - 2\eta^2) t \frac{\Delta \varphi^2}{2} - \eta^2 \frac{\Delta \varphi^3}{6} + \eta^2 \frac{\Delta \varphi^4}{24} t.$$

But since

$$\begin{aligned} 1 + \eta^2 &= V^2, \quad \text{and hence with sufficient approximation} \\ 1 - \eta^2 &= \frac{1}{V^2} \quad \text{and} \quad 1 - 2\eta^2 = (1 - \eta^2)^2 = \frac{1}{V^4}, \end{aligned}$$

then we will have

$$\tan \delta_2 = \frac{\eta^2}{V^2} \Delta \varphi - \frac{\eta^2}{V^4} t \frac{\Delta \varphi^2}{2} - \eta^2 \frac{\Delta \varphi^3}{6} + \eta^2 t \frac{\Delta \varphi^4}{24} \quad (23)$$

or else

$$\delta_2 = \eta^2 \frac{\Delta \varphi}{V^2} - \frac{\eta^2 t}{2} \frac{\Delta \varphi^2}{V^4} - \frac{\eta^2}{6} \frac{\Delta \varphi^3}{V^6} + \frac{\eta^2 t}{24} \frac{\Delta \varphi^4}{V^8}. \quad (24)$$

Then we have the three formulae for δ_2 :

$$\delta_2 = \eta^2 \frac{\Delta \varphi}{V^2} \left\{ 1 - \frac{t}{2} \frac{\Delta \varphi}{V^2} - \frac{1}{6} \left(\frac{\Delta \varphi}{V^2} \right)^2 + \frac{t}{24} \left(\frac{\Delta \varphi}{V^2} \right)^3 \right\} \quad (25)$$

$$\delta_2 = \eta_1^2 \frac{\Delta \varphi}{V_1^2} \left\{ 1 - \frac{3}{2} \frac{t_1}{V_1^2} \frac{\Delta \varphi}{V_1^2} - \frac{2}{3} \left(\frac{\Delta \varphi}{V_1^2} \right)^2 + \frac{t_1^2}{2} \left(\frac{\Delta \varphi}{V_1^2} \right)^2 + \frac{5}{8} \frac{t_1}{V_1^2} \left(\frac{\Delta \varphi}{V_1^2} \right)^3 \right\} \quad (26)$$

$$\delta_2 = \eta_2^2 \frac{\Delta \varphi}{V_2^2} \left\{ 1 + \frac{t_2}{2} \frac{\Delta \varphi}{V_2^2} - \frac{1}{6} \left(\frac{\Delta \varphi}{V_2^2} \right)^2 - \frac{t_2}{24} \left(\frac{\Delta \varphi}{V_2^2} \right)^3 \right\}. \quad (27)$$

These formulae can also be used for the determination of δ_1 , for it can already be seen from (3) and (4) that δ_1 and $-\delta_2$ mutually transformed into one another, if we mutually interchange φ_1 and φ_2 , therefore also take $\Delta \varphi$ negatively. In this way we obtain from (25), (26), (27):

$$\delta_1 = \eta^2 \frac{\Delta \varphi}{V^2} \left\{ 1 + \frac{t}{2} \frac{\Delta \varphi}{V^2} - \frac{1}{6} \left(\frac{\Delta \varphi}{V^2} \right)^2 - \frac{t}{24} \left(\frac{\Delta \varphi}{V^2} \right)^3 \right\} \quad (25^*)$$

$$\delta_1 = \eta_2^2 \frac{\Delta \varphi}{V_2^2} \left\{ 1 + \frac{3}{2} \frac{t_2}{V_2^2} \frac{\Delta \varphi}{V_2^2} - \frac{2}{3} \left(\frac{\Delta \varphi}{V_2^2} \right)^2 + \frac{t_2^2}{2} \left(\frac{\Delta \varphi}{V_2^2} \right)^2 - \frac{5}{8} \frac{t_2}{V_2^2} \left(\frac{\Delta \varphi}{V_2^2} \right)^3 \right\} \quad (26^*)$$

$$\delta_1 = \eta_1^2 \frac{\Delta \varphi}{V_1^2} \left\{ 1 - \frac{t_1}{2} \frac{\Delta \varphi}{V_1^2} - \frac{1}{6} \left(\frac{\Delta \varphi}{V_1^2} \right)^2 + \frac{t_1}{24} \left(\frac{\Delta \varphi}{V_1^2} \right)^3 \right\}. \quad (27^*)$$

When δ_1 and δ_2 are computed, then the expressions $\frac{S_1}{N_1}$ and $\frac{S_2}{N_2}$ can easily be computed, for we have, e.g., according to Fig. 2:

$$\frac{S_2}{N_2} = \frac{\sin(90^\circ - \varphi_2)}{\sin(90^\circ - (\varphi_2 - \delta_2))} = \frac{\cos \varphi_2}{\cos(\varphi_2 - \delta_2)}. \quad (28)$$

But it is better to use development in series also for this, for which the equations (5) and (6) are well suited. If we introduce the auxiliary quantity p according to (7) again, then we have for $\frac{S_2}{N_2}$ according to (6):

$$\left(\frac{S_2}{N_2} \right)^2 = 1 - 2e^2 p \sin \varphi_2 + e^4 p^2.$$

For the development in series of this rigorously correct expression we will limit ourselves to terms of the fourth order and can take first from the first half-volume, section 37, p. 41, equation (5),

$$e^2 = e'^2 - e'^4 + \dots, \quad e^4 = e'^4 + \dots$$

With this we will have

$$\left(\frac{S_2}{N_2}\right)^2 = 1 - 2 e'^2 p \sin \varphi_2 + 2 e'^4 p \sin \varphi_2 + e'^4 p^2 + \dots$$

With the help of (18) and (22) the individual terms of the above equation can easily be indicated. If we take into account that

$$e'^2 \cos^2 \varphi = \eta^2 \quad \frac{1}{V^2} = 1 - \eta^2 + \dots,$$

then we will have

$$\begin{aligned} -2 e'^2 p \sin \varphi_2 &= -2 \Delta \varphi \eta^2 t + 2 \Delta \varphi \eta^4 t - \Delta \varphi^2 \eta^2 - 2 e'^2 \Delta \varphi \eta^2 t + \frac{\Delta \varphi^3}{3} \eta^2 t + \frac{\Delta \varphi^4}{12} \eta^2 \\ + 2 e'^4 p \sin \varphi_2 &= + 2 e'^2 \Delta \varphi \eta^2 t + e'^2 \Delta \varphi^2 \eta^2 \\ + e'^4 p^2 &= + e'^2 \Delta \varphi^2 \eta^2, \end{aligned}$$

and with these, we have

$$\left(\frac{S_2}{N_2}\right)^2 = 1 - 2 \Delta \varphi \eta^2 t (1 - \eta^2) - \Delta \varphi^2 \eta^2 + \frac{\Delta \varphi^3}{3} \eta^2 t + \frac{\Delta \varphi^4}{12} \eta^2 + 2 e'^2 \Delta \varphi^2 \eta^2.$$

By using the series

$$\sqrt{1-x} = 1 - \frac{1}{2} x - \frac{1}{8} x^2 - \dots$$

we have then from the first half-volume, section 34, p. 20,

$$\frac{S_2}{N_2} = 1 - \Delta \varphi \eta^2 t (1 - \eta^2) - \frac{\Delta \varphi^2}{2} \eta^2 (1 - 2 e'^2 + \eta^2 t^2) + \frac{\Delta \varphi^3}{6} \eta^2 t + \frac{\Delta \varphi^4}{24} \eta^2.$$

As can easily be seen, we have, if we introduce $e'^2 = \frac{\eta^2}{\cos^2 \varphi}$,

$$1 - 2 e'^2 + \eta^2 t^2 = 1 - 2 \eta^2 - \eta^2 t^2 = (1 - \eta^2)^2 (1 - \eta^2 t^2),$$

and since

$$1 - \eta^2 = \frac{1}{V^2},$$

then we will have

$$\frac{S_2}{N_2} = 1 - \Delta \varphi \frac{\eta^2}{V^2} t - \frac{1}{2} \left(\frac{\Delta \varphi}{V^2}\right)^2 \eta^2 (1 - \eta^2 t^2) + \frac{1}{6} \left(\frac{\Delta \varphi}{V^2}\right)^3 \eta^2 t + \frac{1}{24} \left(\frac{\Delta \varphi}{V^2}\right)^4 \eta^2. \quad (29)$$

In the last two terms we have added arbitrarily the denominator V^2 again, because of symmetry, whereby only the terms of fifth and higher order are influenced.

By change of sign, we obtain therefrom also:

$$\frac{S_1}{N_1} = 1 + \Delta \varphi \frac{\eta^2}{V^2} t - \frac{1}{2} \left(\frac{\Delta \varphi}{V^2} \right)^2 \eta^2 (1 - \eta^2 t^2) - \frac{1}{6} \left(\frac{\Delta \varphi}{V^2} \right)^3 \eta^2 t + \frac{1}{24} \left(\frac{\Delta \varphi}{V^2} \right)^4 \eta^2. \quad (30)$$

In these formulae also, $\eta^2 = e'^2 \cos^2 \varphi$ and $t = \tan \varphi$ refer to the *mean* latitude φ .
As a numerical example we take:

$$\left. \begin{array}{ll} \varphi_1 = 49^\circ 30' & \varphi_2 = 50^\circ 30' \\ \varphi = 50^\circ 00' & \Delta \varphi = 1^\circ = 3600'' \end{array} \right\} \quad (31)$$

With these, the computation was carried out according to the closed formulae as well as according to the development in series:

$$\delta_1 = 10.069\ 58'' \quad \delta_2 = 9.86\ 285'' \quad (32)$$

$$\left. \begin{array}{ll} \frac{S_1}{N_1} = 1.000\ 057\ 1638 & \frac{S_2}{N_2} = 0.999\ 941\ 9985 \\ \log \frac{S_1}{N_1} = 0.000\ 0248\ 252 & \log \frac{S_2}{N_2} = 9.999\ 9748\ 096 \end{array} \right\} \quad (33)$$

Later we still shall make use of these numbers several times.

Section 2. Elliptic Arc of a Normal Section

In Fig. 1 we have drawn a part from the previous Fig. 2, section 1, p. 2, namely all which concerns the normal section AB which passes through the normal AK_a and the point B .

We know that the arc AB is an *elliptic* arc from the nature of the ellipsoid, which can be cut by a plane AK_aB in such an arc. For the computation of the length of arc, the geographic latitude φ_1 of A as well as the azimuth α is to be regarded as constant, as well as all magnitudes dependent thereupon, such as N_1 , η_1^2 , V_1 , etc., s is then merely a function of the angle σ .

For the development of s , S_2 shall at first be expressed in terms of $\Delta \varphi$ and then $\Delta \varphi$ in terms of σ so that we obtain S_2 as a function of σ ; s is then easily found by integration. The former problem could be attached to equation (29), section 1, p. 8; but since in this η^2 , V^2 and t are referred to the mean latitude, then they are again dependent on $\Delta \varphi$. We reach our goal more quickly by the relation resulting directly from Fig. 2, section 1, p. 2:

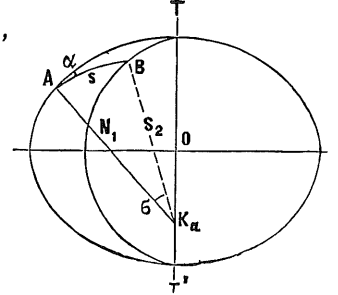


Fig. 1.

$$\frac{S_2}{N_2} = \frac{\sin(90^\circ - \varphi_2)}{\sin(90^\circ - (\varphi_2 - \delta_2))} = \frac{\cos \varphi_2}{\cos(\varphi_2 - \delta_2)}. \quad (1)$$

Since $\varphi_2 = \varphi_1 + \Delta \varphi$, then we obtain by development in series according to the compilation of formulae of the first half-volume, section 34, p. 18:

$$\frac{S_2}{N_2} = \frac{\cos \varphi_1 - \Delta \varphi \sin \varphi_1 - \frac{\Delta \varphi^2}{2} \cos \varphi_1 + \frac{\Delta \varphi^3}{6} \sin \varphi_1 + \dots}{\cos \varphi_1 - (\Delta \varphi - \delta_2) \sin \varphi_1 - \frac{1}{2} (\Delta \varphi - \delta_2)^2 \cos \varphi_1 + \frac{1}{6} (\Delta \varphi - \delta_2)^3 \sin \varphi_1 + \dots}$$

or, if we set again $\tan \varphi_1 = t_1$:

$$\frac{S_2}{N_2} = \frac{1 - \Delta \varphi t_1 - \frac{1}{2} \Delta \varphi^2 + \frac{1}{6} \Delta \varphi^3 t_1 + \dots}{1 - (\Delta \varphi - \delta_2) t_1 - \frac{1}{2} (\Delta \varphi - \delta_2)^2 + \frac{1}{6} (\Delta \varphi - \delta_2)^3 t_1 + \dots} \quad (2)$$

If we calculate the fraction on the right-hand side, then at once there follows a considerable simplification by the circumstance that all terms free of δ_2 must be omitted, since for $\delta_2 = 0$ there must become $\frac{S_2}{N_2} = 1$ according to (1). If we retain only the terms of fifth order, then we will have, since δ_2 is of the third order,

$$\frac{S_2}{N_2} = 1 - t_1 \delta_2 - \Delta \varphi \delta_2 (1 + t_1^2) - \Delta \varphi^2 \delta_2 (1 + t_1^2) t_1. \quad (3)$$

To this, we take the value of δ_2 from equation (26), section 1, p. 7:

$$\delta_2 = \Delta \varphi \frac{\eta_1^2}{V_1^2} - \frac{3}{2} \Delta \varphi^2 \frac{\eta_1^2}{V_1^4} t_1 - \frac{2}{3} \Delta \varphi^3 \frac{\eta_1^2}{V_1^6} + \frac{1}{2} \Delta \varphi^3 \frac{\eta_1^2}{V_1^6} t_1^2 \quad (4)$$

and obtain after simple conversions:

$$\frac{S_2}{N_2} = 1 - \Delta \varphi \frac{\eta_1^2}{V_1^2} t_1 - \frac{1}{2} \Delta \varphi^2 \frac{\eta_1^2}{V_1^4} (2 - t_1^2) + \frac{7}{6} \Delta \varphi^3 \frac{\eta_1^2}{V_1^6} t_1. \quad (5)$$

In order to express also N_2 by $\Delta \varphi$, we take from the first half-volume, section 40, p. 63, equation (o)

$$\frac{N_2}{N_1} = 1 + \Delta \varphi \frac{\eta_1^2}{V_1^2} t_1 + \frac{1}{2} \Delta \varphi^2 \frac{\eta_1^2}{V_1^4} (1 - t_1^2) - \frac{2}{3} \Delta \varphi^3 \frac{\eta_1^2}{V_1^6} t_1 \quad (6)$$

and obtain then very easily by multiplication of (5) and (6)

$$\frac{S_2}{N_1} = 1 - \frac{1}{2} \Delta \varphi^2 \frac{\eta_1^2}{V_1^4} + \frac{1}{2} \Delta \varphi^3 \frac{\eta_1^2}{V_1^6} t_1,$$

or else, since terms of the sixth order are neglected,

$$\frac{S_2}{N_1} = 1 - \frac{1}{2} \Delta \varphi^2 \eta_1^2 + \frac{1}{2} \Delta \varphi^3 \eta_1^2 t_1. \quad (7)$$

We obtain the necessary relation between $\Delta \varphi$ and σ from the spherical formulae of the first half-volume, section 63. For if we describe a sphere around K_a in Fig. 1, p. 9, then the three straight lines $K_a A$, $K_a B$ and $K_a T$ yield a spherical triangle which corresponds to the triangle $PP'N$ in the first half-volume, section 63, p. 176, Fig. 1. We are to bear in mind, however, that the geographic latitude $\varphi_2 - \delta_2$ is assigned to the ray $K_a B$ in Fig. 1, p. 9. Then we have according to the first half-volume, section 63, p. 178, equation (27), in which φ' is to be replaced by $\varphi_2 - \delta_2$,

$$\varphi_2 - \delta_2 - \varphi_1 = \sigma \cos \alpha_1 - \frac{1}{2} \sigma^2 \sin^2 \alpha_1 t_1.$$

Since only terms of the second order are retained here, and δ_2 is already of third order, then we can write offhand:

$$\Delta \varphi = \sigma \cos \alpha_1 - \frac{1}{2} \sigma^2 \sin^2 \alpha_1 t_1. \quad (8)$$

This set into (7) yields

$$\frac{S_2}{N_1} = 1 - \frac{1}{2} \sigma^2 \eta_1^2 \cos^2 \alpha_1 + \frac{1}{2} \sigma^3 \eta_1^3 \sin^2 \alpha_1 \cos \alpha_1 t_1 + \frac{1}{2} \sigma^3 \eta_1^2 \cos^3 \alpha_1 t_1,$$

or else

$$\frac{S_2}{N_1} = 1 - \frac{1}{2} \sigma^2 \eta_1^2 \cos^2 \alpha_1 + \frac{1}{2} \sigma^3 \eta_1^2 t_1 \cos \alpha_1, \quad (9)$$

in which the terms up to the fifth order are correct.

Now we are sufficiently prepared to compute the length of arc s . By means of the equation (9) the curve AB is determined in polar coordinates from which we can find s by integration. To this, Fig. 2 yields the differential equation:

$$ds^2 = (S_2 d\sigma)^2 + (dS_2)^2. \quad (10)$$

From (9) there follows

$$\begin{aligned} S_2 d\sigma &= N_1 \left(1 - \frac{1}{2} \sigma^2 \eta_1^2 \cos^2 \alpha_1 + \frac{1}{2} \sigma^3 \eta_1^2 t_1 \cos \alpha_1 \right) d\sigma \\ dS_2 &= -N_1 \sigma \eta_1^2 \cos \alpha_1 \left(\cos \alpha_1 - \frac{3}{2} \sigma t_1 \right) d\sigma. \end{aligned} \quad (10a)$$

We see therefrom that dS_2 in (10) yields only terms with $\sigma^2 \eta_1^4$, which go beyond the fifth order. Consequently, we will have

$$ds = S_2 d\sigma = N_1 \left(1 - \frac{1}{2} \sigma^2 \eta_1^2 \cos^2 \alpha_1 + \frac{1}{2} \sigma^3 \eta_1^2 t_1 \cos \alpha_1 \right) d\sigma$$

and the integration yields

$$s = N_1 \left(\sigma - \frac{1}{6} \sigma^3 \eta_1^2 \cos^2 \alpha_1 + \frac{1}{8} \sigma^4 \eta_1^2 t_1 \cos \alpha_1 \right), \quad (11)$$

or else

$$s = N_1 \sigma \left(1 - \frac{1}{6} \sigma^2 \eta_1^2 \cos^2 \alpha_1 + \frac{1}{8} \sigma^3 \eta_1^2 t_1 \cos \alpha_1 \right). \quad (12)$$

This formula can also be directly inverted and yields

$$\sigma = \frac{s}{N_1} \left(1 + \frac{1}{6} \frac{s^2}{N_1^2} \eta_1^2 \cos^2 \alpha_1 - \frac{1}{8} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 \right). \quad (13)$$

In the case of $\sigma = 1^\circ = 3600''$ at the latitude of 52° , and hence in the case of $s = 106$ km, the last term in (12) and (13) becomes only equal to 0.2 mm, in the maximum, or equal to 0.000 0065'', and hence, can be neglected in most cases.

A further question concerns the difference in longitude of the two elliptic arcs AB and BA in Fig. 2, section 1, p. 2, which we will discuss, however, only in section 9 in connection with the geodetic line.

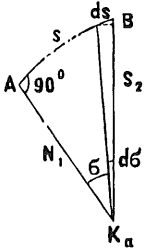


Fig. 2.

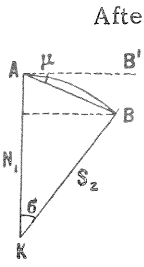


Fig. 3.

or

$$\tan (90^\circ - \mu) = \frac{S_2 \sin \sigma}{N_1 - S_2 \cos \sigma},$$

$$\tan \mu \sin \sigma = \frac{N_1}{S_2} - \cos \sigma. \quad (14)$$

If we introduce the value of $\frac{N_1}{S_2}$ from (9) and introduce also the development in series for $\sin \sigma$ and $\cos \sigma$, then we have

$$\tan \mu \left(\sigma - \frac{\sigma^3}{6} + \dots \right) = \frac{1}{2} \sigma^2 (1 + \eta_1^2 \cos^2 \alpha_1) - \frac{1}{2} \sigma^3 \eta_1^2 t_1 \cos \alpha_1 - \frac{1}{24} \sigma^4 + \dots,$$

or

$$\left(\mu + \frac{\mu^3}{3} \right) \left(1 - \frac{\sigma^2}{6} \right) = \frac{1}{2} \sigma (1 + \eta_1^2 \cos^2 \alpha_1) - \frac{1}{2} \sigma^2 \eta_1^2 t_1 \cos \alpha_1 - \frac{1}{24} \sigma^3 + \dots$$

Since we have in the first approximation

$$\mu = \frac{\sigma}{2} + \dots, \quad \text{and hence} \quad \mu^3 = \frac{\sigma^3}{8} + \dots$$

then we obtain easily

$$\mu = \frac{1}{2} \sigma (1 + \eta_1^2 \cos^2 \alpha_1) - \frac{1}{2} \sigma^2 \eta_1^2 t_1 \cos \alpha_1. \quad (15)$$

The first approximation $\mu = \frac{\sigma}{2}$ corresponds to the assumption that AB is an arc of a circle with the center point K .

In the foregoing expression for μ we can replace also σ by s , for which equation (13) can directly be used. Then there results at once the expression

$$\mu = \frac{s}{2 N_1} (1 + \eta_1^2 \cos^2 \alpha_1) - \frac{s^2}{2 N_1^2} \eta_1^2 t_1 \cos \alpha_1, \quad (16)$$

in which the terms of the fifth and higher order are neglected.

The length of the chord AB

The foregoing Fig. 3 at the same time offers also the opportunity of computing the length of the chord AB , for we have, if we set $AB = k$,

$$\frac{k}{N_1} = \frac{\sin \sigma}{\sin (\sigma + 90^\circ - \mu)}, \quad \text{and hence} \quad \frac{k}{N_1} = \frac{\sin \sigma}{\cos (\sigma - \mu)}. \quad (17)$$

If we retain the terms up to the sixth order in this expression, then the development in series yields

$$\frac{k}{N_1} = \sigma \left(1 - \frac{\sigma^2}{6} + \frac{\sigma^4}{120} \right) \left(1 + \frac{1}{2} (\sigma - \mu)^2 + \frac{5}{24} (\sigma - \mu)^4 \right). \quad (18)$$

But according to (15) we have

$$\sigma - \mu = \frac{1}{2} \sigma (1 - \eta_1^2 \cos^2 \alpha_1) + \frac{1}{2} \sigma^2 \eta_1^2 t_1 \cos \alpha_1,$$

and hence

$$(\sigma - \mu)^2 = \frac{1}{4} \sigma^2 (1 - 2 \eta_1^2 \cos^2 \alpha_1) + \frac{1}{2} \sigma^3 \eta_1^2 t_1 \cos \alpha_1$$

and

$$(\sigma - \mu)^4 = \frac{1}{16} \sigma^4.$$

With this, we obtain from (18)

$$\frac{k}{N_1} = \sigma \left\{ 1 - \frac{1}{24} \sigma^2 (1 + 6 \eta_1^2 \cos^2 \alpha_1) + \frac{1}{4} \sigma^3 \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{1920} \sigma^4 \right\}. \quad (19)$$

Also in this we can introduce the length s of the vertical section with the help of equation (13), p. 11, which we will no longer give in detail now. We find easily

$$k = s \left\{ 1 - \frac{1}{24} \frac{s^2}{N_1^2} (1 + 2 \eta_1^2 \cos^2 \alpha_1) + \frac{1}{8} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{1920} \frac{s^4}{N_1^4} \right\}. \quad (20)$$

If we compare equation (19) with equation (12), then we have first approximation

$$s = k + \frac{1}{24} N_1 \sigma^3, \quad (21)$$

which corresponds to an arc of a circle of length s and radius N_1 , as well as to its chord k .

Radius of curvature of the arc AB

We already know the radius of curvature of the arc of a section AB at the starting point A from the general considerations of the first half-volume, section 39, and we have developed in (4), section 39, p. 54:

$$R_1 = \frac{N_1}{1 + \eta_1^2 \cos^2 \alpha_1}. \quad (22)$$

We also can easily set up an expression for the radius of curvature R_2 at the point B (Fig. 3, p. 12) by using the polar coordinates S_2 and σ of Fig. 3 for the point B .

For a curve given in polar coordinates, analytical geometry offers the rigorous expression

$$R_2 = S_2 \frac{\left\{ 1 + \left(\frac{1}{S_2} \frac{dS_2}{d\sigma} \right)^2 \right\}^{\frac{3}{2}}}{1 + 2 \left(\frac{1}{S_2} \frac{dS_2}{d\sigma} \right)^2 - \frac{1}{S_2} \frac{d^2 S_2}{d\sigma^2}} \quad (23)$$

If we neglect here again the terms $\left(\frac{dS_2}{d\sigma}\right)^2$, which, according to (10a), p. 11, are of the order $\eta_1^4 \sigma^2$, and introduce (9) with the same omission, then we obtain finally

$$R_2 = R_1 \left(1 + 3 \eta_1^2 \sigma \cos \alpha_1 t_1 - \eta_1^2 \frac{\sigma^2}{2} \cos^2 \alpha_1 + \dots \right). \quad (24)$$

In this, R_1 is the radius of curvature of a normal section of the ellipsoid, at a point with the latitude φ_1 in the azimuth α_1 and R_2 is the radius of curvature of the same arc of a section at the distance σ from A .

Section 3. Convergence of the Two Normal Sections

Now while we study more closely the two normal sections between the two points A and B , we examine at first the small azimuth distance $\alpha_1 - \alpha_1'$, which is produced at the one point A .

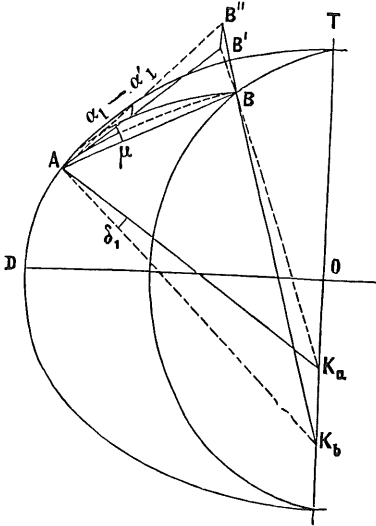


Fig. 1.

For this purpose, we have repeated in Fig. 1 essentially the relations of the former Fig. 2, section 1, p. 2, with the introduction of the azimuth difference $\alpha_1 - \alpha_1'$ and of the angle of depression μ . The azimuth α_1' is given by the tangent AB'' , which we have to imagine to be placed at the normal section BA at A .

The next Fig. 2, p. 15, is generated by describing, around the point A , a sphere with any arbitrary radius on which each ray emanating from A appears as a point and each plane starting from A appears as an arc of a great circle.

A part of Fig. 2 is represented once again in Fig. 3, p. 15, with the same denotations, however with a few completions, and the meaning of the individual denotations will be easy to understand by comparison of the three figures. We only add that n is the angle of inclination of the normal AK_a with the counter-section plane BAK_b and n' is the angle of inclination of the straight line AK_b with the normal-section plane ABK_a on this side. In Fig. 3 at B there is visible the angle ν which the two normal-section planes make with one another.

In the spherical triangles of Fig. 3 we can derive all relations which exist between the individual angles by spherical trigonometry.

At first we aim to determine the difference $\alpha_1 - \alpha_1''$; to this, one of the cotangent equations of the first half-volume, section 33, p. 16, applied to the triangle BK_aK_b , yields:

$$\begin{aligned} \cot (90^\circ - \mu) \sin \delta_1 &= \cos \delta_1 \cos (180^\circ - \alpha_1) + \sin (180^\circ - \alpha_1) \cot \alpha_1'' \\ \tan \mu \sin \delta_1 &= -\cos \delta_1 \cos \alpha_1 + \sin \alpha_1 \cot \alpha_1'' \\ \tan \mu \sin \delta_1 &= \left(2 \sin^2 \frac{\delta_1}{2} - 1 \right) \cos \alpha_1 + \frac{\sin \alpha_1 \cos \alpha_1''}{\sin \alpha_1''} \\ \tan \mu \sin \delta_1 - 2 \cos \alpha_1 \sin^2 \frac{\delta_1}{2} &= \frac{\sin (\alpha_1 - \alpha_1'')}{\sin \alpha_1''}. \end{aligned} \quad (1)$$

$\alpha_1 - \alpha_1''$ is thereby determined; and in order to find also the difference $\alpha_1'' - \alpha_1'$, we examine the small right triangle K_aK_bN , for which we have according to the first half-volume, p. 15:

$$\begin{aligned} \cot \alpha_1'' \cot (90^\circ - \alpha_1') &= \cos \delta_1 \\ \cos \alpha_1'' \sin \alpha_1' &= \sin \alpha_1'' \cos \alpha_1' \left(1 - 2 \sin^2 \frac{\delta_1}{2} \right) \\ \sin (\alpha_1'' - \alpha_1') &= 2 \sin^2 \frac{\delta_1}{2} \sin \alpha_1'' \cos \alpha_1'. \end{aligned} \quad (2)$$

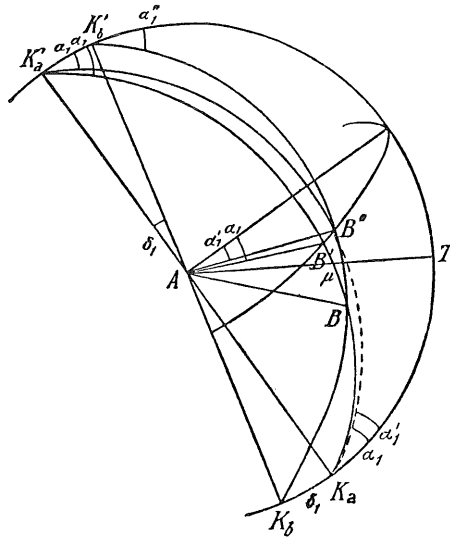


Fig. 2.

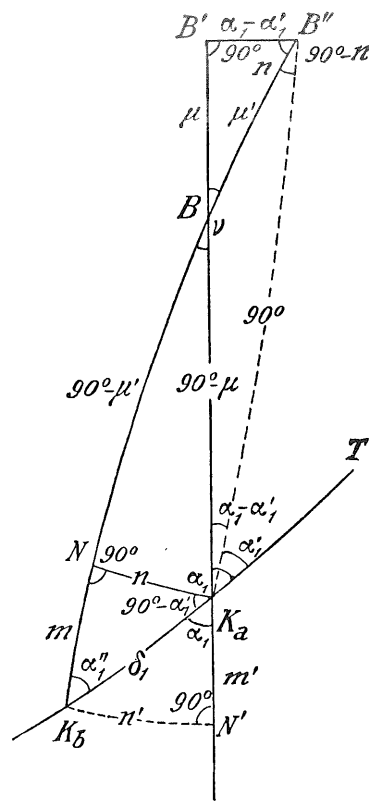


Fig. 3.

After $\alpha_1 - \alpha'_1$ is determined also by means of (1) and (2), then we obtain from the upper right triangle $B B' B''$

$$\tan \nu = \frac{\tan (\alpha_1 - \alpha'_1)}{\sin \mu} \quad (3)$$

and

$$\tan n = \frac{\sin (\alpha_1 - \alpha'_1)}{\tan \mu}, \quad (4)$$

or from the triangle $B K_a N$:

$$\sin n = \sin \nu \cos \mu. \quad (5)$$

n thereby is doubly determined, as in general many additional control equations can be read off from Fig. 3 (to which also μ' , m and m' are noted). In order to express also n and n' directly in δ_1 , we have from the small right triangles $K_a K_b N$ and $K_a K_b N'$:

$$\left. \begin{aligned} \sin n &= \sin \delta_1 \sin \alpha''_1 \\ \sin n' &= \sin \delta_1 \sin \alpha'_1 \end{aligned} \right\} \quad (6)$$

All angles can be computed according to these closed formulae; however it is more convenient to change again to developments in series also here, which we will carry out in the following.

Development in series for $\alpha_1 - \alpha'_1$

We begin with the development of $\alpha_1 - \alpha'_1$ and have from equation (1), p. 14, correct to the fifth order:

$$\alpha_1 - \alpha_1'' = \mu \delta_1 \sin \alpha_1''.$$

By introducing for δ_1 the value from equation (27*), section 1, p. 7, and replacing on the right-hand side also α_1'' by α_1 , which is admissible, since $\alpha_1 - \alpha_1''$ is already of the fourth order, as we see, we obtain:

$$\alpha_1 - \alpha_1'' = \sin \alpha_1 \frac{\sigma}{2} \left(\eta_1^2 \Delta \varphi - \eta_1^2 \Delta \varphi^2 \frac{t_1}{2} \right).$$

According to (8), section 2, p. 10, we have

$$\Delta \varphi = \sigma \cos \alpha_1 - \frac{1}{2} \sigma^2 t_1 \sin^2 \alpha_1 \quad \Delta \varphi^2 = \sigma^2 \cos^2 \alpha_1,$$

and with these we find easily

$$\alpha_1 - \alpha_1'' = \sin \alpha_1 \eta_1^2 \frac{\sigma^2}{2} \left(\cos \alpha_1 - \frac{t_1}{2} \sigma \right).$$

With the help of equation (13), section 2, p. 11, we also can replace σ by s and then have

$$\alpha_1 - \alpha_1'' = \sin \alpha_1 \frac{\eta_1^2}{2} \frac{s^2}{N_1^2} \left(\cos \alpha_1 - \frac{t_1}{2} \frac{s}{N_1} \right). \quad (7)$$

This expression can still be simplified a little if we reduce all this to the mean latitude $\frac{\varphi_1 + \varphi_2}{2} = \varphi$ and introduce also the mean azimuth $\frac{\alpha_1 + \alpha_2}{2} = \alpha$. According to the Taylor series we have

$$\begin{aligned} \sin \alpha_1 &= \sin \alpha - \frac{\alpha_2 - \alpha_1}{2} \cos \alpha \\ \cos \alpha_1 &= \cos \alpha + \frac{\alpha_2 - \alpha_1}{2} \sin \alpha, \end{aligned}$$

to which we take from the first half-volume, section 63, p. 179, equation (29) approximated:

$$\alpha_2 - \alpha_1 = \sigma \sin \alpha t.$$

We have further, likewise according to the Taylor series,

$$\eta_1^2 = \eta^2 (1 + t \Delta \varphi).$$

If we introduce all this into equation (7), in the last term of which t_1 can be directly replaced by t , then there results easily

$$\alpha_1 - \alpha_1'' = \frac{1}{2} \eta^2 \frac{s^2}{N^2} \sin \alpha \cos \alpha, \quad (8)$$

in which everything is referred to the mean latitude φ .

After $\alpha_1 - \alpha_1''$ is thus determined, we still want to develop $\alpha_1'' - \alpha_1'$ according to (2). But since δ_1 is of the third order, we see from (2) that $\alpha_1'' - \alpha_1'$ is of the sixth order, and hence must be neglected. Therefore,

we have the following expression for the azimuth difference between the two vertical sections at A and at the same time also at B :

$$\alpha_1 - \alpha_1' = \alpha_2 - \alpha_2' = -\frac{1}{2} \frac{s_1^2}{N_1^2} \eta_1^2 \sin \alpha_1 \cos \alpha_1 - \frac{1}{4} \frac{s_1^3}{N_1^3} \eta_1^2 t_1 \sin \alpha_1, \quad (9)$$

or else

$$\alpha_1 - \alpha_1' = \alpha_2 - \alpha_2' = \frac{1}{2} \frac{s^2}{N^2} \eta^2 \sin \alpha \cos \alpha. \quad (9^*)$$

In the former equation everything is referred to the latitude φ_1 , whereas in the latter equation all magnitudes are to be computed for the mean latitude φ . In (8) and (9) N_1 could be replaced directly by N since only terms of the sixth order are neglected here.

Transverse distance of the vertical sections

For the angle ν , which the two vertical sections make with one another in the chord AB , we can likewise undertake a development in series, which we will limit, however, to a first approximation. According to equation (3) we have, accurate to terms of the fourth order,

$$\nu = \frac{\alpha_1 - \alpha_1'}{\mu},$$

and hence with the help of the foregoing equation (9) and of the former equation (15) from section 2, p. 12, into which we introduce for σ the approximate value $\frac{s}{N_1}$,

$$\nu = \eta^2 \frac{s}{N} \sin \alpha \cos \alpha. \quad (10)$$

This angle of intersection ν is illustrated in Fig. 5. In order to arrive therefrom at the transverse distance q , we can regard the two elliptic arcs AB and BA as equal arcs, and this we already have established at the end of the previous section 2, pp. 13 and 14. The height of the two arcs is then

$$p = \frac{s^2}{8N},$$

and consequently

$$q = p\nu = \frac{\eta^2}{8} \frac{s^3}{N^2} \sin \alpha \cos \alpha. \quad (11)$$

The calculation of the transverse distance q for a distance $s = 100,000$ m at the azimuth $\alpha = 45^\circ$ and the latitude $\varphi = 45^\circ$ yields

$$q = 0.005 \text{ m.}$$

Since q does not have any practical significance, the foregoing formula of approximation suffices for all estimates.

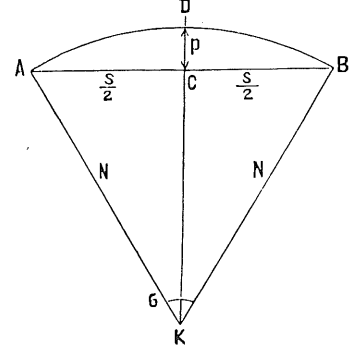


Fig. 4.



Fig. 5.

Section 4. Influence of the Height of the Target Points on the
Measurement of the Horizontal Angles

In the case of angle measurement for triangulation, azimuth measurement and the like, as a rule, the station points and target points do not lie directly on the surface of the ellipsoid of rotation, but at a smaller or greater height above it. The question is how much the measurement of horizontal angles is influenced by this.

At first we have to state that the height of the station point above the surface of the ellipsoid is without influence, since with the theodolite there are measured the angles between vertical planes which pass through the vertical line of the station point. We shall therefore assume in Fig. 1 that the station point A lies directly on the ellipsoid of rotation. Let the target point H , however, be at the height h above the ellipsoid, and let its projection be B' by means of the normal to the ellipsoid HK_b .

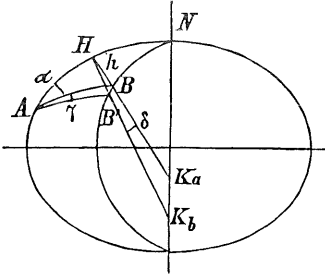


Fig. 1.

Now if we measure from A to B by means of the theodolite, then this is done on the plane AK_a , which is normal at A , and the theodolite does not project the point H to B' , but to B .

In order to determine the azimuth error $BAB' = \gamma$ thereby produced, we have at first to determine the small difference of latitude $B'B$. The small angle $K_aHK_b = \delta'$ is to be replaced directly by the angle δ_2 of Fig. 2, section 1,

p. 2, if magnitudes of the order $\frac{h}{N}$ are neglected in it; and hence, according to

equation (26), section 1, p. 7, we have, accurate to terms of the fourth order,

$$\delta' = \eta_1^2 \Delta \varphi \left(1 - \frac{3}{2} \tan \varphi_1 \Delta \varphi \right) \quad (1)$$

and with the same accuracy $B'B = h \delta'$.

If we denote the latitudes of B and B' by φ_2 and φ_2' , then we have

$$\varphi_2 - \varphi_2' = \frac{h \delta'}{M}.$$

Since δ' is of the third order and terms of the fifth order are neglected, then we can set also according to the first half-volume, section 38, p. 51, equation (29),

$$\varphi_2 - \varphi_2' = \frac{h \delta'}{N_1}. \quad (2)$$

In Fig. 2, section 1, p. 2, now we have to examine further the trihedron ABT with the apex K_a , to which we can apply spherical trigonometry. In this, the three sides are σ_1 , $90^\circ - \varphi_1$ and $90^\circ - (\varphi_2 - \delta_2)$, while of the angles only l and α are of interest to us.

According to the formulae of cotangents of the summary in the first half-volume, section 33, p. 16, we have then

$$\cot \alpha \sin l = \cos \varphi_1 \tan (\varphi_2 - \delta_2) - \sin \varphi_1 \cos l. \quad (3)$$

If instead of the point B the point B' with the latitude φ_2' is introduced, then only the angle α changes in the foregoing equation, and we have

$$\cot \alpha' \sin l = \cos \varphi_1 \tan (\varphi_2' - \delta_2) - \sin \varphi_1 \cos l. \quad (4)$$

Therefore we obtain

$$(\cot \alpha' - \cot \alpha) \sin l = (\tan (\varphi_2' - \delta_2) - \tan (\varphi_2 - \delta_2)) \cos \varphi_1. \quad (5)$$

In the same trihedron we have

$$\sin l \cos (\varphi_2 - \delta_2) = \sin \sigma_1 \sin \alpha, \quad (6)$$

and from (5) and (6) we will have

$$\cot \alpha' - \cot \alpha = (\tan (\varphi_2' - \delta_2) - \tan (\varphi_2 - \delta_2)) \frac{\cos \varphi_1 \cos (\varphi_2 - \delta_2)}{\sin \alpha \sin \sigma_1}.$$

Now in order to pass over to the differences $\alpha' - \alpha$ and $\varphi_2' - \varphi_2$, we have

$$\cot \alpha' - \cot \alpha = -\frac{1}{\sin^2 \alpha} (\alpha' - \alpha) + \dots$$

$$\tan (\varphi_2' - \delta_2) - \tan (\varphi_2 - \delta_2) = \frac{1}{\cos^2 (\varphi_2 - \delta_2)} (\varphi_2' - \varphi_2) + \dots$$

and with these

$$\alpha' - \alpha = -\frac{\sin \alpha \cos \varphi_1}{\sin \sigma_1 \cos (\varphi_2 - \delta_2)} (\varphi_2' - \varphi_2). \quad (7)$$

If in this we develop $\frac{\cos \varphi_1}{\cos (\varphi_2 - \delta_2)}$ in a series and hereby retain only terms with $\varphi_2 - \varphi_1 = \Delta \varphi$, then we can neglect δ_2 which belongs to the third order, and we have

$$\cos \varphi_2 = \cos \varphi_1 - \sin \varphi_1 \Delta \varphi$$

or

$$\frac{\cos \varphi_1}{\cos \varphi_2} = 1 + \tan \varphi_1 \Delta \varphi,$$

and hence, we have

$$\alpha' - \alpha = -\frac{\sin \alpha}{\sin \sigma_1} (1 + \tan \varphi_1 \Delta \varphi) (\varphi_2' - \varphi_2). \quad (8)$$

If we introduce in this the values of (2) and (1), then we will have

$$\alpha' - \alpha = \frac{\sin \alpha}{\sin \sigma_1} \frac{h}{N_1} \eta_1^2 \Delta \varphi (1 + \tan \varphi_1 \Delta \varphi) \left(1 - \frac{3}{2} \tan \varphi_1 \Delta \varphi\right). \quad (9)$$

In this we still can introduce the value from the first half-volume, section 63, p. 178, equation (27), for $\Delta \varphi$, and if we set, at the same time, σ_1 instead of $\sin \sigma_1$, then we obtain

$$\alpha' - \alpha = \frac{h}{N_1} \eta_1^2 \sin \alpha \cos \alpha \left(1 - \frac{1}{2} \sigma_1 \tan \alpha \sin \alpha \tan \varphi_1\right) \\ (1 + \sigma_1 \cos \alpha \tan \varphi_1) \left(1 - \frac{3}{2} \sigma_1 \cos \alpha \tan \varphi_1\right),$$

or by neglecting the terms of the fifth order

$$\alpha' - \alpha = \frac{h}{N_1} \eta_1^2 \left(\sin \alpha \cos \alpha - \frac{1}{2} \sigma \sin \alpha \tan \varphi_1 \right).$$

In the second term there can be set also $\sigma = \frac{s}{N_1}$, and if we add, at the same time, the factor ρ , then we obtain:

$$\alpha' - \alpha = \frac{h}{N_1} \eta_1^2 \rho \left(\sin \alpha \cos \alpha - \frac{s}{2 N_1} \sin \alpha \tan \varphi_1 \right), \quad (10)$$

where α denotes the measured azimuth and α' denotes the azimuth reduced to the surface of the ellipsoid.

As a numerical example for the use of formula (10) we will compute the reduction for the longest side of the quadrangle measured for the connection of Spain with North Africa (cf. the first half-volume, p. 22*), for which we take the necessary numerical data from *Zeitschr. f. Verm.*, 1882, pp. 304-305.

For the direction measured from Filhaussen to Mulhacen we have

$$\begin{aligned} \varphi_1 &= 35^\circ 01' & h &= 3,482 \text{ m} \\ \alpha &= 327 \quad 40 & s &= 269,926 \end{aligned}$$

By computing the two parts of equation (10) separately, we obtain

$$\alpha' - \alpha = -0.2291'' + 0.0040'' = -0.2251''.$$

Therefore, the reduction can by no means be neglected in the case of great heights of the target point.

Effect of refraction. Due to the flattening of the earth and of the level layers of the atmosphere, a deflection of a light ray by refraction does not only take place in the vertical sense but also in the horizontal sense. Every two consecutive elements of a light ray lie on a plane which contains the vertical line of the surface of separation between two differently dense and differently refracting layers of the atmosphere. Since the surface of separation of two such layers must be perpendicular to the direction of the vertical, we find for the course of a light curve through the atmosphere a similar law as for the geodetic line (cf. Fig. 2 in section 6, p. 25), namely that the osculating plane of the light curve must everywhere be the normal plane of the refractive surface or contain the vertical line of this surface.

On the basis of this law, investigations about the azimuthal deflection of the light ray have been made by Andrae, Sonderhof and Helmert (see Helmert, *Höhere Geodäsie II*, p. 565), in a similar way as for the course of the geodetic line on the terrestrial ellipsoid between the two normal sections. Since the light curve in its main curvature is much flatter than the geodetic line on the earth, then also the transverse distance between the normal sections becomes smaller in the same ratio, and likewise also the small angles of the two light curves become smaller than for the geodetic line on the earth. The ratio of curvature between the light line and an earth line is the so-called coefficient of refraction, on the average about $k = 0.13$, and accordingly, the azimuthal deflection of the light ray is to the corresponding geodetic reduction as 0.13 is to 1. Now since this geodetic reduction itself is very small; then, according to the cited investigations by Andrae, Sonderhof and Helmert, the deflection of the light can be neglected.

* Not translated.

The developments of the foregoing sections 1 through 4 enable us to reduce the solution of the spheroidal polar triangle to a spherical problem by computing, according to section 1, the small angles δ_1 or δ_2 , according to section 2, the angle at center σ_1 or σ_2 , and according to section 3, the angle α' or β' . After these preparatory computations, the geodetic main problem in the trihedron ABT in Fig. 2, section 1, p. 2, with the apex K_a or K_b can be solved with the formulae of spherical trigonometry.

It is evident that instead of the closed spherical-trigonometric formulae it is better to apply the development in series of the first half-volume, section 63, p. 176, for this, and now we will transform this development by introducing the auxiliary quantities δ , σ and α' or, as the case may be, β' for the direct computation.

As spherical auxiliary triangle we take the one which in Fig. 2, section 1, p. 2, has the center point of the sphere K_a with the angle at center σ_1 , and which thus contains

$$\begin{array}{llll} \text{the angles} & l & \alpha & \beta' \\ \text{and the sides} & \sigma_1 & 90^\circ - (\varphi_2 - \delta_2) & 90^\circ - \varphi_1. \end{array}$$

Computation of the difference of latitude

For the spherical auxiliary triangle just mentioned, we have, according to the first half-volume, section 63, p. 178, equation (27), accurate to terms of the fourth order,

$$\begin{aligned} \varphi_2 - \delta_2 - \varphi_1 &= \sigma_1 \cos \alpha - \frac{1}{2} \sigma_1^2 \sin^2 \alpha t_1 - \frac{1}{6} \sigma_1^3 \sin^2 \alpha \cos \alpha (1 + 3 t_1^2) \\ &+ \frac{1}{24} \sigma_1^4 \sin^4 \alpha (1 + 3 t_1^2) t_1 - \frac{1}{6} \sigma_1^4 \sin^2 \alpha \cos^2 \alpha (2 + 3 t_1^2) t_1. \end{aligned} \quad (1)$$

To this we take the value for δ_2 from equation (26), section 1, p. 7:

$$\delta_2 = \frac{\Delta \varphi}{V_1^2} \eta_1^2 - \frac{3}{2} \frac{\Delta \varphi^2}{V_1^4} \eta_1^2 t_1 - \frac{2}{3} \frac{\Delta \varphi^3}{V_1^6} \eta_1^2 + \frac{1}{2} \frac{\Delta \varphi^3}{V_1^6} \eta_1^2 t_1^2 + \frac{5}{8} \frac{\Delta \varphi^4}{V_1^8} \eta_1^2 t_1. \quad (2)$$

If we set this into (1) and multiply $\varphi_2 - \varphi_1$ by the unit $\frac{1 + \eta_1^2}{V_1^2}$, then the first term in δ_2 vanishes and we obtain:

$$\begin{aligned} \frac{\varphi_2 - \varphi_1}{V_1^2} &= \sigma_1 \cos \alpha - \frac{1}{2} \sigma_1^2 \sin^2 \alpha t_1 - \frac{1}{6} \sigma_1^3 \sin^2 \alpha \cos \alpha (1 + 3 t_1^2) \\ &+ \frac{1}{24} \sigma_1^4 \sin^4 \alpha (1 + 3 t_1^2) t_1 - \frac{1}{6} \sigma_1^4 \sin^2 \alpha \cos^2 \alpha (2 + 3 t_1^2) t_1 \\ &- \frac{3}{2} \frac{\Delta \varphi^2}{V_1^4} \eta_1^2 t_1 - \frac{2}{3} \frac{\Delta \varphi^3}{V_1^6} \eta_1^2 + \frac{1}{2} \frac{\Delta \varphi^3}{V_1^6} \eta_1^2 t_1^2 + \frac{5}{8} \frac{\Delta \varphi^4}{V_1^8} \eta_1^2 t_1. \end{aligned} \quad (3)$$

From this equation $\frac{\varphi_2 - \varphi_1}{V_1^2} = \frac{\Delta \varphi}{V_1^2}$ is to be determined by the method of gradual approximation.

At first we have the term of first order:

$$\frac{\varphi_2 - \varphi_1}{V_1^2} = \sigma_1 \cos \alpha \quad \text{and therefore} \quad \frac{\Delta \varphi^2}{V_1^4} = \sigma_1^2 \cos^2 \alpha + \dots,$$

in which only terms of the third order are neglected.

With this we find, accurate to the second order:

$$\frac{\varphi_2 - \varphi_1}{V_1^2} = \sigma_1 \cos \alpha - \frac{1}{2} \sigma_1^2 \sin^2 \alpha t_1 - \frac{3}{2} \eta_1^2 t_1 \sigma_1^2 \cos^2 \alpha$$

and therefrom, accurate to the third order:

$$\begin{aligned} \frac{\Delta \varphi^2}{V_1^4} &= \sigma_1^2 \cos^2 \alpha - \sigma_1^3 \sin^2 \alpha \cos \alpha t_1 \\ \frac{\Delta \varphi^3}{V_1^6} &= \sigma_1^3 \cos^3 \alpha. \end{aligned}$$

With this, we already can compute the terms of third order in $\frac{\varphi_2 - \varphi_1}{V_1^2}$, namely:

$$\begin{aligned} \frac{\varphi_2 - \varphi_1}{V_1^2} &= \sigma_1 \cos \alpha - \frac{1}{2} \sigma_1^2 \sin^2 \alpha t_1 - \frac{3}{2} \eta_1^2 t_1 \sigma_1^2 \cos^2 \alpha - \frac{1}{6} \sigma_1^3 \sin^2 \alpha \cos \alpha (1 + 3 t_1^2) \\ &\quad + \frac{3}{2} \eta_1^2 t_1^2 \sigma_1^3 \sin^2 \alpha \cos \alpha - \frac{1}{6} \eta_1^2 (4 - 3 t_1^2) \sigma_1^3 \cos^3 \alpha. \end{aligned}$$

With this value, the higher powers of $\frac{\Delta \varphi}{V_1^2}$ can already be computed finally, and we obtain:

$$\begin{aligned} \frac{\Delta \varphi^2}{V_1^4} &= \sigma_1^2 \cos^2 \alpha - \sigma_1^3 \sin^2 \alpha \cos \alpha t_1 + \frac{1}{4} \sigma_1^4 \sin^4 \alpha t_1 - \frac{1}{3} \sigma_1^4 \sin^2 \alpha \cos^2 \alpha (1 + 3 t_1^2) \\ \frac{\Delta \varphi^3}{V_1^6} &= \sigma_1^3 \cos^3 \alpha - \frac{3}{2} \sigma_1^4 \sin^2 \alpha \cos^2 \alpha t_1 \\ \frac{\Delta \varphi^4}{V_1^8} &= \sigma_1^4 \cos^4 \alpha. \end{aligned}$$

If we introduce these three expressions into equation (3), then we find after collecting the terms:

$$\begin{aligned} \frac{\varphi_2 - \varphi_1}{V_1^2} &= \sigma_1 \cos \alpha - \frac{1}{2} t_1 \sigma_1^2 \sin^2 \alpha - \frac{3}{2} \eta_1^2 t_1 \sigma_1^2 \cos^2 \alpha - \frac{1}{6} \eta_1^2 (4 - 3 t_1^2) \sigma_1^3 \cos^3 \alpha \\ &\quad - \frac{1}{6} (1 + 3 t_1^2 - 9 \eta_1^2 t_1^2) \sigma_1^3 \sin^2 \alpha \cos \alpha + \frac{1}{24} t_1 (1 + 3 t_1^2 - 9 \eta_1^2 t_1^2) \sigma_1^4 \sin^4 \alpha \\ &\quad + \frac{5}{8} \eta_1^2 t_1 \sigma_1^4 \cos^4 \alpha - \frac{1}{12} t_1 (4 + 6 t_1^2 - 18 \eta_1^2 - 9 \eta_1^2 t_1^2) \sigma_1^4 \sin^2 \alpha \cos^2 \alpha. \end{aligned} \quad (4)$$

Now we still have to introduce the length s of the arc of the vertical section, instead of the angle at center σ_1 , for which we take from equation (13), section 2, p. 11:

$$\sigma_1 = \frac{s}{N_1} + \frac{1}{6} \frac{s^3}{N_1^3} \eta_1^2 \cos^2 \alpha - \frac{1}{8} \frac{s^4}{N_1^4} \eta_1^2 t_1 \cos \alpha. \quad (5)$$

With this we will have:

$$\begin{aligned} \frac{\varphi_2 - \varphi_1}{V_1^2} &= \frac{s}{N_1} \cos \alpha - \frac{1}{2} \frac{s^2}{N_1^2} t_1 \sin^2 \alpha - \frac{3}{2} \frac{s^2}{N_1^2} \eta_1^2 t_1 \cos^2 \alpha - \frac{1}{2} \frac{s^3}{N_1^3} \eta_1^2 (1 - t_1^2) \cos^3 \alpha \\ &\quad - \frac{1}{6} \frac{s^3}{N_1^3} (1 + 3 t_1^2 - 9 \eta_1^2 t_1^2) \sin^2 \alpha \cos \alpha - \frac{1}{8} \frac{s^4}{N_1^4} \eta_1^2 t_1 \cos^2 \alpha \\ &\quad + \frac{1}{24} \frac{s^4}{N_1^4} t_1 (1 + 3 t_1^2 - 9 \eta_1^2 t_1^2) \sin^4 \alpha - \frac{1}{12} \frac{s^4}{N_1^4} t_1 (4 + 6 t_1^2 \\ &\quad - 16 \eta_1^2 - 9 \eta_1^2 t_1^2) \sin^2 \alpha \cos^2 \alpha + \frac{5}{8} \frac{s^4}{N_1^4} \eta_1^2 t_1 \cos^4 \alpha. \end{aligned} \quad (6)$$

Finally, we set again for abbreviation:

$$\frac{s}{N_1} \sin \alpha = v \quad \text{and} \quad \frac{s}{N_1} \cos \alpha = u \quad (7)$$

and obtain finally:

$$\begin{aligned} \frac{\varphi_2 - \varphi_1}{V_1^2} = & u - \frac{1}{2} v^2 t_1 - \frac{3}{2} u^2 \eta_1^2 t_1 - \frac{1}{2} u^3 \eta_1^2 (1 - t_1^2) - \frac{1}{6} v^2 u (1 + 3 t_1^2 - 9 \eta_1^2 t_1^2) \\ & + \frac{1}{2} u^4 \eta_1^2 t_1 - \frac{1}{24} v^2 u^2 t_1 (8 + 12 t_1^2 - 29 \eta_1^2 - 18 \eta_1^2 t_1^2) \\ & + \frac{1}{24} v^4 t_1 (1 + 3 t_1^2 - 9 \eta_1^2 t_1^2). \end{aligned} \quad (8)$$

Computation of the difference of longitude

For the spheroidal difference of longitude l we can use, without change, the spherical formula (28) of the first half-volume, section 63, p. 179, and only have to replace the angle at center σ_1 by the length s of the arc of the vertical section according to (5). Then we will have:

$$\begin{aligned} l \cos \varphi_1 = & v + v u t_1 - \frac{1}{3} v^3 t_1^2 + \frac{1}{6} v u^2 (2 + 6 t_1^2 + \eta_1^2) - \frac{1}{24} v^3 u t_1 (8 + 24 t_1^2 + 3 \eta_1^2) \\ & + \frac{1}{24} v u^3 t_1 (16 + 24 t_1^2 + 5 \eta_1^2). \end{aligned} \quad (9)$$

Computation of the difference of azimuth

The third equation to be converted, (29) of the first half-volume, section 63, p. 179, yields the azimuth difference $\beta' - \alpha$ in the spherical triangle. If we introduce here again the length s of the arc of the vertical section according to (5), then we obtain, if we neglect terms with $\frac{s^4}{N_1^4} \eta_1^2$,

$$\begin{aligned} \beta' - \alpha = & \frac{s}{N_1} \sin \alpha t_1 + \frac{1}{2} \frac{s^2}{N_1^2} \sin \alpha \cos \alpha (1 + 2 t_1^2) + \frac{1}{6} \frac{s^3}{N_1^3} \sin \alpha \cos^2 \alpha t_1 (5 + \eta_1^2 + 6 t_1^2) \\ & - \frac{1}{6} \frac{s^3}{N_1^3} \sin^3 \alpha t_1 (1 + 2 t_1^2) + \frac{1}{24} \frac{s^4}{N_1^4} \sin \alpha \cos^3 \alpha (5 + 28 t_1^2 + 24 t_1^4) \\ & - \frac{1}{24} \frac{s^4}{N_1^4} \sin^3 \alpha \cos \alpha (1 + 20 t_1^2 + 24 t_1^4). \end{aligned} \quad (10)$$

To this we take equation (9), section 3, p. 17, in which there is to be set β and β' instead of α_2 and α_2' as well as α and α' instead of α_1 and α_1' , and hence:

$$\beta - \beta' = \frac{1}{2} \frac{s^2}{N_1^2} \eta_1^2 \sin \alpha \cos \alpha - \frac{1}{4} \frac{s^3}{N_1^3} \eta_1^2 t_1 \sin \alpha, \quad (11)$$

and (10) and (11) yield then together:

$$\begin{aligned} \beta - \alpha = & \frac{s}{N_1} \sin \alpha t_1 + \frac{1}{2} \frac{s^2}{N_1^2} \sin \alpha \cos \alpha (1 + 2 t_1^2 + \eta_1^2) \\ & + \frac{1}{12} \frac{s^3}{N_1^3} \sin \alpha \cos^2 \alpha t_1 (10 - \eta_1^2 + 12 t_1^2) - \frac{1}{12} \frac{s^3}{N_1^3} \sin^3 \alpha t_1 (2 + 4 t_1^2 - 3 \eta_1^2) \\ & + \frac{1}{24} \frac{s^4}{N_1^4} \sin \alpha \cos^3 \alpha (5 + 28 t_1^2 + 24 t_1^4) - \frac{1}{24} \frac{s^4}{N_1^4} \sin^3 \alpha \cos \alpha (1 + 20 t_1^2 + 24 t_1^4). \end{aligned} \quad (12)$$

If we introduce also u and v according to (7), then we finally obtain

$$\begin{aligned}\beta - \alpha &= v t_1 + \frac{1}{2} v u (1 + 2 t_1^2 + \eta_1^2) + \frac{1}{12} v u^2 t_1 (10 + 12 t_1^2 - \eta_1^2) \\ &\quad - \frac{1}{12} v^3 t_1 (2 + 4 t_1^2 + 2 \eta_1^2) + \frac{1}{24} v u^3 (5 + 28 t_1^2 + 24 t_1^4) \\ &\quad - \frac{1}{24} v^3 u (1 + 20 t_1^2 + 24 t_1^4).\end{aligned}\quad (13)$$

According to Fig. 2, section 1, p. 2, β is here the azimuth of the arc of the vertical section at the end point B to the starting point A .

According to this, we have as the result of the above developments the following group of formulae:

$$\begin{aligned}\frac{\varphi_2 - \varphi_1}{V_1^2} &= u - \frac{1}{2} v^2 t_1 - \frac{3}{2} u^2 \eta_1^2 t_1 - \frac{1}{2} u^3 \eta_1^2 (1 - t_1^2) - \frac{1}{6} v^2 u (1 + 3 t_1^2 - 9 \eta_1^2 t_1^2) \\ &\quad + \frac{1}{2} u^4 \eta_1^2 t_1 - \frac{1}{24} v^2 u^2 t_1 (8 + 12 t_1^2 - 29 \eta_1^2 - 18 \eta_1^2 t_1^2) \\ &\quad + \frac{1}{24} v^4 t_1 (1 + 3 t_1^2 - 9 \eta_1^2 t_1^2).\end{aligned}\quad (14)$$

$$\begin{aligned}l \cos \varphi_1 &= v + v u t_1 - \frac{1}{3} v^3 t_1^2 + \frac{1}{6} v u^2 (2 + 6 t_1^2 + \eta_1^2) - \frac{1}{24} v^3 u t_1 (8 + 24 t_1^2 + 3 \eta_1^2) \\ &\quad + \frac{1}{24} v u^3 t_1 (16 + 24 t_1^2 + 5 \eta_1^2).\end{aligned}\quad (15)$$

$$\begin{aligned}\beta - \alpha &= v t_1 + \frac{1}{2} v u (1 + 2 t_1^2 + \eta_1^2) + \frac{1}{24} v u^2 t_1 (10 + 12 t_1^2 - \eta_1^2) \\ &\quad - \frac{1}{12} v^3 t_1 (2 + 4 t_1^2 + 3 \eta_1^2) + \frac{1}{24} v u^3 (5 + 28 t_1^2 + 24 t_1^4) \\ &\quad - \frac{1}{24} v^3 u (1 + 20 t_1^2 + 24 t_1^4).\end{aligned}\quad (16)$$

Section 6. The Geodetic Line

After having convinced ourselves in section 1 that between two points A and B of the ellipsoidal surface of the earth in general there exist *two* different normal sections in which the sighting lines are found for theodolite observations from A to B and from B to A , we can also indicate what line is obtained if we stake off several consecutive points A, B, C (Fig. 1) as a straight line in the plane by continued lining up of the theodolite.

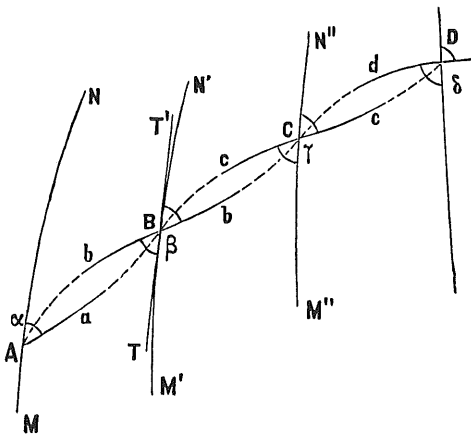


Fig. 1.

In Fig. 1 let a theodolite with a vertical axis with which a distant point B is aimed at or lined up be at A , whereby the sight AaB takes place. After this, we go to B with the theodolite, put it up there likewise with a vertical axis, aim back at A , which gives the sight BbA , then turn by 180° and obtain the new sight BbC . After this, we go to C , take again the sight backward CcB , and having turned by 180° forward, obtain CcD , and so on.

The theory of section 1 has shown us that in the case of this method, of course, *two* connecting lines between every two points A and B , B and C , and so on, are in general involved, namely AaB from A to B and BbA from B to A , and so on; however, the deviations between a and b , c and d , and so on, are so small that even in the case of triangle sides of 100,000 m on our terrestrial surface they still can be neglected.

If our earth were more strongly flattened, then these deviations also would be stronger, and in the sense of the theory, the

point in question is not now the size of the deviations, but that the mathematical law of the line A, B, C, D is understood.

At any rate, in the case of curvature, like our terrestrial ellipsoid, the deviations ab become increasingly smaller if the aiming ranges AB, BC , and so on, are continuously shortened. The small azimuth displacements aAb , and so on, increase according to (9*), section 3, p. 17, only with the square of the aiming ranges; and if we let the aiming ranges $AB, BC \dots$ themselves become infinitesimally small (in the sense of differential calculus), then the loops Aa, Bb , and so on, will close, and instead of the loop line we obtain one constant line $ABCD$, which is called the *geodetic line*, and which in general is a curve of double curvature.

As the direction angle or, as the case may be, azimuth of the geodetic line which results in Fig. 1 after the coinciding of the loops there are to be regarded the angles $\alpha, \beta, \gamma, \delta$ or, more exactly, the limiting values toward which these angles $\alpha, \beta, \gamma, \delta$ converge in the case of unlimitedly decreasing distances AB, BC , and so on.

With the concepts of field- and land-survey in the plane, we can explain the geodetic line briefly in such a way:

We do exactly the same on the ellipsoidal surface of the earth what the land surveyor does when he stakes off, piece by piece, a very long straight line AD in the plane by setting up his theodolite at first at A , then at B, C, D , and setting off each time an angle of 180° .

Or: with regard to continued lining up with short aiming ranges, a geodetic line is the same on a curved surface as is a straight extended traverse line with only angles of 180° in the plane.

It is therefore an excellent nomenclature which Soldner uses in the *Monatliche Korrespondenz zur Beförderung der Erd- und Himmelskunde*, 1805 [Monthly correspondence for the promotion of geography and astronomy, 1805], in which he says: a "geodätisch gerade Linie" [a geodetically straight line].

In the indicated relation, a geodetic line on any curved surface is the same as is an arc of a great circle on a spherical surface.

According to this, if the staking off in small partial stretches in the plane for the straight line, on the sphere for the great circle and on the ellipsoid or any other curved surface for the geodetic line, are analogous to one another, then, on the other hand, for the staking off or sighting on the total length, this analogy no longer exists, and this shall be explained more closely by means of Fig. 2.

Let in Fig. 2 a geodetic line $Aabc \dots ghB$ be obtained by stepwise staking off with the small aiming ranges $Aa = ab = bc$, and so on, whereby the theodolite at a, b, c , and so on, always shows angles of 180° .

But if we set up the theodolite again at A vertically after staking off the individual points, and aim at once to the end point B (as far as the earth's curvature permits this), then we obtain a quite different line of sight than before, namely $AA'B$ as the vertical section from A to B now, and likewise at B the sight $BB'A$ as the vertical section from B to A .

In order to show this still more clearly, in Fig. 3, p. 26, we have represented the two normal sections (vertical sections) between two points A and B , as well as the geodetic line running between on an ellipsoid of rotation with the flattening 1:3.

This Fig. 3, p. 26, is made after a model whose major semiaxis is $a = 15$ cm and whose minor semiaxis is $b = 10$ cm. Therefore, the flattening is $a = \frac{a-b}{a} = \frac{1}{3}$, the eccentricity $e = \sqrt{\frac{a^2 - b^2}{a^2}} = 0.745$ and $e' = \sqrt{\frac{a^2 - b^2}{b^2}} = 1.118$. The normal sections and the geodetic line are constructed according to mathematical laws.

Osculating plane and vertical azimuths

If we put the explanation of the geodetic line derived above from the concepts of field- and land-survey into a more abstract form, then we need the concept of the *osculating plane* [Schmiegungeebene

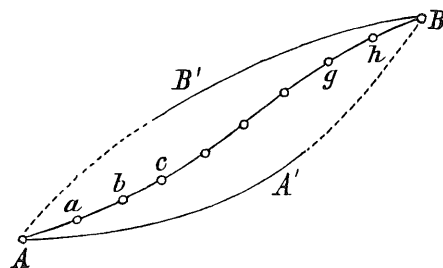


Fig. 2.

(Oskulationsebene)], i.e., of a plane which goes through two consecutive elements of a curve, in our case two consecutive elements of the geodetic line.

According to our explanation of staking off in the field, this is the plane in which at each point there lies the backward and forward directed ray of an angle of 180° , and since this plane goes through the vertical axis of the theodolite, it is the normal plane of the curved surface on which the geodetic line is imagined staked off; and therefore the following fundamental theorem holds true:

The osculating plane of the geodetic line is everywhere the normal plane of the curved surface on which the geodetic line runs.

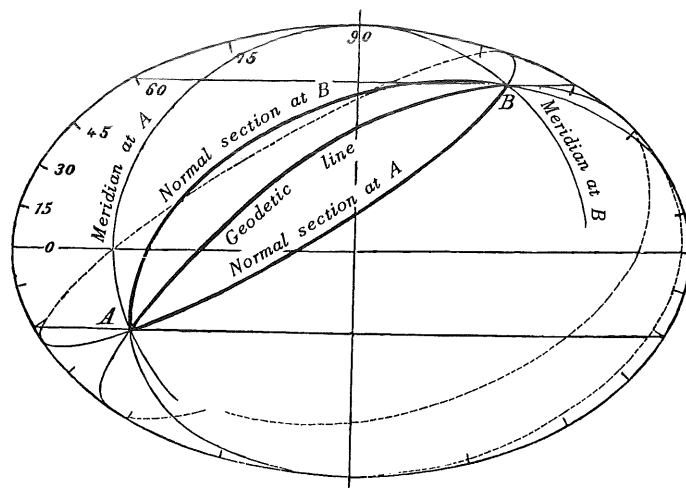


Fig. 3.

If at any arbitrary point of the geodetic line, e.g. B , Fig. 1, p. 24, any arbitrary surface tangent TT' is drawn, then the two vertical angles $bBT = \beta$ and $T'Bb = \beta$, which the geodetic line forms with this tangent, are equal to one another.

At first sight it might seem as if this were self-evident and the case with all curves; it is true, vertical angles between two straight lines, and hence, also between two tangents to a curve are equal to one another at one point, and therefore we would have at the point B for each curve $\beta = \beta'$, if ABC , Fig. 1, p. 24, holds true as *one* element of the curve; however, if the part of the curve ABC is assumed to consist of two or more elements, or with other words, if we aim to examine the curvature of the curve ABC at the point B , then the two angles denoted with β are equal only for the case in which the two elements of the curve coinciding at the point B lie commonly in *one* plane which contains also the normal to the surface of the point B , i.e. which is the osculating plane at B , so that also this normal to the surface at B appears then as a section of the osculating plane and of the plane of the normal to the surface passing through TT' .

All this is satisfied now in the case of the geodetic line, and therefore, if we intersect a geodetic line $ABCD \dots$ (Fig. 1, p. 24) by a family of other geodetic lines MN , $M'N'$, and so on, then all intersection angles occurring here β and β at B , γ and γ at C , and so on, are equal to one another.

We shall later find as geodetic lines MN , $M'N'$, and so on, of Fig. 1, p. 24, all of which are intersected by a geodetic line $ABCD$, especially the meridians of the ellipsoid of rotation, where α , β , γ are the azimuths, and therefore, we express what was recognized about the angles β , β as well as γ , γ , etc., at once in the theorem: *the geodetic line intersects every meridian at equal vertical azimuths.*

Lines of curvature

For the general clarification of the concepts it is proper to mention also the line of curvature in addition to the geodetic line. A line of curvature drawn on a curved surface has the property that every two consecutive surface normals pertaining to it intersect, which is not the case with the geodetic line, as, e.g., is seen from the two points K_a and K_b , Fig. 1, section 1, p. 1.

A line of curvature always follows the greatest or the smallest curvature whose directions are perpendicular to one another according to Euler's theorem (first half-volume, section 39, p. 53); and therefore, all lines of curvature of a surface form two families of curves which intersect reciprocally at right angles everywhere.

A surface point at which the two main radii of curvature (and with it also all radii of curvature of normal sections) are equal is called an "umbilical point" of the surface, e.g., the two poles of the ellipsoid of rotation are umbilical points in this sense; the meridians are lines of curvature of one family, and the parallels are lines of curvature of the second family. The ray-shaped emanating of the meridians as the first family from the pole as the umbilical point is however only a special case and does no longer take place, e.g., in the case of the four umbilical points of the triaxial ellipsoid.

If a line of curvature is at the same time supposed to be a geodetic line, then it must lie entirely in a plane, because each normal to the surface must be intersected by the two neighboring surface normals and must lie also in the plane of two neighboring curve elements. This is possible only in the case of a plane curve; conversely, on the other hand, a line of curvature which lies in a plane is therefore not necessarily a geodetic line.

On the ellipsoid of rotation (as well as on every other surface of rotation) every meridian is a geodetic line and a line of curvature; a parallel is a line of curvature, but not a geodetic line.

Section 7. Differential Equations of the Geodetic Line

After we have become acquainted, in the foregoing section 6, with the geometric properties of the geodetic line, now we pass over to setting up its differential equations.

To this, we examine in Fig. 1 an element ds of the geodetic line between the two points P and P_1 , which have the difference of latitude $d\varphi$ and the difference of longitude $d\lambda$. By the two parallels and the meridians of P and P_1 the small triangle PP_1P' in Fig. 1 is formed. With the principal radii of curvature M and N of the latitude φ we have, since the radius of the parallel is $p = N \cos \varphi$,

$$PP_1 = M d\varphi \quad P_1P' = N \cos \varphi d\lambda.$$

If α is the azimuth of the linear element ds , then we have in the small triangle PP_1P'

$$PP_1 = ds \cos \alpha \quad P_1P' = ds \sin \alpha.$$

And hence, we obtain the two equations

$$ds \cos \alpha = M d\varphi \quad (1)$$

$$ds \sin \alpha = N \cos \varphi d\lambda. \quad (2)$$

Since with regard to ds we have not introduced any further conditions as yet, then the two equations (1) and (2) hold for any arbitrary curve on the ellipsoid.

In order to express the special properties of the geodetic line, in Fig. 1 we have represented a second linear element $P'P''$ whereby the three points P, P', P'' are supposed to lie on a geodetic line. This requirement is satisfied if the three points lie in a vertical plane of the ellipsoid passing through P' , which is then the osculating plane of the geodetic line at P' . Let the azimuth of the geodetic line at P' be equal to $\alpha + d\alpha$.

For the further study, in Fig. 1 we have laid, around the parallel $AP'C$ with the latitude φ , a cone tangent to the ellipsoid whose apex S lies on the axis of rotation. Since the distances of the points P and P'' from P' are infinitesimally small, then P and P'' can be regarded likewise as lying on the cone. We draw further the two tangents SP_1 and SP' and imagine the envelope of the cone developed, so that we obtain a plane triangle PSP' . Because of the equality of the two angles at P' the three points P, P' and P'' lie then in the plane on a straight line. Since the angle $\alpha + d\alpha$ is the exterior angle of the triangle PSP' , then it follows that the acute angle at S is equal to $d\alpha$.

With this we can easily determine the size of the angle $d\alpha$. We have:

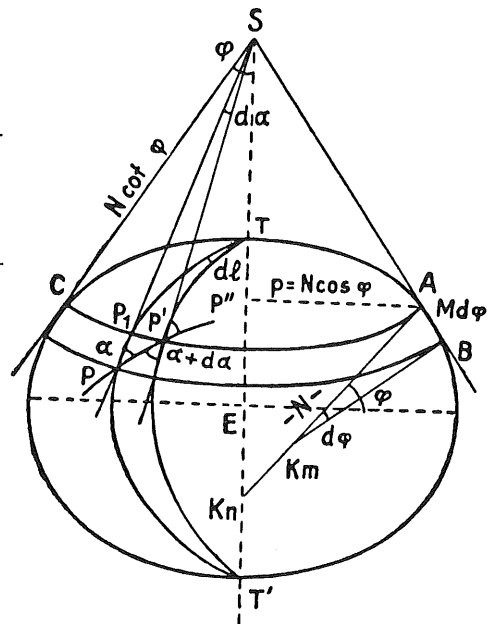


Fig. 1.

$$P_1 P^1 = p dl = N \cos \varphi dl \quad \text{and} \quad d\alpha = \frac{P_1 P^1}{P_1 S} = \frac{N \cos \varphi dl}{N \cot \varphi}$$

or

$$d\alpha = \sin \varphi dl. \quad (3)$$

With the equations (1), (2) and (3) we have already found the differential equations of the geodetic line.

A further property of the geodetic line results with the help of the radius of the parallel $p = N \cos \varphi$.

We obtain by differentiation

$$dp = -N \sin \varphi d\varphi + \cos \varphi dN. \quad (4)$$

But according to the first half-volume, section 38, p. 50, equation (26), we have

$$N = \frac{c}{V} \quad \text{and hence} \quad \frac{dN}{d\varphi} = -\frac{c}{V^2} \frac{dV}{d\varphi},$$

and according to section 40, p. 62, equation (h),

$$\frac{dV}{d\varphi} = -\frac{\eta^2 t}{V} \quad \text{and hence} \quad \frac{dN}{d\varphi} = \frac{c}{V^2} \frac{\eta^2 t}{V} = N \frac{\eta^2 t}{V^2}.$$

Therefore we will have

$$dp = -N \sin \varphi d\varphi + N \frac{\eta^2}{V^2} \sin \varphi d\varphi$$

or

$$dp = -N \sin \varphi \left(1 - \frac{\eta^2}{V^2}\right) d\varphi = -\frac{N \sin \varphi}{V^2} d\varphi.$$

But this is according to the first half-volume, section 38, p. 51, equation (29),

$$dp = -M \sin \varphi d\varphi. \quad (5)$$

Now if we write equation (3) in the form

$$d\alpha = \frac{M \sin \varphi d\varphi}{M d\varphi} dl,$$

then we obtain with the help of (5) and (1)

$$d\alpha = -\frac{dp}{M d\varphi} dl = -\frac{dp}{\cos \alpha} \frac{dl}{ds}.$$

Since according to (2) we have

$$dl = \frac{\sin \alpha}{N \cos \varphi} ds = \frac{\sin \alpha}{p} ds;$$

then we will have

$$d\alpha = -\frac{\sin \alpha dp}{\cos \alpha p}$$

or

$$p \cos \alpha d\alpha + \sin \alpha dp = 0. \quad (6)$$

But this is the complete differential of

$$p \sin \alpha = k \text{ (constant),} \quad (7)$$

and with this we have found the original equation of the geodetic line set up first by Clairaut. In words, the theorem expressed by (7) reads:

The product of the radius of the parallel p by the sine of the azimuth α is constant for the whole course of the geodetic line.

This theorem, to which the law of sines of spherical trigonometry corresponds on the sphere, gives information at once about the entire course of a geodetic line on the ellipsoid of rotation.

The two factors p and $\sin \alpha$, whose product must remain constant $= k$ according to (7), themselves vary between easily assignable limits. The azimuth α can in general not become $=$ zero (which corresponds to the special case of the meridian), but has its smallest value when p has its largest value, i.e. at the equator, where $p = a$; and hence:

$$\sin \alpha_{min} = \frac{k}{a}. \quad (8)$$

The largest value of α , i.e. 90° , corresponds to the smallest value of p , i.e. with $\sin \alpha = 1$, we have:

$$p_{min} = k. \quad (9)$$

The constant k of the formula (7) is therefore the radius of the northernmost or southernmost parallel which the geodetic line can reach; and by this there is also determined a certain extreme geographic latitude beyond which a geodetic line cannot go.

In Fig. 3, section 6, p. 26, this extreme latitude is $= 60^\circ$. The geodetic line touches alternately the northern and the southern extreme parallel, and since it does in general not return into itself, it encompasses the spheroid in an infinite number of spirally curved winding lines between the extreme parallels mentioned.

The question under which conditions geodetic lines return into themselves after a number of circuits around the terrestrial ellipsoid is examined by Schmehl in *Zeitschr. f. Verm.*, 1930, pp. 1-10.

Summary of the principal formulae

We will compile once again our formulae found which are needed further:

$$(1) \quad ds \cos \alpha = M d\varphi \quad (\varphi)$$

$$(2) \quad ds \sin \alpha = N \cos \varphi dl \quad (l)$$

$$(3) \text{ and } (2) \quad d\alpha = dl \sin \varphi \quad \text{or} \quad d\alpha = \frac{ds}{N} \sin \alpha \tan \varphi \quad (\alpha)$$

$$(7) \quad p \sin \alpha = k \quad (p = N \cos \varphi). \quad (\psi)$$

Here is M the radius of curvature in the meridian, N the radius of curvature in the prime vertical and p the radius of the parallel for the latitude φ .

We have denoted the last of the foregoing equations by (ψ) , because it is applied later to the "reduced latitude" ψ .

We examine an arbitrary curve on the ellipsoid which shall be determined by the equation

$$\varphi = f(l). \quad (1)$$

For an element ds of this curve we find from (1) and (2), section 7, p. 27,

$$ds^2 = M^2 d\varphi^2 + N^2 \cos^2 \varphi dl^2$$

or

$$ds = \sqrt{M^2 \left(\frac{d\varphi}{dl}\right)^2 + N^2 \cos^2 \varphi} dl. \quad (2)$$

If we set in condensed denotation

$$ds = U dl, \quad (3)$$

then U is a function of φ and $\frac{d\varphi}{dl}$ or of φ and φ' , as we see from (2).

Between two points P_1 and P_2 of the curve which have the longitudes l_1 and l_2 there results then

$$s = \int_{l_1}^{l_2} U dl. \quad (4)$$

Thus far we have presupposed an arbitrary curve. Now we will determine the function $f(l)$ in such a way that the curve forms the shortest connection of the two points P_1 and P_2 on the ellipsoid.

For the solution of this problem we introduce, instead of the function $\varphi = f(l)$, a new function $\varphi = f(l, \varepsilon)$ in which ε denotes a variable parameter. But here there shall exist the two equations

$$f(l_1, \varepsilon) = \varphi_1 \quad \text{and} \quad f(l_2, \varepsilon) = \varphi_2 \quad (5)$$

in which φ_1 and φ_2 are the latitudes of the two points P_1 and P_2 . If we assume different values for ε , then we obtain another function each time; therefore, the function $\varphi = f(l, \varepsilon)$ represents a family of curves which all pass through the points P_1 and P_2 , and for $\varepsilon = 0$ the function shall be the required one. Also the integral (4), which we will denote by s_ε now, assumes different values for different values of ε , and it

shall have a minimum for $\varepsilon = 0$. This will be the case if $\frac{\partial s_\varepsilon}{\partial \varepsilon}$ for $\varepsilon = 0$ becomes likewise equal to zero.

And hence, the function $f(l, \varepsilon)$ must be determined in such a way that we will have

$$\frac{\partial s_\varepsilon}{\partial \varepsilon} = 0 \quad \text{for} \quad \varepsilon = 0. \quad (6)$$

Then $f(l, 0)$ is the required function.

The differentiation of the definite integral s_ε yields

$$\frac{\partial s_\varepsilon}{\partial \varepsilon} = \int_{l_1}^{l_2} \frac{\partial U}{\partial \varepsilon} dl,$$

and since U is a function of φ and of $\varphi' = \frac{\partial \varphi}{\partial l}$, then we have

$$\frac{\partial s_\varepsilon}{\partial \varepsilon} = \int_{l_1}^{l_2} \left(\frac{\partial U}{\partial \varphi} \frac{\partial \varphi}{\partial \varepsilon} + \frac{\partial U}{\partial \varphi'} \frac{\partial^2 \varphi}{\partial l \partial \varepsilon} \right) dl. \quad (7)$$

For the transformation of this expression we set up the following total differential:

$$d \left(\frac{\partial U}{\partial \varphi'} \frac{\partial \varphi}{\partial \varepsilon} \right) = \frac{\partial U}{\partial \varphi'} \frac{\partial^2 \varphi}{\partial \varepsilon \partial l} dl + \frac{\partial \varphi}{\partial \varepsilon} d \frac{\partial U}{\partial \varphi'}$$

or

$$\frac{d}{dl} \left(\frac{\partial U}{\partial \varphi'} \frac{\partial \varphi}{\partial \varepsilon} \right) = \frac{\partial U}{\partial \varphi'} \frac{\partial^2 \varphi}{\partial \varepsilon \partial l} + \frac{\partial \varphi}{\partial \varepsilon} \frac{d}{dl} \frac{\partial U}{\partial \varphi'},$$

and hence we have

$$\frac{\partial U}{\partial \varphi'} \frac{\partial^2 \varphi}{\partial \varepsilon \partial l} = \frac{d}{dl} \left(\frac{\partial U}{\partial \varphi'} \frac{\partial \varphi}{\partial \varepsilon} \right) - \frac{\partial \varphi}{\partial \varepsilon} \frac{d}{dl} \frac{\partial U}{\partial \varphi'}. \quad (8)$$

Introduced into (7) this yields:

$$\frac{\partial s_\varepsilon}{\partial \varepsilon} = \int_{l_1}^{l_2} \left\{ \frac{\partial U}{\partial \varphi} \frac{\partial \varphi}{\partial \varepsilon} + \frac{d}{dl} \left(\frac{\partial U}{\partial \varphi'} \frac{\partial \varphi}{\partial \varepsilon} \right) - \frac{\partial \varphi}{\partial \varepsilon} \frac{d}{dl} \frac{\partial U}{\partial \varphi'} \right\} dl.$$

But we have

$$\int_{l_1}^{l_2} \frac{d}{dl} \left(\frac{\partial U}{\partial \varphi'} \frac{\partial \varphi}{\partial \varepsilon} \right) dl = \left[\frac{\partial U}{\partial \varphi'} \frac{\partial \varphi}{\partial \varepsilon} \right]_{l_1}^{l_2},$$

and hence we obtain

$$\frac{\partial s_\varepsilon}{\partial \varepsilon} = \left[\frac{\partial U}{\partial \varphi'} \frac{\partial \varphi}{\partial \varepsilon} \right]_{l_1}^{l_2} + \int_{l_1}^{l_2} \left(\frac{\partial U}{\partial \varphi} - \frac{d}{dl} \frac{\partial U}{\partial \varphi'} \right) \frac{\partial \varphi}{\partial \varepsilon} dl. \quad (9)$$

Since we will limit ourselves to such curves which pass through the two points P_1 and P_2 , so that φ_1 and φ_2 are constant quantities, then we have

$$\frac{\partial \varphi_1}{\partial \varepsilon} = 0 \text{ and } \frac{\partial \varphi_2}{\partial \varepsilon} = 0 \text{ and hence also } \left[\frac{\partial U}{\partial \varphi'} \frac{\partial \varphi}{\partial \varepsilon} \right]_{l_1}^{l_2} = 0.$$

Now if we set at the same time $\frac{\partial s_\varepsilon}{\partial \varepsilon} = 0$, then there remains from (9)

$$\frac{\partial s_\varepsilon}{\partial \varepsilon} = \int_{l_1}^{l_2} \left(\frac{\partial U}{\partial \varphi} - \frac{d}{dl} \frac{\partial U}{\partial \varphi'} \right) \frac{\partial \varphi}{\partial \varepsilon} dl = 0.$$

Since the differential quotient $\frac{\partial \varphi}{\partial \varepsilon}$ is quite an arbitrary value here, which in general is not equal to zero, then we must have

$$\frac{\partial U}{\partial \varphi} - \frac{d}{dl} \frac{\partial U}{\partial \varphi'} = 0 \quad (10)$$

This differential equation of the second order is to be integrated. In this

$$\frac{\partial U}{\partial \varphi'} = \frac{\partial U}{\partial \frac{d\varphi}{dl}}$$

is a function of φ and φ' , and hence, we have

$$\frac{d}{dl} \frac{\partial U}{\partial \varphi'} = \frac{\partial}{\partial \varphi} \frac{\partial U}{\partial \varphi'} \frac{d\varphi}{dl} + \frac{\partial}{\partial \varphi'} \frac{\partial U}{\partial \varphi'} \frac{d\varphi'}{dl} \quad (11)$$

and with this, (10) changes to

$$\frac{\partial U}{\partial \varphi} = \frac{\partial}{\partial \varphi} \frac{\partial U}{\partial \varphi'} \frac{d\varphi}{dl} + \frac{\partial}{\partial \varphi'} \frac{\partial U}{\partial \varphi'} \frac{d\varphi'}{dl}. \quad (12)$$

We have further

$$\frac{dU}{dl} = \frac{\partial U}{\partial \varphi} \frac{d\varphi}{dl} + \frac{\partial U}{\partial \varphi'} \frac{d\varphi'}{dl}$$

and if we introduce in this the value of $\frac{\partial U}{\partial \varphi}$ from (12)

$$\frac{dU}{dl} = \frac{\partial}{\partial \varphi} \frac{\partial U}{\partial \varphi'} \left(\frac{d\varphi}{dl} \right)^2 + \frac{\partial}{\partial \varphi'} \frac{\partial U}{\partial \varphi'} \frac{d\varphi'}{dl} \frac{d\varphi}{dl} + \frac{\partial U}{\partial \varphi'} \frac{d\varphi'}{dl}. \quad (13)$$

The first two terms on the right-hand side in (13) are equal to the two terms on the right-hand side in (11) if we multiply the latter equation by $\frac{d\varphi}{dl}$; and hence we have

$$\frac{dU}{dl} = \frac{d}{dl} \frac{\partial U}{\partial \varphi'} \frac{d\varphi}{dl} + \frac{\partial U}{\partial \varphi'} \frac{d\varphi'}{dl} = \frac{d}{dl} \frac{\partial U}{\partial \varphi'} \frac{d\varphi}{dl} + \frac{\partial U}{\partial \varphi'} \frac{d^2 \varphi}{dl^2}$$

or

$$\frac{dU}{dl} = \frac{d}{dl} \left(\frac{\partial U}{\partial \varphi'} \frac{d\varphi}{dl} \right).$$

Integrated this yields

$$U = \frac{\partial U}{\partial \varphi'} \frac{d\varphi}{dl} + k, \quad (14)$$

where k denotes the constant of integration.

Now we have according to (2) and (3), p. 30,

$$U = \sqrt{M^2 \left(\frac{d\varphi}{dl} \right)^2 + N^2 \cos^2 \varphi} = \sqrt{M^2 \varphi'^2 + p^2}, \quad (15)$$

and hence, we will have

$$\frac{\partial U}{\partial \varphi'} = \frac{M^2 \varphi'}{\sqrt{M^2 \varphi'^2 + p^2}}. \quad (16)$$

If we set (15) and (16) into (14), then there follows

$$\sqrt{M^2 \varphi'^2 + p^2} - \frac{M^2 \varphi'}{\sqrt{M^2 \varphi'^2 + p^2}} = k$$

or

$$\frac{p^2}{\sqrt{M^2 \varphi'^2 + p^2}} = k. \quad (17)$$

From the two equations (1) and (2), section 7, p. 27, there follows

$$\tan \alpha = \frac{p dl}{M d\varphi} \quad \text{and hence} \quad \frac{d\varphi}{dl} = \varphi' = \frac{p}{M} \cot \alpha,$$

and with this (17) changes to

$$\frac{p^2}{\sqrt{p^2 \cot^2 \alpha + p^2}} = k \quad \text{or} \quad \frac{p}{\sqrt{\cot^2 \alpha + 1}} = k.$$

There follows hence

$$p \sin \alpha = k. \quad (18)$$

In (18) we have found equation (7), section 7, p. 29, of the geodetic line again and thus established that the geodetic line is at the same time the shortest line of connection of two points on the ellipsoid.

Geodetic circle and geodetic parallel

From the concept of the shortest line there can be derived, by simple geometric consideration, two theorems concerning the "geodetic circle" and the "geodetic parallel." Gauss has represented this in the treatise "Disquisitiones generales circa superficies curvas," Arts. 15 and 16, thusly:

Geodetic circle. If on a curved surface an infinite number of shortest lines all of equal length emanate from a starting point, then the line connecting their ends is normal to all individual ones.

Let in Fig. 1 AB and AB' be two equally long shortest lines which include between them the infinitely small angle at A ; and let us at first assume that the two angles at B and B' are *not* both $= 90^\circ$ but deviate from 90° by a finite quantity, so that according to the law of continuity one would be larger, the other smaller than 90° , e.g. $B = 90^\circ - \omega$. Then we assume on the line BA a point C in such a way that we will have $BC = BB' \operatorname{cosec} \omega$; and if the infinitely small triangle $BB'C$ can be regarded as plane, there follows hence $CB' = BC \cos \omega$ and further

$$AC + CB' = AC + BC \cos \omega = AB - BC (1 - \cos \omega).$$

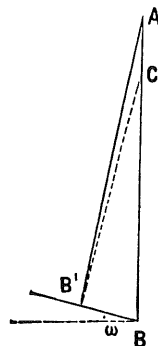


Fig. 1.

But it is assumed from the outset that $AB = AB'$; therefore:

$$AC + CB' = AB' - BC (1 - \cos \omega),$$

According to this, we would have from A to B' a *shorter* path through C than directly through AB' , and this is in contradiction to the assumption that AB' itself is a shortest line. Therefore, ω cannot be a finite quantity, but the angles at B and at B' are both $= 90^\circ$.

Geodetic parallel. If on a curved surface there is drawn an arbitrary line from whose individual points an infinite number of shortest lines of equal length start, perpendicularly to the line and toward the same side, then the curve which connects the other end points of the shortest lines intersects them all at right angles.

We can prove this similarly as in the case of Fig. 1, p. 33, by introducing again a small angle ω and treating two infinitely closely neighboring geodetic lines like two straight lines in the plane.

This second theorem about the geodetic parallel is more general than the first theorem of the geodetic circle, which is contained also in the second theorem if we only assume an infinitely small circle described around A as the given line.

The system of meridians and parallels on the sphere or on the ellipsoid of rotation (and on other surfaces of rotation) offers an obvious example for geodetic circles and parallels. The parallels are geodetic circles with regard to the pole as center point of the meridians and are geodetic parallels with regard to any arbitrary parallel.

Just as these parallels themselves are *not* geodetic lines, the geodetic circles and geodetic parallels themselves are in general not geodetic lines.

Section 9. Comparison of the Geodetic Line with the Normal Sections

In the sense of field survey, the geodetic line appears at a short stretch in all parts like a *straight line*; if we surveyed it at short stretches as a traverse line, then we would find only angles of 180° , as in the case of a straight line in the plane.

On the other hand, a normal section at greater distance appears as a straight line, in the sense of field survey, only in the immediate neighborhood of the point at which it is normal (through whose plumb line it passes); in its further course, by a traverse, it would no longer yield angles of 180° , but produce an image like a flat curve in the plane.

We will aim at determining the *radius of curvature* of a normal section as a curve in this sense; by so doing it will be possible to compare a geodetic line with the normal sections with regard to azimuths as well as with regard to the linear extent.

In Fig. 1 we have illustrated all relations: The broken line AB represents a geodetic line between two points at A and B ; AaB is the normal section at A and BbA the normal section at B . The denotation of the azimuths of these three curves at the points A and B results from Fig. 1.

At first we need the inclination n , which a normal section plane forms with the normal to the surface of the ellipsoid at the distance s from the point at which it is normal itself. With this inclination, we have occupied ourselves already in section 3, and we have found there in equation (6), p. 15:

$$\sin n = \sin \delta_1 \sin \alpha'' \quad \text{or approximately} \quad n = \delta_1 \sin \alpha. \quad (1)$$

Here n is the angle of inclination of the vertical section BbA with respect to the direction of the vertical at A .

Accordingly, we have

$$n' = \delta_2 \sin \beta \quad (2)$$

where, in contrast to section 3, p. 15, n' now denotes the angle of inclination of the vertical section AaB

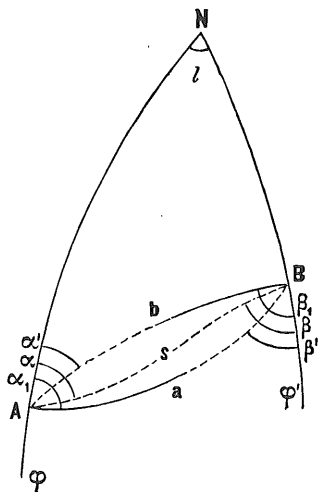


Fig. 1.

with respect to the direction of the vertical at B .

To this we have according to (15), section 1, p. 4,

$$\delta_2 = \eta_1^2 \frac{\Delta \varphi}{V_1^2} - \frac{3}{2} \eta_1^2 \left(\frac{\Delta \varphi}{V_1^2} \right)^2 t_1 - \dots \quad (3)$$

Now we have from the developments in series (6) and (12) of section 5, pp. 22 and 23:

$$\frac{\Delta \varphi}{V_1^2} = \frac{s}{N_1} \cos \alpha_1 - \frac{1}{2} \frac{s^2}{N_1^2} \sin^2 \alpha_1 t_1 + \dots \quad (4)$$

and

$$\beta_1 = \alpha_1 + \frac{s}{N_1} \sin \alpha_1 t_1 + \dots \quad (5)$$

In these two equations (4) and (5) we still have retained the correct azimuths α_1 and β_1 . However, since it is only a question of the evaluation of the approximate equation (2), then in (4) and (5) we can also replace α_1 and β_1 by α and β . Then we will have

$$\sin \beta = \sin \alpha + \frac{s}{N_1} \sin \alpha \cos \alpha t_1. \quad (6)$$

If we set (3), (4) and (6) into (2), then we obtain:

$$n' = \eta_1^2 \frac{s}{N_1} \sin \alpha \cos \alpha - \eta_1^2 \frac{s^2}{2 N_1^2} \sin \alpha t_1 \quad (7)$$

This is the inclination which the vertical section $A a B$ forms with the normal to the surface of B at the opposite point B .

We still denote the radius of curvature of the arc of the vertical section at A with R_1 and at B with R_2 and then we can use for R_2 the equation (24) from section 2, p. 14:

$$R_2 = R_1 (1 + 3 \eta_1^2 \Delta \varphi t_1 + \dots). \quad (8)$$

With this, we now can easily indicate the radius of curvature ρ of the above-defined curve of intersection, i.e., of the curve which results by projection of the curve of intersection $A a B$ on the plane of tangency of B .

If we imagine a cylinder surface standing vertically at B laid through the arc $A a B$, then the plane of tangency at B intersects this cylinder surface perpendicularly, and the curve of intersection is the curve whose radius of curvature ρ we aim to determine.

According to the theorem of Meusnier from analytical geometry, the radius of curvature R_2 is equal to the projection ρ on the plane of the arc of intersection $A a B$, and hence

$$R_2 = \rho \sin n' \quad \text{or} \quad \rho = \frac{R_2}{\sin n'}. \quad (9)$$

Therefore, we obtain from (7) and (8), if terms of the fifth order are neglected:

$$\frac{1}{\rho} = \eta_1^2 \frac{s}{R_1 N_1} \sin \alpha \cos \alpha - \eta_1^2 \frac{s}{2 R_1 N_1^2} \sin \alpha t_1. \quad (10)$$

Now we consider the surface strip which contains the three curves, and whose width according to (11),

section 3, p. 17, is only equal to the product from s and terms of fourth order, fixed at one end, for instance at B , and twisted at the other end in such a way that the direction of the vertical at A lies in a plane with the direction of the vertical at B . According to (7), for $s = 100,000$ m and $\alpha = 45^\circ$ there is necessary to this a twisting $n' = 4.5''$ at A . From the size of this angle we recognize that an assignable change of form of the curves within the surface strip is not caused by the twisting. The surface strip forms then the element of a cylinder surface while the geodetic line represents the curve of intersection of the cylinder surface and of a plane lying normally to the axis of the latter.

Now we still can go a step further and spread the cylinder surface out in a plane so that we obtain the picture represented in Fig. 2. The distance between the two end points A and B will change here only by approximately 1.03 m according to (21), section 2, p. 13, if we assume again a distance of 100 km, so that the radius of curvature ρ in (10) will hold true also for the plane curve of Fig. 2. The angles at A and B between the geodetic line now appearing as a straight line and the two arcs of vertical sections will remain unchanged here.

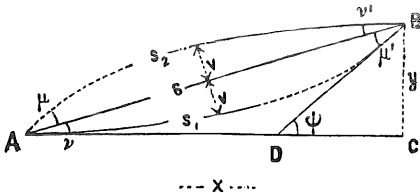


Fig. 2.

For this plane representation we introduce a rectangular system of coordinates for which the tangent AC of the normal section As_1B is taken as the axis of abscissae.

At the point A the reciprocal radius of curvature of the curve As_1B is equal to the second differential quotient of the curve; but since it is a question of a very flat curve, then we can regard this relation as valid for the whole curve from A to B .

Since in (10) we can replace also further s by x , then we have

$$\frac{d^2 y}{dx^2} = Px + Q \frac{x^2}{2}, \quad (11)$$

where we set for abbreviation

$$P = \frac{\eta_1^2}{R_1 N_1} \sin \alpha \cos \alpha \quad \cdot \quad Q = - \frac{\eta_1^2}{R_1 N_1^2} \sin \alpha t_1. \quad (12)$$

We can integrate the curve equation (11) twice:

$$\frac{dy}{dx} = \frac{Px^2}{2} + \frac{Qx^3}{6} \quad (13)$$

$$y = \frac{Px^3}{6} + \frac{Qx^4}{24}. \quad (14)$$

Now it is easy to indicate the angles ν and μ' which the curve As_1B forms on both ends with the geodetic line AB .

From (14) we find with $x = s$

$$\nu = \frac{y}{x} = \frac{Ps^2}{6} + \frac{Qs^3}{24},$$

and likewise from (13):

$$\psi = \nu + \mu' = \frac{Ps^2}{2} + \frac{Qs^3}{6}, \quad \text{and hence} \quad \mu' = \frac{Ps^2}{3} + \frac{Qs^3}{8}.$$

By introduction of the coefficients P and Q of (12) these two equations yield:

$$\nu = \frac{\eta_1^2}{6} \frac{s^2}{R_1 N_1} \sin \alpha \cos \alpha - \frac{\eta_1^2}{24} \frac{s^3}{R_1 N_1^2} \sin \alpha t_1 \quad (15)$$

$$\mu' = \frac{\eta_1^2}{3} \frac{s^2}{R_1 N_1} \sin \alpha \cos \alpha - \frac{\eta_1^2}{8} \frac{s^3}{R_1 N_1^2} \sin \alpha t_1 . \quad (16)$$

Besides, we have from the first half-volume, section 39, p. 54, equation (4)

$$\frac{1}{R_1} = \frac{1}{N_1} (1 + \eta_1^2 \cos^2 \alpha),$$

and if terms of the sixth order are to be neglected, then we can also write (15), with regard to Fig. 1, p. 34:

$$\alpha_1 - \alpha = \varrho \frac{\eta_1^2}{6} \frac{s^2}{N_1^2} \sin \alpha \cos \alpha - \varrho \frac{\eta_1^2}{24} \frac{s^3}{N_1^3} \sin \alpha t_1 . \quad (17)$$

In this, α_1 is the azimuth of the vertical section AB , and hence, at the point A the astronomical azimuth to the point B , while α is the azimuth of the geodetic line from A to B . Therefore, by means of this formula (17) we can reduce the directions measured with the theodolite to the geodetic line.

Accordingly, (16) yields the formula

$$\beta - \beta' = \varrho \frac{\eta_1^2}{3} \frac{s^2}{N_1^2} \sin \alpha \cos \alpha - \varrho \frac{\eta_1^2}{8} \frac{s^3}{N_1^3} \sin \alpha t_1 , \quad (18)$$

which has no practical significance, however.

Now we pass over to the determination of the difference of the length of the geodetic line and of the arc of the vertical section.

The length s_1 of the curve As_1B is found by forming $AC = c$, as a first approximation:

$$s_1 = \int_0^c \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^c \left(1 + \frac{P^2 x^4}{8}\right) dx = c + \frac{P^2 c^5}{40} . \quad (19)$$

For comparison, we have the length of the geodetic line AB in Fig. 2, p. 36,

$$s = \sqrt{c^2 + \left(\frac{P c^3}{6}\right)^2} = c + \frac{P^2 c^5}{72} . \quad (20)$$

From (17) and (18) we have the difference by setting $c = AC$ sufficiently $= s$ now:

$$s_1 - s = \frac{P^2 s^5}{90} = \frac{\eta_1^4}{90} \frac{s^5}{N_1^4} \sin^2 \alpha \cos^2 \alpha . \quad (21)$$

We find also easily the transverse distance ν as the difference of the mean ordinates as a first approximation:

$$\nu = \frac{1}{2} \frac{P s^3}{6} - \frac{P}{6} \left(\frac{s}{2}\right)^3 = \frac{1}{16} P s^3 = \frac{\eta_1^2}{16} \frac{s^3}{N_1^2} \sin \alpha \cos \alpha . \quad (22)$$

In the case of these formulae (21) and (22) we have neglected the second terms with Q , which are still retained in (17) and (18). Besides, the expression (22) agrees with the former equation (11) in section 3, p. 17.

Now if we aim to have also the angles ν' and μ of Fig. 2, p. 36, then we must repeat the foregoing development from the initial equations (2) and (3) for the case of the second curve, which we will not discuss here. Besides, we see at once that the first term in the expressions for ν' and μ will nearly agree with the first term of ν and μ' .

These formulae let us recognize that the geodetic line in general lies *between* the two normal sections, as in Fig. 2, p. 36.

However, the special case is to be borne in mind here that the points *A* and *B*, between which the geodetic line and the two normal sections are drawn lie on the same latitude, as is suggested in Fig. 3.

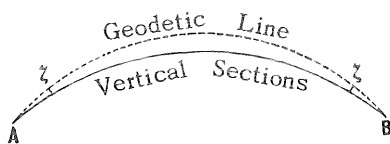


Fig. 3.

In this case, the two normal sections (vertical sections) coincide, and the geodetic line can therefore no longer lie between them. It is understood immediately that the geodetic line itself cannot coincide with these two plane sections insofar as the geodetic line cannot in this case be itself a plane curve.

In the former third edition of this volume (1890), p. 381, it was shown that in this case, according to Fig. 3, the geodetic line lies above the vertical section.

For a first numerical example we will take $s = 100,000$ m and $\varphi = 45^\circ$ as well as $\alpha = 45^\circ$; with this, we will have $\log r = 6.8046$, $\eta^2 = 0.00336$, and with this

$$\begin{aligned}\mu = \mu' &= 0.028'' & \nu = \nu' &= 0.014'' \\ 2\nu &= 0.005 \text{ m} & s_1 - s = s_2 - s &= 0.00000 \ 00002 \text{ m.}\end{aligned}$$

The value $2\nu = 5$ mm agrees with the former (11), section 3, p. 17. A second example, used several times by us, with the mean latitude $\varphi = 50^\circ$, the two end latitudes $49^\circ 30'$, and $50^\circ 30'$ and $l = 1^\circ$, yields: $s = 132,315$ m $\alpha = 32^\circ 48'$

$$\begin{aligned}\log r &= 6.80489 & \log \eta^2 &= 7.44345 \\ \mu = \mu' &= 0.0373'' & \nu = \nu' &= 0.0187'' \\ 2\nu &= 0.009 \text{ m} & s_1 - s = s_2 - s &= 0.00000 \ 00004 \text{ m.}\end{aligned}$$

These numerical examples show that for the common triangle sides the small angles μ and ν can be neglected in the case of the present-day state of surveying. The difference $s_1 - s$ would not even be measurable microscopically.

On the relations between the vertical sections and the geodetic line see also W. Grossmann, "Reihenentwicklungen zur Theorie der Vertikalschnitte" [developments in series to the theory of vertical sections], *Zeitschr. f. Verm.*, 1935, pp. 33-46.

Significance of the geodetic line for practical surveys

The geodetic line is never the object of direct measurement, but only of computation, and thereby indirectly an auxiliary means for extended geodetic measurements.

In the case of the measurement of the individual triangles, we do not speak of geodetic lines, for the sights of the measurement with theodolite take place undoubtedly in vertical sections, not in geodetic lines; and the astronomical azimuth measurements also do not refer to the geodetic line, but likewise to vertical sections.

We can reduce the measured azimuths and the measured horizontal angles from the vertical sections to the geodetic lines, as has been shown in the above; the reduction amounts to very little, namely only to $0.04''$ in the azimuth for 45° latitude and azimuth 45° in the case of a distance of $100,000$ m, so that this reduction is mostly neglected.

Let it be now that we neglect these small reductions (besides others, e.g. reduction of elevation of section 4), or take them into account; in any case, we *can* do it, and instead of a triangulation net measured in normal sections, now we can thus set a triangulation net whose sides are geodetic lines, and whose angles are enclosed by the horizontal tangents of the geodetic lines at the corner points.

We will not learn until the following Chapter II how we can compute such a spheroidal triangulation net of geodetic lines in theoretical rigorousness; in practice, the spherical triangle computation is nearly always sufficient.

About the significance of the geodetic line for practical geodesy, in general this much can be said: The introduction of the theory of the geodetic line in geodesy is not a necessity, as is, e.g., the theory of the rectilinear plane triangles for the plane triangulation; we could treat the problems of higher geodesy, for instance, also by chord triangles and polyhedric-spatial point determinations and in still many other ways; but the geodetic line has proven until now to be the best means for establishing the necessary mathematical relations between the direct geodetic and astronomic measurements on one hand and the assumptions about the earth's surface, on the other.

COMPUTATION OF THE SPHEROIDAL TRIANGLE

Section 10. The Reduced Length of the Geodetic Line

Although the spherical formulae of the first half-volume,* sections 46-48, will suffice, as a rule, for the computation of triangulation nets, in this chapter we will treat also the computation of spheroidal triangles, partly in order to arrive at formulae of general validity, partly also to test the applicability of the former spherical formulae.

To this, we have to develop a few other auxiliary theorems.

If in a plane system of polar coordinates (Fig. 1) the direction angle α_1 is enlarged by the small angle $d\alpha_1$, then in the case of a constant radius vector, the end point P_2 describes a linear element

$$P_2 P'_2 = dp = s d\alpha_1 . \quad (1)$$

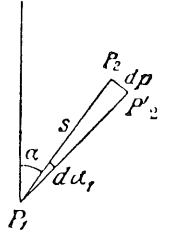


Fig. 1.

For a spherical system of polar coordinates with the radius of a sphere R the radius vector is to be replaced by an arc of a great circle, and if the central angle of this arc is equal to σ , then the displacement of the point P_2 will be

$$dp = R \sin \sigma d\alpha_1 . \quad (2)$$

By setting $R \sin \sigma = m$, we obtain also for the spherical surface the simple equation:

$$dp = m d\alpha_1 , \quad (3)$$

in which m , however, is no longer constant now, but is a function of σ .

This idea can be transferred also to the surface of the ellipsoid, as well as in general to any arbitrary mathematical surface, where the radius vector shall be formed by a geodetic line. In this case α_1 is the azimuth and s the length of the geodetic line, and we can express the displacement of P_2 again by equation (3), in which m will be a function of s and α_1 as well as of the position of the point P_1 . We designate this function m , which we will develop in the following as the *reduced length* of the geodetic line.

For the computation of the reduced length we examine at the point P_1 the vertical section which passes through P_2 . A rotation of the vertical section about the direction of the vertical at P_1 by the angle $d\alpha'_1$ corresponds to the rotation of the geodetic line by the angle $d\alpha_1$. If k is the length of the chord $P_1 P_2$ and μ its angle of inclination to the horizontal line at P_1 , then the distance of the point P_2 from the vertical line at P_1 is equal to $k \cos \mu$, and consequently

$$dp = k \cos \mu d\alpha'_1 . \quad (4)$$

Comparing (3) and (4) we obtain:

$$m = k \cos \mu \frac{d\alpha'_1}{d\alpha_1} . \quad (5)$$

For the product $k \cos \mu$ we can use the equations (16) and (20) in section 2, pp. 12 and 13, which we write here once again; we have

$$k = s \left\{ 1 - \frac{1}{24} \frac{s^2}{N_1^2} (1 + 2 \eta_1^2 \cos^2 \alpha_1) + \frac{1}{8} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{1920} \frac{s^4}{N_1^4} \right\} \quad (6)$$

$$\mu = \frac{s}{2 N_1} \left(1 + \eta_1^2 \cos^2 \alpha_1 - \frac{s}{N_1} \eta_1^2 t_1 \cos \alpha_1 \right). \quad (7)$$

Strictly speaking, α_1 is here the azimuth of the arc of the vertical section. But the difference $\alpha_1' - \alpha_1$ is of the fourth order, as results from the following equation (10), and for this reason we can also regard α_1 in (6) and (7) directly as the azimuth of the geodetic line.

Since

$$\cos \mu = 1 - \frac{\mu^2}{2} + \frac{\mu^4}{24} - \dots$$

then we obtain from (7):

$$\cos \mu = 1 - \frac{1}{8} \frac{s^2}{N_1^2} (1 + 2 \eta_1^2 \cos^2 \alpha_1) + \frac{1}{4} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{384} \frac{s^4}{N_1^4} \quad (8)$$

and the product of (6) and (8) yields:

$$k \cos \mu = s \left\{ 1 - \frac{1}{6} \frac{s^2}{N_1^2} (1 + 2 \eta_1^2 \cos^2 \alpha_1) + \frac{3}{8} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{120} \frac{s^4}{N_1^4} \right\}. \quad (9)$$

For the computation of the quotient $\frac{d \alpha_1'}{d \alpha_1}$ in (5) we take the angle between the geodetic line and the vertical section from (17), section 9, p. 37. According to this we have

$$\alpha_1' - \alpha_1 = \frac{1}{6} \frac{s^2}{N_1^2} \eta_1^2 \sin \alpha_1 \cos \alpha_1 - \frac{1}{24} \frac{s^3}{N_1^3} \eta_1^2 t_1 \sin \alpha_1, \quad (10)$$

and therefore we will have:

$$d \alpha_1' = d \alpha_1 \left\{ 1 + \frac{1}{6} \frac{s^2}{N_1^2} \eta_1^2 \cos 2 \alpha_1 - \frac{1}{24} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 \right\}. \quad (11)$$

With this, we are ready for the computation of m , and now according to (5), we only have to form the product of (9) and (11). If all terms of the sixth order are omitted again here and $\cos 2 \alpha_1$ is replaced by $2 \cos^2 \alpha_1 - 1$, then there follows easily:

$$m = s \left\{ 1 - \frac{1}{6} \frac{s^2}{N_1^2} (1 + \eta_1^2) + \frac{1}{3} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{120} \frac{s^4}{N_1^4} \right\}. \quad (12)$$

In this development the quantity s is equal to the length of the arc of the vertical section from P_1 to P_2 . If in (12) we wish to replace s by the length of the geodetic line, then, according to (21), section 9, p. 37, in the parenthesis of (12) there would have to be added only terms of the eighth order, which we have not taken into account from the outset, however. In (12) we can therefore understand by s without further ado the length of the geodetic line.

We will once again transform the foregoing equation (12) by introducing, instead of the latitude φ_1 of the one end point P_1 of the geodetic line, the mean latitude as argument. Therefore, we set:

$$\frac{\varphi_1 + \varphi_2}{2} = \varphi \quad \frac{\varphi_2 - \varphi_1}{2} = \frac{\Delta \varphi}{2}$$

and denote the auxiliary quantities referred to this mean latitude by N , η , t , and so on, without index.

If we take into account again only terms of the fifth order, then we can replace at once the quantities N_1 , η_1 , and so on, by N , η in the last two terms of (12). For the second term in (12) we have according to the first half-volume, section 40, p. 63, equations (m) and (o):

$$\begin{aligned} \frac{1}{N_1} &= \frac{1}{N} \left(1 + \frac{1}{2} \frac{\eta^2 t}{V^2} \Delta \varphi - \frac{1}{8} \frac{\eta^2}{V^4} (1 - t^2) \Delta \varphi^2 + \dots \right) \\ \frac{1}{N_1^2} &= \frac{1}{N^2} \left(1 + \frac{\eta^2 t}{V^2} \Delta \varphi + \dots \right) \end{aligned}$$

and

$$\eta_1^2 = \eta^2 + \eta^2 t \Delta \varphi \dots$$

Therefore, we will have

$$-\frac{1}{6} \frac{s^2}{N_1^2} (1 + \eta_1^2) = -\frac{1}{6} \frac{s^2}{N^2} \left(1 + \frac{\eta^2 t}{V^2} \Delta \varphi \right) \left(1 + \eta^2 + \eta^2 t \Delta \varphi \right),$$

and if we also set $\frac{\Delta \varphi}{V^2}$ instead of $\Delta \varphi$ in the last term, then we have

$$-\frac{1}{6} \frac{s^2}{N_1^2} (1 + \eta_1^2) = -\frac{1}{6} \frac{s^2}{N^2} (1 + \eta^2) - \frac{1}{3} \frac{s^2}{N^2} \eta^2 t \frac{\Delta \varphi}{V^2}.$$

But now according to equation (6), section 5, p. 22, we have:

$$\frac{\Delta \varphi}{V^2} = \frac{s}{N} \cos \alpha + \dots,$$

and if all this is set into equation (12), then we obtain to terms of the fifth order inclusive:

$$m = s \left\{ 1 - \frac{1}{6} \frac{s^2}{N^2} (1 + \eta^2) + \frac{1}{120} \frac{s^4}{N^4} + \dots \right\}. \quad (13)$$

Since $1 + \eta^2 = V^2$, then we have according to the first half-volume, p. 51,

$$\frac{1 + \eta^2}{N^2} = \frac{1}{MN} = \frac{1}{r^2}.$$

Likewise we have

$$\frac{1}{N^4} = \frac{1}{r^4} - \frac{1}{(1 + \eta^2)^2},$$

where the second factor can be neglected, however, since we have retained only the terms of the fifth order. Therefore, we may set in place of (13) with the same accuracy:

$$m = s \left\{ 1 - \frac{1}{6} \frac{s^2}{r^2} + \frac{1}{120} \frac{s^4}{r^4} + \dots \right\},$$

or also

$$\frac{m}{r} = \sin \frac{s}{r}. \quad (14)$$

We find the same expression also when we compute the reduced length of the geodetic line for the second end point P_2 . We conclude therefrom that the reduced length of a geodetic line has the same value for both end points.

We have proved this proposition here only for the development in series up to the fifth order, but we must still add that it holds true quite generally for the geodetic line. With regard to the general proof we refer to "Helmert, *Die math. u. phys. Theor. d. höh. Geod.*," Bd. I, 1880, p. 273.

It is to be noted that the expression (14) according to (2), p. 41, corresponds to the purely spherical calculation for a sphere of the radius r .

The measure of curvature

According to the first half-volume, section 38, p. 51, by the mean radius of curvature for any arbitrary point of the terrestrial ellipsoid we have understood the geometric mean of the two main radii of curvature, and hence the quantity \sqrt{MN} . According to C. F. Gauss, we denote the reciprocal value of the square of the mean radius of curvature as the *measure of curvature* of the surface at the point in question. If we introduce the notation K for this, then we have accordingly:

$$\text{Measure of curvature } K = \frac{1}{MN}. \quad (15)$$

Gauss' definition of the measure of curvature refers to quite arbitrary mathematical surfaces in the case of which M and N denote then the largest and the smallest radius of curvature at an arbitrary point. Likewise for arbitrary surfaces, a simple relation between the measure of curvature K and the reduced length of the geodetic line can be found now. We will at first find this relation for the simplest case, for a sphere of radius R , for which we have $K = \frac{1}{R^2}$. Let the arc of a great circle s , which takes here the place of the geodetic line, have the central angle σ ; then according to (2), p. 41, its reduced length is

$$m = R \sin \sigma,$$

while

$$s = R \sigma.$$

Hence there follows

$$\begin{aligned} \frac{dm}{ds} &= \cos \sigma \\ \frac{d^2 m}{ds^2} &= -\sin \sigma \frac{d\sigma}{ds} = -\frac{1}{R} \sin \sigma = -\frac{m}{R^2}, \end{aligned}$$

or

$$\frac{1}{m} \frac{d^2 m}{ds^2} = -K. \quad (16)$$

The validity of this simple equation (16) for arbitrary surfaces has been proved by C. F. Gauss. It would lead us too far to present this proof in a generally valid form even only for the ellipsoid of rotation; however, we will show the proof of the validity for the terrestrial ellipsoid at least within the limits of our former developments in series.

From (12) we obtain by differentiation:

$$\frac{d m}{d s} = 1 - \frac{s^2}{2 N_1^2} (1 + \eta_1^2) + \frac{4}{3} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{24} \frac{s^4}{N_1^4} \quad (17)$$

and further:

$$\frac{d^2 m}{d s^2} = - \frac{s}{N_1^2} (1 + \eta_1^2) + 4 \frac{s^2}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{6} \frac{s^3}{N_1^4}. \quad (18)$$

To this, we form, in addition, the reciprocal value of m , namely according to (12):

$$\frac{1}{m} = \frac{1}{s} \left\{ 1 + \frac{1}{6} \frac{s^2}{N_1^2} (1 + \eta_1^2) - \frac{1}{3} \frac{s^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 - \frac{7}{360} \frac{s^4}{N_1^4} \right\}, \quad (19)$$

and the product of (18) and (19) yields:

$$\frac{1}{m} \frac{d^2 m}{d s^2} = - \left\{ \frac{s}{N_1^2} (1 + \eta_1^2) - 4 \frac{s^2}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 \right\}. \quad (20)$$

On the other hand, we have according to the first half-volume, section 38, p. 51, equation (28),

$$K_1 = \frac{V_1^4}{c^2} \quad \text{and} \quad K_2 = \frac{V_2^4}{c^2} \quad \text{with} \quad c = \frac{a^2}{b}, \quad (21)$$

where the indices refer to the points P_1 and P_2 .

Now we have according to the Taylor series

$$V_2^4 = V_1^4 + 4 V_1^3 \frac{d V}{d \varphi} (\varphi_2 - \varphi_1) + \dots,$$

or according to the first half-volume, section 40, p. 63, equation (n),

$$V_2^4 = V_1^4 - 4 V_1^2 \eta_1^2 t_1 (\varphi_2 - \varphi_1).$$

But according to (6), section 5, p. 22, we have

$$\frac{\varphi_2 - \varphi_1}{V_1^2} = \frac{s}{N_1} \cos \alpha_1 + \dots,$$

and hence we will have:

$$V_2^4 = V_1^4 - 4 \frac{s}{N_1} V_1^4 \eta_1^2 t_1 \cos \alpha_1 + \dots \quad (22)$$

With the help of (19) and (15) we obtain therefore:

$$K_2 = K_1 - 4 K_1 \frac{s}{N_1} \eta_1^2 t_1 \cos \alpha_1 + \dots$$

or else, since

$$K = \frac{1}{M N} = \frac{1}{N^2} (1 + \eta^2)$$

$$K_2 = \frac{1}{N_1^2} (1 + \eta_1^2) - 4 \frac{s}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 . \quad (23)$$

From (23) and (20) there results:

$$\frac{1}{m} \frac{d^2 m}{ds^2} = -K_2 , \quad (24)$$

with which the above indicated relation (16) is found also for the terrestrial ellipsoid, if terms of the sixth and higher order are disregarded.

A further important theorem, whose validity is proven likewise by Gauss for arbitrary surfaces, exists for a rectangular system of coordinates formed by geodetic lines. This theorem also can be indicated at once for the sphere.

For if in a rectangular-spherical system of coordinates two points P and P' have the same ordinate y and the difference of abscissae Δx , then we set the quotient

$$\frac{PP'}{\Delta x} = n . \quad (25)$$

If the ordinate y has the central angle σ , and hence $y = R \sigma$, then we have, as we see at once:

$$n = \cos \sigma . \quad (26)$$

Then we will have:

$$\frac{dn}{dy} = -\sin \sigma \frac{d\sigma}{dy} = -\frac{1}{R} \sin \sigma$$

$$\frac{d^2 n}{dy^2} = -\frac{1}{R} \cos \sigma \frac{d\sigma}{dy} = -\frac{1}{R^2} \cos \sigma = -\frac{n}{R^2} . \quad (27)$$

There follows hence:

$$K = -\frac{1}{n} \frac{d^2 n}{dy^2} . \quad (28)$$

We must exclude also the general proof of this proposition for the ellipsoid of rotation, but we will show at least as in the case of equation (24) that our developments in series correspond with this equation within the terms carried.

To this, we have to premise a few more general considerations.

In Fig. 1 we have drawn once again, at a larger scale, the right-hand part of the former Fig. 1 from section 10, p. 41, and added also the two meridians passing through P_2 and P_2' . If we also lay a line parallel to the arc P_2P_2' , through the point Q , then there results the point Q' .

We denote the reduced length of the geodetic line P_1P_2 (Fig. 1, p. 41) by m as previously, and have then the following expressions for the two linear elements P_2P_2' and QQ' according to (3), section 10, p. 41:

$$\left. \begin{aligned} dp &= m d\alpha_1 \\ dp' &= \left(m + \frac{dm}{ds} ds \right) d\alpha_1, \end{aligned} \right\} \quad (1)$$

or

$$\frac{dp'}{ds} = \frac{m}{ds} d\alpha_1 + \frac{dm}{ds} d\alpha_1. \quad (2)$$

From the small triangle $P_2'Q'Q$, which we can regard as a plane triangle, there follows with the notations of Fig. 1:

$$\frac{dp'}{ds} = \frac{\sin(\alpha_2 + d\alpha_2)}{\sin(90^\circ - (\alpha_2 + \delta\alpha_2))} = \frac{\sin(\alpha_2 + d\alpha_2)}{\cos(\alpha_2 + \delta\alpha_2)}. \quad (3)$$

For $\sin(\alpha_2 + d\alpha_2)$ and $\cos(\alpha_2 + \delta\alpha_2)$ we use development in series and have then

$$\frac{dp'}{ds} = \frac{\sin \alpha_2 + \cos \alpha_2 d\alpha_2}{\cos \alpha_2 - \sin \alpha_2 \delta\alpha_2} = \frac{\tan \alpha_2 + d\alpha_2}{1 - \tan \alpha_2 \delta\alpha_2},$$

or

$$\frac{dp'}{ds} = \tan \alpha_2 + d\alpha_2 + \tan^2 \alpha_2 \delta\alpha_2. \quad (4)$$

Further we have from the triangle P_2QP_2'

$$\frac{dp}{ds} = \tan(\alpha_2 + \delta\alpha_2) = \tan \alpha_2 + \frac{1}{\cos^2 \alpha_2} \delta\alpha_2, \quad (5)$$

or according to (1)

$$\frac{m d\alpha_1}{ds} = \tan \alpha_2 + \frac{1}{\cos^2 \alpha_2} \delta\alpha_2, \quad (6)$$

We set equations (4) and (6) into (2) and obtain

$$d\alpha_2 + \tan^2 \alpha_2 \delta\alpha_2 = \frac{1}{\cos^2 \alpha_2} \delta\alpha_2 + \frac{dm}{ds} d\alpha_1,$$

or

$$d\alpha_2 = \frac{dm}{ds} d\alpha_1 + \delta\alpha_2. \quad (7)$$

We can easily express the quantity $\delta\alpha_2$ by ds by using the differential equations of the geodetic

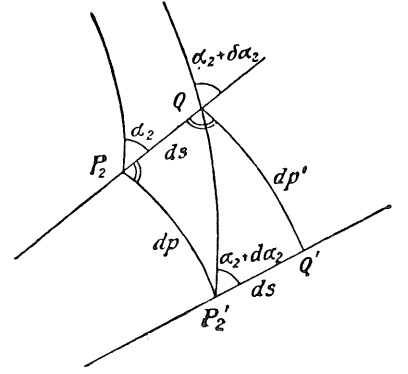


Fig. 1.

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$$ds = dp \cos \Theta \quad m d\alpha_1 = dp \sin \Theta ,$$

and this introduced into (10) yields at once

$$d\alpha_2 = \frac{1}{m} \frac{dm}{ds} \sin \Theta dp + \frac{1}{N_2} \sin(\alpha_2 + \Theta) \tan \varphi_2 dp . \quad (11)$$

But from Fig. 2, p. 48, there follows that

$$d\Theta = d\alpha_2' - d\alpha_2 , \quad (12)$$

and since the differential equation (α) of section 7, p. 29, yields again the relation

$$d\alpha_2' = \frac{1}{N_2} \sin \alpha_2' \tan \varphi_2 dp \quad (13)$$

then we have, if (11) and (13) are introduced into (12):

$$\frac{d\Theta}{dp} = - \frac{1}{m} \frac{dm}{ds} \sin \Theta , \quad (14)$$

an equation which is entirely free from the azimuth α_2 .

Application to spheroidal-rectangular coordinates

If we assume on the ellipsoid of rotation a geodetic line as axis of abscissae and denote as ordinate of a point the geodetic line passing through the point and intersecting the axis of abscissae at right angles, then we have the most general case of a spheroidal-rectangular system of coordinates.

This is represented in Fig. 3, in which p and q are the rectangular coordinates of the point P . Now we imagine the ordinate q rotated by an infinitesimally small angle around the point P , so that the foot point F moves to F' , while at the same time the length PF' will agree with PF up to magnitudes of higher order.

To this rotation we can apply at once equation (14) in which ds is to be replaced by dq now. Then we obtain, since $\Theta = 90^\circ$

$$\frac{d\Theta}{dq} = - \frac{1}{m} \frac{dm}{dq} . \quad (15)$$

Now a second rotation of the line $F'P$ around F' is to take place until this line lies again perpendicular to the axis of abscissae, for which the angle of rotation must be equal to $d\Theta$. By so doing, the point P moves to P' , whose coordinates are $p + dp$ and q . For this latter rotation we have

$$PP' = - m d\Theta , \quad (16)$$

where the negative sign is chosen, because the rotation of the first is opposite.

Corresponding to the function m defined by the equation (3), section 10, p. 41, we will introduce a new function n now by setting

$$PP' = n dp , \quad (17)$$

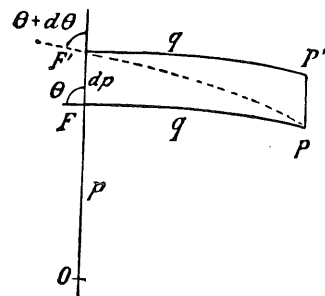


Fig. 3.

where n thus depends on the position of the point P in the system of coordinates. Since according to (15) and (17)

$$-m d\Theta = \frac{dm}{dq} dp = n dp$$

then we will have

$$n = \frac{dm}{dq}. \quad (18)$$

After having found in this important equation a relation between the polar coordinates and the rectangular coordinates of a point, we will introduce also the measure of curvature with the help of equation (24), section 10, p. 46. The latter equation reads with regard to Fig. 3, p. 49:

$$\frac{d^2 m}{dq^2} = -mK,$$

or after repeated differentiation and comparison with (18):

$$\frac{d^3 m}{dq^3} = -\frac{dm}{dq} K = -nK. \quad (19)$$

On the other hand, there follows from equation (18) by differentiation twice done

$$\frac{d^2 n}{dq^2} = \frac{d^3 m}{dq^3}, \quad (20)$$

consequently we obtain from (19) and (20):

$$K = -\frac{1}{n} \frac{d^2 n}{dq^2}. \quad (21)$$

The foregoing equation (21) which, relying on Helmholtz, *Die math. u. physik. Theorien der höheren Geodäsie*, Bd. I, Leipzig, 1880, Kap. 8 and 9, we have derived from the developments in series for the surface of the terrestrial ellipsoid, was developed quite generally for arbitrary mathematical surfaces in the treatise, "Disquisitiones generales circa superficies curvas," auctore Carolo Friederico Gauss, Göttingae, 1828 (societati regiae oblatae d. 8. Okt. 1827). This treatise is contained in *Carl Friedrich Gauss' Werke*, Bd. IV, Göttingen, 1873, pp. 217-258, and in German translation, "Allgemeine Flächentheorie" von Carl Friedrich Gauss, published by A. Wangerin, *Ostwalds Klassiker der exakten Wissenschaften*, Nr. 5, Second Edition, Leipzig, 1900.

In the following there is represented in substance the last part of Gauss' treatise (Article 20-28) in a simple form.

In Fig. 1 O is the starting point of two spheroidal systems of coordinates consisting of geodetic lines, a rectangular system and a polar system, so that, e.g., the point A has the rectangular coordinates p, q and the polar coordinates s, α . Accordingly, the point B has the rectangular coordinates $p + dp, q + dq$ and the polar coordinates $s + ds, \alpha + d\alpha$. But the axis of abscissae or, as the case may be, zero direction OP shall not coincide with the direction of the meridian here but shall have an arbitrary direction, so that the direction angles α are not to be regarded as azimuths.

In addition to the direction angle α of the polar system there is introduced the angle β , which the ray s and the ordinate q make at A .

Between the two ordinates q and $q + dq$ let the transverse distance at A be $AD = n dp$ and, accordingly, $AC = m d\alpha$ be the transverse distance at A between the two rays OA and OB .

n is here the function introduced in the previous section 11, while m is the reduced length of the geodetic line.

The expression for the function n results in consequence of equation (18), in section 11, p. 50, from equation (17) in section 10, p. 45, in which s is to be replaced by q . If the quantities N_1, η_1 and t_1 refer to the geographic latitude of the foot point of ordinates P , and α_1 denotes the azimuth of the geodetic line PA at P , then we have accordingly

$$n = 1 - \frac{1}{2} \frac{q^2}{N_1^2} (1 + \eta_1^2) + \frac{4}{3} \frac{q^3}{N_1^3} \eta_1^2 t_1 \cos \alpha_1 + \frac{1}{24} \frac{q^4}{N_1^4} + \dots, \quad (1)$$

or with a simple designation of the coefficients of q^2, q^3 and $q^4 \dots$

$$n = 1 + f q^2 + g q^3 + h q^4 + \dots \quad (2)$$

If we move the ordinate to the zero point O , then we obtain a corresponding equation

$$n_0 = 1 + f_0 q^2 + g_0 q^3 + h_0 q^4 + \dots,$$

and since in the case of this shifting a constant change of the coefficients $f, g, h \dots$ is to be assumed, then we can set up the relations:

$$\left. \begin{aligned} f &= f_0 + f_1 p + f_2 p^2 + \dots \\ g &= g_0 + g_1 p + g_2 p^2 + \dots \\ h &= h_0 + h_1 p + h_2 p^2 + \dots \end{aligned} \right\} \quad (3)$$

If we set these expressions (3) into (2), and thereby retain all terms up to the fourth order inclusive, then we obtain:

$$\left. \begin{aligned} n &= 1 + f_0 q^2 + f_1 p q^2 + f_2 p^2 q^2 + \dots \\ &\quad + g_0 q^3 + g_1 p q^3 + \dots \\ &\quad + h_0 q^4 + \dots \end{aligned} \right\} \quad (4)$$

Now if we partially differentiate (2) twice with respect to q , then we obtain:

$$\frac{\partial^2 n}{\partial q^2} = 2f + 6gq + 12hq^2 + \dots \quad (5)$$

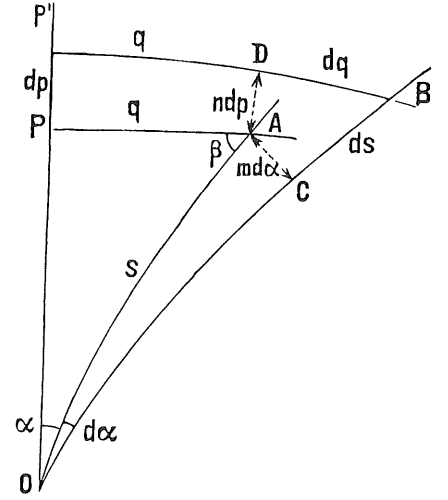


Fig. 1.

Now we can set up the expressions for the measure of curvature at the point A according to equation (21), section 11, p. 50. According to this we have

$$\begin{aligned} -K &= \frac{1}{n} \frac{d^2 n}{d q^2} = \frac{2f + 6gq + 12hq^2 + \dots}{1 + f q^2 + \dots} \\ \text{or} \quad -K &= 2(f + 3gq + 6(h - f^2)q^2) + \dots \end{aligned} \quad (6)$$

If we introduce again the coefficients (4), then we obtain up to the second order inclusive:

$$\left. \begin{aligned} K &= -2f_0 - 2f_1 p - 6g_0 q \\ &\quad - 2f_2 p^2 - 6g_1 p q - (12h_0 - 2f_0^2) q^2. \end{aligned} \right\} \quad (7)$$

But now we shall limit K to a linear function, i.e. we shall set:

$$K = -2f_0 - 2f_1 p - 6g_0 q. \quad (8)$$

This assumption includes that we have in (7):

$$f_2 = 0, \quad g_1 = 0, \quad 12h_0 = 2f_0^2 \quad \text{or} \quad h_0 = \frac{1}{6} f_0^2. \quad (9)$$

With this, the former n of (4) also is reduced to:

$$n = 1 + f_0 q^2 + f_1 p q^2 + g_0 q^3 + \frac{1}{6} f_0^2 q^4. \quad (10)$$

In the following, we also need $\frac{1}{n}$ several times, and for this reason we develop the reciprocal value according to the first half-volume, p. 20:

$$\frac{1}{n} = 1 - f_0 q^2 - f_1 p q^2 - g_0 q^3 + \frac{5}{6} f_0^2 q^4. \quad (11)$$

After these preparations we can pass over to the theory of the geodetic triangle.

In Fig. 1, section 12, p. 51, we study the small quadrilateral $ACBD$, which we have represented once again in the following Fig. 1. While we treat this quadrilateral, perpendicular at C and D , as plane in the differential sense, we take the following equations from it by transformation of coordinates:

$$ds = n dp \sin \beta + dq \cos \beta \quad (1)$$

$$m d\alpha = dq \sin \beta - n dp \cos \beta \quad (2)$$

or in the form of differential equations:

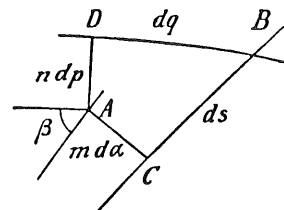


Fig. 1.

$$\frac{\partial s}{\partial p} = n \sin \beta \quad \frac{\partial s}{\partial q} = \cos \beta \quad (3)$$

$$\frac{\partial \alpha}{\partial p} = -\frac{n}{m} \cos \beta \quad \frac{\partial \alpha}{\partial q} = \frac{1}{m} \sin \beta. \quad (4)$$

From the two equations (3) there follows by squaring:

$$n^2 = \left(\frac{\partial s}{\partial p} \right)^2 + n^2 \left(\frac{\partial s}{\partial q} \right)^2; \quad (5)$$

further from (3) and (4) by multiplication:

$$\frac{\partial s}{\partial p} \frac{\partial \alpha}{\partial p} = -n^2 \frac{\partial s}{\partial q} \frac{\partial \alpha}{\partial q}. \quad (6)$$

In order to introduce in (5), instead of the derivatives of s with respect to p and with respect to q , the corresponding derivatives of s^2 , we have:

$$\frac{\partial (s^2)}{\partial p} = \frac{\partial (s^2)}{\partial s} \frac{\partial s}{\partial p} = 2s \frac{\partial s}{\partial p} \text{ and } \frac{\partial (s^2)}{\partial q} = 2s \frac{\partial s}{\partial q}, \quad (7)$$

and hence:

$$n^2 = \left(\frac{1}{2s} \frac{\partial (s^2)}{\partial p} \right)^2 + n^2 \left(\frac{1}{2s} \frac{\partial (s^2)}{\partial q} \right)^2,$$

or

$$4s^2 = \left(\frac{1}{n} \frac{\partial (s^2)}{\partial p} \right)^2 + \left(\frac{\partial (s^2)}{\partial q} \right)^2. \quad (8)$$

If we imagine s^2 developed as a series according to powers of p and q , then this series will start with the terms $p^2 + q^2$, because for infinitely small values p and q , Fig. 2, p. 55, shrinks to a plane right triangle. Therefore, we will assume at first the series for s^2 in the following form:

$$s^2 = p^2 + q^2 + A p^2 q + B p q^2, \quad (9)$$

where A and B are coefficients assumed undetermined for the time being, which must be determined by comparison with (8). At first we have from (9):

$$\left. \begin{aligned} \frac{\partial (s^2)}{\partial p} &= 2p + 2A p q + B q^2 + \dots, \\ \frac{1}{n} &= 1 - f_0 q^2 - \dots \end{aligned} \right\} \quad (10)$$

to this from (11), section 12, p. 52,

Further from (9):

$$\frac{\partial (s^2)}{\partial q} = 2q + A p^2 + 2B p q + \dots \quad (11)$$

From (10) and (11) we can put together the term (8); we obtain to the third power:

$$\text{from (8):} \quad 4s^2 = 4p^2 + 4q^2 + 12A p^2 q + 12B p q^2,$$

$$\text{on the other hand from (9):} \quad 4s^2 = 4p^2 + 4q^2 + 4A p^2 q + 4B p q^2.$$

From these two equations there follows as comparison of coefficients $12A = 4A$ and $12B = 4B$, i.e. $A = 0$ and $B = 0$.

And hence we find that in series (9) only the first two terms exist, and that the terms with third powers $p^2 q$ and $p q^2$ vanish. Therefore, we make an assumption with terms of fourth order now, i.e. instead of (9) let there be now:

$$s^2 = p^2 + q^2 + A p^3 q + B p^2 q^2 + A' p q^3. \quad (12)$$

This yields:

$$\left. \begin{aligned} \frac{\partial (s^2)}{\partial p} &= 2p + 3A p^2 q + 2B p^2 q^2 + A' q^3, \\ \frac{1}{n} &= 1 - f_0 q^2 - \dots \end{aligned} \right\} \quad (13)$$

to this from (11), section 12, p. 52,

Further from (12):

$$\frac{\partial (s^2)}{\partial p} = 2p + A p^3 + 2B p^2 q + 3A' p q^2. \quad (14)$$

If we put together the formula (8) from (13) and (14), then we find from (8):

$$4s^2 = 4p^2 + 4q^2 + 16A p^3 q + 8(2B - f_0) p^2 q^2 + 16A' p q^3,$$

on the other hand from (12): $4s^2 = 4p^2 + 4q^2 + 4A p^3 q + 4B p^2 q^2 + 4A' p q^3.$

The comparison of coefficients in these two equations yields:

$$16A = 4A \quad 16B - 8f_0 = 4B \quad 16A' = 4A',$$

$$\text{i.e.:} \quad A = 0 \quad B = \frac{2}{3}f_0 \quad A' = 0.$$

Now, consequently, according to (12):

$$s^2 = p^2 + q^2 + \frac{2}{3}f_0 p^2 q^2. \quad (15)$$

In this way, progressing one step, we obtain to the fifth order:

$$s^2 = p^2 + q^2 + \frac{2}{3}f_0 p^2 q^2 + \frac{1}{2}f_1 p^3 q^2 + \frac{1}{2}g_0 p^2 q^3. \quad (16)$$

In order to arrive at the sixth order, we add to (16) the following indefinite terms (where A, B, C have new meanings again):

$$+ A p^6 + B p^5 q + C p^4 q^2 + D p^3 q^3 + C' p^2 q^4 + B' p q^5 + A' q^6. \quad (16a)$$

If we form the expression (8) with this, then we obtain:

$$\begin{aligned} 4 s^2 = & 4 p^2 + 4 q^2 + \frac{8}{3} f_0 p^2 q^2 + 2 f_1 p^3 q^2 + 2 g_0 p^2 q^3 \\ & + 24 A p^6 + 24 B p^5 q + \left(24 C + \frac{16}{9} f_0^2 \right) p^4 q^2 + 24 D p^3 q^3 \\ & + 24 A' q^6 + 24 B' p q^5 + \left(24 C' + \frac{16}{9} f_0^2 \right) p^2 q^4. \end{aligned}$$

If we compare this with (16) and (16a) with respect to the coefficients, then we see that A, B, D, B', A' all become $= 0$, and we will have:

$$C = C' = -\frac{4}{45} f_0^2.$$

With this, series (16) and (16a), completed to the sixth order, yields:

$$s^2 = p^2 + q^2 + \frac{1}{6} (4 f_0 p^2 q^2 + 3 f_1 p^3 q^2 + 3 g_0 p^3 q^3) - \frac{4}{45} f_0^2 (p^4 q^2 + p^2 q^4). \quad (17)$$

For the linear function K , $f_2 = 0$, $g_1 = 0$ and $h_0 = \frac{1}{6} f_0^2$ is set here, according to (9), section 12,

p. 52. (If we do not make these limiting assumptions for f_2 , g_1 and h_0 , instead of the foregoing formula (17) we obtain the more general formula [1] of Art. 24 of the "Disquisitiones generales, etc.")

Introduction of the measure of curvature

The general linear expression for the measure of curvature is

$$K = -2 f_0 - 2 f_1 p - 6 g_0 q \quad (18)$$

according to (8), section 12, p. 52.

If we apply this function to our case according to the indication of Fig. 2, then we obtain

$$\left. \begin{aligned} K_\alpha &= -2 f_0 \\ K_{90} &= -2 f_0 - 2 f_1 p \\ K_\beta &= -2 f_0 - 2 f_1 p - 6 g_0 q \end{aligned} \right\} \quad (19)$$

$$K_\alpha + 2 K_{90} + K_\beta = -(8 f_0 + 6 f_1 p + 6 g_0 q).$$

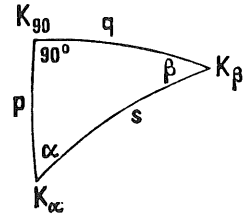


Fig. 2.

This is just the expression which occurs in the large parentheses of (17), and if we set, at the same time, the value K_α^2 from (19) instead of $4 f_0^2$ in the last parentheses of (17), or in short K instead of K_α , because it is in the last term of a converging series, then (17) changes to:

$$s^2 = p^2 + q^2 - \frac{K_\alpha + 2 K_{90} + K_\beta}{12} p^2 q^2 - \frac{K^2}{45} p^2 q^2 (p^2 + q^2), \quad (20)$$

where the coefficients f_0, f_1 and g_0 are replaced completely by the measures of curvature.

Developments in series for $s \sin \alpha$ and $s \cos \alpha$

Of the general differential formulae, developed at the beginning of the section, we have (3), p. 53:

$$n \sin \beta = \frac{\partial s}{\partial p} \quad \text{and} \quad \cos \beta = \frac{\partial s}{\partial q},$$

or with introduction of the variable s^2 , as above in the case of (7):

$$\begin{aligned} n \sin \beta &= \frac{1}{2s} \frac{\partial (s^2)}{\partial p} & \cos \beta &= \frac{1}{2s} \frac{\partial (s^2)}{\partial q} \\ 2s \sin \beta &= \frac{1}{n} \frac{\partial (s^2)}{\partial p} & 2s \cos \beta &= \frac{\partial (s^2)}{\partial q}. \end{aligned} \quad (21)$$

Now s^2 has been determined according to (17), from which we find by differentiation:

$$\left. \begin{aligned} \frac{\partial (s^2)}{\partial p} &= 2p + \frac{4}{3} f_0 p q^2 + \frac{3}{2} f_1 p^2 q^2 + g_0 p q^3 - \frac{16}{45} f_0^2 p^3 q^2 - \frac{8}{45} f_0^2 p q^4. \\ \frac{1}{n} &= 1 - f_0 q^2 - f_1 p q^2 - g_0 q^3 + \frac{5}{6} f_0^2 q^4. \end{aligned} \right\} \quad (22)$$

from (11), p. 52,

Further the derivative with respect to q from (17):

$$\frac{\partial (s^2)}{\partial q} = 2q + \frac{4}{3} f_0 p^2 q + f_1 p^3 q + \frac{3}{2} g_0 p^2 q^2 - \frac{8}{45} f_0^2 p^4 q - \frac{16}{45} f_0^2 p^2 q^3. \quad (23)$$

The two equations (21) can be carried out according to (22) and (23), and yield:

$$s \sin \beta = p - \frac{1}{3} f_0 p q^2 - \frac{1}{4} f_1 p^2 q^2 - \frac{1}{2} g_0 p q^3 - \frac{8}{45} f_0^2 p^3 q^2 + \frac{7}{90} f_0^2 p q^4 \quad (24)$$

$$s \cos \beta = q + \frac{2}{3} f_0 p^2 q + \frac{1}{2} f_1 p^3 q + \frac{3}{4} g_0 p^2 q^2 - \frac{4}{45} f_0^2 p^4 q - \frac{8}{45} f_0^2 p^2 q^3. \quad (25)$$

If we introduce again the measures of curvature according to (19), then we bring (24) and (25) to these forms:

$$s \sin \beta = p + \frac{K_\alpha + K_{90} + 2 K_\beta p q^2}{4} - \frac{K^2}{360} p q^2 (16 p^2 - 7 q^2) \quad (26)$$

$$s \cos \beta = q - \frac{2 K_\alpha + 3 K_{90} + 3 K_\beta p^2 q}{8} - \frac{K^2}{45} p^2 q (p^2 + 2 q^2). \quad (27)$$

These equations hold, of course, also for the other angle α , and yield with the corresponding interchange of the p 's, q 's and K 's:

$$s \sin \alpha = q + \frac{K_\beta + K_{90} + 2 K_\alpha p^2 q}{4} - \frac{K^2}{360} p^2 q (16 q^2 - 7 p^2) \quad (28)$$

$$s \cos \alpha = p - \frac{2 K_\beta + 3 K_{90} + 3 K_\alpha p q^2}{8} - \frac{K^2}{45} p q^2 (q^2 + 2 p^2). \quad (29)$$

For a check, we also can calculate:

$$(s \sin \beta)^2 + (s \cos \beta)^2 = s^2 \quad \text{or} \quad (s \sin \alpha)^2 + (s \cos \alpha)^2 = s^2.$$

By so doing, we shall find the same expression for s^2 , as in the case of (20) already.

We can also invert the series (26) to (29) (as in the first half-volume, section 50, p. 105, the series for $s \sin \alpha$ and $s \cos \alpha$ were inverted). We find:

$$p = s \cos \alpha + \frac{s^3}{3} \sin^2 \alpha \cos \alpha \frac{3 K_\alpha + 3 K_{90} + 2 K_\beta}{8} + \frac{K^2}{15} p q^2 (-p^2 + 2 q^2) \quad (30)$$

$$q = s \sin \alpha - \frac{s^3}{6} \sin \alpha \cos^2 \alpha \frac{2 K_\alpha + K_{90} + K_\beta}{4} + \frac{K^2}{120} p^2 q (p^2 - 8 q^2). \quad (31)$$

In the last term $p = s \cos \alpha$ and $q = s \sin \alpha$ are taken here.

Besides, we can also form the following formulae by interchange of signs:

$$p = s \sin \beta - \frac{s^3}{6} \sin \beta \cos^2 \beta \frac{K_\alpha + K_{90} + 2 K_\beta}{4} + \frac{K^2}{120} p q^2 (q^2 - 8 p^2) \quad (32)$$

$$q = s \cos \beta + \frac{s^3}{3} \sin^2 \beta \cos \beta \frac{2 K_\alpha + 3 K_{90} + 3 K_\beta}{8} + \frac{K^2}{15} p^2 q (2 p^2 - q^2). \quad (33)$$

Spheroidal excess of the right triangle

According to Fig. 2, p. 55, we have:

$$\varepsilon = (\alpha + \beta + 90^\circ) - 180^\circ = \alpha + \beta - 90^\circ \quad (34)$$

$$\sin \varepsilon = -\cos (\alpha + \beta) = \sin \alpha \sin \beta - \cos \alpha \cos \beta. \quad (35)$$

Since we have the series for $s \sin \alpha$, $s \sin \beta$, as well as $s \cos \alpha$, $s \cos \beta$ in (26) to (29), we can form the two products, needed for (35), namely:

$$\begin{aligned} s^2 \sin \alpha \sin \beta &= p q + \frac{p^3 q}{6} \frac{2 K_\alpha + K_{90} + K_\beta}{4} + \frac{K^2}{360} p^3 q (7 p^2 - 16 q^2) \\ &\quad + \frac{p q^3}{6} \frac{K_\alpha + K_{90} + 2 K_\beta}{4} + \frac{K^2}{36} p^3 q^3 \\ &\quad + \frac{K^2}{360} p q^3 (-16 p^2 + 7 q^2) \\ s^2 \cos \alpha \cos \beta &= p q - \frac{p q^3}{3} \frac{3 K_\alpha + 3 K_{90} + 3 K_\beta}{8} - \frac{K^2}{360} p q^3 (16 p^2 + 8 q^2) \\ &\quad - \frac{p^3 q}{3} \frac{2 K_\alpha + 3 K_{90} + 3 K_\beta}{8} + \frac{K^2}{9} p^3 q^3 \\ &\quad - \frac{K^2}{360} p^3 q (8 p^2 + 16 q^2). \end{aligned}$$

If we subtract these two expressions from one another, and if we arrange the homogeneous terms together here, then we obtain:

$$s^2 \sin \varepsilon = \frac{pq}{2} \frac{K_\alpha + K_{90} + K_\beta}{3} (p^2 + q^2) + \frac{15}{360} K^2 pq (p^2 - q^2)^2. \quad (36)$$

To this we have from (20):

$$\frac{p^2 + q^2}{s^2} = 1 + \frac{K_\alpha + 2K_{90} + K_\beta}{12s^2} p^2 q^2 + K^2 \dots$$

If we set this into (36) and no longer distinguish the individual K_α 's, K_β 's, K_{90} 's in the terms with K^2 (as the distinction was also not made in the case of previous formulae in the same case), and if we neglect entirely the terms of the order of K^3 (like thus far), then we obtain from (36):

$$\varepsilon = \frac{pq}{2} \frac{K_\alpha + K_{90} + K_\beta}{3} + \frac{pq}{24} K^2 (p^2 + q^2). \quad (37)$$

This formula, which with $\frac{K_\alpha + K_{90} + K_\beta}{3} = \frac{1}{r^2}$ passes over into the previous spherical formula, the first half-volume, p. 104, equation (3), says in words that we obtain the spheroidal excess of a right geodetic triangle if we calculate the triangle like a spherical triangle whose spherical radius r corresponds to the arithmetic mean of the measures of curvature $K_\alpha, K_{90}, K_\beta$ at the three corners of the triangle.

This theorem can also easily be extended to an arbitrary oblique-angle triangle, as we shall see forthwith in the next section 14.

Section 14. Calculation of the General (Oblique-Angle) Spheroidal Triangle

With the developments in series for the right spheroidal triangle, which we have learned in the foregoing section 13, we also can bring about the trigonometric calculation of general spheroidal triangles by putting together two right triangles as a general (oblique-angle) triangle.

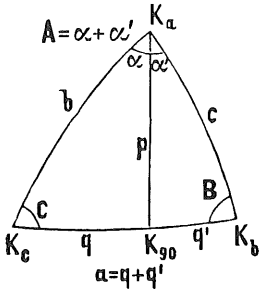


Fig. 1.

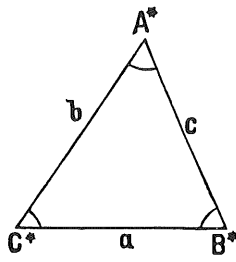


Fig. 2.

We shall proceed here in the same way as previously in the first half-volume, section 50, where we have derived Legendre's extended theorem with Fig. 3, p. 107, from the formulae for two right spherical triangles. Likewise, we shall treat now the formulae by means of which Gauss in 1827 in Art. 25 of the "Disquisitiones generales circa superficies curvas" has made the significant step from Legendre's theorem on the sphere to trigonometry of an arbitrary curved surface.

While we essentially retain the former denotations, in Fig. 1 we form a spheroidal triangle with the sides b, c and $a = q + q'$, while a perpendicular p divides the triangle b, c, a into two right triangles p, q , as well as p, q' .

If ε_1 and ε_2 are the spheroidal excesses of the two right subtriangles, then we have according to (37) of the previous section 13, above:

$$\varepsilon_1 = \frac{pq}{2} \frac{K_a + K_{90} + K_c}{3} + K^2 \dots \quad (1)$$

$$\varepsilon_2 = \frac{pq'}{2} \frac{K_a + K_{90} + K_b}{3} + K^2 \dots \quad (2)$$

By leaving aside at first the terms of order K^2 , we can easily convince ourselves that the excess $\varepsilon_1 + \varepsilon_2 = \varepsilon$ of the whole triangle is computed as a first approximation (i.e. excepting the terms with K^2) in the following way:

$$\varepsilon_1 + \varepsilon_2 = \varepsilon = \frac{p(q+q')}{2} \frac{K_a + K_b + K_c}{3}. \quad (3)$$

In order to prove the agreement of this formula (3) with the sum of (1) and (2), we only need to introduce the assumption on which the whole theory is based that the measure of curvature K shall be a linear function of the coordinates on the surface, and hence shall change in proportion to the lengths q and q' on the line $a = q + q'$, i.e. we must have:

$$K_{90} = K_c + \frac{q}{q+q'}(K_b - K_c) \quad \text{or} \quad K_{90}(q+q') = K_c q' + K_b q \quad (4)$$

and with these, the sum of (1) and (2) changes to (3). Therefore, we can write the equation (3) now thusly:

$$\varepsilon = \frac{ap}{2} \frac{K_a + K_b + K_c}{3} + K^2 \dots \quad \text{or} \quad \varepsilon = \Delta \frac{K_a + K_b + K_c}{3} + K^2 \dots \quad (5)$$

Δ shall be an approximate value for the area of the triangle here; e.g., Δ instead of $= \frac{ap}{2}$ can be also the area of a plane triangle, which we construct from the three side lengths a, b, c . If the terms of the order K^2 neglected in (5) are to be taken into account however, then the meaning of Δ must also be distinguished, e.g., whether it shall be $= \frac{ap}{2}$ or equal to the area of the plane triangle a, b, c , because according to this distinction the higher terms of the order K^2 also turn out differently.

In Fig. 2, p. 58, we have drawn a plane triangle which has the same sides a, b, c as the spheroidal triangle Fig. 1, p. 58, but, therefore, must have different angles A^*, B^*, C^* whose sum is $= 180^\circ$, and whose differences compared to the spheroidal angles A, B, C shall now be investigated.

From (26) to (29), section 13, p. 56, we have the following four equations:

$$\left. \begin{aligned} b \cos \alpha &= p - \frac{pq^2}{3} \frac{3K_a + 3K_{90} + 2K_c}{8} \\ c \cos \alpha' &= p - \frac{pq'^2}{3} \frac{3K_a + 3K_{90} + 2K_b}{8} \end{aligned} \right\} \quad (6)$$

$$\left. \begin{aligned} b \sin \alpha &= q + \frac{p^2 q}{6} \frac{2K_a + K_{90} + K_c}{4} \\ c \sin \alpha' &= q' + \frac{p^2 q'}{6} \frac{2K_a + K_{90} + K_b}{4} \end{aligned} \right\} \quad (7)$$

From these we form:

$$\begin{aligned} bc \cos \alpha \cos \alpha' &= p^2 - \frac{p^2 q^2}{3} \frac{3K_a + 3K_{90} + 2K_c}{8} - \frac{p^2 q'^2}{3} \frac{3K_a + 3K_{90} + 2K_b}{8} \\ bc \sin \alpha \sin \alpha' &= q q' + \frac{p^2 q q'}{6} \frac{2K_a + K_{90} + K_c}{4} + \frac{p^2 q q'}{6} \frac{2K_a + K_{90} + K_b}{4}. \end{aligned}$$

Since we have $\alpha + \alpha' = A$, and hence $\cos \alpha \cos \alpha' - \sin \alpha \sin \alpha' = \cos A$, we obtain therefrom:

$$b c \cos A = p^2 - q q' - \frac{p^2 K_a}{24} (3 q^2 + 3 q'^2 + 4 q q') - \frac{p^2 K_c}{24} (2 q^2 + q q') - \frac{p^2 K_{90}}{24} (3 q^2 + 3 q'^2 + 2 q q') - \frac{p^2 K_b}{24} (2 q'^2 + q q') . \quad (8)$$

Between the various triangle sides there exist relations, namely according to (20), section 13, p. 55:

$$b^2 = p^2 + q^2 - \frac{p^2 q^2}{3} \frac{K_a + 2 K_{90} + K_c}{4} \quad (9)$$

$$c^2 = p^2 + q'^2 - \frac{p^2 q'^2}{3} \frac{K_a + 2 K_{90} + K_b}{4} . \quad (10)$$

Now, according to Fig. 2, p. 58, we examine the plane triangle which has the same side lengths b , c and $a = q + q'$ as the spheroidal triangle Fig. 1, p. 58, while the angles become different ones, namely A^* , B^* , C^* .

This triangle, Fig. 2, yields the equation:

$$a^2 = (q + q')^2 = b^2 + c^2 - 2 b c \cos A^* . \quad (11)$$

Therefore, we have from (11), (10) and (9):

$$\begin{aligned} 2 b c \cos A^* &= b^2 + c^2 - (q^2 + q'^2 + 2 q q') \\ &= 2 p^2 - 2 q q' - \frac{K_a p^2}{12} (q^2 + q'^2) - \frac{K_{90} p^2}{6} (q^2 + q'^2) - \frac{K_c p^2}{12} q^2 - \frac{K_b p^2}{12} q'^2 . \end{aligned}$$

If we compare this with (8), then we obtain:

$$\begin{aligned} b c (\cos A^* - \cos A) &= \frac{p^2 K_a}{24} (2 q^2 + 2 q'^2 + 4 q q') + \frac{p^2 K_{90}}{24} (q^2 + q'^2 + 2 q q') \\ &\quad + \frac{p^2 K_c}{24} (q^2 + q q') + \frac{p^2 K_b}{24} (q'^2 + q q') \\ b c (\cos A^* - \cos A) &= \frac{p^2 (q + q')}{24} \left(2 K_a (q + q') + K_{90} (q + q') + K_c q + K_b q' \right) . \end{aligned} \quad (12)$$

The measure of curvature K_{90} is to be eliminated here again with the help of equation (4); by so doing we obtain from (12):

$$b c (\cos A^* - \cos A) = \frac{p^2 (q + q')^2}{24} (2 K_a + K_b + K_c) . \quad (13)$$

Here we have:

$$\cos A^* - \cos A = (A - A^*) \sin A^*$$

and

$$b c \sin A^* = p (q + q') = 2 \triangle ;$$

where \triangle shall be an approximate value for the area of the triangle. With this (13) yields:

$$A - A^* = \frac{\triangle}{12} (2 K_a + K_b + K_c)$$

While we must make the reservation, made already in the case of (5), p. 59, here also, in regard to the

meaning of Δ as a first approximation for the area of the plane or curved triangle surface, we write all three equations of the kind of those just now found together:

$$\left. \begin{aligned} A - A^* &= \frac{\Delta}{12} (2 K_a + K_b + K_c) \\ B - B^* &= \frac{\Delta}{12} (K_a + 2 K_b + K_c) \\ C - C^* &= \frac{\Delta}{12} (K_a + K_b + 2 K_c) \end{aligned} \right\} \quad (14)$$

$$\text{Sum: } \varepsilon = \Delta \frac{K_a + K_b + K_c}{3}. \quad (15)$$

This is again the same equation as that found already in the case of (5), p. 59, and if we wish to neglect the terms of the order K^2 , then the spheroidal triangle computation is brought to a close by the formulae (14) and (15), just as the spherical triangle computation was determined by the simple Legendre theorem, first half-volume, section 47, p. 95, equations (11) to (12), to the order $\frac{1}{r^2}$ inclusive, but excluding $\frac{1}{r^4}$.

Now in our case in order to find also the terms of the order K^2 (corresponding to $\frac{1}{r^4}$), we can repeat the whole foregoing development (6) to (15) with the addition of all terms of the order K^2 , and we are to note here, for instance, only one thing in particular: that then the triangle area Δ must no longer be set at will $= \frac{ap}{2}$ or $= \frac{bc}{2} \sin A^*$.

While we establish now for the development with terms K^2 that Δ shall be the area of the plane auxiliary triangle, Fig. 2, p. 58, to be constructed from the three side lengths a, b, c , we obtain a relation between $p (q + q')$ and Δ , at first by the further use of the equations (6) and (7), namely:

$$\begin{aligned} \sin A &= \sin (\alpha + \alpha') = \sin \alpha \cos \alpha' + \cos \alpha \sin \alpha' \\ b c \sin A &= p (q + q') \left(1 + \frac{K}{6} (p^2 - 2 q q') \right) \\ b c \sin A &= b c \left(\sin A^* + \frac{\Delta}{3} K \cos A^* \right), \text{ and so on.} \end{aligned}$$

Since the path for the development of $A - A^*$ to terms in K^2 inclusive is set down with these, we limit ourselves in the following to communicate here the final result of the development, all the more because the terms with K^2 – if we no longer make a distinction between K_a, K_b, K_c in them – assume merely spherical forms, and are nothing else but the terms of order $\frac{1}{r^4}$ in the formulae (31), (32), (33), first half-volume, section 50, p. 110, which we can add directly to the formulae (14).

Either by such an addition, or by the further development for the spheroidal triangle up to K^2 we find:

$$\left. \begin{aligned} A - A^* &= \frac{\Delta}{3} \frac{2 K_a + K_b + K_c}{4} + \frac{\Delta}{24} K^2 \frac{a^2 + 7 b^2 + 7 c^2}{15} \\ B - B^* &= \frac{\Delta}{3} \frac{K_a + 2 K_b + K_c}{4} + \frac{\Delta}{24} K^2 \frac{7 a^2 + b^2 + 7 c^2}{15} \\ C - C^* &= \frac{\Delta}{3} \frac{K_a + K_b + 2 K_c}{4} + \frac{\Delta}{24} K^2 \frac{7 a^2 + 7 b^2 + c^2}{15} \end{aligned} \right\} \quad (16)$$

$$\text{Sum: } \varepsilon = \Delta \frac{K_a + K_b + K_c}{3} + \frac{\Delta}{8} K^2 \frac{a^2 + b^2 + c^2}{3}. \quad (17)$$

By means of these formulae (16) and (17), the solution of a spheroidal triangle is reduced to the solution of a plane triangle, therefore completely taken care of, and further formulae are not needed.

If we examine the spherical models of our formulae in the first half-volume, section 50, pp. 110 and 111, however, then we find that we miss the analogue to the equations (35), (38) and (39), p. 111, of that place, something which is not required for practical computing, it is true, but is so interesting that we shall occupy ourselves with it in the next section 15.

Section 15. Curved Surface of the Spheroidal Triangle

In Fig. 1, we take up again the combination of a rectangular and a polar system of geodetic coordinates on the spheroidal surface.

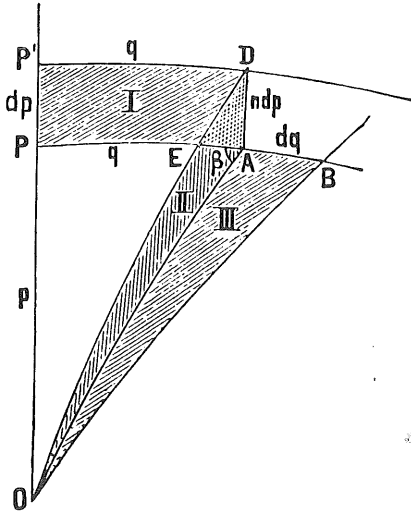


Fig. 1.

In the rectangular geodetic system of coordinates with the origin O , the point A has the abscissa $OP = p$ and the ordinate $PA = q$, and let the curved surface of the right triangle OPA thus determined have the area F .

In order to determine the differential dF , we investigate how the area F changes if p and q are changed by dp and dq , respectively.

If P alone changes, then the point A moves on the geodetic parallel from A to D , and the change of area is $= I - II$, where the strip $PP'DA$ is denoted by I and the narrow triangle ODA by II , and the very small triangle with the legs AD and DE is neglected. Therefore we can write:

$$\frac{\partial F}{\partial p} dp = I - II. \quad (1)$$

In the same way we also have:

$$\frac{\partial F}{\partial q} dq = III, \quad (2)$$

if the narrow triangle OAB is denoted by III .

In order to investigate the three surface parts I, II, III more closely, we begin with I , this is

$$I = \int n dp dq = dp \int n dq. \quad (3)$$

dp as the base PP' of the strip I is constant here.

The triangle II can be brought into relation to III by the ratio $EA:AB$, namely:

$$II:III = EA:AB.$$

Here we have $AB = dq$ and $EA = n dp \cot \beta$; there follows hence with respect to (2):

$$II = \frac{\partial F}{\partial q} n dp \cot \beta. \quad (4)$$

And hence, we have now from (1), (3) and (4):

$$\frac{\partial F}{\partial p} dp = dp \int n dq - n dp \cot \beta \frac{\partial F}{\partial q}.$$

The factor dp is omitted, and then we have:

$$\sin \beta \frac{\partial F}{\partial p} + \cos \beta n \frac{\partial F}{\partial q} = \sin \beta \int n dq$$

or, because $\sin \beta$ and $\cos \beta$ of the former developments are existing only in the products $s \sin \beta$ and $s \cos \beta$, we write:

$$s \sin \beta \frac{\partial F}{\partial p} + n s \cos \beta \frac{\partial F}{\partial q} = s \sin \beta \int n dq. \quad (5)$$

This equation serving for the determination of F shall be brought into agreement with the following equation whose coefficients A, B, C, D are introduced at first indeterminately:

$$F = \frac{1}{2} p q + A p^2 + B q^2 + C p^2 q + D p q^2. \quad (6)$$

According to the instruction of (5) we form hence:

$$\begin{aligned} \frac{\partial F}{\partial p} &= \frac{1}{2} q + 2 A p + 2 C p q + D q^2 \\ \frac{\partial F}{\partial q} &= \frac{1}{2} p + 2 B q + C p^2 + 2 D p q. \end{aligned}$$

To this we take as a first approximation of (24) and (25), p. 56, and (10), p. 52:

$$s \sin \beta = p \quad s \cos \beta = q \quad n = 1 + f_0 q^2.$$

If we form equation (5) with this and compare it with (6), then we find $A = 0, B = 0, C = 0, D = 0$, i.e. the series for F has no terms of the form $p^2, q^2, p^2 q, p q^2$.

After this is recognized, we assume as a new form:

$$F = \frac{1}{2} p q + A p^3 q + B p^2 q^2 + C p q^3. \quad (7)$$

Hence we form $\frac{\partial F}{\partial p}$ and $\frac{\partial F}{\partial q}$, to this as a second approximation of (24), (25), section 13, p. 56, besides (10), section 12, p. 52,

$$\begin{aligned} s \sin \beta &= p - \frac{1}{3} f_0 p q^2 & s \cos \beta &= q + \frac{2}{3} f_0 p^2 q \\ n &= 1 + f_0 q^2. \end{aligned}$$

This is set into (5); the resulting expression is compared with (7), whereby there will follow:

$$4 A = -\frac{1}{3} f_0 \quad 4 B = 0 \quad 4 C = -\frac{1}{3} f_0.$$

And if we also set, in addition, $2 f_0 = -K$, according to (19), section 13, p. 55, then (7) yields:

$$F = \frac{1}{2} p q + \frac{p q}{24} K (p^2 + q^2). \quad (8)$$

In the next step we have added four more indeterminate terms of the form $A p^4 q + B p^3 q^2 + B' p^2 q^3 + A' p q^4$, for which $s \sin \beta$, $s \cos \beta$ and n were also to be taken accordingly higher. The execution and comparison of coefficients according to the method hitherto used yielded:

$$F = \frac{1}{2} p q + \frac{p q}{120} \left(-10 f_0 p^2 - 10 f_0 q^2 - 6 f_1 p^3 - 9 g_0 p^2 q - 7 f_1 p q^2 - 12 g_0 q^3 \right).$$

If we introduce the measures of curvature according to (19), section 13, p. 55, here again, then we can bring the foregoing expression for F to the following form:

$$F = \frac{p q}{2} + \frac{p q}{240} \left\{ K_\alpha (4 p^2 + 3 q^2) + K_{90} (3 p^2 + 3 q^2) + K_\beta (3 p^2 + 4 q^2) \right\}. \quad (9)$$

If we set the various K_α 's, K_β 's, K_{90} 's equal to one another here, putting them simply $= K$, then we obtain again equation (8).

Now it is required to pass over from the area F of a right spheroidal triangle to the area of a general triangle with arbitrary angles. The way to this is already indicated by the development of section 14 with Fig. 1, p. 58; again we shall divide the general triangle into two right triangles and have then the following, if we denote the areas of the two right triangles by F_1 and F_2 , as used hitherto, and assume the symbol F for the whole area:

$$F = F_1 + F_2. \quad (10)$$

If we apply the formula (9) for the area F of a right triangle to the two parts of Fig. 1, then we have:

$$F_1 = \frac{p q}{2} + \frac{p q}{240} \left(K_a (4 p^2 + 3 q^2) + K_{90} (3 p^2 + 3 q^2) + K_c (3 p^2 + 4 q^2) \right)$$

$$F_2 = \frac{p q'}{2} + \frac{p q'}{240} \left(K_a (4 p^2 + 3 q'^2) + K_{90} (3 p^2 + 3 q'^2) + K_b (3 p^2 + 4 q'^2) \right).$$

To this we have according to (30) to (33), section 13, p. 57,

$$p = b \cos \alpha \left(1 + \frac{q^2}{3} \frac{3 K_a + 3 K_{90} + 2 K_c}{8} \right), \quad p = c \cos \alpha' \left(1 + \frac{q'^2}{3} \frac{3 K_a + 3 K_{90} + 2 K_b}{8} \right)$$

$$q = b \sin \alpha \left(1 - \frac{p^2}{6} \frac{2 K_a + K_{90} + K_c}{4} \right), \quad q' = c \sin \alpha' \left(1 - \frac{p'^2}{6} \frac{2 K_a + K_{90} + K_b}{4} \right)$$

$$p = c \cos \alpha' \left(1 + \frac{q'^2}{3} \frac{3 K_a + 3 K_{90} + 2 K_b}{8} \right), \quad p = b \cos \alpha \left(1 + \frac{q^2}{3} \frac{3 K_a + 3 K_{90} + 2 K_c}{8} \right)$$

By these, there is shown the way in which we can at first express the area of the curved triangle surface F in p, q, q' besides the various K 's. Then we have various geometric relations in the triangle itself, e.g. $p^2 + q^2 = b^2$, $p^2 + q'^2 = c^2$ as first approximations, and so on. If we execute the calculation according to these indications, then we will obtain:

$$F = \frac{b c \sin A}{2} \left\{ 1 + \frac{1}{120} K_a (3 b^2 + 3 c^2 - 12 b c \cos A) + \frac{1}{120} K_b (3 b^2 + 4 c^2 - 9 b c \cos A) \right. \\ \left. + \frac{1}{120} K_c (4 b^2 + 3 c^2 - 9 b c \cos A) \right\}. \quad (11)$$

This equation (11), which we can also write down in two other forms with $ac \sin B$ and with $ab \sin C$, is not symmetrical because one of the three elements A, B, C or, as the case may be, a, b, c is preferred. Therefore, we will express $bc \sin A$ by Δ , and to this, we have from (14), section 14, p. 61:

$$A = A^* + \frac{\Delta}{3} \frac{2K_a + K_b + K_c}{4}$$

and in the plane triangle:

$$2bc \cos A^* = b^2 + c^2 - a^2.$$

Since we may always interchange A with A^* in the terms of second order of (11), by means of the two equations just written we can bring equation (11) into the following form:

$$F = \Delta \left\{ 1 + \frac{1}{120} K_a (a^2 + 2b^2 + 2c^2) + \frac{1}{120} K_b (2a^2 + b^2 + 2c^2) + \frac{1}{120} K_c (2a^2 + 2b^2 + c^2) \right\}. \quad (12)$$

If we set $K_a = K_b = K_c = K$ here, then we obtain:

$$F = \Delta \left(1 + \frac{1}{24} K (a^2 + b^2 + c^2) \right). \quad (13)$$

This corresponds to the former (36) in the first half-volume, p. 111, and with this, we can also easily transfer the equations of that place (31) to (42) of pp. 110-112 to our case, and this shall be done in the compilation of formulae of section 16.

In the former 3rd edition of this volume, 1890, pp. 480-488, we had inserted here a second foundation of the basic formulae of geodetic triangles (given also in a partly different form previously in the *Zeitschr. f. Verm.*, 1889, pp. 295-304), which, founded on the principle of the reduced latitude (Chapter III), uses a spherical auxiliary triangle. This may be passed over here.

Section 16. Practical Application of the General Theory of the Spheroidal Triangles

(Notation according to Figs. 1 and 2, p. 58)

We will at first compile the various formulae of sections 14 and 15 suited for practical application and introduce to this also some condensed notation. If the measures of curvature at the three corners of a triangle are denoted by K_a, K_b, K_c , then we take in place of this a mean value

$$\frac{K_a + K_b + K_c}{3} = K_0. \quad (1)$$

This value K_0 corresponds to the center of gravity of the triangle and to the arithmetic mean of the geographic latitudes of the three corner points of the triangle.

If the three sides of a geodetic triangle have the lengths a, b, c , then we compute the mean square of the sides:

$$\frac{a^2 + b^2 + c^2}{3} = m^2. \quad (2)$$

The angles of the geodetic triangle are A, B, C , and the angles of a plane triangle, which has the

same side lengths a, b, c as the geodetic triangle, are A^*, B^*, C^* . The sum of angles of the plane triangle, i.e. $A^* + B^* + C^* = 180^\circ$, and the sum of the angles of the geodetic triangle, i.e. $A + B + C = 180^\circ + \varepsilon$, where ε is called the geodetic excess of the triangle.

Let the area of the geodetic triangle, measured on the curved surface, be F , and the area of the plane triangle with the sides a, b, c be $= \Delta$.

With these denotations we have from (16) and (17), section 14, p. 61, with the addition of the necessary ρ 's:

$$A - A^* = \frac{\Delta}{3} \rho \frac{2K_a + K_b + K_c}{4} + \frac{\Delta}{24} \rho K^2 \frac{a^2 + 7b^2 + 7c^2}{15}, \quad (3)$$

or with the introduction of K_0 and of m^2 in triple form:

$$\left. \begin{aligned} A - A^* &= \frac{\Delta}{3} \rho K_0 + \frac{\Delta}{12} \rho (K_a - K_0) + \frac{\Delta}{120} \rho K^2 (7m^2 - 2a^2) \\ B - B^* &= \frac{\Delta}{3} \rho K_0 + \frac{\Delta}{12} \rho (K_b - K_0) + \frac{\Delta}{120} \rho K^2 (7m^2 - 2b^2) \\ C - C^* &= \frac{\Delta}{3} \rho K_0 + \frac{\Delta}{12} \rho (K_c - K_0) + \frac{\Delta}{120} \rho K^2 (7m^2 - 2c^2) \end{aligned} \right\} \quad (4)$$

$$\text{Sum: } \varepsilon = \Delta \rho K_0 + 0 + \frac{\Delta}{8} \rho K^2 m^2. \quad (5)$$

For the sake of theoretical completeness, we add here also the formula for the area of the curved surface F , according to (13), section 15, p. 65:

$$F = \Delta + \frac{\Delta}{8} K m^2, \quad (6)$$

and as a series from (5) and (6):

$$\varepsilon = F \rho K_0 + K^3 \dots \quad (7)$$

Finally, by the elimination of Δ we form from (4) and (5) the difference:

$$A - A^* = \frac{\varepsilon}{3} \frac{2K_a + K_b + K_c}{4K_0} + \varepsilon \frac{K}{180} (-2a^2 + b^2 + c^2), \quad (8)$$

or with the introduction of the mean values K_0 and m^2 according to (1) and (2) in triple form:

$$\left. \begin{aligned} A - A^* &= \frac{\varepsilon}{3} + \frac{\varepsilon}{12} \left(\frac{K_a - K_0}{K_0} \right) + \frac{\varepsilon K}{60} (m^2 - a^2) \\ B - B^* &= \frac{\varepsilon}{3} + \frac{\varepsilon}{12} \left(\frac{K_b - K_0}{K_0} \right) + \frac{\varepsilon K}{60} (m^2 - b^2) \\ C - C^* &= \frac{\varepsilon}{3} + \frac{\varepsilon}{12} \left(\frac{K_c - K_0}{K_0} \right) + \frac{\varepsilon K}{60} (m^2 - c^2) \end{aligned} \right\} \quad (9)$$

Sum: $\varepsilon = \varepsilon$.

Where in the higher terms of these formulae K stands simply, the individual K_a, K_b, K_c are no longer distinguished, and then we can take at will, for instance $K = K_0$.

We can take the numerical values of K from the auxiliary table, first half-volume, pp. [12] to [33] of

our Appendix, for we have:

$$K = \frac{1}{r^2}, \quad \log K = \log \frac{1}{r^2}.$$

We may also be permitted, if it is a question of the mean value of K_0 according to (1), to let hold the mean of the different $\log K$'s as $\log K_0$ instead of the mean of the K 's themselves. Or we also can take $\log K_0$ to the arithmetic mean of the latitudes φ of the three corners of the triangle if we can assume, for not too far an extent, the differences between the latitudes φ , between the values K and the values $\log K$ all nearly proportional to each other.

If the proportionality between $\Delta\varphi$ and ΔK does no longer hold, then the assumption based upon the whole theory that K is a linear function of the surface coordinates is no longer satisfied [cf. (7) and (8), section 12, p. 52, and (18), section 13, p. 55].

As a numerical example we take first again the classical triangle Inselsberg-Hohehagen-Brocken, which has already served as a computational example several times, first half-volume on pp. 90 and 112.

According to p. 90 we take first again the approximate geographic latitudes of the three corner points of the triangle, and, after this, take the measures of curvature from the first half-volume, p. [24] of the Appendix:

Point	Latitude	$\log K = \log \frac{1}{r^2}$	(10)
Inselsberg	50° 51' 9"	$\log K_a = 6.390\,1277\cdot8$	
Hohehagen	51 28 31	$\log K_b = 6.390\,0659\cdot4$	
Brocken	51 48 2	$\log K_c = 6.390\,0337\cdot4$	
Mean	51° 22' 34"	$\log K_0 = 6.390\,0758\cdot2$	

In so doing, we have calculated exceptionally rigorously, i.e. we have interpolated at first $\log r$ from p. [24] and formed therefrom $\log r^2$ and $\log K$. With these, we will calculate the angles to an accuracy of 0.000 001", which has only a formal meaning as an example of comparison.

By using the former numerical values of the first half-volume, pp. 90 and 112, we have:

$$\begin{aligned} a &= 69.194 \text{ km} & b &= 105.973 \text{ km} & c &= 84.941 \text{ km} \\ a^2 &= 4787.8 \text{ gkm} & b^2 &= 11230.2 \text{ gkm} & c^2 &= 7215.0 \text{ gkm} & m^2 &= 7744.3 \text{ gkm} \\ \log \triangle &= 9.467\,2167\cdot6 & \triangle &= 2,932,356,450 \text{ gm}, \end{aligned}$$

$$\text{with these, according to (5):} \quad \varepsilon = 14.849\,701'' + 0.000\,353'' = 14.850\,054''; \quad (11)$$

$$\text{then according to (6):} \quad F = 2,932,356,450 \text{ gm} + 69,693 \text{ gm} = 2,932,426,143 \text{ gm};$$

$$\text{and, with this, } \varepsilon \text{ according to (7):} \quad \varepsilon = 14.850\,054'' \quad [\text{agrees with (11)}].$$

The group (4) and (5) yields:

$$\left. \begin{aligned} A - A^* &= 4.949\,900'' - 0.000\,148'' + 0.000\,136'' = 4.949\,888'' \\ B - B^* &= 4.949\,900 + 0.000\,028 + 0.000\,096 = 4.950\,024 \\ C - C^* &= 4.949\,900 + 0.000\,120 + 0.000\,121 = 4.950\,141 \end{aligned} \right\} \quad (12)$$

$$\varepsilon = 14.849\,700'' + 0 + 0.000\,353'' = 14.850\,053''.$$

Further, the group (9) yields:

$$\left. \begin{aligned} A - A^* &= 4.950\,018'' + 0.000\,148'' + 0.000\,018'' = 4.950\,184'' \\ B - B^* &= 4.950\,018 - 0.000\,028 - 0.000\,021 = 4.949\,969 \\ C - C^* &= 4.950\,018 - 0.000\,120 + 0.000\,003 = 4.949\,901 \end{aligned} \right\} \quad (13)$$

$$\varepsilon = 14.850\,054'' + 0 + = 14.850\,054''.$$

If we compare these angles (12) and (13) with the previous spherical data of the first half-volume, p. 112, then we find only differences of approximately 0.0001%, from which there is seen that in this case of a very large triangle, the computation according to the spheroidal formulae does not involve a noticeable deviation from the spherical computation.

The numerical values which we have computed here in (12) and (13) do not agree with the values which Gauss himself has given in Art. 28 of the "Disquisitiones generales, etc."; for Gauss' data are:

$$\left. \begin{array}{rcl} \text{Inselsberg} & A - A^* = & 4.95131'' \\ \text{Hohehagen} & B - B^* = & 4.95113 \\ \text{Brocken} & C - C^* = & 4.95104 \\ \hline & \varepsilon = & 14.85348'' \end{array} \right\} \quad (14)$$

This lies in the fact that in 1827 Gauss based his calculation on other terrestrial dimensions than Bessel's dimensions of the earth, which were derived first in 1841, on which our calculations are based.

The angle reductions computed above are independent of the reductions between the geodetic lines and the vertical sections, which are determined by the previous formulae of section 9. That reduction must already be applied before the geodetic theory of sections 12 to 15 is brought into use.

We will show this on the basis of a bigger example which is represented in Fig. 1, p. 69.

For this, we take one of the large triangles which were established in 1879 by Ibanez and Perrier for the trigonometric connection between Spain and Algeria, as we already have reported in general previously in the first half-volume, p. 22*.

These large triangles are very well suited as numerical examples for the use of the geodetic formulae with spheroidal terms, and in this sense, a computation according to Helmert's formulae has already been communicated by Fenner in *Zeitschr. f. Verm.*, 1882, pp. 303-308. Furthermore, we have the original source papers: "Enlace geodésico y astronómica de Europa y Africa, Madrid 1880" and in *Generalbericht d. europ. Gradm. für 1880*, pp. 44-57: "Jonction géodésique et astronomique de l'Algérie avec l'Espagne," and in final computation in the work, *Memorias del instituto geográfico y estadístico*, Tomo VII, Madrid, 1888, pp. 97-111.

According to these papers and some side computations, we have compiled the main values of the latitudes and of the azimuths necessary for the computation, in Fig. 1, which yields now all which is essential in connection with Fig. 3, p. 22, from the first half-volume.

We will calculate here only one of the four connecting triangles, i.e. the largest: Mulhacen, M'Sabiha, Filhaoussen.

The measured angles are the following (*Memorias*, etc., p. 100):

$$\left. \begin{array}{rcl} \text{Mulhacen} & A = & 22^\circ 28' 45.269'' \\ \text{M'Sabiha} & B = & 78 \ 48 \ 45.563 \\ \text{Filhaoussen} & C = & 78 \ 43 \ 39.198 \\ \hline & \text{Sum} = & 180^\circ \ 0' \ 70.030''. \end{array} \right\} \quad (15)$$

For our purposes, however, we will better represent these three angles in the form of six directions, and in fact so that the directions become nearly equal to the azimuths of the sides in question:

$$\left. \begin{array}{rcl} \text{Mulhacen} & \text{M'Sabiha} & \text{Filhaoussen} \\ (A \ B) = 124^\circ 15' \ 0.000'' & (B \ C) = 226^\circ 53' \ 0.000'' & (C \ A) = 327^\circ 40' \ 0.000'' \\ (A \ C) = 146 \ 43 \ 45.269 & (B \ A) = 305 \ 41 \ 45.563 & (C \ B) = 46 \ 23 \ 39.198 \\ \hline A = 22^\circ 28' 45.269'' & B = 78^\circ 48' 45.563'' & C = 78^\circ 43' 39.198''. \end{array} \right\} \quad (16)$$

The differences A, B, C are again the same as in the case of (15). The fact that $(A \ B)$ and $(B \ A)$, etc., do not differ by about 180° , although the directions themselves are azimuths to an accuracy of

* Not translated.

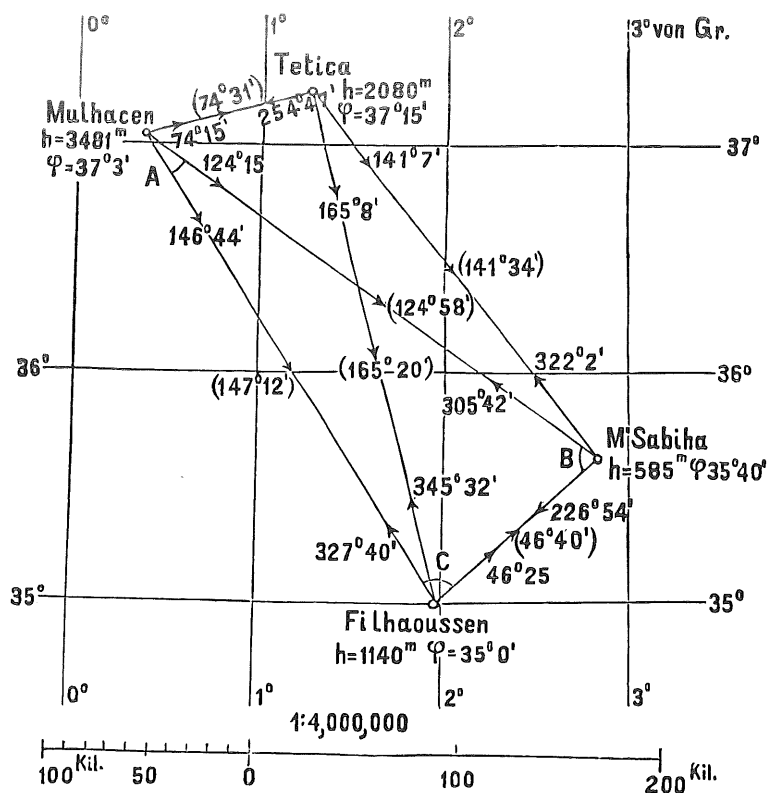


Fig. 1 (cf. first half-volume, Fig. 3, p. 22).
Trigonometric connection between Spain and Algeria.

approximately $1'$, results from the meridian convergences; the mean values of two such counterdirections are entered in Fig. 1 approximated as mean azimuths, e.g. $(147^{\circ}12')$ as the mean of $146^{\circ}44'$ and $327^{\circ}40' \pm 180^{\circ}$.

Now the measured directions (16) must at first be reduced in a double manner because of the flattening of the earth.

First, the reduction is carried out because of the elevation of the target points above the sea; according to formula (10) in section 4, p. 20, the reduction for a target point which is aimed at the height h above the sea at the azimuth α is in seconds:

$$\gamma = \eta^2 \frac{h}{N} \rho \sin \alpha \cos \alpha. \quad (17)$$

Second, a reduction is necessary from the vertical sections at which the directions (16) are measured on the geodetic lines; for this, we have, according to (17), p. 37, in addition to Fig. 2, p. 36, the reduction to a sufficient approximation:

$$\nu = -\frac{1}{6} \eta^2 \frac{s^2}{N^2} \rho \sin \alpha \cos \alpha. \quad (18)$$

The factor η^2 occurring in (17) and (18) has the meaning $\eta^2 = e'^2 \cos^2 \varphi$, as usual. Since the latitudes φ of the three corner points are indicated in Fig. 1, we have with this also the mean latitudes for the three sides, and to this, the three values $\log \eta^2$ can be computed, as well as the necessary $\log N$ taken from the table, p. [20], in the Appendix of the first half-volume:

M'Sabiha 35° 40'	Mulhacen 37° 3'	Mulhacen 37° 3'
Filhaoussen 35 0	Filhaoussen 35 0	M'Sabiha 35 40
Mean latitude $\varphi = 35^\circ 20'$	$36^\circ 2'$	$36^\circ 22'$
$\log \eta^2$ 7.65049	7.64286	7.63917
$\log N$ 6.80513	6.80514	6.80515.

} (19)

Since, for the remaining part, nothing further is to be remarked to the calculation according to the formulae (17) and (18), the necessary elements partly being given in (19), partly being entered in Fig. 1, p. 69, then we communicate the results of these computations at once:

Direction	Mulhacen (A B) (A C)	M'Sabiha (B C) (B A)	Filhaoussen (C A) C B)
γ	— 0.039", — 0.074"	+ 0.082", — 0.230"	— 0.225", + 0.042"
ν	+ 0.126 , + 0.123	— 0.021 , + 0.126	+ 0.123 , — 0.021
$\gamma + \nu$	+ 0.087", + 0.049" — 0.038"	+ 0.061", — 0.104" — 0.165"	— 0.102", + 0.021" + 0.123".

} (20)

By adding these reductions (20) to the measured directions (16), we obtain the following new table of the directions which, in order to distinguish from (A B), etc., we will denote now by [A B], etc.:

[A B] = 124° 15' 0.087"	[B C] = 226° 53' 0.061"	[C A] = 327° 39' 59.898"
[A C] = 146 43 45.318	[B A] = 305 41 45.459	[C B] = 46 23 39.219
A' = 22° 28' 45.231"	B' = 78° 48' 45.398"	C' = 78° 43' 39.321".

(21)

Now in order to reduce the triangle to which these angles belong to a plane triangle with equally long sides, or in order to find the angles denoted formerly by A^* , B^* , C^* , we have to apply again the formulae valid for this, which we have compiled at the beginning of this section 16 under (1) to (9), pp. 65 and 66.

The measures of curvature K needed for this or, as the case may be, the corresponding $\log K \rho$'s are:

A, Mulhacen $\varphi = 37^\circ 3'$	$\log K \rho = \log \frac{\rho}{r^2} = 1.705\ 9395$
B, M'Sabiha . . 35 40 1.706 0732
C, Filhaoussen . . 35 1 1.706 1356.

As for the remaining computation, we have based it on the side $AC = b = 269,926$ m, adjusted the angles A, B, C at first preliminarily to 180° , and obtained, with this, first approximations of A^*, B^*, C^* , from which there followed further: $BC = a = 105,173.9$ m and $AB = c = 269,845.7$ m. With this, we could compute further $\log \Delta = 10.143\ 6726$ and $\varepsilon = 70.7607''$, and finally:

$$A' - A^* = 23.5866'' \quad B' - B^* = 23.5866'' \quad C' - C^* = 23.5875'' . \quad (22)$$

If we subtract these (22)'s from the A', B', C' in (21), then we obtain:

A* = 22° 28' 21.644"
B* = 78 48 21.811
C* = 78 43 15.733
Sum = 179° 59' 59.188"
w = — 0.812".

(23)

This discrepancy now still remaining $w = -0.812''$ is attributable to the observational errors; the geodetic angle reduction in itself is completed with this.

If we raise the practical question whether the small reductions (20) and (22), with which we have occupied ourselves here, are to be taken into account in the case of triangulations, then, at the present state of the art of observation, we will answer this question in the negative for the ordinary small triangles and minor elevations; in the case of large proportions, as those of the triangulation between Spain and Algeria, however, the corrections for the elevations of the target points and for the reduction to geodetic lines, which we have obtained under (20), are not to be neglected.

However, the spheroidal reduction of the triangles to plane triangles according to the theory of the foregoing sections 13 to 15 yields, in contrast to the purely spherical reduction, only differences of $0.001''$, which at present hardly need to be taken into account.

Chapter III

SPHEROIDAL COORDINATES

Preliminary remark. In this chapter, we shall treat in substance, with the help of the geodetic line, on the ellipsoid the problems which, in Chapter V of the first-half-volume, were solved on the sphere with the help of the arc of a great circle. While there, however, rectangular coordinates were of special importance for practical use, on the ellipsoid geographic coordinates play the main role for the practice of the triangulation. Therefore, we shall treat the theory of geographic coordinates in the first place, and then join rectangular coordinates.

Section 17. Spheroidal Polar Triangle

In Fig. 1, A denotes a point of the ellipsoid with the latitude φ , and accordingly, B a point with the latitude φ' ; let the difference of longitude of these two points, i.e. the angle which their meridian planes NA and NB include, be l (counted positively from west to east). The two points are connected by a geodetic line AB whose linear quantity $= s$, and which has the azimuths α and α' at A and B .

In general, we count the azimuths from north through east, like α at the point A ; and the azimuth, counted likewise from north to east, at the point B is thus $= \alpha' \pm 180^\circ$, if α' is the angle entered in Fig. 1.

Between these six quantities, φ , φ' , l , s , α , α' , there exist relations of a similar kind as for the *spherical* triangle in the first half-volume, section 59, p. 158, Fig. 1, which are expressed mainly in two forms of problems, namely *first*: given φ , φ' and l , required s , α and α' , or *second*: given φ , s and α , required φ' , l and α' .

The solutions of these two problems repeatedly overlap one another.

Before we go to the various solutions of the problem itself, we set up in advance a few examples for it (as we have done this also for the spherical problem). We cannot now as yet prove that these examples are correct in themselves; this will result from the agreeing computation according to the various methods to be developed later.

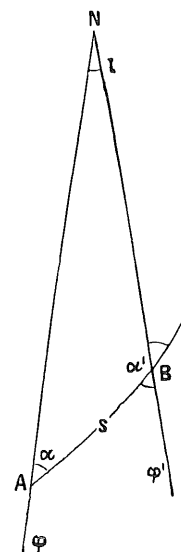


Fig. 1.

1. *Small spheroidal normal example*

$\varphi = 49^\circ 30' 0''$	$\varphi' = 50^\circ 30' 0''$	$l = 1^\circ 0' 0''$	}	(1)
$\frac{\varphi' + \varphi}{2} = 50 \quad 0 \quad 0$	$\varphi' - \varphi = 1 \quad 0 \quad 0$			
$\frac{\alpha' + \alpha}{2} = 32^\circ 48' 20.4580''$	$\alpha' - \alpha = 0^\circ 45' 57.8942''$			
$\frac{\alpha' - \alpha}{2} = 0 \quad 22 \quad 58.9471$	$\log s = 5.121 \, 6103.131$			
$\alpha' = 33^\circ 11' 19.4051''$	$s = 132,315.375 \, \text{m.}$			
$\alpha = 32 \quad 25 \quad 21.5109$				

II. Large spheroidal normal example

$$\left. \begin{array}{ll}
 \varphi = 45^{\circ} 0' 0'' & \varphi' = 55^{\circ} 0' 0'' \quad l = 10^{\circ} 0' 0'' \\
 \frac{\varphi' + \varphi}{2} = 50 \ 0 \ 0 & \varphi' - \varphi = 10 \ 0 \ 0 \\
 \frac{\alpha' + \alpha}{2} = 32^{\circ} 54' 11.4302'' & \alpha - \alpha = 7^{\circ} 41' 51.9408'' \\
 \frac{\alpha' - \alpha}{2} = 3 \ 50 \ 55.9704 & \log s = 6.120 \ 6674.805 \\
 \hline
 \alpha' = 36^{\circ} 45' 7.4006'' & s = 1,320,284.366 \text{ m.} \\
 \alpha = 29 \ 3 \ 15.4598 &
 \end{array} \right\} \quad (2)$$

An example which lies between the two preceding ones has been computed as a control diagonal over the whole country by the geodesists of Mecklenburg (*Zeitschr. f. Verm.*, 1896, pp. 240-242). It yields with the denotations of Fig. 1 the following:

III. Diagonal of Mecklenburg

$$\left. \begin{array}{lll}
 \varphi = 53^{\circ} 0' & \varphi' = 54^{\circ} 30' & l = 3^{\circ} 30' \\
 \frac{\varphi' + \varphi}{2} = 53 \ 45 & \varphi' - \varphi = 1 \ 30 & l = 12,600'' \\
 \frac{\alpha' + \alpha}{2} = 54^{\circ} 8' 20.77402'' & \alpha' - \alpha = 2^{\circ} 48' 23.18112'' & \\
 \frac{\alpha' - \alpha}{2} = 1 \ 24 \ 41.59056 & \log s = 5.454 \ 5946.712 & \\
 \hline
 \alpha' = 55^{\circ} 33' 2.36458'' & s = 284,835.8642 \text{ m.} & \\
 \alpha = 52 \ 43 \ 39.18346 & &
 \end{array} \right\} \quad (3)$$

We take a small example with numbers not rounded off from Bohnenberger's triangulation of Württemberg:

IV. $P = \text{Hornisgrinde}$ $P' = \text{Tübingen}$

$$\left. \begin{array}{ll}
 \varphi = 48^{\circ} 36' 21.8966'' & \varphi' = 48^{\circ} 31' 12.4000'' \\
 l = 0^{\circ} 50' 55.5537'' = 3055.5537'' & \\
 \alpha = 98^{\circ} 21' 29.9583'' & \alpha' = 98^{\circ} 59' 40.6800'' \\
 \log s = 4.801 \ 8443.0 & s = 63,364.218 \text{ m.}
 \end{array} \right\} \quad (4)$$

Finally, we take a still larger example with numbers not rounded off, which has already been used elsewhere several times.

V. $P = \text{Berlin}$ $P' = \text{Königsberg}$

$$\left. \begin{array}{ll}
 \varphi = 52^{\circ} 30' 16.7'' & \varphi' = 54^{\circ} 42' 50.6'' \\
 l = 7^{\circ} 6' 0'' = 25 \ 560'' & \\
 \alpha = 59^{\circ} 33' 0.6892'' & \alpha' = 65^{\circ} 16' 9.3650'' \\
 \log s = 5.724 \ 2591.353 & s = 529,979.578 \text{ m.}
 \end{array} \right\} \quad (5)$$

(Denotations according to Fig. 1, p. 73.)

The three differential equations which we have developed in section 7, p. 29, for the geodetic line are the following:

$$ds \cos \alpha = M dl \quad (1)$$

$$ds \sin \alpha = N \cos \varphi dl \quad (2)$$

$$d\alpha = dl \sin \varphi. \quad (3)$$

As usual, M is here the radius of curvature in the meridian and N the radius of curvature in the prime vertical for the latitude φ :

$$M = \frac{c}{V^3}, \quad N = \frac{c}{V}, \quad \text{where} \quad V = \sqrt{1 + e'^2 \cos^2 \varphi}. \quad (4)$$

If we introduce this denotation V and eliminate, at the same time, dl from (3) by means of (2), then we obtain from (1), (2) and (3):

$$\frac{d\varphi}{ds} = \frac{1}{c} V^3 \cos \alpha \quad (5)$$

$$\frac{dl}{ds} = \frac{1}{c} V \frac{\sin \alpha}{\cos \varphi} \quad (6)$$

$$\frac{d\alpha}{ds} = \frac{1}{c} V \sin \alpha \tan \varphi. \quad (7)$$

On this we can found a development according to Maclaurin's series, entirely in conformity with the previous spherical development of the first half-volume, section 63. We have to the fifth power:

$$\varphi' - \varphi = \frac{d\varphi}{ds} s + \frac{d^2\varphi}{ds^2} \frac{s^2}{2} + \frac{d^3\varphi}{ds^3} \frac{s^3}{6} + \frac{d^4\varphi}{ds^4} \frac{s^4}{24} + \frac{d^5\varphi}{ds^5} \frac{s^5}{120} + \dots \quad (8)$$

$$l = \frac{dl}{ds} s + \frac{d^2l}{ds^2} \frac{s^2}{2} + \frac{d^3l}{ds^3} \frac{s^3}{6} + \frac{d^4l}{ds^4} \frac{s^4}{24} + \frac{d^5l}{ds^5} \frac{s^5}{120} + \dots \quad (9)$$

$$\alpha' - \alpha = \frac{d\alpha}{ds} s + \frac{d^2\alpha}{ds^2} \frac{s^2}{2} + \frac{d^3\alpha}{ds^3} \frac{s^3}{6} + \frac{d^4\alpha}{ds^4} \frac{s^4}{24} + \frac{d^5\alpha}{ds^5} \frac{s^5}{120} + \dots \quad (10)$$

Since we always need, in the case of successive differentiations, also the derivative of V (cf. to this also first half-volume, section 40, p. 62), we send this up first:

$$V = \sqrt{1 + e'^2 \cos^2 \varphi}, \quad \frac{dV}{d\varphi} = - \frac{e'^2 \sin \varphi \cos \varphi}{V}. \quad (11)$$

For abbreviation, we shall always write:

$$e'^2 \cos^2 \varphi = \eta^2 \quad \text{and} \quad \tan \varphi = t, \quad (12)$$

and with this, (11) becomes:

$$V^2 = 1 + \eta^2 \quad \frac{dV}{d\varphi} = - \frac{\eta^2}{V} t \quad (13)$$

$$\frac{dV}{ds} = \frac{dV}{d\varphi} \frac{d\varphi}{ds} = - \eta^2 \frac{V^2}{c} \cos \alpha t. \quad (14)$$

Now we differentiate (5) further:

$$\frac{d\varphi}{ds} = \frac{V^3}{c} \cos \alpha, \quad \frac{d^2\varphi}{ds^2} = \frac{3}{c} \frac{V^2}{ds} \cos \alpha - \frac{V^3}{c} \sin \alpha \frac{d\alpha}{ds},$$

and hence, because of (14) and (7):

$$\begin{aligned} \frac{d^2\varphi}{ds^2} &= -3\eta^2 \frac{V^4}{c^2} \cos^2 \alpha t - \frac{V^4}{c^2} \sin^2 \alpha t \\ \frac{d^2\varphi}{ds^2} &= -\frac{V^4}{c^2} (\sin^2 \alpha t + 3 \cos^2 \alpha \eta^2 t). \end{aligned} \quad (15)$$

If we differentiate this further, then it is useful to treat the function η^2 , which according to (12) is a function of φ , always thusly (just as first half-volume, p. 62).

$$\frac{d\eta^2}{d\varphi} = -2\eta^2 t \quad \text{more generally} \quad \frac{d\eta^n}{d\varphi} = -n\eta^n t. \quad (16)$$

In this manner, we differentiate (15) once again [bearing in mind that $V^3 = V(1 + \eta^2)$]:

$$\begin{aligned} \frac{d^3\varphi}{ds^3} &= -\frac{4}{c^2} \frac{V^3}{c} \left(-\eta^2 \frac{V^2}{c} \cos \alpha t \right) \left\{ \sin^2 \alpha t + 3 \cos^2 \alpha \eta^2 t \right\} \\ &\quad - \frac{V^4}{c^2} \left\{ 2 \sin \alpha \cos \alpha \frac{V}{c} \sin \alpha t^2 + \sin^2 \alpha (1 + t^2) \frac{V}{c} \cos \alpha (1 + \eta^2) \right. \\ &\quad \left. - 6 \cos \alpha \sin \alpha \frac{V}{c} \sin \alpha t \eta^2 t + 3 \cos^2 \alpha (-2\eta^2 t^2 + \eta^2 (1 + t^2)) \frac{V}{c} \cos \alpha (1 + \eta^2) \right\}. \end{aligned}$$

If we collect terms, then we find:

$$\frac{d^3\varphi}{ds^3} = -\frac{V^5 \cos \alpha}{c^3} \left\{ \sin^2 \alpha (1 + 3t^2 + \eta^2 - 9\eta^2 t^2) + \cos^2 \alpha (3\eta^2 - 3\eta^2 t^2 + 3\eta^4 - 15\eta^4 t^2) \right\}. \quad (17)$$

In the same manner, also the other derivatives are treated so that we obtain to the third order inclusive:

$$\frac{d^2 l}{ds^2} = \frac{2}{c^2} \frac{V^2}{\cos \varphi} \sin \alpha \cos \alpha t \quad (18)$$

$$\frac{d^3 l}{ds^3} = \frac{2}{c^3} \frac{V^3}{\cos \varphi} \left\{ \sin \alpha \cos^2 \alpha (1 + 3t^2 + \eta^2) - \sin^3 \alpha t^2 \right\} \quad (19)$$

$$\frac{d^2 \alpha}{ds^2} = \frac{V^2}{c^2} \sin \alpha \cos \alpha (1 + 2t^2 + \eta^2) \quad (20)$$

$$\frac{d^3 \alpha}{ds^3} = \frac{V^3}{c^3} \left\{ \sin \alpha \cos^2 \alpha t (5 + 6t^2 + \eta^2 - 4\eta^4) - \sin^3 \alpha t (1 + 2t^2 + \eta^2) \right\}. \quad (21)$$

Before we develop further, we will introduce abbreviating denotations, where we have to bear in mind that $\frac{c}{V} = N$ is the radius of curvature in the prime vertical for the latitude φ . We set then:

$$\frac{\rho}{N} s \sin \alpha = \frac{\rho}{c} V s \sin \alpha = v \quad (22)$$

$$\frac{\rho}{N} s \cos \alpha = \frac{\rho}{c} V s \cos \alpha = u. \quad (23)$$

Here s is the geodetic line measured linearly (in meters) and according to the first half-volume, p. 44:

$$\log \frac{\rho}{c} = 8.508\ 3274 \cdot 897, \quad \log \frac{c}{\rho} = 1.491\ 6725 \cdot 103. \quad (24)$$

Further developments to the fifth order

Without indicating the details of the differentiations, in the following we compile the further differential quotients, and in fact to the fourth order with all terms which occur at all, in the case of the fifth order only with the terms without η^2 , i.e. with the spherical terms. In order to be able to use the abbreviations v and u according to (22) and (23), we always add at the left-hand side s , s^2 , s^3 , and so on, as a factor, and also take over to the left-hand side the constant factor V^2 in the case of φ and $\cos \varphi$ in the case of l .

$$\frac{d\varphi}{ds} \frac{s}{V^2} = +u$$

$$\frac{d^2\varphi}{ds^2} \frac{s^2}{V^2} = -v^2 t - u^2 (3\eta^2 t)$$

$$\frac{d^3\varphi}{ds^3} \frac{s^3}{V^2} = -v^2 u (1 + 3t^2 + \eta^2 - 9\eta^2 t^2) - 3u^3 \eta^2 (1 - t^2 + \eta^2 - 5\eta^2 t^2)$$

$$\frac{d^4\varphi}{ds^4} \frac{s^4}{V^2} = +v^4 t (1 + 3t^2 + \eta^2 - 9\eta^2 t^2) - 2v^2 u^2 t (4 + 6t^2 - 13\eta^2 - 9\eta^2 t^2 - 17\eta^4 + 45\eta^4 t^2) + u^4 t \eta^2 (12 + 69\eta^2 - 45\eta^2 t^2 + 57\eta^4 - 105\eta^4 t^2)$$

$$\frac{d^5\varphi}{ds^5} \frac{s^5}{V^2} = +v^4 u (1 + 30t^2 + 45t^4) - 2v^2 u^3 (4 + 30t^2 + 30t^4)$$

$$\frac{dl}{ds} s \cos \varphi = +v$$

$$\frac{d^2 l}{ds^2} s^2 \cos \varphi = +2vut$$

$$\frac{d^3 l}{ds^3} s^3 \cos \varphi = +2vu^2 (1 + 3t^2 + \eta^2) - 2v^3 t^2$$

$$\frac{d^4 l}{ds^4} s^4 \cos \varphi = 8vu^3 t (2 + 3t^2 + \eta^2 - \eta^4) - 8v^3 ut (1 + 3t^2 + \eta^2)$$

$$\frac{d^5 l}{ds^5} s^5 \cos \varphi = 8vu^4 (2 + 15t^2 + 15t^4) - 8v^3 u^2 (1 + 20t^2 + 30t^4) + 8v^5 t^2 (1 + 3t^2)$$

$$\frac{d\alpha}{ds} s = vt$$

$$\frac{d^2\alpha}{ds^2} s^2 = vu (1 + 2t^2 + \eta^2)$$

$$\frac{d^3\alpha}{ds^3} s^3 = vu^2 t (5 + 6t^2 + \eta^2 - 4\eta^4) - v^3 t (1 + 2t^2 + \eta^2)$$

$$\frac{d^4\alpha}{ds^4} s^4 = vu^3 (5 + 28t^2 + 24t^4 + 6\eta^2 + 8\eta^2 t^2 - 3\eta^4 + 4\eta^4 t^2 - 4\eta^6 + 24\eta^6 t^2) - v^3 u (1 + 20t^2 + 24t^4 + 2\eta^2 + 8\eta^2 t^2 + \eta^4 - 12\eta^4 t^2)$$

$$\frac{d^5\alpha}{ds^5} s^5 = vu^4 t (61 + 180t^2 + 120t^4) - v^3 u^2 t (58 + 280t^2 + 240t^4) + v^5 t (1 + 20t^2 + 24t^4).$$

One will hardly ever need more than these terms. Besides, in the former third edition, 1890, p. 392, we have given the terms to the fifth order with all additions η^2 , and so on, and then also to the sixth order at least spherically, i.e. without η^2 .

For abbreviation we will omit all which goes beyond the fifth order, where η^2 is treated as a quantity of the second order.

With this, we obtain the following formulae prepared for practical application, in which u and v have the meanings of (22) and (23):

$$\left. \begin{aligned} \frac{\varphi' - \varphi}{V^2} = & u - \frac{1}{2\varrho} v^2 t - \frac{3}{2\varrho} u^2 \eta^2 t \\ & - \frac{v^2 u}{6\varrho^2} (1 + 3t^2 + \eta^2 - 9\eta^2 t^2) - \frac{u^3}{2\varrho^2} \eta^2 (1 - t^2) \\ & + \frac{v^4}{24\varrho^3} t (1 + 3t^2 + \eta^2 - 9\eta^2 t^2) - \frac{v^2 u^2}{12\varrho^3} t (4 + 6t^2 - 13\eta^2 \\ & \quad - 9\eta^2 t^2) + \frac{u^4}{2\varrho^3} \eta^2 t \\ & + \frac{v^4 u}{120\varrho^4} (1 + 30t^2 + 45t^4) - \frac{v^2 u^3}{30\varrho^4} (2 + 15t^2 + 15t^4) \end{aligned} \right\} \quad (25)$$

$$\left. \begin{aligned} l \cos \varphi = & v + \frac{1}{\varrho} v u t \\ & - \frac{v^3}{3\varrho^2} t^2 + \frac{v u^2}{3\varrho^2} (1 + 3t^2 + \eta^2) \\ & - \frac{v^3 u}{3\varrho^3} t (1 + 3t^2 + \eta^2) + \frac{v u^3}{3\varrho^3} t (2 + 3t^2 + \eta^2) \\ & + \frac{v^5}{15\varrho^4} t^2 (1 + 3t^2) + \frac{v u^4}{15\varrho^4} (2 + 15t^2 + 15t^4) - \frac{v^3 u^2}{15\varrho^4} (1 + 20t^2 + 30t^4) \end{aligned} \right\} \quad (26)$$

$$\left. \begin{aligned} \alpha' - \alpha = & v t + \frac{v u}{2\varrho} (1 + 2t^2 + \eta^2) \\ & - \frac{v^3}{6\varrho^2} t (1 + 2t^2 + \eta^2) + \frac{v u^2}{6\varrho^2} t (5 + 6t^2 + \eta^2 - 4\eta^4) \\ & - \frac{v^3 u}{24\varrho^3} (1 + 20t^2 + 24t^4 + 2\eta^2 + 8\eta^2 t^2) + \frac{v u^3}{24\varrho^3} (5 + 28t^2 \\ & \quad + 24t^4 + 6\eta^2 + 8\eta^2 t^2) \\ & + \frac{v^5}{120\varrho^4} t (1 + 20t^2 + 24t^4) - \frac{v^3 u^2}{120\varrho^4} t (58 + 280t^2 + 240t^4) \\ & \quad + \frac{v u^4}{120\varrho^4} t (61 + 180t^2 + 120t^4). \end{aligned} \right\} \quad (27)$$

The constant logarithms needed here are:

$$\left. \begin{aligned} \log \frac{1}{\varrho} &= 4.685\,5749 - 10, & \log \frac{1}{2\varrho} &= 4.384\,5449 - 10, & \log \frac{3}{2\varrho} &= 4.861\,6661 - 10 \\ \log \frac{1}{2\varrho^2} &= 9.070\,120 - 20, & \log \frac{1}{3\varrho^2} &= 8.894\,028 - 20, & \log \frac{1}{6\varrho^2} &= 8.592\,998 - 20 \\ \log \frac{1}{3\varrho^3} &= 3.57960 - 20, & \log \frac{1}{6\varrho^3} &= 3.27857 - 20, & \log \frac{1}{24\varrho^3} &= 2.67651 - 20 \\ \log \frac{1}{15\varrho^4} &= 7.5662 - 30, & \log \frac{1}{30\varrho^4} &= 7.2652 - 30, & \log \frac{1}{120\varrho^4} &= 6.6631 - 30 \end{aligned} \right\} \quad (28)$$

If we omit in (25), (26), (27) all η^2 's, then we obtain again the spherical formulae (27) to (29) from the first half-volume, section 63, pp. 178 and 179, as it must be.

With these constant coefficients we can also use our auxiliary tables, pages [1] to [4] of the Appendix. We will compute our small spheroidal normal example (1), section 17, p. 73, in this manner:

$$\text{Given} \quad \varphi = 49^\circ 30' 0'' \quad \alpha = 32^\circ 25' 21.5109'' \quad \log s = 5.121\,6103\cdot 1,$$

to this from the first half-volume, page [25] of the Appendix:

$$\log [2] = 8.508\,9420\cdot 3 \quad \text{and} \quad \log V^2 = 0.001\,2290\cdot 7.$$

As to the rest, the calculation yields according to the indicated method in a similar manner as in the spherical example, first half-volume, section 63, p. 179:

$\log [2]$	8.508 9420.3	$\log [2]$	8.508 9420.3
$\log s$	5.121 6103.1	$\log s$	5.121 6103.1
$\log \sin \alpha$	6.729 2947.4	$\log \cos \alpha$	9.926 4021.9
$\log v$	3.359 8470.8	$\log u$	3.556 9545.3

The further calculation has yielded the following:

Latitude		Longitude		Azimuth	
$+ V^2 u =$	$+ 3615.6269''$	$v =$	$+ 3526.1653''$	$vt =$	$+ 2681.3172''$
$- v^2 \dots$	$- 14.9269$	$+ vu \dots$	$+ 72.1660$	$+ vu \dots$	$+ 74.9467$
$- u^2 \eta^2 \dots$	$- 0.3146$	$- v^3 \dots$	$- 0.1986$	$- v^3 \dots$	$- 0.2063$
$- v^2 u \dots$	$- 0.3774$	$+ v u^2 \dots$	$+ 1.8371$	$+ v u^2 \dots$	$+ 1.8061$
$+ u^3 \eta^2 \dots$	$+ 0.0006$	$- v^3 u \dots$	$- 0.0152$	$- v^3 u \dots$	$- 0.0152$
$+ v^4 \dots$	$+ 0.0008$	$+ v u^3 \dots$	$+ 0.0452$	$+ v u^3 \dots$	$+ 0.0455$
$- v^2 u^2 \dots$	$- 0.0093$	$+ v^5 \dots$	0.0000	$+ v^5 \dots$	0.0000
$+ v^4 u \dots$	$+ 0.0001$	$+ v u^4 \dots$	$+ 0.0007$	$- v u^4 \dots$	$+ 0.0011$
$- v^2 u^3 \dots$	$- 0.0002$	$- v^3 u^2 \dots$	$- 0.0005$	$- v^3 u^2 \dots$	$- 0.0008$
$\varphi' - \varphi =$	$3600.0000''$	$l =$	$+ 3600.0000''$	$\alpha' - \alpha =$	$2757.8943''$
$= 1^\circ 0' 0.0000$		$= 1^\circ 0' 0.0000$		$= 45' 57.8943$	
should be	0.0000	should be	0.0000	should be	57.8942

If we compare the foregoing developments with the previous formulae (14) to (16) from section 5, p. 24, which refer to the vertical section, then we are in a position to compute once again the difference of direction between the vertical section and the geodetic line. Since we have established in section 9, p. 38, that the difference in the linear length of the arc of the vertical section and of the geodetic line between two points of the terrestrial ellipsoid is only of the eighth order, we are justified in assuming that the difference of the formulae of section 5 in comparison with the foregoing formulae (25) to (27) is caused only by the difference of azimuth of the vertical section and of the geodetic line. If we denote the difference of azimuth by $\Delta \alpha$ and the difference of the two differences of latitude in (25) and in (14), section 5, p. 24, by $\Delta \left(\frac{\varphi' - \varphi}{V^2} \right)$, then we obtain at first from both equations in agreement:

$$\Delta \left(\frac{\varphi' - \varphi}{V^2} \right) = - (v + v u t - 3 v u \eta^2 t + \dots) \Delta \alpha$$

or

$$\Delta \left(\frac{\varphi' - \varphi}{V^2} \right) = - v (1 + u t (1 - 3 \eta^2)) \Delta \alpha. \quad (29)$$

But, on the other hand, the comparison of the foregoing equation (25) with equation (14), section 5, p. 24, shows that:

$$\Delta \left(\frac{\varphi' - \varphi}{V^2} \right) = - \frac{1}{6} v^2 u \eta^2 + \frac{1}{24} v^4 \eta^2 t - \frac{1}{8} v^2 u^2 \eta^2 t \quad (30)$$

It is true that in equation (25) the purely spherical terms of the fifth order are still indicated; for the comparison, however, they are not involved since the spherical terms for the vertical section and for the geodetic line must be the same. We thus obtain from (29) and (30):

$$(1 + u t (1 - 3 \eta^2)) \Delta \alpha = \frac{1}{6} v u \eta^2 - \frac{1}{24} v^3 t \eta^2 + \frac{1}{8} v u^2 t \eta^2$$

or, if we do not go beyond the fifth order

$$\begin{aligned} \Delta \alpha &= \frac{1}{6} v u \eta^2 - \frac{1}{24} v^3 t \eta^2 + \frac{1}{8} v u^2 t \eta^2 - \frac{1}{6} v u^2 t \eta^2 + \dots \\ \Delta \alpha &= \frac{1}{6} v u \eta^2 - \frac{1}{24} v^3 t \eta^2 - \frac{1}{24} v u^2 t \eta^2. \end{aligned} \quad (31)$$

As a check, we will carry out the same calculation once again with the expression for the difference of longitude. According to (26) we will have

$$\Delta (l \cos \varphi) = (u + (u^2 - v^2) t) \Delta \alpha, \quad (32)$$

and as the difference of the foregoing equation (26) with equation (15) of section 5, p. 24, we obtain:

$$\Delta (l \cos \varphi) = \frac{1}{6} v u^2 \eta^2 - \frac{5}{24} v^3 u \eta^2 t + \frac{1}{8} v u^3 \eta^2 t. \quad (33)$$

We obtain hence, again to an accuracy of terms of the fifth order,

$$\Delta \alpha \left(1 + \frac{u^2 - v^2}{u} t \right) = \frac{1}{6} v u \eta^2 - \frac{5}{24} v^3 \eta^2 t + \frac{1}{8} v u^2 \eta^2 t$$

or

$$\Delta \alpha = \frac{1}{6} v u \eta^2 - \frac{1}{24} v^3 \eta^2 t - \frac{1}{24} v u^2 \eta^2 t, \quad (34)$$

which agrees with (31).

In order to achieve also complete agreement of equation (31) or (34) with the previous equation (17), section 9, p. 37, we still have to set:

$$v = \frac{s}{N} \sin \alpha \quad \text{and} \quad u = \frac{s}{N} \cos \alpha$$

with which we obtain

$$\Delta \alpha = \frac{1}{6} \frac{s^2}{N^2} \eta^2 \sin \alpha \cos \alpha - \frac{1}{24} \frac{s^3}{N^3} \eta^2 t \sin \alpha \quad (35)$$

in agreement with (17), section 9, p. 37.

With this, we have derived once more the reduction of the arc of the vertical section to the direction of the geodetic line independently of the geometric considerations of section 9.

Length of the arc of the meridian

Our formulae contain also the special case of the rectification of the arc of the meridian if the azimuth α becomes zero. If we then set also the pertinent value $s = m$, then we will obtain from (25) the following to the third order:

$$\frac{\varphi' - \varphi}{V^2} = \frac{m}{N} - \frac{3}{2} \eta^2 t \frac{m^2}{N^2} + \frac{m^3}{N^3} \frac{\eta^2}{2} (t^2 - 1 - \eta^2 + 5 \eta^2 t^2). \quad (36)$$

This is the inversion of the previous formula (38) in the first half-volume, section 41, p. 73, as it is shown more clearly if we write that formula thusly:

$$\frac{m}{N} = \frac{\varphi' - \varphi}{V^2} + \left(\frac{\varphi' - \varphi}{V^2} \right)^2 \frac{3}{2} \eta^2 t - \left(\frac{\varphi' - \varphi}{V^2} \right)^3 \frac{\eta^2}{2} (t^2 - 1 - \eta^2 - 4 \eta^2 t^2). \quad (37)$$

We can easily prove by an approximate solution that these two formulae (36) and (37) agree with one another.

The first development according to powers of the geodetic line (to s^3 inclusive) for the transfer of latitudes, longitudes and azimuths is given by Legendre in the *Memoirs of the Paris Academy of 1806*. These formulae by Legendre have been used in the land survey of Baden. Helmert, *Höhere Geodäsie*, I, 1880, pp. 296-300, gives the developments to the third order with e^2 and then, in addition, fourth to fifth order spherically, with reference to the literature on p. 300. In order to apply in practice, on a large scale, the developments in series treated above, one would have to produce convenient and accurate tables of coefficients, where in the series (25) to (27) the coefficients would have to be introduced with all terms in η^2 .

In order to facilitate the development of the higher terms, which become already very unwieldy in the case of the fifth order, L. Grabowski has derived general formulae which make it possible to find from the n th differential quotients for φ , l and α the $(n+1)$ th differential quotients. Cf. to this L. Grabowski, "Über die Potenzreihen zur sogenannten 'geodätischen Hauptaufgabe,'" in *Österr. Zeitschr. f. Verm.*, 1917, pp. 133-139 and pp. 198-208. However, we will seldom have occasion to carry the developments beyond the fifth order, since the practical use of the formulae is then no longer convenient. In such cases, one will prefer the computing method of the following sections 19-20.

The foregoing developments in series form the foundation for newer solutions of the problem of the transfer of geographic coordinates, of which we will communicate the most important ones in the following.

For the Danish Land Survey Andrae gave a solution of the problem in *Den danske Gradmaaling*, Vol. III, Copenhagen, 1878, pp. 281-316. The same formulae are found also in Zachariae, *Die geodätischen Hauptpunkte und ihre Koordinaten*, German by E. Lamp, Berlin, 1878, p. 198. In section 27, p. 136, we bring details about Andrae's formulae.

Formulae which were developed by Schreiber for the trigonometric section of the Prussian Land Survey are especially important for practical application. The formulae are published in connection with detailed auxiliary tables in "Formeln und Tafeln zur Berechnung der geographischen Koordinaten. Erste Ordnung." Berlin, 1878. In section 19 we bring the development of these formulae.

Further solutions of the problem are contained in Helmert, *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, Bd. I, Leipzig, 1880, pp. 451-459, where detailed historical notes are also communicated, and in Börsch, *Anleitung zur Berechnung geodätischer Koordinaten*, Cassel, 1885, p. 72 and following. Another derivation of the Andrae-Helmert formulae is given by L. Krüger in *Zeitschr. f. Verm.*, 1921, pp. 1-8, 33-38, 65-80.

We have a critical comparison of the various systems of formulae by L. Krüger in the treatise, "Die Formeln von C. G. Andrae, O. Schreiber, F. R. Helmert und O. Börsch für geographische Koordinaten und Untersuchung ihrer Genauigkeit" in *Zeitschr. f. Verm.*, 1921, pp. 547-557, 579-588. In the place of the four solutions, Krüger puts a new one, which we will develop in section 20.

Finally, we further mention a new work, A. Tonolo, "Trasporto delle coordinate geografiche e dell'azimut lungo un arco di linea qualunque di un ellissoide di rotazione," *Atti della Reale Acc. Naz. dei Lincei*, 1939 (XVII), Vol. XXIX, Roma, 1939, pp. 573-580.

Section 19. The Formulae of Schreiber for the Transfer of Geographic Coordinates

The convergence of the power series developed in the foregoing section 18 is substantially improved if we introduce the computation of rectangular ellipsoidal coordinates as intermediate terms in the case of the transfer of geographic coordinates. If we place in Fig. 1 through P' a geodetic line, which intersects the meridian of P at right angles at P_1 , then we can apply Legendre's simple theorem to the right triangle PP_1P' , if $PP' = s$ is not greater than about 150 km. The triangle is regarded here as a spherical triangle, for which we use, as spherical radius, the mean radius of curvature \sqrt{MN} . Let α be again the azimuth of PP' .

For the computation of the spherical excess, we can neglect the curvature of the surface of the triangle and have then

$$x = s \cos \alpha \quad y = s \sin \alpha.$$

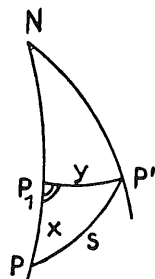


Fig. 1.

With these, we will have

$$\varepsilon = \frac{s^2 \sin \alpha \cos \alpha}{2 MN} \rho. \quad (1)$$

Now, in Fig. 2, we consider the plane triangle PP_1P' with the same sides x , y and s , however with the angles

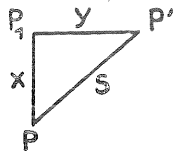


Fig. 2.

$$\begin{aligned} P: \quad \alpha - \frac{\varepsilon}{3} \quad \quad P_1: \quad 90^\circ - \frac{\varepsilon}{3} \\ P': \quad 90^\circ - \left(\alpha - \frac{2}{3} \varepsilon\right). \end{aligned}$$

The solution of the plane triangle, Fig. 2, yields then

$$\begin{aligned} x = s \frac{\sin \left(90^\circ - \left(\alpha - \frac{2}{3} \varepsilon\right)\right)}{\sin \left(90^\circ - \frac{\varepsilon}{3}\right)} \quad \quad y = s \frac{\sin \left(\alpha - \frac{\varepsilon}{3}\right)}{\sin \left(90^\circ - \frac{\varepsilon}{3}\right)} \\ \text{or} \\ x = s \frac{\cos \left(\alpha - \frac{2}{3} \varepsilon\right)}{\cos \frac{\varepsilon}{3}} \quad \quad y = s \frac{\sin \left(\alpha - \frac{\varepsilon}{3}\right)}{\cos \frac{\varepsilon}{3}}. \end{aligned} \quad (2)$$

The denominator in (2) is so little different from unity, in view of the above assumed value of s , that it does not numerically influence the values of x and y . Hence, we can write also

$$\begin{aligned} x = s \cos \left(\alpha - \frac{2}{3} \varepsilon\right) \quad \quad y = s \sin \left(\alpha - \frac{\varepsilon}{3}\right) \\ \text{or} \\ x = s \cos \alpha + s \sin \alpha \frac{2 \varepsilon}{3} \quad \quad y = s \sin \alpha - s \cos \alpha \frac{\varepsilon}{3} \end{aligned} \quad (3)$$

Here and in the following, we will also use, everywhere, the denotations introduced by Schreiber and set

$$s \sin \alpha = v \quad \quad s \cos \alpha = u. \quad (4)$$

[We must note to this that these u 's and v 's are not the same as our u and v from (22) and (23), section 18, p. 76.]

We use the values of x and y computed with the help of (3) for the transfer of the geographic coordinates from P to P' , by computing from x the difference of latitude of P and P_1 , from y the difference of latitude of P_1 and P' , as well as the difference of longitude of P_1 and P' , which agrees with the difference of longitude of P and P' .

For the difference of latitude PP_1 we use the formula (25), section 18, p. 78, in which we are to set $\alpha = 0$ and $s = x$. Then we obtain to the third order

$$\frac{\varphi_1 - \varphi}{\rho^2} = \frac{x}{N} - \frac{3}{2} \frac{x^2}{N^2} \eta^2 t - \frac{1}{2} \frac{x^3}{N^3} \eta^3 (1 - t^2)$$

and with the above value of x

$$\frac{\varphi_1 - \varphi}{V^2} = \frac{u}{N} \left(1 + \frac{v^2}{3MN} \right) - \frac{3}{2} \frac{u^2}{N^2} \eta^2 t - \frac{1}{2} \frac{u^3}{N^3} \eta^2 (1 - t^2)$$

$$\varphi_1 - \varphi = u \frac{V^2}{N} \left\{ 1 + \frac{1}{3} \frac{v^2}{MN} - \frac{3}{2} \frac{u}{N} \eta^2 t - \frac{1}{2} \frac{u^2}{N^2} \eta^2 (1 - t^2) \right\}.$$

Now we still use the relation $\frac{N}{M} = V^2$ and simply replace in the last term N by the equatorial radius α , so that we have

$$\varphi_1 - \varphi = \frac{u}{M} \left\{ 1 + \frac{1}{3} \frac{v^2}{MN} - \frac{3}{2} \frac{u}{M V^2} \eta^2 t - \frac{1}{2} \frac{u}{\alpha^2} \eta^2 (1 - t^2) \right\}. \quad (5)$$

Now, in addition, in accordance with Schreiber's procedure, V shall be replaced by W and e' by e . To this, we have

$$\eta^2 = e'^2 \cos^2 \varphi \quad 1 + e'^2 = \frac{1}{1 - e^2} \quad e'^2 = \frac{e^2}{1 - e^2}.$$

Since $V^2 (1 - e^2) = W^2$, then we have for the third term in (5)

$$\frac{\eta^2}{V^2} = \frac{e^2 \cos^2 \varphi}{(1 - e^2) V^2} = \frac{e^2 \cos^2 \varphi}{W^2}$$

and in the last term of (5) we can set $\eta^2 = e^2 \cos^2 \varphi$. Then we will have, if we write, at the same time, everywhere $\tan \varphi$ for t :

$$\varphi_1 - \varphi = \frac{u}{M} \left\{ 1 + \frac{1}{3} \frac{v^2}{MN} - \frac{3}{2} \frac{u}{M W^2} e^2 \cos^2 \varphi \tan \varphi - \frac{1}{2} \frac{u^2}{\alpha^2} e^2 \cos^2 \varphi (1 - \tan^2 \varphi) \right\}$$

or

$$\varphi_1 - \varphi = \frac{u}{M} \left\{ 1 + \frac{1}{3} \frac{v^2}{MN} - \frac{3}{4} \frac{u}{M W^2} e^2 \sin 2\varphi - \frac{1}{2} \frac{u^2}{\alpha^2} e^2 \cos 2\varphi \right\} \quad (6)$$

Without changing anything in the accuracy of this formula, we can write it also in the following form, where we add ρ at the same time,

$$\varphi_1 - \varphi = \frac{u \rho}{M} \left(1 + \frac{1}{3} \frac{v^2}{MN} \right) \left(1 - \frac{3}{4} \frac{u}{M W^2} e^2 \sin 2\varphi \right) \left(1 - \frac{1}{2} \frac{u^2}{\alpha^2} e^2 \cos 2\varphi \right). \quad (7)$$

This expression can easily be put into logarithmic form and yields

$$\log(\varphi_1 - \varphi) = \log \frac{u}{M} \rho + \frac{1}{3} \frac{v^2 \mu}{MN} - \frac{3}{4} \frac{u \mu}{M W^2} e^2 \sin 2\varphi - \frac{1}{2} \frac{u^2 \mu}{\alpha^2} e^2 \cos 2\varphi. \quad (8)$$

For the coefficients occurring here, which are dependent on the geographic latitude, there has been issued by the Prussian Land Survey a collection of tables, which we shall indicate at the end of this section. The following denotations hold in it

$$\varphi_1 - \varphi = b \quad \frac{\rho}{M} = (1) \quad (1) u = \beta \quad \frac{\mu}{3MN} = \frac{\mu}{3r^2} = (5)$$

$$\frac{3\mu e^2 \sin 2\varphi}{4W^2M} = (4) \quad -\frac{\mu e^2 \cos 2\varphi}{2\alpha^2} = (6)$$

and equation (8) reads with these

$$\left. \begin{aligned} \log b &= \log \beta - (4) u + (5) v^2 + (6) u^2 \\ \varphi_1 &= \varphi + b. \end{aligned} \right\} \quad (9)$$

Determination of the difference of longitude l. For the computation of the difference of longitude we use equation (26), section 18, p. 78, where we still carry the terms of fifth order. Now we have to set in this equation $\alpha = 90^\circ$ and $s = y$. If we take into account further that the argument φ_1 now holds, then we have

$$l \cos \varphi_1 = \frac{y}{N_1} \varrho - \frac{1}{3} \frac{y^3}{N_1^3} \varrho t_1^2 + \frac{1}{15} \frac{y^5}{N_1^5} \varrho t_1^2 (1 + 3 t_1^2) \quad (10)$$

We set in this $\frac{y}{N_1} \varrho = c$, with which (10) passes over into

$$l \cos \varphi_1 = c - \frac{1}{3} \frac{c^3}{\varrho^2} \tan^2 \varphi_1 + \frac{1}{15} \frac{c^5}{\varrho^4} \tan^2 \varphi_1 (1 + 3 \tan^2 \varphi_1)$$

or

$$l = c \sec \varphi_1 \left\{ 1 - \frac{1}{3} \frac{c^2}{\varrho^2} \tan^2 \varphi_1 + \frac{1}{15} \frac{c^4}{\varrho^4} \tan^2 \varphi_1 (1 + 3 \tan^2 \varphi_1) \right\}. \quad (11)$$

For the conversion to logarithmic form of this equation, we have, according to the first half-volume, section 34, p. 21, the logarithmic series

$$\log (1 + x) = \mu x - \frac{\mu x^2}{2} + \dots$$

In our case we have

$$\begin{aligned} x &= -\frac{1}{3} \frac{c^2}{\varrho^2} \tan^2 \varphi_1 + \frac{1}{15} \frac{c^4}{\varrho^4} \tan^2 \varphi_1 (1 + 3 \tan^2 \varphi_1) \\ \frac{x^2}{2} &= +\frac{1}{18} \frac{c^4}{\varrho^4} \tan^4 \varphi_1. \end{aligned}$$

With this, we will have

$$\begin{aligned} \log l &= \log c \sec \varphi_1 - \frac{1}{3} \frac{\mu c^2}{\varrho^2} \tan^2 \varphi_1 + \frac{1}{15} \frac{\mu c^4}{\varrho^4} \tan^2 \varphi_1 \\ &\quad + \frac{1}{5} \frac{\mu c^4}{\varrho^4} \tan^4 \varphi_1 - \frac{1}{18} \frac{\mu c^4}{\varrho^4} \tan^4 \varphi_1 \end{aligned}$$

or

$$\log l = \log c \sec \varphi_1 - \frac{1}{3} \frac{\mu c^2}{\varrho^2} \tan^2 \varphi_1 + \frac{1}{90} \frac{\mu c^4}{\varrho^4} \tan^2 \varphi_1 (6 + 13 \tan^2 \varphi_1).$$

We can write the expression within parentheses of the last term also in the form

$$\frac{6}{\cos^2 \varphi_1} - 6 \tan^2 \varphi_1 + 13 \tan^2 \varphi_1$$

and obtain with this

$$\log l = \log c \sec \varphi_1 - \frac{\mu}{3} \frac{c^2}{\varrho^2} \tan^2 \varphi_1 + \frac{\mu}{15} \frac{c^4}{\varrho^4} \frac{\tan^2 \varphi_1}{\cos^2 \varphi_1} + \frac{7\mu}{90} \frac{c^4}{\varrho^4} \tan^2 \varphi_1. \quad (12)$$

Now we set for further simplification

$$c \sec \varphi_1 = \lambda \quad \text{and} \quad c \tan \varphi_1 = \tau. \quad (13)$$

Then we will have

$$\log l = \log \lambda - \frac{\mu}{3} \frac{\tau^2}{\rho^2} + \frac{\mu}{15} \frac{\lambda^2 \tau^2}{\rho^4} + \frac{7\mu}{90} \frac{\tau^4}{\rho^4}. \quad (14)$$

We collect the constants occurring here under the following denotations:

$$\frac{\mu}{3\rho^2} = \nu \quad \frac{\mu}{15\rho^4} = \nu_1 \quad \frac{7\mu}{90\rho^4} = \nu_2. \quad (15)$$

Then the final expression for $\log l$ is

$$\log l = \log \lambda - \nu \tau^2 + \nu_1 \lambda^2 \tau^2 + \nu_2 \tau^4. \quad (16)$$

Determination of the difference of latitude $\varphi_1 - \varphi'$. We use once more equation (25), section 18, p. 78, in which we have to set now $\alpha = 90^\circ$ and $s = y$. Then we have to terms of fourth order

$$\frac{\varphi' - \varphi_1}{V_1^3} = -\frac{1}{2} \frac{y^2}{N_1^2} \rho t_1 + \frac{1}{24} \frac{y^4}{N_1^4} \rho t_1 (1 + 3 t_1^2 + \eta_1^2 - 9 \eta_1^2 t_1^2).$$

If we set again for simplification $\frac{y}{N_1} \rho = c$, then we will have

$$\frac{\varphi' - \varphi_1}{V_1^2} = -\frac{1}{2} \frac{c^2}{\rho} t_1 + \frac{1}{24} \frac{c^4}{\rho^3} t_1 (1 + 3 t_1^2 + \eta_1^2 - 9 \eta_1^2 t_1^2). \quad (17)$$

In the last term we can set, as before, with sufficient accuracy, $\eta^2 = e^2 \cos^2 \varphi$, and we write the expression within parentheses in the form

$$1 + 3 t_1^2 + e^2 - e^2 \sin^2 \varphi_1 - 9 e^2 \sin^2 \varphi_1 + 3 e^2 t_1^2 - 3 e^2 t_1^2.$$

If we neglect terms with e^4 , then we can set for this also

$$(1 + 3 t_1^2 - 10 e^2 \sin^2 \varphi_1 - 3 e^2 t_1^2) (1 + e^2)$$

or

$$(1 + 3 t_1^2 - 10 e^2 \sin^2 \varphi_1 - 3 e^2 t_1^2) \frac{1}{1 - e^2}.$$

We introduce this into (17), and if we pass over now from $\varphi' - \varphi_1$ to $\varphi_1 - \varphi'$, then we obtain

$$\varphi_1 - \varphi' = \frac{1}{2} \frac{c^2}{\rho} t_1 V_1^2 \left\{ 1 - \frac{1}{12} \frac{c^2}{\rho^2 (1 - e^2)} (1 + 3 t_1^2 - e^2 (10 \sin^2 \varphi_1 + 3 t_1^2)) \right\} \quad (18)$$

The expression within parentheses of the second term can be simplified still further, if we set

$$1 + 3 t_1^2 = \frac{1}{\cos^2 \varphi_1} + 2 \tan^2 \varphi_1 \quad \text{and} \quad \sin^2 \varphi_1 = \tan^2 \varphi_1 (1 - \sin^2 \varphi_1).$$

The expression within parentheses then passes over into

$$\frac{1}{\cos^2 \varphi_1} + 2 \tan^2 \varphi_1 - e^2 \tan^2 \varphi_1 (13 - 10 \sin^2 \varphi_1).$$

Finally, we set again

$$c \sec \varphi_1 = \lambda \quad c \tan \varphi_1 = \tau$$

and obtain then from (18)

$$\varphi_1 - \varphi' = \frac{1}{2} \frac{V_1^2}{e} c \tau \left\{ 1 - \frac{\lambda^2}{12 (1 - e^2) e^2} - \frac{\tau^2}{6 (1 - e^2) e^2} + \frac{e^2 \tau^2}{12 (1 - e^2) e^2} (13 - 10 \sin^2 \varphi_1) \right\} \quad (19)$$

We can put also (19) into logarithmic form in the same manner, as was previously shown for equation (6), p. 83, and use for this the additional denotations introduced by Schreiber

$$\begin{aligned} \frac{V_1^2}{2e} &= \frac{1}{2e} \frac{N_1}{M_1} = (3) \quad c \tau = \delta \\ \frac{\mu}{6(1-e^2)e^2} &= \mu \quad \frac{1}{2} \mu e^2 (13 - 10 \sin^2 \varphi_1) = (8) \quad \varphi_1 - \varphi' = d. \end{aligned}$$

It is to be noted here that μ hitherto denoted the logarithmic modulus, but now, as a new coefficient, obtains another meaning.

Then we find from (19)

$$\log d = \log \delta - \mu \tau^2 - \frac{1}{2} \mu \lambda^2 + (8) \tau^2. \quad (20)$$

Determination of the azimuth ($P_1 P$). At first we apply equation (27), section 18, p. 78, to the geodesic line $P_1 P$. If we set here according to Schreiber $\alpha' - \alpha = t$, then (27), section 18, p. 78, yields for $\alpha = 90^\circ$ and $s = y$

$$t = \frac{3}{N_1} e \tan \varphi_1 - \frac{1}{6} \frac{y^3}{N_1^3} e \tan \varphi_1 (1 + 2 \tan^2 \varphi_1 + \eta_1^2)$$

and with

$$\begin{aligned} \frac{y}{N_1} e &= c \\ t &= c \tan \varphi_1 \left\{ 1 - \frac{c^2}{6e^2} (1 + 2 \tan^2 \varphi_1 + \eta_1^2) \right\}. \end{aligned} \quad (21)$$

Since

$$\eta_1^2 = e'^2 \cos^2 \varphi_1 = \frac{e^2}{1 - e^2} \cos^2 \varphi_1,$$

then we can bring the expression within parentheses of the second term into the form

$$\frac{1}{1-e^2} (1 - e^2 + e^2 - e^2 \sin^2 \varphi_1 + 2(1 - e^2) \tan^2 \varphi_1)$$

or

$$\frac{1}{1-e^2} (1 + 2 \tan^2 \varphi_1 - e^2 (\sin^2 \varphi_1 + 2 \tan^2 \varphi_1)).$$

This is synonymous with

$$\frac{1}{1-e^2} \left(\frac{1}{\cos^2 \varphi_1} + \tan^2 \varphi_1 - e^2 (\cos^2 \varphi_1 \tan^2 \varphi_1 + 2 \tan^2 \varphi_1) \right)$$

or

$$\frac{1}{1-e^2} \left(\frac{1}{\cos^2 \varphi_1} + \tan^2 \varphi_1 - e^2 \tan^2 \varphi_1 (3 - \sin^2 \varphi_1) \right).$$

Equation (21) then changes to

$$t = c \tan \varphi_1 \left\{ 1 - \frac{c^2}{6 \varrho^2 (1-e^2)} \left(\frac{1}{\cos^2 \varphi_1} + \tan^2 \varphi_1 - e^2 \tan^2 \varphi_1 (3 - \sin^2 \varphi_1) \right) \right\}. \quad (22)$$

Finally, if we introduce again

$$c \cos \varphi_1 = \lambda \quad \text{and} \quad c \tan \varphi_1 = \tau$$

then we have

$$t = \tau \left\{ 1 - \frac{\lambda^2}{6 (1-e^2) \varrho^2} - \frac{\tau^2}{6 (1-e^2) \varrho^2} + \frac{\tau^2}{6 (1-e^2) \varrho^2} e^2 (3 - \sin^2 \varphi_1) \right\} \quad (23)$$

For the conversion to logarithmic form of this term, we use again the coefficient μ , introduced on p. 86, and set, in addition,

$$\mu e^2 (3 - \sin^2 \varphi_1) = (7)$$

and then obtain easily

$$\log t = \log \tau - \mu \lambda^2 - \mu \tau^2 + (7) \tau^2. \quad (24)$$

Now in order to find the azimuth α' of the geodetic line PP' at P' , we have represented, in Fig. 3, the two meridians of P and P' as well as the geodetic line PP' , so that at P' the angle $90^\circ + t$ becomes visible. If we draw, in addition, a geodetic parallel to the meridian PN through P' , then we have the angle $N'P'N = t$, and we see hence that t is the meridian convergence at P' with respect to the meridian of P .

If we denote the angle at P' in the triangle PP_1P' by P' , then we have

$$P' = 180^\circ + \varepsilon - \alpha - 90^\circ = 90^\circ - (\alpha - \varepsilon);$$

therefore, we will have the azimuth $(P'P)$ or

$$\begin{aligned} \alpha' &= 90^\circ + t + 180^\circ - (90^\circ - \alpha + \varepsilon) \\ \alpha' &= \alpha \pm 180^\circ + t - \varepsilon. \end{aligned} \quad (25)$$

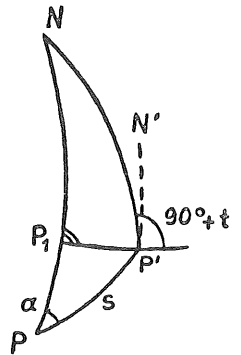


Fig. 3.

Since we have developed above completely Schreiber's method for the transfer of geographic coordinates, we compile once more all the formulae for computation.

Compilation of formulae

Given φ, L, α, s . Required φ', L', α' .

$$\begin{aligned}
 u &= s \cos \alpha & v &= s \sin \alpha & \beta &= (1) u \\
 \log b &= \log \beta - (4) u + (5) v^2 + (6) u^2 \\
 \varphi_1 &= \varphi + b & \gamma &= (2)_1 v \\
 \log c &= \log (\gamma) - \frac{1}{2} (5) u^2 \\
 \tau &= c \tan \varphi_1 & \lambda &= \frac{c}{\cos \varphi_1} & \delta &= (3)_1 c \tau & \varepsilon &= \frac{b c}{2 \varrho} \\
 \log t &= \log \tau - \mu \lambda^2 - \mu \tau^2 + (7)_1 \tau^2 \\
 \log l &= \log \lambda - \nu \tau^2 + \nu_1 \lambda^2 \tau^2 + \nu_2 \tau^4 \\
 \log d &= \log \delta - \mu \tau^2 - \frac{1}{2} \mu \lambda^2 + (8)_1 \tau^2 \\
 \varphi' &= \varphi_1 - d \\
 \alpha' &= \alpha \pm 180^\circ + t - \varepsilon \\
 L' &= L + l.
 \end{aligned}$$

The coefficients have the following meaning:

$$\begin{aligned}
 (1) &= \frac{\varrho}{M} & (2)_1 &= \frac{\varrho}{N_1} & (3)_1 &= \frac{1}{2 \varrho} \frac{N_1}{M_1} \\
 (4) &= \frac{\text{Mod } 3 e^2 \sin 2 \varphi}{4 W^2 M} & (5) &= \frac{\text{Mod}}{3 M N} \\
 (6) &= -\frac{\text{Mod } e^2 \cos 2 \varphi}{2 a^2} & (7)_1 &= \mu e^2 (3 - \sin^2 \varphi_1) \\
 (8)_1 &= \frac{1}{2} \mu e^2 (13 - 10 \sin^2 \varphi_1) \\
 \mu &= \frac{\text{Mod}}{6 (1 - e^2) \varrho^2} & \nu &= \frac{\text{Mod}}{3 \varrho^2} & \nu_1 &= \frac{\text{Mod}}{15 \varrho^4} & \nu_2 &= \frac{7}{90} \frac{\text{Mod}}{\varrho^4} \\
 \log \mu &= 5.233\,6912 - 10 & \log \nu &= 5.531\,8128 - 10 \\
 \log \frac{1}{2 \varrho} &= 4.384\,5449 - 10 & \log \nu_1 &= 4.20399 - 20 \\
 & & \log \nu_2 &= 4.27094 - 20.
 \end{aligned}$$

In this compilation, we have denoted the logarithmic modulus by *Mod* in order to avoid mistaking it for Schreiber's coefficient μ .

For the practical execution of the computations according to the foregoing formulae, auxiliary tables have been published by the Prussian Land Survey:

Rechnungsvorschriften für die Trigonometrische Abteilung der Landesaufnahme. Formeln und Tafeln zur Berechnung der geographischen Koordinaten aus den Richtungen und Längen der Dreiecksseiten, Erste Ordnung, Berlin 1878.

These tables are adapted for eight-place logarithmic computation.

For the computation of the second order with seven-place logarithms, auxiliary tables have likewise been published under the same title. A few simplifications are made here in the formulae: In $\log b$, $\log t$ and $\log d$, the last term is omitted; in $\log l$ the last two terms are omitted; in addition, μ is replaced by $\frac{\nu}{2}$.

For the land survey in Finland, auxiliary tables which are likewise based on the formulae by Schreiber have been published by the Finnish Geodetic Institute. However, these tables which hold for the geographic latitudes from 59° to 71° are based on the terrestrial dimensions of the International Ellipsoid, which we have indicated in the first half-volume, section 37, p. 46. The tables are published under the title, *Veröffentlichungen des Finnischen Geodätischen Instituts*, Nr. 1, "Tafeln für geodätische Berechnungen nach den Erddimensionen von Hayford," by Y. Väisälä, Helsinki, 1923. The Baltic Geodetic Commission has proposed an extension of these tables to the south for the computation of the triangulations running around the Baltic Sea.

Section 20. Krüger's Formulae for the Transfer of Geographic Coordinates

We arrive at a simple numerical computation for the transfer of coordinates, if we solve the problem at first spherically and then transfer the results onto the ellipsoid. We already have treated thoroughly the spherical computation in the first half-volume, section 64, p. 181, and, therefore, need to occupy ourselves only with the difference of the spherical and spheroidal transfer of coordinates.

We imagine that on the ellipsoid a geodetic line, of length s , with azimuth α , starts from a point with the geographic latitude φ . For the end point we obtain then the values φ' , l and α' according to the formulae (25) to (27) of section 18, p. 78. Now we imagine the same computation carried out on a sphere with the radius N , taking the same starting values as a basis. Let the final values be denoted for the sphere by ψ' , λ and α' . We would obtain these final values if we omitted in the spheroidal formulae (25) to (27), section 18, p. 78, all terms which are multiplied by η^2 or by η^4 and, besides, replaced V^2 by unity. However, we will not make use of this, since the spherical computation turns out very conveniently according to the formulae (19), first half-volume, section 64, p. 183. On the other hand, we will use now the terms with η^2 and η^4 in (25) to (27), section 18, p. 78, for the transfer from the spherical to the spheroidal end values. Then we obtain at once the following equations for the differences of the spheroidal and the spherical end values:

$$\left. \begin{aligned} (\varphi' - \varphi) - V^2(\psi' - \varphi) &= -V^2\eta^2 \left(\frac{3}{2\varrho} u^2 t + \frac{v^2 u}{6\varrho^2} (1 - 9t^2) - \frac{u^3}{2\varrho^2} (1 - t^2) \right) \\ (l - \lambda) \cos \varphi &= \frac{v u^2}{3\varrho^2} \eta^2 \\ (\alpha' - \alpha) &= \eta^2 \left(\frac{v u}{2\varrho} - \frac{v^3 t}{6\varrho^2} + \frac{v u^2}{6\varrho^2} t \right), \end{aligned} \right\} \quad (1)$$

In the first equation (1) we can replace, on the left-hand side, V^2 by $1 + \eta^2$ and obtain then

$$(\varphi' - \psi') - \eta^2(\psi' - \varphi)$$

or, if we set the spherical difference of latitude $\psi' - \varphi = \Delta \psi$,

$$\varphi' - \psi' - \eta^2 \Delta \psi. \quad (2)$$

Before we introduce this, we will develop the expression $\cos^2 \left(\varphi + \frac{3}{4} \Delta \psi \right)$ in a series in order to achieve a simplification of the first equation (1). We have to this

$$\frac{d(\cos^2 \varphi)}{d\varphi} = -2 \cos^2 \varphi \tan \varphi \quad \frac{d^2(\cos^2 \varphi)}{d\varphi^2} = 2(\sin^2 \varphi - \cos^2 \varphi)$$

and with this, we will have

$$\cos^2 \left(\varphi + \frac{3}{4} \Delta \psi \right) = \cos^2 \varphi \left\{ 1 - \frac{3}{2\varrho} t \Delta \psi - \frac{9}{16\varrho^2} (1 - t^2) \Delta \psi^2 + \dots \right\} \quad (3)$$

or

$$e'^2 \cos^2 \left(\varphi + \frac{3}{4} \Delta \psi \right) \Delta \psi = \eta^2 \left\{ \Delta \psi - \frac{3}{2\varrho} t \Delta \psi^2 - \frac{9}{16\varrho^2} (1-t^2) \Delta \psi^3 + \dots \right\} \quad (4)$$

Now we take, in addition, from (27), first half-volume, section 63, p. 178,

$$\Delta \psi = u - \frac{1}{2\rho} v^2 t + \dots \quad (5)$$

If we use this in order to eliminate $\Delta \psi$ in the last two terms of (4), then we have

$$e'^2 \cos^2 \left(\varphi + \frac{3}{4} \Delta \psi \right) \Delta \psi = \eta^2 \left\{ \Delta \psi - \frac{3}{2\varrho} t u^2 + \frac{3}{2\varrho^2} t u v^2 - \frac{9}{16} (1-t^2) u^3 + \dots \right\} \quad (6)$$

Now we turn back to the first equation (1), in which we replace the left-hand side by the expression (2)

$$\varphi' - \psi' - \eta^2 \Delta \psi = -\eta^2 \left\{ \frac{3}{2\varrho} t u^2 + \frac{1}{6\varrho^2} (1-9t^2) u v^2 + \frac{1}{2\varrho^2} (1-t^2) u^3 + \frac{3}{2\varrho} t \eta^2 u^2 \right\}.$$

This yields together with equation (6):

$$\varphi' - \psi' = e'^2 \Delta \psi \cos^2 \left(\varphi + \frac{3}{4} \Delta \psi \right) - \frac{3}{2\varrho} t \eta^4 u^2 + \frac{1}{16\varrho^2} (1-t^2) \eta^2 u^3 - \frac{1}{6\varrho^2} \eta^2 u v^2. \quad (7)$$

The second and the third equation (1) do not need any further conversion, so that we have the following for the computation of l and α' from λ and a'

$$\left. \begin{aligned} (l - \lambda) \cos \varphi &= \frac{1}{3\varrho^2} \eta^2 v u^2 \\ \alpha' - a' &= \frac{1}{2\varrho} \eta^2 v u - \frac{1}{6\varrho^2} t \eta^2 v^3 + \frac{1}{6\varrho^2} t \eta^2 v u^2. \end{aligned} \right\} \quad (8)$$

With this, the problem of the transfer of the geographic coordinates is completely solved, and now we will summarize the total computational procedure with the pertinent formulae from the first half-volume, section 64, p. 183.

Given φ , α and s . Required φ' , α' and l .

$$\text{We set } \frac{\rho}{N} s = \sigma \quad \frac{\rho}{N} s \sin \alpha = v \quad \frac{\rho}{N} s \cos \alpha = u.$$

With this, we are to compute:

$$\log x = \log u + \tau_\sigma - \tau_x \quad \tau_\sigma = \frac{\mu}{3\rho^2} \sigma^2 \quad \tau_x = \frac{\mu}{3\rho^2} x^2$$

$$\log s = \log \frac{v}{2\rho} + \frac{1}{4} \tau_\sigma - \frac{1}{2} \tau_x \quad \log \frac{\mu}{3\rho^2} = 5.53180$$

$$\log \gamma = \log v \tan (\varphi + x) - \frac{1}{2} \tau_\sigma - \tau_\gamma$$

$$\log \lambda = \log \frac{v}{\cos (\varphi + x)} - \frac{1}{2} \tau_x - \tau_\gamma$$

$$\log \delta = \log \frac{v \gamma}{2\rho} + \frac{1}{4} \tau_\sigma + \frac{1}{4} \tau_\gamma - \frac{3}{4} \tau_x$$

$$\psi' = \varphi + x - \delta \quad a' = \alpha + \gamma - s$$

$$x - \delta = \Delta \psi$$

$$\varphi' = \psi' + e'^2 \Delta \psi \cos^2 \left(\varphi + \frac{3}{4} \Delta \psi \right) - \frac{3}{2\rho} t \eta^4 u^2 + \frac{1}{16\rho^2} (1 - t^2) \eta^2 u^3 - \frac{1}{6\rho^2} \eta^2 u v^2$$

$$l = \lambda + \frac{1}{3\rho^2} \frac{\eta^2}{\cos \varphi} v u^2$$

$$l = \lambda + \frac{e'^2}{3\rho^2} \cos \varphi v u^2$$

$$\alpha' = \alpha' + \frac{1}{2\rho} \eta^2 v u - \frac{1}{6\rho^2} t \eta^2 v^3 + \frac{1}{6\rho^2} t \eta^2 v u^2.$$

or

We will add, at the same time, the logarithms of coefficients for the numerical computation and have then the computing formulae

$$\begin{aligned} \varphi' &= \psi' + [7.827319] \Delta \psi \cos^2 \left(\varphi + \frac{3}{4} \Delta \psi \right) - [0.51630] \sin \varphi \cos^3 \varphi u^2 \\ &\quad + [5.9944] \cos^2 \varphi u^3 - [5.9944] \sin^2 \varphi u^3 - [6.4203] \cos^2 \varphi u v^2 \\ l &= \lambda + [6.72135] \cos \varphi v u^2 \\ \alpha' &= \alpha' + [2.21186] \cos^2 \varphi v u - [6.4203] \sin \varphi \cos \varphi v^3 + [6.4203] \sin \varphi \cos \varphi v u^2. \end{aligned}$$

The application of the foregoing formulae by Krüger proves especially convenient with the "Tafeln für die Berechnung der geodätischen Linie und der Additamente für den Übergang von \log auf $\log \sin$ und $\log \tan$," bearbeitet von A. G a l l e, Berlin, 1920, published by the Geodätisches Institut in Potsdam.

As a numerical example, we use our small spheroidal standard example (1), section 17, p. 73.

$$\text{Given} \quad \varphi = 49^\circ 30' 00'' \quad \alpha = 32^\circ 25' 21.5109'' \quad \log s = 5.1216103 \cdot 1.$$

To this, we take at once from the numerical computation from section 18, p. 79:

$$\begin{aligned} \log [2] s &= \log \sigma = 3.6305523 \cdot 4 \\ \log v &= 3.3598470 \cdot 8 \\ \log u &= 3.5569545 \cdot 3. \end{aligned}$$

With this, we obtain the following computation:

$\log u$	3.5569545·3	$\log v$	3.3598470·8
$+ \tau_\sigma$	+ 620·7	$\log x$	3.5569723·7
	3.5570166·0	$\log 1:2\rho$	4.3845448·7
$- \tau_x$	- 442·3	$+ \frac{1}{4} \tau_\sigma$	+ 155·2
$\log x$	3.5569723·7	$- \frac{1}{2} \tau_x$	- 221·2
$x = 3605.5570''$		$\log s$	1.3013577·2
$x = 1^\circ 00' 05.5570''$		$s = 20.0151''$	
$\varphi + x = 50^\circ 30' 05.5570''$			

$\log v$	3.359 8470.8
$\log \tan (\varphi+x)$	0.083 9193.7
$-\frac{1}{2} \tau \sigma$	— 310.4
<hr/>	
	3.443 7354.1
$-\tau \gamma$	— 262.6
$\log \gamma$	3.443 7091.5
$\gamma = 2777.8523''$	
$\gamma = 46' 17.8523''$	
$\log v$	3.359 8470.8
$\log \gamma$	3.443 7091.5
$\log 1:2 \varrho$	4.384 5448.7
$+\frac{1}{4} \tau \sigma$	+ 155.2
$+\frac{1}{4} \tau \gamma$	+ 65.6
$-\frac{3}{4} \tau x$	— 331.7
<hr/>	
$\log \delta$	1.188 0900.1
$\delta = 15.4202''$	
$\varphi + \frac{3}{4} \Delta \psi$	$= 50^{\circ} 14' 52.6026''$
$\psi' = 50^{\circ} 29' 50.1368''$	
$+\Delta \psi \dots$	+ 9.86430
$-u^2 \dots$	— 0.00089
$+u^3 \dots$	+ 0.00020
$-u^3 \dots$	— 0.00027
$-u v^2 \dots$	— 0.00021
<hr/>	
$\varphi' = 50^{\circ} 29' 59.9999''$	
$\lambda = 0^{\circ} 59' 59.9989''$	
$+v u^2 \dots$	+ 0.00102
<hr/>	
$l = 0^{\circ} 59' 59.9999''$	

$\log v$	3.359 8470.8
$\log \sec (\varphi+x)$	0.196 5036.7
$-\frac{1}{2} \tau x$	— 221.2
$-\tau \gamma$	— 262.6
<hr/>	
$\log \lambda$	3.556 3023.7
$\lambda = 3599.9989''$	
$\lambda = 59' 59.9989''$	
$\varphi = 49^{\circ} 30' 00''$	
$x = 1 \ 00 \ 05.5570$	
$-\delta =$	15.4202
<hr/>	
$\psi' = 50^{\circ} 29' 50.1368''$	
$\Delta \psi = 0^{\circ} 59' 50.1368''$	
$\alpha = 32^{\circ} 25' 21.5109''$	
$\gamma = 46 \ 17.8523$	
$-\varepsilon =$	— 20.0151
<hr/>	
$a' = 33^{\circ} 11' 19.3481''$	

$a' = 33^{\circ} 11' 19.3481''$	
$+v u \dots$	+ 0.05672
$-v^3 \dots$	— 0.00016
$+v u^2 \dots$	+ 0.00039
<hr/>	
$\alpha' = 33^{\circ} 11' 19.4050''$	
$\alpha = 32^{\circ} 25' 21.5109''$	
<hr/>	
$\alpha' - \alpha = 0^{\circ} 45' 57.8941''$	

These results agree sufficiently with those of section 17, p. 74.

The foregoing solution of the geodetic main problem is published by L. Krüger in "Neue Formeln zur Übertragung geographischer Koordinaten durch Hauptdreiecksseiten," *Zeitschr. f. Verm.*, 1919, pp. 281-294. For the practical use of the formulae, the tables already mentioned in the first half-volume, p. 184, and on the preceding p. 91 were published by A. Galle.

In conclusion we further mention: Ö. Burrau, "Einfache Methode zur sukzessiven Berechnung der geographischen Koordinaten der Dreieckspunkte," *Verh. d. 6. Tagung der Balt. Geod. Komm.* 1932, Helsinki, 1933, pp. 216-223.

Now we will develop Gauss' formulae of the mean latitude found in the first half-volume, section 61, for the transfer of geographic coordinates for the sphere, also for the ellipsoid with the help of the geodetic line.

In Fig. 1 the two points P_1 and P_2 with the latitudes φ_1 and φ_2 and the longitudes L_1 and L_2 are connected by a geodetic line of the length s , which has the azimuths α_1 and α_2 at the end points.

As auxiliary point, we use, in addition, the center point of the geodetic line whose distance from the two end points is thus equal to $\frac{s}{2}$. We denote

the latitude and longitude of the point by φ_m and L_m and the azimuth of the geodetic line by α_m . Starting from this point, we can develop L_1 , φ_1 and α_1 as well as L_2 , φ_2 and α_2 by means of Maclaurin's series according to the pattern of the series (8), (9) and (10) in section 18, p. 75. If we begin with the geographic length, then we have to set in (9), section 18, p. 75, instead of s first $-\frac{s}{2}$, and second $+\frac{s}{2}$ and obtain then

$$L_1 - L_m = -\left(\frac{dl}{ds}\right)_m \frac{s}{2} + \frac{1}{2} \left(\frac{d^2l}{ds^2}\right)_m \frac{s^2}{4} - \frac{1}{6} \left(\frac{d^3l}{ds^3}\right)_m \frac{s^3}{8} + \dots \quad (1)$$

$$L_2 - L_m = +\left(\frac{dl}{ds}\right)_m \frac{s}{2} + \frac{1}{2} \left(\frac{d^2l}{ds^2}\right)_m \frac{s^2}{4} + \frac{1}{6} \left(\frac{d^3l}{ds^3}\right)_m \frac{s^3}{8} + \dots \quad (2)$$

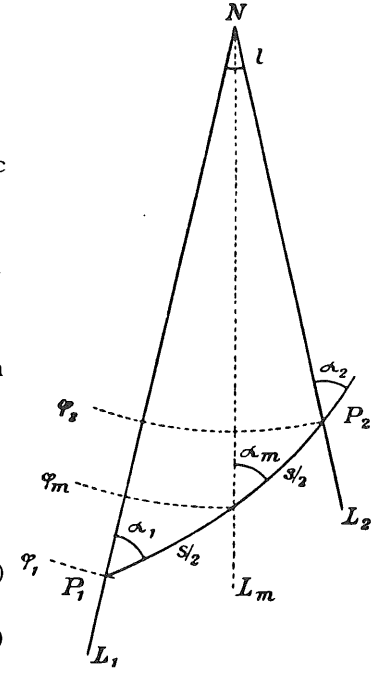


Fig. 1.

In this, the differential quotients are to be computed with the values φ_m and α_m , which are not as yet known to us, however.

From (1) and (2) there follows the difference

$$L_2 - L_1 = l = \left(\frac{dl}{ds}\right)_m s + \frac{1}{24} \left(\frac{d^3l}{ds^3}\right)_m s^3 + \dots, \quad (3)$$

in which only the terms in s^5 are neglected.

In addition to the values L_m , φ_m and α_m we will now introduce the mean values

$$L = \frac{L_1 + L_2}{2} \quad \varphi = \frac{\varphi_1 + \varphi_2}{2} \quad \alpha = \frac{\alpha_1 + \alpha_2}{2} \quad (4)$$

It is true that we also do not know as yet these mean values, but we shall see later that we can compute them with sufficient accuracy. We obtain at once a relation between L and L_m by addition of the two equations (1) and (2):

$$L - L_m = \frac{1}{8} \left(\frac{d^2l}{ds^2}\right)_m s^2 + \dots, \quad (5)$$

in which the terms in s^4 are neglected.

If we imagine the two equations (1) and (2) set up also for φ and α , then we obtain, in accordance with (5),

$$\varphi - \varphi_m = \frac{1}{8} \left(\frac{d^2\varphi}{ds^2}\right)_m s^2 + \dots \quad (6)$$

$$\alpha - \alpha_m = \frac{1}{8} \left(\frac{d^2\alpha}{ds^2}\right)_m s^2 + \dots \quad (7)$$

The differential quotients of L , φ and α with respect to s occurring in the equations set up thus far are functions of φ and α , and we can again set up a series according to Maclaurin's formula. With this, we obtain for $\frac{d l}{d s}$

$$\left(\frac{d l}{d s}\right)_m = \frac{d l}{d s} + \frac{\partial \frac{d l}{d s}}{\partial \varphi} (\varphi_m - \varphi) + \frac{\partial \frac{d l}{d s}}{\partial \alpha} (\alpha_m - \alpha) + \dots \quad (8)$$

According to (6) and (4), section 18, p. 75, we have

$$\frac{d l}{d s} = \frac{\sin \alpha}{N \cos \varphi} \quad \frac{d \frac{1}{N}}{d \varphi} = \frac{1}{c} \frac{d V}{d \varphi} = -\frac{\eta^2}{c V} t, \quad (9)$$

hence we have

$$\frac{\partial \frac{d l}{d s}}{\partial \varphi} = -\frac{\eta^2 t \sin \alpha}{c V \cos \varphi} + \frac{\sin \alpha t}{N \cos \varphi} = -\frac{\eta^2 t \sin \alpha}{N V^2 \cos \varphi} + \frac{V^2 t \sin \alpha}{N V^2 \cos \varphi},$$

and since $V^2 = 1 + \eta^2$, then we will have

$$\frac{\partial \frac{d l}{d s}}{\partial \varphi} = \frac{t \sin \alpha}{N V^2 \cos \varphi}. \quad (10)$$

On the other hand, we have from (9)

$$\frac{\partial \frac{d l}{d s}}{\partial \alpha} = \frac{\cos \alpha}{N \cos \varphi}. \quad (11)$$

In order to be able to compute also, $\varphi_m - \varphi$ and $\alpha_m - \alpha$ according to (5) and (6), we take from (15) and (20), section 18, p. 76, by introducing, at the same time, $\frac{1}{N} = \frac{V}{c}$,

$$\begin{aligned} \frac{d^2 \varphi}{d s^2} &= -\frac{V^2}{N^2} (\sin^2 \alpha t + 3 \cos^2 \alpha \eta^2 t) \\ \frac{d^2 \alpha}{d s^2} &= \frac{1}{N^2} \sin \alpha \cos \alpha (1 + 2 t^2 + \eta^2). \end{aligned}$$

Therefore, we have

$$\varphi_m - \varphi = \frac{V^2 s^2}{8 N^2} (\sin^2 \alpha t + 3 \cos^2 \alpha \eta^2 t) \quad (12)$$

$$\alpha_m - \alpha = -\frac{s^2}{8 N^2} \sin \alpha \cos \alpha (1 + 2 t^2 + \eta^2). \quad (13)$$

With the help of equations (9) to (13), we can compute $\left(\frac{d l}{d s}\right)_m$ according to (8). Then, in order to set up equation (3), we still lack the third differential quotient $\frac{d^3 l}{d s^3}$. According to (19), section 18, p. 76, we have

$$\frac{d^3 l}{ds^3} = \frac{2}{N^3 \cos \varphi} \left\{ \sin \alpha \cos^2 \alpha (1 + 3 t^2 + \eta^2) - \sin^3 \alpha t^2 \right\}. \quad (14)$$

This expression can be introduced at once also for $\left(\frac{d^3 l}{ds^3}\right)_m$, since higher terms have been neglected from the outset.

Now if we set all the foregoing into (3), then we obtain

$$\begin{aligned} l = & \frac{s \sin \alpha}{N \cos \varphi} + \frac{s^3}{8 N^3} \frac{\sin \alpha}{\cos \varphi} t^2 (\sin^2 \alpha + 3 \eta^2 \cos^2 \alpha) \\ & - \frac{s^3}{8 N^3} \frac{\sin \alpha}{\cos \varphi} \cos^2 \alpha (1 + 2 t^2 + \eta^2) \\ & + \frac{s^3}{12 N^3} \frac{\sin \alpha}{\cos \varphi} (\cos^2 \alpha (1 + 3 t^2 + \eta^2) - \sin^2 \alpha t^2). \end{aligned}$$

This can be contracted further, and if we further add ρ at the same time, then we find easily

$$l = \rho \frac{s \sin \alpha}{N \cos \varphi} \left\{ 1 + \frac{s^2}{24 N^2} (\sin^2 \alpha t^2 - \cos^2 \alpha (1 + \eta^2 - 9 \eta^2 t^2)) \right\}. \quad (15)$$

Now we have to adopt the same way also for the computation of $\varphi_2 - \varphi_1 = b$ and of $\alpha_2 - \alpha_1$, and therefore we can be brief in the light of the foregoing. After the two equations (1) and (2) for φ_1 and φ_2 are set up, we obtain

$$b = \left(\frac{d \varphi}{ds}\right)_m s + \frac{1}{24} \left(\frac{d^3 \varphi}{ds^3}\right)_m s^3 + \dots, \quad (16)$$

in which we have again

$$\begin{aligned} \left(\frac{d \varphi}{ds}\right)_m &= \frac{d \varphi}{ds} + \frac{\partial \frac{d \varphi}{ds}}{\partial \varphi} (\varphi_m - \varphi) + \frac{\partial \frac{d \varphi}{ds}}{\partial \alpha} (\alpha_m - \alpha) + \dots \\ \left(\frac{d^3 \varphi}{ds^3}\right)_m &= \frac{d^3 \varphi}{ds^3} + \dots \end{aligned}$$

From (5), section 18, p. 75, we find $\frac{d \varphi}{ds} = \frac{V^2}{N} \cos \alpha$; hence we have

$$\frac{\partial \frac{d \varphi}{ds}}{\partial \varphi} = -\frac{3 \eta^2}{N} t \cos \alpha \quad \frac{\partial \frac{d \varphi}{ds}}{\partial \alpha} = -\frac{V^2}{N} \sin \alpha.$$

We have further according to (17), section 18, p. 76,

$$\begin{aligned} \frac{d^3 \varphi}{ds^3} = & -\frac{V^2}{N^3} \cos \alpha \left\{ \sin^2 \alpha (1 + \eta^2 + 3 t^2 - 9 \eta^2 t^2) \right. \\ & \left. + \cos^2 \alpha (3 \eta^2 - 3 \eta^2 t^2 + 3 \eta^4 - 15 \eta^4 t^2) \right\}, \end{aligned}$$

and if we further add the values of $\varphi_m - \varphi$ and $\alpha_m - \alpha$ from (12) and (13), then we obtain according to (16)

$$\begin{aligned}
b = & \frac{V^2}{N} s \cos \alpha - \frac{3}{8} \frac{V^2}{N^3} s^3 \eta^2 t^2 \sin^2 \alpha \cos \alpha - \frac{9}{8} \frac{V^2}{N^3} s^3 \eta^4 t^2 \cos^3 \alpha \\
& + \frac{1}{8} \frac{V^2}{N^3} s^3 \sin^2 \alpha \cos \alpha (1 + 2 t^2 + \eta^2) \\
& - \frac{1}{24} \frac{V^2}{N^3} s^3 \cos \alpha \left\{ \sin^2 \alpha (1 + \eta^2 + 3 t^2 - 9 \eta^2 t^2) \right. \\
& \left. + \cos^2 \alpha (3 \eta^2 - 3 \eta^2 t^2 + 3 \eta^4 - 15 \eta^4 t^2) \right\}.
\end{aligned}$$

This collected, yields

$$b = e \frac{V^2}{N} s \cos \alpha \left\{ 1 + \frac{s^2}{24 N^2} (\sin^2 \alpha (2 + 3 t^2 + 2 \eta^2) + 3 \cos^2 \alpha \eta^2 (t^2 - 1 - \eta^2 - 4 \eta^2 t^2)) \right\}$$

or also

$$b = \frac{e}{M} s \cos \alpha \left\{ 1 + \frac{s^2}{24 N^2} (\sin^2 \alpha (2 + 3 t^2 + 2 \eta^2) + 3 \cos^2 \alpha \eta^2 (t^2 - 1 - \eta^2 - 4 \eta^2 t^2)) \right\}. \quad (17)$$

For the development of $\alpha_2 - \alpha_1$, we have according to the above

$$\alpha_2 - \alpha_1 = \left(\frac{d \alpha}{d s} \right)_m s + \frac{1}{24} \left(\frac{d^3 \alpha}{d s^3} \right)_m s^3 + \dots \quad (18)$$

To this, we need again

$$\begin{aligned}
\left(\frac{d \alpha}{d s} \right)_m &= \frac{d \alpha}{d s} + \frac{\partial \frac{d \alpha}{d s}}{\partial \varphi} (\varphi_m - \varphi) + \frac{\partial \frac{d \alpha}{d s}}{\partial \alpha} (\alpha_m - \alpha) + \dots \\
\left(\frac{d^3 \alpha}{d s^3} \right)_m &= \frac{d^3 \alpha}{d s^3} + \dots,
\end{aligned}$$

and since we have according to (7), section 18, p. 75,

$$\frac{d \alpha}{d s} = \frac{1}{N} \sin \alpha t,$$

then we will have

$$\frac{\partial \frac{d \alpha}{d s}}{\partial \varphi} = \frac{\sin \alpha}{N V^2} (1 + \eta^2 + t^2) \quad \frac{\partial \frac{d \alpha}{d s}}{\partial \alpha} = \frac{\cos \alpha}{N} t.$$

If we still take from (21), section 18, p. 76,

$$\frac{d^3 \alpha}{d s^3} = \frac{1}{N^3} \sin \alpha \cos^2 \alpha t (5 + \eta^2 - 4 \eta^4 + 6 t^2) - \frac{1}{N^3} \sin^3 \alpha t (1 + 2 t^2 + \eta^2),$$

then (18) yields

$$\begin{aligned}
\alpha_2 - \alpha_1 &= \frac{s}{N} \sin \alpha t + \frac{s^3}{8 N^3} \sin \alpha t (3 \eta^2 \cos^2 \alpha + \sin^2 \alpha) (1 + \eta^2 + t^2) \\
&\quad - \frac{s^3}{8 N^3} \sin \alpha \cos^2 \alpha t (1 + 2 t^2 + \eta^2) \\
&\quad - \frac{s^3}{24 N^3} \sin^3 \alpha t (1 + 2 t^2 + \eta^2) + \frac{s^3}{24 N^3} \sin \alpha \cos^2 \alpha t (5 + \eta^2 - 4 \eta^4 + 6 t^2)
\end{aligned}$$

$$\alpha_2 - \alpha_1 = \varrho \frac{s}{N} \sin \alpha t \left\{ 1 + \frac{s^2}{24 N^2} t (\cos^2 \alpha (2 + 7 \eta^2 + 9 \eta^2 t^2 + 5 \eta^4) + \sin^2 \alpha (2 + 2 \eta^2 + t^2)) \right\}. \quad (19)$$

With the three equations (15), (17) and (19) the problem is solved. Now, in addition, we will form a new equation for $\alpha_2 - \alpha_1$ by dividing (19) by (15). There follows hence at once

$$\alpha_2 - \alpha_1 = l \sin \varphi \left\{ 1 + \frac{s^2}{24 N^2} (\sin^2 \alpha (2 + 2 \eta^2) + \cos^2 \alpha (3 + 8 \eta^2 + 5 \eta^4)) \right\}. \quad (20)$$

In all these equations we have $\alpha = \frac{\alpha_1 + \alpha_2}{2}$ and $\varphi = \frac{\varphi_1 + \varphi_2}{2}$; also the quantities N , t and η^2 refer to this mean latitude.

For later use, we will compute, in addition, the difference of the longitude $L = \frac{L_1 + L_2}{2}$ and of the longitude L_0 of the point of the geodetic

line which has the mean latitude $\varphi = \frac{\varphi_1 + \varphi_2}{2}$ (cf. Fig. 2). If we avail ourselves again of the center point of the geodetic line according to Fig. 1, p. 93, then we have, on the one hand, according to (5), p. 93,

$$L = L_m + \frac{1}{8} \frac{d^2 l}{d s^2} s^2 + \dots$$

or according to (18), section 18, p. 76,

$$L_m = L - \frac{1}{4 N^2} \frac{t}{\cos \varphi} s^2 \sin \alpha \cos \alpha. \quad (21)$$

On the other hand, we have

$$L_0 = L_m + \frac{d l}{d \varphi} (\varphi - \varphi_m).$$

According to (5) and (6), section 18, p. 75, we will have

$$\frac{d l}{d \varphi} = \frac{M}{N \cos \varphi} t = \frac{t}{V^2 \cos \varphi},$$

and since we have according to (12)

$$\varphi - \varphi_m = -\frac{V^2 s^2}{8 N^2} (\sin^2 \alpha t + 3 \cos^2 \alpha \eta^2 t)$$

then we obtain

$$L_0 = L_m - \frac{s^2}{8 N^2} \frac{t \sin^3 \alpha}{\cos \varphi \cos \alpha} - \frac{3 s^2 t \sin \alpha \cos \alpha}{8 N^2 \cos \varphi}. \quad (22)$$

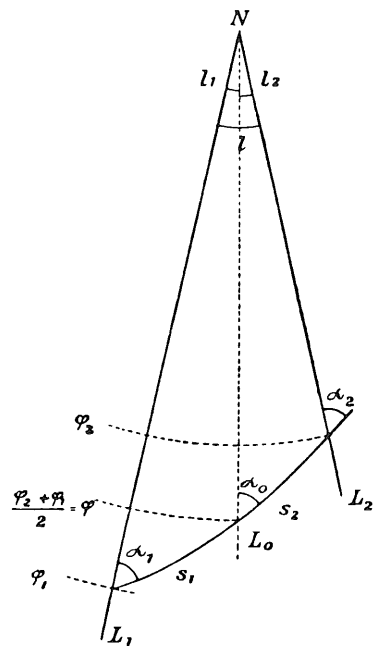


Fig. 2.

The two equations (21) and (22) yield the required result

$$L - L_0 = \frac{s^2}{8 N^2 \cos \varphi} \frac{\sin^3 \alpha}{\cos \alpha} t + \frac{s^2}{8 N^2 \cos \varphi} \sin \alpha \cos \alpha t (2 + 3 \eta^2). \quad (23)$$

With this we can also indicate the difference $l_2 - l_1$ according to Fig. 2, p. 97, for we will have

$$(l_2 - l_1) \cos \varphi = \frac{s^2}{4 N^2} \frac{\sin^3 \alpha}{\cos \alpha} t + \frac{s^2}{4 N^2} \sin \alpha \cos \alpha t (2 + 3 \eta^2). \quad (24)$$

In the later section 26 we shall make use of this equation.

Now we will return once more to our equations (15), (17) and (20), in whose higher terms we can set

$$\frac{s^2}{N^2} \sin^2 \alpha = l^2 \cos^2 \varphi \quad \frac{s^2}{N^2} \cos^2 \alpha = \frac{b^2}{V^4}. \quad (25)$$

With this, equations (15), (17) and (20) can be written thusly:

$$l \cos \varphi = \frac{\varrho}{N} s \sin \alpha \left(1 + \frac{l^2 \sin^2 \varphi}{24 \varrho^2} - \frac{b^2}{24 \varrho^2} \frac{(1 + \eta^2 - 9 \eta^2 t^2)}{V^4} \right) \quad (26)$$

$$b = \frac{\varrho}{M} s \cos \alpha \left(1 + \frac{l^2 \cos^2 \varphi}{24 \varrho^2} (2 + 3 t^2 + 2 \eta^2) + \frac{b^2}{8 \varrho^2} \eta^2 \frac{(t^2 - 1 - \eta^2 - 4 \eta^2 t^2)}{V^4} \right) \quad (27)$$

$$\alpha_2 - \alpha_1 = l \sin \varphi \left(1 + \frac{l^2 \cos^2 \varphi}{12 \varrho^2} V^2 + \frac{b^2}{24 \varrho^2} \frac{(3 + 8 \eta^2 + 5 \eta^4)}{V^4} \right). \quad (28)$$

Before we introduce simple denotations for the individual coefficients, we will further transform equation (27) somewhat. To do so, we write the second term of the large parentheses in the form

$$\frac{l^2 \cos^2 \varphi}{24 \varrho^2} (3 + 3 t^2 - 1 + 2 \eta^2) \quad \text{or} \quad \frac{l^2 \cos^2 \varphi}{8 \varrho^2} (1 + t^2) - \frac{l^2 \cos^2 \varphi}{24 \varrho^2} (1 - 2 \eta^2).$$

But this is

$$\frac{l^2}{8 \varrho^2} - \frac{l^2 \cos^2 \varphi}{24 \varrho^2} (1 - 2 \eta^2).$$

With this, equation (27) then becomes

$$b = \frac{\varrho}{M} s \cos \alpha \left(1 + \frac{l^2}{8 \varrho^2} - \frac{l^2 \cos^2 \varphi}{24 \varrho^2} (1 - 2 \eta^2) + \frac{b^2}{8 \varrho^2} \eta^2 \frac{(t^2 - 1 - \eta^2 - 4 \eta^2 t^2)}{V^4} \right)$$

and with the same accuracy

$$b = \frac{\varrho}{M} s \cos \alpha \left(1 + \frac{l^2}{8 \varrho^2} \right) \left(1 - \frac{l^2 \cos^2 \varphi}{24 \varrho^2} (1 - 2 \eta^2) + \frac{b^2}{8 \varrho^2} \eta^2 \frac{(t^2 - 1 - \eta^2 - 4 \eta^2 t^2)}{V^4} \right).$$

Since we have carried the developments in series everywhere only to the second order, then we can set

$$1 + \frac{l^2}{8 \varrho^2} = \frac{1}{\cos \frac{l}{2}}$$

and have then in the place of equation (27):

$$b \cos \frac{l}{2} = \frac{\rho}{M} s \cos \alpha \left(1 - \frac{l^2 \cos^2 \varphi}{24 \rho^2} (1 - 2 \eta^2) + \frac{b^2 \eta^2 (l^2 - 1 - \eta^2 - 4 \eta^2 l^2)}{8 \rho^2 V^4} \right). \quad (29)$$

Now we will give the coefficients of the formulae found a special notation. At first we take for the terms of first order the main coefficients, introduced several times already for other purposes,

$$\frac{\rho}{N} = [2] \quad \text{and} \quad \frac{\rho}{M} = [1].$$

We will determine further the following additional coefficients:

$$\left. \begin{aligned} [3] &= \frac{\mu}{24 \rho^2} & [4] &= \frac{\mu}{24 \rho^2} \frac{1 + \eta^2 - 9 \eta^2 l^2}{V^4} \\ [5] &= \frac{\mu}{24 \rho^2} (1 - 2 \eta^2) & [6] &= \frac{\mu}{8 \rho^2} \frac{\eta^2 (l^2 - 1 - \eta^2 - 4 \eta^2 l^2)}{V^4} \\ [7] &= \frac{\mu}{12 \rho^2} V^2 & [8] &= \frac{\mu}{24 \rho^2} \frac{3 + 8 \eta^2 + 5 \eta^4}{V^4} \end{aligned} \right\} \quad (30)$$

or also to the same accuracy

$$[8] = \frac{\mu}{24 \rho^2} \frac{3 + 5 \eta^2}{V^2}.$$

μ means here the logarithmic modulus for units of the seventh place $\log \mu = 6.637\,7843$, and for this, we can thus calculate immediately:

$$\log \frac{\mu}{8 \rho^2} = 5.105\,8441, \quad \log \frac{\mu}{12 \rho^2} = 4.929\,7528, \quad \log \frac{\mu}{24 \rho^2} = 4.628\,7228. \quad (31)$$

With these abbreviations, the formulae (26), (28) and (29) are represented thusly:

$$L_2 - L_1 = l = [2] \frac{s \sin \alpha}{\cos \varphi} \left(1 + \frac{[3]}{\mu} l^2 \sin^2 \varphi - \frac{[4]}{\mu} b^2 \right) \quad (32)$$

$$\varphi_2 - \varphi_1 = b = [1] \frac{s \cos \alpha}{\cos \frac{l}{2}} \left(1 - \frac{[5]}{\mu} l^2 \cos^2 \varphi + \frac{[6]}{\mu} b^2 \right) \quad (33)$$

$$\alpha_2 - \alpha_1 = \Delta \alpha = l \sin \varphi \left(1 + \frac{[7]}{\mu} l^2 \cos^2 \varphi + \frac{[8]}{\mu} b^2 \right) \quad (34)$$

and

$$\Delta \alpha = [2] s \sin \alpha \tan \varphi \left(1 + \frac{[7]}{\mu} l^2 \cos^2 \varphi + \frac{[8]}{\mu} b^2 + \frac{[3]}{\mu} l^2 \sin^2 \varphi - \frac{[4]}{\mu} b^2 \right). \quad (35)$$

To this, we have also the inversions:

$$s \sin \alpha = \frac{l \cos \varphi}{[2]} \left(1 - \frac{[3]}{\mu} l^2 \sin^2 \varphi + \frac{[4]}{\mu} b^2 \right) \quad (36)$$

$$s \cos \alpha = \frac{b \cos \frac{l}{2}}{[1]} \left(1 + \frac{[5]}{\mu} l^2 \cos^2 \varphi - \frac{[6]}{\mu} b^2 \right) \quad (37)$$

$$\Delta \alpha = l \sin \varphi \left(1 + \frac{[7]}{\mu} l^2 \cos^2 \varphi + \frac{[8]}{\mu} b^2 \right). \quad (38)$$

There follow hence the formulae for use, as are indicated in the numerical example on pp. 102 and 103.

If φ_1 , α_1 and s are given and φ_2 , α_2 and l required, then we cannot apply the formulae of the mean latitude directly, but indirectly by the introduction of approximate values, as is already indicated, with Gauss' own words, at the end of the first half-volume, section 61, p. 172, in the case of the spherical formulae of the mean latitude. As for the approximate values needed, we can take the longitudes and latitudes already from the picture of the triangulation net and, with this, also the meridian convergences $= l \sin \varphi$ to the same accuracy; but we will assume that we have computed the whole net preliminarily according to the formulae of section 18, p. 78, abbreviated to terms of third order, which has about the same significance as the preliminary computation of a triangulation for purposes of centerings, spherical excesses and the like. In short, we will assume that we have the latitudes, longitudes and azimuths to an accuracy of approximately 0.1", and then one or, at the most, two computations according to p. 103 are sufficient in order to make all agree to within 0.0001".

At any rate, we can take immediately from the auxiliary tables of pp. [34] to [39] of the first half-volume and pp. [8] to [10] of this half-volume all logarithms of coefficients $\log [1]$, $\log [2]$, and so on, with the preliminary mean latitude φ to a sufficient accuracy and, with this, carry out the computation of p. 103.

The final values still turn out with errors within 0.003" on p. 103, which must be eliminated completely by a repeated computation.

According to this, it might seem that the procedure is detailed and troublesome, but this is not the case, for the repetition is extended only to the three logarithms $\log \sin \alpha$, $\log \cos \alpha$, $\log \tan \varphi$; all the rest, especially the corrections of second order, remain unchanged. Not until latitude and azimuth agree is the longitude taken up also.

The example on p. 103 shows that with each computation we go further by about two places, and moreover the example is an extensive one, with $\Delta \varphi$ and $l = 1^\circ$ and $s = 132$ km; in practice, the sides are much shorter and then the approximate convergence proceeds also much faster.

The development of spheroidal formulae of the mean latitude and their inversion form the contents of "Untersuchungen über Gegenstände der höheren Geodäsie" by Gauss, zweite Abhandlung, Göttingen, 1846. Gauss very successfully applied here to geodesy the exceedingly useful fundamental theorem of the mean argument in the case of developments in series and gave two independent proofs, first by the conformal projection of surfaces, and second by the immediate development in series according to powers of the geodetic line.

But the table of coefficients appended by Gauss extends only from 51° to 54° of latitude. We gave an extension of this table from 45° to 55° in the former editions of this book, and a table of Gauss' logarithms of coefficients $\log (1)$, $\log (2)$, . . . , $\log (6)$ for the whole extent from $\varphi = 34^\circ$ to $\varphi = 70^\circ$ was computed by Biek, and published in the Russian translation of Jordan, *Handbuch der Vermessungskunde*, pp. 652-665 (cf. the more exact quotation in the first half-volume, p. 86).

What is given in the foregoing rests on Gauss' idea, but is brought into another form according to the development and representation of coefficients, because it seemed to us that Gauss' form of the correction terms with *three* elements s , β and τ (τ = meridian convergence) without λ is not favorable in some respects.

Consideration of terms of fourth order in the formulae of the mean latitude

The foregoing formulae (36) to (38) or, as the case may be, the practical logarithmic formulae on p. 102 are sufficient only for lengths as they occur in measurable triangles, say, up to 150 km, if we wish to have exact the eighth decimal place in the logarithms of the distances and a thousandth of a second in the azimuths.

We obtain a considerably higher accuracy if we take in the development the terms of fifth order, so that in equations (32) to (38) also the terms of fourth order still occur. The development of these terms becomes quite troublesome; however, since we can neglect in them all which is multiplied by η^2 , we can take them immediately from the former spherical computation of the first half-volume, section 62, p. 175. The

supplementary terms to the three formulae for use on p. 102 for $\log s \sin \alpha$, $\log s \cos \alpha$ and $\log \Delta \alpha$ are then with simplified notation of the coefficients

$$\left. \begin{aligned} \Delta (\log s \sin \alpha) &= I b^4 - II b^2 l^2 - III l^4 \\ \Delta (\log s \cos \alpha) &= -IV b^2 l^2 - V l^4 \\ \Delta (\log \Delta \alpha) &= VI b^2 - VII b^2 l^2 + VIII l^4 \end{aligned} \right\} \quad (39)$$

The coefficients have here the following meaning:

$$\left. \begin{aligned} I &= \frac{\mu}{2880 \varrho^4} & II &= \frac{\mu}{1440 \varrho^4} (4 + 15 t^2) \cos^2 \varphi & III &= \frac{\mu}{2880 \varrho^4} (12 t^2 + t^4) \cos^4 \varphi \\ IV &= \frac{\mu}{1440 \varrho^4} (4 + 15 t^2) \cos^2 \varphi & V &= \frac{\mu}{2880 \varrho^4} (14 + 40 t^2 + 15 t^4) \cos^4 \varphi \\ VI &= \frac{\mu}{192 \varrho^4} & VII &= \frac{\mu}{48 \varrho^4} \sin^2 \varphi & VIII &= \frac{\mu}{1440 \varrho^4} (7 - 6 t^2) \cos^4 \varphi \end{aligned} \right\} \quad (40)$$

In addition, we will group together the constants occurring here, and in fact for b and l in seconds and for units in the 7th place of logarithms.

$$\left. \begin{aligned} \log \frac{\mu}{48 \varrho^4} &= 3.698\,843 - 20 & \log \frac{\mu}{192 \varrho^4} &= 3.096\,783 - 20 \\ \log \frac{\mu}{1440 \varrho^4} &= 2.221\,721 - 20 & \log \frac{\mu}{2880 \varrho^4} &= 1.920\,691 - 20 \end{aligned} \right\} \quad (41)$$

What amounts the terms of fourth order can reach can be determined from the tables of the first half-volume, pp. 175 and 176.

The serviceability of the formulae of the mean latitude is increased quite considerably by adding the terms of fourth order. We are to bear in mind here that in reality these are already the terms of fifth order, and that due to the peculiarity of the formulae of the mean latitude the terms of sixth order are likewise already taken into account, just as the simpler equations (32) to (38), pp. 99 and 100, do not only reach to the third, but actually to the fourth order. Consequently, with the help of the supplementary terms we still obtain, in the case of distances of 400 km, the logarithms of $s \sin \alpha$ and $s \cos \alpha$ to an accuracy of 10 places, whereas in $\Delta \alpha$ the 5th decimal place in the seconds is still correct.

New formulae of the mean latitude likewise with supplementary terms of fourth order are developed by L. Krüger in the treatise, "Die kürzeste Entfernung und ihre Azimute zwischen zwei gegebenen Punkten des Erdellipsoids," *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, Mathem.-Physikal. Klasse 1918*. The formulae (12) in connection with (12*) of that place agree exactly with our practical formulae on p. 102 with the supplementary terms (39). Krüger's formulae differ in form from ours only by the fact that in them a part of the terms of fourth order is already contained in the principal terms.

Adapting to our notation, Krüger's formulae read:

$$\left. \begin{aligned} \log \Delta \alpha &= \log \gamma + [1] l^2 + [4] \beta^2 \\ \log s \sin \alpha &= \log \lambda + [2] l^2 - [5] \beta^2 \\ \log s \cos \alpha &= \log \beta + [3] l^2 + [6] \beta^2 \end{aligned} \right\} \quad (42)$$

$$\left. \begin{aligned} \gamma &= l \sin \varphi \sec \frac{b}{2} \\ \beta &= \frac{M}{\varrho} b \cos \frac{l}{2} \\ \lambda &= \frac{N}{\varrho} l \cos \varphi \left(\sec \frac{b}{2} \cos \frac{l}{2} \right)^{1/3} \end{aligned} \right\} \quad (43)$$

Spheroidal formulae of the mean latitude

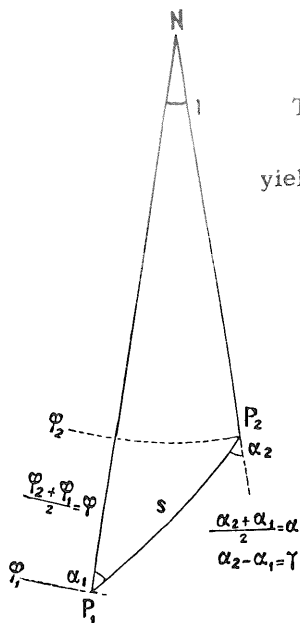


Fig. 3.

Given $\varphi_1 = 49^\circ 30'$ $\varphi_2 = 50^\circ 30'$ $l = 1^\circ 0'$
 $\varphi = 50^\circ 0' 0''$ $b = 1^\circ = 3600''$ $l = 1^\circ = 3600''$.

There are required s , α_1 and α_2 .

The auxiliary tables in the Appendix of the first half-volume yield with $\varphi = 50^\circ 0'$:

Page [36]: $\log [1] = 8.510\ 1335.3$ $\log [2] = 8.508\ 9295.0$.

Page [10] of this half-volume yield with $\varphi = 50^\circ 0'$:

$\log [3] = 4.6287$ $\log [4] = 4.6119$ $\log [5] = 4.6263$ $\log [6] = 2.151$
 $\log [7] = 4.9310$ $\log [8] = 5.1066$.

Formulae for use

$$\log s \sin \alpha = \log \frac{l \cos \varphi}{[2]} - [3] l^2 \sin^2 \varphi + [4] b^2$$

$$\log s \cos \alpha = \log \frac{b \cos \frac{l}{2}}{[1]} + [5] l^2 \cos^2 \varphi - [6] b^2$$

$$\log \Delta \alpha = \log l \sin \varphi + [7] l^2 \cos^2 \varphi + [8] b^2.$$

Longitude		Latitude		Azimuth	
$\log l$	3.556 3025.0	$\log b$	3.556 3025.0	$\log l$	3.556 3025.0
$\log \cos \varphi$	9.808 0675.0	$\log \cos \frac{l}{2}$	9.999 9834.6	$\log \sin \varphi$	9.884 2539.7
$\log l \cos \varphi$	3.364 3700.0	$\log b \cos \frac{l}{2}$	3.556 2859.6	$\log l \sin \varphi$	3.440 5564.7
$\log [2]$	8.508 9295.0	$\log [1]$	8.510 1335.3		
$\log \frac{l \cos \varphi}{[2]}$	4.855 4405.0	$b \cos \frac{l}{2}$	5.046 1524.3	$\log l^2 \sin^2 \varphi$	6.8811
$\log l^2 \cos^2 \varphi$	6.7287	$\log \frac{b \cos \frac{l}{2}}{[1]}$	5.046 1524.3	$l^2 \cos^2 \varphi$	6.7287
$l^2 \sin^2 \varphi$	6.8811			b^2	7.1126
$- [3]$	4.6287	$\log b^2$	7.1126	$+ [7]$	4.9310
$+ [4]$	4.6119	$l^2 \cos^2 \varphi$	6.7287	$+ [8]$	5.1066
	1.5098	$+ [5]$	4.6263		1.6597
	32.35	$+ 22.65$	1.3550		45.68
	53.02	$- 0.18$	9.2636		165.67
	20.67	$+ 22.47$			211.35
	4.855 4405.0				3.440 5564.7
$s \sin \alpha$	4.855 4425.7	$s \cos \alpha$	5.046 1546.8		3.440 5776.0
$s \cos \alpha$	5.046 1546.8	$\sin \alpha$	9.733 8322.5	$\Delta \alpha = 2757.8942''$	
$\tan \alpha$	9.809 2878.9	$\cos \alpha$	9.924 5443.6	$\Delta \alpha = 0^\circ 45' 57.8942''$	
$\alpha = 32^\circ 48' 20.460''$		$\log s$	5.121 6103.2	$\frac{\Delta \alpha}{2} = 0\ 22\ 58.9471$	
$\frac{\Delta \alpha}{2} = 0\ 22\ 58.947$		s	132,315.38 m		
$\alpha_2 = 33^\circ 11' 19.407''$					
$\alpha_1 = 32\ 25\ 21.513$					

Spheroidal formulae of the mean latitude
with indirect solution

Given $\log s = 5.121\ 6103.2$	$\varphi_1 = 49^\circ\ 30'\ 0.0000''$	$\alpha_1 = 32^\circ\ 25'\ 21.513''$
Approximate $l = 1^\circ\ 0'\ 0.1''$	$\varphi_2 = 50^\circ\ 30'\ 0.1''$	$\alpha_2 = 33^\circ\ 11'\ 19.5''$
$= 3600.1''$	$\varphi = 50^\circ\ 0'\ 0.05''$	$\alpha = 32^\circ\ 48'\ 20.5065''$.

With $\varphi = 50^\circ\ 0'\ 0.0''$ the auxiliary tables in the Appendix of the first half-volume yield

Page [36]: $\log [1] = 8.510\ 1335.3$ $\log [2] = 8.508\ 9295.0$,

and the auxiliary table on p. [10] of this half-volume yields:

$\log [3] = 4.6287$ $\log [4] = 4.6119$ | $\log [5] = 4.6263$ $\log [6] = 2.151$ | $\log [7] = 4.9310$ $\log [8] = 5.1066$.

Formulae for use

$$\log l = \log \left(\frac{[2] s \sin \alpha}{\cos \varphi} \right) + [3] l^2 \sin^2 \varphi - [4] b^2$$

$$\log b = \log \left(\frac{[1] s \cos \alpha}{\cos \frac{l}{2}} \right) - [5] l^2 \cos^2 \varphi + [6] b^2$$

$$\log \Delta \alpha = \log \left([2] s \sin \alpha \tan \varphi \right) + [7] l^2 \cos^2 \varphi + [8] b^2 + [3] l^2 \sin^2 \varphi - [4] b^2.$$

Longitude			Latitude			Azimuth		
$\log s$	5.121 6103.2		$\log s$	5.121 6103.2		$\log s$	5.121 6103.2	
$\log [2]$	8.508 9295.0		$\log [1]$	8.510 1335.3		$\log [2]$	8.508 9295.0	
$\log \sin \alpha$	9.733 8324.1		$\log \cos \alpha$	9.924 5443.0		$\log \sin \alpha$	9.733 8324.1	
$[2] s \sin \alpha$	3.364 3722.3		$[1] s \cos \alpha$	3.556 2881.5		$\log \tan \varphi$	0.076 1866.8	
$\log \cos \varphi$	9.808 0673.7					$\log (\Delta \alpha)$	3.440 5589.0	
$\log (l)$	3.556 3048.6		$\log \cos \frac{l}{2}$	9.999 9834.6		$= \log l \sin \varphi$		
$\log ([2] s \sin \alpha)^2 = \log l^2 \cos^2 \varphi = 6.7287$			$\log (b)$	3.556 3046.9		$\log l^2 \sin^2 \varphi = 6.8811$		
			$\log b^2$	7.1126				
$l^2 \sin^2 \varphi$	6.8811	b^2 7.1126	$l^2 \cos^2 \varphi$	6.7287	b^2 7.1126	$l^2 \cos^2 \varphi$	6.7287	b^2 7.1126
$+ [3]$	4.6287	$- [4]$ 4.6119 _n	$- [5]$	4.6263 _n	$- [6]$ 2.151	$+ [7]$	4.9310	$+ [8]$ 5.1066
	1.5098	1.7245 _n		1.3550 _n	9.264		1.6597	2.2192
$+ 32.35$		$- 53.02$	$- 22.65$		$+ 0.18$	$+ 45.68$		$+ 165.67$
		$- 20.67$			$- 22.47$	$+ 211.35$		$\} + 190.68$
						$- 20.67$		
$\log (l)$	3.556 3048.6		$\log (b)$	3.556 3046.9		$\log (\Delta \alpha)$	3.440 5589.0	
	$- 20.7$			$- 22.5$			$+ 190.7$	
$\log l$	3.556 3027.9		$\log b$	3.556 3024.4		$\log \Delta \alpha$	3.440 5779.7	
$l = 3600.0022''$			$b = 3599.9994''$			$\Delta \alpha = 2757.897''$		
should be 0.0000			should be 0.0000			should be .894		

$$\left. \begin{aligned} [1] &= \frac{\mu}{12} \frac{1}{MN} & [2] &= \frac{\mu}{24} \frac{1}{N^2} & [3] &= \frac{\mu}{24} \frac{1-2\eta^2}{N^2} & [4] &= \frac{\mu}{12} \frac{\eta^2}{MN} \\ [5] &= \frac{\mu}{24} \frac{\eta^2}{N^2} (1+\eta^2+9t^2) & [6] &= \frac{\mu}{8} \frac{\eta^2}{N^2} (1+\eta^2-t^2+4\eta^2 t^2) \\ [1] + [3] &= 3[2] & \varphi &= \frac{\varphi_1 + \varphi_2}{2} & \alpha &= \frac{\alpha_1 + \alpha_2}{2} \end{aligned} \right\} \quad (44)$$

The supplementary terms are then added to these:

$$\left. \begin{aligned} \text{in } \log A \alpha: & \quad -\frac{\mu}{1440} \left(\frac{s}{N}\right)^4 \left\{ (10+30t^2) \sin^2 \alpha - (27+34t^2) \sin^4 \alpha \right\} \\ \text{, } \log s \sin \alpha: & \quad -\frac{\mu}{2880} \left(\frac{s}{N}\right)^4 \left\{ 4 + (10+30t^2) \sin^2 \alpha - (29+38t^2+4t^4) \sin^4 \alpha \right\} \\ \text{, } \log s \cos \alpha: & \quad -\frac{\mu}{2880} \left(\frac{s}{N}\right)^4 \left\{ (18+30t^2) \sin^2 \alpha - (29+30t^2) \sin^4 \alpha \right\}. \end{aligned} \right\} \quad (45)$$

Detailed auxiliary tables are computed for Krüger's formulae by A. Galle in *Veröffentlichung des Preuss. Geodätischen Instituts*, Neue Folge Nr. 83, "Tafeln für die Berechnung der geodätischen Linie und der Additamente für den Übergang von \log auf $\log \sin$ und $\log \tan$," Berlin, 1920.

In the above-mentioned treatise by Krüger a further solution of the problem is developed, in which also the terms of fourth order are still spheroidal, whereas in our supplementary terms (39), p. 101, everything multiplied by η^2 is neglected. These formulae are therefore applicable to still considerably greater distances and are not inferior to the formulae of the later section 26.

Another transformation of the spheroidal formulae of the mean latitude is given by R. A. Hirvonen in *Verhandlungen der in Kaunas im Juni 1938 abgehaltenen 10. Tagung der Baltischen Geodätischen Kommission*. The international ellipsoid is taken as a basis here for the numerical application.

Section 22. The Reduced Latitude

We obtain a new treatment of the geodetic line if we introduce the *reduced latitude*.

In Fig. 1 a circle with the equatorial radius is described around the meridian ellipse, and the ordinate of an arbitrary point is extended to the intersection with the circle. Then there results the angle ψ , the reduced latitude, entered in Fig. 1, whereas φ is as always the geographic latitude. If we denote the abscissa and the ordinate of the point by x and y , then we have according to Fig. 1

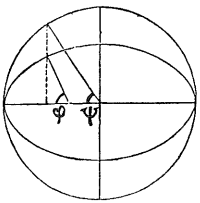


Fig. 1.

and since the ellipse equation

$$\frac{x}{a} = \cos \psi, \quad (1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

holds for the point, we have

$$\frac{y}{b} = \sin \psi. \quad (2)$$

On the other hand, we have according to the first half-volume, section 38, p. 49, equations (14) and (15)

$$\frac{x}{a} = \frac{\cos \varphi}{W} \quad \frac{y}{b} = \frac{\sin \varphi}{W} \sqrt{1-e^2}, \quad (3)$$

and hence, we have

$$\cos \psi = \frac{\cos \varphi}{W}, \quad \sin \psi = \frac{\sin \varphi}{W} \sqrt{1 - e^2}. \quad (4)$$

We have a second occasion for the introduction of the reduced latitude in the equation of the geodetic line, which we have found in equation (7), section 7, p. 29, namely

$$p \sin \alpha = k. \quad (5)$$

In here the radius of the parallel p is identical with the abscissa x ; we have therefore with the first equation of (3) and (4)

$$p = \frac{a \cos \varphi}{W} = a \cos \psi. \quad (6)$$

If we have two points of the geodetic line with the reduced latitudes ψ and ψ' as well as with the azimuths α and α' , then we have according to (5) and (6)

$$\cos \psi \sin \alpha = \cos \psi' \sin \alpha' = k. \quad (7)$$

We shall make use of this important equation later for computations on the terrestrial ellipsoid.

Now we will set up further relations between the reduced and the geographic latitude for later use.

Introducing in equations (4) $V = \frac{W}{\sqrt{1 - e^2}}$ according to the first half-volume, section 40, p. 56, then

we have

$$\cos \psi = \frac{\cos \varphi}{V \sqrt{1 - e^2}} \quad \sin \psi = \frac{\sin \varphi}{V} \quad (8)$$

$$\tan \psi = \tan \varphi \sqrt{1 - e^2}. \quad (9)$$

In order to obtain also formulae for the computation of φ from ψ , we express at first the auxiliary quantity V by ψ . We have

$$V^2 = 1 + e'^2 \cos^2 \varphi = 1 + e'^2 - e'^2 \sin^2 \varphi$$

and according to (8)

$$V^2 = 1 + e'^2 - e'^2 V^2 \sin^2 \psi,$$

and there follows hence

$$V^2 = \frac{1 + e'^2}{1 + e'^2 \sin^2 \psi} \quad \text{and} \quad V^2 = \frac{1}{1 - e'^2 \cos^2 \varphi}. \quad (10)$$

Now we obtain from (8)

$$\cos \varphi = \frac{\cos \psi \sqrt{1 - e^2} \sqrt{1 + e'^2}}{\sqrt{1 + e'^2 \sin^2 \psi}} = \frac{\cos \psi}{\sqrt{1 + e'^2 \sin^2 \psi}} \quad (11)$$

and

$$\sin \varphi = V \sin \psi = \frac{\sqrt{1 + e'^2} \sin \psi}{\sqrt{1 + e'^2 \sin^2 \psi}} \quad (12)$$

therefore also

$$\tan \varphi = \sqrt{1 + e'^2} \tan \psi. \quad (13)$$

Now we assemble once more the relations found between φ and ψ :

$$\left. \begin{aligned} \sin \varphi &= \frac{\sqrt{1 - e^2}}{W} \sin \varphi = \frac{1}{V} \sin \varphi \\ \cos \varphi &= \frac{1}{W} \cos \varphi = \frac{1}{V\sqrt{1 - e^2}} \cos \varphi \\ \tan \psi &= \sqrt{1 - e^2} \tan \varphi \end{aligned} \right\} \quad (14)$$

$$\left. \begin{aligned} \sin \varphi &= \frac{\sqrt{1 + e'^2}}{\sqrt{1 + e'^2 \sin^2 \psi}} \sin \psi \\ \cos \varphi &= \frac{1}{\sqrt{1 + e'^2 \sin^2 \psi}} \cos \psi \\ \tan \varphi &= \sqrt{1 + e'^2} \tan \psi. \end{aligned} \right\} \quad (15)$$

In addition, we also need the differential relation between φ and ψ , which results in the simplest way from (9); namely:

$$\frac{d\psi}{\cos^2 \psi} = \frac{d\varphi \sqrt{1 - e^2}}{\cos^2 \varphi},$$

and hence, because of (8):

$$\frac{d\varphi}{d\psi} = V^2 \sqrt{1 - e^2}. \quad (16)$$

The formulae (10), which yields V^2 or, as the case may be, V as a function of ψ , is important for later use.

Numerical computation of $\varphi - \psi$

In order to compute for a given φ the pertinent ψ or conversely, we can first use equation (9):

$$\tan \psi = \tan \varphi \sqrt{1 - e^2} \quad (\log \sqrt{1 - e^2} = 9.998\,5458 \cdot 202). \quad (17)$$

But if we desire special numerical accuracy, then it will be better to aim directly at the difference $\varphi - \psi$, and for this, we have according to (14):

$$\sin \varphi = V \sin \psi \quad \sin \psi = \frac{\sin \varphi}{V} \quad (18)$$

$$\cos \varphi = V \sqrt{1 - e^2} \cos \psi \quad \cos \psi = \frac{\cos \varphi}{V \sqrt{1 - e^2}}. \quad (19)$$

Now we have $\sin(\varphi - \psi) = \sin \varphi \cos \psi - \cos \varphi \sin \psi$; we can apply this in a twofold manner to (18) and (19), whereby we find:

$$\sin(\varphi - \psi) = \sin 2\varphi \frac{1 - \sqrt{1 - e^2}}{2 V \sqrt{1 - e^2}} \quad (20)$$

$$\sin(\varphi - \psi) = \sin 2\psi \frac{V}{2} (1 - \sqrt{1 - e^2}). \quad (21)$$

V is used here either, as usual, as a function of φ , or, according to (10), as a function of ψ . For the application of (20) and (21) we have from the first half-volume, p. 42, equation (7), and from p. 45

$$\log(1 - \sqrt{1 - e^2}) = \log a = 7.524\ 1069.093. \quad (22)$$

By using, in addition, for the passage from $\log \sin(\varphi - \psi)$ to $\log(\varphi - \psi)$ the formula for $\log \sin x$ (first half-volume, p. 24), we obtain from (20) and (22):

$$\log(\varphi - \psi) = \log \frac{\sin 2\varphi}{V} + 2.538\ 9562.266 + [5.23078](\varphi - \psi)^2, \quad (23)$$

where [5.23078] is the logarithm of the coefficient to the seventh place of logarithms.

According to this, we can compute with extreme accuracy, e.g.

$$\text{Given Berlin} \quad \varphi = 52^\circ 30' 16.7'' \quad 2\varphi = 105^\circ 0' 33.4''.$$

With this, the auxiliary table in the first half-volume on p. [5] of the Appendix yields $\log V = 0.000\ 5399.278$, and if we compute, moreover, according to the foregoing formula (23), then we obtain:

$\log \sin 2\varphi$	2.538 9562.266
$\log 1: V$	9.984 9249.285
	9.999 4600.722
	2.523 3412.273
adding to this the last term of (23):	+ 1.894
$\log(\varphi - \psi)$	2.523 3414.167
$\varphi - \psi =$	333.68864''
$\varphi - \psi = 5' 33.68864''$	}
$\varphi = 52^\circ 30' 16.70000''$	
$\psi = 52^\circ 24' 43.01136''$	

(24)

In general, one will of course not compute with so many decimals, but we have carried here this rigorous computation in order to use it at the same time as a check for the following method of approximation.

We obtain the best form for the numerical computation of $\varphi - \psi$ from given φ 's or ψ 's by a development in series according to the fundamental theorem of the mean argument (first half-volume, section 35, p. 29). According to (9) we have:

$$\tan \psi = \sqrt{1 - e^2} \tan \varphi = \left(1 - \frac{e^2}{2} - \frac{e^4}{8}\right) \tan \varphi,$$

or

$$\tan \varphi - \tan \psi = \left(\frac{e^2}{2} + \frac{e^4}{8}\right) \tan \varphi.$$

On the other hand, we have according to the first half-volume, p. 29, neglecting the terms of third order:

$$\tan \varphi - \tan \psi = \frac{\varphi - \psi}{\cos^2 \mu}, \quad \text{where} \quad \mu = \frac{\varphi + \psi}{2}.$$

With this, we will have

$$\varphi - \psi = (\tan \varphi - \tan \psi) \cos^2 \mu = \left(\frac{e^2}{2} + \frac{e^4}{8} \right) \tan \varphi \cos^2 \mu. \quad (25)$$

But we have

$$\varphi = \frac{\varphi + \psi}{2} + \frac{\varphi - \psi}{2} = \mu + \frac{\varphi - \psi}{2} = \mu + \frac{e^2}{4} \tan \varphi \cos^2 \mu,$$

where e^4 is neglected. Now we will have

$$\begin{aligned} \tan \varphi &= \tan \mu + \frac{1}{\cos^2 \mu} \frac{\varphi - \psi}{2} \\ \tan \varphi &= \tan \mu + \frac{1}{\cos^2 \mu} \frac{e^2}{4} \tan \varphi \cos^2 \mu = \tan \mu + \frac{e^2}{4} \tan \varphi. \end{aligned}$$

Therefore, we have

$$\tan \mu = \left(1 - \frac{e^2}{4} \right) \tan \varphi$$

or with the same accuracy

$$\tan \varphi = \left(1 + \frac{e^2}{4} \right) \tan \mu.$$

With this, we will have according to (25)

$$\varphi - \psi = \left(\frac{e^2}{2} + \frac{e^4}{8} \right) \left(1 + \frac{e^2}{4} \right) \sin \mu \cos \mu = \left(\frac{e^2}{2} + \frac{e^4}{8} + \frac{e^4}{8} \right) \frac{1}{2} \sin 2\mu$$

or

$$\varphi - \psi = \left(\frac{e^2}{4} + \frac{e^4}{8} \right) \sin(\varphi + \psi). \quad (26)$$

We can carry this further to one more degree, which is not indicated here in detail, by which we obtain (with the addition of ρ):

$$\varphi - \psi = \left(\frac{e^2}{4} + \frac{e^4}{8} + \frac{5}{64} e^6 \right) \varrho \sin(\varphi - \psi) + \frac{1}{384} e^6 \varrho \sin^3(\varphi + \psi). \quad (27)$$

With Bessel's eccentricity $\log e^2 = 7.824\,4104\cdot237$ the calculation yields:

$$\begin{aligned} \varphi - \psi &= 345.325\,3808 \sin(\varphi + \psi) + 0.000\,160 \sin^3(\varphi + \psi) \\ (\log &= 2.538\,2285\cdot0) \quad (\log = 6.2033). \end{aligned} \quad (28)$$

The second term, which is at the most $0.00016''$, is ordinarily neglected. In order to apply the formula (28) conveniently, we must first have an approximate value of $\varphi - \psi$, and this is furnished by our auxiliary table on page [11] of the Appendix; an example may illustrate the application:

Given Berlin

To this, according to p. [11]: $\varphi - \psi = 53.65$ approximate

$$\begin{array}{r|l} \psi = 52^\circ 24' 43.05'' & \\ \varphi + \psi = 104 & 54 \ 59.75 \\ \log \sin(\varphi + \psi) & 9.985 \ 1126.8 \\ \log 345.3 \dots & 2.538 \ 2285.0 \\ \hline \log 345.3 \dots \sin(\varphi + \psi) & 2.523 \ 3411.8 \\ 345.3 \dots \sin(\varphi + \psi) = & 333.68847''. \end{array}$$

To this, the second term of (25): $+0.00014''$

$$\begin{array}{r|l} (\varphi - \psi) = 333.68861'' & \\ = 5' 33.68861'' & \\ \text{Originally given } \varphi = 52^\circ 30' 16.70000'' & \end{array}$$

And hence: $\psi = 52^\circ 24' 43.01139''$.

(29)

This agrees sufficiently with the value (24) calculated more rigorously.

The question of how accurately we must have the approximate value $\varphi + \psi$ in order to attain a certain final accuracy can be answered by differentiating (28); we find that an error of $1''$ in the approximate value produces only an error of about $0.001''$, and for this reason an accuracy of $0.1''$ in the approximate value (as the auxiliary table on page [11] affords) is sufficient for the final calculation.

For the spheroidal standard examples, which we have put ahead in (1) to (5), section 17, pp. 73 and 74, the geographic latitudes φ and the corresponding reduced latitudes ψ are the following:

$\varphi = 45^\circ \ 0' \ 0''$	$\psi = 44^\circ \ 54' \ 14.67493''$		Mecklenburg	
49 30 0	49 24 18.83709		$\varphi = 53^\circ \ 0'$	$\psi = 52^\circ \ 54' \ 27.89895''$
50 0 0	49 54 19.82230		54 30	54 24 33.31059
50 30 0	50 24 20.91117			
55 0 0	54 54 35.31462			
Tübingen	$\varphi = 48^\circ \ 31' \ 12.4000''$		$\psi = 48^\circ \ 25' \ 29.6082''$	
Hornisgrinde	$\varphi = 48 \ 36 \ 21.8966$		$\psi = 48 \ 30 \ 30.2280$	
Berlin	$\varphi = 52 \ 30 \ 16.7''$		$\psi = 52 \ 24 \ 43.0014$	
Königsberg	$\varphi = 54 \ 42 \ 50.6$		$\psi = 54 \ 37 \ 24.7564$	

Section 23. The Spherical Auxiliary Triangle with Reduced Latitudes

We tie to the equation found in the foregoing section 22, (9), p. 105:

$$\cos \psi \sin \alpha = \cos \psi' \sin \alpha'. \quad (1)$$

The following Fig. 2 corresponds to this equation.

In the following Fig. 1 P and P' are two points on the ellipsoid, s the connecting geodetic line with the azimuths α and α' . The two points P and P' have the geographic latitudes φ and φ' and the difference of longitude l .

In Fig. 2 a corresponding spherical triangle TQQ' is drawn, whose arc QQ' has the same azimuths α and α' as the geodetic line PP' . The arc QQ' is denoted by $a\sigma$, since the spherical radius is assumed $= a$ (equatorial radius of the ellipsoid) and the central angle $= \sigma$. The difference of longitude between Q and Q' is $= \lambda$, which differs from l . Also the spherical latitudes ψ and ψ' are different from the ellipsoidal ones; the reduced latitudes pertaining to φ and φ' are the following, i.e., the following relations exist according to (14) and (15), section 22, p. 106:

$$\tan \psi = \tan \varphi \sqrt{1 - e^2} \quad \tan \psi' = \tan \varphi' \sqrt{1 - e^2}. \quad (2)$$

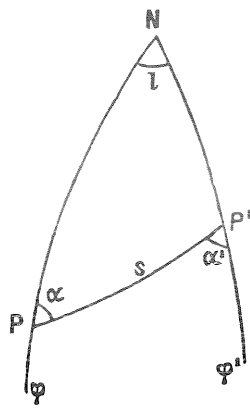


Fig. 1. Ellipsoid.

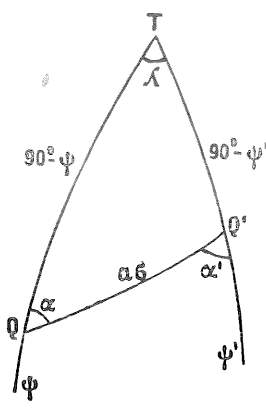


Fig. 2. Sphere.

The correctness of all these relations is proved by the spherical equation (1), and we know now that a spherical triangle $TQ Q'$ always corresponds to a geodetic polar triangle NPP' on the ellipsoid, with the same azimuths α, α' and with reduced latitudes ψ, ψ' , which pertain to φ, φ' . Against this, the remaining elements, the distance between the two points and their difference of longitude, are different in the two triangles.

Now the point in question is to establish a relation between s and σ and a relation between l and λ , for since relations already exist between all remaining elements of Fig. 1 and Fig. 2 according to equations (1) and (2), we shall be able to pass then over in all parts from the spheroidal triangle to the spherical triangle and conversely.

Now we use the differential equations (1) and (2), section 18, p. 75, and the corresponding equations for the sphere and have then

Ellipsoid	Sphere	
$ds \cos \alpha = M d\varphi$	$a d\sigma \cos \alpha = a d\psi$	(3)
$ds \sin \alpha = N \cos \varphi dl$	$a d\sigma \sin \alpha = a \cos \psi d\lambda$	(4)

Hence by division:

$$\frac{ds}{a d\sigma} = \frac{M d\varphi}{a d\psi} \quad \frac{dl}{d\lambda} = \frac{M \cos \varphi d\varphi}{N \cos \psi d\psi}.$$

According to (8) and (16), section 22, pp. 105 and 106, we have here:

$$\frac{\cos \varphi}{\cos \psi} = V \sqrt{1 - e^2} \quad \text{and} \quad \frac{d\varphi}{d\psi} = V^2 \sqrt{1 - e^2}. \quad (5)$$

From the first half-volume, section 38, pp. 48-51, we have $\frac{M}{a} = \frac{1 - e^2}{V^3} = \frac{1}{V^3 \sqrt{1 - e^2}}$ and $\frac{M}{N} = \frac{1}{V^2}$, with which the two equations (5) assume the following simple form:

$$ds = a d\sigma \frac{1}{V} \quad (6)$$

$$dl = d\lambda \frac{1}{V}. \quad (7)$$

The quantity V , occurring here twice, is the function of latitude always used by us, which, expressed either in φ or in ψ is according to (10), section 22, p. 105:

$$V = \sqrt{1 + e'^2 \cos^2 \varphi} \quad \text{or} \quad \frac{1}{V} = \sqrt{1 - e^2 \cos^2 \psi}. \quad (8)$$

We have already indicated the geometric meaning of V in the first half-volume, section 38, p. 51, for V^2 is the ratio of the two main radii of curvature N and M .

Section 24. Integration of the Differential Equations of the Polar Triangle

From the foregoing section 23, (6) and (8), we have the differential equation:

$$ds = a d\sigma \sqrt{1 - e^2 \cos^2 \psi'}. \quad (1)$$

This equation refers to Figs. 1 and 2 indicated below, since ds is the differential of the geodetic line s in Fig. 1 and $a d\sigma$ the differential of the spherical arc σ (referred to the radius a) of Fig. 2; also ψ' is the spherical latitude of a point on the arc σ .

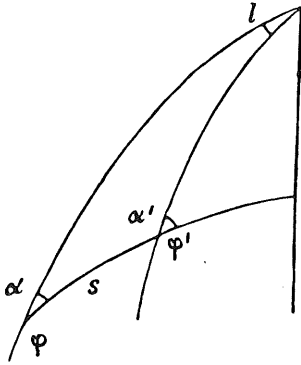


Fig. 1. Ellipsoid.

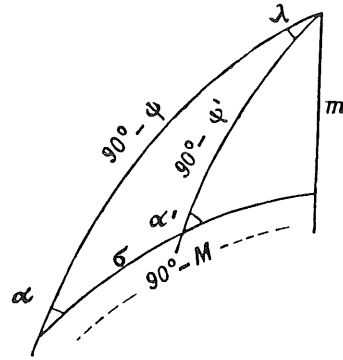


Fig. 2. Sphere.

In order to be able to integrate equation (1) with respect to σ , we must express ψ' in terms of σ , for which the formulae of spherical trigonometry, which we have already indicated in the first half-volume, section 59, pp. 162 and 163, equations (20) to (22), are used, namely transferring to our new Fig. 2:

$$\sin m = \cos \psi \sin \alpha, \quad \tan M = \frac{\sin \psi}{\cos \psi \cos \alpha} \quad (2)$$

$$\cos m = \frac{\sin \psi}{\sin M} = \frac{\cos \psi \cos \alpha}{\cos M} \quad (3)$$

$$\sin \psi' = \cos m \sin (M + \sigma). \quad (4)$$

Now we set for abbreviation in the following:

$$M + \sigma = x, \quad \text{where } M \text{ is constant; therefore } d\sigma = dx, \quad (5)$$

and hence: $\sin^2 \psi' = \cos^2 m \sin^2 x$ and $\cos^2 \psi' = 1 - \cos^2 m \sin^2 x. \quad (6)$

Set into (1) this yields:

$$ds = a \sqrt{1 - e^2 + e^2 \cos^2 m \sin^2 x} d\sigma$$

$$\frac{e^2}{1 - e^2} = e'^2, \text{ hence } ds = a \sqrt{1 - e^2} \sqrt{1 + e'^2 \cos^2 m \sin^2 x} d\sigma. \quad (7)$$

$$a \sqrt{1 - e^2} = b, \quad e' \cos m = k, \quad ds = b \sqrt{1 + k^2 \sin^2 x} dx. \quad (8)$$

Now we develop according to the usual series of the first half-volume, pp. 20 and 28:

$$\begin{aligned} \sqrt{1 + k^2 \sin^2 x} &= 1 + \frac{1}{2} k^2 \sin^2 x - \frac{1}{8} k^4 \sin^4 x \\ \sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x, \quad \sin^4 x = \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x. \end{aligned} \quad (9)$$

This together yields:

$$\sqrt{1 + k^2 \sin^2 x} = \left(1 + \frac{1}{4} k^2 - \frac{3}{64} k^4\right) + \left(-\frac{1}{4} k^2 + \frac{1}{16} k^4\right) \cos 2x - \frac{k^4}{64} \cos 4x. \quad (10)$$

For integration we have:

$$\begin{aligned} \int \cos 2x \, dx &= \frac{1}{2} \sin 2x, & \int \cos 4x \, dx &= \frac{1}{4} \sin 4x \\ \int_M^{M+\sigma} \cos 2x \, dx &= \frac{1}{2} \left(\sin (2M + 2\sigma) - \sin 2M \right) = \sin \sigma \cos (2M + \sigma) \end{aligned} \quad (11)$$

$$\int_M^{M+\sigma} \cos 4x \, dx = \frac{1}{4} \left(\sin (4M + 4\sigma) - \sin 4M \right) = \frac{1}{2} \sin 2\sigma \cos (4M + 2\sigma). \quad (12)$$

With this, we can assemble the integrals of (9), i.e. also the integration of (8), whereby we obtain an expression of this form:

$$s = A b \sigma - B b \sin \sigma \cos (2M + \sigma) - C b \sin 2\sigma \cos (4M + 2\sigma). \quad (13)$$

The coefficients A , B and C have the following meanings here:

$$A = \left(1 + \frac{1}{4} k^2 - \frac{3}{64} k^4\right), \quad B = \left(\frac{1}{4} k^2 - \frac{1}{16} k^4\right), \quad C = \left(\frac{1}{128} k^4\right). \quad (14)$$

The inversion of (13) yields:

$$\sigma = \alpha \frac{s}{b} + \beta \sin \sigma \cos (2M + \sigma) + \gamma \sin 2\sigma \cos (4M + 2\sigma), \quad (15)$$

where the newly introduced coefficients are (with the addition of the necessary ρ 's):

$$\alpha = \frac{1}{A} \varrho, \quad \beta = \frac{B}{A} \varrho, \quad \gamma = \frac{C}{A} \varrho. \quad (16)$$

In the same manner, we also treat the differential equation for the difference of longitude, namely according to (7) and (8), section 23, pp. 110 and 111:

$$dl = \sqrt{1 - e^2 \cos^2 \psi} \, d\lambda. \quad (17)$$

We develop here:

$$\sqrt{1 - e^2 \cos^2 \psi} = 1 - \frac{e^2}{2} \cos^2 \psi - \frac{e^4}{8} \cos^4 \psi - \frac{e^6}{16} \cos^6 \psi. \quad (18)$$

Here we have spherical equations, according to the first half-volume, section 60, p. 165, equation (1):

$$d\lambda \cos \psi = d\sigma \sin \alpha,$$

then according to equations (2) pertaining to Fig. 2, p. 111:

$$\sin \alpha = \frac{\sin m}{\cos \psi}, \quad \text{and hence} \quad d\lambda = \frac{\sin m}{\cos^2 \psi} d\sigma. \quad (19)$$

With this, we can transform (17) into an integration with respect to σ , namely taking into account (18):

$$l = \lambda - e^2 \sin m \int \left(\frac{1}{2} + \frac{e^2}{8} \cos^2 \psi + \frac{e^4}{16} \cos^4 \psi + \dots \right) d\sigma. \quad (20)$$

Now we have again according to (6):

$$\begin{aligned} \cos^2 \psi &= 1 - \cos^2 m \sin^2 x \\ \cos^4 \psi &= 1 - 2 \cos^2 m \sin^2 x + \cos^4 m \sin^4 x. \end{aligned}$$

Besides, we have expressed $\sin^2 x$ and $\sin^4 x$ in terms of $\cos 2x$ and $\cos 4x$ by (9), and all this together reduces function (20), which is integrated, to a series which progresses toward $\cos 2x$, $\cos 4x$, etc., i.e. (20) will be:

$$l = \lambda - e^2 \sin m \int (A' + B' \cos 2x + C' \cos 4x + \dots) dx. \quad (21)$$

The coefficients have the following meanings here:

$$\left. \begin{aligned} A' &= \frac{1}{2} + \frac{e^2}{8} + \frac{e^4}{16} - \frac{e^2}{16} \cos^2 m - \frac{e^4}{16} \cos^2 m + \frac{3}{128} e^4 \cos^4 m \\ B' &= \frac{e^2}{16} \cos^2 m + \frac{e^4}{16} \cos^2 m - \frac{e^4}{32} \cos^4 m \\ C' &= \frac{e^4}{128} \cos^4 m. \end{aligned} \right\} \quad (22)$$

If we imagine these coefficients set into (21), integrated, and the limits introduced just as before in the case of (11) and (12), then we see easily at a glance that the following is obtained:

$$l = \lambda - e^2 \sin m \left(A' \sigma + B' \sin \sigma \cos (2M + \sigma) + \frac{C'}{2} \sin 2\sigma \cos (4M + 2\sigma) + \dots \right). \quad (23)$$

Here we set, in addition, at B' and C' the factor ρ ; by doing this, and also including e^2 in the parentheses, we form from (23) this last form:

$$l = \lambda - \sin m (\alpha' \sigma + \beta' \sin \sigma \cos (2M + \sigma) + \gamma' \sin 2\sigma \cos (4M + 2\sigma)). \quad (24)$$

Here we have:

$$\alpha' = A' e^2, \quad \beta' = B' e^2 \rho, \quad \gamma' = \frac{C' e^2}{2} \rho. \quad (25)$$

In the foregoing development we have retained only so many terms as we can comprehend conveniently at a glance, and as many as are usually needed.

For a sure judgment about the influence of the higher terms it is necessary to make the further development of the foregoing series. We insert here only the final results of the development in series.

Series (15) receives one more term, and is then:

$$\sigma = \alpha \frac{s}{b} + \beta \sin \sigma \cos (2M + \sigma) + \gamma \sin 2\sigma \cos (4M + 2\sigma) + \delta \sin 3\sigma \cos (6M + 3\sigma). \quad (26)$$

Following are the pertinent coefficients with $k = e' \cos m$:

$$\left. \begin{aligned} \alpha &= \frac{1}{A} e, & A &= \left(1 + \frac{k^2}{4} - \frac{3}{64} k^4 + \frac{5}{256} k^6 - \frac{175}{16384} k^8 \right) \\ \beta &= \frac{B}{A} e, & B &= \left(\frac{k^2}{4} - \frac{k^4}{16} + \frac{15}{512} k^6 - \frac{35}{2048} k^8 \right) \\ \gamma &= \frac{C}{A} e, & C &= \left(\frac{k^4}{128} - \frac{3}{512} k^6 + \frac{35}{8192} k^8 \right) \\ \delta &= \frac{D}{A} e, & D &= \left(\frac{k^6}{1536} - \frac{5}{6144} k^8 \right) \\ \varepsilon &= \frac{E}{A} e, & E &= \left(\frac{5}{65536} k^8 \right). \end{aligned} \right\} \quad (27)$$

Also series (24) receives a further term and becomes:

$$l = \lambda - \sin m (\alpha' \sigma + \beta' \sin \sigma \cos (2M + \sigma) + \gamma' \sin 2\sigma \cos (4M + 2\sigma) + \delta' \sin 3\sigma \cos (6M + 3\sigma)). \quad (28)$$

Following are the pertinent coefficients:

$$\left. \begin{aligned} \alpha' &= \frac{e^2}{2} \left(1 + \frac{e^2}{4} + \frac{e^4}{8} + \frac{5e^6}{64} \right) - \frac{e^4 \cos^2 m}{16} \left(1 + e^2 + \frac{15}{16} e^4 \right) \\ &\quad + \frac{3}{128} e^6 \cos^4 m \left(1 + \frac{15}{8} e^2 \right) - \frac{25}{2048} e^8 \cos^6 m \\ \beta' &= e \left(\frac{e^4}{16} \cos^2 m \left(1 + e^2 + \frac{15}{16} e^4 \right) - \frac{e^6}{32} \cos^4 m \left(1 + \frac{15}{8} e^2 \right) + \frac{75}{4096} e^8 \cos^6 m \right) \\ \gamma' &= e \left(\frac{e^6}{256} \cos^4 m \left(1 + \frac{15}{8} e^2 \right) - \frac{15}{4096} e^8 \cos^6 m \right) \\ \delta' &= e \left(\frac{5}{12288} e^8 \cos^6 m \right). \end{aligned} \right\} \quad (28a)$$

If we calculate here all constant parts with Bessel's eccentricity e ($\log e^2 = 7.824\,4104\,237$ according to the first half-volume, p. 44), then we obtain:

$$\left. \begin{aligned} \alpha' &= 0.003\,342\,773\,183 - [4.447\,6079] \cos^2 m + [1.84854] \cos^4 m - [9.3843] \cos^6 m \\ \beta' &= [9.762\,0330] \cos^2 m - [7.28791] \cos^4 m + [4.87477] \cos^6 m \\ \gamma' &= [6.38482] \cos^4 m - [4.17580] \cos^6 m \\ \delta' &= [3.22156] \cos^6 m. \end{aligned} \right\} \quad (29)$$

These series exceed by far the usual need. In the case of geodetic lines of an extent of several degrees, we mostly need from (29) only α' and β' and, besides, only the first two terms of α' and the last term of β' .

We usually need a little more in the case of series (26) with the coefficients (27), but mostly also only α , β and γ only, say, to within k^4 . A calculation to, say, 8 places of logarithms is assumed here. With the coefficients (27) and (29) we can calculate even the largest cases to 10 places.

Section 25. Bessel's Formulae for the Transfer of Geographic Coordinates

By means of equations (26) and (28) of the foregoing section 24 with the pertinent coefficients α , β , γ , α' , β' , γ' , etc., the required relations between Fig. 1 and Fig. 2 are established and with this we can solve the polar triangle in the following manner:

From a point of the ellipsoid with the geographic latitude φ there starts a geodetic line s with the azimuth α ; we shall determine the latitude φ' of the end point of this geodetic line as well as the azimuth α' , and the difference of longitude l of the two points.

From the given latitude φ we compute the pertinent reduced latitude ψ according to the equation $\tan \psi = \sqrt{1 - e^2} \tan \varphi$ (or according to another method indicated in section 22). With this ψ and the azimuth α we can determine the two auxiliary quantities m and M in the spherical right triangle in Fig. 2 and solve with this equation (15) or (26), section 24, pp. 112 and 114, with respect to σ .

We thus have three elements ψ , α , σ , with which the oblique-angled spherical triangle of Fig. 2 can be solved, so that we find the spherical latitude ψ' from the other side and the spherical difference of longitude λ .

From the spherical (reduced) latitude ψ' we go back to the actual latitude φ' by the equation $\tan \varphi' = \tan \psi' \sqrt{1 + e'^2}$ (or by another method indicated in section 22), and from the spherical longitude λ we arrive at the spheroidal longitude l by equation (24) or (28), section 24, pp. 113 and 114, with which the solution of the whole problem is complete.

As a numerical example for this we will take according to (5), section 17, p. 74:

$$\text{Berlin } \varphi = 52^\circ 30' 16.7000'' \quad (1)$$

$$\text{Berlin-Königsberg } \alpha = 59 \quad 33 \quad 0.6892 \quad \log s = 5.724 \quad 2591.353. \quad (2)$$

We already have treated the calculation of the reduced latitude of Berlin in (27), section 22, p. 108, and found:

$$\text{Berlin } \psi = 52^\circ 24' 43.0114''. \quad (3)$$

Now comes the computation of m and M according to equations (2) and (3), section 24, p. 111:

$$m = 31^\circ 43' 31.13'' \quad M = 68^\circ 41' 19.95''. \quad (4)$$

We need further the coefficients for the computation of σ , and, in fact, first $k = e' \cos m$ according to (8), section 24, p. 111; we have

$$\log e' \cos m = \log k = 8.843 \quad 3740,$$

and, with this, sufficiently accurate according to (14) and (16), section 24, p. 112, without the further development (27), p. 114:

$$\begin{array}{lll} \log A = 0.000 \quad 5270.0 & \log B = 7.084 \quad 1599.2 & \log C = 3.266 \quad 286 \\ \log \alpha = 5.313 \quad 8981.0 & \log \beta = 2.398 \quad 0580.5 & \log \gamma = 8.580 \quad 184. \end{array}$$

With these coefficients α , β , γ we can solve equation (15), section 24, p. 112, with respect to σ , however not directly, because σ itself occurs on the right-hand side; but the series (15), section 24, p. 112, converges very rapidly, so that it is sufficient to compute a first approximate value of σ only from the first

term of (15), section 24, p. 112, i.e. to set $\sigma = \frac{\alpha s}{b}$ with which we can calculate also the following terms; or

briefly, we solve equation (15), section 24, p. 112, by approximation indirectly, step by step with respect to σ . This method yielded in our case:

$$\begin{array}{ll} \text{first approximation} & \alpha \frac{s}{b} = \sigma = 4^\circ 46' 17.8'' \\ \text{adding to this} & \beta \sin \sigma \cos (2M + \sigma) = - \quad 16.4 \\ \text{second approximation} & \sigma = 4^\circ 46' 14''. \end{array}$$

With this, we can calculate the second and third term of (15), section 24, p. 112, and have then altogether:

$$\begin{array}{ll} \alpha \frac{s}{b} = 4^\circ 46' 17.8176'' \\ \beta \sin \sigma \cos (2M + \sigma) = - \quad 16.4086 \\ \gamma \sin 2\sigma \cos (4M + 2\sigma) = + \quad 0.0015 \\ \text{finally} & \sigma = 4^\circ 46' 14.105''. \end{array} \quad (5)$$

Now we assemble from (3), (2), (5):

$$\psi = 52^\circ 24' 43.0114'' \quad \alpha = 59^\circ 33' 0.6892'' \quad \sigma = 4^\circ 46' 14.105''. \quad (6)$$

With this, we can solve the spherical triangle, which yields ψ' , α' and λ ; the calculation according to the formulae (14) and (15) in the first half-volume, section 59, p. 161 (in the same manner as the numerical example of that place), has yielded:

$$\psi' = 54^\circ 37' 24.7566'' \quad \alpha' = 65^\circ 16' 9.3655'' \quad (7)$$

$$\lambda = 7^\circ 6' 30.1340''. \quad (8)$$

The spherical value ψ' thus found is the reduced latitude of Königsberg, from which we calculate the actual latitude according to section 22, namely:

$$\varphi' = 54^\circ 42' 50.6002''. \quad (9)$$

Now, in addition, we have the problem of converting the spherical difference of longitude λ of (8) into the spheroidal difference of longitude l , for which equation (28) with the coefficients (29), section 24, p. 114, is used. We calculate according to (29), section 24, p. 114, however only with the terms to within $\cos^4 m$:

$$\log \alpha' = 7.523 \ 8439 \quad \log \beta' = 9.62045 \quad \log \gamma' = 6.098.$$

Therefore, (24), section 24, p. 113:

$$l = \lambda - 30.1479'' + 0.0144'' + 0.0000 \dots = \lambda - 30.1335'',$$

and hence, according to (8):

$$l = 7^\circ 6' 30.1340'' - 30.1335'' = 7^\circ 6' 0.0005''. \quad (10)$$

Now we have the accurate solution in (9), (7), (10):

$$\left. \begin{array}{l} \text{Königsberg} \quad \varphi' = 54^\circ 42' 50.6002'' \\ \text{Königsberg-Berlin} \quad \alpha' = 65 \ 16 \ 9.3655, \quad l = 7^\circ 6' 0.0005''. \end{array} \right\} \quad (11)$$

With the extended formulae (26) to (29), section 24, p. 114, we will calculate, in addition, the large standard example (2), section 17, p. 74, for which the principal numbers are the following:

$$\text{Given } \varphi = 45^\circ 0' 0'' \quad \alpha = 29^\circ 3' 15.4598'' \quad (12)$$

$$\log s = 6.120\ 6674\ 805. \quad (13)$$

The calculation starts with the reduced latitude at $\varphi = 45^\circ$:

$$\psi = 44^\circ 54' 14.67493''. \quad (14)$$

The right spherical auxiliary triangle yields:

$$m = 20^\circ 7' 8.712'' \quad M = 48^\circ 44' 46.551''. \quad (15)$$

The coefficients for the computation of σ become according to (27), section 24, p. 114:

$$\log \alpha = 5.313\ 7831\ 066, \quad \log \beta = 2.483\ 7124, \quad \log \gamma = 8.749\ 94, \quad \log \delta = 5.445,$$

and with this, σ itself to four terms:

$$\begin{aligned} \sigma &= 42,782.021\ 652'' - 20.794\ 012'' - 0.017\ 667'' + 0.000\ 012'' \\ \sigma &= 11^\circ 52' 41.20998''. \end{aligned} \quad (16)$$

With ψ , α and σ of (14), (12) and (16) the spherical triangle is solved; it yields:

$$\psi' = 54^\circ 54' 35.3145'' \quad \alpha' = 36^\circ 45' 7.4006'' \quad (17)$$

$$\lambda = 10^\circ 0' 49.11952''. \quad (18)$$

The reduced latitude ψ' is converted into the latitude φ' , according to section 22, namely:

$$\varphi' = 54^\circ 59' 59.9999'' \text{ (should be } 55^\circ 0' 00''). \quad (19)$$

The azimuth α' according to (17) is already also a spheroidal azimuth; therefore, in order to complete the solution, we only have to convert, in addition, λ of (18) into l , for which equation (28), section 24, p. 114, with the coefficients (29) of that place is used.

The calculation of coefficients according to (29), section 24, p. 114, yields:

$$\log \alpha' = 7.523\ 7864\ 329 \quad \log \beta' = 9.706\ 0633 \quad \log \gamma' = 6.27300,$$

and with this we will have:

$$l = \lambda - 49.131\ 513'' + 0.011\ 935'' + 0.000\ 020'' = \lambda - 49.119\ 558'';$$

$$\text{therefore, according to (18): } l = 9^\circ 59' 59.99996'' \text{ (should be } = 10^\circ 0' 0''). \quad (20)$$

In α' , φ' and l of (17), (19) and (20) we have the complete solution of the problem set forth in sufficient agreement with the assertions of (2), section 17, p. 74.

Inversion of the problem

If φ , α and s are not given, but φ , φ' and l , so that s , α and α' are sought for, then we can still apply the method treated in the foregoing, but only indirectly and clumsily, because the spherical angles m and M , or in the first approximation at least m , are previously needed for the reduction of l to λ .

However, we have for the case in which φ , φ' and l are given, and s , α and α' sought for the more convenient solution of our following section 26.

Comparison of our formulae with Bessel's method

The fundamental idea of the solution of a spheroidal polar triangle by means of a spherical auxiliary triangle with reduced latitudes is treated by Bessel in a treatise, "Über die Berechnung der geographischen Längen und Breiten aus geodätischen Vermessungen," *Astronomische Nachrichten*, Nr. 86, 4. Band, 1826, pp. 241-254, together with "Tafeln zur Berechnung der geodätischen Vermessungen."

This theory of Bessel with the auxiliary tables forms also a part of the work, *Das Messen auf der sphäroidischen Oberfläche*, etc., by J. J. Baeyer, Berlin, 1862.

In order to compare Bessel's method with its auxiliary tables with the formulae of our foregoing section 25, we note at first that our coefficients α , β , γ according to (27), section 24, p. 114, are the same as Bessel's coefficients α , β , γ , whose logarithms are contained in Bessel's first auxiliary table; however, the form of the computation is different in both cases.

The coefficients α' , β' , γ' of the second part of Bessel's auxiliary table are not immediately identical with our coefficients α' , β' , γ' of (29), section 24, p. 114, but they are proportional to them. In Bessel's case, a constant factor, which we will call here F , is taken into the computation.

$$F = \frac{e^2}{\sqrt{1 - 0.75 e^2}} \quad (\log F = 7.825\,1369\,0) . \quad (21)$$

Denoting Bessel's coefficients by α'' , β'' , γ'' , for the next purpose of the comparison, and the coefficients of our development according to (29), p. 114, by α' , β' , γ' , we have:

$$F \alpha'' = \alpha', \quad F \beta'' = \beta', \quad F \gamma'' = \gamma'. \quad (22)$$

As the argument for Bessel's first table, $\log \alpha$, $\log \beta$, $\log \gamma$, there is used the logarithm of the modulus $k = e' \cos m$, which is also the argument of our α , β , γ according to (8) and (27), section 24, and hence, the first argument $= \log e' \cos m$. However, for the second part of Bessel's table, there is used as the argument a quantity $\log k'$, where k' has the meaning:

$$k' = \frac{e \sqrt{0.75}}{\sqrt{1 - 0.75 e^2}} \cos m \quad \left(\log \frac{e \sqrt{0.75}}{\sqrt{1 - 0.75 e^2}} = 8.850\,8255\,6 \right). \quad (23)$$

With these relations we can take our coefficients α , β , γ , α' , β' , γ' also from Bessel's auxiliary table, instead of computing them according to (27) and (29), section 24, p. 114, by entering with the argument $\log e' \cos m$ in the first part and with the argument $\log k'$ according to (23) in the second part of Bessel's table, whereupon the constant $\log F$ according to (21) is still added to the already found $\log \alpha'$ and $\log \beta'$ in order to obtain our α' , β' .

Bessel's tables are also contained in the work, Th. Albrecht, *Formeln und Hilfstafeln für geographische Ortsbestimmungen*, 4th Edition, Leipzig, 1908, pp. 279-286.

A solution of the problem of the transfer of coordinates, starting likewise from Bessel's principles, is given by A. Wirowetz in *Verh. der 8. Tagung der Balt. Geod. Kommission*, Helsinki, 1936, pp. 196-206.

(Notation according to Figs. 1 and 2, section 23, p. 110)

As we have already mentioned at the end of the previous section 25, Bessel's solution of the problem of the transfer of geographic coordinates for the inverse problem, the computation of the distance and azimuths from the given coordinates of two points, is not very convenient. For this problem, Jordan has given a special solution, which we will develop in the following.

We take up once more the two basic differential formulae according to (6), (7), section 23, p. 110, namely:

$$d\sigma = \frac{V}{a} ds \quad (1)$$

$$d\lambda = V dl, \quad (2)$$

where V has the following meaning, as always:

$$V = \sqrt{1 + e'^2 \cos^2 \varphi} \quad (e'^2 \cos^2 \varphi = \eta^2). \quad (3)$$

If the two equations (1) and (2) are integrated, then all relations between a geodetic polar triangle and a spherical polar auxiliary triangle (Figs. 1 and 2, p. 110) are known, and we can solve the problem, as we have discussed in section 23, p. 110.

We will now carry out the integration of the fundamental equations (1) and (2) by development according to Maclaurin's series, i.e. at first to within the third power, by developing the series:

$$\sigma = \left[\frac{d\sigma}{ds} \right] s + \left[\frac{d^2\sigma}{ds^2} \right] \frac{s^2}{2} + \left[\frac{d^3\sigma}{ds^3} \right] \frac{s^3}{6} \quad (4)$$

$$\lambda = \left[\frac{d\lambda}{dl} \right] l + \left[\frac{d^2\lambda}{dl^2} \right] \frac{l^2}{2} + \left[\frac{d^3\lambda}{dl^3} \right] \frac{l^3}{6}. \quad (5)$$

At first, we study (4) more closely, and since we have according to the first half-volume, section 37, p. 42, equation (9), $a = c \sqrt{1 - e^2}$, we have from (1):

$$\sqrt{1 - e^2} \sigma = \frac{1}{c} \left\{ V s + \left[\frac{dV}{ds} \right] \frac{s^2}{2} + \left[\frac{d^2V}{ds^2} \right] \frac{s^3}{6} \right\}. \quad (6)$$

We carry out the derivations needed here in the same form and treatment as previously in section 18 for φ , l and α . We take also from there (5), (6), (7), section 18, p. 75, with $\tan \varphi = t$:

$$\frac{d\varphi}{ds} = \frac{V^3}{c} \cos \alpha, \quad \frac{d}{ds} = \frac{V \sin \alpha}{c \cos \varphi}, \quad \frac{d\alpha}{ds} = \frac{V}{c} \sin \alpha t \quad (7)$$

(13), (14), p. 75:

$$\frac{dV}{d\varphi} = -\frac{\eta^2}{V} t, \quad \frac{dV}{ds} = -\eta^2 \frac{V^2}{c} \cos \alpha t. \quad (8)$$

We differentiate further:

$$\frac{d^2V}{ds^2} = -\eta^2 \frac{V^3}{c^2} \left\{ \cos^2 \alpha (1 - t^2 + \eta^2 - 3\eta^2 t^2) - \sin^2 \alpha t^2 \right\}. \quad (9)$$

Now we can already assemble formula (6), and we note that s with V and c occur here always in the same combination, as also in the former series [cf. (22), section 18, p. 76]; therefore, we set for radian measure (without ρ):

$$\frac{V}{c} s = S, \quad (10)$$

and with this, (6), (8) and (9) yield:

$$\sigma \sqrt{1-e^2} = S - \frac{S^2}{2} \cos \alpha \eta^2 t - \frac{S^3}{6} \eta^2 \left(\cos^2 \alpha (1 - t^2 + \eta^2 - 3 \eta^2 t^2) - \sin^2 \alpha t^2 \right). \quad (11)$$

In the same manner, we have to form also the longitude formula, namely at first (2) and (5):

$$\lambda = V l + \left[\frac{dV}{dl} \right] \frac{l^2}{2} + \left[\frac{d^2V}{dl^2} \right] \frac{l^3}{6}. \quad (12)$$

The derivatives needed for this are:

$$\begin{aligned} \frac{dV}{dl} &= \frac{dV}{d\varphi} \frac{d\varphi}{dl}, \quad \frac{d\varphi}{dl} = V^2 \cot \alpha \cos \varphi, \quad \frac{d\alpha}{dl} = \sin \varphi \\ \frac{dV}{dl} &= -\eta^2 V \cot \alpha \sin \varphi \quad \text{or} \quad = -\eta^2 V \frac{\cos \varphi}{\sin \alpha} \cos \alpha t \end{aligned} \quad (13)$$

$$\frac{d^2V}{dl^2} = -\eta^2 V \frac{\cos^2 \varphi}{\sin^2 \alpha} \left\{ \cos^2 \alpha (1 - 3t^2 + \eta^2 - 3\eta^2 t^2) - \sin^2 \alpha t^2 \right\}. \quad (14)$$

With this, we can combine (12)

$$\lambda = V \left\{ l - \frac{l^2 \cos \varphi}{2 \sin \alpha} \eta^2 \cos \alpha t - \frac{l^3 \cos^2 \varphi}{6 \sin^2 \alpha} \eta^2 (\cos^2 \alpha (1 - 3t^2 + \eta^2 - 3\eta^2 t^2) - \sin^2 \alpha t^2) \right\}. \quad (15)$$

The formulae (11) and (15) yield σ and λ as a function of the starting latitude φ and the starting azimuth α of the geodetic line; but now we will apply the principle of the mean argument, which has already been very useful in section 21.

For this purpose, we assume the notation of Fig. 1 in the margin, i.e. we take three points at equal intervals of latitude:

$$\varphi_2 - \varphi = \varphi - \varphi_1, \quad \frac{\varphi_1 + \varphi_2}{2} = \varphi. \quad (16)$$

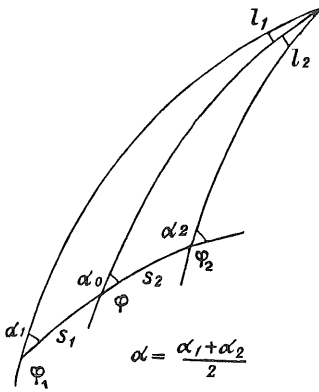


Fig. 1.

From the mean latitude φ there starts a geodetic line s_2 with the azimuth α_0 , and a second geodetic line s_1 with the azimuth $\alpha_0 \pm 180^\circ$. According to (10), two quantities S_2 and S_1 with the sum $S_2 + S_1 = S$ correspond to the geodetic lines s_2 and s_1 , whose sum we let be $s_2 + s_1 = s$. With this, the double use of formula (11) yields:

$$\sigma \sqrt{1-e^2} = S - \frac{S_2^2 - S_1^2}{2} \cos \alpha_0 \eta^2 t - \frac{S_2^3 + S_1^3}{6} \eta^2 (\cos^2 \alpha_0 (1 - t^2 + \eta^2 - 3 \eta^2 t^2) - \sin^2 \alpha_0 t^2)$$

Applying (25), section 18, p. 78, to s_1 and s_2 there follows:

$$S_2 - S_1 = \frac{S^2}{4} t \left(\frac{\sin^2 \alpha_0}{\cos \alpha_0} + 3 \eta^2 \cos \alpha_0 \right).$$

If we set this into the foregoing, then we may also write everywhere α instead of α_0 , and with this, we obtain:

$$\sigma \sqrt{1 - e^2} = S - \frac{S^2}{24} \eta^2 \left\{ \sin^2 \alpha \, 2 t^2 + \cos^2 \alpha (1 - t^2 + \eta^2 + 6 \eta^2 t^2) \right\}. \quad (17)$$

Now we apply also equation (15) to Fig. 1 in a twofold manner and obtain with $l_2 + l_1 = l$, $\lambda_2 + \lambda_1 = \lambda$:

$$\lambda = V \left\{ l - \frac{l_2^2 - l_1^2}{2} \frac{\cos \varphi}{\sin \alpha_0} \eta^2 \cos \alpha_0 t - \frac{l_2^3 + l_1^3}{6} \frac{\cos^2 \varphi}{\sin^2 \alpha_0} \eta^2 (\cos^2 \alpha_0 (1 - 3 t^2 + \eta^2 - 3 \eta^2 t^2) - \sin^2 \alpha_0 t^2) \right\}.$$

To this, we have from (24), section 21, p. 98:

$$(l_2 - l_1) \cos \varphi = \frac{S^2 \sin^3 \alpha}{4 \cos \alpha} t + \frac{S^2}{4} \sin \alpha \cos \alpha t (2 + 3 \eta^2).$$

We can set $S \sin \alpha = l \cos \varphi$ here, and hence:

$$(l_2 - l_1) = \frac{l^3 \sin \alpha}{4 \cos \alpha} \cos \varphi t + \frac{l^2 \cos \alpha}{4 \sin \alpha} \cos \varphi t (2 + 3 \eta^2).$$

We set this into the previous formula for λ , where α_0 and α can be interchanged again; we obtain thereby:

$$\lambda = V \left\{ l - \frac{l^3 \cos^2 \varphi}{24 \sin^2 \alpha} \eta^2 \left(\sin^2 \alpha \, 2 t^2 + \cos^2 \alpha (1 + 3 t^2 + \eta^2 + 6 \eta^2 t^2) \right) \right\}. \quad (18)$$

Equations (17) and (18) contain already the solution of our problem, if we presuppose S , α and l as given, at least approximately; however, it is more convenient to reduce everything to the difference of latitude b and the difference of longitude l . We have here for the correction terms:

$$S \sin \alpha = l \cos \varphi, \quad S \cos \alpha = l \cos \varphi \cot \alpha = \frac{b}{V^2}.$$

This introduced into (17) and (18) yields:

$$\sigma = \frac{S}{\sqrt{1 - e^2}} \left\{ 1 - \frac{\eta^2}{24} \left(\frac{b^2}{V^4} (1 - t^2 + \eta^2 + 6 \eta^2 t^2) + 2 l^2 \sin^2 \varphi \right) \right\} \quad (19)$$

$$\lambda = V l \left\{ 1 - \frac{\eta^2}{24} \left(\frac{b^2}{V^4} (1 + 3 t^2 + \eta^2 + 6 \eta^2 t^2) + 2 l^2 \sin^2 \varphi \right) \right\}. \quad (20)$$

We will pick out the coefficients and form the following formulae for use (taking into account the necessary ρ 's):

$$\sigma = Us \left\{ 1 + (\sigma_1) b^2 + (\sigma_2) l^2 \sin^2 \varphi \right\} \quad (21)$$

$$\lambda = Vl \left\{ 1 + (\lambda_1) b^2 + (\lambda_2) l^2 \sin^2 \varphi \right\}. \quad (22)$$

We can apply these formulae also in logarithmic form, e.g. if $\log \sigma$ is given and $\log s$ is to be determined, we have by inversion of (21) in logarithmic form:

$$\log s = (\log \sigma - \log U) - \mu (\sigma_1) b^2 - \mu (\sigma_2) l^2 \sin^2 \varphi. \quad (23)$$

V is here the function denoted hitherto always by V

and

$$V = \sqrt{1 + e'^2 \cos^2 \varphi} \quad (24)$$

$$U = \frac{V}{e \sqrt{1 - e^2}} e \quad \text{or} \quad = \frac{V}{a} e$$

$$\log U = \log V + 8.509\,7816\,695. \quad (25)$$

We can take the value $\log V$ or, as the case may be, $\log V^2$ to ten places from the first half-volume, auxiliary table, pp. [2] to [7] of our Appendix, and according to (25) we have then also $\log U$.

For the coefficients (σ_1) , (σ_2) , (λ_1) , (λ_2) in (21) and (22), the following meanings result by comparison with (19) and (20):

$$(\sigma_1) = + \frac{\eta^2}{24 \varrho^2 V^4} (t^2 - (1 + \eta^2 + 6 \eta^2 t^2)), \quad (\sigma_2) = - \frac{\eta^2}{12 \varrho^2} \quad (26)$$

$$(\lambda_1) = - \frac{\eta^2}{24 \varrho^2 V^4} (3 t^2 + 1 + \eta^2 + 6 \eta^2 t^2), \quad (\lambda_2) = - \frac{\eta^2}{12 \varrho^2}. \quad (27)$$

The constant logarithms of the coefficients are here:

$$\log \frac{1}{24 \varrho^2} = 7.990\,9385 - 20, \quad \log \frac{1}{12 \varrho^2} = 8.291\,9685 - 20.$$

Further development to the fifth order

With the formulae hitherto developed we can already compute geodetic lines of several degrees extent, as is seen from the comparison of the following numerical examples with the results of section 25.

However, we have the best means for forming an opinion about the method hitherto treated and about the possibility of its extension in the further development by one step higher, i.e. to the fifth order.

We carried out this development and indicated the principal intermediate stages in *Zeitschrift für Vermessungswesen*, 1883, pp. 72-76; since very long collections of formulae, which can hardly be reproduced in print and whose fundamental mathematical idea is made already completely clear by the foregoing, are thereby involved, we give here only the final results.

The formulae (21) and (22) are extended thusly [cf. (31) and (32)]:

$$\sigma = Us \left\{ 1 + (\sigma_1) b^2 + (\sigma_2) l^2 \sin^2 \varphi + (\sigma_3) b^4 + (\sigma_4) b^2 l^2 \cos^2 \varphi + (\sigma_5) l^4 \cos^4 \varphi \right\} \quad (28)$$

$$\lambda = Vl \left\{ 1 + (\lambda_1) b^2 + (\lambda_2) l^2 \sin^2 \varphi + (\lambda_3) b^4 + (\lambda_4) b^2 l^2 \cos^2 \varphi + (\lambda_5) l^4 \cos^4 \varphi \right\}. \quad (29)$$

If we aim to apply the formulae (28) and (29) inversely, i.e. if, for instance, we aim to compute s from σ , then we need not take into account the terms $(\sigma_1)^2 b^4$, $(\sigma_1)(\sigma_2) b^2 l^2 \sin^2 \varphi$ and $(\sigma_2)^2 l^4 \sin^4 \varphi$, which occur at first in the development of series, because the coefficients (σ_1) and (σ_2) both have the factor η^2

and terms of the order η^4 are altogether neglected in the coefficients (σ_3) , (σ_4) and (σ_5) .

And hence, even if we aim to calculate logarithmically, we can invert (28) briefly in the following way:

$$\log s = (\log \sigma - \log U) - \mu(\sigma_1) b^2 - \mu(\sigma_2) l^2 \sin^2 \varphi - \mu(\sigma_3) b^4 - \mu(\sigma_4) b^2 l^2 \cos^2 \varphi - \mu(\sigma_5) l^4 \cos^4 \varphi. \quad (30)$$

In these formulae (28), (29), (30), the coefficients (σ_1) , (σ_2) , (λ_1) , (λ_2) are the same as was already indicated in the case of (26) and (27); the remaining ones have the following meanings to an accuracy of η^2 inclusive:

$$\left. \begin{aligned} (\sigma_3) &= \frac{\eta^2}{480 \varrho^4} (1 - t^2) &= [3.88838] \cos^2 \varphi (1 - t^2) \\ (\sigma_4) &= \frac{\eta^2}{720 \varrho^4} (-1 + 2 t^2 + 15 t^4) &= [3.712 286] \cos^2 \varphi (-1 + 2 t^2 + 15 t^4) \\ (\sigma_5) &= -\frac{\eta^2}{720 \varrho^4} (9 t^2 - 5 t^4) &= [3.712 286_n] \cos^2 \varphi (9 t^2 - 5 t^4) \end{aligned} \right\} \quad (31)$$

$$\left. \begin{aligned} (\lambda_3) &= -\frac{\eta^2}{1440 \varrho^4} (1 + 15 t^2) &= [3.411 256_n] \cos^2 \varphi (1 + 15 t^2) \\ (\lambda^4) &= \frac{\eta^2}{720 \varrho^4} (-1 - 10 t^2 + 15 t^4) &= [3.712 286] \cos^2 \varphi (-1 - 10 t^2 + 15 t^4) \\ (\lambda_5) &= \frac{\eta^2}{240 \varrho^4} (-3 t^2 + t^4) &= [4.189 407] \cos^2 \varphi (-3 t^2 + t^4). \end{aligned} \right\} \quad (32)$$

The numbers in brackets are logarithms here, and the attached n means that the pertinent number is negative. As always, we have the meanings $\eta^2 = e'^2 \cos^2 \varphi$ and $t = \tan \varphi$.

According to these, we have computed a table of coefficients and given it on pp. [12] to [13] of the Appendix. The tables [1] to [4] of the Appendix can be used in addition.

In the foregoing formulae there occur various constants, which we group together here for use:

$$\left. \begin{aligned} \log(1: 12 \varrho^2) &= 8.291 9684.9 - 20 & \log(e'^2: 12 \varrho^2) &= 6.119 2832.7 - 20 \\ \log(1: 24 \varrho^2) &= 7.990 9384.9 - 20 & \log(e'^2: 24 \varrho^2) &= 5.818 2572.8 - 20 \\ \log(1: 240 \varrho^4) &= 6.362 0882 - 30 & \log(e'^2: 240 \varrho^4) &= 4.189 4070 - 30 \\ \log(1: 480 \varrho^4) &= 6.061 0582 - 30 & \log(e'^2: 480 \varrho^4) &= 3.888 3770 - 30 \\ \log(1: 720 \varrho^4) &= 5.884 9670 - 30 & \log(e'^2: 720 \varrho^4) &= 3.712 2858 - 30 \\ \log(1: 1440 \varrho^4) &= 5.583 9370 - 30 & \log(e'^2: 1440 \varrho^4) &= 3.411 2558 - 30 \end{aligned} \right\} \quad (33)$$

In order to obtain a general view of the amount of the terms of fifth order in our latitudes, we have calculated the following two tabular summaries for the total amount of the three end terms in (28) and (29).

1. Terms of fifth order in formula (28) for σ , with $\varphi = 50^\circ$

$b =$	$l = 2^\circ$	$l = 4^\circ$	$l = 6^\circ$	$l = 8^\circ$	$l = 10^\circ$
2°	+ 0.00000''	+ 0.00000''	+ 0.00001''	+ 0.00001''	+ 0.00001''
4	+ 0.00000	+ 0.00002	+ 0.00005	+ 0.00010	+ 0.00016
6	+ 0.00001	+ 0.00006	+ 0.00014	+ 0.00030	+ 0.00048
8	+ 0.00001	+ 0.00012	+ 0.00031	+ 0.00061	+ 0.00103
10	+ 0.00002	+ 0.00018	+ 0.00056	+ 0.00111	+ 0.00186

$b =$	$l = 2^\circ$	$l = 4^\circ$	$l = 6^\circ$	$l = 8^\circ$	$l = 10^\circ$
2°	— 0.00000"	— 0.00000"	— 0.00001"	— 0.00003"	— 0.00012"
4	— 0.00001	— 0.00001	— 0.00001	— 0.00001	— 0.00004
6	— 0.00004	— 0.00006	— 0.00006	— 0.00005	— 0.00005
8	— 0.00011	— 0.00020	— 0.00025	— 0.00026	— 0.00022
10	— 0.00028	— 0.00053	— 0.00070	— 0.00079	— 0.00078

As first application of the developed formulae we will take our fifth standard example, (5), section 17, p. 74, in this manner:

$$\left. \begin{array}{l} \text{Given Berlin } \varphi_1 = 52^\circ 30' 16.7'' \\ \text{Königsberg } \varphi_2 = 54 \ 42 \ 50.6 \end{array} \right\} l = 7^\circ 6' 0''. \quad (36)$$

We shall compute the geodetic line s between the two points and the two azimuths α_1 and α_2 . At first we form the mean of the given latitudes:

$$\varphi = 53^\circ 36' 33.65''. \quad (37)$$

With this, we enter into the auxiliary tables in the Appendix, p. [5], of the first half-volume and [12] to [13] of this half-volume and take the coefficients:

$$\begin{array}{ccccc} \log V = 0.000 \ 5129 \cdot 683 & & & & \\ \log (\lambda_1) & \log (\lambda_2) & \log (\lambda_3) & \log (\lambda_4) & \log (\lambda_5) \\ 6.17908_n & 5.66582_n & 4.414_n & 4.756 & 4.065_n \end{array} \quad (38)$$

With this, we calculate according to formula (29) with $l = 7^\circ 6' 0'' = 25,560''$; the principal term becomes $25,590.208 \ 116''$, then the five correction terms:

$$\begin{array}{l} - 0.024 \ 452'', \quad - 0.050 \ 187'', \quad - 0.000 \ 003'', \quad + 0.000 \ 021'', \quad - 0.000 \ 016'', \\ \lambda = 25,590.208 \ 116'' - 0.074 \ 637'' = 25,590.133 \ 479'' \\ \lambda = 7^\circ 6' 30.133 \ 479''. \end{array} \quad (39)$$

We have calculated here with six decimals of seconds, in order to see how far the last three terms become noticeable altogether; since they amount to only $0.000 \ 002''$, we could omit them entirely.

Now we take the reduced latitudes to (36) as well as λ of (39) together:

$$\left. \begin{array}{l} \text{Berlin } \psi_1 = 52^\circ 24' 43.01137'' \\ \text{Königsberg } \psi_2 = 54 \ 37 \ 24.75639 \end{array} \right\} \lambda = 7^\circ 6' 30.13348''. \quad (40)$$

We have solved the spherical triangle thereby determined, according to Gauss' formulae (4), (5) in the first half-volume, section 59, p. 159, whereby we found:

$$\left. \begin{array}{l} \alpha_1 = 59^\circ 33' 0.6889'' \quad \alpha_2 = 65^\circ 16' 9.3650'' \\ \sigma = 4^\circ 46' 1.41023'' = 17,161.41023''. \end{array} \right\} \quad (41)$$

In order to reduce σ to s , we need again coefficients, at first $\log U$ according to formula (25) with the use of $\log V$ computed already in (38):

$$\log U = 8.510 \ 2946 \cdot 378.$$

From the auxiliary table, pp. [12] to [13], we take, with the argument $\varphi = 53^\circ 36' 33.65''$ of (37), the five logarithms of coefficients for s :

$\log (\sigma_1)$	$\log (\sigma_2)$	$\log (\sigma_3)$	$\log (\sigma_4)$	$\log (\sigma_5)$
5.27256	5.66582 _n	3.360 _n	4.987	2.836.

With this, we compute according to formula (30) and have at first the principal term 5.724 2583.351 and the five logarithmic correction terms:

$$- 0.5146, \quad + 8.5179, \quad + 0.0000, \quad - 0.0061, \quad - 0.0002.$$

This yields in all:

$$\log s = 5.724\ 2583.351 + 7.997 = 5.724\ 2591.348 \quad s = 529,979.578\ \text{m.} \quad (42)$$

The length s and the two azimuths of (41) represent the solution, which agrees sufficiently with the corresponding values (2) and (11) of the previous section 25, pp. 115 and 116.

After this, we will treat, in addition, our large standard example (2), section 17, p. 74.

$$\left. \begin{array}{l} \text{Given} \quad \varphi_1 = 45^\circ 0' 0'' \\ \quad \quad \varphi_2 = 55 \quad 0 \quad 0 \quad l = 10^\circ 0' 0'' \\ \text{Mean} \quad \varphi = 50 \quad 0 \quad 0 \end{array} \right\} \quad (43)$$

With this, we enter into the auxiliary tables, p. [5] of the first half-volume and pp. [12] to [13] of this half-volume and take the coefficients:

$$\log V = 0.000\ 6020.131 \quad \log U = 8.510\ 3836.826 \quad (44)$$

$$\left. \begin{array}{l} \log (\sigma_1) \quad \log (\sigma_2) \quad \log (\sigma_3) \quad \log (\sigma_4) \quad \log (\sigma_5) \end{array} \right\} \quad (45)$$

$$\begin{array}{l} 5.02731 \quad 5.73542_n \quad 3.128_n \quad 4.835 \quad 3.759_n \end{array}$$

$$\left. \begin{array}{l} \log (\lambda_1) \quad \log (\lambda_2) \quad \log (\lambda_3) \quad \log (\lambda_4) \quad \log (\lambda_5) \end{array} \right\} \quad (46)$$

$$\begin{array}{l} 6.155215_n \quad 5.73542_n \quad 4.376_n \quad 4.506 \quad 4.156_n. \end{array}$$

The reduction for λ according to formula (29) yields the principal term 36,049.93731" and the five correction terms:

$$- 0.667\ 923'', \quad - 0.149\ 088'', \quad - 0.001\ 438'', \quad + 0.000\ 802'', \quad - 0.000\ 148''.$$

$$\text{Altogether:} \quad \lambda = 36,049.93731'' - 0.817\ 795'' = 36,049.11952''. \quad (47)$$

The two reduced latitudes are:

$$\psi_1 = 44^\circ 54' 14.67493'' \quad \psi_2 = 54^\circ 54' 35.31462''. \quad (48)$$

These, ψ_1 and ψ_2 as well as λ of (47), determine a spherical triangle whose solution yields:

$$\begin{array}{l} \alpha_1 = 29^\circ 3' 15.45983'' \quad \alpha_2 = 36^\circ 45' 7.40055'' \\ \sigma_1 = 11^\circ 52' 41.20996'' = 42,761.20996''. \end{array} \quad (49)$$

For the reduction of σ from s we have formula (30) with the coefficients (45); the principal term becomes 6.120 6663.024 and the five correction terms:

$$- 5.994, \quad + 17.961, \quad + 0.010, \quad - 0.206, \quad + 0.007.$$

Altogether:

$$\log s = 6.120\ 6663 \cdot 024 + 11 \cdot 778 = 6.120\ 6674 \cdot 802 \quad s = 1,320,284.365 \text{ m.} \quad (50)$$

The values (49) and (50) represent the solution of the problem, which, compared with (12), (13), (17) of the previous section 25, p. 117, agree sufficiently.

In *Zeitschr. f. Verm.*, 1883, pp. 81-82, we have given a table of coefficients for the formulae (28) and (29), which is not the same as the newly computed table, pp. [12] to [13] of our Appendix. Only the coefficients (σ_1) , (σ_3) , (λ_1) , (λ_3) are identical with the former [1], [3], (1), (3), apart from a small difference in the last places of $\log(\sigma_1)$ and $\log(\lambda_1)$, originating from the fact that formerly, there was set $\frac{1}{v^4} = 1 - 2\eta^2$, which contains the omission of η^4 , which no longer occurs in the new coefficients (σ_1) , (σ_2) , (λ_1) , (λ_2) . Besides, there is the difference that the functions $\sin^2 \varphi$, $\cos^2 \varphi$, $\cos^4 \varphi$, which formerly were drawn into the coefficients, now remain in the formula, in order that the table of coefficients may have smaller differences. Only the factor $\cos^2 \varphi$ contained in η^2 is drawn into the coefficients, because the modulus $\eta^2 = e'^2 \cos^2 \varphi$ is found best analytically, and also formally, compared with the factors t^2 , contributes to equilibrium in the coefficients.

As an additional example for the application of the method treated in the foregoing section 26, we can mention the diagonal of Mecklenburg, which, included already among our standard examples in section 17, p. 74, was computed in *Zeitschr. f. Verm.*, 1896, pp. 240-248.

L. Krüger gives another derivation of Jordan's formulae with the help of the formulae of the mean latitude in the treatise, mentioned already on p. 101: "Die kürzeste Entfernung und ihre Azimute zwischen zwei gegebenen Punkten des Erdellipsoids," in *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen*, 1918.

We mention in addition: H. Andoyer, "Formule donnant la longueur de la géodésique joignant 2 points de l'ellipsoïde donnés par leurs coordonnées géographiques," *Bulletin géodésique*, 1932, pp. 77-81.

Section 27. Rectangular Coordinates on the Ellipsoid

In the same manner as we have introduced a rectangular system of coordinates on the sphere in the first half-volume, section 52, p. 116, we will now examine also a rectangular system of coordinates on the ellipsoid. As the axis of abscissae we use again an arbitrary meridian on which the zero point lies, and then we obtain the coordinates of a point if we lay, through it, a geodetic line which intersects the zero meridian at right angles. The abscissa of the point is the meridian arc from the zero point to the point of intersection with the geodetic line, whereas the ordinate is equal to the distance of the point from the zero meridian, measured on the geodetic line.

Therefore, the spheroidal rectangular coordinates are defined just as the spherical ones; however, the geodetic line takes the place of the great circle of ordinates.

All which we have said in the first half-volume, Chapter V, about the spherical rectangular coordinates will nearly always suffice for geodetic practice. We also have treated there already, in part, the relations between rectangular and geographic coordinates. Nevertheless, we will resume the subject once more, since we have found an exact mathematical foundation for it in section 18.

Computation of geographic coordinates from spheroidal rectangular coordinates

We denote by x and y the spheroidal rectangular coordinates of a point whose geographic coordinates are φ and l . For the sake of simplicity, the longitude l shall thereby be referred to the meridian of the zero point of coordinates. The latitude of the zero point shall further be denoted by φ_0 and the latitude of the foot-point of ordinates by φ_1 .

We start from the formulae (25) to (27), section 18, p. 78, in which we set the starting azimuth α equal to 0° and equal to 90° .

If we set at first in equation (25), section 18, p. 78, $\alpha = 0^\circ$ and substitute for φ' and φ , respectively, φ_1 and φ_0 , then we will have, if we disregard ρ :

$$v = \frac{s \sin \alpha}{N} = 0 \quad \text{and} \quad u = \frac{s \cos \alpha}{N} = \frac{x}{N}, \quad (1)$$

and then we have accurate to the fifth order:

$$\frac{\varphi_1 - \varphi_0}{V_0^2} = \frac{x}{N_0} - \frac{3x^2}{2N_0^2} \eta_0^2 t_0 + \frac{x^3}{2N_0^3} \eta_0^2 (t_0^2 - 1). \quad (2)$$

If we set further in the equations (25) to (27), section 18, p. 78, $\alpha = 90^\circ$ and substitute for φ' and φ now φ and φ_1 , then we will have:

$$v = \frac{s \sin \alpha}{N} = \frac{y}{N} \quad \text{and} \quad u = \frac{s \cos \alpha}{N} = 0, \quad (3)$$

and hence there follows then likewise accurate to the fifth order:

$$\frac{\varphi - \varphi_1}{V_1^2} = -\frac{y^2 t_1}{2N_1^2} + \frac{y^4 t_1}{24N_1^4} (1 + 3t_1^2) \quad (4)$$

$$l \cos \varphi_1 = \frac{y}{N_1} - \frac{y^3}{3N_1^3} t_1^2 + \frac{y^5 t_1^2}{15N_1^5} (1 + 3t_1^2) \quad (5)$$

$$\gamma = \frac{y}{N_1} t_1 - \frac{y^3 t_1}{6N_1^3} (1 + 2t_1^2 + \eta_1^2) + \frac{y^5 t_1}{120N_1^5} (1 + 20t_1^2 + 24t_1^4). \quad (6)$$

V_1^2 and N_1 as well as $t_1 = \tan \varphi_1$ all pertain here to the latitude of the foot-point φ_1 .

The foregoing formulae (2) and (4) to (6) suffice for the computation of the geographic coordinates φ and l from the rectangular spheroidal coordinates x and y , if the latitude of the zero point φ_0 is given.

Now we will still add the factors ρ and obtain then the following grouping:

Given x, y together with φ_0

Required φ, l, γ .

$$\varphi_1 = \varphi_0 + \frac{\rho}{N_0} V_0^2 x - \frac{3x^2}{2N_0^2} \rho \eta_0^2 t_0 + \frac{x^3}{2N_0^3} \rho \eta_0^2 (t_0^2 - 1) \quad (7)$$

$$\varphi = \varphi_1 - \frac{y^2 V_1^2}{2N_1^2} \rho t_1 + \frac{y^4}{24N_1^4} \rho t_1 (1 + 3t_1^2) \quad (8)$$

$$l \cos \varphi_1 = \frac{y}{N_1} \rho - \frac{y^3}{3N_1^3} \rho t_1^2 + \frac{y^5}{15N_1^5} \rho t_1^2 (1 + 3t_1^2) \quad (9)$$

$$\gamma = \frac{y}{N_1} \rho t_1 - \frac{y^3}{6N_1^3} \rho t_1 (1 + 2t_1^2 + \eta_1^2) + \frac{y^5}{120N_1^5} \rho t_1 (1 + 20t_1^2 + 24t_1^4). \quad (10)$$

Sometimes it is more convenient to compute $\log(\varphi_1 - \varphi_0)$, $\log(\varphi - \varphi_1)$, $\log l \cos \varphi$ and $\log \gamma$ and therefore we will transform the foregoing formulae once more.

Instead of equation (7) we can write:

$$\varphi_1 - \varphi_0 = \frac{\rho}{N_0} V_0^2 x \left(1 - \frac{3x}{2N_0} \eta_0^2 t_0\right) \left(1 + \frac{x^2}{2N_0^2} \eta_0^2 (t_0^2 - 1)\right).$$

If we do not go beyond the fifth order, then we obtain hence, according to the first half-volume, 'section 34, p. 21:

$$\log (\varphi_1 - \varphi_0) = \log \frac{\varrho}{N_0} V_0^2 x - \frac{3}{2} \frac{\mu}{N_0} x \eta_0^2 t_0 + \frac{\mu}{2 N_0^2} x^2 \eta_0^2 (t_0^2 - 1). \quad (11)$$

Equation (8) can likewise be easily transformed; we will have

$$\varphi_1 - \varphi = \frac{y^2 V_1^2}{2 N_1^2} \varrho t_1 \left(1 - \frac{y^2}{12 N_1^2} (1 + 3 t_1^2) \right)$$

and

$$\log (\varphi_1 - \varphi) = \log \frac{y^2 V_1^2}{2 N_1^2} \varrho t_1 - \frac{\mu}{12 N_1^2} y^2 (1 + 3 t_1^2), \quad (12)$$

where $\frac{V_1^2}{N_1^2} = \frac{1}{c^2}$ and $c = \frac{a^2}{b}$.

For the computation of the difference of longitude l we write equation (9) in the following form:

$$l = \frac{y}{N_1} \frac{\varrho}{\cos \varphi_1} \left(1 - \frac{y^2}{3 N_1^2} t_1^2 \right) \left(1 + \frac{y^4}{15 N_1^4} t_1^2 (1 + 3 t_1^2) \right).$$

Now we have according to the first half-volume, section 34, p. 21:

$$\begin{aligned} \log \left(1 - \frac{y^2}{3 N_1^2} t_1^2 \right) &= - \frac{\mu}{3 N_1^2} y^2 t_1^2 - \frac{\mu}{18 N_1^4} y^4 t_1^4 \\ \log \left(1 + \frac{y^4}{15 N_1^4} t_1^2 (1 + 3 t_1^2) \right) &= + \frac{\mu}{15 N_1^4} y^4 t_1^2 (1 + 3 t_1^2), \end{aligned}$$

and consequently, we will have:

$$\log l = \log \frac{y}{N_1} \frac{\varrho}{\cos \varphi_1} - \frac{\mu}{3 N_1^2} y^2 t_1^2 + \frac{\mu}{90 N_1^4} y^4 t_1^2 (6 + 13 t_1^2). \quad (13)$$

If we transform also equation (10) in the same manner, which we do no longer wish to indicate in detail, then we have, in this connection, the following formula:

Given x, y together with φ_0

Required φ, l, γ .

$$\log (\varphi_1 - \varphi_0) = \log \frac{\varrho}{N_0} V_0^2 x - \frac{3}{2} \frac{\mu}{N_0} x \eta_0^2 t_0 + \frac{\mu}{2 N_0^2} x^2 \eta_0^2 (t_0^2 - 1) \quad (14)$$

$$\log (\varphi_1 - \varphi) = \log \frac{V_1^2 \varrho}{2 N_1^2} y^2 t_1 - \frac{\mu}{12 N_1^2} y^2 (1 + 3 t_1^2) \quad (15)$$

$$\log l = \log \frac{y}{N_1} \frac{\varrho}{\cos \varphi_1} - \frac{\mu}{3 N_1^2} y^2 t_1^2 + \frac{\mu}{90 N_1^4} y^4 t_1^2 (6 + 13 t_1^2) \quad (16)$$

$$\log \gamma = \log \frac{y}{N_1} \varrho t_1 - \frac{\mu}{6 N_1^2} y^2 (1 + 2 t_1^2 + \eta_1^2) - \frac{\mu}{180 N_1^4} y^4 (1 - 20 t_1^2 - 26 t_1^4). \quad (17)$$

In this, $\log \frac{\rho}{N} = \log [2]$ is to be taken from our auxiliary table in the first half-volume, pp. [12] to [33] of the Appendix. Further we have

$$\log \rho = 5.314\ 4251 \cdot 3 \quad \log \mu = 9.63778 - 10,$$

whereas $\log V^2$ and $\log N$ are contained likewise in the mentioned auxiliary table of the Appendix.

Computation of rectangular coordinates from geographic coordinates

The formulae (2) and (4) to (6) can also be directly inverted; however, there is to be taken into account that the unknown foot-point latitude is replaced everywhere by the given latitude φ .

We start with equation (5):

$$l \cos \varphi_1 = \frac{y}{N_1} - \frac{y^3}{3 N_1^3} t_1^2 + \frac{y^5}{15 N_1^5} t_1^2 (1 + 3 t_1^2) \quad (18)$$

and have as a first approximation:

$$\frac{y}{N_1} = l \cos \varphi_1 + \dots \quad \frac{y^3}{N_1^3} = l^3 \cos^3 \varphi_1 + \dots$$

With this, we will have

$$l \cos \varphi_1 = \frac{y}{N_1} - \frac{1}{3} l^3 \cos^3 \varphi_1 t_1^2 + \dots$$

and this yields as a second approximation:

$$\begin{aligned} \frac{y}{N_1} &= l \cos \varphi_1 + \frac{1}{3} l^3 \cos^3 \varphi_1 t_1^2 + \dots \\ \frac{y^3}{N_1^3} &= l^3 \cos^3 \varphi_1 + l^5 \cos^5 \varphi_1 t_1^2 + \dots \\ \frac{y^5}{N_1^5} &= l^5 \cos^5 \varphi_1 + \dots \end{aligned}$$

If we introduce this into (18), then there follows easily:

$$y = N_1 l \cos \varphi_1 + \frac{1}{3} N_1 l^3 \sin^2 \varphi_1 \cos \varphi_1 - \frac{1}{15} N_1 l^5 \sin^2 \varphi_1 \cos^3 \varphi_1 (1 - 2 t_1^2). \quad (19)$$

Before we can pass over from φ_1 to φ in this formula, we still must develop an expression for $\varphi - \varphi_1$ from equation (4). According to (19) we have

$$\begin{aligned} \frac{y^2}{N_1^2} &= l^2 \cos^2 \varphi_1 + \frac{2}{3} l^4 \sin^2 \varphi_1 \cos^2 \varphi_1 + \dots \\ \frac{y^4}{N_1^4} &= l^4 \cos^4 \varphi_1 + \dots \end{aligned}$$

and this, introduced into (4), yields after an easy conversion:

$$\varphi_1 - \varphi = \frac{1}{2} V_1^2 l^2 \sin \varphi_1 \cos \varphi_1 + \frac{1}{24} l^4 \sin \varphi_1 \cos^3 \varphi_1 (5 t_1^2 - 1) \dots \quad (20)$$

For the passage from φ_1 to φ we have according to the first half-volume, section 34, p. 18:

$$\left. \begin{aligned} \sin \varphi_1 &= \sin \varphi + (\varphi_1 - \varphi) \cos \varphi - \frac{1}{2} \sin \varphi (\varphi_1 - \varphi)^2 \\ \cos \varphi_1 &= \cos \varphi + (\varphi_1 - \varphi) \sin \varphi - \frac{1}{2} \cos \varphi (\varphi_1 - \varphi)^2, \end{aligned} \right\} \quad (21)$$

and hence

$$\sin \varphi_1 \cos \varphi_1 = \sin \varphi \cos \varphi + (\varphi_1 - \varphi) (\cos^2 \varphi - \sin^2 \varphi) - \frac{3}{2} (\varphi_1 - \varphi)^2 \sin \varphi \cos \varphi. \quad (22)$$

From the first half-volume, section 40, p. 63, equation (n), it follows that

$$V_1 = V \left(1 - \frac{\varphi_1 - \varphi}{V^2} \eta^2 t - \dots \right)$$

and since $\varphi_1 - \varphi$ is already of the second order, then we have to sufficient accuracy

$$V_1 = V (1 - \dots),$$

and hence, in (20) V_1 can be directly replaced by V . If we set (22) into (20), then we will have

$$\begin{aligned} \varphi_1 - \varphi &= \frac{1}{2} V^2 l^2 \sin \varphi \cos \varphi + \frac{1}{2} V^2 l^2 (\varphi_1 - \varphi) (\cos^2 \varphi - \sin^2 \varphi) \\ &\quad + \frac{1}{24} V^2 l^4 \sin \varphi \cos^3 \varphi (5 t^2 - 1). \end{aligned}$$

For the conversion of the second term, we have hence

$$\varphi_1 - \varphi = \frac{1}{2} V^2 l^2 \sin \varphi \cos \varphi + \dots,$$

and with this, we find easily with the addition of ρ :

$$\varphi_1 - \varphi = \frac{1}{2} \frac{V^2}{\varrho} l^2 \sin \varphi \cos \varphi + \frac{1}{24} \frac{l^4}{\varrho^3} \sin \varphi \cos^3 \varphi (5 - t^2) + \dots \quad (23)$$

This equation for $\varphi_1 - \varphi$ puts us at the same time in a position to convert also the formula (19) for y from φ_1 to φ . For this, we write equation (19) in the following form:

$$y = N_1 l \cos \varphi_1 \left\{ 1 + \frac{1}{3} l^2 \sin^2 \varphi_1 - \frac{1}{15} l^4 \sin^2 \varphi_1 \cos^2 \varphi_1 (1 - 2 t_1^2) \right\}, \quad (24)$$

and can in the last term replace φ_1 forthwith by φ . For the second term of the expression within brackets, we have according to (21) and (23):

$$\begin{aligned} \sin \varphi_1 &= \sin \varphi + \frac{1}{2} l^2 \sin \varphi \cos^2 \varphi + \dots, \\ \sin^2 \varphi_1 &= \sin^2 \varphi + l^2 \sin^2 \varphi \cos^2 \varphi + \dots, \end{aligned}$$

and hence

and with this, we obtain for the whole expression within brackets in (24) after simple conversions:

$$\left\{ \dots \right\} = 1 + \frac{1}{3} l^2 \sin^2 \varphi + \frac{1}{15} l^4 \sin^2 \varphi \cos^2 \varphi (4 + 2 t^2). \quad (25)$$

For the conversion of the product $N_1 \cos \varphi_1$ in (24), we have already developed in (21) $\cos \varphi_1$. Since we have further in the first half-volume, section 40, p. 63,

$$N_1 = N (1 + (\varphi_1 - \varphi) \eta^2 t + \dots) \quad (25a)$$

then we obtain, if we introduce, at the same time, the value of $\varphi_1 - \varphi$ from (23):

$$N_1 \cos \varphi_1 = N \cos \varphi - \frac{1}{2} N l^2 \sin \varphi \cos^2 \varphi t - \frac{1}{24} N l^4 \sin^2 \varphi \cos^3 \varphi (8 - t^2). \quad (26)$$

Now the product of (26) and (25) is still to be formed, as well as the factor l to be inserted. The calculation of this product yields:

$$y = l N \cos \varphi \left\{ 1 - \frac{1}{2} l^2 \sin \varphi \cos \varphi t - \frac{1}{24} l^4 \sin^2 \varphi \cos^2 \varphi (8 - t^2) + \frac{1}{3} l^2 \sin \varphi \cos \varphi t - \frac{1}{6} l^4 \sin^2 \varphi \cos^2 \varphi t^2 + \frac{1}{15} l^4 \sin^2 \varphi \cos^2 \varphi (4 + 2 t^2) \right\},$$

or after combining the terms of the same kind and inserting the factor ρ :

$$y = \frac{l}{\rho} N \cos \varphi - \frac{1}{6} N \frac{l^3}{\rho^3} \sin^2 \varphi \cos \varphi - \frac{1}{120} N \frac{l^5}{\rho^5} \sin^2 \varphi \cos^3 \varphi (8 - t^2). \quad (27)$$

Likewise, we have still to invert formula (6) for the meridian convergence now, where y can be replaced by the foregoing value (27). For the second and third terms in (6) we have:

$$y^3 = N^3 l^3 \cos^3 \varphi - \frac{1}{2} N^3 l^5 \sin^2 \varphi \cos^3 \varphi$$

$$y^5 = N^5 l^5 \cos^5 \varphi.$$

According to the first half-volume, section 40, p. 63, we have

$$\frac{1}{N_1} = \frac{1}{N} (1 - (\varphi_1 - \varphi) \eta^2 t + \dots),$$

or with the help of (23)

$$\frac{1}{N_1} = \frac{1}{N} \left(1 - \frac{1}{2} l^2 \eta^2 \sin^2 \varphi \right).$$

Finally, we have according to the first half-volume, section 40, p. 63:

$$t_1 = t + (\varphi_1 - \varphi) (1 + t^2) + (\varphi_1 - \varphi)^2 t (1 + t^2) + \dots,$$

and after introducing the value of $\varphi_1 - \varphi$ from (23), this transforms to

$$t_1 = t + \frac{1}{2} l^2 \sin \varphi \cos \varphi (1 + t^2 + \eta^2 + \eta^2 t^2) + \frac{1}{24} l^4 \sin \varphi \cos^3 \varphi (5 + 10 t^2 + 5 t^4).$$

All this is to be set into equation (6), and we find after some collecting:

$$\gamma = l \sin \varphi + \frac{1}{3} \frac{l^3}{\rho^2} \sin \varphi \cos^2 \varphi (1 + \eta^2) + \frac{1}{15} \frac{l^5}{\rho^4} \sin \varphi \cos^4 \varphi (2 - t^2). \quad (28)$$

The abscissa x is the meridian arc between the latitudes φ_0 and φ_1 , for which we have already developed a convenient formula in the first half-volume, section 41, p. 74, equation (40). According to this we have, if terms of the sixth order are neglected:

$$x = \frac{M}{\rho} (\varphi_1 - \varphi_0) + \frac{M}{8 \rho^3} (\varphi_1 - \varphi_0)^3 \eta^2 (1 - t^2). \quad (29)$$

We also can disregard at first the assumption of a definite zero point on the zero meridian and count the abscissae from the equator. If we denote by B the meridian arc from the equator to the latitude φ , then we have according to the first half-volume, section 41, p. 74, equation (40), and the foregoing equation (23), if the relation $M V^2 = N$ is taken into account,

$$x = B + \frac{1}{2} N \frac{l^2}{\rho^2} \sin \varphi \cos \varphi + \frac{1}{24} N \frac{l^4}{\rho^4} \sin \varphi \cos^3 \varphi (5 - t^2) + \dots \quad (29a)$$

For the computation of the rectangular coordinates x , y and the meridian convergence γ from φ and φ_0 we thus have the following group of formulae:

Given φ , l and φ_0
Required x , y and γ .

$$\varphi_1 - \varphi = \frac{1}{2} \frac{V^2}{\rho} l^2 \sin \varphi \cos \varphi + \frac{1}{24} \frac{l^4}{\rho^3} \sin \varphi \cos^3 \varphi (5 - t^2) + \dots \quad (30)$$

$$x = \frac{M}{\rho} (\varphi_1 - \varphi_0) + \frac{M}{8 \rho^3} (\varphi_1 - \varphi_0)^3 \eta^2 (1 - t^2) \quad (31)$$

$$y = \frac{l}{\rho} N \cos \varphi - \frac{1}{6} N \frac{l^3}{\rho^3} \sin^2 \varphi \cos \varphi - \frac{1}{120} N \frac{l^5}{\rho^5} \sin^2 \varphi \cos^3 \varphi (8 - t^2) \quad (32)$$

$$\gamma = l \sin \varphi + \frac{1}{3} \frac{l^3}{\rho^2} \sin \varphi \cos^2 \varphi (1 + \eta^2) + \frac{1}{15} \frac{l^5}{\rho^4} \sin \varphi \cos^4 \varphi (2 - t^2). \quad (33)$$

We will bring also these formulae into logarithmic form for the numerical computation. But since the necessary development has already been presented in detail for equations (14) to (17) on p. 128, the result can be communicated here at once.

Given φ , l and φ_0
Required x , y and γ .

$$\log (\varphi_1 - \varphi) = \log \frac{V^2}{2 \rho} l^2 \sin \varphi \cos \varphi + \frac{\mu}{12 \rho^2} l^2 \cos^2 \varphi (5 - t^2) \quad (34)$$

$$\log x = \log \frac{M}{\rho} (\varphi_1 - \varphi_0) - \frac{\mu}{8 \rho^2} \eta^2 (t^2 - 1) (\varphi_1 - \varphi_0)^2 \quad (35)$$

$$\log y = \log \frac{N}{\rho} l \cos \varphi - \frac{\mu}{6 \rho^2} l^2 \sin^2 \varphi - \frac{\mu}{180 \rho^4} l^4 \sin^2 \varphi \cos^2 \varphi (12 + t^2) \quad (36)$$

$$\log \gamma = \log l \sin \varphi + \frac{\mu}{3 \rho^2} l^2 \cos^2 \varphi (1 + \eta^2) + \frac{\mu}{90 \rho^4} l^4 \cos^4 \varphi (7 - 6 t^2). \quad (37)$$

In the equations for $\varphi_1 - \varphi$, for y and for γ all auxiliary quantities are to be computed for the latitude φ in both systems; in the equations for x , however, the quantities M , η^2 and t^2 are valid for the mean latitude $\frac{\varphi_0 + \varphi_1}{2}$.

The values of $\log \frac{\rho}{M} = \log [1]$, $\log \frac{\rho}{N} = \log [2]$ and $\log V^2$ are contained in the Appendix of the first half-volume, pp. [12] to [39], whereas the values of η^2 , t^2 and t^4 can be taken from pp. [1] to [4] of this half-volume. In addition, we insert here the important numerical values

$$\log \frac{1}{\rho} = 4.685\,5748\cdot7 \quad \log \mu = 9.63778 - 10$$

for the computation of the coefficients.

Numerical example

For the application of the foregoing formulae we use the small standard example (1) from section 17, p. 73, whose elements are grouped on pp. 73 and 74 on the basis of exact computation.

Given $\varphi = 50^\circ 30' 00''$ $l = 1^\circ 00' 00'' = 3600''$ $\varphi_0 = 49^\circ 30' 00''$.

Required x , y and γ .

1. Computation of $\varphi_1 - \varphi$ according to (34), p. 132.

Argument: $\varphi = 50^\circ 30' 00''$

$$\begin{array}{rcl} \log \frac{V^2}{2e} = 4.385\,7239\cdot3 & \log \frac{\mu}{12e^2} = 4.92975 & \\ 5 - t^2 = 3.5284 & & \\ \begin{array}{r|l} \log \frac{V^2}{2e} & 4.385\,7239\cdot3 \\ \log l^2 & 7.112\,6050\cdot0 \\ \log \sin \varphi & 9.887\,4060\cdot6 \\ \log \cos \varphi & 9.803\,5105\cdot3 \\ \hline & 1.189\,2455\cdot2 \\ \text{Corr.} & + 157\cdot3 \\ \hline \log (\varphi_1 - \varphi) & 1.189\,2612\cdot5 \end{array} & \begin{array}{r|l} \log \frac{\mu}{12e^2} & 4.92975 \\ \log l^2 & 7.11260 \\ \log \cos^2 \varphi & 9.60702 \\ \log (5 - t^2) & 0.54758 \\ \hline & 2.19695 \\ & + 157\cdot3 \\ \hline & 2.19852 \end{array} & \\ \varphi_1 - \varphi = 15.4618'' & \varphi_1 = 50^\circ 30' 15.4618'' & \end{array}$$

2. Computation of x according to (35), p. 132.

Argument: $\frac{\varphi_0 + \varphi_1}{2} = 50^\circ 00' 07.7309'' = 50^\circ 00.12\,885'$

$$\begin{array}{rcl} \log \frac{M}{e} = 1.489\,8666\cdot3 & \log \frac{\mu}{8e^2} = 5.10584 & \\ t^2 - 1 = 0.4205 & \eta^2 = 0.00278 & \\ \varphi_1 - \varphi_0 = 1^\circ 00' 15.4618'' = 3615.4618'' & & \\ \begin{array}{r|l} \log \frac{M}{e} & 1.489\,8666\cdot3 \\ \log (\varphi_1 - \varphi_0) & 3.558\,1637\cdot8 \\ \hline & 5.048\,0304\cdot1 \\ \text{Corr.} & - 0.2 \\ \hline \log x & 5.048\,0303\cdot9 \end{array} & \begin{array}{r|l} \log \frac{\mu}{8e^2} & 5.10584 \\ \log \eta^2 & 7.44404 \\ \log (t^2 - 1) & 9.62377 \\ \log (\varphi_1 - \varphi_0)^2 & 7.11633 \\ \hline & 9.28998 \end{array} & \\ & 0.2 & \end{array}$$

$$x = +111,694.141 \text{ m.}$$

3. Computation of y according to (36), p. 132.

Argument: $\varphi = 50^\circ 30' 00''$

$\log \frac{N}{e} = 1.491\ 0829.8$	$\log \frac{\mu}{6e^2} = 5.23\ 078$	$\log \frac{\mu}{180e^4} = 3.1248$
	$12 + t^2 = 13.472$	
$\log \frac{N}{e}$	$\log \frac{\mu}{6e^2}$	$\log \frac{\mu}{45e^4}$
1.491 0829.8	5.23078	3.1248
$\log l$	$\log l^2$	$\log l^4$
3.556 3025.0	7.11260	4.2252
$\log \cos \varphi$	$\log \sin^2 \varphi$	$\log \sin^2 \varphi$
9.803 5105.3	9.77481	9.7748
		$\log \cos^2 \varphi$
		9.6070
		$\log (12 + t^2)$
		1.1294
1. Corr.	131.3	
2. Corr.		7.8612
$\log y$		0.0
4.850 8828.8		

$y = + 70,938.644\text{ m.}$

4. Computation of γ according to (37), p. 132.

Argument: $\varphi = 50^\circ 30' 00''$

$\log \frac{\mu}{3e^2} = 5.53\ 181$	$\log \frac{\mu}{90e^4} = 3.4258$	
$1 + \eta^2 = 1.00272$	$7 - 6t^2 = -1.834$	
$\log l$	$\log \frac{\mu}{3e^2}$	$\log \frac{\mu}{90e^4}$
3.556 3025.0	5.53 181	3.4258
$\log \sin \varphi$	$\log l^2 \cos^2 \varphi$	$\log l^4 \cos^4 \varphi$
9.887 4060.6	6.71 962	3.4392
	$\log (1 + \eta^2)$	$\log (7 - 6t^2)$
	0.00 118	0.2634 _n
1. Corr.	2.25 261	7.1184 _n
2. Corr.		- 0.0
$\log \gamma$	178.9	
3.443 7264.5		

$\gamma = 2777.9630'' = 46' 17.9630''.$

We thus obtain as result of the computation:

$x = + 111,694.141\text{ m} \quad y = 46' 17.9630''$

$y = + 70,938.644\text{ m.}$

We insert here, in addition, the inverse problem, where we use the system of formulae (14) to (17) of p. 128.

Given $x = + 111,694.141\text{ m} \quad y = + 70,938.644\text{ m}$

$\varphi_0 = 49^\circ 30' 00''$

Required φ , l and γ .

1. Computation of $\varphi_1 - \varphi_0$ according to (14), p. 128.

Argument: $\varphi_0 = 49^\circ 30' 00''$

$\log \frac{e}{N} V^2 = 8.510\ 1711.0$	$\log \frac{3\mu}{2N} = 0.00\ 839$
$\log \eta^2 = 7.45\ 240$	$\log \frac{\mu}{2N^2} = 2.72\ 58$
$t^2 - 1 = 0.37\ 088$	

$\log \frac{\rho}{N} V^2$	8.510 1711.0	$\log \frac{3\mu}{2N}$	0.00 839	$\log \frac{\mu}{2N^2}$	2.7258
$\log x$	5.048 0303.9	$\log x$	5.04 803	$\log x^2$	0.0961
	3.558 2014.9	$\log \eta^2$	7.45 240	$\log \eta^2$	7.4524
1. Corr.	— 377.8	$\log t$	0.06 850	$\log (t^2 - 1)$	9.5692
2. Corr.	+ 0.7		2.57 732		9.8435
$\log (\varphi_1 - \varphi_0)$	3.558 1637.8	377.8			0.7
$\varphi_1 - \varphi_0 = 3615.4618''$					
$\varphi_1 - \varphi_0 = 1^\circ 00' 15.4618''$					
$\varphi_0 = 49^\circ 30' 00''$					
$\varphi_1 = 50^\circ 30' 15.4618''$					

2. Computation of $\varphi_1 - \varphi$ according to (15), p. 128.

Argument: $\varphi_1 = 50^\circ 30' 15.4618''$

$\log \frac{V^2 \rho}{2N^2} = 1.403\ 5575.2$	$\log \frac{\mu}{12N^2} = 1.94\ 758$
	$1 + 3t^2 = 5.41\ 630$
$\log \frac{V^2 \rho}{2N^2}$	$\log \frac{\mu}{12N^2}$
$\log y^2$	$\log y^2$
$\log t$	$\log (1 + 3t^2)$
	2.38 305
Corr.	241.6
$\log (\varphi_1 - \varphi)$	
$\varphi_1 - \varphi = 15.4618''$	
$\varphi_1 = 50^\circ 30' 15.4618''$	
$\varphi_1 = 50^\circ 30' 00''$	

3. Computation of l according to (16), p. 128.

Argument: $\varphi_1 = 50^\circ 30' 15.4618''$

$\log \frac{\rho}{N} = 8.508\ 9169.1$	$\log \frac{\mu}{3N^2} = 2.54964$
$6 + 13t^2 = 25.1373$	$\log \frac{\mu}{90N^4} = 7.46150$
$\log \frac{\rho}{N}$	$\log \frac{\mu}{2N^2}$
$\log y$	$\log y^2$
$\log 1 : \cos \varphi$	$\log t^2$
	2.41 933
1. Corr.	262.6
2. Corr.	
$\log l$	
$l = 3600.0000'' = 1^\circ 00' 00''$	

4. Computation of γ according to (17), p. 128.

Argument: $\varphi_1 = 50^\circ 30' 15.4618''$

$\log \frac{\rho}{N} = 8.508\ 9169.1$	$\log \frac{\mu}{6N^2} = 2.24\ 861$
$1 + 2t^2 + \eta^2 = 3.9469$	$\log \frac{\mu}{180N^4} = 7.1605$
$1 - 20t^2 - 26t^4 = -84.7802$	

$\log \frac{e}{N}$	8.508 9169.1	$\log \frac{\mu}{6 N^2}$	2.24 861	$\log \frac{\mu}{180 N^4}$	7.1605
$\log y$	4.850 8828.8	$\log y^2$	9.70 177	$\log y^4$	9.4035
$\log t$	0.083 9618.6	$\log (1+2t^2+\eta^2)$	0.59 626	$\log (1-20t^2-26t^4)$	1.9283 _n
	3.443 7616.5		2.54 664		8.4923 _n
1. Corr.	— 352.1	352.1		— 0.0	
3. Corr.	+ 0.0				
$\log \gamma$	3.443 7264.4				

$$\gamma = 2777.9629'' = 46' 17.9629''.$$

We also summarize the result of this second computation and have accordingly:

$$\varphi = 50^\circ 30' 00.0000''$$

$$l = 1 \ 00 \ 00.0000$$

$$\gamma = 46 \ 17.9629 ,$$

which agrees well with the starting values, or the first computation.

Andrae's formulae for the transfer of geographic coordinates

By means of the foregoing formulae for the spheroidal rectangular coordinates we also can indicate the transfer of geographic coordinates according to the method of Andrae, already referred to in the notes of section 18, p. 81.

Let PN and $P'N'$ denote the meridians of the two points P and P' . If we assume P as the zero point of a rectangular spheroidal system of coordinates, in which x and y are the coordinates of the point P' , then we have at first, according to the first half-volume, (26), section 52, p. 122:

$$\left. \begin{aligned} y &= s \sin (\alpha - \varepsilon) \\ x &= s \cos (\alpha - 2 \varepsilon), \end{aligned} \right\} \quad (38)$$

if 3ε denotes the excess of the triangle PP_1P' . Further we have

$$3\varepsilon = \frac{xy}{2MN} \rho$$

or to sufficient accuracy:

$$\varepsilon = \frac{s^2 \rho}{6MN} \sin \alpha \cos \alpha. \quad (39)$$

For the computation of φ_1 and φ there suffices the simple equation (6) in the first half-volume, section 56, p. 139:

$$\varphi_1 = \varphi + \frac{x}{M} \rho$$

or

$$\varphi_1 = \varphi + \frac{s}{M} \rho \cos (\alpha - 2 \varepsilon), \quad (40)$$

where M holds good for the mean latitude $\frac{\varphi + \varphi_1}{2}$ and, therefore, makes a first approximate computation of φ_1 necessary.

The computation of φ' from φ_1 is carried out with the help of equation (15), p. 128, in which we set the value of y from (38). Thus we have:

$$\log(\varphi_1 - \varphi') = \log \frac{V_1^2 \varrho}{2 N_1^2} s^2 \sin^2(\alpha - \varepsilon) \tan \varphi_1 - \frac{\mu}{12 N_1^2} s^2 \sin^2(\alpha - \varepsilon) - \frac{\mu}{4 N_1^2} s^2 \sin^2(\alpha - \varepsilon) \tan^2 \varphi_1. \quad (41)$$

For $\log l$ we have equation (16), p. 128, of which the first two terms are sufficient, however. With (38) we will then have:

$$\log l = \log \frac{s \varrho}{N_1 \cos \varphi_1} \sin(\alpha - \varepsilon) - \frac{\mu}{3 N_1^2} s^2 \sin^2(\alpha - \varepsilon) \tan^2 \varphi_1. \quad (42)$$

For the computation of $\log \gamma$, in (17), p. 128, we can omit, likewise, the last term completely and η_1^2 in the second term, so that we obtain with (38):

$$\log \gamma = \log \frac{s}{N_1} \varrho \sin(\alpha - \varepsilon) \tan \varphi_1 - \frac{\mu}{6 N_1^2} s^2 \sin^2(\alpha - \varepsilon) - \frac{\mu}{3 N_1^2} s^2 \sin^2(\alpha - \varepsilon) \tan^2 \varphi_1. \quad (43)$$

In order to find the azimuth α' with the help of the meridian convergence γ , we are to bear in mind that in the triangle PP_1P' the angle at P' is equal to $180^\circ + 3\varepsilon - 90^\circ - \alpha$. Consequently, we will have:

$$180^\circ + 3\varepsilon - 90^\circ - \alpha + 90^\circ + \alpha' - \gamma = 180^\circ$$

or

$$\alpha' = \alpha - 3\varepsilon + \gamma. \quad (44)$$

Now we group the formulae found once more, introducing at the same time the coefficients $[1] = \frac{\rho}{M}$

and $[2] = \frac{\rho}{N}$.

Given φ , s and α . Required φ' , l , γ and α' .

$$\left. \begin{aligned} \varepsilon &= \frac{[1][2]}{6 \varrho} s^2 \sin \alpha \cos \alpha \\ \varphi_1 &= \varphi + [1] s \cos(\alpha - 2\varepsilon) \\ \log(\varphi_1 - \varphi') &= \log \frac{[1][2]}{2 \varrho} s^2 \sin^2(\alpha - \varepsilon) \tan \varphi_1 - \frac{\mu [2]^2}{12 \varrho^2} s^2 \sin^2(\alpha - \varepsilon) - \frac{\mu [2]^2}{4 \varrho^2} s^2 \sin^2(\alpha - \varepsilon) \tan^2 \varphi_1 \\ \log l &= \log [2] \frac{s \sin(\alpha - \varepsilon)}{\cos \varphi_1} - \frac{\mu [2]^2}{3 \varrho^2} s^2 \sin^2(\alpha - \varepsilon) \tan^2 \varphi_1 \\ \log \gamma &= \log [2] s \sin(\alpha - \varepsilon) \tan \varphi_1 - \frac{\mu [2]^2}{6 \varrho^2} s^2 \sin^2(\alpha - \varepsilon) - \frac{\mu [2]^2}{3 \varrho^2} s^2 \sin^2(\alpha - \varepsilon) \tan^2 \varphi_1 \\ \alpha' &= \alpha - 3\varepsilon + \gamma. \end{aligned} \right\} \quad (45)$$

The coefficients $[1]$ and $[2]$ are to be used in the formulae for ε and φ_1 with the argument $\frac{\varphi + \varphi_1}{2}$, in the remaining ones with the argument φ_1 .

The foregoing formulae were developed by Andrae in *Problèmes de haute géodésie. Extraits de l'ouvrage danois: Den danske Gradmaaling. 2^e cahier: Calcul des latitudes, des longitudes et des azimuts sur le sphéroïde*, Copenhagen, 1882, p. 7. The same formulae are also found in Zachariae, *Die geodätischen Hauptpunkte und ihre Koordinaten*, Deutsch von E. Lamp, Berlin, 1878, p. 198.

Section 28. Power Series for Rectangular and Geographic Coordinates

Now we will utilize the developments of the foregoing section in still another way, namely for setting up formulae which give φ , l and γ only as an algebraic function of the variables x and y , while all coefficients depend only on the latitude of the zero point. In the case of large abscissae, these formulae are not directly useful; we shall however show later how we can arrive at a very favorable form of computation by shifting the zero point on its meridian.

We start with the formulae (2) and (4) to (6), section 27, p. 127; in the first, the argument φ_0 occurs already everywhere, whereas in the latter the latitude of the foot-point φ_1 occurs. Therefore, t_1 , N_1 and η_1 are to be expressed by t_0 , N_0 and η_0 .

For $t_1 = \tan \varphi_1$ we have according to the first half-volume, section 34, p. 18,

$$t_1 = t_0 + (\varphi_1 - \varphi_0)(1 + t_0^2) + (\varphi_1 - \varphi_0)^2 t_0(1 + t_0^2) + \frac{1}{3}(\varphi_1 - \varphi_0)^3(1 + 4t_0^2 + 3t_0^4) + \dots$$

For $\varphi_1 - \varphi_0$ we can substitute the value (2), section 27, p. 127, where we must take into account that $V_0^2 = 1 + \eta_0^2$. Then there follows:

$$\begin{aligned} t_1 = t_0 + \frac{x}{N_0}(1 + t_0^2 + \eta_0^2 + \eta_0^2 t_0^2) + \frac{x^2}{2N_0^2}t_0(2 + 2t_0^2 + \eta_0^2 + \eta_0^2 t_0^2) \\ + \frac{x^3}{3N_0^3}(1 + 4t_0^2 + 3t_0^4) + \dots \end{aligned} \quad (1)$$

We have further according to the first half-volume, section 40, p. 63, equation (o):

$$\frac{N_0}{N_1} = 1 - \frac{\varphi_1 - \varphi_0}{V_0^2} \eta_0^2 t_0 - \frac{(\varphi_1 - \varphi_0)^2}{2} \eta_0^2 (1 - t_0^2) + \frac{2(\varphi_1 - \varphi_0)^3}{3} \eta_0^2 t_0 + \dots$$

If we introduce here likewise (2), section 27, p. 127, then we obtain:

$$\frac{N_0}{N_1} = 1 - \frac{x}{N_0} \eta_0^2 t_0 - \frac{x^2}{2N_0^2} \eta_0^2 (1 - t_0^2) + \frac{2x^3}{3N_0^3} \eta_0^2 t_0 + \dots \quad (2)$$

As the last preparation for the transformation of equation (4), section 27, p. 127, we form from (1) and

(2) the products $\frac{t_1}{N_1^2}$ and $\frac{t_1}{N_1^4}$ and find for them:

$$\begin{aligned} \frac{t_1}{N_1^2} = \frac{1}{N_0^2} \left\{ t_0 + \frac{x}{N_0}(1 + t_0^2 + \eta_0^2 - \eta_0^2 t_0^2) + \frac{x^2}{N_0^2} t_0(1 + t_0^2) \right. \\ \left. + \frac{x^3}{3N_0^3}(1 + 4t_0^2 + 3t_0^4) + \dots \right\} \end{aligned} \quad (3)$$

$$\frac{t_1}{N_1^4} = \frac{1}{N_0^4} \left\{ t_0 + \frac{x}{N_0}(1 + t_0^2) + \dots \right\}. \quad (4)$$

With the equations (1), (3) and (4) the former equation (4), section 27, p. 127, passes over into:

$$\begin{aligned} \frac{\varphi - \varphi_1}{V_1^2} = & -\frac{y^2}{2N_0^2}t_0 - \frac{y^2x}{2N_0^3}(1+t_0^2+\eta_0^2-\eta_0^2t_0^2) + \frac{y^4}{24N_0^4}t_0(1+3t_0^2) \\ & - \frac{y^2x^2}{2N_0^4}t_0(1+t_0^2) - \frac{y^2x^3}{6N_0^5}(1+4t_0^2+3t_0^4) + \frac{y^4x}{24N_0^5}(1+10t_0^2+9t_0^4) + \dots \end{aligned} \quad (5)$$

Now, in addition, on the left-hand side of the foregoing equation (5) V_1^2 is to be replaced by V_0^2 . But since on the right-hand side only terms of the second and of higher order occur, then we can limit ourselves in V_1^2 to terms of third order. According to the first half-volume, section 40, p. 63, equation (o), we have

$$V_1^2 = V_0^2 \frac{N_0^2}{N_1^2} = V_0^2 \left(1 - 2 \frac{x}{N_0} \eta_0^2 t_0\right), \quad (6)$$

and with this, we will have then:

$$\begin{aligned} \frac{\varphi - \varphi_1}{V_0^2} = & -\frac{y^2}{2N_0^2}t_0 - \frac{y^2x}{2N_0^3}(1+t_0^2+\eta_0^2-3\eta_0^2t_0^2) + \frac{y^4}{24N_0^4}t_0(1+3t_0^2) \\ & - \frac{y^2x^2}{2N_0^4}t_0(1+t_0^2) - \frac{y^2x^3}{6N_0^5}(1+4t_0^2+3t_0^4) + \frac{y^4x}{24N_0^5}(1+10t_0^2+9t_0^4) + \dots \end{aligned} \quad (7)$$

To this we can add immediately equation (2), section 27, p. 127, and obtain then the difference of latitude $\varphi - \varphi_0$. We will have:

$$\begin{aligned} \frac{\varphi - \varphi_0}{V_0^2} = & \frac{x}{N_0} - \frac{y^2}{2N_0^2}t_0 - \frac{3x^2}{2N_0^2}\eta_0^2t_0 + \frac{x^3}{2N_0^3}\eta_0^2(t_0^2-1) - \frac{y^2x}{2N_0^3}(1+t_0^2+\eta_0^2-3\eta_0^2t_0^2) \\ & + \frac{y^4}{24N_0^4}t_0(1+3t_0^2) - \frac{y^2x^2}{2N_0^4}t_0(1+t_0^2) - \frac{y^2x^3}{6N_0^5}(1+4t_0^2+3t_0^4) \\ & + \frac{y^4x}{24N_0^5}(1+10t_0^2+9t_0^4). \end{aligned} \quad (8)$$

For the numerical computation, we have to insert, in addition, the factor ρ on the right-hand side everywhere. If we still take into account at the same time that

$$V_0^2 = \frac{N_0}{M_0} \quad \text{and} \quad M_0 N_0 = r^2$$

then we will have

$$\begin{aligned} \varphi - \varphi_0 = & \frac{x}{M_0} \rho - \frac{y^2 \rho}{2r_0^2} t_0 - \frac{3}{2} \frac{x^2 \rho}{r_0^2} \eta_0^2 t_0 + \frac{x^3 \rho}{2r_0^2 N_0} \eta_0^2 (t_0^2 - 1) \\ & - \frac{y^2 x \rho}{2r_0^2 N_0} (1 + t_0^2 + \eta_0^2 - 3\eta_0^2 t_0^2) + \frac{y^4 \rho}{24r_0^2 N_0^2} t_0 (1 + 3t_0^2) - \frac{y^2 x^2 \rho}{2r_0^2 N_0^2} t_0 (1 + t_0^2). \end{aligned} \quad (9)$$

In equation (5), section 27, p. 127, which we will also reduce to the argument φ_0 , there occur likewise the quantities t_1 and N_1 , which we can replace by t_0 and N_0 with the help of (1) and (2). The result of this substitution is:

$$l \cos \varphi_1 = \frac{y}{N_0} - \frac{yx}{N_0^2} \eta_0^2 t_0 + \frac{yx^2}{2N_0^3} \eta_0^2 (t_0^2 - 1) - \frac{y^3}{3N_0^3} t_0^2 + \frac{2yx^3}{3N_0^4} \eta_0^2 t_0 - \frac{2y^3 x}{3N_0^4} t_0 (1 + t_0^2). \quad (10)$$

In order to obtain therefrom l , we have to multiply all terms on the right-hand side by the factor $\frac{1}{\cos \varphi_1}$, which we also transform beforehand. According to the first half-volume, section 34, p. 18, we have

$$\frac{1}{\cos \varphi_1} = \frac{1}{\cos \varphi_0} \left\{ 1 + (\varphi_1 - \varphi_0) t_0 + \frac{1}{2} (\varphi_1 - \varphi_0)^2 (1 + 2 t_0^2) + \frac{1}{6} (\varphi_1 - \varphi_0)^3 t_0 (5 + 6 t_0^2) \right\}.$$

If we introduce here the value of $(\varphi_1 - \varphi_0)$ from (2), section 27, p. 127, and set, at the same time, $V_0^2 = 1 + \eta_0^2$, then we obtain:

$$\frac{1}{\cos \varphi_1} = \frac{1}{\cos \varphi_0} \left\{ 1 + \frac{x}{N_0} t_0 (1 + \eta_0^2) + \frac{x^2}{2 N_0^2} (1 + 2 \eta_0^2 + 2 t_0^2 + \eta_0^2 t_0^2) + \frac{y^3}{6 N_0^3} t_0 (5 + 6 t_0^2) \right\}, \quad (11)$$

This yields in equation (10), if ρ is inserted at the same time,

$$l = \frac{\rho}{\cos \varphi_0} \left\{ \frac{y}{N_0} + \frac{yx}{N_0^2} t_0 + \frac{y x^2}{2 N_0^3} (1 + \eta_0^2 + 2 t_0^2) - \frac{y^3}{3 N_0^3} t_0^2 + \frac{y x^3}{6 N_0^4} t_0 (5 + 6 t_0^2) - \frac{y^3 x}{3 N_0^4} t_0 (2 + 3 t_0^2) \right\}. \quad (12)$$

Now equation (6), section 27, p. 127, is still to be transformed, for which we use again equations (1) and (2). For η_1^2 occurring in the second term of (6), section 27, p. 127, η_0^2 can immediately be substituted, since it belongs to a term of fifth order. We will not carry out the details of the conversion, which does not present difficulties, and give at once the result:

$$\begin{aligned} \gamma = \frac{y \rho}{N_0} t_0 + \frac{y x \rho}{N_0^2} (1 + \eta_0^2 + t_0^2) - \frac{y^3 \rho}{6 N_0^3} t_0 (1 + \eta_0^2 + 2 t_0^2) + \frac{y x^2 \rho}{N_0^3} t_0 (1 - \eta_0^2 + t_0^2) \\ + \frac{y x^3 \rho}{3 N_0^4} (1 + 4 t_0^2 + 3 t_0^4) - \frac{y^3 x \rho}{6 N_0^4} (1 + 7 t_0^2 + 6 t_0^4). \end{aligned} \quad (13)$$

Equations (9), (12) and (13) make possible the computation of the geographic coordinates φ and l and of the meridian convergence γ from the rectangular coordinates y and x if the coefficients of the equations are computed for the latitude of the zero point φ_0 . Corresponding formulae can also be set up for the inverse problem, the computation of the rectangular coordinates from the geographic coordinates. We arrive at these formulae in the easiest way if we solve the equations (8) and (12) for x and y by successive approximation.

For this, we obtain as a first approximation:

$$\frac{x}{N_0} = \frac{\varphi - \varphi_0}{V_0^2} + \dots \quad \frac{y}{N_0} = l \cos \varphi_0 + \dots$$

Consequently, we will have

$$\begin{aligned} \frac{y^2}{N_0^2} &= l^2 \cos^2 \varphi_0 + \dots & \frac{x^2}{N_0^2} &= \frac{(\varphi - \varphi_0)^2}{V_0^4} + \dots \\ \frac{xy}{N_0^2} &= \frac{l(\varphi - \varphi_0)}{V_0^2} \cos \varphi_0 + \dots \end{aligned}$$

With this, there follows the second approximation:

$$\begin{aligned} \frac{x}{N_0} &= \frac{\varphi - \varphi_0}{V_0^2} + \frac{1}{2} l^2 \cos^2 \varphi_0 t_0 + \frac{3}{2} \frac{(\varphi - \varphi_0)^2}{V_0^4} \eta_0^2 t_0 + \dots \\ \frac{y}{N_0} &= l \cos \varphi_0 - \frac{l(\varphi - \varphi_0)}{V_0^2} \cos \varphi_0 t_0 + \dots \end{aligned}$$

With these values we can compute the terms of second and third order in (8) and (12) and obtain then as a third approximation:

$$\begin{aligned}\frac{x}{N_0} &= \frac{\varphi - \varphi_0}{V_0^2} + \frac{1}{2} l^2 \cos^2 \varphi_0 t_0 + \frac{3}{2} \frac{(\varphi - \varphi_0)^2}{V_0^4} \eta_0^2 t_0 + \frac{1}{2} \frac{\varphi - \varphi_0}{V_0^2} l^2 \cos^2 \varphi_0 (1 - t_0^2 + \eta_0^2) \\ &\quad - \frac{1}{2} \frac{(\varphi - \varphi_0)^3}{V_0^6} \eta_0^2 (t_0^2 - 1) + \dots \\ \frac{y}{N_0} &= l \cos \varphi_0 - \frac{\varphi - \varphi_0}{V_0^2} l \cos \varphi_0 t_0 - \frac{1}{6} l^3 \cos^3 \varphi_0 t_0^2 \\ &\quad - \frac{1}{2} \frac{(\varphi - \varphi_0)^2}{V_0^4} l \cos \varphi_0 (1 + \eta_0^2 + 3 \eta_0^2 t_0^2) + \dots\end{aligned}$$

If we continue this procedure until we have found also the terms of the fourth order, then we obtain, finally, the following two equations, in which we set $\varphi - \varphi_0 = \Delta \varphi$ for the sake of simplicity:

$$\begin{aligned}x &= \frac{N_0}{V_0^2 \varrho} \Delta \varphi + \frac{N_0}{2 \varrho^2} l^2 \cos^2 \varphi_0 t_0 + \frac{3 N_0}{2 V_0^4 \varrho^2} \Delta \varphi^2 \eta_0^2 t_0 - \frac{N_0}{2 \varrho^3} \Delta \varphi^3 \eta_0^2 (t_0^2 - 1) \\ &\quad - \frac{N_0}{2 V_0^2 \varrho^3} \Delta \varphi l^2 \cos^2 \varphi_0 (t_0^2 - \eta_0^2 - 1) - \frac{N_0}{\varrho^4} \Delta \varphi^2 l^2 \cos^2 \varphi_0 t_0 \\ &\quad + \frac{N_0}{24 \varrho^4} l^4 \cos^4 \varphi_0 t_0 (5 - t_0^2) + \dots\end{aligned}\quad (14)$$

$$\begin{aligned}y &= \frac{N_0}{\varrho} l \cos \varphi_0 - \frac{N_0}{\varrho^2 V_0^2} \Delta \varphi l \cos \varphi_0 t_0 - \frac{N_0}{6 \varrho^3} l^3 \cos^3 \varphi_0 t_0^2 - \frac{N_0}{2 \varrho^3 V_0^4} \Delta \varphi^2 l \cos \varphi_0 (1 + \eta_0^2 + 3 \eta_0^2 t_0^2) \\ &\quad - \frac{N_0}{6 \varrho^4} \Delta \varphi l^3 \cos^3 \varphi_0 t_0 (2 - t_0^2) + \frac{N_0}{6 \varrho^4} \Delta \varphi^3 l \cos \varphi_0 t_0 + \dots\end{aligned}\quad (15)$$

The expression for the meridian convergence γ can be found at once with the help of equations (14) and (15), if we set these values of x and y into equation (13). Then we obtain:

$$\begin{aligned}\gamma &= l \sin \varphi_0 + \frac{\Delta \varphi l}{\varrho} \cos \varphi_0 + \frac{l^3}{3 \varrho^2} V_0^2 \sin \varphi_0 \cos^2 \varphi_0 - \frac{\Delta \varphi^2 l}{2 \varrho^2} \sin \varphi_0 \\ &\quad - \frac{\Delta \varphi l^3}{3 \varrho^3} \cos^3 \varphi_0 (2 t_0^2 - 1) - \frac{\Delta \varphi^3 l}{6 \varrho^3} \cos \varphi_0 + \dots\end{aligned}\quad (16)$$

In the case of the relatively small lengths of ordinates of local cadastral systems, the neglected terms of fifth order will ever hardly exceed 0.0001" or, as the case may be, 0.001 m. It is very advantageous not to compute the coefficients of the foregoing power series for the zero point of coordinates, but for a geographic latitude which is not much different from the latitudes of the points to be converted. Then we have to add to all computed abscissae the meridional arc from the auxiliary latitude to the latitude of the zero point.

For this kind of computation, the meridional arcs from the equator to the latitude φ for $\varphi = 45^\circ$ to $\varphi = 57^\circ$ are indicated on pp. [41] to [44] of the first half-volume.

Since an even value can thereby be assumed for the auxiliary latitude, then the coefficients of the power series can be put together for various even values of φ , whereby we arrive then at a very convenient form of computation.

We group once more the above developed equations with a simplified notation for the coefficients:

Computation of geographic coordinates from rectangular coordinates

$$\left. \begin{aligned}\Delta \varphi &= a x - b x^2 - c y^2 - d y^2 x + e x^3 - f y^2 x^2 + g y^4 \\ l &= h y + i y x + k y x^2 - l y^3 - m y^3 x + n y x^3 \\ \gamma &= p y + q y x + r y x^2 - s y^3 - t y^3 x + u y x^3.\end{aligned}\right\}\quad (17)$$

The coefficients have the following meaning, as follows by comparison with the formulae (9), (12) and (13):

$$\begin{aligned}
a &= [1], & b &= \frac{3}{2\varrho} [2]^2 V^2 \eta^2 t, & c &= \frac{[2]^2 t V^2}{2\varrho}, \\
d &= \frac{[2]^3}{2\varrho^2} V^2 (1 + t^2 + \eta^2 - 3\eta^2 t^2), & e &= \frac{[2]^3}{2\varrho^2} V^2 \eta^2 (t^2 - 1), \\
f &= \frac{[2]^4 t V^2}{2\varrho^3 \cos^2 \varphi} = \frac{[2]^4 t V^2}{2\varrho^3} (1 + t^2), & g &= \frac{[2]^4 t V^2 (1 + 3t^2)}{24\varrho^3} \\
h &= \frac{[2]}{\cos \varphi}, & i &= \frac{[2]^2 t}{\varrho \cos \varphi}, & k &= \frac{[2]^3}{2\varrho^2 \cos \varphi} (1 + 2t^2 + \eta^2), & l &= \frac{[2]^3 t^2}{3\varrho^2 \cos \varphi} \\
m &= \frac{[2]^4 t}{3\varrho^2 \cos \varphi} (2 + 3t^2), & n &= \frac{[2]^4 t}{6\varrho^3 \cos \varphi} (5 + 6t^2), \\
p &= [2] t, & q &= \frac{[2]^2}{\varrho} (1 + t^2 + \eta^2), & r &= \frac{[2]^3 t}{\varrho^2} (1 + t^2 - \eta^2), \\
s &= \frac{[2]^3 t}{6\varrho^2} (1 + 2t^2 + \eta^2), & t &= \frac{[2]^4}{6\varrho^3} (1 + 7t^2 + 6t^4), & u &= \frac{[2]^4}{3\varrho^3} (1 + 4t^2 + 3t^4).
\end{aligned}$$

The logarithms of these coefficients have been given for the latitudes 47° to 55° for every $30'$ on pp. [16] to [17] and [19] of the Appendix.

Computation of rectangular coordinates from geographic coordinates

$$\left. \begin{aligned}
x &= A \Delta \varphi + B \Delta \varphi^2 + C l^2 - D \Delta \varphi l^2 - E \Delta \varphi^3 - F \Delta \varphi^2 l^2 + G l^4 \\
y &= H l - J \Delta \varphi l - K \Delta \varphi^2 l - L l^3 - M \Delta \varphi l^3 + N \Delta \varphi^3 l \\
\gamma &= P l + Q \Delta \varphi l - R \Delta \varphi^2 l + S l^3 - T \Delta \varphi l^3 - U \Delta \varphi^3 l.
\end{aligned} \right\} \quad (18)$$

The coefficients have the following meaning, as the comparison with equations (14) to (16) shows:

$$\begin{aligned}
A &= \frac{1}{[1]}, & B &= \frac{3}{2\varrho} \frac{\eta^2 t}{[1] V^2}, & C &= \frac{1}{2\varrho [2]} \sin \varphi \cos \varphi, \\
D &= \frac{\cos^2 \varphi}{2[1]\varrho^2} (t^2 - 1 - \eta^2), & E &= \frac{1}{2[1]\varrho^2} \frac{\eta^2}{V^4} (t^2 - 1 - \eta^2 - 4\eta^2 t^2), \\
F &= \frac{1}{[2]\varrho^3} \sin \varphi \cos \varphi, & G &= \frac{(5 - t^2)}{24\varrho^3 [2]} \sin \varphi \cos^3 \varphi, \\
H &= \frac{\cos \varphi}{[2]}, & J &= \frac{\sin \varphi}{\varrho [1]}, & K &= \frac{\cos \varphi}{2\varrho^2 [1] V^2} (1 + \eta^2 + 3\eta^2 t^2), \\
L &= \frac{1}{6\varrho^2} \frac{\sin^2 \varphi \cos \varphi}{[2]}, & M &= \frac{(2 - t^2)}{6\varrho^3 [2] V^2} \sin \varphi \cos^2 \varphi, & N &= \frac{\sin \varphi}{6\varrho^3 [2]}, \\
P &= \sin \varphi, & Q &= \frac{\cos \varphi}{\varrho}, & R &= \frac{\sin \varphi}{2\varrho^2}, \\
S &= \frac{\sin \varphi \cos^2 \varphi V^2}{3\varrho^2}, & T &= \frac{\cos^3 \varphi}{3\varrho^3} (2t^2 - 1), & U &= \frac{\cos \varphi}{6\varrho^3}.
\end{aligned}$$

The logarithms of these coefficients have been given for the latitudes 47° to 55° for every $30'$ on pp. [14] to [15] and [18] of the Appendix.

In the tables there are contained, in addition, coefficients for conformal coordinates, to which we shall return in the later section 34.

Numerical example

We will apply the foregoing formulae likewise to the small standard example (1) of section 17, p. 73:

$$\varphi_0 = 49^\circ 30' \quad \Delta \varphi = + 1^\circ = 3600'' \quad l = + 1^\circ = 3600''.$$

We assume $\varphi = 50^\circ 00'$ as auxiliary latitude and have the value $\Delta x = 55,605.814$ m for the meridional arc from $\varphi = 49^\circ 30'$ to $\varphi = 50^\circ 00'$ according to p. [42] of the Appendix to the first half-volume. $\Delta \varphi = 1800''$ is then introduced in the formulae (18).

$$\begin{array}{rcl}
 + A \Delta \varphi & = + & 55,608.2177 \text{ m} \\
 + B \Delta \varphi^2 & = + & 2.4016 \\
 + C l^2 & = + & 479.2277 \\
 - D \Delta \varphi l^2 & = & -1.4610 \text{ m} \\
 - E \Delta \varphi^3 & = & -0.0023 \\
 - F \Delta \varphi^2 l^2 & = & -0.0730 \\
 + G l^4 & = + & 0.0180 \\
 \hline
 & + & 56,089.8650 \text{ m} - 1.5363 \text{ m} \\
 \Delta x & = + & 55,605.814 \\
 \hline
 x & = + & 111,694.143 \text{ m} \\
 \\
 + H l & = + & 71,687.0132 \text{ m} \\
 - J \Delta \varphi l & = & -743.4819 \text{ m} \\
 - K \Delta \varphi^2 l & = & -2.7542 \\
 - L l^3 & = & -2.1357 \\
 - M \Delta \varphi l^3 & = & -0.0090 \\
 + N \Delta \varphi^3 l & = + & 0.0094 \\
 \hline
 & + & 71,687.0226 \text{ m} - 748.3808 \text{ m} \\
 y & = + & 70,938.642 \text{ m} \\
 + P l & = + & 2757.76000'' \\
 + Q \Delta \varphi l & = + & 20.19377 \\
 - R \Delta \varphi^2 l & = & -0.10 501'' \\
 + S l^3 & = + & 0.11602 \\
 - T \Delta \varphi l^3 & = & -0.00 156 \\
 - U \Delta \varphi^3 l & = & -0.00 026 \\
 \hline
 & + & 2778.06979'' - 0.10 683'' \\
 \gamma & = + & 2777.9630'' = + 0^\circ 46' 17.9630''.
 \end{array}$$

For the inversion of the problem we have:

$$\varphi_0 = 49^\circ 30' \quad x = + 111,694.143 \text{ m} \quad y = + 70,938.642 \text{ m}.$$

If we use the auxiliary latitude $\varphi = 50^\circ 00'$ again, then $\Delta x = 55,605.814$ m is to be deducted from the abscissa, and hence, the value $x = + 56,088.329$ m is to be introduced in the formulae (17):

$$\begin{array}{rcl}
 + a x & = + & 1815.54088'' \\
 - b x^2 & = & -0.07909'' \\
 - c y^2 & = & -15.19008 \\
 - d x y^2 & = & -0.26977 \\
 + e x^3 & = + & 0.00008 \\
 - f x^2 y^2 & = & -0.00283 \\
 + g y^4 & = + & 0.00082 \\
 \hline
 & + & 1815.54178'' - 15.54177'' \\
 \Delta \varphi & = & 1800.0000''
 \end{array}$$

$$\begin{aligned}
+ h y &= + 3562.41805'' \\
+ i x y &= + 37.26562 \\
+ k x^2 y &= + 0.52744 \\
- l y^3 &= - 0.20786'' \\
- m x y^3 &= - 0.00958 \\
+ n x^3 y &= + 0.00647 \\
\hline
&+ 3600.21758'' - 0.21744'' \\
l &= + 3600.0001''
\end{aligned}$$

$$\begin{aligned}
+ p y &= + 2728.97048'' \\
+ q x y &= + 48.70260 \\
+ r x^2 y &= + 0.50830 \\
- s y^3 &= - 0.21544'' \\
- t x y^3 &= - 0.00951 \\
+ u x^3 y &= + 0.00657 \\
\hline
&+ 2778.18795'' - 0.22495'' \\
\gamma &= 2777.9630''
\end{aligned}$$

There follows from these computations that the formulae (17) and (18) for the conversion of geographic coordinates to rectangular coordinates give an accuracy of 1 mm.

Section 29. Rectangular Transverse-Axis Coordinates

In the first half-volume, section 57, we have treated a spherical transverse-axis system of coordinates, in which case we used as axis of abscissae a great circle which is tangent to the parallel of the zero point at the latter. The ordinates are formed by great circles which lie at right angles to the axis of abscissae.

If we pass over now to a transverse-axis system of coordinates on the ellipsoid, then we choose, in accordance with the above, as axis of abscissae the geodetic line which passes through the zero point and is tangent to its parallel. Likewise, geodetic lines which lie at right angles to the axis of abscissae take the place of the circles of ordinates of the sphere.

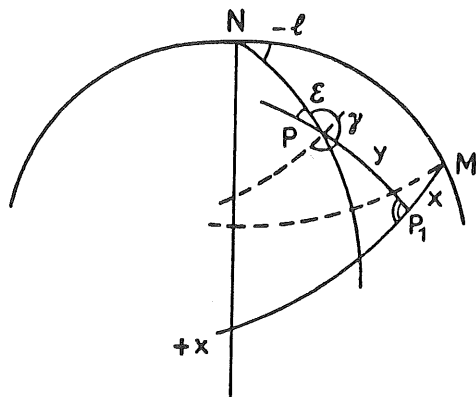


Fig. 1.

In Fig. 1 we have reproduced a sector from the corresponding Fig. 1 in the first half-volume, p. 146, in which, in addition to the zero point M , the point of tangency of its parallel and the line of abscissae, there are indicated the abscissa x and the ordinate y as well as the meridian for a point P .

In contrast to section 57 of the first half-volume, we will now count the geographic longitudes *east* of the zero meridian in the positive sense, as is the case everywhere in this half-volume.

For the computation of the geographic coordinates from the rectangular coordinates of the point P we start from the developments in series indicated in section 18, which we have to apply here twice.

At first we determine the coordinates of the foot-point P_1 , for which we are to set in equations (25) to (27), section 18, p. 78, $\alpha = 270^\circ$ and $s = x$. We denote further the geographic latitude of M by φ_0 and the geographic coordinates of P_1 by φ_1 and l_1 ; then we will have in (25) to (27), p. 78:

$$u = 0 \quad v = -\frac{x}{N_0} \quad t = t_0$$

and we have to the third order

$$\frac{\varphi_1 - \varphi_0}{l_0^2} = -\frac{x^2}{2N_0^2} t_0$$

$$l_1 \cos \varphi_0 = -\frac{x}{N_0} + \frac{x^3}{3 N_0^3} t_0^2 \quad (2)$$

$$\alpha_1 = 270^\circ - \frac{x}{N_0} t_0 + \frac{x^3}{6 N_0^3} t_0 (1 + 2 t_0^2 + \eta_0^2), \quad (3)$$

where α_1 is the azimuth of the axis of abscissae at P_1 .

For the passage from P_1 to P we have to set further in equations (25) to (27), section 18, p. 78, $s = y$, $\alpha = \alpha_1 + 90^\circ$, and if we denote this latter value, the azimuth of the ordinate y at the foot-point P_1 , by α_1' , then we have according to (3)

$$\alpha_1' = -\frac{x}{N_0} t_0 + \frac{x^3}{6 N_0^3} t_0 (1 + 2 t_0^2 + \eta_0^2). \quad (4)$$

If we form, with this, the two quantities

$$v = \frac{s}{N} \sin \alpha_1' \quad \text{and} \quad u = \frac{s}{N} \cos \alpha_1'$$

then we need $\sin \alpha_1'$ and $\cos \alpha_1'$ only to the second order, if we wish to limit ourselves in the whole development to the terms of third order; and hence, we have

$$v = -\frac{y}{N_1} \frac{x}{N_0} t_0 \quad u = \frac{y}{N_1} \left(1 - \frac{x^2}{2 N_0^2} t_0^2\right). \quad (5)$$

With these, we obtain according to (25) to (27), section 18, p. 78:

$$\frac{\varphi - \varphi_1}{V_1^2} = \frac{y}{N_1} \left(1 - \frac{x^2}{2 N_0^2} t_0^2\right) - \frac{3}{2} \frac{y^2}{N_1^2} \eta_1^2 t_1 - \frac{y^3}{2 N_1^3} \eta_1^2 (1 - t_1^2) \quad (6)$$

$$(l - l_1) \cos \varphi_1 = -\frac{y x}{N_1 N_0} t_0 - \frac{y^2 x}{N_1^2 N_0} t_0 t_1 \quad (7)$$

$$\alpha = \alpha_1' - \frac{y x}{N_1 N_0} t_0 t_1 - \frac{y^2 x}{2 N_1^2 N_0} t_0 (1 + 2 t_1^2 + \eta_1^2) \quad (8)$$

where α denotes the azimuth of the extension of the ordinate y at P .

In this second group (6) to (8), we need to reduce now the argument φ_1 to the argument φ_0 . For this, we have according to the first half-volume, section 40, p. 63, equation (n), in connection with (1)

$$V_1 = V_0 \left(1 - \frac{\varphi_1 - \varphi_0}{V_0^2} \eta_0^2 t_0\right) \quad V_1 = V_0 \left(1 + \frac{x^2}{2 N_0^2} \eta_0^2 t_0\right) \quad (9)$$

and according to the first half-volume, section 40, p. 63, equation (o),

$$N_1 = N_0 \left(1 - \frac{x^2}{2 N_0^2} \eta_0^2 t_0\right). \quad (10)$$

Then we will have

$$\begin{aligned} \frac{V_1^2}{N_1} &= \frac{V_0^2}{N_0} \left(1 + \frac{x^2}{N_0^2} \eta_0^2 t_0\right) \left(1 + \frac{x^2}{2 N_0^2} \eta_0^2 t_0\right) \\ \text{or} \quad \frac{V_1^2}{N_1} &= \frac{V_0^2}{N_0} \left(1 + \frac{3}{2} \frac{x^2}{N_0^2} \eta_0^2 t_0\right). \end{aligned} \quad (11)$$

In order to reduce (6) to φ_0 we only have to apply (11) to the first term; in the further terms we can set, without further ado, the argument φ_0 instead of the argument φ_1 . We have then

$$\begin{aligned} \varphi - \varphi_1 = \frac{y}{N_0} V_0^2 \left(1 + \frac{3}{2} \frac{x^2}{N_0^2} \eta_0^2 t_0^2 \right) \left(1 - \frac{x^2}{2 N_0^2} t_0^2 \right) \\ - \frac{3}{2} \frac{y^2}{N_0^2} V_0^2 \eta_0^2 t_0 - \frac{y^3}{2 N_0^3} V_0^2 \eta_0^2 (1 - t_0^2). \end{aligned} \quad (12)$$

For the reduction of equation (7) to φ_0 we have

$$\varphi_1 = \varphi_0 + (\varphi_1 - \varphi_0) = \varphi_0 - \frac{x^2 V_0^2}{2 N_0^2} t_0$$

and hence

$$\cos \varphi_1 = \cos \varphi_0 + \frac{x^2 V_0^2}{2 N_0^2} t_0 \sin \varphi_0 = \cos \varphi_0 \left(1 + \frac{x^2 V_0^2}{2 N_0^2} t_0^2 \right).$$

If we set this into (7), then we have

$$(l - l_1) \cos \varphi_0 \left(1 + \frac{x^2 V_0^2}{2 N_0^2} t_0^2 \right) = - \frac{y x}{N_0^2} t_0 - \frac{y^2 x}{N_0^3} t_0^2.$$

If this equation is divided on both sides by the factor of $(l - l_1) \cos \varphi_0$, then on the right-hand side the additional terms appearing are only of the fourth and higher order. Therefore, if we limit ourselves to terms of third order, then we obtain

$$(l - l_1) \cos \varphi_0 = - \frac{y x}{N_0^2} t_0 - \frac{y^2 x}{N_0^3} t_0^2. \quad (13)$$

For the same reasons, we can refer equation (8) without further ado to the argument φ_0 and have then

$$\alpha = \alpha_1' - \frac{y x}{N_0^2} t_0^2 - \frac{y^2 x}{2 N_0^3} t_0 (1 + 2 t_0^2 + \eta_0^2). \quad (14)$$

We arrive now at the final equations for the computation of geographic coordinates from rectangular, if we combine equations (1), (2) and (4), pp. 144 and 145, with equations (12) to (14). At the same time we will replace everywhere V_0^2 by $1 + \eta_0^2$ and neglect everywhere terms with η_0^4 . Then we will have

$$\begin{aligned} \varphi - \varphi_0 = \frac{y}{N_0} (1 + \eta_0^2) - \frac{x^2}{2 N_0^2} t_0 (1 + \eta_0^2) - \frac{3}{2} \frac{y^2}{N_0^2} \eta_0^2 t_0 \\ - \frac{y^3}{2 N_0^3} \eta_0^2 (1 - t_0^2) - \frac{y x^2}{2 N_0^3} t_0^2 (1 - 2 \eta_0^2) \end{aligned} \quad (15)$$

$$l \cos \varphi_0 = - \frac{x}{N_0} - \frac{y x}{N_0^2} t_0 + \frac{x^3}{3 N_0^3} t_0^2 - \frac{y^2 x}{N_0^3} t_0^2 \quad (16)$$

$$\alpha = - \frac{x}{N_0} t_0 - \frac{y x}{N_0^2} t_0^2 + \frac{x^3}{6 N_0^3} t_0 (1 + 2 t_0^2 + \eta_0^2) - \frac{y^2 x}{2 N_0^3} t_0 (1 + 2 t_0^2 + \eta_0^2). \quad (17)$$

These equations (15) to (17) correspond to the former spherical equations (17) to (19) in the first half-volume, section 57, pp. 151 and 152; in the foregoing, however, only the third order has been taken into account. The geographic longitude l east of the zero point M is counted here also in the positive sense, in contrast to the former formulae of the first half-volume, as is mentioned above. The azimuth α in the

equation (17) is the azimuth of the extension of the ordinate y at the point P ; the angle α corresponds thus to the meridian convergence of the rectangular Soldner coordinates in section 27.

*Computation of rectangular transverse-axis coordinates from
geographic coordinates*

Now it is a question of solving the equations (15) to (17) for x and y , which must be done by successive approximation. In the first approximation, we have from (15) and (16), if we set $\varphi - \varphi_0 = \Delta \varphi$

$$\frac{y}{N_0} = \frac{\Delta \varphi}{V_0^2} \quad \frac{x}{N_0} = -l \cos \varphi_0 \quad (18)$$

$$\frac{y^2}{N_0^2} = \frac{\Delta \varphi^2}{V_0^4} \quad \frac{x^2}{N_0^2} = l^2 \cos^2 \varphi_0 \quad \frac{y x}{N_0^2} = -\frac{\Delta \varphi l}{V_0^2} \cos \varphi_0. \quad (19)$$

If we set (19) into (15) and (16), then we obtain in the second approximation

$$\left. \begin{aligned} \frac{y}{N_0} &= \frac{\Delta \varphi}{V_0^2} + \frac{1}{2} l^2 \cos^2 \varphi_0 t_0 + \frac{3}{2} \Delta \varphi^2 \eta_0^2 t_0 \\ \frac{x}{N_0} &= -l \cos \varphi_0 + \frac{\Delta \varphi l}{V_0^2} \cos \varphi_0 t_0 \end{aligned} \right\} \quad (20)$$

and this yields, accurate to terms of third order,

$$\left. \begin{aligned} \frac{y^2}{N_0^2} &= \frac{\Delta \varphi^2}{V_0^4} + \frac{\Delta \varphi l^2}{V_0^2} \cos^2 \varphi_0 t_0 + 3 \Delta \varphi^2 \eta_0^2 t_0 \\ \frac{x^2}{N_0^2} &= l^2 \cos^2 \varphi_0 - \frac{2 \Delta \varphi l^2}{V_0^2} \cos^2 \varphi_0 t_0 \\ \frac{y x}{N_0^2} &= -\frac{\Delta \varphi l}{V_0^2} \cos \varphi_0 - \frac{1}{2} l^3 \cos^3 \varphi_0 t_0 + \frac{1}{2} \Delta \varphi^2 l \cos \varphi_0 t_0 (2 - 7 \eta_0^2). \end{aligned} \right\} \quad (21)$$

For the terms of third order in (15) to (17) we can use, without further ado, the above values (18).

If we set this at first into (15), then we obtain

$$\begin{aligned} \varphi - \varphi_0 &= \frac{y}{N_0} V_0^2 - \frac{1}{2} l^2 V_0^2 \cos^2 \varphi_0 t_0 - \frac{3}{2} \Delta \varphi^2 \eta_0^2 t_0 - \frac{1}{2} \Delta \varphi^3 \eta_0^2 (1 - t_0^2) \\ &\quad + \frac{1}{2} \Delta \varphi l^2 \cos^2 \varphi_0 t_0^2 \\ \frac{\varphi - \varphi_0}{V_0^2} &= \frac{y}{N_0} - \frac{1}{2} l^2 \cos^2 \varphi_0 t_0 - \frac{3}{2} \Delta \varphi^2 \eta_0^2 t_0 - \frac{1}{2} \Delta \varphi^3 \eta_0^2 (1 - t_0^2) \\ &\quad + \frac{1}{2} \frac{\Delta \varphi l^2}{V_0^2} \cos^2 \varphi_0 t_0^2 \end{aligned}$$

and this yields

$$\begin{aligned} \frac{y}{N_0} &= \frac{\Delta \varphi}{V_0^2} + \frac{1}{2} l^2 \cos^2 \varphi_0 t_0 + \frac{3}{2} \Delta \varphi^2 \eta_0^2 t_0 + \frac{1}{2} \Delta \varphi^3 \eta_0^2 (1 - t_0^2) \\ &\quad - \frac{1}{2} \Delta \varphi l^2 \cos^2 \varphi_0 t_0^2 (1 - \eta_0^2) \end{aligned} \quad (22)$$

Equation (16) yields in the same manner

$$\begin{aligned} l \cos \varphi_0 &= -\frac{x}{N_0} + \frac{\Delta \varphi l}{V_0^2} \cos \varphi_0 t_0 + \frac{1}{2} l^3 \cos^3 \varphi_0 t_0^2 - \frac{1}{2} \Delta \varphi^2 l \cos \varphi_0 t_0^2 (2 - 7 \eta_0^2) \\ &\quad - \frac{1}{3} l^3 \cos^3 \varphi_0 t_0^2 + \frac{\Delta \varphi^2 l}{V_0^4} \cos \varphi_0 t_0^2 \end{aligned}$$

or

$$\begin{aligned} \frac{x}{N_0} = & -l \cos \varphi_0 + \Delta \varphi l \cos \varphi_0 t_0 (1 - \eta_0^2) + \frac{1}{6} l^3 \cos^3 \varphi_0 t_0^2 \\ & + \frac{3}{2} \Delta \varphi^2 l \cos \varphi_0 \eta_0^2 t_0^2. \end{aligned} \quad (23)$$

Finally, we can also invert the equation (17) in order to represent α as a function of $\Delta \varphi$ and l . There follows with the help of (20), (21) and (23)

$$\begin{aligned} \alpha = & l \cos \varphi_0 t_0 - \frac{1}{6} l^3 \cos^3 \varphi_0 t_0 (1 + \eta_0^2) + \frac{1}{2} \Delta \varphi^2 l \cos \varphi_0 t_0 (1 - \eta_0^2) \\ \text{or} \\ \alpha = & l \sin \varphi_0 - \frac{1}{6} l^3 \sin \varphi_0 \cos^2 \varphi_0 (1 + \eta_0^2) + \frac{1}{2} \Delta \varphi^2 l \sin \varphi_0 (1 - \eta_0^2). \end{aligned} \quad (24)$$

As a check, we can also reconvert this equation (24) into (17) again, with the help of (15) and (16), which will agree.

Now we have in (15) to (17) and in (22) to (24) all necessary formulae to the third order.

To these, we will set down, in addition, the purely spherical terms of fourth order, which we have found in the first half-volume, section 57, equations (17) to (19), pp. 151 and 152.

The angle α is the azimuth of the extension of the ordinate y at the point P ; it corresponds to the meridian convergence in section 27, p. 126, as we have already mentioned on pp. 146 and 147. The meridian convergence treated there could also be defined as the azimuth of the parallel line to the axis of abscissae at the point P , and if we use this latter definition also for the transverse-axis system, then we have according to Fig. 1, p. 144, in which an auxiliary angle ε is indicated further,

$$\varepsilon = 360^\circ - \alpha \quad \text{and} \quad \gamma = 270^\circ - \varepsilon \quad \text{or} \quad \gamma = 270^\circ + \alpha.$$

If we insert now, in addition, the necessary ρ 's everywhere, then we obtain the following six formulae for use:

I. Computation of geographic coordinates

$$\begin{aligned} \varphi - \varphi_0 = & \frac{y}{N_0} \varrho (1 + \eta_0^2) - \frac{x^2}{2 N_0^2} \varrho t_0 (1 + \eta_0^2) - \frac{3}{2} \frac{y^2}{N_0^2} \varrho \eta_0^2 t_0 - \frac{y^3}{2 N_0^3} \varrho \eta_0^2 (1 - t_0^2) \\ & - \frac{y x^2}{2 N_0^3} \varrho t_0^2 (1 - 2 \eta_0^2) - \frac{y^2 x^2}{2 N_0^4} \varrho t_0^3 + \frac{x^4}{24 N_0^4} \varrho t_0 (1 + 3 t_0^2) \end{aligned} \quad (25)$$

$$\begin{aligned} l \cos \varphi_0 = & -\frac{x}{N_0} \varrho - \frac{y x}{N_0^2} \varrho t_0 + \frac{x^3}{3 N_0^3} \varrho t_0^2 - \frac{y^2 x}{N_0^3} \varrho t_0^2 - \frac{y^3 x}{3 N_0^4} \varrho t_0 (1 + 3 t_0^2) \\ & + \frac{y x^3}{6 N_0^4} \varrho t_0 (1 + 6 t_0^2) \end{aligned} \quad (26)$$

$$\begin{aligned} \gamma = 270^\circ - & \frac{x}{N_0} \varrho t_0 - \frac{y x}{N_0^2} \varrho t_0^2 + \frac{x^3}{6 N_0^3} \varrho t_0 (1 + 2 t_0^2 + \eta_0^2) - \frac{y^2 x}{2 N_0^3} \varrho t_0 (1 + 2 t_0^2 + \eta_0^2) \\ & + \frac{y x^3}{3 N_0^4} \varrho t_0^2 (2 + 3 t_0^2) - \frac{y^3 x}{6 N_0^4} \varrho t_0^2 (5 + 6 t_0^2). \end{aligned} \quad (27)$$

II. Computation of rectangular coordinates

$$\begin{aligned} y = & \frac{\Delta \varphi}{\varrho} \frac{N_0}{V_0^2} + \frac{l^2}{2 \varrho^2} N_0 \cos^2 \varphi_0 t_0 + \frac{3}{2} \frac{\Delta \varphi^2}{\varrho^2} N_0 \eta_0^2 t_0 + \frac{\Delta \varphi^3}{2 \varrho^3} N_0 \eta_0^2 (1 - t_0^2) \\ & - \frac{\Delta \varphi l^2}{2 \varrho^3} N_0 \cos^2 \varphi_0 t_0^2 (1 - \eta_0^2) - \frac{l^4}{24 \varrho^4} N_0 \cos^2 \varphi_0 t_0 \end{aligned} \quad (28)$$

$$x = -\frac{l}{\varrho} N_0 \cos \varphi_0 + \frac{\Delta \varphi l}{\varrho^2} \cos \varphi_0 t_0 (1 - \eta_0^2) + \frac{l^3}{6 \varrho^3} N_0 \cos^3 \varphi_0 t_0^2$$

$$+ \frac{3}{2} \frac{\Delta \varphi^2 l}{\varrho^3} N_0 \cos \varphi_0 \eta_0^2 t_0^2 + \frac{\Delta \varphi^3 l}{3 \varrho^4} N_0 \cos \varphi_0 t_0 - \frac{\Delta \varphi l^3}{6 \varrho^4} N_0 \cos \varphi_0 t_0$$

$$(29)$$

$$\gamma = 270^\circ + l \sin \varphi_0 - \frac{l^3}{6 \varrho^2} \sin \varphi_0 \cos^2 \varphi_0 (1 + \eta_0^2) + \frac{\Delta \varphi^2 l}{2 \varrho^2} \sin \varphi_0 (1 - \eta_0^2)$$

$$+ \frac{\Delta \varphi l^3}{2 \varrho^3} \sin^2 \varphi_0 \cos \varphi_0.$$

$$(30)$$

In the foregoing, we have limited ourselves to the development of the basic formulae for the transverse-axis coordinates on the ellipsoid. In practice, these coordinates may be used only in the case of countries which extend mainly in the west-east direction and have only a small extent in the north-south direction. In this case, we can regard the axis of abscissae and the ordinates as straight lines and treat the whole like a plane coordinate system.

The system of the transverse-axis coordinates obtains increased significance if it is projected conformally on the plane, which we will treat in detail in the next Chapter IV, section 39.

Coordinates of Anhalt

Transverse-axis coordinates were introduced with conformal representation in 1896 for the former Duchy of Anhalt. However, the Anhalt system is rotated by 90° with respect to ours, so that our x 's agree with the negative y 's and our y 's with the x 's of the Anhalt system, as Fig. 2 shows. The latitude of the zero point is $\varphi_0 = 51^\circ 50'$.

After these definitions, we further indicate the base numbers and the formulae for the spheroidal coordinates of the Anhalt system:

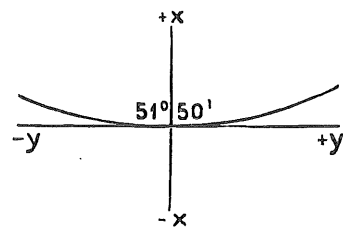


Fig. 2.

$$\begin{aligned} \log \cos \varphi_0 &= 9.790\,9541\cdot080 & \log \sin \varphi_0 &= 9.895\,5421\cdot736 \\ \log \cos^2 \varphi_0 &= 9.581\,9082\cdot160 & \log \sin^2 \varphi_0 &= 9.791\,0843\cdot472 \\ \log e'^2 &= 7.827\,3187\cdot833 \\ \log \eta^2 &= \log e'^2 \cos^2 \varphi_0 = 7.409\,2269\cdot993 & \eta^2 &= 0.002\,5658\cdot248 \\ \log \eta^2 t^2 &= \log e'^2 \sin^2 \varphi_0 = 7.618\,4031\cdot305 & \eta^2 t^2 &= 0.004\,1533\cdot940 \\ V^2 &= 1 + \eta^2 = 1.002\,565824805 \\ \tan \varphi_0 &= t, \log t &= 0.104\,5880\cdot656 & t^2 &= 1.618\,7363\cdot954 \\ & \log t^2 &= 0.209\,1761\cdot312 \\ \log \varrho &= 5.314\,4251\cdot332 & \log \frac{1}{\varrho} &= 4.685\,5748\cdot668 \\ \log N &= 6.805\,5411\cdot953 & \log N^2 &= 3.611\,0823\cdot906 \\ \log N^3 &= 0.416\,6235\cdot9 & \log N^4 &= 7.222\,1647\cdot8. \end{aligned}$$

If we introduce these constants into the foregoing formulae, then we obtain

$$\Delta \varphi = [8.509\,9968\cdot343] x - [1.508\,0137\cdot1] y^2 - [9.394\,3620] x^2$$

$$+ [1.811\,208] x^3 - [4.803\,7047] x y^2 - [8.10277] x^2 y^2 + [7.58202] y^4 \quad \left. \vphantom{\Delta \varphi} \right\} \quad (25a)$$

$$l = [8.717\,9298\cdot299] y + [2.016\,9767] x y + [5.316\,0226] x^2 y$$

$$- [4.838\,9023] y^3 + [8.69416] x^3 y - [8.65540] x y^3 \quad \left. \vphantom{l} \right\} \quad (26a)$$

$$\gamma = [8.613\,4720\cdot035] y + [1.912\,5188\cdot8] x y + [5.328\,6062] x^2 y$$

$$- [4.851\,4850] y^3 - [8.6582] x y^3 + [8.688\,74] x^3 y \quad \left. \vphantom{\gamma} \right\} \quad (27a)$$

$$x = [1.490\,0031\cdot657] \Delta \varphi + [5.562\,1572\cdot1] l^2 - [0.351\,2073] \Delta \varphi l^2$$

$$+ [3.864\,3715] \Delta \varphi^2 - [7.744\,955] \Delta \varphi^3 - [3.854\,68] l^4 \quad \left. \vphantom{x} \right\} \quad (28a)$$

$$y = [1.282\,0701\cdot701] l - [6.071\,1202\cdot1] \Delta \varphi l - [9.666\,1530] l^3$$

$$- [8.445\,4885] \Delta \varphi^2 l - [4.96682] \Delta \varphi^3 l + [4.66579] \Delta \varphi l^3 \quad \left. \vphantom{y} \right\} \quad (29a)$$

$$\gamma = [9.895\,5421\cdot736] l - [8.071\,5618] l^3 + [8.964\,5490] \Delta \varphi^2 l$$

$$+ [3.33773] \Delta \varphi l^3 \quad \left. \vphantom{\gamma} \right\} \quad (30a)$$

By γ we understand here the azimuth of the geodetic line on which the abscissa of the point is measured. And hence, γ agrees with $\alpha - 270^\circ$ of formulae (27) and (30).

For a first application of these formulae we will take in round numbers:

$$x = 50,000 \text{ m} \quad y = 50,000 \text{ m.} \quad (31)$$

We obtain hence:

$$\left. \begin{aligned} \Delta \varphi &= 1609.761\,561'' = 26' 49.761\,561'' \\ l &= 2637.728\,348 = 43\, 57.728\,348 \\ \gamma &= 2073.867\,723 = 34\, 33.867\,723 \end{aligned} \right\} \quad (32)$$

and the reversion:

$$\left. \begin{aligned} x &= 50000.00015 \text{ m} \\ y &= 50000.00063 \\ \gamma &= 2073.867\,640'' = 34' 33.867\,640''. \end{aligned} \right\} \quad (33)$$

The checks agree in x to 0.15 mm, in y to 0.63 mm and γ to 0.000083", therefore everywhere satisfactorily.

In the following Chapter IV, sections 39 to 40, we shall treat in detail the conformal representation of the transverse-axis coordinates.

Details in reference to the Anhalt coordinates are indicated in *Zeitschr. f. Verm.*, 1896, pp. 88 and 89.

THE PROJECTION OF THE TERRESTRIAL ELLIPSOID

Section 30. The Fundamental Equations of the Projection

For the projection of the terrestrial ellipsoid we shall use, as a rule, the plane as the image surface; however, just as in the case of the terrestrial sphere (first half-volume, Chapter VI), we also consider the projection on a cylinder surface and a conic surface whose development yields then the plane projection. But, finally, the ellipsoidal surface can also be projected on a spherical surface at first; this projection can then be transferred to the plane according to the methods treated in the first half-volume, Chapter VI, so that a double projecting process takes place in this case.

If we limit ourselves at first to the plane projection of the terrestrial ellipsoid, then we suppose that a point be given on the ellipsoid by the geographic coordinates φ and l , and that an image point with the rectangular coordinates x and y correspond to it on the plane. If we have then two equations of the form,

$$x = f_1(\varphi, l) \quad \text{and} \quad y = f_2(\varphi, l), \quad (1)$$

then a point of the plane corresponds to every point of the ellipsoid; and hence, the law of projection is given by the equations (1). Because of the curvature of the ellipsoidal surface, neither here nor in the case of the sphere is a perfectly true projection on the plane possible; rather, we must put up with distortions, and now the problem arises of choosing both functions (1) in such a way that the distortions are as small as possible.

Let there be given on the ellipsoid two points P and P' lying infinitely near one another with the coordinates

$$\varphi, l \quad \text{and} \quad \varphi + d\varphi, \quad l + dl;$$

we suppose that on the plane the two points with the coordinates

$$x, y \quad \text{and} \quad x + dx, \quad y + dy$$

correspond to them.

In Fig. 1, which represents the position of the two points on the ellipsoid, a small rectangular triangle with the legs $PQ = N \cos \varphi dl$, $QP' = M d\varphi$ and the hypotenuse $d\sigma$ is formed by the arc element $PP' = d\sigma$ by the meridian of P' and by the parallel of P . Then we have

$$d\sigma^2 = M^2 d\varphi^2 + N^2 \cos^2 \varphi dl^2. \quad (2)$$

On the other hand, on the plane we have for the corresponding linear element, which we will denote by ds , the equation

$$ds^2 = dx^2 + dy^2. \quad (3)$$

For the scale factor m of the projection we obtain then further the expression

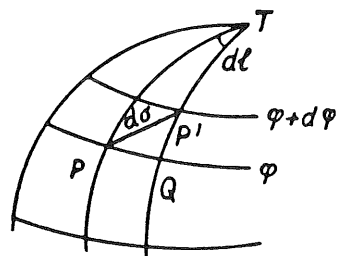


Fig. 1.

$$m^2 = \frac{dx^2 + dy^2}{M^2 d\varphi^2 + N^2 \cos^2 \varphi dl^2} \quad (4)$$

If we introduce here the values of dx and dy , as they result from the equations (1), then we obtain m as a function of φ and l .

The isometric latitude

For the further investigations, it is appropriate to write equation (2) in another form, which we obtain in the following way. We have

$$d\sigma^2 = N^2 \cos^2 \varphi \left(\frac{M^2}{N^2 \cos^2 \varphi} d\varphi^2 + dl^2 \right).$$

If we introduce a new quantity q by means of the differential equation

$$\frac{M}{N \cos \varphi} d\varphi = dq, \quad (5)$$

then (2) passes over to

$$d\sigma^2 = N^2 \cos^2 \varphi (dq^2 + dl^2). \quad (6)$$

We designate the form (6) for the linear element on the ellipsoid the *isometric* form, and we call the quantity q , which is only a function of the geographic latitude, as we see from (5), the *isometric latitude*.

The value of q can be easily determined by the integration of (5). Since

$$M = \frac{a(1-e^2)}{(1-e^2 \sin^2 \varphi)^{3/2}} \quad N = \frac{a}{(1-e^2 \sin^2 \varphi)^{1/2}}$$

then there follows at first from (5)

$$dq = \frac{(1-e^2)d\varphi}{(1-e^2 \sin^2 \varphi) \cos \varphi} = \frac{d\varphi}{\cos \varphi} - e^2 \frac{\cos \varphi d\varphi}{1-e^2 \sin^2 \varphi}. \quad (7)$$

The integration of the first term yields between the limits 0 and φ

$$\int_0^\varphi \frac{d\varphi}{\cos \varphi} = l \tan \left(45^\circ + \frac{\varphi}{2} \right). \quad (8)$$

For the integration of the second term in (7) we set

$$\sin \varphi = x \quad \cos \varphi d\varphi = dx$$

and obtain then for the second term in (7) the integral determined between the limits 0 and φ

$$e^2 \int \frac{dx}{1-e^2 x^2} = \frac{e}{2} l \frac{1+ex}{1-ex} = l \left(\frac{1+e \sin \varphi}{1-e \sin \varphi} \right)^{e/2}, \quad (9)$$

where the symbol l in (8) and (9) means the natural logarithm.

The equations (8) and (9) yield then together

$$q = l \tan \left(45^\circ + \frac{\varphi}{2} \right) - l \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{e/2}$$

or

$$q = l \tan \left(45^\circ + \frac{\varphi}{2} \right) \left(\frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right)^{e/2}. \quad (10)$$

There follows hence

$$e^q = \tan \left(45^\circ + \frac{\varphi}{2} \right) \left(\frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right)^{e/2}. \quad (10a)$$

The corresponding isometric latitude q can be computed with this for every value of φ .

An auxiliary table for the computation of the isometric latitude has been published by L. Grabowski in *Zeitschr. f. Verm.*, 1929, pp. 35-44, for the geographic latitudes 40° - 60° .

Since we have obtained, with this, a closed formula for the computation of the isometric from the geographic latitude, we will carry out, in addition, the conversion of a small geographic difference of latitude into the corresponding isometric difference of latitude. For this, we have according to Maclaurin's theorem

$$\Delta q = \frac{dq}{d\varphi} \Delta \varphi + \frac{1}{2} \frac{d^2 q}{d\varphi^2} \Delta \varphi^2 + \frac{1}{6} \frac{d^3 q}{d\varphi^3} \Delta \varphi^3 + \frac{1}{24} \frac{d^4 q}{d\varphi^4} \Delta \varphi^4 + \dots \quad (11)$$

Now we have according to (5)

$$\frac{dq}{d\varphi} = \frac{M}{N \cos \varphi} = \frac{1}{V^2 \cos \varphi}. \quad (12)$$

The second differential quotient follows hence at once with $V^2 = 1 + \eta^2$ and $\frac{d\eta^2}{d\varphi} = -2\eta^2 t$

$$\frac{d^2 q}{d\varphi^2} = \frac{1}{V^4 \cos^2 \varphi} (2\eta^2 t \cos \varphi + V^2 \sin \varphi)$$

and

$$\frac{d^2 q}{d\varphi^2} = \frac{t}{V^4 \cos \varphi} (1 + 3\eta^2). \quad (13)$$

If we pass over to the third differential quotient, then we will have, if we take into account that

$$\frac{dt}{d\varphi} = \frac{1}{\cos^2 \varphi} = 1 + t^2,$$

$$\begin{aligned} \frac{d^3 q}{d\varphi^3} = \frac{1}{V^8 \cos^2 \varphi} \bigg\{ & V^4 \cos \varphi ((1 + t^2)(1 + 3\eta^2) - 6\eta^2 t^2) \\ & + t(1 + 3\eta^2)(4V^2 \eta^2 t \cos \varphi - V^4 \sin \varphi) \bigg\} \end{aligned}$$

and this can be reduced to

$$\frac{d^3 q}{d \varphi^3} = \frac{1}{V^6 \cos \varphi} (1 + 2 t^2 + 4 \eta^2 + 6 \eta^2 t^2 + 3 \eta^4 + 12 \eta^4 t^2) \quad (14)$$

In order to arrange the setting up of the fourth differential quotient more clearly, we divide (14) into two parts by setting

$$\frac{d^3 q}{d \varphi^3} = \frac{1}{V^6 \cos \varphi} (1 + 2 t^2 + 4 \eta^2) + \frac{3}{V^6 \cos \varphi} (2 \eta^2 t^2 + \eta^4 + 4 \eta^2 t^2) = I + II.$$

We have then

$$\frac{d(I)}{d \varphi} = \frac{1}{V^{12} \cos^2 \varphi} \left\{ V^6 \cos \varphi (4 t + 4 t^3 - 8 \eta^2 t) \right. \\ \left. + (1 + 2 t^2 + 4 \eta^2) (6 V^4 \cos \varphi \eta^2 t + V^6 \cos \varphi t) \right\}$$

or

$$\frac{d(I)}{d \varphi} = \frac{t}{V^8 \cos \varphi} (5 + 6 t^2 + 7 \eta^2 + 18 \eta^2 t^2 + 20 \eta^4) \quad (15)$$

and further

$$\frac{d(II)}{d \varphi} = \frac{3}{V^{12} \cos^2 \varphi} \left\{ V^6 \cos \varphi (-4 \eta^2 t^3 + 4 \eta^2 t + 4 \eta^2 t^3 - 4 \eta^4 t - 16 \eta^4 t^3 + 8 \eta^4 t + 8 \eta^4 t^3) \right. \\ \left. + (2 \eta^2 t^2 + \eta^4 + 4 \eta^4 t^2) (6 V^4 \cos \varphi \eta^2 t + V^6 \cos \varphi t) \right\},$$

which can be reduced to

$$\frac{d(II)}{d \varphi} = \frac{3 t}{V^8 \cos \varphi} (4 \eta^2 + 2 \eta^2 t^2 + 9 \eta^4 + 10 \eta^4 t^2 + 11 \eta^6 + 20 \eta^6 t^2). \quad (16)$$

Both expressions (15) and (16) yield together, if we neglect the terms with η^6

$$\frac{d^4 q}{d \varphi^4} = \frac{t}{V^8 \cos \varphi} (5 + 6 t^2 + 19 \eta^2 + 24 \eta^2 t^2 + 47 \eta^4 + 30 \eta^4 t^2). \quad (17)$$

With the four differential quotients found, (12), (13), (14), (17), we can now assemble the series (11) for Δq and obtain at once

$$\Delta q \cos \varphi = \frac{1}{V^2} \Delta \varphi + \frac{1}{2 V^4} t (1 + 3 \eta^2) \Delta \varphi^2 + \frac{1}{6 V^6} (1 + 2 t^2 + 4 \eta^2 + 6 \eta^2 t^2 \\ + 3 \eta^4 + 12 \eta^4 t^2) \Delta \varphi^3 + \frac{1}{24 V^8} t (5 + 6 t^2 + 19 \eta^2 + 24 \eta^2 t^2 + 47 \eta^4 + 30 \eta^4 t^2) \Delta \varphi^4 \quad (18)$$

A further development of the foregoing series to the sixth order is given by Wl. Hristow in *Zeitschr. f. Verm.*, 1935, pp. 649-650.

Meridional arc between the isometric latitudes q_1 and q_2

In accordance with the development in the first half-volume, section 41, p. 73, we will set up the meridional arc between the isometric latitudes q_1 and q_2 or q_1 and $q_1 + \Delta q$. According to Maclaurin's series

we have, if we denote this meridional arc by b ,

$$b = \frac{d b}{d q} \Delta q + \frac{1}{2} \frac{d^2 b}{d q^2} \Delta q^2 + \frac{1}{6} \frac{d^3 b}{d q^3} \Delta q^3 + \frac{1}{24} \frac{d^4 b}{d q^4} \Delta q^4 + \dots \quad (19)$$

Here we have

$$\frac{d b}{d q} = \frac{d b}{d \varphi} \frac{d \varphi}{d q} \quad \frac{d b}{d \varphi} = M \text{ and according to (5) } \frac{d \varphi}{d q} = \frac{N \cos \varphi}{M} = V^2 \cos \varphi,$$

therefore

$$\frac{d b}{d q} = N \cos \varphi. \quad (20)$$

For the setting up of the further differential quotients we take, in addition, from the first half-volume, pp. 51, 60 and 62:

$$N = \frac{c}{V}, \quad \frac{N}{M} = V^2 = 1 + \eta^2, \quad \frac{d \eta^2}{d \varphi} = -2 \eta^2 t, \quad \frac{d V}{d \varphi} = -\frac{\eta^2 t}{V}.$$

Then we have further

$$\frac{d^2 b}{d q^2} = \frac{d(N \cos \varphi)}{d \varphi} \frac{d \varphi}{d q} = \left(\frac{c}{V^3} \eta^2 t \cos \varphi - N \sin \varphi \right) V^2 \cos \varphi$$

or

$$\frac{d^2 b}{d q^2} = N \eta^2 t \cos^2 \varphi - N \frac{(1 + \eta^2)}{\sin \varphi \cos \varphi}$$

and this yields

$$\frac{d^2 b}{d q^2} = -N t \cos^2 \varphi. \quad (21)$$

Differentiating again yields

$$\frac{d^3 b}{d q^3} = \left(-\frac{c}{V^3} \eta^2 t \sin \varphi \cos \varphi - N \cos^2 \varphi + N \sin^2 \varphi \right) \frac{d \varphi}{d q}$$

and with the above value of $\frac{d \varphi}{d q}$

$$\frac{d^3 b}{d q^3} = -N \eta^2 \sin^2 \varphi \cos \varphi - N(1 + \eta^2) \cos^3 \varphi + N(1 + \eta^2) \sin^2 \varphi \cos \varphi$$

or

$$\begin{aligned} \frac{d^3 b}{d q^3} &= N t^2 \cos^3 \varphi - N(1 + \eta^2) \cos^3 \varphi \\ \frac{d^3 b}{d q^3} &= -N(1 - t^2 + \eta^2) \cos^3 \varphi. \end{aligned} \quad (22)$$

Finally, we have for the fourth differential quotient the following development:

$$\frac{d^4 b}{d q^4} = \left\{ - (1 - t^2 + \eta^2) \cos^3 \varphi \frac{c}{V^3} \eta^2 t + N \cos^3 \varphi 2 \frac{t}{\cos^2 \varphi} + N \cos^3 \varphi 2 \eta^2 t \right. \\ \left. + N (1 - t^2 + \eta^2) 3 \cos^2 \varphi \sin \varphi \right\} (1 + \eta^2) \cos \varphi.$$

Since we do not go beyond η^4 , then we can set here $\frac{1}{V^2} = 1 - \eta^2$, and if we introduce further

$$\frac{1}{\cos^2 \varphi} = 1 + t^2, \text{ then we find easily}$$

$$\frac{d^4 b}{d q^4} = N t \cos^3 \varphi \left\{ 5 + 4 \eta^2 - t^2 + t^2 \eta^2 - \eta^4 t^2 \right\} (1 + \eta^2) \cos \varphi$$

and ultimately

$$\frac{d^4 b}{d q^4} = N t \cos^4 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4). \quad (23)$$

If we set the differential quotients (20) to (23) into (19), then we obtain at once

$$b = N \cos \varphi \Delta q - \frac{1}{2} N t \cos^2 \varphi \Delta q^2 - \frac{1}{6} N (1 - t^2 + \eta^2) \cos^3 \varphi \Delta q^3 \\ + \frac{1}{24} N (5 - t^2 + 9 \eta^2 + 4 \eta^4) t \cos^4 \varphi \Delta q^4. \quad (24)$$

Section 31. The Conformal Projection of the Terrestrial Ellipsoid

The ideas put forward at the beginning of the preceding section 30 hold good for arbitrary methods of projection, which we obtain if we introduce arbitrary functions for the equations (1), p. 151. In the first half-volume, Chapter VI, we already have seen that for geodetic purposes the conformal projections are of special importance, and now in the following we will study more closely the methods of the conformal projection of the terrestrial ellipsoid.

These investigations turn out especially simple if we introduce, instead of the geographic latitude φ , the isometric latitude q . Instead of the projection equations (1), section 30, p. 151, we obtain then the new projection equations

$$x = f_1' (q, l) \quad y = f_2' (q, l). \quad (1)$$

There follows hence

$$dx = \frac{\partial f_1' (q, l)}{\partial q} dq + \frac{\partial f_1' (q, l)}{\partial l} dl$$

or in a simplified manner of writing

$$dx = \frac{\partial x}{\partial q} dq + \frac{\partial x}{\partial l} dl \quad (2)$$

and, correspondingly,

$$dy = \frac{\partial y}{\partial q} dq + \frac{\partial y}{\partial l} dl \quad (3)$$

and for ds there follows then according to (3), section 30, p. 151,

$$ds^2 = \left(\left(\frac{\partial x}{\partial q} \right)^2 + \left(\frac{\partial y}{\partial q} \right)^2 \right) dq^2 + \left(\left(\frac{\partial x}{\partial l} \right)^2 + \left(\frac{\partial y}{\partial l} \right)^2 \right) dl^2 + 2 \left(\frac{\partial x}{\partial q} \frac{\partial x}{\partial l} + \frac{\partial y}{\partial q} \frac{\partial y}{\partial l} \right) dq dl \quad (4)$$

If we introduce three quantities E , F and G whose meaning is

$$E = \left(\frac{\partial x}{\partial q} \right)^2 + \left(\frac{\partial y}{\partial q} \right)^2, \quad F = \frac{\partial x}{\partial q} \frac{\partial x}{\partial l} + \frac{\partial y}{\partial q} \frac{\partial y}{\partial l}, \quad G = \left(\frac{\partial x}{\partial l} \right)^2 + \left(\frac{\partial y}{\partial l} \right)^2 \quad (5)$$

then we have in a brief manner of writing

$$ds^2 = E dq^2 + 2 F dq dl + G dl^2. \quad (6)$$

The three quantities E , F and G are referred to as basic quantities of first order.

Now we will set the three values (5) also into the expression (4), section 30, p. 152, for the scale factor and have then

$$m^2 = \frac{E dq^2 + 2 F dq dl + G dl^2}{N^2 \cos^2 \varphi (dq^2 + dl^2)} \quad (7)$$

This can also be put into another form by introducing the azimuth α of the linear element $d\sigma$ instead of the two quantities dq and dl . According to Fig. 1, which shows a part of the previous Fig. 1 in section 30, p. 151, we have

$$\tan (90^\circ - \alpha) = \frac{QP'}{PQ} = \frac{M dq}{N \cos \varphi dl}$$

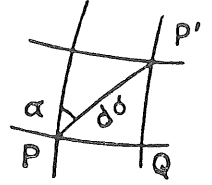


Fig. 1.

or

$$\tan \alpha = \frac{N \cos \varphi dl}{M dq} \quad \text{and according to (5), section 30, p. 152,} \quad \tan \alpha = \frac{dl}{dq}. \quad (8)$$

With this, the denominator of (7) becomes

$$N^2 \cos^2 \varphi (dq^2 + \tan^2 \alpha dq^2) = N^2 \cos^2 \varphi dq^2 (1 + \tan^2 \alpha) = N^2 \cos \varphi \frac{dq^2}{\cos^2 \alpha},$$

and if we eliminate, likewise, dl in the numerator of (7) with the help of (8), then we will have

$$m^2 = \frac{(E dq^2 + 2 F dq^2 \tan \alpha + G dq^2 \tan^2 \alpha) \cos^2 \alpha}{N^2 \cos^2 \varphi dq^2}$$

or

$$m^2 = \frac{E \cos^2 \alpha + 2 F \sin \alpha \cos \alpha + G \sin^2 \alpha}{N^2 \cos^2 \varphi}. \quad (9)$$

We have found in the first half-volume, p. 190, as a characteristic of the conformal projection that

between the original and the image a similarity exists in the smallest parts, or, which means the same, that the scale factor at a point of the image with respect to the original is the same in all directions.

If we start from this second definition, then the expression (9) must be independent of the azimuth α , and for this it is necessary that $F = 0$ and $E = G$; for in this case

$$m^2 = \frac{E}{N^2 \cos^2 \varphi} = \frac{G}{N^2 \cos^2 \varphi} \quad (10)$$

becomes independent of α .

The two conditions $F = 0$ and $E = G$ yield according to (5)

$$\left. \begin{aligned} \frac{\partial x}{\partial q} \frac{\partial x}{\partial l} + \frac{\partial y}{\partial q} \frac{\partial y}{\partial l} &= 0 \\ \left(\frac{\partial x}{\partial q} \right)^2 + \left(\frac{\partial y}{\partial q} \right)^2 &= \left(\frac{\partial x}{\partial l} \right)^2 + \left(\frac{\partial y}{\partial l} \right)^2 \end{aligned} \right\} \quad (11)$$

From the first equation (11) we obtain

$$\frac{\partial y}{\partial l} = - \frac{\frac{\partial x}{\partial q} \frac{\partial x}{\partial l}}{\frac{\partial y}{\partial q}}$$

and this set into the second equation yields

$$\left(\frac{\partial x}{\partial q} \right)^2 + \left(\frac{\partial y}{\partial q} \right)^2 = \frac{\left(\frac{\partial x}{\partial l} \right)^2}{\left(\frac{\partial y}{\partial q} \right)^2} \left(\left(\frac{\partial x}{\partial q} \right)^2 + \left(\frac{\partial y}{\partial q} \right)^2 \right).$$

This equation can exist only if either

$$\left(\frac{\partial x}{\partial q} \right)^2 + \left(\frac{\partial y}{\partial q} \right)^2 = 0 \quad \text{or} \quad \left(\frac{\partial x}{\partial l} \right)^2 = \left(\frac{\partial y}{\partial q} \right)^2$$

Since the first equation leads to imaginary quantities, we will limit ourselves to the second equation and have then

$$\frac{\partial x}{\partial l} = \pm \frac{\partial y}{\partial q}. \quad (12)$$

If we consider at first the upper sign here, i.e. the equation

$$\frac{\partial x}{\partial l} = + \frac{\partial y}{\partial q},$$

and set this value into the first equation (11), then we obtain

$$\frac{\partial x}{\partial q} = - \frac{\partial y}{\partial l}.$$

On the other hand, if we take the lower sign in (12), and hence the equation

$$\frac{\partial x}{\partial l} = -\frac{\partial y}{\partial q},$$

then the first equation (11) yields

$$\frac{\partial x}{\partial q} = +\frac{\partial y}{\partial l}.$$

We thus have found that a projection is conformal if the differential equations

$$\frac{\partial x}{\partial l} = +\frac{\partial y}{\partial q} \quad \text{and} \quad \frac{\partial x}{\partial q} = -\frac{\partial y}{\partial l} \quad (13)$$

or

$$\frac{\partial x}{\partial l} = -\frac{\partial y}{\partial q} \quad \text{and} \quad \frac{\partial x}{\partial q} = +\frac{\partial y}{\partial l} \quad (14)$$

are satisfied. We call these two pairs of equations the Cauchy-Riemann differential equations.

We can put the conditions for the conformal projection yet into another form. For this, we form the complex quantities

$$w = q + i l \quad (15) \quad \text{and} \quad z = x + i y \quad (16)$$

and will assume that z is a function of the complex variable w , i.e. we shall have $z = F(w)$. Then

$$\frac{\partial z}{\partial q} = \frac{dz}{dw} \frac{\partial w}{\partial q} \quad \frac{\partial z}{\partial l} = \frac{dz}{dw} \frac{\partial w}{\partial l}.$$

But according to (15) we have

$$\frac{\partial w}{\partial q} = 1 \quad \frac{\partial w}{\partial l} = i,$$

and hence we will have

$$\frac{\partial z}{\partial q} = \frac{dz}{dw} \quad \frac{\partial z}{\partial l} = i \frac{dz}{dw} \quad \text{or} \quad \frac{\partial z}{\partial l} = i \frac{\partial z}{\partial q}. \quad (17)$$

Since according to (16)

$$\frac{\partial z}{\partial l} = \frac{\partial x}{\partial l} + i \frac{\partial y}{\partial l} \quad \frac{\partial z}{\partial q} = \frac{\partial x}{\partial q} + i \frac{\partial y}{\partial q}$$

then we have according to (17)

$$\frac{\partial x}{\partial l} + i \frac{\partial y}{\partial l} = i \frac{\partial x}{\partial q} - \frac{\partial y}{\partial q}.$$

If we separate here the real from the imaginary parts, then we obtain

$$\frac{\partial x}{\partial l} = -\frac{\partial y}{\partial q} \quad \frac{\partial y}{\partial l} = +\frac{\partial x}{\partial q},$$

which agrees with equations (14).

On the other hand, we can start from the two complex quantities

$$w = q - i l \quad (15^*) \quad \text{and} \quad z = x + i y, \quad (16^*)$$

where $z = F(w)$ shall again be a function of the complex variable w . If we then carry out the above development once more, then there follows

$$\frac{\partial x}{\partial l} = + \frac{\partial y}{\partial q}, \quad \frac{\partial y}{\partial l} = - \frac{\partial x}{\partial q},$$

and this agrees with the previous equations (13).

We can thus formulate the condition for the conformal projection quite generally also in such a way that the expression $z = x + i y$ must be a function of the variable $w = q \pm i l$.

Besides, the function

$$z = F(w) \quad \text{or} \quad x + i y = F(q \pm i l) \quad (18)$$

can then be chosen in such a way that it corresponds to further given conditions.

In this section we only have developed the conformal projection of the terrestrial ellipsoid on the plane. The projection equation (18) found hereby has further significance, however. In the following Chapter V, we shall further consider the special case of the conformal projection of the ellipsoid on the sphere. Denoting on the sphere the isometric latitude by ω and the longitude by λ , then, instead of the foregoing equation (18), we have the equation

$$\omega + i \lambda = F(q + i l) \quad (19)$$

as the general projection equation for the conformal projection of the ellipsoid on the sphere, as can be shown by analogy with the foregoing development.

Section 32. The Gauss-Krüger Projection

The projection by Gauss-Krüger is characterized by the requirements that the axis of abscissae of the plane coordinate system is the image of a meridian, the main meridian, and that this main meridian is projected at true scale. Besides, the projection is conformal.

We start from the basic equation (18), section 31, above, of the conformal projection

$$x + i y = F(q + i l) \quad (1)$$

and use the main meridian at the same time as zero meridian for the counting of longitude. The axis of abscissae is then the image of the main meridian if we set $l = 0$ and $y = 0$, whereby we obtain

$$x = F(q). \quad (2)$$

Since the main meridian shall further be projected at true scale, then, counting the abscissae from the equator, x must be equal to the meridional arc from the equator to the isometric latitude q . If we denote this meridional arc by B , then we have

$$x = B \quad \text{or} \quad F(q) = B. \quad (3)$$

Function (1) is therefore the same as the one by which the dependence of the meridional arc B on the isometric latitude is expressed.

If we limit ourselves in the case of the Gauss-Krüger projection to the projection of a narrow strip of the terrestrial ellipsoid at both sides of the main meridian, then l is a small quantity, and we can develop the function $F(q + il)$ in a series according to Taylor's theorem. We obtain then for the points on the parallel q

$$F(q + il) = F(q) + il \frac{dF(q)}{dq} + \frac{1}{2} (il)^2 \frac{d^2 F(q)}{dq^2} + \frac{1}{6} (il)^3 \frac{d^3 F(q)}{dq^3} + \frac{1}{24} (il)^4 \frac{d^4 F(q)}{dq^4} + \dots$$

or according to (1) and (3)

$$x + iy = B + il \frac{dB}{dq} + \frac{1}{2} (il)^2 \frac{d^2 B}{dq^2} + \frac{1}{6} (il)^3 \frac{d^3 B}{dq^3} + \frac{1}{24} (il)^4 \frac{d^4 B}{dq^4} + \dots$$

Now we have

$$i^2 = -1 \quad i^3 = -i \quad i^4 = +1, \text{ and so on,}$$

and hence, we have

$$x + iy = B + il \frac{dB}{dq} - \frac{1}{2} l^2 \frac{d^2 B}{dq^2} - \frac{1}{6} il^3 \frac{d^3 B}{dq^3} + \frac{1}{24} l^4 \frac{d^4 B}{dq^4} + \dots$$

and if we separate in this equation the real parts from the imaginary ones, then there follows, if we go at the same time as far as the sixth order,

$$x = B - \frac{l^2}{2} \frac{d^2 B}{dq^2} + \frac{l^4}{24} \frac{d^4 B}{dq^4} - \frac{l^6}{720} \frac{d^6 B}{dq^6} + \dots \quad (4)$$

$$y = l \frac{dB}{dq} - \frac{l^3}{6} \frac{d^3 B}{dq^3} + \frac{l^5}{120} \frac{d^5 B}{dq^5} - \dots \quad (5)$$

The form of these two series is evident at once, because the meridian taken as x -axis is an axis of symmetry. As a power series, the value $x - B$ can only contain the even powers l^2, l^4, \dots , and with $l = 0$ we must also have $x - B = 0$, i.e. $x = B$. Just as certainly, with $l = 0$ we must also have $y = 0$, and since y and l have like signs, and besides, y for $\pm l$ taken absolutely must remain the same, series (5) can only contain the odd powers l, l^3, \dots .

The derivatives of B with respect to q must be carried out, for which we have:

$$dB = M dq = \frac{c}{V^3} dq \quad \text{and} \quad \frac{dq}{d\varphi} = \frac{1}{V^2 \cos \varphi}, \quad (6)$$

and hence

$$\begin{aligned} \frac{dB}{dq} &= \frac{c}{V} \cos \varphi \\ \frac{d^2 B}{dq dq} &= -\frac{c}{V^2} \frac{dV}{d\varphi} \cos \varphi - \frac{c}{V} \sin \varphi. \end{aligned} \quad (7)$$

We used already previously (section 30, p. 155)

$$\frac{dV}{d\varphi} = -\frac{\eta^2}{V} t;$$

therefore

$$\begin{aligned}\frac{d^2 B}{d q d \varphi} &= \frac{c}{V^3} (\eta^2 \sin \varphi - V^2 \sin \varphi) = \frac{c}{V^3} \sin \varphi (\eta^2 - (1 + \eta^2)) \\ \frac{d^2 B}{d q^2} &= \frac{-c}{V^3} \sin \varphi \frac{d \varphi}{d q} = -\frac{c \sin \varphi \cos \varphi}{V}.\end{aligned}\quad (8)$$

If we differentiate further in these formulae, then we obtain:

$$\frac{d^3 B}{d q^3} = -\frac{c \cos^3 \varphi}{V} (1 - t^2 + \eta^2) \quad (9)$$

$$\frac{d^4 B}{d q^4} = +\frac{c}{V} \sin \varphi \cos^3 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4). \quad (10)$$

In the next derivatives we will already neglect the terms with η^4 :

$$\frac{d^5 B}{d q^5} = +\frac{c}{V} \cos^5 \varphi (5 - 18 t^2 + t^4 + 14 \eta^2 - 58 \eta^2 t^2) \quad (11)$$

$$\frac{d^6 B}{d q^6} = -\frac{c}{V} \sin \varphi \cos^5 \varphi (61 - 58 t^2 + t^4 + 270 \eta^2 - 330 \eta^2 t^2). \quad (12)$$

From here on we only need to retain the spherical terms, i.e. the terms without η^2 :

$$\frac{d^7 B}{d q^7} = -\frac{c}{V} \cos^7 \varphi (61 - 479 t^2 + 179 t^4 - t^6) \quad (13)$$

$$\frac{d^8 B}{d q^8} = +\frac{c}{V} \sin \varphi \cos^7 \varphi (1385 - 3111 t^2 + 543 t^4 - t^6). \quad (14)$$

Now we can put together the formulae for x and y according to (4) and (5), where we will include, however, only the terms of sixth order; if needed, the terms of seventh and eighth order can easily be inserted according to the foregoing. Taking into account that $\frac{c}{V} = N$ and inserting the necessary ρ 's, then we have:

$$x = B + \frac{l^2 N}{2 \rho^2} \sin \varphi \cos \varphi + \frac{l^4 N}{24 \rho^4} \sin \varphi \cos^3 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4) + \frac{l^6 N}{720 \rho^6} \sin \varphi \cos^5 \varphi (61 - 58 t^2 + t^4) \quad (15)$$

$$y = l \frac{N}{\rho} \cos \varphi + \frac{l^3 N}{6 \rho^3} \cos^3 \varphi (1 - t^2 + \eta^2) + \frac{l^5 N}{120 \rho^5} \cos^5 \varphi (5 - 18 t^2 + t^4). \quad (16)$$

In the formulae (15) and (16) φ and l are the given geographic coordinates of a point, l counted positive to the east from an arbitrary meridian which is assumed as x -axis of a rectangular conformal system of coordinates. B means the meridian arc from the equator to the latitude φ ; x and y are the plane conformal coordinates sought for, x counted like B from the equator of the earth, y perpendicular to x , positive to the east like l .

Since B and x become very large numbers in this manner, we can abbreviate them arbitrarily or count them from an arbitrary zero point in the survey region itself. In theory this is not important, however, because only the difference $x - B$ always occurs in the formulae, and therefore we calculate in the simplest manner with B itself in the formulae.

Hitherto we have assumed that the main meridian is projected at true scale on the axis of abscissae.

We will now extend this requirement so that the lengths on the axis of abscissae are related to the corresponding lengths of the main meridian at a definite constant ratio m_0 . Equation (3) on p. 160 is then replaced by the equation

$$F(q) = m_0 B, \quad (17)$$

and if we carry out the foregoing development once again with this, then we find that the expressions (15) and (16) are likewise to be multiplied by m_0 . We will denote by x_0 and y_0 the coordinates thus reduced and then have to set

$$x_0 = m_0 x \quad y_0 = m_0 y. \quad (18)$$

The introduction of the factor m_0 , to be chosen arbitrarily, has the advantage that we can influence, in a favorable sense, the scale of the image which changes within the whole territory projected. While for $m_0 = 1$ the starting meridian is projected at true scale, and with the distance from the starting meridian the projection becomes more and more magnified, we can reach by a suitable choice of m_0 the result that the maximum magnification occurs at the starting meridian and at the edges of the projection and becomes considerably smaller than the maximum magnification there.

For practical use, we will in addition bring equations (15) and (16) into logarithmic form and write for this

$$x - B = \frac{l^2 N}{2 \varrho^2} \sin \varphi \cos \varphi \left\{ 1 + \frac{l^2}{12 \varrho^2} \cos^2 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4) + \frac{l^4}{360 \varrho^4} \cos^4 \varphi (61 - 58 t^2 + t^4) \right\} \quad (19)$$

$$y = l \frac{N}{\varrho} \cos \varphi \left\{ 1 + \frac{l^2}{6 \varrho^2} \cos^2 \varphi (1 - t^2 + \eta^2) + \frac{l^4}{120 \varrho^4} \cos^4 \varphi (5 - 18 t^2 + t^4) \right\}. \quad (20)$$

From the logarithmic series of the first half-volume, p. 21,

$$\log(1+x) = \mu x - \mu \frac{x^2}{2} + \dots,$$

there follows

$$\log(1+ax+bx^2) = \mu ax - \mu \frac{a^2-2b}{2} x^2 + \dots, \quad (21)$$

and according to this we obtain from (19) and (20)

$$\log(x-B) = \log \frac{l^2 N}{2 \varrho^2} \sin \varphi \cos \varphi + \frac{\mu}{12 \varrho^2} l^2 \cos^2 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4) + \frac{\mu}{1440 \varrho^4} l^4 \cos^4 \varphi (119 - 182 t^2 - t^4) \quad (22)$$

$$\log y = \log l \frac{N}{\varrho} \cos \varphi + \frac{\mu}{6 \varrho^2} l^2 \cos^2 \varphi (1 - t^2 + \eta^2) + \frac{\mu}{180 \varrho^4} l^4 \cos^4 \varphi (5 - 22 t^2 - t^4) \quad (23)$$

$$x_0 = m_0 x \quad y_0 = m_0 y.$$

A further conversion of these logarithmic formulae will be given at the end of the following section. We will there, at the same time, treat another solution of the problem by power series.

Historical remarks

The method of projection treated in this chapter was used for the first time by C. F. Gauss about

1820-1830 in order to project, for the Hannover Land Survey, the surface of the ellipsoid on the plane so that the whole triangle computation could be carried out on the plane, and rectangular coordinates could also be computed for all points. The fundamentals of this conformal projection are contained in the paper:

“Allgemeine Auflösung der Aufgabe, die Teile einer gegebenen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Teilen ähnlich wird,” von C. F. Gauss. Als Beantwortung der von der Königlichen Sozietät der Wissenschaften in Kopenhagen für 1822 gestellten Preisaufgabe, veröffentlicht in *Schumachers Astronomischen Abhandlungen*, Heft 3, Altona 1825.

[“General solution of the problem of projecting the parts of a given surface so that the projection becomes similar to the projected surface in the smallest parts,” by C. F. Gauss. In reply to a prize contest set up by the Royal Society of Sciences in Copenhagen for 1822, published in *Schumachers Astronomische Abhandlungen*, Heft 3, Altona, 1825.]

Otherwise, Gauss did not publish anything, however, on the theory of the method of projection applied in the land survey of Hannover. On the other hand, we possess a later representation in the work, *Theorie der Projektionsmethode der hannoverschen Landesvermessung*, von Oskar Schreiber, Hauptmann im Kgl. Hann. 1. Jägerbataillon, Hannover, Hahnsche Hofbuchhandlung 1866 [Theory of the method of projection of the land survey of Hannover by Oskar Schreiber, Captain at the Royal First Battalion of Riflemen of Hannover, Hannover, Hahn's Booksellers to the Court, 1866], with a preface by Wittstein.

Not until 1912 was a thorough working up of Gauss' coordinates given by L. Krüger in the publication of the Geodetic Institute in Potsdam: *Konforme Abbildung des Erdellipsoids in der Ebene*, von Prof. Dr. L. Krüger, Abteilungsvorsteher im Kgl. Preuss. Geod. Institut., Potsdam 1912.

In 1927 the conformal coordinates were introduced as Gauss-Krüger coordinates for the whole German Reich in the form of six systems of coordinates whose axes of abscissae coincide with the meridians $6^\circ, 9^\circ, 12^\circ, 15^\circ, 18^\circ, 21^\circ$ east of Greenwich. We have already indicated a few things concerning this in Vol. II, 1, 1931,* section 145; we will bring further details later at the end of this chapter.

An additional publication has appeared for the computations of the land survey, *Formeln zur konformen Abbildung des Ellipsoids in der Ebene*, von Prof. Dr. L. Krüger, Geh. Regierungsrat, herausgegeben von der Preussischen Landesaufnahme, Berlin 1919, im Selbstverlage; to be obtained through E. S. Mittler & Sohn, book-sellers.

Section 33. Computation of Geographic Coordinates from Conformal Plane Coordinates

The formulae (15) and (16) of the preceding section 32, p. 162, from which we will start, read

$$x - B = \frac{l^2 N}{2 \varrho^2} \sin \varphi \cos \varphi + \frac{l^4 N}{24 \varrho^4} \sin \varphi \cos^3 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4) + \frac{l^6 N}{720 \varrho^6} \sin \varphi \cos^5 \varphi (61 - 58 t^2 + t^4) \quad (1)$$

$$y = l \frac{N}{\varrho} \cos \varphi + \frac{l^3 N}{6 \varrho^3} \cos^3 \varphi (1 - t^2 + \eta^2) + \frac{l^5 N}{120 \varrho^5} \cos^5 \varphi (5 - 18 t^2 + t^4). \quad (2)$$

For the computation of l and φ from x and y we will invert these formulae directly, which we will carry out now. We begin with (2) and have as a first approximation:

$$l = \frac{y}{N \cos \varphi} + \dots, \text{ and hence } l^3 = \frac{y^3}{N^3 \cos^3 \varphi} + \dots \quad (3)$$

and this set into the second term of (2) yields:

$$y = l N \cos \varphi + \frac{y^3}{6 N^2} (1 - t^2 + \eta^2) + \dots \quad (4)$$

Hence we obtain the second approximation:

$$l = \frac{y}{N \cos \varphi} - \frac{y^3}{6 N^3 \cos \varphi} (1 - t^2 + \eta^2) + \dots \quad (5)$$

and this is sufficient in order to compute the second and third term in (2) finally. There follows with this:

$$y = l N \cos \varphi + \left(\frac{y^3}{N^3 \cos^3 \varphi} - \frac{y^5}{2 N^5 \cos^3 \varphi} (1 - t^2 + \eta^2) \right) \frac{N}{6} \cos^3 \varphi (1 - t^2 + \eta^2) + \frac{y^5}{120 N^4} (5 - 18 t^2 + t^4)$$

and after simple conversions:

$$l = \frac{y}{N \cos \varphi} - \frac{y^3}{6 N^3 \cos \varphi} (1 - t^2 + \eta^2) + \frac{y^5}{120 N^5 \cos \varphi} (5 - 2 t^2 + 9 t^4). \quad (6)$$

We can use this equation (6) immediately in order to replace in (1) l by y . We have for this:

$$\begin{aligned} l^2 &= \frac{y^2}{N^2 \cos^2 \varphi} - \frac{y^4}{3 N^4 \cos^2 \varphi} (1 - t^2 + \eta^2) + \frac{y^6}{36 N^6 \cos^2 \varphi} (1 - t^2)^2 \\ &\quad + \frac{y^6}{60 N^6 \cos^2 \varphi} (5 - 2 t^2 + 9 t^4) \\ l^4 &= \frac{y^4}{N^4 \cos^4 \varphi} - \frac{2 y^6}{3 N^6 \cos^4 \varphi} (1 - t^2) \\ l^6 &= \frac{y^6}{N^6 \cos^6 \varphi}, \end{aligned}$$

and if we set all this into (1), then we obtain easily:

$$x - B = \frac{y^2}{2 N} t + \frac{y^4}{24 N^3} t (1 + 3 t^2 + 5 \eta^2) + \frac{y^6}{720 N^5} t (1 + 30 t^2 + 45 t^4). \quad (7)$$

This equation can be used now to develop the difference of latitude $\varphi_1 - \varphi$, for which we take equation (38), p. 73, from the first half-volume, section 41. Since we shall see at once that $\varphi_1 - \varphi$ is of the second order, then the following expression is sufficient:

$$x - B = M (\varphi_1 - \varphi) + \frac{3 M}{2 V^2} \eta^2 t (\varphi_1 - \varphi)^2 \quad (8)$$

and from (7) and (8) there follows:

$$\begin{aligned} M (\varphi_1 - \varphi) + \frac{3 M}{2 V^2} \eta^2 t (\varphi_1 - \varphi)^2 &= \frac{y^2}{2 N} t + \frac{y^4}{24 N^3} t (1 + 3 t^2 + 5 \eta^2) \\ &\quad + \frac{y^6}{720 N^5} t (1 + 30 t^2 + 45 t^4). \end{aligned} \quad (9)$$

$\varphi_1 - \varphi$ is found again by successive approximation by setting down first

$$\varphi_1 - \varphi = \frac{y^2}{2 M N} t + \frac{y^4}{24 M N^3} t (1 + 3 t^2 + 5 \eta^2)$$

Then we have

$$(\varphi_1 - \varphi)^2 = \frac{y^4}{4 M^2 N^2} t^2 + \frac{y^6}{24 M^2 N^4} t^2 (1 + 3 t^2 + 5 \eta^2),$$

and if this is set into (9), then we find by simple collection of the terms

$$\begin{aligned} \varphi_1 - \varphi = & \frac{y^2}{2 M N} t + \frac{y^4}{24 M N^3} t (1 + 3 t^2 + 5 \eta^2 - 9 \eta^2 t^2) \\ & + \frac{y^6}{720 M N^5} t (1 + 30 t^2 + 45 t^4). \end{aligned} \quad (10)$$

With equations (6) and (10) the inversion of the two equations (1) and (2) is complete. These equations (6) and (10), however, are not yet usable in practice, since all coefficients in them refer to the unknown latitude φ . For the given meridional arc x the latitude φ_1 may be taken from the auxiliary tables, first half-volume, pp. [41] to [44]; at any rate, the foot-point latitude φ_1 is thus to be regarded as known, and therefore we must reduce all coefficients in (6) and (10) to φ_1 .

We begin with the coefficients $t = \tan \varphi$, for which we have according to the first half-volume, section 34, p. 18:

$$t = t_1 - (1 + t_1^2) (\varphi_1 - \varphi) + t_1 (1 + t_1^2) (\varphi_1 - \varphi)^2. \quad (11)$$

If we set this into (10), then we find at first:

$$\begin{aligned} \varphi_1 - \varphi = & \frac{y^2}{2 M N} (t_1 - (1 + t_1^2) (\varphi_1 - \varphi) + t_1 (1 + t_1^2) (\varphi_1 - \varphi)^2) \\ & + \frac{y^4}{24 M N^3} (t_1 (1 + 5 \eta^2) + t_1^3 (3 - 9 \eta^2) - (1 + 10 t_1^2 + 9 t_1^4) (\varphi_1 - \varphi)) \\ & + \frac{y^6}{720 M N^5} (t_1 + 30 t_1^3 + 45 t_1^5), \end{aligned}$$

and can again solve this equation for $\varphi_1 - \varphi$. Since the successive approximation required for this has been carried out now several times already, we will at once give the result here:

$$\begin{aligned} \varphi_1 - \varphi = & \frac{y^2}{2 M N} t_1 - \frac{y^4}{24 M N^3} t_1 (5 + 3 t_1^2 + \eta^2 + 15 t_1^2 \eta^2) \\ & + \frac{y^6}{720 M N^5} t_1 (61 + 90 t_1^2 + 45 t_1^4). \end{aligned} \quad (12)$$

By going a step further, we will replace M and N by M_1 and N_1 . For this we have according to the first half-volume, section 38, pp. 49 and 50, equations (21) and (26):

$$M N = \frac{c^2}{V^4} \quad M_1 N_1 = \frac{c^2}{V_1^4} \quad \frac{M_1 N_1}{M N} = \frac{V^4}{V_1^4},$$

and hence, according to the first half-volume, section 40, p. 63, equation (o),

$$\frac{1}{M N} = \frac{1}{M_1 N_1} \left(1 + \frac{4 \eta^2 t_1}{V^2} (\varphi_1 - \varphi) \right),$$

where we have set immediately t_1 instead of t , since it is a term of sixth order. If we introduce further

for $\varphi_1 - \varphi$ the value $\frac{y^2}{2 M_1 N_1} t_1$, then we have

$$\frac{1}{M N} = \frac{1}{M_1 N_1} \left(1 + \frac{2 y^2}{N_1^2} t_1^2 \eta^2 \right). \quad (13)$$

In the second and third terms of (12), M and N can be replaced immediately by M_1 and N_1 . With this, (12) passes over into

$$\begin{aligned} \varphi_1 - \varphi = & \frac{y^2}{2 M_1 N_1} t_1 - \frac{y^4}{24 M_1 N_1^3} t_1 (5 + 3 t_1^2 + \eta^2 - 9 t_1^2 \eta^2) \\ & + \frac{y^6}{720 M_1 N_1^5} t_1 (61 + 90 t_1^2 + 45 t_1^4). \end{aligned} \quad (14)$$

Now there remains only the reduction of η to η_1 with the help of equation (m) in the first half-volume, section 40, p. 63; we see however that this conversion has only an influence on the terms of eighth order.

In the same way we have to reduce equation (6), p. 165, to the latitude φ_1 . If we begin also here with t , then we have according to (11) and (14):

$$t = t_1 - (1 + t_1^2) (\varphi_1 - \varphi) = t_1 - \frac{y^2}{2 M N} t_1 (1 + t_1^2) \quad (15)$$

and

$$t^2 = t_1^2 - \frac{y^2}{M N} t_1^2 (1 + t_1^2).$$

Since the second term of it, after setting t^2 into (6), p. 165, yields already a term of fifth order, then it is permitted to write also immediately:

$$t^2 = t_1^2 - \frac{y^2}{N_1^2} t_1^2 (1 + t_1^2). \quad (16)$$

Instead of η^2 we can set at once η_1^2 , as was already established.

We have further according to equation (o) in the first half-volume, section 40, p. 63:

$$\frac{1}{N} = \frac{1}{N_1} \left(1 + \frac{\eta^2 t}{V} (\varphi_1 - \varphi) \right) = \frac{1}{N_1} \left(1 + \frac{y^2}{2 N_1^2} t_1^2 \eta_1^2 \right). \quad (17)$$

If we set (16) and (17) into (6), then we obtain:

$$l = \frac{y}{N_1 \cos \varphi} - \frac{y^3}{6 N_1^3 \cos \varphi} (1 - t_1^2 + \eta_1^2 - 3 t_1^2 \eta_1^2) + \frac{y^5}{120 N_1^5 \cos \varphi} (5 - 22 t_1^2 - 11 t_1^4). \quad (18)$$

There remains further the conversion of $\frac{1}{\cos \varphi}$ to $\frac{1}{\cos \varphi_1}$, for which we take at first from the first half-volume, section 34, p. 18:

$$\frac{1}{\cos \varphi} = \frac{1}{\cos \varphi_1} \left(1 - t_1 (\varphi_1 - \varphi) + \frac{1}{2} (1 + 2 t_1^2) (\varphi_1 - \varphi)^2 \right).$$

We set into this the value of $(\varphi_1 - \varphi)$ according to (10), and if we set down at the same time

$$\frac{1}{M} = \frac{V^2}{N} = \frac{1}{N}(1 + \eta^2),$$

then we will have

$$\frac{1}{\cos \varphi} = \frac{1}{\cos \varphi_1} \left(1 - \frac{y^2}{2 N_1^2} t_1^2 (1 + \eta_1^2) + \frac{y^4}{24 N_1^4} t_1^2 (8 + 9 t_1^2) \right). \quad (19)$$

Combining (18) and (19) no longer causes any difficulty, and therefore we give the result without an intermediate computation:

$$l = \frac{y}{N_1 \cos \varphi_1} - \frac{y^3}{6 N_1^3 \cos \varphi_1} (1 + 2 t_1^2 + \eta_1^2) + \frac{y^5}{120 N_1^5 \cos \varphi_1} (5 + 28 t_1^2 + 24 t_1^4). \quad (20)$$

Finally, we shall put together once again the two equations (14) and (20), which represent the solution of the problem of converting conformal plane coordinates into geographic coordinates, with the insertion of ρ :

$$\begin{aligned} \varphi_1 - \varphi = \frac{y^2 \rho}{2 M_1 N_1} t_1 - \frac{y^4 \rho}{24 M_1 N_1^3} t_1 (5 + 3 t_1^2 + \eta_1^2 - 9 t_1^2 \eta_1^2) \\ + \frac{y^6 \rho}{720 M_1 N_1^5} t_1 (61 + 90 t_1^2 + 45 t_1^4) \end{aligned} \quad (21)$$

$$l = \frac{y \rho}{N_1 \cos \varphi_1} - \frac{y^3 \rho}{6 N_1^3 \cos \varphi_1} (1 + 2 t_1^2 + \eta_1^2) + \frac{y^5 \rho}{120 N_1^5 \cos \varphi_1} (5 + 28 t_1^2 + 24 t_1^4). \quad (22)$$

These formulae correspond to the formulae of Schreiber (11), p. 25, and Wittstein, p. X.

We will in addition bring these two equations (21) and (22) into logarithmic form and write at first

$$\begin{aligned} \varphi_1 - \varphi = \frac{y^2 \rho}{2 M_1 N_1} t_1 \left\{ 1 - \frac{y^2}{12 N_1^2} (5 + 3 t_1^2 + \eta_1^2 - 9 t_1^2 \eta_1^2) \right. \\ \left. + \frac{y^4}{360 N_1^4} (61 + 90 t_1^2 + 45 t_1^4) \right\} \end{aligned} \quad (23)$$

$$l = \frac{y \rho}{N_1 \cos \varphi_1} \left\{ 1 - \frac{y^2}{6 N_1^2} (1 + 2 t_1^2 + \eta_1^2) + \frac{y^4}{120 N_1^4} (5 + 28 t_1^2 + 24 t_1^4) \right\}. \quad (24)$$

If we apply to this equation (21) of the previous section 32, p. 163, then there follows at once

$$\begin{aligned} \log (\varphi_1 - \varphi) = \log \frac{y^2 \rho}{2 M_1 N_1} t_1 - \frac{\mu y^2}{12 N_1^2} (5 + 3 t_1^2 + \eta_1^2 - 9 t_1^2 \eta_1^2) \\ + \frac{\mu y^4}{1440 N_1^4} (119 + 210 t_1^2 + 135 t_1^4) \end{aligned} \quad (25)$$

$$\log l = \log \frac{y \rho}{N_1 \cos \varphi_1} - \frac{\mu y^2}{6 N_1^2} (1 + 2 t_1^2 + \eta_1^2) + \frac{\mu y^4}{180 N_1^4} (5 + 28 t_1^2 + 24 t_1^4). \quad (26)$$

The foregoing formulae (23) and (24) as well as formulae (15) and (16) in section 32, p. 162, have been developed further by Wl. Hristow in *Zeitschrift für Vermessungswesen*, 1938, pp. 598-600.

In the paper by Krüger, *Formeln zur konformen Abbildung des Erdellipsoids in der Ebene* [Formulae for the conformal projection of the terrestrial ellipsoid on the plane], mentioned at the end of section 32, p. 164, the equations (22) and (23), section 32, p. 163, and the foregoing equations (25) and (26) are brought into another form which permits the use of auxiliary tables.

Equations (22) and (23), section 32, p. 163, can be written also in the following form:

$$\begin{aligned}\log(x - B) &= \log \frac{N}{2 \varrho^2} l^2 \sin \varphi \cos \varphi + \frac{\mu}{2 \varrho^2} l^2 \cos^2 \varphi - \frac{\mu}{12 \varrho^2} l^2 + \frac{9}{12} \frac{\mu}{\varrho^2} \eta^2 l^2 \cos^2 \varphi \\ \log y &= \log \frac{N}{\varrho} l \cos \varphi + \frac{\mu}{3 \varrho^2} l^2 \cos^2 \varphi - \frac{\mu}{6 \varrho^2} l^2 + \frac{\mu}{6 \varrho^2} \eta^2 l^2 \cos^2 \varphi \\ &\quad - \frac{\mu}{180 \varrho^4} \cos^4 \varphi (t^4 + 22 t^2 - 5) l^4.\end{aligned}$$

For these, the following auxiliary quantities are introduced:

$$\nu' = \frac{\mu}{6 \varrho^2} \eta^2 l^2 \cos^2 \varphi \quad \kappa' = \frac{\mu}{180 \varrho^4} \cos^4 \varphi (t^4 + 22 t^2 - 5), \quad (27)$$

with which we obtain

$$\log(x - B) = \frac{N}{2 \varrho^2} l^2 \sin \varphi \cos \varphi + \frac{\mu}{2 \varrho^2} l^2 \cos^2 \varphi - \frac{\mu}{12 \varrho^2} l^2 + \frac{9}{2} \nu' \quad (28)$$

$$\log y = \log \frac{N}{\varrho} l \cos \varphi + \frac{\mu}{3 \varrho^2} l^2 \cos^2 \varphi - \frac{\mu}{6 \varrho^2} l^2 + \nu' - \kappa' l^4. \quad (29)$$

These equations agree with the system of formulae (6) of Schreiber, *ibid.*, p. 12.

For the inverse formulae (25) and (26), p. 168, we set at first in (25)

$$-\frac{\mu}{12 N_1^2} y^2 (5 + 3 t_1^2 + \eta_1^2 - 9 \eta_1^2 t_1^2) = -\frac{\mu}{6 N_1^2} y^2 - \frac{\mu}{4 N_1^2} \frac{y^2}{\cos^2 \varphi_1} - \frac{\mu}{12 N_1^2} \eta_1^2 y^2 (1 - 9 t_1^2)$$

as well as

$$119 + 210 t_1^2 + 135 t_1^4 = \frac{1}{\cos^4 \varphi_1} (156 - 120 \cos^2 \varphi_1 - 21 + 60 \cos^2 \varphi_1 + 44 \cos^4 \varphi_1).$$

Krüger neglects, in the place cited, the last three terms of the foregoing expression within parentheses, but, on the other hand, includes a term with η_1^2 which we have already neglected in the foregoing developments.

Likewise we use in (26), p. 168, the following conversions:

$$\begin{aligned}-(1 + 2 t_1^2 + \eta_1^2) &= \left(1 - \frac{2}{\cos^2 \varphi_1} - \eta_1^2\right), \\ (5 + 32 t_1^2 + 26 t_1^4) &= \frac{1}{\cos^4 \varphi_1} (26 - 20 \cos^2 \varphi_1 - \cos^4 \varphi_1).\end{aligned}$$

If we use further the auxiliary quantities

$$\left. \begin{aligned}\mu_1 &= \frac{\mu}{6 N_1^2} \eta_1^2 y^2 & \tau &= \frac{\mu}{3 N_1^2} \frac{y^2}{\cos^2 \varphi_1} - \frac{\mu}{90 N_1^4} \frac{y^4}{\cos^4 \varphi_1} (13 - 10 \cos^2 \varphi_1) \\ \mu_2 &= (2 + 4 \eta_1^2) \mu_1 & \mu_3 &= (-2.5 - 2 \eta_1^2 + 4.5 t_1^2) \mu_1 & \mu'_3 &= \mu_2 + \mu_3,\end{aligned} \right\} \quad (30)$$

then we obtain the conversion formulae

$$\log(\varphi_1 - \varphi) = \log \frac{\rho}{2 M_1 N_1} y^2 t_1 - \frac{\mu}{6} \frac{y^2}{N_1^2} - \frac{3}{4} \tau + \mu'_3 \quad (31)$$

$$\log l = \log \frac{\rho}{N_1 \cos \varphi_1} y + \frac{\mu}{6 N_1^2} y^2 - \tau - \mu_1 - \frac{\mu}{180 N_1^4} y^4, \quad (32)$$

in agreement with the system of formulae (13) in Krüger, *ibid.*, p. 20.

Section 34. Meridian Convergence and Scale Factor. Power Series.

If in Fig. 1 NAS is the conformal image of a meridian and WAE is the conformal image of a parallel circle, whereby these two lines intersect at a point A through which we also draw AB and AC parallel to the coordinate axes, then there arises a small angle γ , which Gauss calls the "meridian convergence."

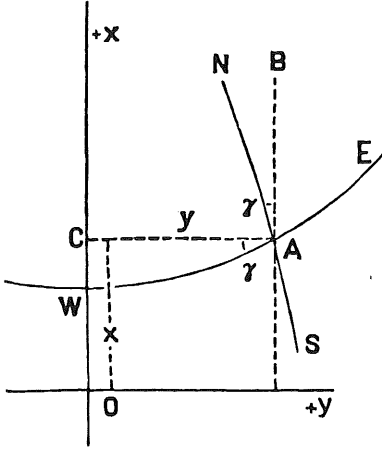


Fig. 1. Plane.

If we can set up the equation of the image of the parallel circle WAE as a function of the plane rectangular coordinates x and y , then we only need to form, in addition, $\frac{dx}{dy}$ in order to have $\tan \gamma$.

In order to form, in this sense, the equation of the parallel circle, we only have to imagine φ constant, and l alone variable, i.e. we differentiate equations (15) and (16), section 32, p. 162, partially with respect to l and thus obtain:

$$\frac{\partial x}{\partial l} = l N \sin \varphi \cos \varphi + \frac{l^3}{6} N \sin \varphi \cos^3 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4) + \frac{l^5}{120} N \sin \varphi \cos^5 \varphi (61 - 58 t^2 + t^4) \quad (1)$$

$$\frac{\partial y}{\partial l} = N \cos \varphi + \frac{l^2}{2} N \cos^3 \varphi (1 - t^2 + \eta^2) + \frac{l^4}{24} N \cos^5 \varphi (5 - 18 t^2 + t^4). \quad (2)$$

The division of (1) by (2) yields:

$$\tan \gamma = \frac{\partial x}{\partial l} : \frac{dy}{dl}.$$

From (2) we obtain:

$$1 : \frac{\partial y}{\partial l} = \frac{1}{N \cos \varphi} \left(1 - \frac{l^2}{2} \cos^2 \varphi (1 - t^2 + \eta^2) + \frac{l^4}{24} \cos^4 \varphi (1 + 6 t^2 + 5 t^4) \right),$$

and if we multiply this by (1), then there follows:

$$\tan \gamma = l \sin \varphi + \frac{l^3}{3} \sin \varphi \cos^2 \varphi (1 + t^2 + 3 \eta^2 + 2 \eta^4) + \frac{l^5}{15} \sin \varphi \cos^4 \varphi (2 + 4 t^2 + 2 t^4). \quad (3)$$

Now we have according to the arc tangent series of the first half-volume, section 34, p. 23:

$$\gamma = \tan \gamma - \frac{1}{3} \tan^3 \gamma + \frac{1}{5} \tan^5 \gamma,$$

and if we set in this the expression for $\tan \gamma$, then there follows easily:

$$\gamma = l \sin \varphi + \frac{l^3}{3 \varrho^2} \sin \varphi \cos^2 \varphi (1 + 3 \eta^2 + 2 \eta^4) + \frac{l^5}{15 \varrho^4} \sin \varphi \cos^4 \varphi (2 - t^2). \quad (4)$$

This equation is to be used when a point is given by φ and l ; but if x and y are given, then it is advisable first to express l by y and second also to reduce everything which depends on φ to φ_1 , i.e. to the foot-point latitude.

For the first we have from (22), section 33, p. 168:

$$\begin{aligned} l &= \frac{y}{N_1 \cos \varphi_1} - \frac{y^3}{6 N_1^3 \cos \varphi_1} (1 + 2 t_1^2 + \eta_1^2) + \frac{y^5}{120 N_1^5 \cos \varphi_1} (5 + 28 t_1^2 + 24 t_1^4) \\ l^3 &= \frac{y^3}{N_1^3 \cos^3 \varphi_1} - \frac{y^5}{2 N_1^5 \cos^3 \varphi_1} \\ l^5 &= \frac{y^5}{N_1^5 \cos^5 \varphi_1}. \end{aligned}$$

For the reduction of the coefficients of (4) to φ_1 a special conversion is still necessary for $\sin \varphi$. We will have:

$$\sin \varphi = \sin \varphi_1 - (\varphi_1 - \varphi) \cos \varphi_1 - \frac{(\varphi_1 - \varphi)^2}{2} \sin \varphi_1,$$

and if we introduce the value of $(\varphi_1 - \varphi)$ according to (14), section 33, p. 167:

$$\sin \varphi = \sin \varphi_1 - \frac{y^2 t_1}{2 M_1 N_1} \cos \varphi_1 + \frac{y^4 t_1}{24 M_1 N_1^3} \cos \varphi_1 (5 + 3 t_1^2) - \frac{y^4 t_1^2}{8 M_1^2 N_1^2} \sin \varphi_1.$$

Here we must take into account that

$$\frac{1}{M} = \frac{V^2}{N} = \frac{1}{N} (1 + \eta^2)$$

with which we obtain easily:

$$\sin \varphi = \sin \varphi_1 - \frac{y^2}{2 N_1^2} \sin \varphi_1 (1 + \eta_1^2) + \frac{5 y^4}{24 N_1^4} \sin \varphi_1.$$

From equation (19), section 33, p. 168, found previously already we have further:

$$\cos^2 \varphi = \cos^2 \varphi_1 \left(1 + \frac{y^2}{N_1^2} t_1^2 \right),$$

and if we set everything into (4) now, then we obtain after collecting the terms of the same order:

$$\gamma = \frac{y \varrho}{N_1} t_1 - \frac{y^3 \varrho}{3 N_1^3} t_1 (1 + t_1^2 - \eta_1^2) + \frac{y^5 \varrho}{15 N_1^5} t_1 (2 + 5 t_1^2 + 3 t_1^4). \quad (5)$$

The symbols N_1 and t_1 indicate that these values, e.g. $t_1 = \tan \varphi_1$, are to be taken as a function of the foot-point latitude φ_1 which corresponds to the length of the arc of meridian x counted from the

equator. Even in the higher terms of formula (5) we have written throughout t_1, η_1^2 , and so on, in this sense, although in the previous development φ and φ_1 were no longer distinguished in the higher terms because by neglecting the terms of seventh order, this is no longer of any consequence.

Equation (5) agrees with Schreiber, formula c, p. 31, if we neglect consistently in the latter the terms of seventh order.

For logarithmic computation we have from (4) and (5) at once:

$$\log \gamma = \log l \sin \varphi + \frac{\mu}{3 \varrho^2} l^2 \cos^2 \varphi (1 + 3 \eta^2 + 2 \eta^4) + \frac{\mu}{90 \varrho^4} l^4 \cos^4 \varphi (7 - 6 t^2) \quad (6)$$

and

$$\log \gamma = \log \frac{\varrho}{N_1} y t_1 - \frac{\mu}{3 N_1^2} y^2 (1 + t_1^2 - \eta_1^2) + \frac{\mu}{90 N_1^4} y^4 (7 + 20 t_1^2 + 13 t_1^4). \quad (7)$$

The latter equation can be brought also into the following form:

$$\log \gamma = \log \frac{\varrho}{N_1} y t_1 - \frac{\mu}{3 N_1^2} \frac{y^2}{\cos^2 \varphi_1} (1 - \eta_1^2 \cos^2 \varphi_1) + \frac{\mu}{90 N_1^4} \frac{y^4}{\cos^4 \varphi_1} (13 - 6 \cos^2 \varphi_1). \quad (8)$$

In Krüger, *Formeln zur konformen Abbildung*, and so on, the auxiliary quantities (27) and (30), section 33, p. 169, are used for this, and then we have:

$$\log \gamma = \log l \sin \varphi + \frac{\mu}{3 \varrho^2} l^2 \cos^2 \varphi + 6 \nu' \quad (9)$$

and

$$\log \gamma = \log \frac{\varrho}{N_1} y t_1 - \tau + \mu_2 + \frac{2 \mu}{45 N_1^4} \frac{y^4}{\cos^2 \varphi_1} \quad (10)$$

in agreement with Krüger, *ibid.*, equation (6), p. 12, and (13), p. 20.

Meridian convergence on the ellipsoid

In addition to the meridian convergence γ on the plane, which we have introduced in this connection in Fig. 1, p. 170, we will now consider the actual meridian convergence, i.e. the spheroidal meridian convergence, which we now denote by γ' . From the previous equation (4) and equation (28), section 27, p. 132, used earlier we obtain

$$\gamma - \gamma' = \frac{2}{3} l^3 \eta^2 \sin \varphi \cos^2 \varphi, \quad (11)$$

in which only terms of the seventh order are neglected. Since η^2 vanishes for the sphere, then (11) shows that the difference $\gamma - \gamma'$ does not exist for the sphere.

Fig. 2 in the margin corresponds to the previous Fig. 1, p. 170; however, everything is now transferred to the ellipsoid. AP'_1 is a line parallel to the meridian of the zero point while AP_1 is the ellipsoidal image of AB parallel to the x -axis (Fig. 1, p. 170). Consequently, the spheroidal and the plane meridian convergence are represented in Fig. 2 by γ' and γ .

In Fig. 2, two further lines $A\varphi'_1$ and $A\varphi_1$ are indicated, of which the first is the spheroidal ordinate of the point A which intersects ON as well as AP'_1 at right angles. $A\varphi_1$, on the other hand, is the ellipsoidal image of the plane ordinate of A , which, because of conformality, must likewise intersect AP_1 and ON at right angles. Therefore, also the small angle $\varphi_1 A \varphi'_1 = \gamma - \gamma'$, and since this is not equal to zero, then we recognize hence that the two foot-points φ_1 and φ'_1 do not coincide exactly.

The scale factor

For the determination of the scale factor of the different parts of the projection against the original there is to be taken into account that we have denoted according to (17), section 32, p. 163, by m_0 the ratio of the abscissae on the plane to the corresponding meridional arcs on the ellipsoid. If a point of the ellipsoid is now projected on the plane by the coordinates

$$x_0 = m_0 x \quad y_0 = m_0 y$$

and from it there starts the infinitely small length dS to which the length $m_0 ds$ corresponds on the plane, then the scale factor of the projection is according to the basic equation (4), section 30, p. 152,

$$m^2 = \frac{m_0^2 ds^2}{dS^2} = \frac{m_0^2 (dx^2 + dy^2)}{(M d\varphi)^2 + (N \cos \varphi dl)^2} \quad (12)$$

or

$$\frac{m^2}{m_0^2} = \frac{dy^2}{dl^2} \frac{1 + \left(\frac{dx}{dy}\right)^2}{N^2 \cos^2 \varphi \left(1 + \left(\frac{M d\varphi}{N \cos \varphi dl}\right)^2\right)}. \quad (13)$$

If we denote by t the direction angle in the plane system and by α the azimuth on the ellipsoid, then we have in part directly, in part according to Fig. 1, section 30, p. 151,

$$\frac{dx}{dy} = \cot t \quad \text{and} \quad \frac{M d\varphi}{N \cos \varphi dl} = \cot \alpha.$$

Herewith (12) becomes

$$\begin{aligned} \frac{m^2}{m_0^2} &= \frac{dy^2}{dl^2} \frac{1 + \cot^2 t}{N^2 \cos^2 \varphi (1 + \cot^2 \alpha)} \\ \frac{m}{m_0} &= \frac{dy}{dl} \frac{1}{N \cos \varphi} \frac{\sin \alpha}{\sin t}. \end{aligned} \quad (14)$$

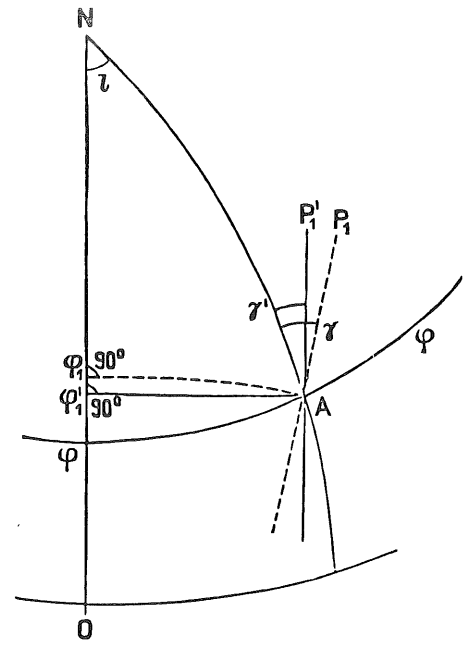


Fig. 2. Ellipsoid.

Now we consider especially the case in which $\alpha = 90^\circ$, i.e. that the arc of the ellipsoid dS lies on a parallel circle, which has as a consequence that φ is constant and further that $t = 90^\circ - \gamma$ if γ is the meridian convergence, which is projected conformally in Fig. 1, p. 170. With this we obtain from (14):

$$\frac{m}{m_0} = \frac{dy}{dl} \frac{\sec \gamma}{N \cos \varphi}. \quad (15)$$

To this we have from (2):

$$\frac{dy}{dl} = N \cos \varphi + \frac{l^2}{2} N \cos^3 \varphi (1 - t^2 + \eta^2),$$

and hence

$$\frac{dy}{dl} \frac{1}{N \cos \varphi} = 1 + \frac{l^2}{2} \cos^2 \varphi (1 - t^2 + \eta^2). \quad (16)$$

We have further from (3):

$$\gamma = l \sin \varphi + l^3 \dots \quad \sec \gamma = 1 + \frac{\gamma^2}{2} = 1 + \frac{l^2 \sin^2 \varphi}{2} + \dots \quad (17)$$

This is sufficient in order to form as a first approximation $\frac{m}{m_0}$, namely as the product of (16) and (17):

$$\begin{aligned} \frac{m}{m_0} &= 1 + \frac{l^2}{2} \cos^2 \varphi (1 - t^2 + \eta^2) + \frac{l^2}{2} \sin^2 \varphi \\ \frac{m}{m_0} &= 1 + \frac{l^2}{2} \cos^2 \varphi (1 + \eta^2). \end{aligned} \quad (18)$$

This is at first only the scale factor in the direction of the parallel circle, and hence perpendicular to the meridian; but since in the case of a conformal projection $\frac{m}{m_0}$ is equal in all directions, we can at once let $\frac{m}{m_0}$ found in (18) hold generally.

Besides, in order to have a check, we will determine $\frac{m}{m_0}$ especially for the meridian and form for this purpose from (12), p. 173,

$$\frac{m^2}{m_0^2} = \frac{dx^2}{d\varphi^2} \frac{1 + \left(\frac{dy}{dx}\right)^2}{M^2 \left(1 + \left(\frac{N \cos \varphi}{M} \frac{dl}{d\varphi}\right)^2\right)}.$$

If we pass over to the meridian, then we will have here according to Fig. 1, p. 170,

$$\frac{dy}{dx} = -\tan \gamma \quad \text{and further} \quad dl = 0;$$

therefore

$$\frac{m}{m_0} = \left(\frac{dx}{d\varphi}\right) \frac{\sec \gamma}{M}.$$

In order to form $\left(\frac{dx}{d\varphi}\right)$, we have from (15), section 32, p. 162:

$$x = B + \frac{l^2}{2} N \sin \varphi \cos \varphi + l^4 \dots$$

$$\frac{dx}{d\varphi} = M + \frac{l^2}{2} \left(\frac{dN}{d\varphi} \sin \varphi \cos \varphi + N \cos^2 \varphi - N \sin^2 \varphi \right).$$

Here we have according to the first half-volume, section 40, equation (h), p. 62:

$$N = \frac{c}{V}, \quad \frac{dN}{d\varphi} = \frac{c}{V^3} \eta^2 t,$$

therefore

$$\begin{aligned} \frac{dx}{d\varphi} &= M + \frac{l^2}{2} \left(\frac{c}{V^3} \eta^2 \sin^2 \varphi + \frac{c}{V} \cos^2 \varphi - \frac{c}{V} \sin^2 \varphi \right) \\ &= M + \frac{c}{V^3} \frac{l^2}{2} \cos^2 \varphi (1 - t^2 + \eta^2); \text{ here is } M = \frac{c}{V^3}, \end{aligned}$$

therefore

$$\frac{dx}{d\varphi} \frac{1}{M} = 1 + \frac{l^2}{2} \cos^2 \varphi (1 - t^2 + \eta^2).$$

This is the same as in the case of (16), and hence the further computation for $\frac{m}{m_0}$ must yield the same value in the direction of the meridian as previously in the case of (16) to (18) in the direction of the parallel circle. Therefore, formula (18) is valid generally, in the direction of the meridian, at right angles to it and in all directions.

In order to develop the formula for $\frac{m}{m_0}$, which goes in (18) only to l^2 , further to l^4 , we must go back again to (2) and take from there:

$$\frac{dy}{dl} \frac{1}{N \cos \varphi} = 1 + \frac{l^2}{2} \cos^2 \varphi (1 - t^2 + \eta^2) + \frac{l^4}{24} \cos^4 \varphi (5 - 18 t^2 + t^4) \quad (19)$$

and from (4):

$$\begin{aligned} \gamma &= l \sin \varphi + \frac{l^3}{3} \sin \varphi \cos^2 \varphi (1 + 3 \eta^2) \\ \sec \gamma &= 1 + \frac{\gamma^2}{2} + \frac{5}{24} \gamma^4 = 1 + \frac{l^2}{2} \sin^2 \varphi + \frac{l^4}{24} \sin^2 \varphi \cos^2 \varphi (8 + 5 t^2). \end{aligned} \quad (20)$$

If we multiply these equations (20) and (19) according to the direction of (15), then we obtain with the insertion of ρ :

$$\frac{m}{m_0} = 1 + \frac{l^2}{2 \rho^2} \cos^2 \varphi (1 + \eta^2) + \frac{l^4}{24 \rho^4} \cos^4 \varphi (5 - 4 t^2). \quad (21)$$

This is the further development of (18) to within l^4 inclusive, but with the omission of all terms in η^2 , and so on, in the coefficients of l^4 . Neglecting these terms, our formula (21) agrees with the corresponding equation of Schreiber, p. 36 (as always after a goniometric conversion).

It is also easy to reduce the formula (21) to y within the accuracy assumed, for we have according to (6), section 33, p. 165:

$$\begin{aligned} l &= \frac{y}{N \cos \varphi} - \frac{y^3}{6 N^3 \cos \varphi} (1 - t^2 + \dots) \\ l^2 &= \frac{y^2}{N^2 \cos^2 \varphi} - \frac{y^4}{3 N^4 \cos^2 \varphi} (1 - t^2 + \dots). \end{aligned}$$

With this, (21) becomes:

$$\frac{m}{m_0} = 1 + \frac{y^2}{2 N^2} (1 + \eta^2) + \frac{y^4}{24 N^4} (1 + \eta^2 \dots). \quad (22)$$

But

$$1 + \eta^2 = V^2 = \frac{N}{M} \quad \text{and} \quad NM = r^2,$$

and hence, we will have

$$\frac{m}{m_0} = 1 + \frac{y^2}{2 r^2} + \frac{y^4}{24 r^4}. \quad (23)$$

The denominator r^4 in the second term holds only approximately, but we may well accept it, since we have neglected all of $1 + \eta^2 \dots$ in the second term anyhow.

An extended development of formulae (21) and (22) is given by Wl. Hristow in *Zeitschr. f. Verm.*, 1938, pp. 595-598.

For logarithmic computation we obtain from (21) easily

$$\log \frac{m}{m_0} = \frac{\mu}{2 \varrho^2} l^2 V^2 \cos^2 \varphi + \frac{\mu}{12 \varrho^4} l^4 \cos^4 \varphi (1 - 2 l^2). \quad (24)$$

In Krüger, *Formeln zur konformen Abbildung*, etc., the last term in (24) is omitted, so that we will have with the auxiliary quantity ν' from (27), section 33, p. 169:

$$\log \frac{m}{m_0} = \frac{\mu}{2 \varrho^2} l^2 \cos^2 \varphi + 3 \nu'. \quad (25)$$

On the other hand, we obtain from (22):

$$\log \frac{m}{m_0} = \frac{\mu}{2 N^2} y^2 (1 + \eta^2) - \frac{\mu}{12 N^4} y^4. \quad (26)$$

For this, Krüger, *ibid.*, p. 20, using the auxiliary quantity μ_1 of equation (30), section 33, p. 169 has

$$\log \frac{m}{m_0} = \frac{\mu}{2 N^2} y^2 + 3 \mu_1 - \frac{\mu}{12 N^4} y^4. \quad (27)$$

Power series for conformal plane coordinates and geographic coordinates

The comparison of the formulae found in the previous sections with the corresponding formulae of section 27 shows that the abscissae of the conformal plane coordinates and the Soldner coordinates agree to the terms of fifth order; and hence, the power series (14), section 28, p. 141, can also be used directly for conformal plane coordinates. But equations (9), (12) and (15), section 28, pp. 139, 140, and 141, in which y occurs, still need a small conversion.

For the moment we will denote by y_s and y_k the spheroidal and the conformal plane ordinate and see then easily from (32), section 27, p. 132, and from (16), section 32, p. 162, that the simple relation

$$y_k = y_s + \frac{y_s^3}{6 N^2} \quad \text{or} \quad y_s = y_k - \frac{y_k^3}{6 N^2}, \quad (28)$$

which we have already known in the first half-volume, p. 197, is correct also for the spheroidal formulae to within the fourth order inclusive. The equations (28) can therefore be used also for the conversion of the power series of section 28.

In the first equation (17), section 28, p. 141, y_s can be replaced immediately by y_k in the terms of third and fourth order. But we will have the term

$$-c y_s^2 = -c y_k^2 + \frac{c y_k^4}{3 N^2},$$

and if the last term of it is combined with $g y^4$, then we obtain:

$$\left(\frac{c}{3 N^2} + g\right) y_k^4 = \left(\frac{\rho t V^2}{6 N^4} + \frac{\rho t V^2}{24 N^4} (1 + 3 t^2)\right) y_k^4 = \frac{\rho t V^2}{24 N^4} (5 + 3 t^2) y_k^4 = g' y_k^4.$$

Also the series for l and y of (17) and (18), section 28, pp. 141 and 142, have been converted in this manner, which we will not indicate further in detail. Therefore, we only summarize the formulae for use, whereby the plane meridian convergence treated above in (4) and (5) is also introduced.

$$\left. \begin{aligned} \Delta \varphi &= a x - b x^2 - c y^2 - d y^2 x + e x^3 - f y^2 x^2 + g' y^4 \\ l &= h y + i y x + k y x^2 - l' y^3 - m' y^3 x + n y x^3 \\ \gamma &= p y + q y x + r y x^2 - s' y^3 - t' y^3 x + u y x^3, \end{aligned} \right\} \quad (29)$$

where the coefficients $a, b, c \dots$ have the meaning indicated in section 28, p. 142, while

$$\begin{aligned} g' &= \frac{[2]^4 t V^2 (5 + 3 t^2)}{24 \rho^3} & l' &= \frac{[2]^3}{6 \rho^2 \cos \varphi} (1 + 2 t^2 + \eta^2) & m' &= n \\ s' &= \frac{[2]^3 t}{3 \rho^2 \cos^2 \varphi} & t' &= u \end{aligned}$$

$$\left. \begin{aligned} x &= A \Delta \varphi + B \Delta \varphi^2 + C l^2 - D \Delta \varphi l^2 - E \Delta \varphi^3 - F \Delta \varphi^2 l^2 + G l^4 \\ y &= H l - J \Delta \varphi l - K \Delta \varphi^2 l - L' l^3 - M' \Delta \varphi l^3 + N \Delta \varphi^3 l \\ \gamma &= P l + Q \Delta \varphi l - R \Delta \varphi^2 l + S' l^3 - T \Delta \varphi l^3 - U \Delta \varphi^3 l, \end{aligned} \right\} \quad (30)$$

where

$$L' = \frac{\cos^3 \varphi}{6 \rho^2 [2]} (t^2 - 1 - \eta^2) \quad M' = \frac{\sin \varphi \cos^2 \varphi}{6 \rho^3 [2] V^2} (5 - t^2) \quad S' = \frac{\sin \varphi \cos^2 \varphi}{3 \rho^2} (1 + 3 \eta^2 + 2 \eta^4).$$

The tables of coefficients of pp. [14] to [19] of the Appendix contain also the above-mentioned new coefficients g', l' , and so on.

With regard to the first publication of these power series we refer to *Zeitschr. f. Verm.*, 1899, pp. 162-176.

The foregoing development could only be carried to the terms of fourth order, since the relations valid for the sphere, which do not permit the carrying further of the formulae, were used here. An immediate development of the power series for the ellipsoid with comprehensive tables of coefficients is given by Wl. Hristow in *Zeitschr. f. Verm.*, 1937, pp. 289-298, and 1938, pp. 617-619.

Section 35. Collection of Formulae and Numerical Examples

For the numerical application we collect the formulae hitherto found once more with the notation of the correction terms introduced by L. Krüger.

$$\begin{aligned}
 1. \quad & \text{Given } \varphi \text{ and } L - L_0 = l \quad \text{Required } x_0, y_0, \frac{m}{m_0} \text{ and } \gamma. \\
 & \nu' = [3.05810] l^2 \cos^4 \varphi \quad \kappa' = [3.12481] (1 + 20 \cos^2 \varphi - 26 \cos^4 \varphi) \\
 & \log(x - B) = \log \frac{N}{2 \varrho^2} l^2 \sin \varphi \cos \varphi + \frac{\mu}{2 \varrho^2} l^2 \cos^2 \varphi - \frac{\mu}{12 \varrho^2} l^2 + \frac{9}{2} \nu' \\
 & \log y = \log \frac{N}{\varrho} l \cos \varphi + \frac{\mu}{3 \varrho^2} l^2 \cos^2 \varphi - \frac{\mu}{6 \varrho^2} l^2 + \nu' - \kappa' l^4 \\
 & \log \gamma = \log l \sin \varphi + \frac{\mu}{3 \varrho^2} l^2 \cos^2 \varphi + 6 \nu' \\
 & \log \frac{m}{m_0} = \frac{\mu}{2 \varrho^2} l^2 \cos^2 \varphi + 3 \nu' \\
 & x_0 = m_0 x \quad y_0 = m_0 y.
 \end{aligned}$$

B denotes the length of the arc of meridian from the equator to the latitude φ (first half-volume, pp. [41] to [44] of the Appendix).

The values of $\log \kappa'$ are to be taken from the following small table.

Auxiliary Table for $\log \kappa'$

	46°	47°	48°	49°	50°	51°	52°	53°	54°	55°
00'	3.7872	3.7948	3.8008	3.8053	3.8083	3.8099	3.8101	3.8090	3.8066	3.8029
10	3.7886	3.7959	3.8017	3.8059	3.8087	3.8101	3.8100	3.8087	3.8061	3.8022
20	3.7899	3.7970	3.8025	3.8065	3.8090	3.8101	3.8099	3.8084	3.8055	3.8014
30	3.7912	3.7980	3.8033	3.8070	3.8093	3.8102	3.8097	3.8080	3.8049	3.8006
40	3.7925	3.7990	3.8040	3.8075	3.8095	3.8102	3.8095	3.8076	3.8043	3.7998
50	3.7937	3.7999	3.8047	3.8079	3.8097	3.8102	3.8093	3.8071	3.8036	3.7989
60	3.7948	3.8008	3.8053	3.8083	3.8099	3.8101	3.8090	3.8066	3.8029	3.7980

$$\begin{aligned}
 \log \frac{1}{2 \varrho} &= 4.384\,5448.7 & \log \frac{\mu}{2 \varrho^2} &= 8.707\,904 & \log \frac{\mu}{6 \varrho^2} &= 5.230\,783 \\
 \log \frac{\mu}{2 \varrho^2} &= 5.707\,904 & \log \frac{\mu}{3 \varrho^2} &= 5.531\,813 & \log \frac{\mu}{12 \varrho^2} &= 4.929\,753.
 \end{aligned}$$

As a numerical example we take from the *Küstenvermessung (Die Kgl. Landes-Triangulation, Hauptdreiecke, 7. Teil., Berlin, 1895, p. 238)* the geographic coordinates of the triangle point Vogelsang:

$$\varphi = 53^\circ 29' 51.3055'' \quad L = 32^\circ 11' 59.8099''.$$

As the zero meridian we assume $L_0 = 31^\circ$; therefore $l = 1^\circ 11' 59.8099''$

$$l = 4319.8099''.$$

Further at the zero meridian we introduce the scale factor:

$$m_0 = 1 - 0.00008.$$

With this, the formulae on p.178 yield the following computational procedure:

$\log(N:\varrho)$	1.491 1568.3	$\mu:2\varrho^2$	5.707 904	$-\mu:12\varrho^2$	4.929 753 _n		
$\log(1:2\varrho)$	4.384 5448.7	l^2	7.270 929	$\log l^2$	7.270 929	l^2	3.05 810
$\log \sin \varphi$	9.905 1651.8	$\cos^2 \varphi$	9.548 825				7.27 093
$\log \cos \varphi$	9.774 4123.4		2.527 658		2.200 682 _n	$\cos^4 \varphi$	9.09 765
$\log l^2$	7.270 9292.6		+ 337.02		- 158.74	ν'	9.42 668
	2.826 2084.8					$\nu' = 0.2671$	
	+ 337.0					$\frac{9}{2} \nu' = + 1.2$	
	- 158.7	$x - B = +$	670.234 m				
	+ 1.2	$B =$	5,929,385.034				
$\log(x-B) =$	2.826 2264.3	$x =$	5,930,055.268 m				
$\log(N:\varrho)$	1.491 1568.3	$\frac{\mu}{3\varrho^2} l^2 \cos^2 \varphi =$	$\frac{2}{3} \times 337.0$			$\nu' = + 0.3$	
$\log l$	3.635 4646.3		= + 224.7			$-\kappa'$	3.8080 _n
$\log \cos \varphi$	9.774 4123.4	$-\frac{\mu}{6\varrho^2} l^2 =$	-2×158.74			l^4	4.5419
	4.901 0338.0		= - 317.5				8.3499 _n
	+ 224.7						- 0.02
	- 317.5						
	+ 0.3						
	- 0.0						
$\log y =$	4.901 0245.5	$y = +$	79,620.436 m				
$\log l$	3.635 4646.3	$\frac{\mu}{3\varrho^2} l^2 \cos^2 \varphi =$	+ 224.7		$\frac{\mu}{2\varrho^2} l^2 \cos^2 \varphi =$	0.000 0337.0	
$\log \sin \varphi$	9.905 1651.8						
	3.540 6298.1	$6 \nu' =$	+ 1.6		$3 \nu' =$	0.8	
	+ 224.7						
	+ 1.6	$\gamma =$	3472.5814''		$\log \frac{m}{m_0} =$	0.000 0337.8	
$\log \gamma =$	3.540 6524.4	$\gamma =$	57' 52.5814''				
		$x =$	5,930,055.268 m		$y = +$	79,620.436 m	
			- 474.404			- 6.370	
		$y_0 =$	5,929,580,864 m		$y_0 = +$	79,614.066 m.	

The more precise computation with ten-place logarithms yielded the values

$$y = +79,620.435 \text{ m} \quad \text{and} \quad y_0 = +79,614.065 \text{ m},$$

which we will retain in the later computations.

2. Given x_0, y_0, L_0 and m_0 Required $\varphi, L = L_0 + l, \gamma$ and $\log \frac{m}{m_0}$.

$$x = \frac{x_0}{m_0} \quad y = \frac{y_0}{m_0}$$

φ_1 = end-point latitude of the meridional arc x starting from the equator.

$$\begin{aligned} \mu_1 &= \frac{\mu}{6 N_1^2} \eta_1^2 y^2 & \tau &= \frac{\mu}{3 N_1^2} \frac{y^2}{\cos^2 \varphi_1} - \frac{\mu}{90 N_1^4} \frac{y^4}{\cos^4 \varphi_1} (13 - 10 \cos^2 \varphi_1) \\ \mu_2 &= (2 + 4 \eta_1^2) \mu_1 & \mu_3 &= (-2.5 - 2 \eta_1^2 + 4.5 t_1^2) \mu_1 & \mu_3' &= \mu_2 + \mu_3 \\ \log(\varphi_1 - \varphi) &= \log \frac{\varrho}{2 M_1 N_1} y^2 t_1 - \frac{\mu}{6 N_1^2} y^2 - \frac{3}{4} \tau + \mu_3' \\ \log l &= \log \frac{\varrho}{N_1 \cos \varphi_1} y + \frac{\mu}{6 N_1^2} y^2 - \tau - \mu_1 - \frac{\mu}{180 N_1^4} y^4 \end{aligned}$$

$$\log \gamma = \log \frac{\varrho}{N_1} y t_1 - \tau + \mu_2 + \frac{2\mu}{45 N_1^4} \frac{y^4}{\cos^2 \varphi_1}$$

$$\log \frac{m}{m_0} = \frac{\mu}{2 N_1^2} y^2 + 3 \mu_1 - \frac{\mu^2}{12 N_1^4} y^4.$$

With these we will compute the inversion of the above numerical example.

$$\begin{aligned} \text{Given } x_0 &= 5,929,580.864 \text{ m} & y_0 &= +79,614.065 \text{ m} \\ L_0 &= 31^\circ & m_0 &= 1 - 0.00008. \end{aligned}$$

By successive approximation we compute from x_0 and y_0

$$x = 5,930,055.268 \text{ m} \quad y = +79,620.435 \text{ m}.$$

For x there follows from the first half-volume, p. [43] of the Appendix, $\varphi_1 = 53^\circ 30' 12.9876''$. Hence the following values follow then for the auxiliary quantities:

$\mu_1 = 0.27 \quad \tau = 635.02 \quad \mu_2 = 0.54 \quad \mu_3 = 1.52 \quad \mu_3' = 2.06$					
$\log(\varrho : M_1 N_1)$	1.704 2919.9	$\log(\varrho : N_1)$	8.508 8430.2	μ	6.63 778
$\log(1 : 2)$	9.698 9700.0	$\log(1 : \cos \varphi)$	0.225 6493.6	1 : 6	9.22 185
$\log y^2$	9.802 0491.0	$\log y$	4.901 0245.5	1 : N_1^2	6.38 884
$\log t_1$	0.130 8483.2		3.635 5169.3	y^2	9.80 205
	1.336 1594.1		+ 112.3		6.76 428
	— 112.3		— 635.0		112.3
	— 476.3		— 0.3		0.0
	+ 2.1		0.0		
$\log(\varphi_1 - \varphi)$	1.336 1007.6	$\log l$	3.635 4646.3		
$\varphi_1 - \varphi$	21.6821"	l	4319.8099"		
φ	53° 29' 51.3055"	L	32° 11' 59.8099"		
$\log(\varrho : N_1)$	8.508 8430.2	$\log \mu$	6.637 784	μ	6.63 778
$\log y$	4.901 0245.5	$\log(1 : 2)$	9.698 970	2	0.30 103
$\log t_1$	0.130 8483.2	$\log(1 : N_1^2)$	6.388 836	1 : 45	8.34 679
	3.540 7158.9	$\log y^2$	9.802 049	1 : N_1^4	2.77 767
	— 635.0		2.527 639	y^4	9.60 410
	+ 0.5		337.0	1 : $\cos^2 \varphi_1$	0.45 130
	+ 0.0		+ 0.8		8.11 867
$\log \gamma$	3.540 6524.4		+ 0.0		0.01
γ	3472.5814"		337.8		
γ	57' 52.5814"	$\log \frac{m}{m_0}$	0.000 0337.8		

The results of both computations thus agree fully.

In the fundamental work by L. Krüger, *Konforme Abbildung des Erdellipsoids in der Ebene*, already mentioned on p. 164, new formulae for the conversion of geographic coordinates into conformal plane coordinates as well as for the computation of the meridian convergence and of the scale factor are developed, to which we will now turn.

In section 32, p. 162, equation (16), we had found:

$$y = \frac{N}{\rho} l \cos \varphi \left\{ 1 + \frac{1}{6 \varrho^2} (1 - t^2 + \eta^2) t^2 \cos^2 \varphi + \frac{1}{120 \varrho^4} (5 - 18 t^2 + t^4) t^4 \cos^4 \varphi \right\}. \quad (1)$$

For this we use the arc sine series of the first half-volume, pp. 23 and 24, and have

$$l = \sin l + \frac{1}{6} \sin^3 l + \frac{3}{40} \sin^5 l + \dots \quad (2)$$

After multiplication by $\cos \varphi$ we obtain from (2) after a simple conversion

$$l \cos \varphi = \sin l \cos \varphi + \frac{1}{6} \sin^3 l \cos^3 \varphi (1 + t^2) + \frac{3}{40} \sin^5 l \cos^5 \varphi (1 + 2 t^2 + t^4), \quad (3)$$

and if we set for the moment

$$\frac{1}{\rho} \sin l \cos \varphi = \lambda, \quad (4)$$

then we obtain from (1) by introducing (3) and (4)

$$\frac{y}{N} = \lambda + \frac{1}{6} (2 + \eta^2) \lambda^3 + \frac{1}{5} \lambda^5 + \dots \quad (5)$$

For the further simplification of this expression we will use the hyperbolic functions whose basic formulae we have given in the first half-volume, pp. 24 and 25. For this purpose we form from (5)

$$i \frac{y}{N} = i \lambda + \frac{1}{6} (2 + \eta^2) i \lambda^3 + \frac{1}{5} i \lambda^5.$$

If we apply to this the tangent series of the first half-volume, p. 23, then there follows, if terms with $\lambda^5 \eta^2$ are omitted,

$$\tan i \frac{y}{N} = i \lambda + \frac{1}{6} i \eta^2 \lambda^3$$

or

$$\frac{1}{i} \tan i \frac{y}{N} = \lambda \left(1 + \frac{1}{6} \eta^2 \lambda^2 \right) \quad (6)$$

and

$$\log \frac{1}{i} \tan i \frac{y}{N} = \log \lambda + \frac{\mu}{6} \eta^2 \lambda^2. \quad (7)$$

According to the fundamental equations for the hyperbolic functions indicated in the first half-volume, pp. 24 and 25, we have

$$\frac{1}{i} \tan i \frac{y}{N} = \sin u,$$

and hence we have

$$\sin u = \lambda + \frac{\eta^2}{6} \lambda^3 \quad (8)$$

and

$$\log \sin u = \log \lambda + \frac{\mu}{6} \eta^2 \lambda^2. \quad (9)$$

We will use equation (8) in order to replace in (5) λ by $\sin u$. We obtain from (8) by inversion

$$\lambda = \sin u - \frac{\eta^2}{6} \sin^3 u,$$

and with this, (5) changes into

$$\frac{y}{N} = \sin u \left(1 + \frac{1}{3} \sin^2 u + \frac{1}{5} \sin^4 u \right)$$

$$\text{and} \quad \log \frac{y}{N} = \log \sin u + \frac{\mu}{3} \sin^2 u + \frac{13}{90} \mu \sin^4 u.$$

For the computation of $\log y$ we have therefore the final equation

$$\log y = \log N \sin u + \frac{\mu}{3} \sin^2 u + \frac{13}{90} \mu \sin^4 u \quad (10)$$

with

$$\log \sin u = \log \sin l \cos \varphi + \frac{\mu}{6} \eta^2 \sin^2 l \cos^2 \varphi. \quad (11)$$

As the next step we undertake the computation of the meridian convergence γ .

According to (3), section 34, p. 170, we have

$$\tan \gamma = l \sin \varphi + \frac{l^3}{3} \sin \varphi \cos^2 \varphi (1 + t^2 + 3\eta^2 + 2\eta^4) + \frac{2l^5}{15} \sin \varphi \cos^4 \varphi (1 + 2t^2 + t^4). \quad (12)$$

To this we take the previous equation (2), which we write in the form:

$$l = \tan l \cos l \left(1 + \frac{1}{6} \sin^2 l + \frac{3}{40} \sin^4 l \right). \quad (13)$$

For $\cos l$ we use the development in series

$$\cos l = 1 - \frac{l^2}{2} + \frac{l^4}{24} - \dots,$$

in which we replace l on the right-hand side again by the value (2). Then we will have

$$\cos l = 1 - \frac{1}{2} \sin^2 l - \frac{1}{8} \sin^4 l. \quad (14)$$

With this we obtain from (13)

$$l = \tan l \left\{ 1 - \frac{1}{3} \sin^2 l \cos^2 \varphi (1 + t^2) - \frac{2}{15} \sin^4 l \cos^4 \varphi (1 + 2t^2 + t^4) \right\}. \quad (15)$$

We set this value into equation (12) for $\tan \gamma$ and find after collecting the individual terms

$$\tan \gamma = \tan l \sin \varphi \left\{ 1 + \left(1 + \frac{2}{3} \eta^2 \right) \eta^2 \sin^2 l \cos^2 \varphi \right\}$$

and

$$\log \tan \gamma = \log \tan l \sin \varphi + \mu \eta^2 \sin^2 l \cos^2 \varphi \left(1 + \frac{2}{3} \eta^2 \right). \quad (16)$$

We can also pass over from $\log \tan \gamma$ to $\log \gamma$. For we have according to the first half-volume, pp. 23 and 24,

$$\gamma = \tan \gamma - \frac{1}{3} \tan^3 \gamma + \frac{1}{5} \tan^5 \gamma - \dots$$

or, if we count at the same time γ on the left-hand side in seconds,

$$\gamma = \varrho \tan \gamma \left(1 - \frac{1}{3} \tan^2 \gamma + \frac{1}{5} \tan^4 \gamma \right).$$

With this, we will have

$$\log \gamma = \log \varrho \tan \gamma - \frac{1}{3} \mu \tan^2 \gamma + \frac{13}{90} \mu \tan^4 \gamma. \quad (17)$$

For small values of l to within $1^\circ 30'$ $\log \gamma$ can be brought also into another form by introducing for $\log \tan \gamma$ the value (16) and setting in the second term $\tan \gamma = \tan l \sin \varphi$. The term with η^4 in (16) as well as the last term in (17) can then be omitted. Then we obtain

$$\log \gamma = \log \varrho \tan l \sin \varphi + \mu \eta^2 \sin^2 l \cos^2 \varphi - \frac{\mu}{3} \tan^2 l \sin^2 \varphi. \quad (18)$$

Before we pass over now to the computation of x , we will carry out a further conversion with the help of equations (1) and (12). From equation (1) we obtain

$$i \frac{y}{2N} = \frac{i}{2} l \cos \varphi \left\{ 1 + \frac{1}{6} (1 - t^2 + \eta^2) l^2 \cos^2 \varphi + \frac{1}{120} (5 - 18 t^2 + t^4) l^4 \cos^4 \varphi \right\}.$$

By using the tangent series we develop hence

$$\begin{aligned} \tan i \frac{y}{2N} = i \frac{l \cos \varphi}{2} \left\{ 1 + \frac{1}{6} (1 - t^2 + \eta^2) l^2 \cos^2 \varphi + \frac{1}{120} (5 - 18 t^2 + t^4) l^4 \cos^4 \varphi \right\} \\ - i \frac{l^3 \cos^3 \varphi}{24} \left\{ 1 + \frac{1}{2} (1 - t^2 + \eta^2) l^2 \cos^2 \varphi \right\} + i \frac{l^5 \cos^5 \varphi}{240}, \end{aligned}$$

and this yields after collecting the homogeneous terms

$$\frac{1}{i} \tan i \frac{y}{2N} = \frac{l \cos \varphi}{2} \left\{ 1 + \frac{1}{12} (1 - 2 t^2 + 2 \eta^2) l^2 \cos^2 \varphi + \frac{1}{120} (1 - 13 t^2 + t^4) l^4 \cos^4 \varphi \right\}. \quad (19)$$

On the other hand, we will develop from (12) an expression for $\tan \frac{\gamma}{2}$. For this we have at first from

the tangent series

$$\tan \frac{\gamma}{2} = \frac{\gamma}{2} + \frac{\gamma^3}{24} + \frac{2\gamma^5}{480} + \dots$$

and since according to the arc tangent series we have again

$$\gamma = \tan \gamma - \frac{1}{3} \tan^3 \gamma + \frac{1}{5} \tan^5 \gamma - \dots$$

then we obtain

$$\tan \frac{\gamma}{2} = \frac{1}{2} \tan \gamma - \frac{1}{8} \tan^3 \gamma + \frac{1}{16} \tan^5 \gamma - \dots$$

If we introduce here the value of $\tan \gamma$ from (12), then we have

$$\begin{aligned} \tan \frac{\gamma}{2} = & \frac{1}{2} l \sin \varphi + \frac{1}{6} l^3 \sin \varphi \cos^2 \varphi (1 + t^2 + 3 \eta^2 + 2 \eta^4) + \frac{1}{15} l^5 \sin \varphi \cos^4 \varphi (1 + 2 t^2 + t^4) \\ & - \frac{1}{8} l^3 \sin^3 \varphi - \frac{1}{8} l^5 \sin^3 \varphi \cos^2 \varphi (1 + t^2) + \frac{1}{16} l^5 \sin^5 \varphi \end{aligned}$$

or

$$\begin{aligned} \tan \frac{\gamma}{2} = & \frac{1}{2} l \sin \varphi \left\{ 1 + \frac{1}{12} l^2 \cos^2 \varphi (4 + t^2 + 12 \eta^2 + 8 \eta^4) \right. \\ & \left. + \frac{1}{120} l^4 \cos^4 \varphi (16 + 2 t^2 + t^4) \right\}. \end{aligned} \quad (20)$$

If we multiply the two equations (19) and (20) by one another, then there follows

$$\begin{aligned} \tan \frac{\gamma}{2} \frac{1}{i} \tan \frac{y}{2N} = & \frac{1}{4} l^2 \cos^2 \varphi \tan \varphi \left\{ 1 + \frac{1}{12} l^2 \cos^2 \varphi (5 - t^2 + 14 \eta^2 + 8 \eta^4) \right. \\ & \left. + \frac{1}{720} l^4 \cos^4 \varphi (122 - 101 t^2 + 2 t^4) \right\}. \end{aligned} \quad (21)$$

After these preparations we can pass over to the computation of the abscissae. To this we append to equation (15), section 32, p. 162 :

$$\begin{aligned} x - B = & \frac{1}{2} N l^2 \cos^2 \varphi \tan \varphi \left\{ 1 + \frac{1}{12} l^2 \cos^2 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4) \right. \\ & \left. + \frac{1}{360} l^4 \cos^4 \varphi (61 - 58 t^2 + t^4) \right\}. \end{aligned}$$

With the help of the tangent series we develop hence

$$\begin{aligned} \tan \frac{x-B}{2N} = & \frac{1}{4} l^2 \cos^2 \varphi \tan \varphi \left\{ 1 + \frac{1}{12} l^2 \cos^2 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4) \right. \\ & \left. + \frac{1}{720} l^4 \cos^4 \varphi (122 - 101 t^2 + 2 t^4) \right\}. \end{aligned} \quad (22)$$

In order to be able to divide this expression by (21), we form at first from (21) the reciprocal value

of the expression within brackets

$$\left\{ 1 + \frac{1}{12} l^2 \cos^2 \varphi (5 - t^2 + 14 \eta^2 + 8 \eta^4) + \frac{1}{720} l^4 \cos^4 \varphi (122 - 101 t^2 + 2 t^4) \right\}$$

for which we obtain

$$1 - \frac{1}{12} l^2 \cos^2 \varphi (5 - t^2 + 14 \eta^2 + 8 \eta^4) + \frac{1}{720} l^4 \cos^4 \varphi (3 + 51 t^2 + 3 t^4).$$

With this, the quotient from (21) and (22) yields

$$\tan \frac{x-B}{2N} = \tan \frac{\gamma}{2} \frac{1}{i} \tan i \frac{y}{2N} \left\{ 1 - \frac{1}{12} l^2 \cos^2 \varphi \eta^2 (5 + 4 \eta^2) \right\}, \quad (23)$$

in which only terms of the sixth order are neglected.

If we introduce further according to (2)

$$l = \sin l + \frac{1}{6} \sin^3 l + \dots,$$

then we will have

$$\tan \frac{x-B}{2N} = \tan \frac{\gamma}{2} \frac{1}{i} \tan i \frac{y}{2N} \left\{ 1 - \frac{1}{12} \eta^2 (5 + 4 \eta^2) \sin^2 l \cos^2 \varphi \right\}, \quad (24)$$

and for this we can also write:

$$\log \tan \frac{x-B}{2N} = \log \tan \frac{\gamma}{2} \frac{1}{i} \tan i \frac{y}{2N} - \frac{\mu}{12} \eta^2 (5 + 4 \eta^2) \sin^2 l \cos^2 \varphi. \quad (25)$$

But now we have according to the fundamental equations for the hyperbolic functions

$$\frac{1}{i} \tan \frac{i y}{2N} = \tan \frac{u}{2};$$

therefore (25) changes into

$$\log \tan \frac{x-B}{2N} = \log \tan \frac{\gamma}{2} \tan \frac{u}{2} - \frac{\mu}{12} \eta^2 (5 + 4 \eta^2) \sin^2 l \cos^2 \varphi. \quad (26)$$

For the change to $\log \frac{x-B}{2N}$ we have according to equation (17)

$$\log \frac{x-B}{2N} = \log \tan \frac{x-B}{2N} - \frac{\mu}{3} \tan^2 \frac{x-B}{2N}. \quad (27)$$

In order to be able to estimate better the order of magnitude of the last term in this equation, we write it for the moment in nonlogarithmic form, namely:

$$\frac{x-B}{2N} = \tan \frac{x-B}{2N} \left(1 - \frac{1}{3} \tan^2 \frac{x-B}{2N} \right).$$

Therefore, the last term is of the third order with respect to $\frac{x-B}{2N}$, and since according to equation (15), section 32, p. 162, the quantity $\frac{x-B}{2N}$ itself is of the second order, then the last term in (27) is thus of the sixth order. The latter can be neglected, since also in (23) we have already neglected the terms of the sixth order. Therefore, we have as the final formula from (26) and (27)

$$\log(x-B) = \log 2N \tan \frac{\gamma}{2} \tan \frac{u}{2} - \frac{\mu}{12} \eta^2 (5 + 4\eta^2) \sin^2 l \cos^2 \varphi. \quad (28)$$

The scale factor

According to (21), section 34, p. 175, we have

$$\frac{m}{m_0} = 1 + \frac{1}{2} l^2 \cos^2 \varphi (1 + \eta^2) + \frac{1}{24} l^4 \cos^4 \varphi (5 - 4\eta^2). \quad (29)$$

For the conversion of this expression we start from the cosine series and the arc tangent series. Accordingly, we have

$$\cos \gamma = 1 - \frac{\gamma^2}{2} + \frac{\gamma^4}{24} - \dots \quad \gamma = \tan \gamma - \frac{1}{3} \tan^3 \gamma + \frac{1}{5} \tan^5 \gamma - \dots$$

and then we will have

$$\cos \gamma = 1 - \frac{1}{2} \tan^2 \gamma + \frac{3}{8} \tan^4 \gamma - \dots$$

If we introduce here the expression for $\tan \gamma$ from (12), then we obtain

$$\cos \gamma = 1 - \frac{1}{2} l^2 \sin^2 \varphi - \frac{1}{3} l^4 \sin^2 \varphi \cos^2 \varphi (1 + \eta^2) + \frac{3}{8} l^4 \sin^4 \varphi$$

and this can be easily converted into

$$\cos \gamma = 1 - \frac{1}{2} l^2 \sin^2 \varphi - \frac{1}{24} l^4 \sin^2 \varphi \cos^2 \varphi (8 - \eta^2).$$

To this we take further from the first half-volume, p. 23, the secant series

$$\frac{1}{\cos l} = 1 + \frac{1}{2} l^2 + \frac{5}{24} l^4 + \dots,$$

so that we obtain then:

$$\frac{\cos \gamma}{\cos l} = 1 + \frac{1}{2} l^2 \cos^2 \varphi + \frac{1}{24} l^4 (5 - 8 \sin^2 \varphi \cos^2 \varphi + \sin^4 \varphi - 6 \sin^2 \varphi)$$

$$\frac{\cos \gamma}{\cos l} = 1 + \frac{1}{2} l^2 \cos^2 \varphi + \frac{1}{24} l^4 \cos^2 \varphi (5 - 9 \sin^2 \varphi)$$

$$\frac{\cos \gamma}{\cos l} = 1 + \frac{1}{2} l^2 \cos^2 \varphi + \frac{1}{24} l^4 \cos^4 \varphi (5 - 4\eta^2).$$

The expression (29) for $\frac{m}{m_0}$ agrees with this up to the term in η^2 , so that we have

$$\frac{m}{m_0} = \frac{\cos \gamma}{\cos l} + \frac{1}{2} \eta^2 l^2 \cos^2 \varphi. \quad (30)$$

In order to arrive further at a value for $\log \frac{m}{m_0}$, we can write the foregoing equation (30) in the form:

$$\frac{m}{m_0} = \frac{\cos \gamma}{\cos l} \left(1 + \frac{1}{2} \eta^2 l^2 \cos^2 \varphi \right),$$

whereby in the last term in the parentheses the factor $\frac{\cos l}{\cos \gamma}$ is neglected. But since

$$\frac{\cos l}{\cos \gamma} = 1 - \frac{1}{2} l^2 \cos^2 \varphi + \dots,$$

then the neglected term is only of the order $\eta^2 l^4$, which is already omitted in (29). In logarithmic form we thus have

$$\log \frac{m}{m_0} = \log \cos \gamma \sec l + \frac{\mu}{2} \eta^2 l^2 \cos^2 \varphi. \quad (31)$$

Summary of the formulae

There are given the geographic coordinates φ and L , the longitude L_0 of the zero meridian and the scale factor m_0 at the zero meridian. There are required the conformal plane coordinates x_0, y_0 , the meridian convergence γ and the scale factor $\frac{m}{m_0}$.

$$\left. \begin{aligned} l &= L - L_0 \\ \log \sin u &= \log \sin l \cos \varphi + v_1 \\ \log y &= \log N \sin u + \frac{\mu}{3} \sin^2 u + \frac{13}{90} \mu \sin^4 u \\ \log \tan \gamma &= \log \tan l \sin \varphi + v_2 \\ \log \gamma &= \log \varrho \tan \gamma - \frac{1}{3} \mu \tan^2 \gamma + \frac{13}{90} \mu \tan^4 \gamma \\ \log (x - B) &= \log 2 N \tan \frac{\gamma}{2} \tan \frac{u}{2} - v_3 \\ \log \frac{m}{m_0} &= \log \cos \gamma \sec l + 3 v_1 \\ y_0 &= m_0 y \quad x_0 = m_0 x. \end{aligned} \right\} \quad (32)$$

The auxiliary quantities have here the following meaning:

$$\left. \begin{aligned} v_1 &= \frac{\mu}{6} 10^7 \eta^2 \sin^2 l \cos^2 \varphi & v_2 &= (6 + 4 \eta^2) v_1 \\ v_3 &= (2.5 + 2 \eta^2) v_1 = \frac{1}{2} (v_2 - v_1). \end{aligned} \right\} \quad (33)$$

Numerical example

Let there be given the geographic coordinates of the point Kleistberg of the Küstenvermessung (*Die Kgl. Landestriangulation, Hauptdreiecke*, 7. Teil., Berlin, 1895, p. 238).

$$\begin{aligned}\varphi &= 53^\circ 28' 20.9266'' & L &= 33^\circ 09' 33.8196'' & L_0 &= 31^\circ \\ m_0 &= 1 - 0.00008 \\ L - L_0 &= l = 2^\circ 09' 33.8196''\end{aligned}$$

With $\eta^2 = 0.002\,3804$ we obtain the auxiliary quantities

$\nu_1 = 0.8666$		$\nu_2 = 5.208$		$\nu_3 = 2.170$	
$\log \sin l$	8.576 1065.061	$\log N$	6.805 5813.515	$\mu : 3$	9.160 663
$\log \cos \varphi$	9.774 6693.638	$\log \sin u$	8.350 7759.566	$\sin^2 u$	6.701 552
ν_1	0.867	$\frac{\mu}{3} \sin^2 u$	+ 728.140		5.862 215
$\log \sin u$	8.350 7759.566	$\frac{13}{90} \mu \sin^4 u$	0.159	13 $\mu : 90$	5.79 748
$u = 1^\circ 17' 06.3397''$		y	5.156 4301.380	$\sin^4 u$	3.40 310
$\frac{u}{2} =$	38' 33.1698''	$y = + 143,360.708$ m			9.20 058
$\log \tan l$	8.576 4150.2	$\mu : 3$	6.160 663	13 $\mu : 90$	5.79 748
$\log \sin \varphi$	9.905 0242.9	$\tan^2 \gamma$	6.962 880	$\tan^4 \gamma$	3.92 576
ν_2	+ 5.2		3.123 543		9.72 324
$\log \tan \gamma$	8.481 4398.3		— 1329.05		+ 0.53
$\log \varrho$	5.314 4251.3	$\gamma = 6247.8718''$			
	— 1329.0	$\gamma = 1^\circ 44' 07.8718''$			
	+ 0.5	$\frac{\gamma}{2} =$	52' 03.9359''		
$\log \gamma =$	3.795 7321.1	$\log \cos \gamma$	9.999 8007.4		
$\log 2$	0.301 0300.0	$\log \sec l$	0.000 3085.2		
$\log N$	6.805 5813.5	$3 \nu_1$	+ 2.6		
$\log \tan \frac{\gamma}{2}$	8.180 3101.9	$\log \frac{m}{m_0}$	0.000 1095.2		
$\log \tan \frac{u}{2}$	8.049 8005.9	$y = + 143,360.708$ m		$x = 5,928,762.568$ m	
$-\nu_3$	— 2.2		— 11.469		— 474.301
$\log (x - B)$	3.336 7219.1	$y_0 = + 143,349.239$ m		$x_0 = 5,928,288.267$ m	
$x - B =$	2171.310 m				
$B =$	5,926,591.258				
$x =$	5,928,762.568 m				

In the case of the projection of two points A and B , which are connected on the ellipsoid by a geodetic line of length S , two different lengths are considered on the plane, namely the projection of the geodetic line itself and the straight line connecting the projections of A and B ; the length of the latter shall be denoted by s , that of the first by σ . If we express for the moment the direction angle which an arbitrary element $d\sigma$ of σ forms with s , by ν , then we have

$$s = \int \cos \nu d\sigma.$$

But since according to the additional investigations of this section 37, pp. 195 and 196, the angle ν is of the second order, then s and σ will differ from each other by terms of the order $\frac{1}{N^4}$.

In the following we will neglect the terms of the order $\frac{1}{N^4}$ and can therefore introduce the straight line connecting A and B directly for the length of the image of the geodetic line. If $\frac{m}{m_0}$ is the scale factor in the differential sense, then we have:

$$S = \int \frac{m_0}{m} ds, \quad (1)$$

to this we have from (23), section 34, p. 176,

$$\frac{m_0}{m} = 1 - \frac{y^2}{2r^2}. \quad (2)$$

For the integration we only have further to bear in mind that r is variable along the length s ; further, y must also be represented as a function of s .

At first we are again concerned with the change of V here. If we give all values relative to the point A the index figure 1, then we have for an arbitrary point of s according to the first half-volume, section 40, pp. 62 and 63,

$$\frac{N_1}{N} = \frac{V}{V_1} = 1 - \frac{(\varphi - \varphi_1)}{V^2} \eta^2 t \quad \text{or} \quad \frac{V^4}{V_1^4} = 1 - \frac{4(\varphi - \varphi_1)}{V^2} \eta^2 t,$$

and since

$$\frac{1}{r^2} = \frac{V^4}{c^2},$$

we have also:

$$\frac{1}{r^2} = \frac{1}{r_1^2} \left(1 - \frac{4(\varphi - \varphi_1)}{V^2} \eta^2 t \right).$$

But we have as a first approximation:

$$\varphi - \varphi_1 = \frac{x - x_1}{M};$$

therefore, likewise as a first approximation:

$$\frac{1}{r^2} = \frac{1}{r_1^2} \left(1 - \frac{4(x-x_1)}{r} \eta^2 t \right). \quad (3)$$

In the second term, we simply set here $V^2 M = N = r$, for which also r_1 can be written. From (2) and (3) we thus have:

$$\frac{m_0}{m} = 1 - \frac{y^2}{2 r_1^2} \left(1 - \frac{4(x-x_1)}{r} \eta^2 t \right). \quad (4)$$

Let this m belong to a point with the coordinates x_y at an arbitrary point of the straight line AB , which has from A the distance p . If the direction angle of the straight line AB on the plane is equal to t_1 , then we have

$$x = x_1 + p \cos t_1 \quad y = y_1 + p \sin t_1, \quad (5)$$

and hence, (4) becomes:

$$\frac{m_0}{m} = 1 - \frac{(y_1 + p \sin t_1)^2}{2 r_1^2} \left(1 - \frac{4 p \cos t_1}{r} \eta^2 t \right). \quad (6)$$

It is to be noted that the last t has the meaning $t = \tan \varphi$ here, as always, while t_1 is the direction angle of AB in the system x_y .

Equation (6) is arranged according to powers of p , and yields thereby:

$$\frac{m_0}{m} = \alpha + \beta p + \gamma p^2 + \delta p^3. \quad (7)$$

The coefficients $\alpha, \beta, \gamma, \delta$ have then the following meanings:

$$\alpha = 1 - \frac{y_1^2}{2 r_1^2} \quad (8)$$

$$\beta = -\frac{y_1 \sin t_1}{r_1^2} + \frac{2 y_1^2 \cos t_1}{r^3} \eta^2 t \quad (9)$$

$$\gamma = -\frac{\sin^2 t_1}{2 r_1^2} + \frac{4 y_1 \sin t_1 \cos t_1}{r^3} \eta^2 t \quad (10)$$

$$\delta = +\frac{2 \sin^2 t_1 \cos t_1}{r^3} \eta^2 t. \quad (11)$$

If we integrate function (7) according to (1) between the boundaries $p = 0$ and $p = s$, then we obtain:

$$\frac{S}{s} = \alpha + \beta \frac{s}{2} + \gamma \frac{s^2}{3} + \delta \frac{s^3}{4}. \quad (12)$$

On the other hand, we introduce three values of $\frac{m_0}{m}$, for the beginning, for the center and for the end point of the line AB , namely:

$$p = 0 \text{ shall give } \frac{m_0}{m_1} = \alpha$$

$$p = \frac{s}{2} \text{ " " } \frac{m_0}{m_m} = \alpha + \beta \frac{s}{2} + \gamma \frac{s^2}{4} + \delta \frac{s^3}{8}$$

$$p = s \text{ " " } \frac{m_0}{m_2} = \alpha + \beta s + \gamma s^2 + \delta s^3.$$

This compared with (12) will yield:

$$\frac{S}{s} = \frac{1}{6} \left(\frac{m_0}{m_1} + \frac{4 m_0}{m_m} + \frac{m_0}{m_2} \right). \quad (13)$$

If we thus calculate the three different $\frac{m_0}{m}$'s according to function (2) not only for the three different y 's but also with respect to the change of r , corresponding to the geographic latitudes $\varphi_1, \varphi_m, \varphi_2$ or the abscissae x_1, x_m, x_2 , then we obtain according to (13) the correct reduction of distance without having needed thereby the coefficients $\alpha, \beta, \gamma, \delta$; it has been sufficient to understand that $\frac{m_0}{m}$ can be expressed by a function of the third degree of the form (7).

But we will nevertheless form also the expression (12) with the introduction of the values of coefficients $\alpha, \beta, \gamma, \delta$ according to (8) to (11) and, in fact, with the transformation $s \sin t_1 = y_2 - y_1$ and $s \cos t_1 = x_2 - x_1$, whereby we obtain:

$$\frac{S}{s} = 1 - \frac{1}{6 r_1^2} (y_1^2 + y_1 y_2 + y_2^2) + \frac{\eta^2 t}{6 r^3} (x_2 - x_1) (y_1^2 + 2 y_1 y_2 + 3 y_2^2). \quad (14)$$

Here we can further reduce r_1 to the mean value r_m according to (3):

$$\frac{1}{r_1^2} = \frac{1}{r_m^2} \left(1 + \frac{2 (x_2 - x_1)}{r} \eta^2 t \right).$$

This combined with (14) yields:

$$\frac{S}{s} = 1 - \frac{1}{6 r_m^2} (y_1^2 + y_1 y_2 + y_2^2) + \frac{\eta^2 t}{6 r^3} (x_2 - x_1) (y_2^2 - y_1^2). \quad (15)$$

r_m holds here as the mean radius of curvature for the mean latitude φ_m or for the mean abscissa x_m of the line AB under consideration.

Equation (15) in logarithmic form is written:

$$\log S - \log s = -\frac{\mu}{6 r_m^2} (y_1^2 + y_1 y_2 + y_2^2) + \frac{\eta^2 t}{6 r^3} (x_2 - x_1) (y_2^2 - y_1^2). \quad (16)$$

The plane distance s refers here to the coordinates x and y of the two points. If the distance for the reduced coordinates $x_0 y_0$ is to be computed, then we insert $\log m_0$, in addition to $\log s$.

This agrees with Schreiber, p. 49, if we make the reciprocal sign transformations as always.

If we neglect in (16) the term with $\frac{\eta^2}{r^3}$, then we can bring the expression also into another form. We have

$$\log S - \log s = -\frac{\mu}{12 r_m^2} (y_1^2 + (y_1 + y_2)^2 + y_2^2).$$

But according to (4) we have by neglecting the same terms

$$\log \frac{m_1}{m_0} = \mu \frac{y_1^2}{2 r_m^2} \quad \log \frac{m_m}{m_0} = \mu \frac{(y_1 + y_2)^2}{8 r_m^2} \quad \log \frac{m_2}{m_0} = \mu \frac{y_2^2}{2 r_m^2},$$

and hence, we will have

$$\log S - \log s = -\frac{1}{6} \left(\log \frac{m_1}{m_0} + 4 \log \frac{m_m}{m_0} + \log \frac{m_2}{m_0} \right). \quad (17)$$

Here also we have to insert $\log m_0$, in addition to $\log s$, if s is to hold for the reduced coordinates x_0, y_0 of the two points.

Reduction of direction

In Figs. 1 and 2 we consider two neighboring points, which are connected on the ellipsoid by a small arc dS ; the latter is projected on the plane by the arc $d\sigma$. We examine the different directions and angles involved here, adding what has already been said in section 34 in regard to Fig. 2, p. 173, about the two meridian convergences γ' on the ellipsoid and γ on the plane.

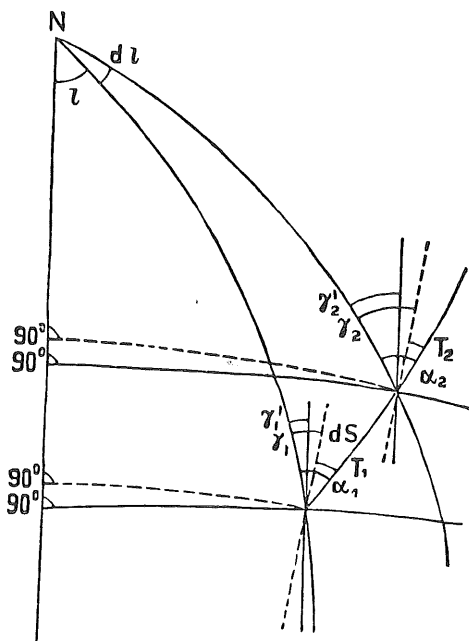


Fig. 1. Ellipsoid.

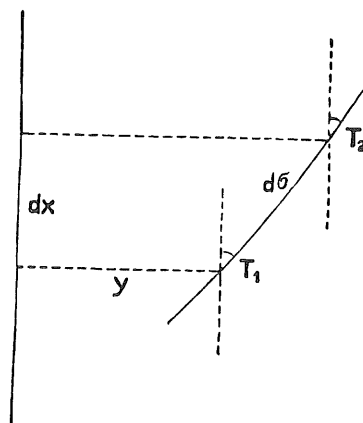


Fig. 2. Plane.

Then we can read immediately the following equations from Fig. 1:

$$\begin{aligned} T_1 &= \alpha_1 - \gamma_1 & T_2 &= \alpha_2 - \gamma_2 \\ T_2 - T_1 &= (\alpha_2 - \alpha_1) - (\gamma_2 - \gamma_1). \end{aligned}$$

Instead of this we also can write

$$T_2 - T_1 = (\alpha_2 - \alpha_1) - (\gamma'_2 - \gamma'_1) - ((\gamma_2 - \gamma'_2) - (\gamma_1 - \gamma'_1))$$

or as a differential

$$dT = d\alpha - d\gamma' - d(\gamma - \gamma'). \quad (18)$$

According to (28), section 27, p. 132, we have on the ellipsoid

$$\begin{aligned} \gamma' &= l \sin \varphi + \frac{1}{3} l^3 \sin \varphi \cos^2 \varphi (1 + \eta^2) + \dots \\ d\gamma' &= dl \sin \varphi + dl^2 \sin \varphi \cos^2 \varphi (1 + \eta^2) + d\varphi l \cos \varphi. \end{aligned} \quad (19)$$

therefore

We have further, according to the differential equation (3), section 7, p. 28, of the geodetic line

$$d\alpha = dl \sin \varphi . \quad (20)$$

If we set (19) and (20) into (18), then we will have

$$dT = -dl l^2 \sin \varphi \cos^2 \varphi (1 + \eta^2) - d\varphi l \cos \varphi - d(\gamma - \gamma') . \quad (21)$$

We have from (11), section 34, p. 172,

$$\gamma - \gamma' = \frac{2}{3} l^3 \eta^2 \sin \varphi \cos^2 \varphi ,$$

and this differentiated yields

$$d(\gamma - \gamma') = 2l^2 \eta^2 \sin \varphi \cos^2 \varphi dl .$$

Therefore, we have now for (21)

$$dT = -d\varphi l \cos \varphi - dl l^2 \sin \varphi \cos^2 \varphi (1 + 3\eta^2) . \quad (22)$$

In this equation, $d\varphi$ and dl must be expressed by dx and dy . For this, we have at first

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy \quad dl = \frac{\partial l}{\partial x} dx + \frac{\partial l}{\partial y} dy . \quad (23)$$

According to (21), section 33, p. 168, we have

$$\frac{\partial \varphi_1}{\partial x} - \frac{\partial \varphi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = -\frac{y t_1}{M_1 N_1}$$

But we have, since φ_1 depends only upon x ,

$$\frac{\partial \varphi_1}{\partial x} = \frac{d\varphi_1}{dx} = \frac{1}{M_1} ,$$

therefore, we obtain from the first equation (23):

$$d\varphi = \frac{1}{M_1} dx - \frac{y t_1}{M_1 N_1} dy . \quad (24)$$

On the other hand, equation (20), section 33, p. 168, yields

$$\frac{\partial l}{\partial x} = 0 \quad \text{and} \quad \frac{\partial l}{\partial y} = \frac{1}{N_1 \cos \varphi_1} ,$$

and we obtain from the second equation (23):

$$dl = \frac{1}{N_1 \cos \varphi_1} dy . \quad (25)$$

In equation (22), the first term is already of the second order, and since $\varphi_1 - \varphi$, according to (21), section 33, p. 168, is likewise of the second order, then we can, if the fourth order is neglected, replace everywhere φ by φ_1 , therefore also η^2 by η_1^2 .

If we introduce (24) and (25) into (22), there follows

$$dT = -\frac{l \cos \varphi_1}{M_1} dx + \frac{l y \sin \varphi_1}{M_1 N_1} dy - \frac{l^2 \sin \varphi_1 \cos \varphi_1}{N_1} (1 + 3 \eta_1^2) dy. \quad (26)$$

According to equation (22), section 33, p. 168, we have

$$l = \frac{y}{N_1 \cos \varphi_1} \text{ therefore, } l^2 = \frac{y^2}{N_1^2 \cos^2 \varphi_1}$$

and thus (26) yields

$$\begin{aligned} dT &= -\frac{y}{M_1 N_1} dx + \frac{y^2 t_1}{M_1 N_1^2} dy - \frac{y^2 t_1}{N_1^3} (1 + 3 \eta_1^2) dy \\ dT &= -\frac{y}{M_1 N_1} dx + \frac{y^2 t_1}{N_1^3} \left(\frac{M_1}{N_1} - (1 + 3 \eta_1^2) \right) dy. \end{aligned}$$

Now since $\frac{N_1}{M_1} = 1 + \eta_1^2$, then dT changes to

$$dT = -\frac{y}{M_1 N_1} dx - \frac{y^2}{N_1^3} 2 \eta_1^2 t_1 dy.$$

Here we have exactly $M_1 N_1 = r_1^2$, and in the second term we can set down approximately $N_1^3 = r_1^3$, and if we now omit the index figure 1 everywhere, then we have

$$dT = -\frac{y}{r^2} dx - \frac{y^2}{r^3} 2 \eta^2 t dy. \quad (27)$$

In Fig. 3 two points A and B are represented at a finite distance, whereby the arc σ is again the plane projection of the geodetic line AB on the ellipsoid. Equation (27) just found represents the change of direction at an arbitrary point of σ , and if we now denote the radius of curvature at this point by R , then we have

$$dT = \frac{d\sigma}{R} \quad \text{or} \quad \frac{1}{R} = \frac{dT}{d\sigma}. \quad (28)$$

For the following, we refer the curve σ to a rectangular system of coordinates p, q , whose p -axis coincides with the connecting straight line AB . Since we are dealing with a very flat curve, and hence $\frac{dq}{dp}$ is very small, then we can set up with sufficient approximation:

$$\frac{1}{R} = \frac{d^2 q}{dp^2}.$$

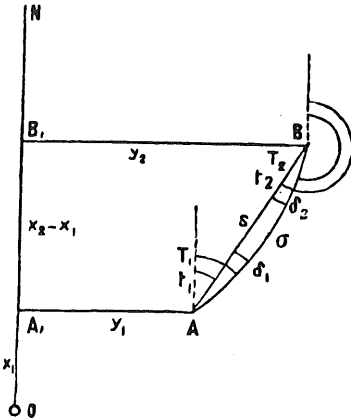


Fig. 3.

With this we will have, if in (28) $d\sigma$ can be replaced by dp , because of the small curvature of the arc σ :

$$\frac{d^2 q}{dp^2} = \frac{dT}{dp} = -\frac{y}{r^2} \frac{dx}{dp} - 2\eta^2 t \frac{y^2}{r^3} \frac{dy}{dp}. \quad (29)$$

We shall assume r^2 as variable again, according to the previous (3), p. 190,

$$\frac{1}{r^2} = \frac{1}{r_1^2} \left(1 - \frac{4(x-x_1)}{r} \eta^2 t \right), \quad (30)$$

therefore, (29) becomes

$$-\frac{d^2 q}{dp^2} = \frac{y}{r_1^2} \left(1 - \frac{4(x-x_1)}{r} \eta^2 t \right) \frac{dx}{dp} + 2\eta^2 t \frac{y^2}{r^3} \frac{dy}{dp}. \quad (31)$$

The conversion of coordinates yields again, just as (5), p. 190:

$$\left. \begin{aligned} x &= x_1 + p \cos t_1 & \text{and} & & y &= y_1 + p \sin t_1 \\ \frac{dx}{dp} &= \cos t_1 & & & \frac{dy}{dp} &= \sin t_1. \end{aligned} \right\} \quad (32)$$

These equations (32) set into (31) will yield again an algebraic function of this form:

$$-\frac{d^2 q}{dp^2} = A + Bp + Cp^2, \quad (33)$$

where the coefficients have the following meanings:

$$A = \frac{1}{r_1^2} y_1 \cos t_1 + \frac{2}{r^3} \eta^2 t y_1^2 \sin t_1 \quad (34)$$

$$B = \frac{1}{r_1^2} \sin t_1 \cos t_1 + \frac{4\eta^2 t y_1}{r^3} (\sin^2 t_1 - \cos^2 t_1) \quad (35)$$

$$C = + \frac{2\eta^2 t}{r^3} \sin t_1 (\sin^2 t_1 - 2\cos^2 t_1). \quad (36)$$

The remaining calculation proceeds again as in the first half-volume, section 68, p. 202, equation (29), namely:

$$\delta_1 = \frac{As}{2} + \frac{Bs^2}{6} + \frac{Cs^3}{12} \quad (37)$$

$$\delta_2 = \frac{As}{2} + \frac{Bs^2}{3} + \frac{Cs^3}{4}. \quad (38)$$

The values of A, B, C from (34) to (36), set into (37) and (38) yield:

$$\delta_1 = \frac{x_2 - x_1}{6r_1^2} (2y_1 + y_2) - \frac{\eta^2 t}{3r^3} (x_2 - x_1)^2 (y_1 + y_2) + \frac{\eta^2 t}{6r^3} (y_2 - y_1) (3y_1^2 + 2y_1 y_2 + y_2^2) \quad (39)$$

$$\delta_2 = \frac{x_2 - x_1}{6r_1^2} (y_1 + 2y_2) - \frac{\eta^2 t}{3r^3} (x_2 - x_1)^2 (y_1 + 3y_2) + \frac{\eta^2 t}{6r^3} (y_2 - y_1) (y_1^2 + 2y_1 y_2 + 3y_2^2). \quad (40)$$

We can bring the equations into a form which is somewhat more useful for practical use if we substitute for r_1 two other values r_{12} and r_{21} , which correspond to the points of the zero point meridian with the abscissae $x_1 + \frac{1}{3}(x_2 - x_1)$ and $x_1 + \frac{2}{3}(x_2 - x_1)$. Then we have according to (30):

$$\frac{1}{r_{12}^2} = \frac{1}{r_1^2} \left(1 - \frac{4}{3} \frac{(x_2 - x_1)}{r_1} \eta^2 t \right), \text{ hence } \frac{1}{r_1^2} = \frac{1}{r_{12}^2} \left(1 + \frac{4}{3} \frac{(x_2 - x_1)}{r_1} \eta^2 t \right) \quad (41)$$

$$\frac{1}{r_{21}^2} = \frac{1}{r_1^2} \left(1 - \frac{8}{3} \frac{(x_2 - x_1)}{r_1} \eta^2 t \right) \quad \frac{1}{r_1^2} = \frac{1}{r_{21}^2} \left(1 + \frac{8}{3} \frac{(x_2 - x_1)}{r_1} \eta^2 t \right). \quad (42)$$

This is to be set into (39) and (40), and if we set down, at the same time, according to Fig. 3

$$T_1 - t_1 = \delta_1 \quad \text{and} \quad T_2 - t_2 = -\delta_2$$

then we obtain:

$$\begin{aligned} T_1 - t_1 = & + \varrho \frac{x_2 - x_1}{6 r_{12}^2} (2 y_1 + y_2) - \varrho \frac{\eta^2 t}{9 r^3} (x_2 - x_1)^2 (y_2 - y_1) \\ & + \varrho \frac{\eta^2 t}{6 r^3} (y_2 - y_1) (3 y_1^2 + 2 y_1 y_2 + y_2^2) \end{aligned} \quad (43)$$

$$\begin{aligned} T_2 - t_2 = & - \varrho \frac{x_2 - x_1}{6 r_{21}^2} (y_1 + 2 y_2) + \varrho \frac{\eta^2 t}{9 r^3} (x_2 - x_1)^2 (y_2 - y_1) \\ & - \varrho \frac{\eta^2 t}{6 r^3} (y_2 - y_1) (y_1^2 + 2 y_1 y_2 + 3 y_2^2). \end{aligned} \quad (44)$$

These formulae agree with Schreiber, p. 46, if we take into account the change of notation.

We obtain a third form of the reductions of direction by substituting in (39) and (40) for r_1 a value r_m , which holds for the mean abscissa $x_m = \frac{x_1 + x_2}{2}$. In (3), p. 190, we will then have $x - x_1 = \frac{x_2 - x_1}{2}$, and if we simply set r again in the term of third order, we will now have

$$\frac{1}{r_1^2} = \frac{1}{r_m^2} + \frac{2 \eta^2 t}{r^3} (x_2 - x_1) \quad (45)$$

and with this we obtain easily from (39) and (40)

$$\begin{aligned} T_1 - t_1 = & + \varrho \frac{x_2 - x_1}{6 r_m^2} (2 y_1 + y_2) + \varrho \frac{\eta^2 t}{3 r^3} (x_2 - x_1)^2 y_1 \\ & + \varrho \frac{\eta^2 t}{6 r^3} (y_2 - y_1) (3 y_1^2 + 2 y_1 y_2 + y_2^2) \end{aligned} \quad (46)$$

$$\begin{aligned} T_2 - t_2 = & - \varrho \frac{x_2 - x_1}{6 r_m^2} (y_1 + 2 y_2) + \varrho \frac{\eta^2 t}{3 r^3} (x_2 - x_1)^2 y_2 \\ & - \varrho \frac{\eta^2 t}{6 r^3} (y_2 - y_1) (y_1^2 + 2 y_1 y_2 + 3 y_2^2). \end{aligned} \quad (47)$$

If we set further for abbreviation

$$\frac{x_2 - x_1}{r_m} = \xi \quad \frac{y_2 - y_1}{r_m} = \eta \quad \frac{y_1 + y_2}{2 r_m} = v. \quad (48)$$

$$\left. \begin{aligned} \varphi_1 = & \frac{\varrho}{2} v \xi + \varrho \eta^2 t v^2 \eta - \frac{\varrho}{12} \eta^2 t (2 \xi^2 - \eta^2) \eta \\ \varphi_2 = & \frac{\varrho}{12} \xi \eta - \frac{\varrho}{3} \eta^2 t (\xi^2 - \eta^2) v, \end{aligned} \right\} \quad (49)$$

then we will have

$$T_1 - t_1 = \varphi_1 - \varphi_2 \quad (50)$$

$$T_2 - t_2 = -(\varphi_1 + \varphi_2). \quad (51)$$

The reductions of direction are contained in this form in L. Krüger, *Formeln zur konformen Abbildung*, etc., p. 37.

Numerical example

In order to be able to show the reduction of distance and direction also by an example, we use the two points Vogelsang and Kleistberg of the Küstenvermessung [survey of the coast], whose rectangular coordinates we have computed in the previous sections 35 and 36, pp. 178 and 188.

With the help of the formulae of the mean latitude, section 21, pp. 99 and 100, we have found at first:

$$\begin{aligned} \log S &= 4.804\,4351\cdot7 \\ \alpha_1 &= 92^\circ\,07'\,34.8826'' \\ \alpha_2 &= 272\,53\,50.8982. \end{aligned}$$

In addition, we take the rectangular coordinates of the two points

Kleistberg: $y_2 = +143,360.708$ m		$x_2 = +5,928,762.568$ m	
Vogelsang: $y_1 = +79,620.436$		$x_1 = +5,930,055.268$	
$\varphi_m = 53^\circ\,29'\,06''$	$\log \frac{\mu}{6 r_m^2} = 2.249\,500$	$m_0 = 1 - 0.00008$	
$\log y_1$ 4.901 024	$\log y_1$ 4.901 024	$\log y_2$ 5.156 430	
$\log y_1$ 4.901 024	$\log y_2$ 5.156 430	$\log y_2$ 5.156 430	
$\log \frac{\mu}{6 r_m^2}$ 2.249 500	$\log \frac{\mu}{6 r_m^2}$ 2.249 500	$\log \frac{\mu}{6 r_m^2}$ 2.249 500	
2.051 548	2.306 954	2.562 360	
- 112.60	- 202.75	- 365.06	
$\log (x_2 - x_1)$ 3.11146 _n	$\log (x_2 - x_1)$ 3.11146 _n		- 112.60
$\log y_2^2$ 0.31280	$\log y_1^2$ 4.80198		- 202.75
$\log \eta^2$ 7.37640	$\log \eta^2$ 7.37640		- 365.06
$\log t$ 0.13056	$\log t$ 0.13056		
$\log \frac{\mu}{6 r^3}$ 5.44444	$\log \frac{\mu}{6 r^3}$ 5.44444		$\log S - \log s = -680.41$
6.37566 _n	5.86484 _n		
- 0.00	- 0.00		
$\log S = 4.804\,4351\cdot7$			
$-(\log S - \log s) = +680.4$			
<hr style="width: 100%;"/>			
$\log s = 4.804\,5032\cdot1$			
$\log m_0 = -347.4$			
<hr style="width: 100%;"/>			
$\log s_0 = 4.804\,4684\cdot7$			

For the computation of the reductions of direction we use equations (48) to (51), pp. 196 and 197.

With

$$\log \frac{1}{r_m} = 3.194\,934$$

we find

$\log \xi = 6.306\ 432_n$			$\log \eta = 7.999\ 348^\#$			$\log \upsilon = 8.242\ 172$		
$\log \varrho : 2$	5.013 395	$\log \varrho$	5.31442	$\log (-\varrho : 6)$	4.53627 _n	$\log \varrho : 12$	4.23524	
$\log \upsilon$	8.242 172	$\log \eta^2$	7.37640	$\log \eta^2$	7.37640	$\log \eta^2$	7.37640	
$\log \xi$	6.306 432 _n	$\log t$	0.13056	$\log t$	0.13056	$\log t$	0.13056	
	9.561 999 _n	$\log \upsilon^2$	6.48434	$\log \xi^2$	2.61286	$\log \eta^3$	3.99804	
	-0.36475"	$\log \eta$	7.99935	$\log \eta$	7.99935			
			7.30507		2.65544		5.74024	
			+ 0.00202"		+ 0.00000"		+ 0.00006"	
$\log \varrho : 12$	4.235 244	$\log (-\varrho : 3)$	4.83724 _n	$\log \varrho : 3$	4.83724		-0.36475"	
$\log \xi$	6.306 432 _n	$\log \eta^2$	7.37640	$\log \eta^2$	7.37640		+ 0.00202	
$\log \eta$	7.999 348	$\log t$	0.13056	$\log t$	0.13056		+ 0.00006	
	8.541 024 _n	$\log \xi^2$	2.61286	$\log \eta^2$	5.99870		$\varphi_1 = -0.36267''$	
	-0.03476"	$\log \upsilon$	8.24217	$\log \upsilon$	8.24217		-0.03476"	
			3.19923 _n		6.58507		+ 0.00038	
			0.00000"		+ 0.00038"		$\varphi_2 = -0.03438''$	
			$T_1 - t_1 = -0.3283''$		$T_2 - t_2 = +0.3970''$			

If we take, in addition to this, the two meridian convergences of pp. 179 and 188,

$$\gamma_1 = 0^\circ 57' 52.5814'' \quad \gamma_2 = 1^\circ 44' 07.8718'' ,$$

then we can compute the two direction angles on the plane:

$$\begin{array}{rcl} \alpha_1 = 92^\circ 07' 34.8826'' & & \alpha_2 = 272^\circ 53' 50.8982'' \\ -(T_1 - t_1) = +0.3283 & & -(T_2 - t_2) = -0.3970 \\ -\gamma_1 = -57\ 52.5814 & & -\gamma_2 = -1\ 44\ 07.8718 \\ \hline \alpha_1^0 = 91^\circ 09' 42.6295'' & & \alpha_2^0 = 271^\circ 09' 42.6294'' . \end{array}$$

The immediate computation from the rectangular coordinates yields:

$$\log s = 4.804\ 5032\cdot 2 \quad \text{or} \quad \log s_0 = 4.804\ 4684\cdot 7$$

$$\alpha_1^0 = \alpha_2^0 \pm 180^\circ = 91^\circ 09' 42.6298'' ,$$

which agrees sufficiently with the previous values.

If we have two neighboring systems of Gauss-Krüger coordinates, for which the difference of longitude of the two zero meridians is equal to Δl , then we will frequently meet with the problem of computing for a point whose coordinates are given in one system the coordinates in the second system.

Without any further theory we can solve the problem in such a way that from the given coordinates of the point we compute the geographic coordinates according to section 33, p. 168, equations (25) and (26), or according to p. 170, equations (31) and (32), and, using the second zero meridian, pass over from them to the rectangular coordinates again.

In the following, however, we will occupy ourselves with the direct solution of the problem.

We find the first treatment of the problem in the work by L. Krüger, *Konforme Abbildung des Erdellipsoids in der Ebene*, Potsdam, 1912, pp. 149-172, mentioned already several times. Further ideas are put forth, in this connection, by W. Grossmann, "Zur Transformation Gauss-Krügerscher Koordinaten mit der Rechenmaschine" [On the transformation of Gauss-Krüger coordinates with the computing machine], *Zeitschr. f. Verm.*, 1935, pp. 353-368 and pp. 385-394. We mention further

Wl. Hristow, "Einige Bemerkungen zu Krügers Koordinaten-Transformation," *Zeitschr. f. Verm.*, 1933, p. 329; 1934, pp. 246 and 402.

Wl. Hristow, "Über die Transformation zwischen zwei Gauss-Krügerschen Streifen," *Zeitschr. f. Verm.*, 1938, pp. 534-540.

A. Hirvonen, "Transformation der Gauss-Krügerschen Koordinaten von einem Streifen zu dem benachbarten," *Zeitschr. f. Verm.*, 1938, pp. 321-326.

In the following discussions we mainly follow the two last-mentioned publications.

The following equation holds for the conformal projection of the surface of the ellipsoid on the plane according to (18), section 31, p. 160,

$$x + iy = F(q + il) \quad \text{or} \quad z = F(w), \quad (1)$$

and hence there follow the equations

$$\frac{\partial x}{\partial l} = -\frac{\partial y}{\partial q} \quad \text{and} \quad \frac{\partial y}{\partial l} = +\frac{\partial x}{\partial q}, \quad (2)$$

which we have already found in (14), section 31, p. 159.

Since x and y are functions of q and l for every projection, then we obtain for function (1) the total differential

$$dz = d(x + iy) = \left(\frac{\partial x}{\partial q} + i \frac{\partial y}{\partial q} \right) dq + \left(\frac{\partial x}{\partial l} + i \frac{\partial y}{\partial l} \right) dl$$

and the differential quotient

$$\begin{aligned} \frac{dz}{dw} = \frac{d(x + iy)}{d(q + il)} &= \frac{\left(\frac{\partial x}{\partial q} + i \frac{\partial y}{\partial q} \right) dq + \left(\frac{\partial x}{\partial l} + i \frac{\partial y}{\partial l} \right) dl}{dq + i dl} \\ &= \frac{\frac{\partial x}{\partial q} + i \frac{\partial y}{\partial q} + i \left(-i \frac{\partial x}{\partial l} + \frac{\partial y}{\partial l} \right) \frac{dl}{dq}}{1 + i \frac{dl}{dq}}. \end{aligned}$$

With equations (2) we obtain

$$\frac{d(x + iy)}{d(q + il)} = \frac{\frac{\partial x}{\partial q} + i \frac{\partial y}{\partial q} + i \left(\frac{\partial x}{\partial q} + i \frac{\partial y}{\partial q} \right) \frac{dl}{dq}}{1 + i \frac{dl}{dq}}$$

and this yields

$$\frac{d(x + iy)}{d(q + il)} = \frac{\partial x}{\partial q} + i \frac{\partial y}{\partial q} = \frac{\partial (x + iy)}{\partial q} \quad (3)$$

or

$$\frac{d\kappa}{dw} = \frac{\partial z}{\partial q} \quad (4)$$

Conversely, we can also regard $w = q + il$ as a function of $z = x + iy$ so that

$$q + il = f(x + iy) \quad \text{or} \quad w = f(z) \quad (5)$$

We obtain then in the same manner

$$\frac{\partial q}{\partial y} = -\frac{\partial l}{\partial x} \quad \text{and} \quad \frac{\partial l}{\partial y} = +\frac{\partial q}{\partial x} \quad (6)$$

and according to the above development

$$\frac{d(q + il)}{d(x + iy)} = \frac{\partial (q + il)}{\partial x} \quad \text{or} \quad \frac{dw}{dz} = \frac{\partial w}{\partial z} \quad (7)$$

If we apply these results (3) or, as the case may be, (4) and (7) to the Gauss-Krüger projection, then function (1) is identical with the function $B = F(q)$, and function (5) is identical with the function $q = f(B)$, as we have already established in section 32, p. 160. Therefore, we have according to (4) and (7)

$$\frac{d\kappa}{dw} = \frac{\partial z}{\partial q} = \frac{dB}{dq} \quad \frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{dq}{dB} \quad (8)$$

According to (7) we have then

$$\frac{d(q + il)}{d(x + iy)} = \frac{\partial f(B)}{\partial x} + i \frac{\partial l}{\partial x}$$

and with the help of the first equation (6), where we now replace q by $f(B)$, we will have

$$\frac{d(q + il)}{d(x + iy)} = \frac{\partial f(B)}{\partial x} - i \frac{\partial f(B)}{\partial y}$$

Since $f(B)$ is at first a function of B , then we can write, instead of this, also

$$\begin{aligned} \frac{d(q + il)}{d(x + iy)} &= \frac{df(B)}{dB} \frac{\partial B}{\partial x} - i \frac{df(B)}{dB} \frac{\partial B}{\partial y} \\ \text{or} \quad \frac{d(q + il)}{d(x + iy)} &= \frac{dq}{dB} \left(\frac{\partial B}{\partial x} - i \frac{\partial B}{\partial y} \right) \end{aligned} \quad (9)$$

For the expression within parentheses in (9) we can find a geometric interpretation with the help of Fig. 1. In Fig. 1 the meridian and the parallel circle as well as lines parallel to the axis of coordinates are

represented for the point P in a Gauss-Krüger system of coordinates. At the same time the meridian convergence γ at P is visible. If P is displaced to P_1 by dx in the direction of the axis of abscissae, then dx corresponds on the zero meridian to the meridional arc dB_x , and we have

$$dB_x = \frac{\partial B}{\partial x} dx.$$

At P the distance between the two parallel circles is equal to $PQ_1 = m dB_x$, where m is the scale factor at P . Therefore, we have for the meridian convergence γ the expression

$$\cos \gamma = \frac{PQ_1}{PP_1} = \frac{m dB_x}{dx} = m \frac{\partial B}{\partial x}. \quad (10)$$

If, on the other hand, P is displaced in the direction of the ordinate axis by dy to P_2 , then dB_y is the corresponding meridional arc of the zero meridian, and we have

$$dB_y = -\frac{\partial B}{\partial y} dy.$$

Furthermore the distance between the two parallel circles of P and P_2 is equal to $PQ_2 = m dB_y$. Therefore, we now have

$$\cos(90^\circ - \gamma) = \frac{PQ_2}{PP_2} = -m \frac{\partial B}{\partial y}$$

or

$$\sin \gamma = -m \frac{\partial B}{\partial y}. \quad (11)$$

With (10) and (11), (9) changes to

$$\frac{d(q + il)}{d(x + iy)} = \frac{dw}{dz} = \frac{1}{m} \frac{dq}{dB} (\cos \gamma + i \sin \gamma) \quad (12)$$

or also

$$\frac{d(q + il)}{d(x + iy)} = \frac{dw}{dz} = \frac{1}{m} \frac{dq}{dB} e^{i\gamma}. \quad (13)$$

Accordingly, we have

$$\frac{d(x + iy)}{d(q + il)} = \frac{dz}{dw} = m \frac{dB}{dq} e^{-i\gamma}. \quad (14)$$

After having found this important result, we will pass over to developments in series.

If we form the difference $z_2 - z_1$ for two points P_1 and P_2 , whose interval shall not exceed the order of magnitude of the ordinates in the Gauss-Krüger system, then the difference $w_2 - w_1$ corresponds to it on the ellipsoid, and since $z_2 - z_1$ and $w_2 - w_1$ are small quantities, then we can set down the development in series

$$z_2 - z_1 = \frac{dz}{dw} (w_2 - w_1) + \frac{1}{2} \frac{d^2 z}{dw^2} (w_2 - w_1)^2 + \frac{1}{6} \frac{d^3 z}{dw^3} (w_2 - w_1)^3 + \dots \quad (15)$$

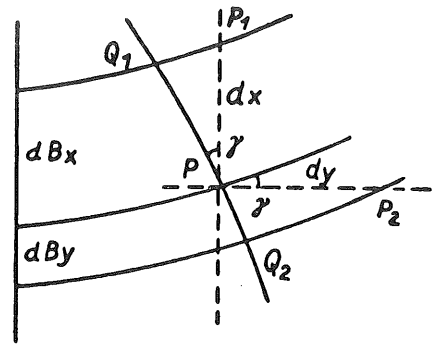


Fig. 1.

or with simplified notation of the coefficients

$$x_2 - x_1 = a_1 (w_2 - w_1) + a_2 (w_2 - w_1)^2 + a_3 (w_2 - w_1)^3 + \dots \quad (15^*)$$

Conversely, we can develop likewise $w_2 - w_1$ as a function of $x_2 - x_1$, for which we have the series

$$w_2 - w_1 = \frac{d w}{d z} (z_2 - z_1) + \frac{1}{2} \frac{d^2 w}{d z^2} (z_2 - z_1)^2 + \frac{1}{6} \frac{d^3 w}{d z^3} (z_2 - z_1)^3 + \dots \quad (16)$$

or again with simplified notation of the coefficients

$$w_2 - w_1 = b_1 (x_2 - x_1) + b_2 (x_2 - x_1)^2 + b_3 (x_2 - x_1)^3 + \dots \quad (16^*)$$

We can easily compute the differential quotients in (15) and (16) or, as the case may be, the coefficients a and b in (15*) and (16*). According to (8) we have

$$\frac{d z}{d w} = \frac{d B}{d q} = \frac{d B}{d \varphi} \frac{d \varphi}{d q} = M \frac{N \cos \varphi}{M} = N \cos \varphi. \quad (17)$$

Hence there follows further

$$\frac{d^2 z}{d w^2} = \frac{d^2 B}{d q^2} = \frac{d (N \cos \varphi)}{d \varphi} \frac{d \varphi}{d q}.$$

If we use the relation $N = \frac{c}{V}$, then we will have with the help of the first half-volume, p. 62, equation (h)

$$\frac{d N}{d \varphi} = + \frac{c}{V^2} \frac{\eta^2 t}{V},$$

hence we have

$$\frac{d^2 z}{d w^2} = \left(\frac{c}{V^3} \eta^2 t \cos \varphi - \frac{c}{V} \sin \varphi \right) \frac{N \cos \varphi}{M},$$

and with

$$V^2 = 1 + \eta^2 \quad \text{and} \quad \frac{c}{V^3} = M$$

$$\frac{d^2 z}{d w^2} = - N \sin \varphi \cos \varphi = - N t \cos^2 \varphi.$$

The derivative of the succeeding differential quotients does not offer any difficulties, and thus we will indicate the results immediately. We obtain the following values for the coefficients of equation (15*)

$$\left. \begin{aligned} a_1 &= + N \cos \varphi \\ a_2 &= - \frac{1}{2} N t \cos^2 \varphi \\ a_3 &= - \frac{1}{6} N \cos^3 \varphi (1 - t^2 + \eta^2) \\ a_4 &= + \frac{1}{24} N \cos^4 \varphi t (5 - t^2 + 9 \eta^2) \\ a_5 &= + \frac{1}{120} N \cos^5 \varphi (5 - 18 t^2 + t^4). \end{aligned} \right\} \quad (18)$$

For the series (16) or, as the case may be, (16*) we have at once from (18)

$$\frac{dw}{dz} = \frac{dq}{dB} = \frac{dq}{d\varphi} \frac{d\varphi}{dB} = \frac{1}{N \cos \varphi},$$

hence we will have

$$\begin{aligned} \frac{d^2 w}{dz^2} &= \frac{d\left(\frac{1}{N \cos \varphi}\right)}{d\varphi} \frac{1}{M} = \left(-\frac{\eta^2 t}{c V \cos \varphi} + \frac{\sin \varphi}{N \cos^2 \varphi}\right) \frac{1}{M} \\ &= \left(-\frac{\eta^2 t}{c V \cos \varphi} + \frac{V^2 \sin \varphi}{c V \cos^2 \varphi}\right) \frac{V^3}{c} = \frac{V^2 t}{c^2 \cos \varphi} = \frac{t}{N^2 \cos \varphi}. \end{aligned}$$

We limit ourselves also here to the first two differential quotients and indicate now immediately also the coefficients b of series (16*). We have

$$\left. \begin{aligned} b_1 &= \frac{1}{N \cos \varphi} \\ b_2 &= \frac{1}{2} \frac{t}{N^2 \cos \varphi} \\ b_3 &= \frac{1}{6} \frac{1}{N^3 \cos \varphi} (1 + 2 t^2 + \eta^2) \\ b_4 &= \frac{1}{24} \frac{1}{N^4 \cos \varphi} t (5 + 6 t^2 + \eta^2) \\ b_5 &= \frac{1}{120} \frac{1}{N^5 \cos \varphi} (5 + 28 t^2 + 24 t^4). \end{aligned} \right\} \quad (19)$$

After these general preparations we pass over to the problem of the conversion of coordinates.

Let there be two neighboring Gauss-Krüger systems of coordinates, which we will designate as west system and as east system.

Let a point P have the ellipsoidal coordinates q, l ,

in the west system, the rectangular coordinates x, y

in the east system, the rectangular coordinates x', y' .

We now introduce a parallel circle with the latitude φ_0 , which lies in the neighborhood of the points to be converted, so that their distances from it are only of the order of magnitude of the ordinates y . Let the points of intersection of this parallel circle with the two zero meridians be P_1 and P_2 , and besides let P_0 be the point of intersection of the central meridian with the same parallel circle.

We have then for the three points P_1, P_2 and P_0

	Ellipsoid	West System	East System
P_1 :	q_0, l_1	x_1, y_1	
P_2 :	q_0, l_2		x_2', y_2'
P_0 :	$q_0, \frac{l_1 + l_2}{2}$	x_0, y_0	x_0', y_0'

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} \quad (20)$$

It shall at first be disregarded that y_1 and y_2' are equal to zero here according to the above assumption.

We form further for P and for the three auxiliary points the complex quantities z and w and have then

Ellipsoid	West System	East System	
$P: w = q + i l$	$z = x + i y$	$z' = x' + i y'$	} (21)
$P_1: w_1 = q_0 + i l_1$	$z_1 = x_1 + i y_1$	$z'_1 = x'_1 + i y'_1$	
$P_2: w_2 = q_0 + i l_2$		$z'_2 = x'_2 + i y'_2$	
$P_0: w_0 = q_0 + i \frac{l_1 + l_2}{2}$	$z_0 = x_0 + i y_0$	$z'_0 = x'_0 + i y'_0$	

The problem to be solved consists in computing from the coordinates x, y of the point P in the west system its coordinates x', y' in the east system.

Since the west system is projected in the east system conformally, then there exists the condition that $z' = x' + i y'$ is a function of $z = x + i y$. If we form the differences $z' - z'_0$ and $z - z_0$, then both are small quantities, and we can develop the function according to Maclaurin's series

$$z' - z'_0 = \left(\frac{dz'}{dz} \right)_0 (z - z_0) + \frac{1}{2} \left(\frac{d^2 z'}{dz^2} \right)_0 (z - z_0)^2 + \frac{1}{6} \left(\frac{d^3 z'}{dz^3} \right)_0 (z - z_0)^3. \quad (22)$$

For the setting up of the individual differential quotients we are to bear in mind that z' is a function of w and that w can be regarded as a function of z . According to the rules for the differentiation of a function of another function we then have

$$\frac{dz'}{dz} = \frac{dz'}{dw} \frac{dw}{dz} \quad (23)$$

$$\frac{d^2 z'}{dz^2} = \frac{d^2 z'}{dw^2} \left(\frac{dw}{dz} \right)^2 + \frac{dz'}{dw} \frac{d^2 w}{dz^2} \quad (24)$$

$$\frac{d^3 z'}{dz^3} = \frac{d^3 z'}{dw^3} \left(\frac{dw}{dz} \right)^3 + 3 \frac{d^2 z'}{dw^2} \frac{dw}{dz} \frac{d^2 w}{dz^2} + \frac{dz'}{dw} \frac{d^3 w}{dz^3}. \quad (25)$$

For the first differential quotient we can use the equations (13) and (14), and hence we have

$$\left(\frac{dz'}{dz} \right)_0 = \left(\frac{dz'}{dw} \right)_0 \left(\frac{dw}{dz} \right)_0 = e^{-i \gamma'_0} e^{i \gamma_0} = e^{i (\gamma_0 - \gamma'_0)}.$$

γ_0 and γ'_0 mean the meridian convergences of the point P_0 in the west and in the east systems. Due to the same distance of the point P_0 from the two zero meridians of the west and of the east systems we have however $-\gamma'_0 = +\gamma_0$, and hence we will have

$$\left(\frac{dz'}{dz} \right)_0 = e^{2i \gamma_0} = \cos 2 \gamma_0 + i \sin 2 \gamma_0. \quad (26)$$

For the setting up of the second and third differential quotients according to (24) and (25) we must first determine the differential quotient

$$\frac{d^2 z'}{dw^2}, \quad \frac{d^2 w}{dz^2}, \quad \frac{d^3 z'}{dw^3} \quad \text{and} \quad \frac{d^3 w}{dz^3} \quad (27)$$

To do so, we apply equation (15*), p. 202, to the two points P and P_2 in the east system. Then we have:

$$x' - x_2' = a_1 (w - w_2) + a_2 (w - w_2)^2 + a_3 (w - w_2)^3 + \dots \quad (28)$$

On the other hand, we apply equation (16*) to the two points P and P_1 in the west system, for which we obtain

$$w - w_1 = b_1 (x - x_1) + b_2 (x - x_1)^2 + b_3 (x - x_1)^3 + \dots \quad (29)$$

Finally, we have further for the two points P_0 and P_1 in the west system according to (15*) the equation

$$x_0 - x_1 = a_1 (w_0 - w_1) + a_2 (w_0 - w_1)^2 + a_3 (w_0 - w_1)^3 + \dots \quad (30)$$

of which we shall make use later.

With equations (28) and (29) we can easily indicate the differential quotients (27). At first we obtain from (28)

$$\left. \begin{aligned} \frac{d x'}{d w} &= a_1 + 2 a_2 (w - w_2) + 3 a_3 (w - w_2)^2 + 4 a_4 (w - w_2)^3 + \dots \\ \frac{d^2 x'}{d w^2} &= 2 a_2 + 6 a_3 (w - w_2) + 12 a_4 (w - w_2)^2 + 20 a_5 (w - w_2)^3 + \dots \\ \frac{d^3 x'}{d w^3} &= 6 a_3 + 24 a_4 (w - w_2) + 60 a_5 (w - w_2)^2 + \dots \end{aligned} \right\} \quad (31)$$

We have likewise from (29)

$$\left. \begin{aligned} \frac{d w}{d x} &= b_1 + 2 b_2 (x - x_1) + 3 b_3 (x - x_1)^2 + 4 b_4 (x - x_1)^3 + \dots \\ \frac{d^2 w}{d x^2} &= 2 b_2 + 6 b_3 (x - x_1) + 12 b_4 (x - x_1)^2 + 20 b_5 (x - x_1)^3 + \dots \\ \frac{d^3 w}{d x^3} &= 6 b_3 + 24 b_4 (x - x_1) + 60 b_5 (x - x_1)^2 + \dots \end{aligned} \right\} \quad (32)$$

With these, we can now assemble the differential quotients (24) and (25). But since they must be computed in equation (22) for the point P_0 , we have to replace w in (31) and (32) everywhere by w_0 and x by x_0 . We have however

$$w_0 - w_2 = (q_0 + i l_0) - (q_0 + i l_2) = i (l_0 - l_2) = -\frac{i}{2} (l_2 - l_1).$$

We set up the difference of longitude of the two zero meridians $l_2 - l_1 = 2p$, so that we have

$$w_0 - w_2 = -i p \quad \text{and} \quad w_0 - w_1 = +i p. \quad (33)$$

With these, the expressions (31) become

$$\left. \begin{aligned} \left(\frac{d x'}{d w} \right)_0 &= a_1 - 2 a_2 p i - 3 a_3 p^2 + 4 a_4 p^3 i + \dots \\ \left(\frac{d^2 x'}{d w^2} \right)_0 &= 2 a_2 - 6 a_3 p i - 12 a_4 p^2 + 20 a_5 p^3 i + \dots \\ \left(\frac{d^3 x'}{d w^3} \right)_0 &= 6 a_3 - 24 a_4 p i - 60 a_5 p^2 + \dots \end{aligned} \right\} \quad (34)$$

In order to be able to introduce the quantity p also into (32), where z is to be replaced likewise by z_0 , we have at first from (30)

$$z_0 - z_1 = a_1 p i - a_2 p^2 - a_3 p i + \dots$$

and then equations (32) pass over into

$$\left. \begin{aligned} \left(\frac{dw}{dz} \right)_0 &= b_1 + 2 a_1 b_2 p i - (2 a_2 b_2 + 3 a_1^2 b_3) p^2 \\ &\quad - (2 a_3 b_2 + 6 a_1 a_2 b_3 + 4 a_1^3 b_4) p^3 i \\ \left(\frac{d^2 w}{dz^2} \right)_0 &= 2 b_2 + 6 a_1 b_3 p i - (6 a_2 b_3 + 12 a_1^2 b_4) p^2 \\ &\quad - (6 a_3 b_3 + 24 a_1 a_2 b_4 + 20 a_1^3 b_5) p^3 i \\ \left(\frac{d^3 w}{dz^3} \right)_0 &= 6 b_3 + 24 a_1 b_4 p i - (24 a_2 b_4 + 60 a_1^2 b_5) p^2. \end{aligned} \right\} \quad (35)$$

We set into these the values of the coefficients a and b from (18) and (19), pp. 202 and 203, and obtain then from (34)

$$\left. \begin{aligned} \left(\frac{dz'}{dw} \right)_0 &= N_0 \cos \varphi_0 + N_0 \cos^2 \varphi_0 t_0 p i + \frac{1}{2} N_0 \cos^3 \varphi_0 (1 - t_0^2 + \eta_0^2) p^2 \\ &\quad + \frac{1}{6} N_0 \cos^4 \varphi_0 t_0 (5 - t_0^2) p^3 i \\ \left(\frac{d^2 z'}{dw^2} \right)_0 &= -N_0 \cos^2 \varphi_0 t_0 + N_0 \cos^3 \varphi_0 (1 - t_0^2 + \eta_0^2) p i \\ &\quad - \frac{1}{2} N_0 \cos^4 \varphi_0 t_0 (5 - t_0^2 + 9 \eta_0^2) p^2 + \frac{1}{6} N_0 \cos^5 \varphi_0 (5 - 18 t_0^2 + t_0^4) p^3 i \\ \left(\frac{d^3 z'}{dw^3} \right)_0 &= -N_0 \cos^3 \varphi_0 (1 - t_0^2 + \eta_0^2) - N_0 \cos^4 \varphi_0 t_0 (5 - t_0^2 + 9 \eta_0^2) p i \\ &\quad - \frac{1}{2} N_0 \cos^5 \varphi_0 (5 - 18 t_0^2 + t_0^4) p^2 \end{aligned} \right\} \quad (36)$$

and from (35)

$$\left. \begin{aligned} \left(\frac{dw}{dz} \right)_0 &= \frac{1}{N_0 \cos \varphi_0} + \frac{1}{N_0} t_0 p i - \frac{1}{2} \frac{\cos \varphi_0}{N_0} (1 + t_0^2 + \eta_0^2) p^2 \\ &\quad - \frac{1}{6} \frac{\cos^2 \varphi_0}{N_0} t_0 (1 + t_0^2) p^3 i \\ \left(\frac{d^2 w}{dz^2} \right)_0 &= \frac{1}{N_0^2 \cos \varphi_0} t_0 + \frac{1}{N_0^2} (1 + 2 t_0^2 + \eta_0^2) p i - 2 \frac{\cos \varphi_0}{N_0^2} t_0 (1 + t_0^2) p^2 \\ \left(\frac{d^3 w}{dz^3} \right)_0 &= \frac{1}{N_0^3 \cos \varphi_0} (1 + 2 t_0^2 + \eta_0^2) + \frac{1}{N_0^3} t_0 (5 + 6 t_0^2 + \eta_0^2) p i \\ &\quad - \frac{1}{2} \frac{\cos \varphi_0}{N_0^3} (5 + 23 t_0^2 + 18 t_0^4) p^2. \end{aligned} \right\} \quad (37)$$

With these, we are in a position to set up the two differential quotients (24) and (25), p. 204. After a simple reduction there follow the values

$$\begin{aligned} \left(\frac{d^2 z'}{dw^2} \right)_0 &= 2 \frac{\cos \varphi_0}{N_0} (1 + \eta_0^2) p i - 6 \frac{\cos^2 \varphi_0}{N_0} t_0 (1 + \eta_0^2) p^2 - \frac{1}{3} \frac{\cos^3 \varphi_0}{N_0} (1 + 31 t_0^2) p^3 i \\ \left(\frac{d^3 z'}{dw^3} \right)_0 &= -2 \frac{\cos \varphi_0}{N_0^2} t_0 (1 + 5 \eta_0^2) p i - 2 \frac{\cos^2 \varphi_0}{N_0^2} (3 - 4 t_0^2) p^2. \end{aligned}$$

In addition, we have from (26), p. 204,

$$\left(\frac{d z'}{d z}\right)_0 = \cos 2 \gamma_0 + i \sin 2 \gamma_0.$$

With the above we have found all coefficients of the series (22), p. 204. Before we set up the latter, we will in addition combine especially the real and the imaginary parts and for this we write the series (22) in the form

$$z' - z_0' = (g_1 + i h_1) (z - z_0) + (g_2 + i h_2) (z - z_0)^2 + (g_3 + i h_3) (z - z_0)^3. \quad (38)$$

Then we have, if we insert ρ at the same time everywhere

$$\left. \begin{aligned} g_1 &= \cos 2 \gamma_0 \\ h_1 &= \sin 2 \gamma_0 \\ g_2 &= -\frac{3}{N_0} \frac{p^2}{\varrho^2} \cos^2 \varphi_0 t_0 (1 + \eta_0^2) \\ h_2 &= \frac{1}{N_0} \frac{p}{\varrho} \cos \varphi_0 (1 + \eta_0^2) - \frac{1}{6 N_0} \frac{p^3}{\varrho^3} \cos^3 \varphi_0 (1 + 31 t_0^2) \\ g_3 &= -\frac{1}{3 N_0^2} \frac{p^2}{\varrho^2} \cos^2 \varphi_0 (3 - 4 t_0^2) \\ h_3 &= -\frac{1}{3 N_0^2} \frac{p}{\varrho} \cos \varphi_0 t_0 (1 + \eta_0^2). \end{aligned} \right\} \quad (39)$$

For the quantities $z' - z_0'$ and $z - z_0$ we have

$$\left. \begin{aligned} z' - z_0' &= (x' - x_0') + i (y' - y_0') = \Delta x' + i \Delta y' \\ z - z_0 &= (x - x_0) + i (y - y_0) = \Delta x + i \Delta y. \end{aligned} \right\} \quad (40)$$

With these, we will have further

$$\begin{aligned} (x - z_0)^2 &= \Delta x^2 + 2 i \Delta x \Delta y - \Delta y^2 \\ (x - z_0)^3 &= \Delta x^3 + 3 i \Delta x^2 \Delta y - 3 \Delta x \Delta y^2 - i \Delta y^3 \end{aligned}$$

and we obtain for the series (38) the expression

$$\begin{aligned} \Delta x' + i \Delta y' &= (g_1 + i h_1) (\Delta x - i \Delta y) + (g_2 + i h_2) (\Delta x^2 + 2 i \Delta x \Delta y - \Delta y^2) \\ &\quad + (g_3 + i h_3) (\Delta x^3 + 3 i \Delta x^2 \Delta y - 3 \Delta x \Delta y^2 - i \Delta y^3). \end{aligned}$$

We can separate in these the real from the imaginary parts and find

$$\left. \begin{aligned} \Delta x' &= g_1 \Delta x - h_1 \Delta y + g_2 (\Delta x^2 - \Delta y^2) - 2 h_2 \Delta x \Delta y \\ &\quad + g_3 (\Delta x^3 - 3 \Delta x \Delta y^2) - h_3 (3 \Delta x^2 \Delta y - \Delta y^3) \\ \Delta y' &= g_1 \Delta y + h_1 \Delta x + 2 g_2 \Delta x \Delta y + h_2 (\Delta x^2 - \Delta y^2) \\ &\quad + g_3 (3 \Delta x^2 \Delta y - \Delta y^3) + h_3 (\Delta x^3 - 3 \Delta x \Delta y^2). \end{aligned} \right\} \quad (41)$$

The problem is herewith solved, for in addition to the given coordinates x, y also x_0, y_0 are known and with this also $\Delta x, \Delta y$. We have further $x_0' = x_0$ and $y_0' = -y_0$, and since we have computed

$\Delta x' = x' - x_0'$ and $\Delta y' = y' - y_0'$, then x' and y' can be found.

We will however go one step further, in order to bring equation (41) into a more convenient form for numerical computation. To do so, we introduce new auxiliary quantities by setting

$$\left. \begin{aligned} s_2 \sin u_2 &= h_2 & s_3 \sin u_3 &= h_3 \\ s_2 \cos u_2 &= g_2 & s_3 \cos u_3 &= g_3 \end{aligned} \right\} \quad (42)$$

$$\left. \begin{aligned} r \sin v &= \Delta y \\ r \cos v &= \Delta x \end{aligned} \right\} \quad (43)$$

We see that the auxiliary quantities s and u remain the same for all points to be converted, while r and v are to be newly computed for each point. Then we will have

$$\begin{aligned} g_2 (\Delta x^2 - \Delta y^2) &= s_2 \cos u_2 r^2 (\cos^2 v - \sin^2 v) = s_2 r^2 \cos u_2 \cos 2v \\ - 2 h_2 \Delta x \Delta y &= - s_2 \sin u_2 r^2 2 \cos v \sin v = - s_2 r^2 \sin u_2 \sin 2v \end{aligned}$$

and

$$g_2 (\Delta x^2 - \Delta y^2) - 2 h_2 \Delta x \Delta y = s_2 r^2 \cos (2v + u_2). \quad (44)$$

We have further

$$\begin{aligned} g_3 (\Delta x^3 - 3 \Delta x \Delta y^2) &= s_3 \cos u_3 (r^3 \cos^3 v - 3 r^3 \cos v \sin^2 v) \\ &= s_3 r^3 \cos u_3 \cos v (\cos 2v - 2 \sin^2 v) \\ - h_3 (3 \Delta x^2 \Delta y - \Delta y^3) &= - s_3 \sin u_3 (3 r^3 \cos^2 v \sin v - r^3 \sin^3 v) \\ &= s_3 r^3 \sin u_3 \sin v (2 \cos^2 v + \cos 2v). \end{aligned}$$

These two expressions yield together

$$g_3 (\Delta x^3 - 3 \Delta x \Delta y^2) - h_3 (3 \Delta x^2 \Delta y - \Delta y^3) = s_3 r^3 \cos (3v + u_3). \quad (45)$$

In the same manner we can treat the second equation (41) and obtain then

$$2 g_2 \Delta x \Delta y + h_2 (\Delta x^2 - \Delta y^2) = s_2 r^2 \sin (2v + u_2) \quad (46)$$

$$g_3 (3 \Delta x^2 \Delta y - \Delta y^3) + h_3 (\Delta x^3 - 3 \Delta x \Delta y^2) = s_3 r^3 \sin (3v + u_3). \quad (47)$$

With these conversions (44) to (47), the transformation equations (41) change to

$$\left. \begin{aligned} \Delta x' &= \cos 2\gamma_0 \Delta x - \sin 2\gamma_0 \Delta y + s_2 r^2 \cos (2v + u_2) + s_3 r^3 \cos (3v + u_3) \\ \Delta y' &= \sin 2\gamma_0 \Delta x + \cos 2\gamma_0 \Delta y + s_2 r^2 \sin (2v + u_2) + s_3 r^3 \sin (3v + u_3) \end{aligned} \right\} \quad (48)$$

Now we can indicate the values of x' and y' . Since the meridian convergence γ_0 is always a small angle, then it is more convenient to use $1 - \cos 2\gamma_0$ instead of $\cos 2\gamma_0$. We obtain

$$\left. \begin{aligned} x' &= x_0 + \Delta x - (1 - \cos 2\gamma_0) \Delta x - \sin 2\gamma_0 \Delta y + s_2 r^2 \cos (2v + u_2) \\ &\quad + s_3 r^3 \cos (3v + u_3) \\ y' &= -y_0 + \Delta y + \sin 2\gamma_0 \Delta x - (1 - \cos 2\gamma_0) \Delta y + s_2 r^2 \sin (2v + u_2) \\ &\quad + s_3 r^3 \sin (3v + u_3) \end{aligned} \right\} \quad (49)$$

For the computation of the coefficients g and h according to (39), p is to be assumed equal to half

the difference of longitude of the Gauss-Krüger system of coordinates, and hence equal to $1^\circ 30' = 5400''$. We then compute the coefficients for various even values of φ_0 , say, at intervals of $30'$, so that we have the coefficients for each point to be converted immediately at our disposal. We will however not tabulate these coefficients g and h , but the values $1 - \cos 2\gamma_0$, $\sin 2\gamma_0$ as well as s_2 , u_2 and s_3 , u_3 , which are to be computed according to (42). Then we have only to compute the quantities r and v for each individual point according to (43) and then we can evaluate the equations (49).

A table of the values of x_0 , y_0 and of the coefficients of the equations (49) is contained on p. [20] of the Appendix for the latitudes $46^\circ 30'$ to $56^\circ 0'$. For the setting up of this table, the tables indicated by Wl. Hristow in *Zeitschr. f. Verm.*, 1938, p. 538, were used.

According to this, the procedure of the computation is the following:

There are given the coordinates x , y in the western system and there are required the coordinates x' , y' in the eastern system. With the value of x we find the nearest value of φ_0 in the table on p. [20] of the Appendix, and we take from it all pertinent numerical values. We then compute $\Delta x = x - x_0$, $\Delta y = y - y_0$, and from the equations

$$r \sin v = \Delta y \quad r \cos v = \Delta x$$

we find v and r , and with these, the individual terms of the end equations (49) can be computed.

For the inverse problem, the passage from an eastern to a western system, nothing changes in the computational procedure; we only have to bear in mind that in this case p becomes negative. With this, also y_0 and γ_0 become negative and h_2 and h_3 change their sign.

Therefore, for the inverse problem, y_0 , $\sin 2\gamma_0$ and u_2 are to be taken with negative sign and u_3 with positive sign from the table on p. [20] of the Appendix.

Example. In Volume II, first half-volume, 1931, p. 160,* the following coordinates are given for the triangle point Grunewald I in the Gauss-Krüger system 12° east of Greenwich with the index 4:

Easting	4,586,714.070 m
Northing	5,814,976.154 .

We wish to compute the coordinates of the point in the system 15° east of Greenwich with the index 5. The following correspond to the above values:

$$x = +5,814,976.154 \text{ m} \quad y = +86,714.070 \text{ m} .$$

With x we find in the table on p. [20] of the Appendix $\varphi_0 = 52^\circ 30'$, and for this we have in the table

$$\begin{array}{ll} x_0 = 5,819,438.155 \text{ m} & \log (1 - \cos 2\gamma_0) = 6.935 \, 9101 \cdot 4 - 10 \\ y_0 = +101,849.888 & \log \sin 2\gamma_0 = 8.618 \, 3774 \cdot 1 - 10 . \\ \log s_2 = 1.3978 - 10 & u_2 = 93^\circ 34.3' \\ \log s_3 = 4.24 - 20 & u_3 = -87^\circ . \end{array}$$

With this, there follows

$$\begin{array}{ll} \Delta x = -4462.001 \text{ m} & \Delta y = -15,135.818 \text{ m} . \\ v = 253^\circ 34.5' & \log r = 4.1981 \\ 2v + u_2 = 240^\circ 43.3' & \\ 3v + u_3 = -46^\circ & \end{array}$$

and the equations (49), p. 208, yield then:

* Not translated.

x_0	5,819,438.155 m	$-y_0$	$\times 898,150.112$ m
Δx	$\times 5,537.999$	Δy	$\times 84,864.182$
$-(1 - \cos 2\gamma_0) \Delta x$	3.850	$\sin 2\gamma_0 \Delta x$	$\times 814.686$
$-\sin 2\gamma_0 \Delta y$	628.613	$-(1 - \cos 2\gamma_0) \Delta y$	13.059
$s_2 r^2 \cos(2v + u_2)$	$\times .696$	$s_2 r^2 \sin(2v + u_2)$	$\times .457$
$s_3 r^3 \cos(3v + u_3)$	0.001	$s_3 r^3 \sin(3v + u_3)$	0
<hr/>		<hr/>	
x'	5,815,608.314 m	y'	$\times 882,841.496$ m
			$+ 500,000.000$
			<hr/>
			382,841.496 m .

Hence, the following values result in the system 15° east of Greenwich with the index 5

Easting 5,382,841.496 m
 Northing 5,815,608.314 .

A second completely new computation can be carried out with a second value of φ_0 . In our example x lies between the two values of x_0 for $\varphi_0 = 52^\circ 0'$ and $52^\circ 30'$. We have repeated the computation with the numerical values for $\varphi_0 = 52^\circ 0'$ and found a complete agreement with the above result.

In order to explain numerically also the inverse problem, we will start from the values found and convert them to the system 12° east of Greenwich. In the system 15° east of Greenwich we thus have

$$x = 5,815,608.314 \text{ m} \quad y = -117,158.504 \text{ m} .$$

The value $\varphi_0 = 52^\circ 30'$ belongs to the value of x according to the table on p. [20] of the Appendix and for this we have, taking into account the signs,

$$\begin{aligned} x_0 &= 5,819,438.155 \text{ m} & \log \cos(1 - 2\gamma_0) &= 6.935\,91014-10 \\ y_0 &= -101,849.888 & \log \sin 2\gamma_0 &= 8.618\,37741_n-10 . \\ \log s_2 &= 1.3978-10 & u_2 &= -93^\circ 34.3' \\ \log s_3 &= 4.24-20 & u_3 &= +87^\circ . \end{aligned}$$

With these tabular values we find

$$\begin{aligned} \Delta x &= -3829.841 \text{ m} & \Delta y &= -15,308.616 \text{ m} \\ v &= 255^\circ 57.3' & \log r &= 4.1981 \\ 2v + u_2 &= 58^\circ 20.3' \\ 3v + u_3 &= 135^\circ \end{aligned}$$

and finally, the equations (49), p. 208, yield

x_0	5,819,438.155 m	$-y_0$	101,849.888 m
Δx	$\times 6,170.159$	Δy	$\times 84,691.384$
$-(1 - \cos 2\gamma_0) \Delta x$	3.304	$\sin 2\gamma_0 \Delta x$	159.059
$-\sin 2\gamma_0 \Delta y$	628.613	$-(1 - \cos 2\gamma_0) \Delta y$	13.208
$s_2 r^2 \cos(2v + u_2)$	$\times .696$	$s_2 r^2 \sin(2v + u_2)$	0.530
$s_3 r^3 \cos(3v + u_3)$	0.001	$s_3 r^3 \sin(3v + u_3)$	0
<hr/>		<hr/>	
x	5,814,976.156 m	y	86,714.069 m

or

Easting 4,586,714.069 m
 Northing 5,814,976.156

in sufficient agreement with the starting values of p. 209.

The transverse-axis conformal projection rests on the transverse-axis coordinate system on the ellipsoid, which we have treated in section 29. It has the characteristic that the geodetic line which is tangent on the ellipsoid to the circle of latitude of the zero point at the meridian of the latter, i.e. the axis of abscissae of the transverse-axis ellipsoidal coordinate system, is projected on the plane at true scale on a straight line. The straight line is at the same time the axis of abscissae of the plane system of coordinates, and now the plane coordinates are to be determined for each point of the ellipsoid in such a way that the projection is a conformal one.

On the ellipsoid, the points are given with respect to the zero point by the quantities $\Delta \varphi$ or, as the case may be, Δq and l , or by rectangular transverse-axis coordinates, which we will denote by ξ and η . Since the longitudes correspond here to the abscissae and the differences of latitude to the ordinates of the plane projection, then we write the general fundamental equation (18), section 31, p. 160, of the conformal projection in the form

$$y + ix = f(\Delta q + il). \quad (1)$$

Now it is a question of determining the function $f(\Delta q + il)$ according to the above conditions. Since a closed expression cannot be indicated for this, we pass over to the development in series. Δq as well as l are to be assumed as small quantities, so that we can develop the function (1) according to Maclaurin's series, and since for $\Delta q = 0$ and $l = 0$ we have also $y = 0$ and $x = 0$, then $f(0) = 0$. Hence, we have

$$y + ix = f'(0)(\Delta q + il) + \frac{1}{2}f''(0)(\Delta q + il)^2 + \frac{1}{6}f'''(0)(\Delta q + il)^3 + \dots \quad (2)$$

For $l = 0$ we have also $x = 0$, and if we denote the value of y belonging to it by y_0 , then we have

$$y_0 = f(\Delta q) = f'(0)\Delta q + \frac{1}{2}f''(0)\Delta q^2 + \frac{1}{6}f'''(0)\Delta q^3 + \frac{1}{24}f^{IV}(0)\Delta q^4 + \dots \quad (3)$$

and hence we form further the differential quotient

$$\frac{dy_0}{d\Delta q} = f'(\Delta q) = f'(0) + f''(0)\Delta q + \frac{1}{2}f'''(0)\Delta q^2 + \frac{1}{6}f^{IV}(0)\Delta q^3 + \dots \quad (4)$$

The values of the individual differential quotients $f'(0)$, $f''(0)$, and so on, are now to be determined. Since the axis of abscissae of the ellipsoidal transverse-axis rectangular system of coordinates (section 29) shall be projected at true scale on a straight line, then, in addition to the geographic coordinates, we introduce further the ellipsoidal rectangular coordinates, which we denote by ξ and η . For this, we consider in Fig. 1 two neighboring points P and P' with the transverse-axis coordinates ξ , η and $\xi + d\xi$, $\eta + d\eta$ on the ellipsoid. Let on the plane correspond to them the conformal coordinates x , y and $x + dx$, $y + dy$, which are represented in Fig. 2.

If we draw in Fig. 1 through P a geodetic line parallel to the axis of abscissae, then there arises a small triangle PP_1P' , to which likewise a triangle PP_1P' corresponds on the plane in Fig. 2.

According to (17), section 11, p. 49, we have then on the ellipsoid

$$PP_1 = n d\xi,$$

where n is dependent only on the length of the ordinate η and on its azimuth at P , as follows from (1), section 12, p. 51.

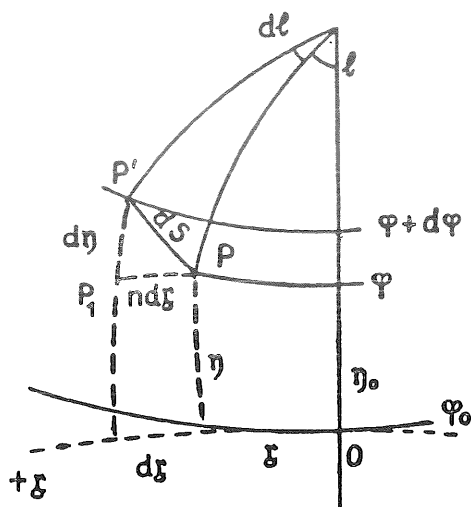


Fig. 1. Ellipsoid.

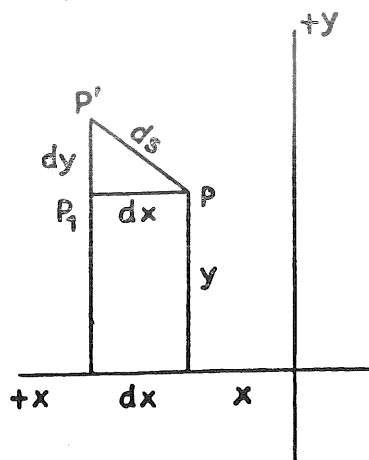


Fig. 2. Plane.

Since we have further $P_1 P' = d\eta$, then we have on the ellipsoid

$$dS^2 = n^2 d\xi^2 + d\eta^2 \quad (5)$$

and accordingly on the plane

$$ds^2 = dx^2 + dy^2. \quad (6)$$

From these two equations we can derive at once the Cauchy-Riemann differential equations as conditions for the conformal projection. We have for the scale factor m the expression

$$m^2 = \frac{ds^2}{dS^2} = \frac{dx^2 + dy^2}{n^2 d\xi^2 + d\eta^2}$$

and we obtain therefrom according to the development in section 31, p. 159, which we need not repeat here, the two equations

$$\frac{\partial y}{\partial \eta} = \frac{\partial x}{n \partial \xi} \quad \frac{\partial y}{n \partial \xi} = -\frac{\partial x}{\partial \eta}. \quad (7)$$

Since the transverse-axis conformal projection has the characteristic that the ξ -axis shall be projected at true scale, then we must have $\frac{\partial x}{\partial \xi} = 1$, and since n is independent of ξ , as we have already established above, then we obtain from the first equation (7) the following relation between the ordinates on the ellipsoid and on the plane

$$\frac{dy}{d\eta} = \frac{1}{n} \quad (8) \quad \text{and for } \xi = 0: \frac{dy_0}{d\eta_0} = \frac{1}{n_0}. \quad (9)$$

For the quantity n we have according to (1), section 12, p. 51, the expression

$$n = 1 - \frac{1}{2} \frac{\eta^2}{N^2} (1 + \eta^2) + \frac{4}{3} \frac{\eta^3}{N^3} \eta^2 t \cos \alpha + \frac{1}{24} \frac{\eta^4}{N^4} + \dots \quad (10)$$

α is here the azimuth of the ordinate η at the point P . Since α is nearly equal to 180° and $\cos \alpha$ deviates from unity only by terms of second and higher order, then we can set $\cos \alpha = -1$.

Further we will reduce the quantities N , η and t to the latitude of the zero point Φ_0 , for which we have from the first half-volume, section 40, p. 63, equation (o):

$$\frac{1 + \eta^2}{N^2} = \frac{V^2}{N^2} = \frac{1 + \eta_0^2}{N_0^2} \left(1 - 4 \eta_0^2 t_0 \frac{\Delta \varphi}{V_0^2} \right).$$

We can set here in the last term

$$\Delta \varphi = \frac{\eta}{M_0} \quad \frac{\Delta \varphi}{V_0^2} = \frac{\eta}{N_0}$$

and then we will have

$$\frac{1 + \eta^2}{N^2} = \frac{1 + \eta_0^2}{N_0^2} \left(1 - 4 \eta_0^2 t_0 \frac{\eta}{N_0} \right)$$

or

$$\frac{1 + \eta^2}{N^2} = \frac{1 + \eta_0^2}{N_0^2} - \frac{4 \eta_0^2 t_0}{N_0^3} \eta. \quad (11)$$

In the last two terms of n in (10), η^2 and N can be replaced immediately by η_0^2 and N_0 . Then we will have

$$n = 1 - \frac{1}{2} (1 + \eta_0^2) \frac{\eta^2}{N_0^2} + 2 \eta_0^2 t_0 \frac{\eta^3}{N_0^3} - \frac{4}{3} \eta_0^2 t_0 \frac{\eta^3}{N_0^3} + \frac{1}{24} \frac{\eta^4}{N_0^4}$$

or

$$n = 1 - \frac{1}{2} (1 + \eta_0^2) \frac{\eta^2}{N_0^2} + \frac{2}{3} \eta_0^2 t_0 \frac{\eta^3}{N_0^3} + \frac{1}{24} \frac{\eta^4}{N_0^4}. \quad (12)$$

For the point of intersection of the zero meridian with the circle of latitude φ , η changes to η_0 , and we have

$$n_0 = 1 - \frac{1}{2} (1 + \eta_0^2) \frac{\eta_0^2}{N_0^2} + \frac{2}{3} \eta_0^2 t_0 \frac{\eta_0^3}{N_0^3} + \frac{1}{24} \frac{\eta_0^4}{N_0^4}. \quad (13)$$

If we form hence the reciprocal value, then we obtain according to (9)

$$\frac{dy_0}{d\eta_0} = \frac{1}{n_0} = 1 + \frac{1}{2} (1 + \eta_0^2) \frac{\eta_0^2}{N_0^2} - \frac{2}{3} \eta_0^2 t_0 \frac{\eta_0^3}{N_0^3} + \frac{5}{24} \frac{\eta_0^4}{N_0^4}$$

or

$$dy_0 = d\eta_0 + \frac{1}{2} (1 + \eta_0^2) \frac{\eta_0^2}{N_0^2} d\eta_0 - \frac{2}{3} \eta_0^2 t_0 \frac{\eta_0^3}{N_0^3} d\eta_0 + \frac{5}{24} \frac{\eta_0^4}{N_0^4} d\eta_0$$

and this integrated yields

$$y_0 = \eta_0 + \frac{1}{6} (1 + \eta_0^2) \frac{\eta_0^3}{N_0^2} - \frac{1}{6} \eta_0^2 t_0 \frac{\eta_0^4}{N_0^3} + \frac{1}{24} \frac{\eta_0^5}{N_0^4}.$$

We obtain hence by inversion by the way of successive approximation

$$\eta_0 = y_0 - \frac{1}{6} (1 + \eta_0^2) \frac{y_0^3}{N_0^2} + \frac{1}{6} \eta_0^2 t_0 \frac{y_0^4}{N_0^3} + \frac{1}{24} \frac{y_0^5}{N_0^4}. \quad (14)$$

On the other hand, we can express η_0 by the difference Δq of the two isometric latitudes, which corresponds to the difference of latitude $\varphi - \varphi_0$. Since η_0 is the meridional arc for Δq , then we have according to section 30, p. 156, equation (24),

$$\begin{aligned} \eta_0 = N_0 \cos \varphi_0 \Delta q - \frac{1}{2} N_0 t_0 \cos^2 \varphi_0 \Delta q^2 - \frac{1}{6} N_0 (1 - t_0^2 + \eta_0^2) \cos^3 \varphi_0 \Delta q^3 \\ + \frac{1}{24} N_0 (5 - t_0^2 + 9 \eta_0^2 + 4 \eta_0^4) t_0 \cos^4 \varphi_0 \Delta q^4. \end{aligned} \quad (15)$$

If we set the expressions (14) and (15) equal to each other, then we can develop $\frac{\eta_0}{N_0}$ from this equation by successive approximation. At first, we have as a first approximation

$$\frac{\eta_0}{N_0} = \cos \varphi_0 \Delta q + \dots$$

There follows hence, accurate to the third order,

$$\frac{\eta_0^3}{N_0^3} = \cos^3 \varphi_0 \Delta q^3 + \dots,$$

and with this, we obtain from (14) and (15)

$$\begin{aligned} \frac{\eta_0}{N_0} - \frac{1}{6} (1 + \eta_0^2) \cos^3 \varphi_0 \Delta q^3 + \dots = \cos \varphi_0 \Delta q - \frac{1}{2} t_0 \cos^2 \varphi_0 \Delta q^2 \\ - \frac{1}{6} (1 - t_0^2 + \eta_0^2) \cos^3 \varphi_0 \Delta q^3 + \dots, \end{aligned}$$

and hence we have as a second approximation

$$\frac{\eta_0}{N_0} = \cos \varphi_0 \Delta q - \frac{1}{2} t_0 \cos^2 \varphi_0 \Delta q^2 + \frac{1}{6} t_0^2 \cos^3 \varphi_0 \Delta q^3 + \dots$$

If we limit ourselves to terms of the fourth order, then we can now set up already finally

$$\frac{\eta_0^3}{N_0^3} = \cos^3 \varphi_0 \Delta q^3 - \frac{3}{2} t_0 \cos^4 \varphi_0 \Delta q^4 \quad \frac{\eta_0^4}{N_0^4} = \cos^4 \varphi_0 \Delta q^4.$$

If we set this into (14), then (14) and (15) yield

$$\begin{aligned} \frac{\eta_0}{N_0} - \frac{1}{6} (1 + \eta_0^2) \cos^3 \varphi_0 \Delta q^3 + \frac{1}{4} t_0 (1 + \eta_0^2) \cos^4 \varphi_0 \Delta q^4 + \frac{1}{6} t_0 \eta_0^2 \cos^4 \varphi_0 \Delta q^4 \\ = \cos \varphi_0 \Delta q - \frac{1}{2} t_0 \cos^2 \varphi_0 \Delta q^2 - \frac{1}{6} (1 - t_0^2 + \eta_0^2) \cos^3 \varphi_0 \Delta q^3 \\ + \frac{1}{12} t_0 (5 - t_0^2 + 9 \eta_0^2) \cos^4 \varphi_0 \Delta q^4, \end{aligned}$$

and we obtain hence for $\frac{y_0}{N_0}$ the value

$$\begin{aligned} \frac{y_0}{N_0} = & \cos \varphi_0 \Delta q - \frac{1}{2} t_0 \cos^2 \varphi_0 \Delta q^2 + \frac{1}{6} t_0^2 \cos^3 \varphi_0 \Delta q^3 \\ & - \frac{1}{24} t_0 (1 + t_0^2 + \eta_0^2) \cos^4 \varphi_0 \Delta q^4. \end{aligned} \quad (16)$$

Finally we have

$$\begin{aligned} \frac{dy_0}{d\Delta q} = & N \cos \varphi_0 - N_0 t_0 \cos^2 \varphi_0 \Delta q + \frac{1}{2} N_0 t_0^2 \cos^3 \varphi_0 \Delta q^2 \\ & - \frac{1}{6} N_0 t_0 (1 + t_0^2 + \eta_0^2) \cos^4 \varphi_0 \Delta q^3. \end{aligned} \quad (17)$$

With this, we can indicate the values of the differential quotients in equation (4), p. 211, for we have

$$\left. \begin{aligned} f'(0) &= N_0 \cos \varphi_0 & f''(0) &= -N_0 t_0 \cos^2 \varphi_0 & f'''(0) &= N_0 t_0^2 \cos^3 \varphi_0 \\ f^{IV} &= -N_0 t_0 (1 + t_0^2 + \eta_0^2) \cos^4 \varphi_0. \end{aligned} \right\} \quad (18)$$

We now can pass over to setting up the projection equation (2), p. 211, and begin with the individual powers of $\Delta q + i l$. We have

$$\left. \begin{aligned} (\Delta q + i l)^2 &= \Delta q^2 + 2 \Delta q i l - l^2 \\ (\Delta q + i l)^3 &= \Delta q^3 + 3 \Delta q^2 i l - 3 \Delta q l^2 - i l^3 \\ (\Delta q + i l)^4 &= \Delta q^4 + 4 \Delta q^3 i l - 6 \Delta q^2 l^2 - 4 \Delta q i l^3 + l^4 \end{aligned} \right\} \quad (19)$$

and with (18) and (19), equation (2), p. 211, yields

$$\begin{aligned} y + ix = & N_0 \cos \varphi_0 (\Delta q + i l) - \frac{1}{2} N_0 t_0 \cos^2 \varphi_0 (\Delta q^2 + 2 \Delta q i l - l^2) + \frac{1}{6} N_0 t_0^2 \cos^3 \varphi_0 (\Delta q^3 \\ & + 3 \Delta q^2 i l - 3 \Delta q l^2 - i l^3) - \frac{1}{24} N_0 t_0 (1 + t_0^2 + \eta_0^2) \cos^4 \varphi_0 (\Delta q^4 + 4 \Delta q^3 i l \\ & - 6 \Delta q^2 l^2 - 4 \Delta q i l^3 + l^4). \end{aligned}$$

After having found, with this, the projection equation for the transverse-axis conformal projection, we obtain the expressions for y and x by separation of the real and of the imaginary parts. There follows

$$\begin{aligned} y = & \Delta q N_0 \cos \varphi_0 - \frac{\Delta q^2}{2} N_0 t_0 \cos^2 \varphi_0 + \frac{l^2}{2} N_0 t_0 \cos^2 \varphi_0 + \frac{\Delta q^3}{6} N_0 t_0^2 \cos^3 \varphi_0 \\ & - \frac{\Delta q l^2}{2} N_0 t_0^2 \cos^3 \varphi_0 - \frac{\Delta q^4}{24} N_0 t_0 (1 + t_0^2 + \eta_0^2) \cos^4 \varphi_0 \end{aligned} \quad (20)$$

$$\begin{aligned} x = & l N_0 \cos \varphi_0 - \Delta q l N_0 t_0 \cos^2 \varphi_0 + \frac{\Delta q^2 l}{2} N_0 t_0^2 \cos^3 \varphi_0 - \frac{l^3}{6} N_0 t_0^2 \cos^3 \varphi_0 \\ & - \frac{\Delta q^3 l}{6} N_0 t_0 (1 + t_0^2 + \eta_0^2) \cos^4 \varphi_0 + \frac{\Delta q l^3}{6} N_0 t_0 (1 + t_0^2 + \eta_0^2) \cos^4 \varphi_0. \end{aligned} \quad (21)$$

In these two fundamental equations for the transverse-axis conformal projection we make use everywhere of the isometric difference of latitude Δq . Since it is more convenient for the practical application of

the formulae to start immediately from the geographic difference of latitude, we will use equation (18), section 30, p. 154, in order to replace in (20) and (21) Δq by $\Delta \varphi$. We obtain from (18), section 30, p. 154,

$$\begin{aligned}\Delta q^2 \cos^2 \varphi_0 &= \frac{1}{V_0^4} \Delta \varphi^2 + \frac{1}{V_0^6} t_0 (1 + 3 \eta_0^2) \Delta \varphi^3 \\ &\quad + \frac{1}{12 V_0^8} (4 + 11 t_0^2 + 16 \eta_0^2 + 42 \eta_0^2 t_0^2 + 12 \eta_0^4 + 75 \eta_0^4 t_0^2) \Delta \varphi^4 \\ \Delta q^3 \cos^3 \varphi_0 &= \frac{1}{V_0^6} \Delta \varphi^3 + \frac{3}{2 V_0^8} t_0 (1 + 3 \eta_0^2) \Delta \varphi^4 \\ \Delta q^4 \cos^4 \varphi_0 &= \frac{1}{V_0^8} \Delta \varphi^4.\end{aligned}$$

Now we set (18), section 30, p. 154, and the above three values into (20) and (21), whereby we neglect, however, η_0^4 in the terms of third order and also η_0^2 in the terms of fourth order. If we insert further the necessary ρ 's, then we obtain without difficulties

$$\begin{aligned}y &= \frac{\Delta \varphi}{\varrho} \frac{N_0}{V_0^2} + \frac{3}{2} \frac{\Delta \varphi^2}{\varrho^2} \frac{N_0}{V_0^4} t_0 \eta_0^2 + \frac{1}{2} \frac{l^2}{\varrho^2} N_0 t_0 \cos^2 \varphi_0 \\ &\quad + \frac{1}{6} \frac{\Delta \varphi^3}{\varrho^3} \frac{N_0}{V_0^6} (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2) - \frac{1}{2} \frac{\Delta \varphi l^2}{\varrho^3} \frac{N_0}{V_0^2} t_0^2 \cos^2 \varphi_0 \\ &\quad + \frac{1}{12} \frac{\Delta \varphi^4}{\varrho^4} \frac{N_0}{V_0^8} t_0 \eta_0^2 + \frac{1}{4} \frac{\Delta \varphi^2 l^2}{\varrho^4} \frac{N_0}{V_0^4} t_0 \cos^2 \varphi_0 - \frac{1}{24} \frac{l^4}{\varrho^4} N_0 t_0 (1 + t_0^2) \cos^4 \varphi_0\end{aligned}\quad (22)$$

$$\begin{aligned}x &= \frac{l}{\varrho} N_0 \cos \varphi_0 - \frac{\Delta \varphi l}{\varrho^2} \frac{N_0}{V_0^2} t_0 \cos \varphi_0 - \frac{3}{2} \frac{\Delta \varphi^2 l}{\varrho^3} \frac{N_0}{V_0^4} t_0^2 \eta_0^2 \cos \varphi_0 - \frac{1}{6} \frac{l^3}{\varrho^3} N_0 t_0^2 \cos^3 \varphi_0 \\ &\quad - \frac{1}{3} \frac{\Delta \varphi^3 l}{\varrho^4} \frac{N_0}{V_0^6} t_0 \cos \varphi_0 + \frac{1}{6} \frac{\Delta \varphi l^3}{\varrho^4} \frac{N_0}{V_0^2} t_0 \cos^3 \varphi_0 (1 + t_0^2).\end{aligned}\quad (23)$$

Computation of geographic coordinates from rectangular coordinates

We could treat the inversion of the problem hitherto solved in the same manner by taking now the equation

$$q + i l = f(y + i x)$$

as a basis instead of the projection equation (1), p. 211. We reach our aim in a simpler manner, however, if we invert the two foregoing equations (22) and (23) by the way of successive approximation. Since we have used this method often already, we will forego here the reproduction of all details. We have from (22) and (23) as a first approximation

$$\frac{\Delta \varphi}{V_0^2} = \frac{y}{N_0} \quad l \cos \varphi_0 = \frac{x}{N_0}.$$

We obtain hence, neglecting the terms of third order:

$$\frac{\Delta \varphi^2}{V_0^4} = \frac{y^2}{N_0^2} \quad l^2 \cos^2 \varphi_0 = \frac{x^2}{N_0^2} \quad \frac{\Delta \varphi}{V_0^2} l \cos \varphi_0 = \frac{x y}{N_0^2}.$$

These values set into (22) and (23) yield, neglecting likewise the terms of third order:

$$\frac{\Delta \varphi}{V_0^2} = \frac{y}{N_0} - \frac{1}{2} \frac{t_0}{N_0^2} x^2 - \frac{3}{2} \frac{t_0 \eta_0^2}{N_0^2} y^2$$

$$l \cos \varphi_0 = \frac{x}{N_0} + \frac{t_0}{N_0^2} x y.$$

With this second approximation we form the following values, neglecting the terms of fourth order:

$$\begin{array}{lll} \frac{\Delta \varphi^2}{V_0^4}, & l^2 \cos^2 \varphi_0, & \frac{\Delta \varphi}{V_0^2} l \cos \varphi_0, \\ \frac{\Delta \varphi^3}{V_0^6}, & l^3 \cos^3 \varphi_0, & \frac{\Delta \varphi^2}{V_0^4} l \cos \varphi_0, \quad \frac{\Delta \varphi}{V_0^2} l^2 \cos^2 \varphi_0 \end{array}$$

and then obtain as a third approximation

$$\begin{aligned} \frac{\Delta \varphi}{V_0^2} &= \frac{y}{N_0} - \frac{3}{2} t_0 \eta_0^2 \frac{y^2}{N_0^2} - \frac{1}{2} t_0 \frac{x^2}{N_0^2} \\ &\quad - \frac{1}{6} (1 + 4 \eta_0^2 - 3 \eta_0^2 t_0^2 + 3 \eta_0^4 - 15 \eta_0^4 t_0^2) \frac{y^3}{N_0^3} - \frac{1}{2} t_0^2 (1 - 3 \eta_0^2) \frac{x^2 y}{N_0^3} \\ l \cos \varphi_0 &= \frac{x}{N_0} + t_0 \frac{x y}{N_0^2} + t_0^2 \frac{x y^2}{N_0^3} - \frac{1}{3} t_0^2 \frac{x^3}{N_0^3}. \end{aligned}$$

If we go one step further in the same way, then we obtain finally also the terms of fourth order, and if we permit ourselves again to neglect the same terms as in the formulae (22) and (23) and at the same time insert everywhere ρ , then we obtain

$$\begin{aligned} \frac{\Delta \varphi}{V_0^2} &= \frac{y}{N_0} \rho - \frac{3}{2} \frac{y^2}{N_0^2} \rho t_0 \eta_0^2 - \frac{1}{2} \frac{x^2}{N_0^2} \rho t_0 - \frac{1}{6} \frac{y^3}{N_0^3} \rho (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2) \\ &\quad - \frac{1}{2} \frac{x^2 y}{N_0^3} \rho t_0^2 (1 - 3 \eta_0^2) - \frac{1}{2} \frac{x^2 y^2}{N_0^4} \rho t_0^3 + \frac{1}{24} \frac{x^4}{N_0^4} \rho t_0 (1 + 3 t_0^3) \end{aligned} \quad (24)$$

$$\begin{aligned} l \cos \varphi_0 &= \frac{x}{N_0} \rho + \frac{x y}{N_0^2} \rho t_0 + \frac{x y^2}{N_0^3} \rho t_0^2 - \frac{1}{3} \frac{x^3}{N_0^3} \rho t_0^2 + \frac{1}{6} \frac{x y^3}{N_0^4} \rho t_0 (1 + 6 t_0^2) \\ &\quad - \frac{1}{6} \frac{x^3 y}{N_0^4} \rho t_0 (1 + 6 t_0^2). \end{aligned} \quad (25)$$

The formulae (22) to (25) agree with the formulae developed in a similar way by Grossmann in *Zeitschr. f. Verm.*, 1934, pp. 498 and 499, in which also the terms of fifth order are taken into account.

In Fig. 1 we have represented the projection of the meridian and of the parallel for the point P in the transverse-axis projection. The angle ε between the meridian and the line parallel to the axis of ordinates is the same angle which we have met in Fig. 1, section 29, p. 144. The angle $360^\circ - \varepsilon$ corresponds to the meridian convergence, as we already indicated on p. 148. If we now aim again to define as meridian convergence γ the azimuth of the line parallel to the axis of abscissae, then we have

$$\gamma = 270^\circ - \varepsilon. \quad (1)$$

In Fig. 1 the angle ε occurs once again as the angle between the line parallel to the axis of abscissae and the image of the parallel circle, and the following expression holds for it

$$\tan \varepsilon = \frac{dy}{dx} = \frac{dy}{dl} : \frac{dx}{dl}. \quad (2)$$

We obtain from (22), section 39, p. 216,

$$\begin{aligned} \frac{dy}{dx} = & l N_0 t_0 \cos^2 \varphi_0 - \frac{\Delta \varphi}{V_0^2} l N_0 t_0^2 \cos^2 \varphi_0 + \frac{1}{2} \frac{\Delta \varphi^2}{V_0^4} l N_0 t_0 \cos^2 \varphi_0 (1 + \eta_0^2 - 3 t_0^2 \eta_0^2) \\ & - \frac{1}{6} l^3 N_0 t_0 \cos^4 \varphi_0 (1 + t_0^2 + \eta_0^2). \end{aligned}$$

Since we obtain, with this, the expression (2) for $\tan \varepsilon$ only to within terms of third order, then it is sufficient to retain the terms of second order in the differential quotient $\frac{dx}{dl}$. The following is then found from (23), section 39, p. 216,

$$\frac{dx}{dl} = N_0 \cos \varphi_0 - \frac{\Delta \varphi}{V_0^2} N_0 t_0 \cos \varphi_0 - \frac{3}{2} \frac{\Delta \varphi^2}{V_0^4} N_0 t_0^2 \eta_0^2 \cos \varphi_0 - \frac{1}{2} l^2 N_0 t_0^2 \cos^3 \varphi_0.$$

In another form these two equations read

$$\begin{aligned} \frac{dy}{dl} = & l N_0 t_0 \cos^2 \varphi_0 \left\{ 1 - \frac{\Delta \varphi}{V_0^2} t_0 + \frac{1}{2} \frac{\Delta \varphi^2}{V_0^4} (1 + \eta_0^2 - 3 t_0^2 \eta_0^2) \right. \\ & \left. - \frac{1}{6} l^2 \cos^2 \varphi_0 (1 + t_0^2 + \eta_0^2) \right\} \end{aligned} \quad (3)$$

$$\frac{dx}{dl} = N_0 \cos \varphi_0 \left\{ 1 - \frac{\Delta \varphi}{V_0^2} t_0 - \frac{3}{2} \frac{\Delta \varphi^2}{V_0^4} t_0^2 \eta_0^2 - \frac{1}{2} l^2 t_0^2 \cos^2 \varphi_0 \right\} \quad (4)$$

and from (4) there follows

$$1 : \frac{dx}{dl} = \frac{1}{N_0 \cos \varphi_0} \left\{ 1 + \frac{\Delta \varphi}{V_0^2} t_0 + \frac{1}{2} \frac{\Delta \varphi^2}{V_0^4} t_0^2 (2 + 3 \eta_0^2) + \frac{1}{2} l^2 t_0^2 \cos^2 \varphi_0 \right\}. \quad (5)$$

With this we obtain according to (2)

$$\tan \varepsilon = l \sin \varphi_0 \left\{ 1 + \frac{1}{2} \frac{\Delta \varphi^2}{V_0^4} (1 + \eta_0^2) - \frac{1}{6} l^2 \cos^2 \varphi_0 (1 - 2 t_0^2 + \eta_0^2) \right\}$$

and since

$$V_0^2 = 1 + \eta_0^2,$$

$$\tan \varepsilon = l \sin \varphi_0 + \frac{1}{2} \frac{\Delta \varphi^2 l}{V_0^2} \sin \varphi_0 - \frac{1}{6} l^3 \sin \varphi_0 \cos^2 \varphi_0 (1 + \eta_0^2) + \frac{1}{3} l^3 \sin^3 \varphi_0.$$

We have

$$\varepsilon = \tan \varepsilon - \frac{1}{3} \tan^3 \varepsilon = \tan \varepsilon - \frac{1}{3} l^3 \sin^3 \varphi_0$$

and with this we will have

$$\varepsilon = l \sin \varphi_0 + \frac{1}{2} \frac{\Delta \varphi^2 l}{V_0^2} \sin \varphi_0 - \frac{1}{6} l^3 V_0^2 \sin \varphi_0 \cos^2 \varphi_0. \quad (6)$$

We therefore obtain for the azimuth of the line parallel to the axis of abscissae at the point P according to (1) the final formula

$$\gamma = 270^\circ - l \sin \varphi_0 - \frac{1}{2} \frac{\Delta \varphi^2 l}{\varrho^2 V_0^2} \sin \varphi_0 + \frac{1}{6} \frac{l^3}{\varrho^2} V_0^2 \sin \varphi_0 \cos^2 \varphi_0. \quad (7)$$

In (6) we can express the geographic coordinates also by the rectangular coordinates x and y by introducing the values (24) and (25), section 39, p. 217. We obtain hence at once

$$\varepsilon = \frac{x}{N_0} t_0 + \frac{xy}{N_0^2} t_0^2 + \frac{1}{2} \frac{xy^2}{N_0^3} t_0 (1 + 2t_0^2 + \eta_0^2) - \frac{1}{6} \frac{x^3}{N_0^3} t_0 (1 + 2t_0^2 + \eta_0^2) \quad (8)$$

and

$$\begin{aligned} \gamma = 270^\circ - \frac{x}{N_0} \varrho t_0 - \frac{xy}{N_0^2} \varrho t_0^2 - \frac{1}{2} \frac{xy^2}{N_0^3} \varrho t_0 (1 + 2t_0^2 + \eta_0^2) \\ + \frac{1}{6} \frac{x^3}{N_0^3} \varrho t_0 (1 + 2t_0^2 + \eta_0^2). \end{aligned} \quad (9)$$

The scale factor

For the development of the scale factor we start from the general equation (4), section 30, p. 152,

$$m^2 = \frac{dx^2 + dy^2}{M^2 d\varphi^2 + N^2 \cos^2 \varphi dl^2} \quad (10)$$

or in another form

$$m^2 = \left(\frac{dx}{dl} \right)^2 \frac{1 + \left(\frac{dy}{dx} \right)^2}{N^2 \cos^2 \varphi \left(1 + \left(\frac{M d\varphi}{N \cos \varphi dl} \right)^2 \right)}. \quad (11)$$

Since m has the same value at the point P in all directions, then we determine the scale factor in the direction of parallel circle, and we have then according to Fig. 1

$$\frac{dy}{dx} = \tan \varepsilon, \quad \text{and hence} \quad 1 + \left(\frac{dy}{dx} \right)^2 = 1 + \tan^2 \varepsilon = \frac{1}{\cos^2 \varepsilon}.$$

With this we will have then

$$m^2 = \left(\frac{dx}{dl} \right)^2 \frac{1}{N^2 \cos^2 \varphi \cos^2 \varepsilon} \quad \text{or} \quad m = \frac{dx}{dl} \frac{1}{N \cos \varphi \cos \varepsilon}. \quad (12)$$

According to (4) and (6) we have

$$\begin{aligned} \frac{dx}{dl} &= N_0 \cos \varphi_0 \left(1 - \frac{\Delta \varphi}{V_0^2} t_0 - \frac{3}{2} \frac{\Delta \varphi^2}{V_0^4} t_0^2 \eta_0^2 - \frac{1}{2} l^2 t_0^2 \cos^2 \varphi_0 \right) \\ \cos \varepsilon &= 1 - \frac{1}{2} l^2 \sin^2 \varphi_0 \quad \text{or} \quad \frac{1}{\cos \varepsilon} = 1 + \frac{1}{2} l^2 \sin^2 \varphi_0 \end{aligned}$$

and then we have

$$\frac{dx}{dl} \frac{1}{\cos \varepsilon} = N_0 \cos \varphi_0 \left(1 - \frac{\Delta \varphi}{V_0^2} t_0 - \frac{3}{2} \frac{\Delta \varphi^2}{V_0^4} t_0^2 \eta_0^2 \right). \quad (13)$$

Before we set up the expression (12) for m , we have further to convert the quantities N and $\cos \varphi$ to the original latitude φ_0 . For this, we take from the first half-volume, section 40, equation (o), p. 63, and section 34, p. 18,

$$\begin{aligned} \frac{1}{N} &= \frac{1}{N_0} \left\{ 1 - \frac{\Delta \varphi}{V_0^2} t_0 \eta_0^2 - \frac{1}{2} \frac{\Delta \varphi^2}{V_0^4} \eta_0^2 (1 - t_0^2 + \eta_0^2) \right\} \\ \frac{1}{\cos \varphi} &= \frac{1}{\cos \varphi_0} \left\{ 1 + \Delta \varphi t_0 + \frac{1}{2} \Delta \varphi^2 (1 + 2 t_0^2) \right\}, \end{aligned}$$

and hence, we will have

$$\frac{1}{N \cos \varphi} = \frac{1}{N_0 \cos \varphi_0} \left\{ 1 + \frac{\Delta \varphi}{V_0^2} t_0 + \frac{1}{2} \frac{\Delta \varphi^2}{V_0^4} (1 + 2 t_0^2 + \eta_0^2 + 3 t_0^2 \eta_0^2) \right\}. \quad (14)$$

If we set (13) and (14) into (12), then we obtain

$$m = 1 + \frac{1}{2} \frac{\Delta \varphi^2}{V_0^4} (1 + \eta_0^2)$$

or, if we insert ρ at the same time,

$$m = 1 + \frac{1}{2} \frac{\Delta \varphi^2}{\rho^2 V_0^2} + \dots \quad (15)$$

We can express $\Delta \varphi$ also by rectangular coordinates with the help of equation (24), section 39, p. 217, and have then to the same accuracy

$$m = 1 + \frac{1}{2} \frac{y^2}{N_0^2} V_0^2$$

or, since $V^2 = \frac{N}{M}$ and $NM = r^2$

$$m = 1 + \frac{y^2}{2r^2} + \dots \quad (16)$$

Here we have developed the scale factor m only to within terms of second order, which will be sufficient for all practical applications. The comparison of formula (16) with equation (23) in section 34, p. 176, shows that within this accuracy the scale factor of the transverse-axis projection agrees with that of the Gauss-Krüger projection.

In the treatise already mentioned by W. Grossmann in *Zeitschr. f. Verm.*, 1934, the formulae for the meridian convergence and the scale factor are carried further to within the terms of fourth order. In this treatise, formulae are also developed for the reduction of distance and direction and for the passage from the transverse-axis to the Gauss-Krüger projection.

By referring to this detailed representation, we will not proceed further in the treatment of the transverse-axis projection, chiefly because it has lost its significance even for Anhalt since the introduction of the Gauss-Krüger projection in Germany.

Section 41. Conformal Conic Projection

For the development of the fundamental formulae we adopt the same way which we have followed in the first half-volume of this volume, section 74, p. 230, for the conformal conic projection of the sphere.

In Fig. 1 the cone is tangent to the surface of the ellipsoid at the normal latitude φ_0 , and there follows hence for the length of the ray of the cone AS

$$R_0 = N_0 \cot \varphi_0. \quad (1)$$

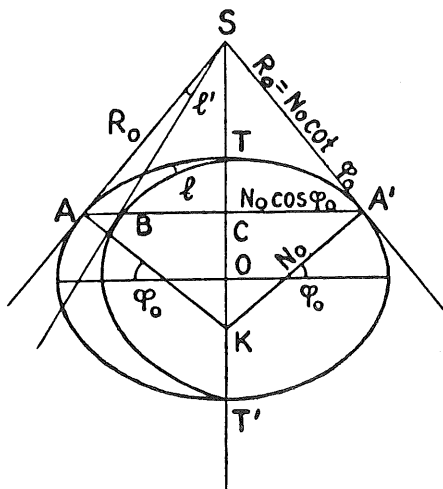


Fig. 1.

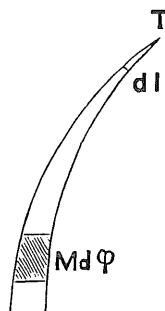


Fig. 2.

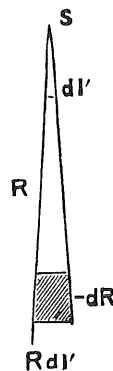


Fig. 3.

For the difference of longitude l' on the cone we have

$$l' = \frac{AB}{R_0} = \frac{l N_0 \cos \varphi_0}{N_0 \cot \varphi_0}$$

Or

$$l' = l \sin \varphi_0. \quad (2)$$

A surface element of the ellipsoid which is bounded by two neighboring arcs of meridians and parallels is represented in Fig. 2. The arc of meridian is equal to $M d\varphi$, while the arc of parallel has the value $N \cos \varphi dl$ at the latitude φ .

Fig. 3 shows the same surface element of the cone. If we denote the distance of this element from the apex of the cone S by R , then its side lengths are $-dR$ and $R dl'$, where dR is provided with the

negative sign because R decreases with increasing latitude.

The condition for the similarity of the projection of the surface element is then

$$\frac{M d\varphi}{N \cos \varphi dl} = - \frac{dR}{R dl} \quad (3)$$

or

$$- \frac{dR}{R} = \sin \varphi_0 \frac{M}{N} \frac{d\varphi}{\cos \varphi} . \quad (4)$$

For $\frac{M}{N}$ we will set according to the first half-volume, p. 51,

$$\frac{M}{N} = \frac{1 - e^2}{W^2} = \frac{1 - e^2}{1 - e^2 \sin^2 \varphi} ;$$

therefore we have according to the above equation (4)

$$- \frac{dR}{R} = \sin \varphi_0 \frac{(1 - e^2) d\varphi}{(1 - e^2 \sin^2 \varphi) \cos \varphi} . \quad (5)$$

The integration is carried out by the way of partial fractions by splitting up at first thusly:

$$\frac{1 - e^2}{(1 - e^2 \sin^2 \varphi) \cos \varphi} = \frac{1}{\cos \varphi} - \frac{1}{2} \frac{e^2 \cos \varphi}{1 + e \sin \varphi} - \frac{1}{2} \frac{e^2 \cos \varphi}{1 - e \sin \varphi} .$$

Consequently, the integral of (5) becomes:

$$\begin{aligned} \int \frac{(1 - e^2) d\varphi}{(1 - e^2 \sin^2 \varphi) \cos \varphi} &= l \tan \left(45^\circ + \frac{\varphi}{2} \right) - \frac{1}{2} e l (1 + e \sin \varphi) + \frac{1}{2} e l (1 - e \sin \varphi) \\ &= l \tan \left(45^\circ + \frac{\varphi}{2} \right) + \frac{1}{2} e l \frac{1 - e \sin \varphi}{1 + e \sin \varphi} . \end{aligned}$$

The left-hand side of (5) integrated yields $-lR$, and by changing on both sides from the natural logarithms l to the common logarithms \log , we thus have as the integration of (5):

$$- \log R = \sin \varphi_0 \left\{ \log \tan \left(45^\circ + \frac{\varphi}{2} \right) + \frac{e}{2} \log \frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right\} - \log A , \quad (6)$$

where the constant of integration is denoted by $-\log A$.

For the determination of the constant of integration we can apply equation (6) to the normal latitude φ_0 and have

$$- \log R_0 = \sin \varphi_0 \left\{ \log \tan \left(45^\circ + \frac{\varphi_0}{2} \right) + \frac{e}{2} \log \frac{1 - e \sin \varphi_0}{1 + e \sin \varphi_0} \right\} - \log A . \quad (7)$$

Since R_0 is known according to (1), then the constant A can be computed from (7).

To equation (6) we now form the antilogarithm

$$\frac{1}{R} = \frac{1}{A} \tan^{\sin \varphi_0} \left(45^\circ + \frac{\varphi}{2} \right) \left(\frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right)^{e/2 \sin \varphi_0} ,$$

and since

$$\tan \left(45^\circ + \frac{\varphi}{2} \right) = \cot \left(45^\circ - \frac{\varphi}{2} \right)$$

then we have also

$$R = A \tan^{\sin \varphi_0} \left(45^\circ - \frac{\varphi}{2} \right) \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{e/2 \sin \varphi_0}. \quad (8)$$

For each latitude φ there could be computed hence the value of R . Meanwhile we will form further the quotient $\frac{R_0}{R}$ for the developments in series of the next section. For this we obtain

$$\frac{R}{R_0} = \left(\frac{\tan \left(45^\circ - \frac{\varphi}{2} \right)}{\tan \left(45^\circ - \frac{\varphi_0}{2} \right)} \right)^{\sin \varphi_0} \cdot \left(\frac{1 - e \sin \varphi_0}{1 + e \sin \varphi_0} \cdot \frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{e/2 \sin \varphi_0}.$$

We can further transform the first factor goniometrically, for we have

$$\begin{aligned} 1 - \cos x &= 2 \sin^2 \frac{x}{2} \\ 1 + \cos x &= 2 \cos^2 \frac{x}{2} \end{aligned} \quad \text{therefore} \quad \frac{1 - \cos x}{1 + \cos x} = \tan^2 \frac{x}{2}.$$

If we set $x = 90^\circ - \varphi$, then we will have

$$\begin{aligned} \frac{1 - \sin \varphi}{1 + \sin \varphi} &= \tan^2 \left(45^\circ - \frac{\varphi}{2} \right) \\ \frac{1 - \sin \varphi_0}{1 + \sin \varphi_0} &= \tan^2 \left(45^\circ - \frac{\varphi_0}{2} \right). \end{aligned}$$

and likewise

With these transformations we obtain

$$\frac{R}{R_0} = \left(\frac{1 + \sin \varphi_0}{1 - \sin \varphi_0} \cdot \frac{1 - \sin \varphi}{1 + \sin \varphi} \right)^{1/2 \sin \varphi_0} \left(\frac{1 - e \sin \varphi_0}{1 + e \sin \varphi_0} \cdot \frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{e/2 \sin \varphi_0}. \quad (9)$$

For the scale factor we have according to (3)

$$m = \frac{R d l'}{N \cos \varphi d l}$$

and this is according to (2)

$$m = \frac{R \sin \varphi_0}{N \cos \varphi}.$$

We will transform this expression for m still further, for we have

$$\begin{aligned} \frac{1}{N} &= \frac{1}{N_0} \frac{V}{V_0}, \\ m &= \frac{R \sin \varphi_0}{N_0 \cos \varphi} \frac{V}{V_0} \end{aligned}$$

therefore

and with the help of (1), this becomes

$$m = \frac{R \cos \varphi_0}{R_0 \cos \varphi} \frac{V}{V_0} = \frac{R \cos \varphi_0}{R_0 \cos \varphi} \frac{\sqrt{1 + e'^2 \cos^2 \varphi}}{\sqrt{1 + e'^2 \cos^2 \varphi_0}}. \quad (10)$$

We can also arrive at the above basic formulae with the help of the general projection equation (18), section 31, p. 160, which we will now write in the form

$$x + i y = f(q - i l). \quad (11)$$

The conic projection has the characteristic that after the development of the envelope of the cone the circles of latitude are represented by concentric circles with the apex of the cone as center, further that on the plane the meridians are straight lines which pass through the apex of the cone, and that the differences of longitude on the plane are proportional to those of the ellipsoid.

According to this, we thus must have $l' = n l$, where n is a constant. In equation (2), p. 221, we have already found that this constant is $n = \sin \varphi_0$.

The radius R of the parallel for the latitude φ must be a function of φ according to the above. In (8) we have found for this the expression

$$R = A \tan^n \left(45^\circ - \frac{\varphi}{2} \right) \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{e/2} n \quad (12)$$

We will introduce in this the isometric latitude q for which we have found in (10a), section 30, p. 153,

$$e^q = \tan \left(45^\circ + \frac{\varphi}{2} \right) \left(\frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right)^{e/2} \quad (13)$$

or also

$$e^{-q} = \tan \left(45^\circ - \frac{\varphi}{2} \right) \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{e/2}. \quad (13a)$$

The expression (12) for R thus changes to

$$R = A e^{-n q}. \quad (14)$$

This equation (14) does not only hold for the special value $n = \sin \varphi_0$, which corresponds to the conic projection with a tangent parallel at the latitude φ_0 , but also for any arbitrary value of n , since R is also in this case a function of φ or, as the case may be, of q .

If we change to rectangular coordinates of the point represented by l' and R , then we have

$$\begin{aligned} x &= R \cos l' & y &= R \sin l' \\ \text{or} \quad x &= A e^{-n q} \cos n l & y &= A e^{-n q} \sin n l \end{aligned}$$

and then we will have

$$x + i y = A e^{-n q} (\cos n l + i \sin n l) = A e^{-n q} e^{i n l}$$

and this is in a different form

$$x + i y = A e^{-n (q - i l)}. \quad (15)$$

The expression (14) for R and the value $l' = n l$ correspond to a conic projection. But it follows from (15) that the expression (14) represents a conformal projection. With this we have found in equation (15) the general projection equation of the conformal conic projection according to equation (11).

The constant A in (15) is nothing more than the constant of integration of equation (6), which we can determine again with the help of the normal latitude φ_0 according to (7).

With Fig. 4 we change now to plane rectangular coordinates of a point P which has the geographic coordinates φ and l . The zero meridian AS shall hereby form on the plane the axis of abscissae, while the tangent to the normal parallel with the latitude φ_0 at A is the axis of ordinates.

According to Fig. 4 we obtain at once:

$$x = R_0 - R \cos l' \quad \text{and} \quad y = R \sin l'. \quad (16)$$

With equations (1), (2), (8) and (16) we are in a position to compute the rectangular coordinates of the point P from its geographic coordinates. However, the computation according to these closed formulae, especially the evaluation of equation (8), p. 223, is not convenient. In the next section we will therefore pass over to developments in series.

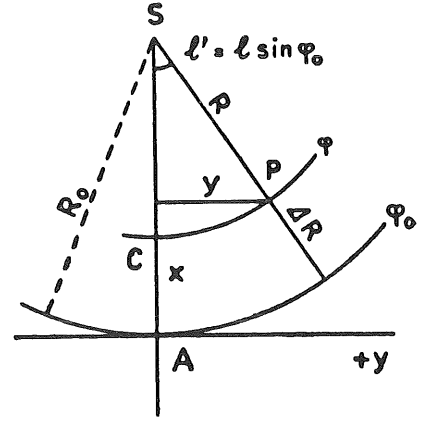


Fig. 4.

Section 42. Developments in Series for the Conformal Conic Projection

We begin with a development in series for the quotient $\frac{R}{R_0}$. For this, we start from the expression (8) in section 41, p. 223, which we write now in abbreviated form

$$\frac{R}{R_0} = \frac{F}{F_0}. \quad (1)$$

Here we have

$$F_0 = \left\{ \left(\frac{1 - \sin \varphi_0}{1 + \sin \varphi_0} \right) \left(\frac{1 + e \sin \varphi_0}{1 - e \sin \varphi_0} \right)^e \right\}^{1/2 \sin \varphi_0} \quad (2)$$

$$F = \left\{ \left(\frac{1 - \sin \varphi}{1 + \sin \varphi} \right) \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^e \right\}^{1/2 \sin \varphi_0}. \quad (3)$$

The factor F_0 , which depends only on φ_0 , is a constant, while F is a function of the geographic latitude φ .

For the development of F we set down $\varphi - \varphi_0 = \Delta \varphi$, and since for $\Delta \varphi = 0$ the function F becomes equal to F_0 , then we obtain according to Taylor's series

$$F = F_0 + \left(\frac{dF}{d\varphi} \right)_0 \Delta \varphi + \frac{1}{2} \left(\frac{d^2 F}{d\varphi^2} \right)_0 \Delta \varphi^2 + \frac{1}{6} \left(\frac{d^3 F}{d\varphi^3} \right)_0 \Delta \varphi^3 + \dots \quad (4)$$

We can take the first differential quotient at once from (5), section 41, p. 222. For according to (1) we have

$$\log R - \log R_0 = \log F - \log F_0,$$

therefore

$$\frac{dR}{R} = \frac{dF}{F}$$

and according to (5), section 41, p. 222,

$$\frac{dF}{d\varphi} = -F \frac{\sin \varphi_0 (1 - e^2)}{(1 - e^2 \sin^2 \varphi) \cos \varphi}.$$

We will now introduce instead of e^2 the second eccentricity e'^2 and have for this, according to the first half-volume, section 40, p. 56, equation (2),

$$\frac{1 - e^2}{1 - e^2 \sin^2 \varphi} = \frac{1}{V^2},$$

$$\frac{dF}{d\varphi} = -F \frac{\sin \varphi_0}{\cos \varphi} \frac{1}{V^2}.$$

we therefore have

We shall set further

$$\frac{1}{V^2 \cos \varphi} = F_1 \quad (5)$$

so that we have then

$$\frac{dF}{d\varphi} = -F F_1 \sin \varphi_0. \quad (6)$$

We obtain for the second differential quotient

$$\frac{d^2 F}{d\varphi^2} = -\sin \varphi_0 \left(\frac{dF}{d\varphi} F_1 + \frac{dF_1}{d\varphi} F \right)$$

and this becomes with (6)

$$\frac{d^2 F}{d\varphi^2} = \sin \varphi_0 F \left(F_1^2 \sin \varphi_0 - \frac{dF_1}{d\varphi} \right). \quad (7)$$

According to (5), we will have

$$\frac{dF_1}{d\varphi} = -\frac{1}{V^4 \cos^2 \varphi} \left(\cos \varphi \frac{dV^2}{d\varphi} - V^2 \sin \varphi \right).$$

But from the first half-volume, p. 62, there follows

$$\frac{dV^2}{d\varphi} = -2\eta^2 t,$$

and with this we will have

$$\frac{dF_1}{d\varphi} = \frac{\sin \varphi_0}{V^4 \cos^2 \varphi} (1 + 3\eta^2). \quad (8)$$

If we introduce this into (7), then we obtain

$$\frac{d^2 F}{d\varphi^2} = F \frac{\sin \varphi}{V^4 \cos^2 \varphi} (\sin \varphi_0 - \sin \varphi (1 + 3\eta^2)). \quad (9)$$

We will in addition treat the third differential quotient in detail and have from (7)

$$\frac{d^3 F}{d \varphi^3} = \sin^2 \varphi_0 \left(F_1^2 \frac{d F}{d \varphi} + 2 F F_1 \frac{d F_1}{d \varphi} \right) - \sin \varphi_0 \left(\frac{d F}{d \varphi} \frac{d F_1}{d \varphi} + F \frac{d^2 F_1}{d \varphi^2} \right).$$

If we introduce here the values of $\frac{d F}{d \varphi}$ and of $\frac{d F_1}{d \varphi}$ according to (6) and (8), then we find easily

$$\frac{d^3 F}{d \varphi^3} = -F F_1^3 \sin^3 \varphi_0 + 3 F F_1 \frac{\sin^2 \varphi_0 \sin \varphi}{V^4 \cos^2 \varphi} (1 + 3 \eta^2) - F \sin \varphi_0 \frac{d^2 F_1}{d \varphi^2}. \quad (10)$$

Now we still lack the differential quotient $\frac{d^2 F_1}{d \varphi^2}$, which we have to form from (8). We obtain at first

$$\begin{aligned} \frac{d^2 F_1}{d \varphi^2} &= \frac{1}{V^8 \cos^4 \varphi} (V^4 \cos^3 \varphi + 4 \sin \varphi \cos^2 \varphi V^2 \eta^2 t + 2 V^4 \sin^2 \varphi \cos \varphi \\ &\quad + 3 \cos^3 \varphi V^4 \eta^2 - 6 V^4 \cos^3 \varphi t^2 \eta^2 + 12 V^2 \cos^3 \varphi t^2 \eta^4 + 6 V^4 \eta^2 \cos^3 \varphi t^2) \\ &= \frac{1}{V^6 \cos \varphi} (V^2 + 4 t^2 \eta^2 + 2 V^2 t^2 + 3 V^2 \eta^2 - 6 V^2 t^2 \eta^2 + 12 t^2 \eta^4 + 6 V^2 t^2 \eta^2) \end{aligned}$$

and if we set in this $V^2 = 1 + \eta^2$, then we find

$$\frac{d^2 F_1}{d \varphi^2} = \frac{1}{V^6 \cos \varphi} (1 + 2 t^2 + 4 \eta^2 + 6 t^2 \eta^2 + 3 \eta^4 + 12 t^2 \eta^4). \quad (11)$$

We set this expression into (10) and obtain

$$\begin{aligned} \frac{d^3 F}{d \varphi^3} &= -F F_1^3 \sin^3 \varphi_0 + 3 F F_1 \frac{\sin^2 \varphi_0 \sin \varphi}{V^4 \cos^2 \varphi} (1 + 3 \eta^2) \\ &\quad - F \frac{\sin \varphi_0}{V^6 \cos \varphi} (1 + 2 t^2 + 4 \eta^2 + 6 t^2 \eta^2 + 3 \eta^4 + 12 t^2 \eta^4). \end{aligned} \quad (12)$$

The three differential quotients (6), (9) and (12) are to be set up for the latitude φ_0 . If we introduce here at the same time the value (5) for F_1 , then we will have

$$\left. \begin{aligned} \left(\frac{d F}{d \varphi} \right)_0 &= -\frac{F_0 t_0}{V_0^2} \\ \left(\frac{d^2 F}{d \varphi^2} \right)_0 &= -\frac{F_0 t_0^2}{V_0^4} 3 \eta_0^2 \\ \left(\frac{d^3 F}{d \varphi^3} \right)_0 &= -\frac{F_0 t_0}{V_0^6} (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2 + 3 \eta_0^4 + 12 t_0^2 \eta_0^4). \end{aligned} \right\} \quad (13)$$

With this we can pass over to the development in series for $\frac{R}{R_0}$. According to (1) we have $\frac{R}{R_0} = \frac{F}{F_0}$; therefore we obtain from (4)

$$\frac{R}{R_0} = 1 + \frac{1}{F_0} \left(\frac{d F}{d \varphi} \right)_0 \Delta \varphi + \frac{1}{2 F_0} \left(\frac{d^2 F}{d \varphi^2} \right)_0 \Delta \varphi^2 + \frac{1}{6 F_0} \left(\frac{d^3 F}{d \varphi^3} \right)_0 \Delta \varphi^3 + \dots \quad (14)$$

We will neglect here the terms with η^4 in the third order, and if we renounce also η^2 in the fourth order, then we can take this term from the corresponding development in the first half-volume, p. 233, equation (22). We obtain then

$$\frac{R}{R_0} = 1 - \frac{t_0^2}{V_0^2} \Delta \varphi - \frac{3 t_0^2 \eta_0^2}{2 V_0^4} \Delta \varphi^2 - \frac{1}{6} \frac{t_0}{V_0^6} (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2) \Delta \varphi^3 - \frac{1}{24} t_0^2 \Delta \varphi^4. \quad (15)$$

We use this series in order to compute the difference

$$\Delta R = R_0 - R.$$

If we write this in the form

$$\frac{R}{R_0} = 1 + \frac{\Delta R}{R_0},$$

then we obtain from (15)

$$\begin{aligned} \Delta R = R_0 \frac{\Delta \varphi}{\varrho} \frac{t_0}{V_0^2} + R_0 \frac{3 \Delta \varphi^2 t_0^2 \eta_0^2}{2 \varrho^2 V_0^4} + R_0 \frac{\Delta \varphi^3 t_0}{6 \varrho^3 V_0^6} (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2) \\ + R_0 \frac{\Delta \varphi^4}{24 \varrho^4} t_0^2 + \dots \end{aligned} \quad (16)$$

We can make use of the above developments in series for the computation of rectangular coordinates from geographic coordinates according to the formulae (16), section 41, p. 225. For this, the value R_0 for the normal latitude φ_0 to be computed according to (1), section 41, p. 221, is to be regarded as known. For the given latitude φ we have then to compute the value of ΔR with $\Delta \varphi = \varphi - \varphi_0$ according to (16), by which $R = R_0 - \Delta R$ is known. We have to compute further $l' = l \sin \varphi_0$ according to (2), section 41, p. 221; and with this, everything is given which is necessary for the computation of x and y according to the formulae (16), p. 225. In the latter we use in the expression for x the conversion $\cos l' = 1 - 2 \sin^2 \frac{l'}{2}$ and have then

$$x = \Delta R + 2 R \sin^2 \frac{l'}{2} \quad y = R \sin l'. \quad (17)$$

Section 43. Developments in Series for the Rectangular Coordinates

The computation of the rectangular coordinates from the geographic coordinates indicated at the end of the preceding section 42 is not inconvenient; one will however prefer in many cases developments in series for x and y to the closed formulae (17) in section 42.

We start for these developments from the formulae (16), section 41, p. 225:

$$x = R_0 - R \cos l' \quad \text{or} \quad \frac{x}{R_0} = 1 - \frac{R}{R_0} \cos l' \quad \text{and} \quad y = R \sin l'. \quad (1)$$

We have at first $l' = l \sin \varphi_0$, and with this, we have the development

$$\cos l' = 1 - \frac{l^2}{2} \sin^2 \varphi_0 + \frac{l^4}{24} \sin^4 \varphi_0.$$

With the help of (15), p. 224, we will thus have:

$$\begin{aligned}
\frac{x}{R_0} &= 1 - \left\{ 1 - \Delta \varphi \frac{t_0}{V_0^2} - \frac{3 \Delta \varphi^2 t_0^2 \eta_0^2}{2 V_0^4} - \frac{\Delta \varphi^3 t_0}{6 V_0^6} (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2) \right. \\
&\quad \left. - \frac{\Delta \varphi^4 t_0^2}{24} \left(1 - \frac{l^2}{2} \sin^2 \varphi_0 + \frac{l^4}{24} \sin^4 \varphi_0 \right) \right\} \\
&= \Delta \varphi \frac{t_0}{V_0^2} + \frac{3 \Delta \varphi^2 t_0^2 \eta_0^2}{2 V_0^4} + \frac{l^2}{2} \sin^2 \varphi_0 - \frac{\Delta \varphi l^2 t_0}{2 V_0^2} \sin^2 \varphi_0 \\
&\quad + \frac{\Delta \varphi^3 t_0}{6 V_0^6} (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2) - \frac{l^4}{24} \sin^4 \varphi_0 + \frac{\Delta \varphi^4 t_0^2}{24}.
\end{aligned} \tag{2}$$

If we set further according to (1), section 41, p. 221, $R_0 = N_0 \cot \varphi_0$, then we obtain for x the final formula:

$$\begin{aligned}
x &= \frac{\Delta \varphi}{\varrho} \frac{N_0}{V_0^2} + \frac{3 \Delta \varphi^2 N_0 t_0 \eta_0^2}{2 \varrho^2 V_0^4} + \frac{l^2}{2 \varrho^2} N_0 t_0 \cos^2 \varphi_0 - \frac{\Delta \varphi l^2 N_0 t_0^2 \cos^2 \varphi_0}{2 \varrho^3 V_0^2} \\
&\quad + \frac{\Delta \varphi^3 N_0 (1 + \eta_0^2 - 3 t_0^2 \eta_0^2)}{6 \varrho^3} + \frac{\Delta \varphi^4 N_0 t_0}{24 \varrho^4} - \frac{l^4}{24 \varrho^4} N_0 t_0^3 \cos^4 \varphi_0.
\end{aligned} \tag{3}$$

For the development of y we set according to (1)

$$y = (R_0 - \Delta R) \sin l' \quad \text{with} \quad l' = l \sin \varphi_0$$

and have then for $\sin l'$ the series

$$\sin l' = l \sin \varphi_0 - \frac{l^3}{6} \sin^3 \varphi_0 + \dots \tag{4}$$

We have further according to (16), section 42, p. 228,

$$R_0 - \Delta R = R_0 - R_0 \Delta \varphi \frac{t_0}{V_0^2} - R_0 \frac{3 \Delta \varphi^2 t_0 \eta_0^2}{2 V_0^4} - R_0 \frac{\Delta \varphi^3 t_0}{6 V_0^6} (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2). \tag{5}$$

From (4) and (5) we obtain:

$$\begin{aligned}
\frac{y}{R_0} &= l \sin \varphi_0 - \Delta \varphi l \frac{t_0}{V_0^2} \sin \varphi_0 - \frac{3 \Delta \varphi^2 l t_0^2 \eta_0^2}{2 V_0^4} \sin \varphi_0 - \frac{\Delta \varphi^3 l t_0}{6} \sin \varphi_0 \\
&\quad - \frac{l^3}{6} \sin^3 \varphi_0 + \frac{\Delta \varphi l^3 t_0}{6 V_0^2} \sin^3 \varphi_0.
\end{aligned} \tag{6}$$

If we set again $R_0 = N_0 \cot \varphi_0$, then we have finally:

$$\begin{aligned}
y &= \frac{l}{\varrho} N_0 \cos \varphi_0 - \frac{\Delta \varphi l}{\varrho^2} \frac{N_0}{V_0^2} t_0 \cos \varphi_0 - \frac{3 \Delta \varphi^2 l}{2 \varrho^3} \frac{N_0}{V_0^4} t_0^2 \eta_0^2 \cos \varphi_0 - \frac{l^3}{6 \varrho^3} N_0 t_0^2 \cos^3 \varphi_0 \\
&\quad - \frac{\Delta \varphi^3 l}{6 \varrho^4} N_0 t_0 \cos \varphi_0 + \frac{\Delta \varphi l^3}{6 \varrho^4} N_0 t_0^3 \cos^3 \varphi_0.
\end{aligned} \tag{7}$$

Numerical example for the computation of rectangular from geographic coordinates

For the application of the above formulae we take from *Grossherzogtl. Mecklenburg. Landesvermessung*, V. Teil, Schwerin, 1895, p. 35, the geographic coordinates of the point Dars (Feuerturm [lighthouse beacon]); the longitude is computed here from the meridian of the castle tower of Schwerin, whereby we count the

longitude l positive to the east, which is, however, in contrast to the publication. We have

$$\varphi = 54^{\circ}28'24.6320'' \quad l = +1^{\circ}4'59.6420''.$$

As normal latitude we assume the latitude $\varphi_0 = 53^{\circ}45'$ so that we have $\Delta\varphi = +43'24.6320''$, and then we will have

$$\Delta\varphi = +2604.6320'' \quad l = +3899.6420''.$$

With $\log \rho = 5.314\,4251.3$ we find hence

$$\log \frac{\Delta\varphi}{\rho} = 8.101\,3212.4 \quad \log \frac{l}{\rho} = 8.276\,5996.1.$$

For the normal latitude $\varphi_0 = 53^{\circ}45'$ we put together the following auxiliary quantities, which we take from the Appendix of the first half-volume:

$$\begin{array}{ll} \log N_0 = 6.805\,5880.8 & \log \cos \varphi_0 = 9.771\,8149.7 \\ \log V_0^2 = 0.001\,0191.2 & \log \tan \varphi_0 = 0.134\,7595.8 \\ \log 1 : V_0^2 = 9.998\,9808.8 & \log \eta_0^2 = 7.370\,9487.1 \\ \log 1 : V_0^4 = 9.997\,9617.7 & \eta_0^2 = 0.002\,349\,3554 \\ & \eta_0^2 l_0^2 = 0.004\,369\,8634. \end{array}$$

With this, we compute the coefficients of $\frac{\Delta\varphi}{\rho}$ and $\frac{l}{\rho}$ for the equations (3) and (7) and find

$$\begin{aligned} x &= [6.804\,5689.6] \frac{\Delta\varphi}{\rho} + [4.485\,3495] \frac{\Delta\varphi^2}{\rho^2} + [6.182\,9476] \frac{l^2}{\rho^2} - [6.316\,688] \frac{\Delta\varphi l^2}{\rho^3} \\ &\quad + [6.022\,738] \frac{\Delta\varphi^3}{\rho^3} + [5.560\,14] \frac{\Delta\varphi^4}{\rho^4} - [4.916\,92] \frac{l^4}{\rho^4} \\ y &= [6.577\,4030.5] \frac{l}{\rho} - [6.711\,1435.1] \frac{\Delta\varphi l}{\rho^2} - [4.391\,924] \frac{\Delta\varphi^2 l}{\rho^3} - [5.612\,401] \frac{l^3}{\rho^3} \\ &\quad - [5.934\,01] \frac{\Delta\varphi^3 l}{\rho^4} + [5.747\,16] \frac{\Delta\varphi l^3}{\rho^4}. \end{aligned}$$

If we introduce the above values of $\frac{\Delta\varphi}{\rho}$ and $\frac{l}{\rho}$ into these formulae, then we obtain

$\dots \Delta\varphi = +80517.484$	$\dots l = +71450.070$
$\dots \Delta\varphi^2 = +4.8752$	$\dots \Delta\varphi l = -1227.6211$
$\dots l^2 = +544.6868$	$\dots \Delta\varphi^2 l = -0.0743$
$\dots \Delta\varphi l^2 = -9.3585$	$\dots l^3 = -2.7682$
$\dots \Delta\varphi^3 = +2.1222$	$\dots \Delta\varphi^3 l = -0.0327$
$\dots \Delta\varphi^4 = +0.0092$	$\dots \Delta\varphi l^3 = +0.0477$
$\dots l^4 = -0.0106$	
$x = +81059.808 \text{ m}$	$y = +70219.621 \text{ m}.$

In *Grossherzogth. Mecklenburg. Landesvermessung*, p. 39, the plane coordinates of the point Dars are indicated likewise. It is to be taken into account, however, that for the normal parallel, therefore also for N_0 , a factor of reduction whose logarithm is equal to 9.999 9821.5 is used there according to p. 5. Our values of x and y are to be multiplied by this factor in order to agree with those of the above work.

In order to be able to compute also the geographic from the rectangular coordinates, we will invert the two equations (2) and (6) by successive approximation. We have for this with the terms of first order

$$\frac{\Delta \varphi}{V_0^2} t_0 = \frac{x}{R_0} + \dots \quad l \sin \varphi_0 = \frac{y}{R_0} + \dots$$

We obtain hence from (2) with the inclusion of the terms of second order:

$$\frac{x}{R_0} = \frac{\Delta \varphi}{V_0^2} t_0 + \frac{3}{2} \frac{x^2}{R_0^2} \eta_0^2 + \frac{1}{2} \frac{y^2}{R_0^2} + \dots$$

and from (6)

$$\frac{y}{R_0} = l \sin \varphi_0 - \frac{x y}{R_0^2} + \dots$$

Therefore, we have

$$\frac{\Delta \varphi}{V_0^2} t_0 = \frac{x}{R_0} - \frac{3}{2} \frac{x^2}{R_0^2} \eta_0^2 - \frac{1}{2} \frac{y^2}{R_0^2} + \dots \quad \text{and} \quad l \sin \varphi_0 = \frac{y}{R_0} + \frac{x y}{R_0^2} + \dots \quad (8)$$

and hence there follows further, if only the terms of the fourth order and, besides, all terms in η_0^4 are neglected,

$$\begin{aligned} \frac{\Delta \varphi^2}{V_0^4} t_0^2 \eta_0^2 &= \frac{x^2}{R_0^2} \eta_0^2 - \frac{2 x y^2}{R_0^3} \eta_0^2 & \Delta \varphi^3 &= \frac{x^3}{R_0^3 t_0^3} \\ \frac{\Delta \varphi}{V_0^2} t_0 l^2 \sin^2 \varphi_0 &= \frac{x y^2}{R_0^3}. \end{aligned}$$

With this, we obtain from (2)

$$\begin{aligned} \frac{\Delta \varphi}{V_0^2} t_0 &= \frac{x}{R_0} - \frac{3}{2} \frac{x^2}{R_0^2} \eta_0^2 - \frac{1}{2} \frac{y^2}{R_0^2} - \frac{1}{2} \frac{x y^2}{R_0^3} (1 - 3 \eta_0^2) \\ &\quad - \frac{1}{6} \frac{x^3}{R_0^3 t_0^3} (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2). \end{aligned}$$

In addition, we also can introduce $R_0 t_0 = N_0$ according to (1), section 41, p. 221. At the same time, we will insert from the spherical formulae of the first half-volume, p. 236, the terms of fourth order, in which we have neglected η_0^2 also so far, and then we obtain

$$\begin{aligned} \frac{\Delta \varphi}{V_0^2} &= \frac{x}{N_0} \varrho - \frac{3}{2} \frac{x^2}{N_0^2} \varrho t_0 \eta_0^2 - \frac{1}{2} \frac{y^2}{N_0^2} \varrho t_0 - \frac{1}{2} \frac{x y^2}{N_0^3} \varrho t_0^2 (1 - 3 \eta_0^2) \\ &\quad - \frac{1}{6} \frac{x^3}{N_0^3} \varrho (1 + 4 \eta_0^2 - 3 t_0^2 \eta_0^2) - \frac{1}{4} \frac{x^2 y^2}{N_0^4} \varrho t_0 (2 t_0^2 - 1) \\ &\quad - \frac{1}{24} \frac{x^4}{N_0^4} \varrho t_0 + \frac{1}{8} \frac{y^4}{N_0^4} \varrho t_0^3. \end{aligned} \quad (9)$$

For the inversion of equation (6) we have from (8) with the inclusion of the terms of third order:

$$\begin{aligned}
-\frac{\Delta \varphi l}{V_0^2} t_0 \sin \varphi_0 &= -\frac{xy}{R_0^2} + \frac{3}{2} \frac{x^2 y}{R_0^3} \eta_0^2 + \frac{1}{2} \frac{y^3}{R_0^3} - \frac{x^2 y}{R_0^3} \\
-\frac{3}{2} \frac{\Delta \varphi^2 l}{V_0^4} t_0^2 \eta_0^2 \sin \varphi_0 &= -\frac{3}{2} \frac{x^2 y}{R_0^3} \eta_0^2 \\
-\frac{1}{6} l^3 \sin^3 \varphi_0 &= -\frac{y^3}{6 R_0^3}.
\end{aligned}$$

With this, (6) yields then

$$l \sin \varphi_0 = \frac{y}{R_0} + \frac{xy}{R_0^2} + \frac{x^2 y}{R_0^3} - \frac{1}{3} \frac{y^3}{R_0^3}.$$

We change again from R_0 to N_0 by introducing $\frac{1}{R_0} = \frac{t_0}{N_0}$ and take also from the first half-volume, p. 236, the spherical terms of fourth order, with which we obtain the final formula

$$l \cos \varphi_0 = \frac{y}{N_0} \varrho + \frac{xy}{N_0^2} \varrho t_0 + \frac{x^2 y}{N_0^3} \varrho t_0^2 - \frac{1}{3} \frac{y^3}{N_0^3} \varrho t_0^2 + \frac{x^3 y}{N_0^4} \varrho t_0^3 - \frac{xy^3}{N_0^4} \varrho t_0^3. \quad (10)$$

Since no terms with η_0^2 occur in this formula, then it must agree completely with the spherical formula (13) in the first half-volume, p. 236.

We shall return once again to the conformal conic projection at the end of the later section 53 and show how we can derive the above developments in series also from the corresponding spherical developments in series.

Reduction of length and direction

In the first half-volume of this volume we have derived the reduction of the lengths as well as those of the directions for the conformal conic projection of the terrestrial sphere. The formulae found there suffice with a very far-reaching accuracy also for the terrestrial ellipsoid.

At first, we have for the scale factor of a point with the plane coordinates xy according to the first half-volume, p. 237, equations (18) and (19),

$$\left. \begin{aligned} m &= 1 + \frac{x^2}{2r^2} - \frac{xy^2}{2r^3} t_0 + \frac{x^3}{6r^3} t_0 \\ \frac{1}{m} &= 1 - \frac{x^2}{2r^2} + \frac{xy^2}{2r^3} t_0 - \frac{x^3}{6r^3} t_0 \end{aligned} \right\} \quad (11)$$

and for practical computation we have, in addition, the logarithmic form

$$\log m = \frac{\mu x^2}{2r^2} - \frac{\mu xy^2}{2r^3} t_0 + \frac{\mu x^3}{6r^3} t_0. \quad (12)$$

$r = \sqrt{M_0 N_0}$ is to be set here for the ellipsoid.

According to the first half-volume, p. 240, equation (7), the reduction of length becomes with this

$$S = \frac{s}{6} \left(\frac{1}{m_1} + \frac{4}{m_0} + \frac{1}{m_2} \right), \quad (13)$$

where S is the length of the geodetic line between two points P_1 and P_2 on the ellipsoid and s the linear distance of the corresponding points on the plane. The values m_1 , m_2 and m_0 are the scale factors at P_1 , P_2 and at the center of the geodetic line.

For the reduction of direction, we can likewise make use of the formulae developed in the first half-volume, p. 243. In Fig. 1, the projection of the geodetic line and the linear connection of two points P_1 and P_2 is represented on the plane. Further, the meridians at the two points as well as the direction of the axis of abscissae are indicated. The designations for the angles in Fig. 1 have the following meaning:

l'_1 and l'_2 are the two plane differences of longitude with respect to the zero meridian, for which we have according to (2), section 41, p. 221,

$$l'_1 = l_1 \sin \varphi_0 \quad l'_2 = l_2 \sin \varphi_0. \quad (14)$$

Since the meridians are projected as straight lines, the angles l'_1 and l'_2 are at the same time the meridian convergences at P_1 and P_2 . Because of the conformal projection, the two angles α_1 and α_2 are the azimuths of the arc P_1P_2 . Further, the direction angles of the arc P_1P_2 and of the straight line P_1P_2 are denoted by α and t . In addition, if we introduce the two auxiliary angles δ_1 and δ_2 , then we have the relations

$$\alpha_1 = \alpha_1 - l'_1 \quad \alpha_2 = \alpha_2 - l'_2 \quad (15)$$

and

$$\alpha_1 = t_1 - \delta_1 \quad \alpha_2 = t_2 + \delta_2. \quad (16)$$

We have developed the values of δ_1 and δ_2 likewise in the first half-volume, p. 243, and we have according to equations (25) and (26) of that place

$$\delta_1 = \frac{\rho}{6r^2} (2x_1 + x_2)(y_2 - y_1) - \frac{t_0 \rho}{12r^3} \left\{ (2y_1^2 + y_2^2)(y_2 - y_1) - (2x_1^2 + x_2^2)(y_2 - y_1) - 2(2x_1y_1 + x_2y_2)(x_2 - x_1) \right\} - \frac{t_0 \rho}{24r^3} \left\{ 3(x_2 - x_1)^2(y_2 - y_1) - (y_2 - y_1)^3 \right\} \quad (17)$$

$$\delta_2 = \frac{\rho}{6r^2} (x_1 + 2x_2)(y_2 - y_1) - \frac{t_0 \rho}{12r^3} \left\{ (2y_2^2 + y_1^2)(y_2 - y_1) - (2x_2^2 + x_1^2)(y_2 - y_1) - 2(x_2y_2 + x_1y_1)(x_2 - x_1) \right\} - \frac{t_0 \rho}{24r^3} \left\{ 3(x_2 - x_1)^2(y_2 - y_1) - (y_2 - y_1)^3 \right\}. \quad (18)$$

[In (18) there are corrected at the same time two sign errors, which equation (26) on p. 243 in the first half-volume contains.]

The mean radius of curvature r holds here for the normal latitude φ_0 of the projection.

Formulae (17) and (18) hold for the terrestrial sphere. The additional investigation shows however that the terms in $\frac{1}{r^2}$ have validity also for the ellipsoid. In the ellipsoidal development, only in the terms with $\frac{1}{r^3}$ there occurs the quantity η^2 , which can be neglected however. Formulae (17) and (18) are therefore sufficient also for all computations on the ellipsoid.

The basic formulae for the conformal conic projection developed in the foregoing were developed by Jordan in 1891 for use in the Mecklenburg Land Survey. The latitude $\varphi_0 = 53^\circ 45'$ is assumed for the normal parallel, while the geographic longitudes are counted from the castle tower in Schwerin - positive to the west. This zero meridian is at the same time the x -axis of the plane coordinates, where the abscissae are counted positive to the south, the ordinates positive to the west, however.

It is mentioned further that the scale factor m is not equal to unity at the normal parallel, but approximately $31'$

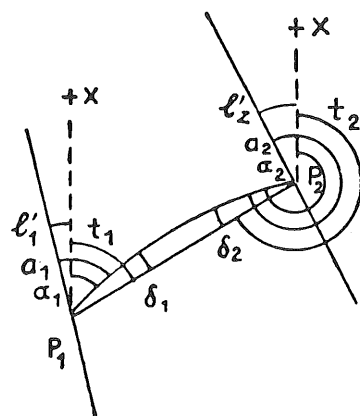


Fig. 1.

to the north and south of the normal parallel, while for the latter the value $m_0 = 1:1.00004\ 11045$ holds. In this manner there was achieved that in the whole survey territory m reaches at the most the value 1.000 041.

A comprehensive representation of the scientific fundamental principles of the Mecklenburg Land Survey is contained in the publication: *Grossherzogliche Mecklenburgische Landesvermessung*, V. Teil, Die konforme Kegelp Projektion und ihre Anwendung auf das trigonometrische Netz I. Ordnung, herausgegeben im Auftrage des Grossherzoglichen Ministeriums des Innern und der Finanzen, Abteilung für Domänen und Forsten von Dr. W. Jordan, Karl Mauck und R. Vogeler, Schwerin 1895.

Formulae for the conversion of the Mecklenburg coordinates to Gauss-Krüger coordinates are given in G. Thilo, "Anschluss der mecklenburgischen Landesvermessung an das deutsche Einheitssystem," Sonderheft zu den *Mitt. d. Reichsamts f. Landesaufn.*, 1926.
Wl. Hristow, "Über die Transformation von Merkator- und Gauss-Krügerschen Koordinaten in mecklenburgische Koordinaten und umgekehrt," *"Zeitschr. f. Verm."*, 1935," pp. 129-137.

Section 44. Conformal Conic Projection with Two Basic Parallels

In addition, we will consider the case, in which two parallels are projected at true scale in a conformal conic projection, and hence, in which the cone intersects the surface of the ellipsoid at these two parallels. This condition can also be formulated in such a way that the ratio of two arcs of parallels on the ellipsoid shall be equal to the ratio of the two corresponding curves on the plane. If φ_1 and φ_2 are the two geographic latitudes, then we have for the difference of longitude l the two arcs on the ellipsoid

$$l N_1 \cos \varphi_1 \quad \text{and} \quad l N_2 \cos \varphi_2. \quad (1)$$

On the plane, the two arcs

$$l' R_1 \quad \text{and} \quad l' R_2 \quad (2)$$

correspond to them.

In section 41, p. 221, we have established that in every conformal conic projection the differences of longitude on the plane are proportional to the ellipsoidal differences of longitude. If we denote the factor of proportionality by n , then we have $l' = n l$. In the case of the conformal conic projection with *one* tangent parallel φ_0 hitherto treated, we had $n = \sin \varphi_0$; in the case under discussion n must be determined specially. Therefore, we have on the plane the two arcs

$$n l R_1 \quad \text{and} \quad n l R_2. \quad (3)$$

For the quantity R we have found in (8), section 41, p. 223, the expression

$$R = A \tan^n \left(45^\circ - \frac{\varphi}{2} \right) \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{\frac{n e}{2}} \quad (4)$$

where $\sin \varphi_0$ is now replaced by the coefficient n .

According to the above, we then have the condition

$$\frac{N_1 \cos \varphi_1}{N_2 \cos \varphi_2} = \frac{R_1}{R_2} = \frac{\tan^n \left(45^\circ - \frac{\varphi_1}{2} \right) \left(\frac{1 + e \sin \varphi_1}{1 - e \sin \varphi_1} \right)^{\frac{n e}{2}}}{\tan^n \left(45^\circ - \frac{\varphi_2}{2} \right) \left(\frac{1 + e \sin \varphi_2}{1 - e \sin \varphi_2} \right)^{\frac{n e}{2}}}. \quad (5)$$

In order to be able to compute therefrom the coefficient n , we introduce two auxiliary angles ζ_1 and ζ_2 by setting generally

$$\tan \left(45^\circ - \frac{\varphi}{2} \right) \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{\frac{e}{2}} = \tan \frac{\zeta}{2}. \quad (6)$$

For the numerical computation of ζ we obtain a simple formula if we set down

$$e \sin \varphi = \cos \omega . \quad (7)$$

Then we have

$$2 \cos^2 \frac{\omega}{2} = 1 + \cos \omega \quad 2 \sin^2 \frac{\omega}{2} = 1 - \cos \omega$$

or

$$\cot \frac{\omega}{2} = \sqrt{\frac{1 + \cos \omega}{1 - \cos \omega}} = \sqrt{\frac{1 + e \sin \varphi}{1 - e \sin \varphi}}$$

and

$$\cot^e \frac{\omega}{2} = \left(\frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{\frac{e}{2}} .$$

Therefore, we have according to (6)

$$\tan \frac{\zeta}{2} = \tan \left(45^\circ - \frac{\varphi}{2} \right) \cot^e \frac{\omega}{2} . \quad (8)$$

With this, we can write the above conditions (5) in the following form

$$\frac{N_1 \cos \varphi_1}{N_2 \cos \varphi_2} = \left(\frac{\tan \frac{\zeta_1}{2}}{\tan \frac{\zeta_2}{2}} \right)^n$$

and from this, there follows for n the formula

$$n = \frac{\log N_1 - \log N_2 + \log \cos \varphi_1 - \log \cos \varphi_2}{\log \tan \frac{\zeta_1}{2} - \log \tan \frac{\zeta_2}{2}} . \quad (9)$$

Now we still lack the constant A in (4), which we can now no longer compute as in section 41, p. 222. With the help of (6), we can write (4) in the form

$$R = A \tan^n \frac{\zeta}{2} . \quad (10)$$

If φ is the geographic latitude, to which the auxiliary angle ζ corresponds, then we have in the case of a true-scale projection of an arc of parallel with the difference of longitude l

$$R n l = N \cos \varphi l$$

or

$$n A \tan^n \frac{\zeta}{2} = N \cos \varphi .$$

For the latitudes φ_1 and φ_2 we thus have the two equations

$$A = \frac{N_1 \cos \varphi_1}{n \tan^n \frac{\zeta_1}{2}} = \frac{N_2 \cos \varphi_2}{n \tan^n \frac{\zeta_2}{2}} \quad (10a)$$

from which A must follow by the agreement of the two sides.

With the two quantities n and A everything is known which is necessary for the computation of R .

For the introduction of rectangular coordinates we use on the plane the mean meridian, from which the longitudes are counted, as axis of abscissae and its point of intersection with an arbitrary parallel φ_0 as zero point of coordinates. Then we have according to (16), section 41, p. 225,

$$x = R_0 - R \cos n l \quad y = R \sin n l. \quad (11)$$

The use of the above closed formulae is not inconvenient; it requires, however, the computation with logarithms to a great number of places. We would arrive at a more convenient computation by development in series, as we have carried out in section 43, for the conformal conic projection with one tangent parallel. Such developments in series, in which the mean latitude $\frac{1}{2}(\varphi_1 + \varphi_2)$ is assumed as normal latitude, are contained in A. E. Young, *Some Investi-*

gations in the Theory of Map Projections, London, Royal Geographical Society, 1920, pp. 59-61. The development in series is not carried out according to powers of the difference of latitude against the normal latitude as in section 43, but according to powers of the arc of meridian, which corresponds to this difference of latitude.

Tables for rectangular coordinates are computed for topographic purposes in O. S. Adams, "Lambert Projection, Tables for the United States," *U. S. Coast and Geodetic Survey. Spec. Publ.*, No. 52, Washington, 1918.

The conformal conic projection was treated for the first time in 1772 by J. H. Lambert in the treatise "Anmerkungen und Zusätze zur Entwerfung der Land- und Himmelscharten." This treatise is contained in: *Beyträge zum Gebrauche der Mathematik und deren Anwendung* durch J. H. Lambert, mit Kupfern und Tafeln, Dritter Teil, Berlin, 1772. A new reprint of the treatise has been published in 1894 by A. Wangerin in volume 54 of *Ostwalds Klassikern der exakten Wissenschaften*.

In sections 47-52 of the treatise, Lambert develops the conformal conic projection with one tangent parallel and in section 53 the conformal conic projection with true-scale projection of two parallels.

Section 45. The Stereographic Projection

In the first half-volume of this volume, we have developed the stereographic projection of the sphere from two different points of view. In section 65, p. 187, we had it as a perspective projection in which the perspective center of the image center lies opposite on the surface of the sphere, while it was represented in section 71, p. 217, as a conformal zenithal projection.

For the ellipsoid, the first definition does not lead to a projection which has the properties of the stereographic projection; but even according to the second definition we obtain for the ellipsoid only an azimuthal projection, which is conformal to a very close approximation but not completely so. (Cf., in this connection, the investigations of the author in *Zeitschr. f. Verm.*, 1936, pp. 160-164, and the investigations by Lehmann in *Zeitschr. f. Verm.*, 1939, p. 430.) A stereographic projection of the ellipsoid according to the above definitions is therefore not possible.

Another conformal projection, based on the stereographic projection of the sphere, was developed in the paper, "Emploi des coordonnées rectangulaires stéréographiques pour le calcul de la triangulation dans un rayon de 560 kilomètres autour de l'origine," par M. H. Roussilhe, ingénieur hydrographe en chef de la marine, Paris, Imprimerie Nationale, 1922, publié dans le Tome I des *Travaux de la Section de géodésie de l'Union géodésique et géophysique internationale*.

The stereographic projection by Roussilhe represents a method of projection in which the meridian of the zero point is projected in the same manner as in the stereographic projection of the sphere. If we apply the law indicated in (5), first half-volume, section 71, p. 218, for the sphere to the ellipsoid, for which an arc of the meridian of the zero point is designated by b , then we have for the ellipsoid

$$x = 2 r_0 \tan \frac{b}{2 r_0}. \quad (1)$$

r_0 is here the mean radius of curvature for the latitude of the zero point. By development in series we obtain from (1)

$$x = b + \frac{b^3}{12 r_0^2} + \frac{b^5}{120 r_0^4} + \dots \quad (2)$$

For the further development we do not follow the original paper by Roussilhe, but a newer representation by Wl. Hristow in *Zeitschr. f. Verm.*, 1937, pp. 84-89.

Since the projection shall be conformal, we have according to (18), section 31, p. 160, the projection equation

$$x + iy = F(q + il) \quad (3)$$

or briefly

$$x + iy = F(w). \quad (4)$$

Let $q = q_0$ and $l = 0$ for the zero point, and since we have for this point $x = 0, y = 0$, then we have also

$$F(w_0) = 0.$$

For an arbitrary point q, l let $q - q_0 = \Delta q$, and we set down $w - w_0 = \Delta w$. We can then develop $F(q + il)$ as a series

$$x + iy = \left(\frac{dF}{dw}\right)_0 \Delta w + \frac{1}{2} \left(\frac{d^2 F}{dw^2}\right)_0 \Delta w^2 + \frac{1}{6} \left(\frac{d^3 F}{dw^3}\right)_0 \Delta w^3 + \dots \quad (5)$$

For the setting up of the differential quotients in (5) we make use of the Gauss-Krüger projection, for which we denote the rectangular coordinates by $x' y'$. If we assume the central point of the stereographic projection also as the zero point of the Gauss-Krüger system of coordinates, then we have according to (3), section 32, p. 160,

$$x' = b, \quad \text{therefore} \quad x = x' + \frac{x'^3}{12 r_0^2} + \frac{x'^5}{120 r_0^4} + \dots \quad (6)$$

Let the projection equation for the conversion from the Gauss-Krüger to the stereographic system be

$$x + iy = f(x' + iy'). \quad (7)$$

For $y = 0$ we also will have $y' = 0$, and hence $x = f(x')$.
We have therefore according to (6)

$$f(x') = x' + \frac{x'^3}{12 r_0^2} + \frac{x'^5}{120 r_0^4}. \quad (8)$$

With this, the mapping function (7) is characterized.

From (8) we obtain for the zero point, i.e. for $x = 0$

$$\left(\frac{dx}{dx'}\right)_0 = 1, \quad \left(\frac{d^2 x}{dx'^2}\right)_0 = 0, \quad \left(\frac{d^3 x}{dx'^3}\right)_0 = \frac{1}{2 r_0^2}, \quad \left(\frac{d^4 x}{dx'^4}\right)_0 = 0 \dots \quad (9)$$

The differentiation of (7) for $x' = 0$ and $y' = 0$ must also lead to these values (9); and hence, the values (9) are at the same time the differential quotients of $x + iy$ with respect to the variable $x' + iy'$ for the zero point of coordinates.

Now we return to equation (3) or (4).

For $l = 0$ we have $y = 0$; therefore $x = F(q)$. In order to find the derivatives of $x + iy$ with respect to $q + il$ for the zero point of coordinates, we therefore need only to set up the derivatives of x with respect to q for the zero point. We obtain according to the general rules for the differentiation of functions of functions:

$$\begin{aligned}
\frac{d F(q)}{d q} &= \frac{d x}{d q} = \frac{d x}{d x'} \frac{d x'}{d q} \\
\frac{d^2 F(q)}{d q^2} &= \frac{d^2 x}{d q^2} = \frac{d^2 x}{d x'^2} \left(\frac{d x'}{d q} \right)^2 + \frac{d x}{d x'} \frac{d^2 x'}{d q^2} \\
\frac{d^3 F(q)}{d q^3} &= \frac{d^3 x}{d q^3} = \frac{d^3 x}{d x'^3} \left(\frac{d x'}{d q} \right)^3 + 3 \frac{d^2 x}{d x'^2} \frac{d x'}{d q} \frac{d^2 x'}{d q^2} + \frac{d x}{d x'} \frac{d^3 x'}{d q^3} \\
\frac{d^4 F(q)}{d q^4} &= \frac{d^4 x}{d q^4} = \frac{d^4 x}{d x'^4} \left(\frac{d x'}{d q} \right)^4 + 6 \frac{d^3 x}{d x'^3} \left(\frac{d x'}{d q} \right)^2 \frac{d^2 x'}{d q^2} + 3 \frac{d^2 x}{d x'^2} \left(\frac{d^2 x'}{d q^2} \right)^2 \\
&\quad + 4 \frac{d^2 x}{d x'^2} \frac{d x'}{d q} \frac{d^3 x'}{d q^3} + \frac{d x}{d x'} \frac{d^4 x'}{d q^4}
\end{aligned}$$

and so on.

In (9) we have already become acquainted with the derivatives of x with respect to x' .

For the derivatives of x' with respect to q we are to take into account that according to (6), $x' = b$; therefore, we can take the derivatives of b with respect to q forthwith from (7) to (14), section 32, pp. 161 and 162. Thus we have

$$\begin{aligned}
\frac{d x'}{d q} &= + N \cos \varphi \\
\frac{d^2 x'}{d q^2} &= - N \sin \varphi \cos \varphi = - N t \cos^2 \varphi \\
\frac{d^3 x'}{d q^3} &= - N \cos^3 \varphi (1 - t^2 + \eta^2) \\
\frac{d^4 x'}{d q^4} &= + N t \cos^4 \varphi (5 - t^2 + 9 \eta^2 + 4 \eta^4).
\end{aligned}$$

If we set these expressions together with (9) into the differential quotients of $F(q)$ with respect to q , where we replace at the same time r^2 by $\frac{N^2}{V^2} = \frac{N^2}{1 + \eta^2}$, then we obtain for the zero point

$$\begin{aligned}
\left(\frac{d F(q)}{d q} \right)_0 &= N_0 \cos \varphi_0 \\
\left(\frac{d^2 F(q)}{d q^2} \right)_0 &= - N_0 t_0 \cos^2 \varphi_0 \\
\left(\frac{d^3 F(q)}{d q^3} \right)_0 &= - \frac{1}{2} N_0 \cos^3 \varphi_0 (1 - 2 t_0^2 + \eta_0^2) \\
\left(\frac{d^4 F(q)}{d q^4} \right)_0 &= N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 + 6 \eta_0^2 + 4 \eta_0^4).
\end{aligned}$$

With this, we obtain for the series (5), p. 237, with $\Delta w = \Delta q + i l$

$$\begin{aligned}
x + i y &= N_0 \cos \varphi_0 (\Delta q + i l) - \frac{1}{2} N_0 t_0 \cos^2 \varphi_0 (\Delta q + i l)^2 - \frac{1}{12} N_0 \cos^3 \varphi_0 (1 - 2 t_0^2 + \eta_0^2) (\Delta q + i l)^3 \\
&\quad + \frac{1}{24} N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 + 6 \eta_0^2 + 4 \eta_0^4) (\Delta q + i l)^4 + \dots
\end{aligned}$$

If we solve for the individual powers of $\Delta q + i l$, then we will have

$$\begin{aligned}
x + i y &= N_0 \cos \varphi_0 (\Delta q + i l) - \frac{1}{2} N_0 t_0 \cos^2 \varphi_0 (\Delta q^2 + 2 i \Delta q l - l^2) \\
&\quad - \frac{1}{12} N_0 \cos^3 \varphi_0 (1 - 2 t_0^2 + \eta_0^2) (\Delta q^3 + 3 i \Delta q^2 l - 3 \Delta q l^2 - i l^3) \\
&\quad + \frac{1}{24} N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 + 6 \eta_0^2 + 4 \eta_0^4) (\Delta q^4 + 4 i \Delta q^3 l - 6 \Delta q^2 l^2 \\
&\quad \quad \quad - 4 i \Delta q l^3 + l^4).
\end{aligned} \tag{10}$$

From this equation (9) we obtain by separating the real and the imaginary parts

$$\begin{aligned}
 x = & N_0 \cos \varphi_0 \Delta q - \frac{1}{2} N_0 t_0 \cos^2 \varphi_0 \Delta q^2 + \frac{1}{2} N_0 t_0 \cos^2 \varphi_0 l^2 - \frac{1}{12} N_0 \cos^3 \varphi_0 (1 - 2 t_0^2 \\
 & + \eta_0^2) \Delta q^3 \\
 & + \frac{1}{4} N_0 \cos^3 \varphi_0 (1 - 2 t_0^2 + \eta_0^2) \Delta q l^2 + \frac{1}{24} N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 + 6 \eta_0^2 + 4 \eta_0^4) \Delta q^4 \\
 & - \frac{1}{4} N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 + 6 \eta_0^2) \Delta q^2 l^2 + \frac{1}{24} N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 \\
 & + 6 \eta_0^2) l^4 + \dots
 \end{aligned} \tag{11}$$

$$\begin{aligned}
 y = & N_0 \cos \varphi_0 l - N_0 t_0 \cos^2 \varphi_0 \Delta q l - \frac{1}{4} N_0 \cos^3 \varphi_0 (1 - 2 t_0^2 + \eta_0^2) \Delta q^2 l \\
 & + \frac{1}{12} N_0 \cos^3 \varphi_0 (1 - 2 t_0^2 + \eta_0^2) l^3 + \frac{1}{6} N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 + 6 \eta_0^2 + 4 \eta_0^4) \Delta q^3 l \\
 & - \frac{1}{6} N_0 t_0 \cos^4 \varphi_0 (2 - t_0^2 + 6 \eta_0^2 + 4 \eta_0^4) \Delta q l^3 + \dots
 \end{aligned} \tag{12}$$

In these expressions (10) and (11) Δq is further to be replaced by $\Delta \varphi$. To this, we have according to (18), section 30, p. 154,

$$\begin{aligned}
 \Delta q \cos \varphi = & \frac{1}{V^2} \Delta \varphi + \frac{1}{2 V^4} t (1 + 3 \eta^2) \Delta \varphi^2 + \frac{1}{6 V^6} (1 + 2 t^2 + 4 \eta^2 + 6 t^2 \eta^2 \\
 & + 3 \eta^4 + 12 t^2 \eta^4) \Delta \varphi^3 + \frac{1}{24 V^8} t (5 + 6 t^2 + 19 \eta^2 + 24 \eta^2 t^2 + 47 \eta^4 \\
 & + 30 t^2 \eta^4) \Delta \varphi^4 + \dots
 \end{aligned}$$

or if we introduce $V^2 = 1 + \eta^2$ and neglect at the same time in the coefficient of $\Delta \varphi^4$ the terms in η^4

$$\begin{aligned}
 \Delta q \cos \varphi = & (1 - \eta^2 + \eta^4) \Delta \varphi + \frac{1}{2} t (1 + \eta^2 - 3 \eta^4) \Delta \varphi^2 + \frac{1}{6} (1 + 2 t^2 + \eta^2 - 3 \eta^4 \\
 & + 6 t^2 \eta^4) \Delta \varphi^3 + \frac{1}{24} t (5 + 6 t^2 - \eta^2) \Delta \varphi^4 + \dots
 \end{aligned}$$

For the higher terms in (10) and (12) we obtain hence

$$\begin{aligned}
 \Delta q^2 \cos^2 \varphi = & (1 - 2 \eta^2 + 3 \eta^4) \Delta \varphi^2 + t (1 - 3 \eta^4) \Delta \varphi^3 + \frac{1}{12} (4 + 11 t^2 - 2 t^2 \eta^2) \Delta \varphi^4 \\
 \Delta q^3 \cos^3 \varphi = & (1 - 3 \eta^2 + 6 \eta^4) \Delta \varphi^3 + \frac{3}{2} t (1 - \eta^2) \Delta \varphi^4 \\
 \Delta q^4 \cos^4 \varphi = & (1 - 4 \eta^2) \Delta \varphi^4.
 \end{aligned}$$

If we set these values into (11) and (12), then there follow the final formulae

$$\begin{aligned}
 x = & N_0 (1 - \eta_0^2 + \eta_0^4) \Delta \varphi + \frac{1}{2} N_0 t_0 (3 \eta_0^2 - 6 \eta_0^4) \Delta \varphi^2 + \frac{1}{2} N_0 t_0 \cos^2 \varphi_0 l^2 \\
 & + \frac{1}{12} N_0 (1 + 4 \eta_0^2 - 6 t_0^2 \eta_0^2 - 9 \eta_0^4 + 42 t_0^2 \eta_0^4) \Delta \varphi^3 \\
 & + \frac{1}{4} N_0 \cos^2 \varphi_0 (1 - 2 t_0^2 + 2 t_0^2 \eta_0^2 - 2 t_0^2 \eta_0^4) \Delta \varphi l^2 - \frac{1}{8} N_0 t_0 \eta_0^2 \Delta \varphi^4 \\
 & - \frac{1}{8} N_0 t_0 \cos^2 \varphi_0 (3 + 2 \eta_0^2 + 6 t_0^2 \eta_0^2) \Delta \varphi^2 l^2 \\
 & + \frac{1}{24} N_0 t_0 \cos^4 \varphi_0 (2 - t^2 + 6 \eta^2) \Delta l^4.
 \end{aligned} \tag{13}$$

$$\begin{aligned}
y = N_0 \cos \varphi_0 l - N_0 t_0 \cos \varphi_0 (1 - \eta_0^2 + \eta_0^4) \Delta \varphi l - \frac{1}{4} N_0 \cos \varphi_0 (1 - \eta_0^2 + 6 t_0^2 \eta_0^2 \\
+ \eta_0^4 - 12 t_0^2 \eta_0^4) \Delta \varphi^2 l + \frac{1}{12} N \cos^3 \varphi_0 (1 - 2 t_0^2 + \eta_0^2) l^3 \\
- \frac{1}{12} N_0 t_0 \cos \varphi_0 (1 + 5 \eta_0^2 - 6 t_0^2 \eta_0^2) \Delta \varphi^3 l - \frac{1}{6} N_0 t_0 \cos^3 \varphi_0 (2 - t_0^2 + 4 \eta_0^2 \\
+ t_0^2 \eta_0^2) \Delta \varphi l^3.
\end{aligned} \tag{14}$$

In the above-mentioned work by Hristow, also the terms of fifth order are developed for the formulae (13) and (14); the inversion of these formulae is likewise indicated there.

Another form of the stereographic projection is treated by L. Krüger in: "Zur stereographischen Projektion," *Veröffentl. d. Preuss. Geod. Inst.*, N. F. Nr. 89, Berlin, 1922. As we have shown in *Zeitschr. f. Verm.*, 1936, pp. 154-158, this method of projection is to be regarded as a conformal double projection. We shall return to this subject in the next chapter at the end of section 53.

CONFORMAL PROJECTION OF THE ELLIPSOID ON THE SPHERE AND CONFORMAL DOUBLE PROJECTION

Section 46. Basic Formulae

In addition to the conformal projection of the ellipsoid on the plane, which we have treated in the preceding chapter, we owe to Gauss also another theory of this kind in the case of which the ellipsoid is projected conformally on a sphere so that the methods of the first half-volume, Chapter VI, can then be applied for the projection of the sphere. Use is made frequently of such a double projection for the purposes of land survey, which we shall discuss later more thoroughly.

In addition to general reference to the literature assembled already in section 32, p. 164, we mention here especially as a source: "Untersuchungen über Gegenstände der höheren Geodäsie" von Carl Friedrich Gauss, erste Abhandlung, der Königl. Sozietät überreicht 1843, Okt. 23. In the total edition, Carl Friedrich Gauss' *Werke*, this treatise is included in volume IV, Göttingen, 1873, pp. 259-300. The treatise is also newly published by J. Frischau in *Ostwalds Klassiker der exakten Wissenschaften*, No. 177, Leipzig, 1910.

The conformal projection of the ellipsoid on the sphere was used by the Prussian Land Survey in 1876-1923 as the foundation for a double projection, for which we give the published literature at this point.

The principal work is: *Die konforme Doppelprojektion der trigonometrischen Abteilung der Königl. Preussischen Landesaufnahme*, Formeln und Tafeln, von Dr. O. Schreiber, Berlin, 1897. The development of the formulae indicated in the above work is given by O. Schreiber in the treatise, "Zur konformen Doppelprojektion der Königl. Preussischen Landesaufnahme," *Zeitschr. f. Verm.*, 1899, pp. 491-502, 593-613; 1900, pp. 257-281, 289-310. Another representation is: v. Schmidt, "Die Projektionsmethode der trigonometrischen Abteilung der Königl. Preussischen Landesaufnahme," in *Zeitschr. f. Verm.*, 1894, pp. 385-401, 409-418. We mention further: L. Krüger, "Transformation der Koordinaten bei der konformen Doppelprojektion des Erdellipsoids auf die Kugel und die Ebene," *Veröff. d. Kgl. Preuss. Geod. Inst.*, Potsdam, 1914; J. Frischau, "Berichtigung eines Druckfehlers von Gauss: Untersuchungen über Gegenstände der höheren Geodäsie," in *Zeitschr. f. Verm.*, 1912, pp. 150-151.

At first, we treat, in the following chapter, the projection of the ellipsoid on the sphere according to Gauss' cited classic original papers.

In our treatment we have retained the notation by Gauss, at any rate the constants P , Q , α , m , and so on, while, for the remaining part, our $V^2 = 1 + \eta^2$ used also in other respects has proved useful.

We have omitted all developments about the third order, referring to the original work.

In section 50 we had to change article 13, which deals with the reduction of azimuth, because Gauss introduces here the geodetic line as the shortest line according to the theory of the calculus of variations, which does not fit into our representation of the geodetic line, and therefore we have put another development in section 50 in the place of article 13.

To this, another general formula, that of Schols, was inserted in section 51.

For the projection of the ellipsoid on the sphere we will further introduce the condition that the meridians and parallels of the ellipsoid are projected on the sphere likewise into meridians and parallels.

In Fig. 1, p. 242, ds designates the differential of a geodetic line on the ellipsoid, and in Fig. 2, p. 242, ds' is the differential of a corresponding arc of a great circle on a sphere of the radius A . Besides, the following notation holds:

	Ellipsoid	Sphere
Point	P	Q
Latitude	φ	u
Difference of longitude	$d\lambda$	$d\lambda$

We have further, corresponding to one another:

$$\text{Arc of parallel} \quad P_1 P' = N \cos \varphi d\lambda \quad Q_1 Q' = A \cos u d\lambda \quad (1)$$

$$\text{Arc of meridian} \quad P P_1 = M d\varphi \quad Q Q_1 = A du. \quad (2)$$

As usual, M and N are here the two main radii of curvature of the ellipsoid of rotation.

Now if $Q Q_1 Q'$ is supposed to be the conformal projection of $P P_1 P'$, then the sides of the two triangles must have a constant relation, which is denoted by m , and hence:

$$\frac{A du}{M d\varphi} = \frac{A \cos u d\lambda}{N \cos \varphi d\lambda} = m. \quad (3)$$

But according to the above condition, u shall be only a function of φ , and therefore we must have in (3) $\frac{d\lambda}{d\lambda} = \alpha$, where α denotes a constant. We have therefore

$$d\lambda = \alpha d\lambda \quad (4)$$

and as the relation between the spherical latitude u and the spheroidal latitude φ the differential equation:

$$\frac{du}{d\varphi} = \alpha \frac{M \cos u}{N \cos \varphi}. \quad (5)$$

The ratio of curvature $M:N$ is introduced according to (29), first half-volume, section 38, p. 51:

$$\frac{M}{N} = \frac{1}{V^2}, \quad \text{therefore} \quad \frac{du}{d\varphi} = \frac{\alpha \cos u}{V^2 \cos \varphi}, \quad (6)$$

or in a different form, with W^2 instead of V^2 according to (29), first half-volume, section 38, p. 51:

$$\frac{du}{\cos u} = \frac{\alpha(1-e^2)}{W^2} \frac{d\varphi}{\cos \varphi} = \frac{\alpha(1-e^2)}{1-e^2 \sin^2 \varphi} \frac{d\varphi}{\cos \varphi}. \quad (7)$$

For the integration we divide into partial fractions:

$$\frac{1-e^2}{(1-e^2 \sin^2 \varphi) \cos \varphi} = \frac{1}{\cos \varphi} - \frac{1}{2} \frac{e^2 \cos \varphi}{1+e \sin \varphi} - \frac{1}{2} \frac{e^2 \cos \varphi}{1-e \sin \varphi}.$$

With this, the integration of (7) yields:

$$\log \tan \left(45^\circ + \frac{u}{2} \right) = \alpha \left\{ \log \tan \left(45^\circ + \frac{\varphi}{2} \right) - \frac{1}{2} e \log (1+e \sin \varphi) + \frac{1}{2} e \log (1-e \sin \varphi) \right\} + \log \frac{1}{k}.$$

$\log \frac{1}{k}$ is inserted here as the constant of integration; with this, the foregoing equation can also be written thusly:

$$\tan \left(45^\circ + \frac{u}{2} \right) = \frac{1}{k} \tan^\alpha \left(45^\circ + \frac{\varphi}{2} \right) \left(\frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right)^{\frac{\alpha e}{2}}. \quad (8)$$

If this relation between u and φ is satisfied, then m is obtained from the two formulae (3) by agreement on both sides, and in fact according to the second form of (3), with the introduction of N according to (26), first half-volume, section 38, p. 50, $N = a:W$:

$$m = \frac{\alpha A \cos u}{N \cos \varphi}, \quad m = \alpha \frac{A \cos u}{a \cos \varphi} W, \quad (9)$$

or

$$\frac{W}{a} = \frac{V}{c}, \quad \text{therefore} \quad m = \frac{A}{c} \frac{\alpha \cos u}{\cos \varphi} V. \quad (10)$$

The relation between the geographic longitudes l and λ follows immediately according to (4), since α is constant:

$$\lambda = \alpha l \quad (11)$$

The constant of integration can be omitted here, since it would only mean a change in the choice of the zero meridian.

Equations (8), (10) and (11) already contain in substance the solution of the stated problem, and in addition we only have to dispose of the three constants α , k and A in order to be able to use the above formulae.

We can indicate the two projection equations (8) and (11) in yet another form if we introduce instead of the geographic latitudes φ and u the isometric latitudes. For the ellipsoid, we have denoted the isometric latitude by q ; for the isometric latitude of the sphere we will use the notation ω .

According to (13), section 41, p. 224, we have

$$e^q = \tan \left(45^\circ + \frac{\varphi}{2} \right) \left(\frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right)^{\frac{e}{2}} \quad (12)$$

and accordingly for the sphere

$$e^\omega = \tan \left(45^\circ + \frac{u}{2} \right). \quad (13)$$

With this, equation (8) obtains the form

$$e^\omega = \frac{1}{k} e^{\alpha q} \quad (14)$$

or in logarithmic form

$$\omega = \alpha q - \log k \quad (15)$$

or, if we set $\log k = q_0$,

$$\omega = \alpha q - q_0. \quad (16)$$

If we include further equation (11), then we can form the expression

$$\omega + i\lambda = \alpha (q + i l) - q_0. \quad (17)$$

This is the mapping equation for the Gauss projection of the ellipsoid on the sphere. We see that, in this case, the complex expression $\omega + i\lambda$ is a linear function of $q + il$. (Cf., in this connection, section 31, p. 160.)

Under these assumptions we will summarize the results hitherto obtained, referring to Figs. 3 and 4:

Fig. 3 represents a geodetic polar triangle on the ellipsoid, with the latitudes φ and φ' and the difference of longitude l ; the geodetic line which connects the two points with the latitudes φ and φ' has the length s and the two azimuths α and α' .

Fig. 4 is the conformal spherical image of Fig. 3; the spherical latitudes u and u' correspond to the latitudes φ and φ' according to equation (8); the spherical difference of longitude $\lambda = \alpha l$ is obtained from the difference of longitude l of the ellipsoid by multiplication by a constant factor α ; and the arc of a great circle s' is related to the geodetic line s by the scale factor m , since we must have $s' = \int m ds$.

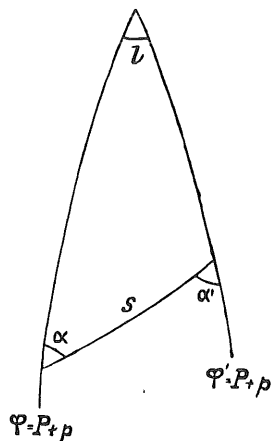


Fig. 3.
Ellipsoid.

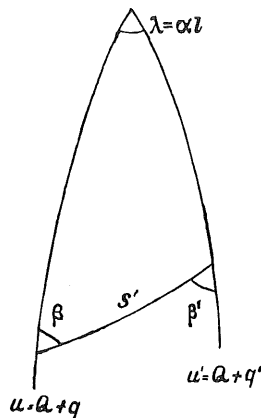


Fig. 4.
Sphere with the radius A .

The azimuths β and β' on the sphere are *not* exactly equal to the azimuths α and α' on the ellipsoid; in the following applications, however, the β 's and α 's will at least be nearly equal to one another.

By the notation of latitude $\varphi = P + p$ and $u = Q + q$ it is indicated that P is a certain normal latitude on the ellipsoid and Q is the corresponding normal latitude on the sphere, as well as that p and q are differences of latitude. [However, we must not mistake here q for the isometric latitude.]

Section 47. Choice of the Constants

The basic equations (8), (10) and (11) which were found in the previous section 46 contain three arbitrary constants, namely α , k and the spherical radius A .

Now we have it in our power to bring about by appropriate determination of these constants α , k and A that for a definite region the deviation of the scale factor m from the value 1 shall become as small as possible.

For this purpose, we assume a value P of the latitude φ , belonging approximately to the center of the region, to which also a certain value Q of the latitude u corresponds on the sphere.

By introducing at the same time also the symbols p and q for the differences of latitude on the ellipsoid and on the sphere, we have the following correlated notation, as is already entered in Figs. 3 and 4 of the previous section 46:

$$\text{Ellipsoidal latitude} \quad \varphi = P + p \quad (1)$$

$$\text{Spherical latitude} \quad u = Q + q. \quad (2)$$

At the normal latitude P or, as the case may be, Q the scale factor shall be $m = 1$, and hence $\log m = 0$, and for any other latitude $\log m$ shall be determined by a series whose first terms shall be the derivatives $\frac{d \log m}{du}$ and $\frac{d^2 \log m}{du^2}$.

We now can dispose of the three constants α , k and A in such a way that these first two derivatives also vanish for the normal latitude; we therefore have the following three conditions for the three constants α , k and A :

$$\text{for } u = Q \text{ we shall have: } 1. m = 1 \text{ or } \log m = 0, \quad (3)$$

$$2. \frac{d \log m}{du} = 0, \quad (4)$$

$$3. \frac{d^2 \log m}{du^2} = 0. \quad (5)$$

According to this, we have to deal at first with the first two derivatives of $\log m$ and take from (10) and (6), section 46, p. 243 and p. 242, the two equations:

$$m = \frac{A}{c} \frac{\alpha \cos u}{\cos \varphi} V, \quad \text{where } V = \sqrt{1 + e'^2 \cos^2 \varphi} \quad (6)$$

$$\text{and} \quad \frac{d \varphi}{du} = \frac{V^2 \cos \varphi}{\alpha \cos u}. \quad (7)$$

From the derivative of V we obtain, just as in the case of (13), section 18, p. 75:

$$\frac{dV}{d\varphi} = -\frac{e'^2}{V} \sin \varphi \cos \varphi = -\frac{\eta^2}{V} \tan \varphi \quad (\text{where } \eta^2 = e'^2 \cos^2 \varphi). \quad (8)$$

Now (6) yields:

$$\begin{aligned} \log m &= \log \frac{A}{c} + \log \cos u - \log \cos \varphi + \log V \\ \frac{d \log m}{du} &= -\tan u + \tan \varphi \frac{V^2 \cos \varphi}{\alpha \cos u} - \frac{\eta^2}{V^2} \tan \varphi \frac{V^2 \cos \varphi}{\alpha \cos u} \\ \frac{d \log m}{du} &= -\tan u + \frac{\sin \varphi}{\alpha \cos u} \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{d^2 \log m}{du^2} &= -\frac{1}{\cos^2 u} + \frac{1}{\alpha \cos^2 u} \left(\cos \varphi \frac{V^2 \cos \varphi}{\alpha} + \sin \varphi \sin u \right) \\ \frac{d^2 \log m}{du^2} &= \frac{1}{\alpha^2 \cos^2 u} (-\alpha^2 + V^2 \cos^2 \varphi + \alpha \sin \varphi \sin u). \end{aligned} \quad (10)$$

Now in order to introduce the conditions (3), (4) and (5), we have to set in (6), (9) and (10): $\varphi = P$ and $u = Q$. This yields

$$\text{from (6):} \quad 1 = \frac{A}{c} \frac{\alpha \cos Q}{\cos P} V \quad (\text{where } V^2 = 1 + e'^2 \cos^2 P), \quad (11)$$

$$\text{from (9):} \quad 0 = -\tan Q + \frac{\sin P}{\alpha \cos Q}, \quad (12)$$

$$\text{from (10):} \quad 0 = -\alpha^2 + V^2 \cos^2 P + \alpha \sin P \sin Q. \quad (13)$$

Now (12) yields at once: $\alpha \sin Q = \sin P.$ (14)

This set into (13) yields, with regard to V^2 in (11):

$$\alpha^2 = 1 + e'^2 \cos^4 P. \quad (15)$$

(14) also yields $\alpha^2 \cos^2 Q = \alpha^2 - \sin^2 P$, and this, in addition to (15), set into (11) yields:

$$A = \frac{c}{V^2} = \frac{c}{1 + e'^2 \cos^2 P}. \quad (16)$$

According to (28), first half-volume, section 38, p. 51, this is the mean radius of curvature at the latitude P . From (14) and (15) we also find:

$$\begin{aligned} \alpha^2 \cos^2 Q &= (1 + e'^2 \cos^4 P) - \sin^2 P = \cos^2 P + e'^2 \cos^4 P \\ &= \cos^2 P (1 + e'^2 \cos^2 P) \\ \alpha \cos Q &= \cos P V, \quad \text{where } V^2 = 1 + e'^2 \cos^2 P. \end{aligned} \quad (17)$$

From (14) and (17) there also follows:

$$V \tan Q = \tan P. \quad (17a)$$

From (15) and (16) we thus have the constants α and A , and through (14), also the third constant k is determined, insofar as P and Q are thereby connected with one another; if we now set in (8), section 46, p. 243, $\varphi = P$ and $u = Q$, i.e., if we apply that equation to the normal latitude, then we obtain:

$$k = \frac{\tan^\alpha \left(45^\circ + \frac{P}{2} \right)}{\tan \left(45^\circ + \frac{Q}{2} \right)} \left(\frac{1 - e \sin P}{1 + e \sin P} \right)^{\frac{\alpha e}{2}}. \quad (18)$$

The following course of the computation now presents itself: We assume arbitrarily a normal latitude P on the ellipsoid, compute with this the mean radius of curvature A according to (16), then α according to (15), Q according to (14) and finally k according to (18); then, for each ellipsoidal latitude φ , we can compute the spherical latitude u corresponding to it and also the scale factor m corresponding to it according to (8) and (9), section 46, p. 243.

But, instead of this, we also can proceed so that we do not assume a normal latitude P on the ellipsoid, but a normal latitude Q on the sphere as arbitrary (round number). In this case, which is not considerably different from the first case, we can however not compute directly according to the formulae (14) and (15), but we must eliminate the latitude P from (14) and (15), in order to express α^2 by Q . If we take to this from (14):

$$\cos^4 P = (1 - \alpha^2 \sin^2 Q)^2 = 1 - 2 \alpha^2 \sin^2 Q + \alpha^4 \sin^4 Q,$$

and if we set this into (15), then we are led to an equation which contains α^2 and α^4 , and this solved for α^2 yields:

$$\alpha^2 = \frac{1 + 2 e'^2 \sin^2 Q - \sqrt{1 + 4 e'^2 \sin^2 Q \cos^2 Q}}{2 e'^2 \sin^4 Q}. \quad (19)$$

This equation (19), in addition to (14), permits then the further computation in the previous manner.

But since formula (19) is very little suited for direct calculation, i.e., applied directly it cannot yield a rigorous computation, it is advisable to develop it into a series according to the first half-volume, p. 20:

$$\left. \begin{aligned} \sqrt{1 + 4 e'^2 \sin^2 Q \cos^2 Q} &= 1 + \frac{4}{2} e'^2 \sin^2 Q \cos^2 Q - \frac{16}{8} e'^4 \sin^4 Q \cos^4 Q \\ &+ \frac{64}{16} e'^6 \sin^6 Q \cos^6 Q - \frac{5}{128} 256 e'^8 \sin^8 Q \cos^8 Q. \end{aligned} \right\}$$

With this, (19) yields a series whose first three terms are:

$$\alpha^2 = 1 + e'^2 \cos^4 Q - 2 e'^4 \sin^2 Q \cos^2 Q + 5 e'^6 \sin^4 Q \cos^8 Q. \quad (20)$$

With this, everything is prepared for use.

We now deal with the introduction of a normal latitude P or Q . The most readily available thing would be to assume the ellipsoidal latitude P as a round number for the center of the geographic region of application; but Gauss has taken a spherical normal value Q as a basis, namely:

$$\text{Sphere } Q = 52^\circ 40' 0''. \quad (21)$$

Besides, Gauss has assumed as the Bessel dimensions of the earth:

$$\log a = 6.514\ 8235 \cdot 337 \text{ for toises}$$

and

$$\log a = 6.804\ 6434 \cdot 637 \text{ for meters} \quad (22)$$

$$\log \sqrt{1 - e^2} = 9.998\ 5458 \cdot 202 \quad (23)$$

$$\log e = 8.912\ 2052 \cdot 079, \quad \log e^2 = 7.824\ 4104 \cdot 158. \quad (24)$$

These values (22) and (23) are the same as those indicated by us in the first half-volume, section 37, p. 45, while $\log e^2$ according to (24) deviates in the last places from our previous data. This originates from the uncertainties which formerly have existed in general with regard to the last places of Bessel's dimensions of the earth (cf. first half-volume, section 37, pp. 42-44).

The trigonometric section of the Prussian Land Survey has carried out, from Gauss' whole theory of the conformal projection of the sphere, with its own constants (i.e. with the bold-face numbers on p. 44 in the first half-volume, a new computation with tables, which are published in "Formeln und Tafeln" already mentioned on p. 241.

As far as we indicate, in the following, computations of our own, we have retained the numbers of the first half-volume, p. 44 and p. 45, namely:

$$\log a = 6.804\ 6434 \cdot 637 \text{ for meters} \quad (25)$$

$$\log c = 6.806\ 0976 \cdot 435 \text{ for meters} \quad (26)$$

$$\log e^2 = 7.824\ 4104 \cdot 237, \quad \log e'^2 = 7.827\ 3187 \cdot 833 \quad (27)$$

$$\log (1 - e^2) = \log \frac{1}{1 + e'^2} = 9.997\ 0916 \cdot 404. \quad (28)$$

With this, we will calculate the remaining constants according to the above formulae. As an arbitrary assumption we base ourselves on the following, as indicated in (21):

$$\text{Normal spherical latitude } Q = 52^\circ 40' 0''. \quad (29)$$

With this, we compute α^2 according to series (20):

$$\begin{aligned} \alpha^2 &= 1.00090\ 88703 - 28399 + 111 = 1.00090\ 60415 \\ \log \alpha &= 0.000\ 1966 \cdot 553 \end{aligned} \quad (30)$$

$$\alpha = 1 + 0.000\ 452\ 918, \quad \frac{1}{\alpha} = 1 - 0.000\ 452\ 713. \quad (31)$$

There follows the computation of P according to (14); we find

$$P = 52^{\circ} 42' 2.53251'' \quad (32)$$

$$\log \sin P = 9.900\ 6297\cdot679, \quad \log \cos P = 9.782\ 4573\cdot113, \quad \log \tan P = 0.118\ 1724\cdot566.$$

With $\cos P$ we also have:

$$\log e^{1/2} \cos^2 P = \log \eta^2 = 7.392\ 2334\cdot059 \quad (33)$$

and with this, we can compute directly $V^2 = 1 + \eta^2$:

$$\log V^2 = 0.001\ 0702\cdot432, \quad \log V = 0.000\ 5351\cdot216. \quad (34)$$

For a check we also can compute $\log V^2$ according to the formula (11), first half-volume, p. 57, or determine $\log V$ by interpolation from the auxiliary table, p. [5] of the Appendix of the first half-volume; both yield the same result as (34).

Before we go further, we also can make the check according to (17), $\alpha \cos Q = V \cos P$, which ends with an error of 0.001, which we do not consider further.

With $\log V^2$ according to (34) we also have the spherical radius A according to (16); the calculation with (26) and (34) yields:

$$\log A = 6.805\ 0274\cdot003. \quad (35)$$

Finally, we also compute k according to (18); we have for this $e \sin P = 0.064\ 988\ 270\ 546$ and further:

$$\left. \begin{array}{l} \log \left(\frac{1 - e \sin P}{1 + e \sin P} \right)^{\frac{\alpha e}{2}} \\ \log \tan^{\alpha} \left(45^{\circ} + \frac{P}{2} \right) \\ \log \cot \left(45^{\circ} + \frac{Q}{2} \right) \end{array} \right\} \begin{array}{l} 9.997\ 6898\cdot845 \\ 0.471\ 9371\cdot356 \\ 9.528\ 7020\cdot994 \end{array} \left\{ \begin{array}{l} \log k = 9.998\ 3291\cdot195 \\ \log \frac{1}{k} = 0.001\ 6708\cdot805. \end{array} \right.$$

Gauss gives $\log \frac{1}{k} = 0.001\ 6708\cdot804. \quad (36)$

We have here the insignificant difference 0.001 compared to the statement by Gauss in Art. 6 of "Untersuchungen über Gegenstände der höheren Geodäsie," while the other constants P , $\log \alpha$, $\log A$ according to (32), (30), (35) agree to within the last decimal with the data by Gauss.

This is an assurance that the difference of the values $\log e^2$ in (24) and (27) is no longer noticeable in the constants P , α , A and k in the case of computing with 10-place logarithms, while in the later computations of coefficients, when the factor η^2 occurs, the small difference in the assumptions of e^2 or, as the case may be, $e^{1/2}$ becomes noticeable.

Earlier, we have calculated a numerical example for the determination of u and m with given φ , according to the basic formulae (8) and (10), section 46, p. 243. The details of this computation were indicated in the former editions, e.g. third edition, 1890, pp. 431 and 432; we will insert here only the result of this computation for the latitude of Karlsruhe:

$$\varphi = 49^{\circ} 0' 0'' \quad u = 48^{\circ} 58' 18.08'' \quad \log m = 0.000\ 0002\cdot7. \quad (37)$$

The more accurate values for this, which we can find from the auxiliary table, p. [16] of the Appendix by interpolation, are:

$$\varphi = 49^{\circ} 0' 0'' \quad u = 48^{\circ} 58' 18.0784'' \quad \log m = 0.000\ 0002\cdot48. \quad (38)$$

The agreement between (37) and (38) is sufficient insofar as the values u and $\log m$ of (37) are computed only with 7-place logarithms (± 0.25).

The computation according to the closed formulae (8) and (10), section 46, p. 243, is troublesome and relatively inaccurate.

We obtain a better computational procedure by developments in series, to which we shall pass over in section 48 and section 49.

Section 48. Development in Series for the Difference of Latitude

The relation between the latitude φ on the ellipsoid and the latitude u on the sphere corresponding to it is given, it is true, by equation (8), section 46, p. 243, which permits the computation of the value u corresponding to each value φ . For immediate application, that closed formula is inconvenient (cf. the numerical example, section 47, p. 248), and can only be used, say, indirectly for the solution with respect to φ with given u .

A more convenient method for the determination of u for a given φ results with the help of equation (15), section 46, p. 243, if we use the auxiliary table by L. Grabowski in *Zeitschr. f. Verm.*, 1929, pp. 35-44 already mentioned on p. 153. We set here $q = \log \tan \left(45^\circ + \frac{\varphi}{2} \right) - \eta$, for which the value of η for the argument φ can be taken from the table. If we have found q in this way, then we obtain from equation (15), p. 243, the isometric latitude ω on the sphere, and then we have according to (13), p. 243:

$$\log \tan \left(45^\circ + \frac{u}{2} \right) = \omega.$$

For many purposes it is desirable to apply development in series, which we will carry out in the following.

Since on the ellipsoid there was assumed a normal latitude P and on the sphere a normal latitude Q , the latitudes shall be expressed generally by their differences from P and Q , i.e. we set according to section 47, (1) and (2), p. 244:

$$\text{Ellipsoid} \quad \varphi = P + p \quad (1)$$

$$\text{Sphere} \quad u = Q + q. \quad (2)$$

Since the relation between P and Q is known, we now deal only with a relation between p and q , which can be set up in two forms, namely:

$$\text{either:} \quad p = \frac{d\varphi}{du} q + \frac{d^2\varphi}{du^2} \frac{q^2}{2} + \frac{d^3\varphi}{du^3} \frac{q^3}{6} + \dots \quad (3)$$

$$\text{or:} \quad q = \frac{du}{d\varphi} p + \frac{d^2u}{d\varphi^2} \frac{p^2}{2} + \frac{d^3u}{d\varphi^3} \frac{p^3}{6} + \dots \quad (4)$$

The sign] shall indicate here that after carrying out the differentiations we are to set $p = 0$ and $q = 0$, or $\varphi = P$ and $u = Q$.

At first, we will take form (4) and have to this from (6), section 46, p. 242:

$$\frac{du}{d\varphi} = \frac{1}{V^2} \frac{\alpha \cos u}{\cos \varphi}. \quad (5)$$

We have here, as indicated already previously in the first half-volume, section 40, p. 62:

$$V = \sqrt{1 + e'^2 \cos^2 \varphi} = \sqrt{1 + \eta^2} \quad (6)$$

$$\frac{dV}{d\varphi} = -\frac{\eta^2}{V}t \quad (t = \tan \varphi) \quad (7)$$

$$\frac{dV^n}{d\varphi} = -n\eta^2 V^{n-2}t \quad \text{and} \quad \frac{d\eta^n}{d\varphi} = -n\eta^n t. \quad (8)$$

We have premised this, because it is needed repeatedly, and treat the derivative of the second factor of (5), likewise occurring frequently, specially:

$$\frac{d}{d\varphi} \left(\frac{\alpha \cos u}{\cos \varphi} \right) = \frac{1}{\cos^2 \varphi} \left(-\alpha \sin u \frac{du}{d\varphi} \cos \varphi + \alpha \cos u \sin \varphi \right). \quad (9)$$

If we introduce (5) here, and take into account $V^2 = 1 + \eta^2$ according to (6), then we will have:

$$\frac{d}{d\varphi} \left(\frac{\alpha \cos u}{\cos \varphi} \right) = \frac{1}{V^2} \frac{\alpha \cos u}{\cos \varphi} \left(-\frac{\alpha \sin u}{\cos \varphi} + t + \eta^2 t \right). \quad (10)$$

Now if once more we take the derivative of (5), then we have at first because of (8):

$$\frac{d^2 u}{d\varphi^2} = \frac{2\eta^2}{V^4} t \frac{\alpha \cos u}{\cos \varphi} + \frac{1}{V^2} \frac{d}{d\varphi} \left(\frac{\alpha \cos u}{\cos \varphi} \right).$$

If we substitute the value already prepared in (9), then we obtain:

$$\frac{d^2 u}{d\varphi^2} = \frac{1}{V^4} \frac{\alpha \cos u}{\cos \varphi} \left(t + 3\eta^2 t - \frac{\alpha \sin u}{\cos \varphi} \right). \quad (11)$$

As preparation for the next derivative of this, we treat at first the last part, and find in a similar way as above in (9) and (10):

$$\frac{d}{d\varphi} \left(\frac{\alpha \sin u}{\cos \varphi} \right) = \frac{1}{V^2} \left(\left(\frac{\alpha \cos u}{\cos \varphi} \right)^2 + \frac{\alpha \sin u}{\cos \varphi} t (1 + \eta^2) \right). \quad (12)$$

Now (11) yields further:

$$\left. \begin{aligned} \frac{d^3 u}{d\varphi^3} &= \frac{4\eta^2}{V^6} t \frac{\alpha \cos u}{\cos \varphi} \left(t + 3\eta^2 t - \frac{\alpha \sin u}{\cos \varphi} \right) \\ &+ \frac{1}{V^4} \frac{d}{d\varphi} \left(\frac{\alpha \cos u}{\cos \varphi} \right) \left(t + 3\eta^2 t - \frac{\alpha \sin u}{\cos \varphi} \right) \\ &+ \frac{1}{V^4} \frac{\alpha \cos u}{\cos \varphi} \left((1 + t^2) - 6\eta^2 t^2 + 3\eta^2 (1 + t^2) - \frac{d}{d\varphi} \left(\frac{\alpha \sin u}{\cos \varphi} \right) \right) \end{aligned} \right\} \quad (13)$$

Since we will stop at the third power, the point in question now is to make all substitutions which are indicated in (3) and (4) by], i.e. to set $\varphi = P$, $u = Q$. But according to (14) and (17), section 47, p. 246, we have $\alpha \sin Q = \sin P$ and $\alpha \cos Q = V \cos P$, and there follows hence:

$$\left[\frac{\alpha \sin u}{\sin \varphi} \right] = 1 \quad \left[\frac{\alpha \sin u}{\cos \varphi} \right] = t \quad \left[\frac{\alpha \cos u}{\cos \varphi} \right] = V \quad (14)$$

and this set into (10) and (12) yields (since $V^2 = 1 + \eta^2$):

$$\left[\frac{d}{d\varphi} \frac{\alpha \cos u}{\cos \varphi} \right] = \frac{\eta^2 t}{V} \quad \left[\frac{d}{d\varphi} \frac{\alpha \sin u}{\cos \varphi} \right] = 1 + t^2. \quad (15)$$

If we substitute these equations (14) and (15) in the three general derivatives (5), (11) and (13), then these derivatives become considerably contracted, and if we collect everything of the same kind, then we obtain:

$$\left[\frac{d u}{d \varphi} \right] = \frac{1}{V}, \quad \left[\frac{d^2 u}{d \varphi^2} \right] = \frac{3 \eta^2}{V^3} t \quad (16)$$

$$\left[\frac{d^3 u}{d \varphi^3} \right] = \frac{3 \eta^2}{V^5} (1 - t^2 + \eta^2 + 4 \eta^2 t^2). \quad (17)$$

With these equations (16) and (17) we can put together formula (4):

$$q = \frac{1}{V} p + \frac{3}{2} \frac{\eta^2 t}{V^3} p^2 + \frac{1}{2} \frac{\eta^2}{V^5} (1 - t^2 + \eta^2 + 4 \eta^2 t^2) p^3. \quad (18)$$

In a similar way as this series, which progresses according to powers of p , we can also find the inverse series (3), which progresses according to powers of q and determines p ; however, if we do not go further than the third order, we obtain the inverse series also by inverting series (18) directly step by step. In the first approximation (18) yields:

$$\begin{aligned} p &= q V + q^2 \dots, & p^2 &= q^2 V^2 + q^3 \dots \\ p &= q V - \frac{3}{2} q^2 \eta^2 t, & p^2 &= q^2 V^2 - 3 q^3 V \eta^2 t + \dots \end{aligned}$$

This p^2 and $p^3 = q^3 V^3$, introduced into (18), and everything arranged according to the same powers, yields at once:

$$p = q V - \frac{3}{2} \eta^2 q^2 t + \frac{1}{2} \frac{\eta^2}{V} (-1 + t^2 - \eta^2 + 5 \eta^2 t^2) q^3. \quad (19)$$

In the series (18) and (19) p and q are understood in radian measure; instead of this, we will now count the independent variable p in (18), q in (19), in degrees and the functions q and p in seconds; series (18) and (19) then assume the following forms:

$$q = \frac{3600}{V} p + \frac{3600}{\rho^{\circ}} \frac{3}{2} \frac{\eta^2 t}{V^3} p^2 - \frac{3600}{\rho^{\circ 2}} \frac{\eta^2}{2 V^5} (-1 + t^2 - \eta^2 + 4 \eta^2 t^2) p^3 \quad (20)$$

$$p = 3600 V q - \frac{3600}{\rho^{\circ}} \frac{3}{2} \eta^2 t q^2 + \frac{3600}{\rho^{\circ 2}} \frac{\eta^2}{2 V} (-1 + t^2 - \eta^2 + 5 \eta^2 t^2) q^3. \quad (21)$$

If we calculate here the coefficients with the constants of section 47, then we obtain:

$$q = 3595.566 \, 945 p + 0.304 \, 138 \, 6587 p^2 - 0.000 \, 946 \, 265 \, 801 p^3 + \dots \quad (22)$$

$$p = 3604.438 \, 521 q - 0.305 \, 264 \, 9836 q^2 + 0.001 \, 002 \, 642 \, 525 q^3 + \dots \quad (23)$$

If we desire to treat these series as converging and breaking off with the third power, then we do not need the coefficients to so many places; we have however worked out many places for comparison with the numerical data by Gauss, who gives the series carried out to the fifth power in Arts. 6 and 8 of "Untersuchungen über Gegenstände der höheren Geodäsie." In particular, the series of Art. 8 indicated by Gauss for the table computation is:

$$\left. \begin{aligned}
 p - q &= 443.852\,122 \frac{q}{100} \\
 &- 3052.649\,780 \left(\frac{q}{100}\right)^2 [3.484\,6769.820] \\
 &+ 1002.642\,506 \left(\frac{q}{100}\right)^3 [3.001\,1461.121] \\
 &+ 4119.589\,282 \left(\frac{q}{100}\right)^4 [3.614\,8539.196] \\
 &- 431.181\,623 \left(\frac{q}{100}\right)^5 [2.634\,661].
 \end{aligned} \right\} \quad (24)$$

The application of this series to $q = -7^\circ$ and $q = +7^\circ$ yields:

$u = Q + q = 45^\circ 40' 0''$	$59^\circ 40' 0''$
$q = -7^\circ$	$q = +7^\circ$
— 31.069 649"	+ 31.069 649"
— 14.957 984	— 14.957 984
— 0.343 906	+ 0.343 906
+ 0.098 911	+ 0.098 911
+ 0.000 725	— 0.000 725
<hr/>	<hr/>
$p - q = -46.271\,903''$	$+ 16.553\,757''$
$p = -7^\circ 0' 46.271\,903''$,	$+ 7^\circ 0' 16.553\,757''$
$P = 52^\circ 42' 2.53251''$	$52^\circ 42' 2.53251''$
$\varphi = P + p = 45^\circ 41' 16.26061''$	$59^\circ 42' 19.08627''$.

in addition

These values lie already beyond the limits of Gauss' table, of which we have given an excerpt somewhat extended at the limits on pages [21] to [22] of the Appendix.

Since the last term of the computation still amounts to 0.0007", and the convergence is not very great, we can conclude that the last places of the above computed values of φ are no longer accurate.

A further development of series (24) has been carried out by Schreiber to within the ninth order in "Formeln und Tafeln" on p. 15, already mentioned on p. 241, and we find therefrom that for $q = \pm 7^\circ$ the term of the sixth order still amounts to 0.00010". This is also confirmed if we take the values of φ directly from Schreiber's tables. Then we have for

$u = 45^\circ 40' 0''$	$\varphi = 45^\circ 41' 16.26050''$
$u = 59^\circ 40' 0''$	$\varphi = 59^\circ 42' 19.08618''$.

The Gauss formulae (24) are therefore no longer sufficient for the argument $\pm 7^\circ$ when an accuracy of 0.00001" is required.

Another method for the determination of the spheroidal difference of latitude from the spherical difference of latitude and vice versa is given by Horvat in *Zeitschr. f. Verm.*, 1939, p. 624. From the spherical difference of latitude there is computed here the ellipsoidal meridional arc, with which we can determine the spheroidal difference of latitude from a table of meridional arcs.

According to (10), section 46, p. 243, the scale factor is

$$m = \frac{A}{c} \frac{\alpha \cos u}{\cos \varphi} V. \quad (1)$$

At the normal latitude $\varphi = P$ (and $u = Q$) this ratio is $m = 1$; and, if any arbitrary latitude on the sphere, u , is set equal to $Q + q$, as thus far, then for such a latitude the ratio m will be able to be represented as a function of q , or the series for $\log m$ has at first this form:

$$\log m = \frac{d \log m}{d q} \Big] q + \frac{d^2 \log m}{d q^2} \Big] \frac{q^2}{2} + \frac{d^3 \log m}{d q^3} \Big] \frac{q^3}{6} + \dots \quad (2)$$

But since the first two derivatives of $\log m$ were set equal to zero [(4) and (5), section 47, p. 245], then (2) becomes contracted to:

$$\log m = \frac{d^3 \log m}{d q^3} \Big] \frac{q^3}{6} + q^4 \dots \quad (3)$$

In addition, we have from (10), section 47, p. 245, the second derivative:

$$\frac{d^2 \log m}{d u^2} = \frac{-\alpha^2 + V^2 \cos^2 \varphi + \alpha \sin \varphi \sin u}{\alpha^2 \cos^2 u} = \frac{Z}{N}, \quad (4)$$

therefore, further:

$$\frac{d^3 \log m}{d u^3} = \frac{1}{N^2} \left(\frac{d Z}{d u} N - \frac{d N}{d u} Z \right). \quad (5)$$

Later when we have to make again the substitutions for the normal latitudes Q and P according to (14), section 48, p. 250, we will find that the numerator Z in (4) vanishes; and hence, there remains of (5) only:

$$\frac{d^3 \log m}{d u^3} \Big] = \frac{1}{N} \frac{d Z}{d u}. \quad (6)$$

Since α^2 in the numerator Z of (4) is also constant, we therefore deal further only with:

$$\begin{aligned} \frac{d Z}{d u} &= \frac{d}{d u} (V^2 \cos^2 \varphi + \alpha \sin \varphi \sin u) \\ &= \left(2 V \frac{d V}{d \varphi} \cos^2 \varphi - 2 V^2 \cos \varphi \sin \varphi \right) \frac{d \varphi}{d u} + \alpha \cos \varphi \frac{d \varphi}{d u} \sin u + \alpha \sin \varphi \cos u. \end{aligned} \quad (7)$$

We have to bear in mind here according to (8) and (7), section 47, p. 245, with $\tan \varphi = t$:

$$\frac{d V}{d \varphi} = -\frac{\eta^2}{V} t \quad \text{and} \quad \frac{d \varphi}{d u} = \frac{V^2 \cos \varphi}{\alpha \cos u}.$$

This substituted in (7) yields:

$$(-2 \eta^2 t \cos^2 \varphi - 2 V^2 \cos \varphi \sin \varphi) \frac{V^2 \cos \varphi}{\alpha \cos u} + \alpha \cos \varphi \frac{V^2 \cos \varphi}{\alpha \cos u} \sin u + \alpha \sin \varphi \cos u.$$

We must now make the substitutions (14), section 48, p. 250, again, whereby $\frac{V^2 \cos \varphi}{\alpha \cos u} = V$, and the above equation thus yields with $t = \tan \varphi$:

$$\left[\frac{dZ}{du} \right] = -2V\eta^2 t \cos^2 \varphi - 2V^3 \cos \varphi \sin \varphi + \alpha \cos \varphi \sin u V + \alpha \sin \varphi \cos u.$$

If we insert further the denominator $N = \alpha^2 \cos^2 u$ from (4) and take into account equations (14), section 48, p. 250, again, then we will obtain altogether:

$$\left[\frac{d^3 \log m}{du^3} \right] = -\frac{2\eta^2}{V} t - 2Vt + \frac{1}{V} t + \frac{1}{V} t,$$

and with $V^2 = 1 + \eta^2$, $t = \tan P$, this becomes contracted to:

$$\left[\frac{d^3 \log m}{du^3} \right] = -\frac{4\eta^2}{V} t = -\frac{4\eta^2}{V} \tan P. \quad (8)$$

The series for $\log m$ sought for is therefore according to (3):

$$\log m = -\frac{2\eta^2}{3V} t q^3 + q^4 \dots \text{ with } t = \tan P. \quad (9)$$

If we wish to stop here, i.e. if we wish to neglect q^4 and p^4 , then we also can easily express $\log m$ by p^3 , for since we have, according to (19), section 48, p. 251, as a first approximation $p = qV$, we can write (9) also thusly:

$$\log m = -\frac{2\eta^2}{3V^4} t p^3 + \dots \text{ with } t = \tan P. \quad (10)$$

In (9) and (10) \log means the natural logarithm; therefore if we wish to use Brigg's common logarithms, then we must insert further the modulus μ , and at the same time, if we aim to set up the formulae for q and p in degrees, then we must further divide by ρ^{03} ; i.e. we obtain from (9):

$$\log m = -\frac{\mu}{\rho^{03}} \frac{2\eta^2}{3V} t q^3 \text{ with } t = \tan P. \quad (11)$$

The calculation with the constants of (25) to (28), section 47, p. 247, yields for units of the seventh place of logarithms:

$$\log m = -0.049\,796\,165\,q^3 + \dots \quad (12)$$

In the same manner, we obtain from (10):

$$\log m = -0.049\,612\,434\,p^3 + \dots \quad (13)$$

In Arts. 7 and 9 of "Untersuchungen über Gegenstände der höheren Geodäsie" Gauss has continued these developments to within the sixth power, whereby he obtained:

$$\log m = -49,796.16394 \left(\frac{q}{100} \right)^3 - 16,150.3076 \left(\frac{q}{100} \right)^4 - 23,973.954 \left(\frac{q}{100} \right)^5 - 125,671.0 \left(\frac{q}{100} \right)^6. \quad (14)$$

Here q is counted in units of 1° and $\log m$ in units of the seventh decimal of logarithms. Our formula is thus only the first approximation of Gauss' formula (14), according to which Gauss' values $\log m$ of our table in the Appendix on pp. [21] to [22] are computed. As an example, we take for $q = -6^\circ$ or $u = 46^\circ 40'$ and for $q = +6^\circ$ or $u = 58^\circ 40'$ from that table $\log m = +10.559$ and $\log m = -10.990$, while the approximate formula (12) gives in both cases only $\log m = +10.7$ and $= -10.8$.

The development for $\log m$ also has been carried further by Schreiber in "Formeln und Tafeln" mentioned on p. 241 to within the tenth order, in which detailed tables for $\log m$ are also contained.

Up to now we always have treated $\log m$ only; we obtain a formula for m itself very simply from the following, since in (9) and (10) natural logarithms hold:

$$m = 1 - \frac{2}{3} \frac{\eta^2}{V} t q^3 + \dots \quad \text{or} \quad m = 1 - \frac{2}{3} \frac{\eta^2}{V^4} t p^3 \quad (15)$$

and conversely (where we always have the meaning $t = \tan P$):

$$\frac{1}{m} = 1 + \frac{2}{3} \frac{\eta^2}{V} t q^3 \quad \text{or} \quad \frac{1}{m} = 1 + \frac{2}{3} \frac{\eta^2}{V^4} t p^3. \quad (16)$$

Reduction of distances

The value m holds only for infinitely small distances, i.e., if dS means a small distance on the ellipsoid and ds the corresponding distance on the sphere, then we have

$$m = \frac{ds}{dS} \quad \text{or} \quad dS = \frac{1}{m} ds,$$

and in order to be able to compare also finite distances s and S , we have to integrate this equation, as has already been done in the first half-volume, section 68, p. 198, and section 70, p. 213.

For this purpose, we count the spherical difference of latitude q from a value q_1 , which corresponds to the beginning of the whole arc s , and we count the length of the arc s itself likewise from the beginning with $+x$ at the azimuth β_1 .

Since the spherical radius is $= A$, we have the difference of latitude $q - q_1$ as a series with respect to powers of x with the starting azimuth β_1 , i.e., we can use for this the previous general developments in series of the first half-volume, section 63, which means we have from equation (27) used there, p. 178, with

$$u = \frac{x}{A} \cos \beta_1 \quad \text{and with} \quad v = \frac{x}{A} \sin \beta_1:$$

$$\text{difference of latitude } q - q_1 = \frac{x}{A} \cos \beta_1 - \frac{x^2}{2 A^2} \sin^2 \beta_1 \tan (Q + q_0).$$

It is sufficient for the following to know that this is a quadratic function of x and that, with this, $\frac{1}{m}$ from (16) can also be developed as a series progressing according to increasing powers of x , exactly as in the case of a previous similar study of the first half-volume, section 70, the expression $\frac{1}{m}$ could be represented as a power series $\alpha + \beta l + \gamma l^2 + \dots$ on p. 213.

This is also sufficient to represent the relation between a geodetic line S on the ellipsoid and its image s on the sphere by an equation which, in accordance with the previous equation (35), first half-volume, section 70, p. 216, or (15), section 68, p. 199, reads in the first approximation thusly:

$$\frac{S}{s} = \frac{1}{6} \left(\frac{1}{m_1} + \frac{4}{m_0} + \frac{1}{m_2} \right), \quad (17)$$

where m_1 means the scale factor at the beginning, m_0 at the center and m_2 at the end.

If the various m 's are not very different, then we can compute with a still greater approximation, and e.g. take logarithmically briefly:

$$\log s - \log S = \frac{\log m_1 + \log m_2}{2}. \quad (18)$$

This is the same as if we write:

$$\frac{s}{S} = \sqrt{m_1 m_2}. \quad (19)$$

In this connection, it is noted once again that S is the geodetic line on the ellipsoid and s the corresponding line on the conformal sphere of radius A .

Section 50. Reduction of Azimuth

When two points of the ellipsoid are projected conformally on the sphere, then we can also consider the connecting lines of the two points, and in fact, we imagine the two points being connected on the ellipsoid by a geodetic line and on the sphere by an arc of a great circle.

But we must not assume that the arc of a great circle is now simply the projection of the geodetic line; this is not the case any more as that in the plane conformal projection of the first half-volume, section 68, where the straight line on the plane must not be taken as the projection of the arc of a great circle, and we shall now also have to undertake a similar study as in the first half-volume, Fig. 3, p. 198, or Fig. 2, p. 211, for the projection of the sphere.

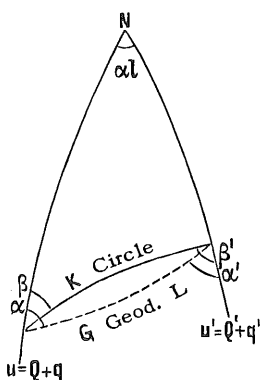


Fig. 1. (Sphere.)

In Fig. 1 in the margin, which refers to the sphere, we consider two points transferred from the ellipsoid with the spherical latitudes u and u' and the difference of longitude αl . Let the connecting arc denoted by K be a great circle arc of the sphere, and in addition, we have drawn a curve G , which is the conformal image on the sphere of the geodetic line of the ellipsoid.

A geodetic line of the ellipsoid does not in general project itself as an arc of a great circle of the sphere, as was already remarked at the beginning, and the point in question now is to determine the azimuth differences $\alpha - \beta$ and $\beta' - \alpha'$ between the image G of the geodetic line and the arc of the great circle K .

According to the principle of conformality, the azimuths α and α' , which the image of the geodetic line on the sphere shows are equal here to the azimuths α and α' of the geodetic line on the ellipsoid, so that the azimuth differences $\alpha - \beta$ and $\beta' - \alpha'$ of the spherical Fig. 1 are what we have to determine.

Our next problem will be to determine the differential of curvature of the line G relatively compared to K , and for this we have drawn separately, in Fig. 2, a differential figure to Fig. 1.

In Fig. 2, we consider the meridian convergence $\alpha_2 - \alpha_1$ for a small portion of the image of the geodetic line, and the meridian convergence $\beta_2 - \beta_1$ for a corresponding portion of the arc of the great circle between the same meridians. Here is dl the difference of longitude on the ellipsoid, and hence, αdl the corresponding difference of longitude on the sphere, where α means the constant for the reduction of length according to (15), section 47, p. 246.

To this, there exist two differential equations:

$$\text{Arc of a great circle } KK' \quad \beta_2 - \beta_1 = \alpha dl \sin u \quad (1)$$

$$\text{Ellipsoidal arc or spherical image } GG' \quad \alpha_2 - \alpha_1 = dl \sin \varphi. \quad (2)$$

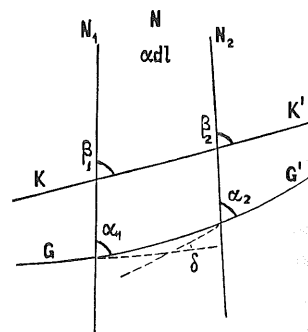


Fig. 2.

$$\delta = (\beta_2 - \beta_1) - (\alpha_2 - \alpha_1) = dl (\alpha \sin u - \sin \varphi). \quad (3)$$

Therefore, the difference:

This difference δ is the curvature of the arc GG' , since the three other sides of the infinitely small quadrilateral of Fig. 2 as arcs of great circles do not have a geodetic curvature and are to be regarded as straight lines in the differential consideration. For we have for the sum of angles of the small quadrilateral of Fig. 2:

$$(180^\circ - \beta_1) + \beta_2 + (180^\circ - \alpha_2) + \alpha_1 = 360^\circ + (\beta_2 - \beta_1) - (\alpha_2 - \alpha_1) = 360^\circ + \delta,$$

and this agrees with the angle δ entered in Fig. 2, as well as with the meaning of δ in equation (3).

For the more rigorous proof of equation (3) it is noted further that the cross-distance y of the lines K and G will be proved later to be very small, only of the order $\eta^2 s^2 q^2$, and therefore, the spherical excess of the small quadrilateral, i.e. the curvature of the surface in addition to the linear curvature, is not considered.

In Fig. 3 we will now assume a spherical system of coordinates x, y , where the arc $Q_1 Q_2$ as x -axis has the meaning of the line K in Fig. 1 and the line L' of Fig. 3 has the same meaning as the line G in Fig. 1, namely the conformal image of the geodetic line.

Since we can now treat the spherical system of coordinates x, y in the first approximation as a plane rectangular system, for the reasons already indicated, we may also apply the differential of curvature found in (3) to the equation:

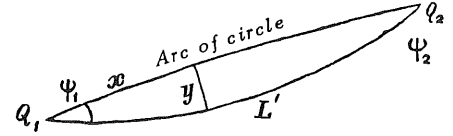


Fig. 3.

$$-\frac{d^2 y}{dx^2} = \frac{\delta}{dx} = (\alpha \sin u - \sin \varphi) \frac{dl}{dx}. \quad (4)$$

We had to write the left side negative here, because in the system of coordinates of Fig. 3 the curve L' is concave with respect to the x -axis.

Now the first thing is to develop the function $\alpha \sin u - \sin \varphi$, and for this, we have according to Figs. 3 and 4, section 46, p. 244:

$$\varphi = P + p \quad u = Q + q;$$

therefore, according to Taylor's series:

$$\sin \varphi = \sin P + p \cos P - \frac{p^2}{2} \sin P \quad (5)$$

$$\alpha \sin u = \alpha \sin Q + \alpha q \cos Q - \alpha \frac{q^2}{2} \sin Q. \quad (6)$$

For the comparison between p and q we have the following series according to (18), section 48, p. 251:

$$q = \frac{p}{V} + \frac{3}{2} \frac{\eta^2}{V^3} p^2 \tan P. \quad (7)$$

Let us bear in mind also the basic formulae for P and Q according to (14) and (17), section 47, p. 246:

$$\alpha \sin Q = \sin P \quad \text{and} \quad \alpha \cos Q = V \cos P. \quad (8)$$

From (5) and (6) we obtain at first, since the first terms cancel each other because of (8):

$$\alpha \sin u - \sin \varphi = \alpha q \cos Q - p \cos P + \frac{p^2 - q^2}{2} \sin P. \quad (9)$$

But because of (8) and (7) we have:

$$\alpha q \cos Q = V \cos P \left(\frac{p}{V} + \frac{3}{2} \frac{\eta^2}{V^3} p^2 \tan P \right),$$

and because of (7):

$$p^2 = q^2 V^2 = q^2 (1 + \eta^2) \quad \text{or} \quad p^2 - q^2 = \frac{p^2}{V^2} \eta^2.$$

With this, we can make (9) into:

$$\begin{aligned} \alpha \sin u - \sin \varphi &= \frac{3}{2} \frac{\eta^2}{V^2} p^2 \sin P + \frac{1}{2} \frac{p^2}{V^2} \eta^2 \sin P \\ \alpha \sin u - \sin \varphi &= \frac{2 \eta^2}{V^2} p^2 \sin P \quad \text{or} \quad = 2 \eta^2 q^2 \sin P. \end{aligned} \quad (10)$$

Reconsidering (4), we thus have the equation of the differential of curvature:

$$-\frac{d^2 y}{dx^2} = 2 \eta^2 q^2 \sin P \frac{dl}{dx}. \quad (11)$$

We have the differential dl of the geographic longitude l according to the general development in series of section 18, p. 77:

$$dl = \frac{dS \sin \alpha}{N \cos \varphi}. \quad (12)$$

Instead of dS for the geodetic line we can set in our case with sufficient accuracy dx , and by taking further in the first approximation $\varphi = P$, we have from (11) and (12):

$$-\frac{d^2 y}{dx^2} = \frac{2 \eta^2}{N} q^2 \sin \alpha \tan P. \quad (13)$$

N , α and P refer to the ellipsoid here, and if we wish to change to the sphere, there is to be set $N = AV$ (since A has the meaning of r in (28), section 38, first half-volume, p. 51). The azimuth α can with sufficient accuracy be set equal to the spherical azimuth β , and according to (8) we have $V \tan Q = \tan P$; consequently, upon changing to the sphere, (13) now yields

$$-\frac{d^2 y}{dx^2} = \frac{2 \eta^2 q^2}{A} \sin \beta \tan Q, \quad (14)$$

where we will write for abbreviation:

$$\frac{2 \eta^2}{A} \sin \beta \tan Q = F, \quad (15)$$

or

$$\frac{2 \eta^2}{AV} \sin \beta \tan P = F. \quad (15a)$$

In order to change to x , we have to set as a first approximation according to Fig. 4, p. 259:

$$q = q_1 + \frac{x}{A} \cos \beta_1 + \dots, \quad \text{therefore} \quad q^2 = q_1^2 + 2 q_1 \frac{x}{A} \cos \beta_1 + \dots \quad (16)$$

Here q_1 is that value of q which belongs to the starting point Q_1 , and q_2 is that value of q which belongs to the end point Q_2 of the arc s under consideration. In the same manner, we also have for the azimuth β , which corresponds to the latitude q and to the abscissa x , according to (29), first half-

volume, p. 179, with $v = \frac{x}{A} \sin \beta_1$:

$$\begin{aligned}\beta &= \beta_1 + \frac{x}{A} \sin \beta_1 \tan Q_1 \\ \sin \beta &= \sin \beta_1 + \frac{x}{A} \sin \beta_1 \cos \beta_1 \tan Q_1.\end{aligned}\quad (17)$$

We thus have from (16) and (17):

$$q^2 \sin \beta = q_1^2 \sin \beta_1 + \frac{x}{A} (2 q_1 \sin \beta_1 \cos \beta_1 + q_1^2 \sin \beta_1 \tan Q_1). \quad (18)$$

This is a linear function of x , which, for temporary abbreviation, may be written thusly:

$$q^2 \sin \beta = f + g x, \quad \text{where} \quad f = q_1^2 \sin \beta_1. \quad (19)$$

With this, we will have according to (14) and (15):

$$-\frac{d^2 y}{d x^2} = F(f + g x) \quad (20)$$

and integrating,

$$-\frac{d y}{d x} = -\psi_1 + F\left(f x + \frac{g x^2}{2}\right) \quad (21)$$

$$-y = -\psi_1 x + F\left(\frac{f x^2}{2} + \frac{g x^3}{6}\right). \quad (22)$$

In (21) $-\psi_1$ is inserted as a constant of integration here, while in (22), in the case of y , which must vanish with $x = 0$, no further constant of integration is added. If $x = s$, then we must have $y = 0$ and $\frac{d y}{d x} = -\psi_2$; this yields from (22) and (21) the following two equations:

$$\begin{aligned}0 &= -\psi_1 s + F\left(f \frac{s^2}{2} + g \frac{s^3}{6}\right) \\ +\psi_2 &= -\psi_1 + F\left(f s + g \frac{s^2}{2}\right).\end{aligned}$$

These two equations are used for the determination of ψ_1 and ψ_2 , and solved for ψ_1 and ψ_2 yield:

$$\psi_1 = F s \left(\frac{f}{2} + g \frac{s}{6} \right) \quad \text{or} \quad = \frac{F s}{6} (2 f + f + g s) \quad (23)$$

$$\psi_2 = F s \left(\frac{f}{2} + g \frac{s}{3} \right) \quad \text{or} \quad = \frac{F s}{6} (f + 2(f + g s)). \quad (24)$$

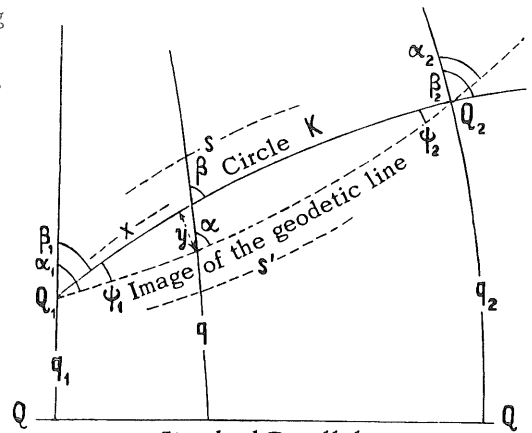


Fig. 4.

The meanings of f and g are to be determined by means of (19):

$$\begin{array}{lll} \text{for } x = 0 & \text{we will have} & q_1^2 \sin \beta_1 = f, \\ \text{for } x = s & \text{we will have} & q_2^2 \sin \beta_2 = f + g s. \end{array}$$

These two equations in connection with (23) and (24) yield:

$$\psi_1 = \frac{F s}{2} \frac{2 q_1^2 \sin \beta_1 + q_2^2 \sin \beta_2}{3} \quad \psi_2 = \frac{F s}{2} \frac{q_1^2 \sin \beta_1 + 2 q_2^2 \sin \beta_2}{3}. \quad (25)$$

In order to introduce the factor F according to (15) or (15a), and to separate later what can be computed by tables, we introduce the function:

$$\eta^2 \tan Q q^2 \quad \text{or} \quad \frac{\eta^2}{V} \tan P q^2 = k. \quad (26)$$

This general k is applied to the starting point and to the end point with:

$$k_1 = \frac{\eta^2}{V} \tan P q_1^2 \quad \text{and} \quad k_2 = \frac{\eta^2}{V} \tan P q_2^2. \quad (27)$$

With this, the formulae (25) and (26) change to the following forms:

$$\alpha_1 - \beta_1 = \psi_1 = \frac{2 k_1 \sin \alpha_1 + k_2 \sin \alpha_2}{3} \frac{s}{A} \quad (28)$$

$$\beta_2 - \alpha_2 = \psi_2 = \frac{k_1 \sin \alpha_1 + 2 k_2 \sin \alpha_2}{3} \frac{s}{A}. \quad (29)$$

For application with numbers it is necessary to set up the function k according to (26) in a definite measure. If we take q in degrees, as up to now, and then the small angles ψ in seconds, then we have to set:

$$k = \frac{\varrho''}{\varrho'^2} \frac{\eta^2}{V} \tan P q^2 = 20 \pi \frac{\eta^2}{V} \tan P q^2,$$

with the constants of section 47 this is calculated:

$$k = 0.203 \, 259 \, 386 \, q^2 \quad (\log = 1.798 \, 1798 \cdot 684).$$

The values k computed according to this are only first approximations which show from the more accurate values k of Gauss' table similar deviations as those between the first approximations and the accurate values of $\log m$, which we have put together in (14), section 49, p. 254.

In the previous section 50 we have derived the differential of curvature δ of the image of a geodetic line by a differential consideration in Fig. 2, p. 256, from the special properties of our case of projection, and it will always be possible to consult the special kind of a conformal projection in order to obtain that differential of curvature.

Thus we have proceeded, e.g., in the first half-volume, section 68, with Fig. 4, p. 201, and section 70 with Fig. 2, p. 211, and further in the case of Gauss' conformal plane projection in section 37, p. 192, of this half-volume.

But there is a quite general relation between the differential dm and the differential of curvature $\delta = \frac{ds}{R}$, with the help of which we also can find δ at once as soon as m , which we must have in any case, is developed.

The general theory for the determination of δ from m is given by Gauss in Arts. 12-13 of "Untersuchungen über Gegenstände der höheren Geodäsie" and in section 14 by Schreiber, *Theorie der Projektionsmethode der hannoverschen Landesvermessung*, Hannover, 1866, where, in both cases, the geodetic line is understood as the shortest line according to the rules of the calculus of variations.

Professor Schols in Delft has given a development and representation of this matter, more illustrative from the geometric point of view, in the treatise: *Annales de l'école polytechnique de Delft*, 1^{re} livraison, Leide, E. J. Brill, 1884, "Sur l'emploi de la projection de Mercator pour le calcul d'une triangulation dans le voisinage de l'équateur," par Ch. M. Schols.

In section 10 of this work, there is developed referring to Fig. 1 in the margin:

Let the rectangle $ABCD$ of the original be formed by infinitely small geodetic lines and be projected conformally as the curvilinear quadrilateral $abcd$. Lines at which the scale factor is constant are drawn through the centers e and f of the sides ab and cd .

(These lines for constant scale factor m are drawn as dotted lines in Fig. 1, where it is noted that e was supposed to be the center of ab which is somewhat misplaced in the wood engraving.)

If dz is the parallel distance of the two dotted lines and β is their angle with ab , then we have:

$$ac = bd = ef = \frac{dz}{\cos \beta}. \quad (1)$$

The short lines ca and db are extended to their intersection o , so that oa is the radius of curvature of the curve ab from which there follows:

$$\frac{oc}{oa} = \frac{oa + ac}{oa} = \frac{cd}{ab}, \quad (2)$$

therefore, from (1) and (2) together:

$$\frac{oa + \frac{dz}{\cos \beta}}{oa} = 1 + \frac{1}{oa} \frac{dz}{\cos \beta} = \frac{cd}{ab}. \quad (3)$$

Now we have to bear in mind that Fig. 1 is a differential figure, and hence, that cd and ab are infinitely small, and hence, that along cd and ab the scale factor m holds as constant, and in fact, let:

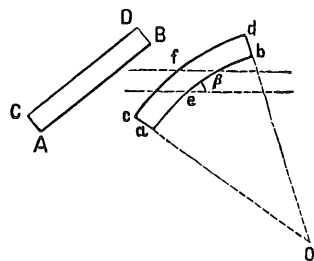


Fig. 1.

$$m = \frac{a b}{A B} \quad \text{and} \quad m + d m = \frac{c d}{C D} = \frac{c d}{A B} \quad (4)$$

$$\frac{c d}{a b} = \frac{m + d m}{m} = 1 + \frac{d m}{m}.$$

Therefore, we have from (3) and (4):

$$1 + \frac{1}{o a} \frac{d z}{\cos \beta} = 1 + \frac{d m}{m}.$$

If we denote the radius of curvature $o a$ by R or the curvature by $1/R$, then we have thus:

$$\text{Curvature} \quad \frac{1}{R} = \frac{d m}{m d z} \cos \beta, \quad (5)$$

or if the natural logarithm of m is denoted by $\log m$:

$$\text{Curvature} \quad \frac{1}{R} = \frac{d \log m}{d z} \cos \beta. \quad (6)$$

As the first application of this general formula we will take the special case of the first half-volume, section 68, p. 197, with:

$$m = 1 + \frac{y^2}{2 r^2} \quad \text{or} \quad \log m = \frac{y^2}{2 r^2}. \quad (7)$$

The direction of the z 's agrees here with the direction y because the lines for constant m run parallel to the x -axis, and β of (6) is the direction angle t_1 of Fig. 4, first half-volume, p. 201; therefore, there results the general equation (6) for our special case:

$$\text{Curvature} \quad \frac{1}{R} = \frac{d \frac{y^2}{2 r^2}}{d y} \cos t_1 = \frac{y}{r^2} \cos t_1. \quad (8)$$

(8) agrees with (21), first half-volume, section 68, p. 201, if we take into account there the expression

$$\frac{d x}{d \xi} = \cos t_1 \quad \text{of p. 201 and also enter the sign } - \text{ according to the position of the system of coordinates.}$$

After this first application of the simple case of the first half-volume, section 68, we will also take up the application of the general formula (6) to our case of the conformal projection of the ellipsoid to the sphere.

Here it is necessary at first to search for the meaning of the angle β , i.e. of the angle which a geodesic line to be projected makes with the lines of constant scale factor m , and since in our case the scale factor m depends only on the geographic latitude, β of the general formula (6) corresponds to $90^\circ - \alpha$, if α is the azimuth counted from the direction of the meridian. In addition, if we set the curvature $1/R = -\frac{d^2 y}{d x^2}$ in the sense of Figs. 3 and 4, section 50, pp. 257 and 259, then we have from (6), at first:

$$-\frac{d^2 y}{d x^2} = \frac{d \log m}{d z} \sin \alpha \quad \text{or} \quad = \frac{d \log m}{d z} \sin \beta, \quad (9)$$

where the azimuths α and β of Fig. 4, section 50, p. 259, can with sufficient accuracy be assumed as

equal. We have further from (9) and (10), section 49, p. 254:

$$\log m = -\frac{2}{3} \frac{\eta^2}{V^4} \tan P p^3 \quad \text{or} \quad = -\frac{2}{3} \frac{\eta^2}{V} \tan P q^3. \quad (10)$$

The differential dx , which occurs in the general formula (6), is to be sought for in the direction of the meridian, i.e. we have $dx = A dq$, where, according to (28) in the first half-volume, p. 51, the mean radius of curvature of the ellipsoid, which is used as radius of the projected sphere, is

$$A = \frac{c}{V^2} \quad \text{or} \quad A = \frac{N}{V},$$

and therefore we now have:

$$\begin{aligned} \frac{d \log m}{dz} &= \frac{d \log m}{dq} \frac{dq}{dz} = -\frac{2 \eta^2}{V} q^2 \tan P \frac{dq}{dz} \\ \frac{d \log m}{dz} &= -\frac{2 \eta^2 q^2}{V A} \tan P = -\frac{2 \eta^2 q^2}{A} \tan Q. \end{aligned} \quad (11)$$

We set here $V \tan Q = \tan P$ according to (17a), section 47, p. 246.

From (6) and (7) we thus have now:

$$-\frac{d^2 z}{dx^2} = \frac{2 \eta^2 q^2}{A} \tan Q \sin \beta. \quad (12)$$

This equation (12) agrees with the previous (14), section 50, p. 258, with which a second derivation of the differential of curvature is thus given, from which the azimuth reductions of section 50 result.

Section 52. Auxiliary Tables and Numerical Examples

Gauss has computed a detailed table for the reduction of spherical latitudes to spheroidal latitudes in addition to $\log m$ and k and given in "Untersuchungen über Gegenstände der höheren Geodäsie," erste Abhandlung, pp. 37-45. (Carl Friedrich Gauss' *Werke*, IV. Band, Göttingen, 1873, pp. 293-300.)

On pp. [21] to [22] of our Appendix we have reprinted an excerpt of Gauss' table, with ten times the interval $\Delta u = 10'$ ($\Delta u = 1'$ in the case of Gauss). In addition, we have appended on p. [23] an auxiliary table for the reduction of the geographic longitudes with the constant α .

Our main table, pp. [21] to [22], requires interpolation with second differences, for which section 36, first half-volume, p. 33, gives the instructions. With this, we obtain nearly the same accuracy as with the original table itself, so that for individual cases the excerpt can be used as a substitute for the original, which is not always accessible. In addition, the excerpt gives a convenient over-all view of the general proportions; e.g., we see that $\log m$ does not go beyond 0.1 in the entire wide zone from $51^\circ 20'$ to $54^\circ 0'$. Conditions are similar with the azimuth corrections, which depend on the tabular quantity k ; we therefore can compute a triangulation spherically on this whole, nearly 3° or approximately 300,000-m-wide zone, without an extra work other than the conversion of the latitudes φ and u by increasing in the table.

As the most important tabular work for the conformal transformation of the ellipsoid to the sphere we have the publication of the Prussian Land Survey already mentioned on p. 241: *Die konforme Doppelprojektion der trigonometrischen Abteilung der Kgl. Preuss. Landesaufnahme*, Formeln und Tafeln, von Dr. O. Schreiber, Generalleutnant z. D., ehem. Chef d. Kgl. Preuss. Landesaufnahme. Herausgegeben von der trigonometrischen Abteilung der Landesaufnahme, Berlin, 1897. In Table I of this work there are indicated for the normal latitude assumed by Gauss the differences $\varphi - u$ and $\log m$ within the latitudes $u = 44^\circ 20'$ and $u = 61^\circ 00'$ with the interval $\Delta u = 1'$. Tables II and III are used for the conversion of ellipsoidal to spherical latitudes and vice versa.

Besides Gauss' table a second table of this kind with a more southern normal latitude, namely $Q = 46^\circ 30'$, has been computed by Marek and Horsky. Like Gauss' table, it is based on Bessel's dimensions of the earth, and it is given in the work by Marek, *Technische Anleitung zur Ausführung der trigonometrischen Operationen des Katasters*, Budapest, 1875, pp. 252-262. We have given a few additional remarks about it earlier in *Zeitschr. f. Verm.*, 1877, pp. 40-46, and our previous second edition, Karlsruhe, 1878, pp. 403-404, gave an excerpt of Marek's table.

An additional table for the normal latitude $Q = 46^\circ 54' 27.83''$ ($P = 46^\circ 57' 8.66''$ = latitude of Berne, Observatory) has been published by Rosenmund in the work, *Die Änderung des Projektionssystems der schweizerischen Landesvermessung*, bearbeitet von Ingenieur M. Rosenmund, Bern 1903. This table is computed likewise on the basis of Bessel's dimensions of the earth and comprises the latitudes $45^\circ 30'$ to $48^\circ 00'$.

As application of Gauss' theory and the auxiliary tables belonging to it we will take up the computation of our small spheroidal normal example (1), section 17, p. 73, in this form:

$$\text{Given} \quad \varphi_1 = 49^\circ 30' 0'' \quad \varphi_2 = 50^\circ 30' 0'' \quad (1)$$

$$l = 1^\circ 0' 0'' \quad (2)$$

Required α_1 , α_2 and s .

The first thing is to transform the latitudes φ_1 and φ_2 to the sphere, i.e. to take the corresponding u_1 's and u_2 's from the table. From p. [21] of our Appendix we have:

u	φ	Differences
$49^\circ 20' 0''$	$49^\circ 21' 44.31358''$	$+ 10' 1.07480'' - 0.01736''$
$49 \ 30 \ 0$	$49 \ 31 \ 45.38838$	
<hr/>		
$10' 0''$		$= 601.07480''$.
$= 600''$		

$\varphi = 49^\circ 30'$ has the following differences with respect to the neighboring values:

$$\text{and} \quad \left. \begin{array}{l} - 8' 15.68642'' = - 495.68642'' \\ + 1 \ 45.38838 = + 105.38838 \end{array} \right\} 601.07480''.$$

The interpolation with respect to the second differences yielded:

$$u = 49^\circ 28' 14.79882''.$$

The computation according to Gauss' original table yielded to $0.00001''$ exactly the same in the listing of all values which are of interest to us here:

Ellipsoid φ	Sphere u	$\log m$	k
$49^\circ 30' 0''$	$u_1 = 49^\circ 28' 14.79881''$	1.609	2.049''
$50 \ 0 \ 0$	$49 \ 58 \ 11.67462$	0.969	1.462
$50 \ 30 \ 0$	$u_2 = 50 \ 28 \ 8.70541$	0.525	6.973.

We form a mean value from the three values of $\log m$ according to the rule of equation (17), section 49, p. 255, which also holds for $\log m$ in the manner indicated there and yields in our case:

$$\log m = \frac{1.609 + 4 \times 0.969 + 0.525}{6} = 1.0017. \quad (4)$$

The difference of longitude $l = 1^\circ 0' 0''$ is reduced to the sphere by multiplication by the constant α or, as the case may be, by the use of the auxiliary table on p. [23] of the Appendix with the result:

$$\lambda = \alpha l = 1^\circ 0' 1.630505''. \quad (5)$$

With u_1 and u_2 from (3) besides λ from (5) we now make a spherical polar triangle computation according to (4) and (5) in the first half-volume, section 59, p. 159, whereby we obtain:

$$\text{Spherical azimuths} \quad \beta_1 = 32^\circ 25' 21.4923'' \quad \beta_2 = 33^\circ 11' 19.4197'' \quad (6)$$

$$\text{and} \quad \log \sin \frac{\sigma}{2} = 8.015\,5452\,409, \quad \frac{\sigma}{2} = 0^\circ 35' 37.85453'',$$

$$s' = A \frac{\sigma}{\rho} \text{ yields } \log s' = 5.121\,6104\,130$$

$$\begin{array}{r} \text{in addition, according} \quad -\log m = \quad -1.002 \\ \text{to (4):} \quad \hline \log s = 5.121\,6103\,128, \quad s = 132,315.375 \text{ m.} \end{array} \quad (7)$$

There follow further the azimuth reductions according to the formulae (28) and (29), section 50, p. 260. For this, we have the k 's already given in (3) and the azimuths rounded off:

$$\begin{array}{ll} k_1 = 2.049'' & k_2 = 0.973'' \\ \alpha_1 = 32^\circ 25' & \alpha_2 = 33^\circ 11'. \end{array}$$

The calculation according to formulae (28) and (29), p. 260, yields:

$$\psi_1 = \alpha_1 - \beta_1 = +0.0189'' \quad \psi_2 = -\alpha_2 + \beta_2 = +0.0149''.$$

These reductions added to β_1 and β_2 in (6) yield the spheroidal azimuths:

$$\alpha_1 = 32^\circ 25' 21.5112'' \quad \alpha_2 = 33^\circ 11' 19.4048''. \quad (8)$$

The solution of the problem set up is contained in these equations (7) and (8), and the latter agree sufficiently with the corresponding numerical values of section 17, (1), p. 73.

Section 53. Conformal Double Projection

We have as the oldest application of Gauss' theory of the projection of the ellipsoid to the sphere the Hungarian operations of triangulation, in which Gauss' conformal projection has been used since 1857. As spherical normal latitude there is assumed the latitude $Q = 46^\circ 30'$ to which the ellipsoidal latitude $P = 46^\circ 32' 43.41041''$ corresponds. In view of the small extent of the country, the reduction to the sphere of the measured angles can be disregarded; therefore, the measured angles also hold immediately for the sphere. Then, there follows further the projection of the sphere on the plane with the help of the stereographic projection, as we have set forth in the first half-volume of this volume in sections 71 to 73.

The computational procedure is thus the following: After having transformed the measured angles to the plane, there follows the adjustment of the triangulation net on the plane and the computation of rectangular coordinates. From the plane coordinates the longitudes and latitudes on the sphere are computed, and these are transformed to the ellipsoid with the help of Gauss' theory.

A detailed report about these Hungarian activities is given in the work already mentioned in section 52, p. 264: *Technische Anleitung zur Ausführung der trigonometrischen Operationen des Katasters*, im Auftrag des Kgl. Ung. Finanzministeriums für den Gebrauch des Kgl. Ung. Triangulierungs-Calcul-Bureaus verfasst von Johann Marek, Vorstand des Kgl. Ung. Triangulierungs-Calcul-Bureaus, Budapest, 1875. This work contains on pp. 252-262 a table computed under the direction of the trigonometrist Horsky in 1857 for the transformation of the spherical latitudes to the spheroid for the spherical latitudes $41^\circ 30'$ to $51^\circ 30'$ or the ellipsoidal latitude $41^\circ 32'$ to $51^\circ 33'$.

Since in the case of the stereographic projection the distortions at the boundaries of the country were quite considerable, three oblique-axis cylinder projections of the sphere were introduced in 1908. For topography, however, after the World War they went back to the stereographic projection, which was sufficient

with respect to the reduced territory of the State and, besides, offers great advantages.

Further details about it are given by Medvey, "Das topographische Kartenwesen Ungarns," *Mitt. d. Reichsamts f. Landesaufn.*, 1932-33, pp. 99-114.

Double projection of the Prussian Land Survey

For the Prussian land triangulation there was introduced by Schreiber in 1876 a conformal double projection, in the case of which the projection of the sphere on the plane takes place by means of Gauss' transverse cylinder projection, which we have treated as Gauss' conformal projection in the first half-volume, section 68, p. 196, and in extended form in section 70, p. 208. The spherical latitude $Q = 52^\circ 40'$ or the ellipsoidal latitude $P = 52^\circ 42' 2.53251''$ was taken as the normal latitude. For the plane projection, the meridian 31° east of Ferro was assumed as the tangent meridian of the cylinder and as the axis of abscissae on the plane. The working up of the main triangulation net was carried out on the ellipsoid, and then, the ellipsoidal longitudes and latitudes were transferred to the sphere. After rectangular plane coordinates were then computed further, the measured directions were likewise reduced to the plane so that the working up of the net of second and third order could be carried out on the plane. After the determination of the final rectangular coordinates of the points of second and third order, all points were transformed again to the sphere and then to the ellipsoid.

Because of the large ordinates which had to occur for the eastern and western territories of the Prussian State and the distortions connected with it, the rectangular coordinates could not be used for cartographic purposes.

All triangulations of second and third order have been worked up by the Prussian Land Survey in the form indicated above until the World War.

We already have indicated the most important literature about the Prussian conformal double projection in section 46, p. 241.

Swiss Land Survey

As the basis for the Swiss map works, in 1903 a conformal double projection was introduced, in the case of which the transformation from the ellipsoid to the sphere is likewise carried out according to the theory treated above. As normal latitude that of the observatory of Berne, $P = 46^\circ 57' 8.660''$ was assumed, for which the spherical latitude $Q = 46^\circ 54' 27.833\ 245''$ results. For the spherical radius the value $\log A = 6.804\ 7400\ 683$ and further $\log \alpha = 0.000\ 3165\ 454$ as well as $\log k = 0.001\ 3318\ 649$ correspond to this latitude. (Cf., in this connection, p. 264.)

The fact that the territory of Switzerland extends mainly in a west-east direction was taken into account in the choice of a system of projection for the projection of the sphere on the plane. The oblique-axis conformal cylinder projection was therefore chosen, in the case of which the meridian of Berne is the axis of abscissae, and the great circle perpendicular to it is used as axis of ordinates. About the theory of the oblique-axis cylinder projection we find all that is required in the first half-volume, where the computation of oblique-axis (transverse-axis) spherical coordinates is treated in section 57, p. 146. The change to conformal plane coordinates and all further details are then carried out in the same manner as in the case of Gauss' conformal projection, first half-volume, sections 68 to 70.

The theoretical fundamental data of the Swiss system of projection are contained in the paper, *Die Änderung des Projektionssystems der schweizerischen Landesvermessung*, im Auftrage der Abt. f. Landestopographie des schweiz. Militärdepartements bearbeitet von Ingenieur M. Rosenmund, Adjunkt des Direktors der Abt. f. Landestopographie, Bern, 1903.

Dutch Land Survey

In the conformal double projection introduced in the Netherlands at the beginning of the twentieth century, the triangulation point Amersfoot with the latitude $P = 52^\circ 9' 22.178''$ holds as central point. We have for it $Q = 52^\circ 7' 15.950''$, $\log A = 6.805\ 0006\ 61$.

The stereographic projection, which offers many advantages for a country with not a large area extent, has been chosen for the projection of the sphere. We have developed the formulae for the stereographic projection in the first half-volume, sections 71 to 73, to which we refer here.

The Netherlands method of projection is represented in the publications: *Nederlandse Rijksdriehoeksmeting*, De stereografische kaartprojectie in hare toepassing bij de Rijksdriehoeksmeting door Hk. J. Heuvelink, Hoogleraar aan de Technische Hoogeschool te Delft, Delft 1918.

Oversluijs, "Die amtlichen niederländischen topographischen Karten," *Mitt. d. R. f. Landesaufn.*, 1932-33, pp. 258-284.

The conformal conic projection

After having treated in detail the conformal conic projection with a tangent parallel in sections 41 to 43, we will now show further that this projection can also be regarded as a conformal double projection.

According to (15), section 41, p. 224, the equation of the conformal conic projection of the ellipsoid is

$$x + i y = A e^{-(q - i l) \sin \varphi_0}. \quad (1)$$

For this, we consider the conformal conic projection of the sphere, for which we will denote the constant of multiplication by B . Let further on the sphere the geographic latitude be denoted by u , the isometric latitude by ω and the longitude by λ . According to (1) we have then for the sphere the mapping equation

$$x + i y = B e^{-(\omega - i \lambda) \sin u_0}. \quad (2)$$

Now let the point u, λ on the sphere correspond to the point φ, l of the ellipsoid according to Gauss' theory so that between the values φ, l and u, λ the relations of sections 46 and 47 exist.

We have according to (14), section 46, p. 243,

$$e^\omega = \frac{1}{k} e^{\alpha q}, \quad (3)$$

where according to (13), section 41, p. 224,

$$e^q = \tan \left(45^\circ + \frac{\varphi}{2} \right) \left(\frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right)^{\frac{e}{2}} \quad (4)$$

and accordingly,

$$e^\omega = \tan \left(45^\circ + \frac{u}{2} \right) \quad (5)$$

For the constant A of the basic parallel in (1) we obtain from (7a), section 41, p. 222, if we replace φ by φ_0 here,

$$A = R_0 \tan \sin \varphi_0 \left(45^\circ + \frac{\varphi_0}{2} \right) \left(\frac{1 - e \sin \varphi_0}{1 + e \sin \varphi_0} \right)^{\frac{e}{2}} \sin \varphi_0 \quad (6)$$

and for the sphere we have for the constant B in (2)

$$B = R_0' \tan \sin u_0 \left(45^\circ + \frac{u_0}{2} \right) \quad (6^*)$$

With the help of (4) and (5) we have then

$$A = R_0 e^{q_0 \sin \varphi_0} \quad B = R_0' e^{\omega_0 \sin u_0}. \quad (7)$$

The values of R_0 and R_0' are according to (1), section 41, p. 221,

$$R_0 = N_0 \cot \varphi_0 \quad R_0' = r \cot u_0. \quad (8)$$

We now assemble further the relations between the values φ_0 , l_0 and u_0 , λ_0 which we have found in section 47, p. 246. We have

$$\left. \begin{aligned} \alpha \sin u_0 &= \sin \varphi_0 \\ \alpha \cos u_0 &= V_0 \cos \varphi_0 \\ r &= \sqrt{M_0 N_0} = \frac{N_0}{V_0} \end{aligned} \quad \begin{aligned} \cot u_0 &= V_0 \cot \varphi_0 \\ \lambda &= \alpha l. \end{aligned} \right\} \quad (9)$$

With this, we obtain at first, according to (8), for R_0' the value

$$R_0' = \frac{N_0}{V_0} V_0 \cot \varphi_0 = N_0 \cot \varphi_0 = R. \quad (10)$$

There follows further, according to (7), for B with the help of (6) the value

$$B = R_0 \frac{1}{k} e^{q_0 \sin \varphi_0}.$$

With this, we obtain from (2)

$$x + iy = R_0 \frac{1}{k} e^{q_0 \sin \varphi_0} e^{-\omega \sin u_0} e^{i \lambda \sin u_0}$$

and with the help of (6) and (9)

$$\begin{aligned} x + iy &= R_0 \frac{1}{k} e^{q_0 \sin \varphi_0} k e^{-q \sin \varphi_0} e^{i l \sin \varphi_0} \\ &= R_0 e^{q_0 \sin \varphi_0} e^{-(q-i l) \sin \varphi_0} \\ x + iy &= A e^{-(q-i l) \sin \varphi_0}. \end{aligned} \quad (11)$$

This agrees with equation (1), p. 267.

We can therefore arrive at the conformal projection of the ellipsoid on a cone tangent at the parallel circle φ_0 also by the method of determining the spherical latitude u_0 from φ_0 according to Gauss' theory and laying a tangent cone to the parallel circle u_0 of the sphere. The projection of the sphere on this cone is in agreement with the immediate projection of the ellipsoid.

We can make use of this result in order to develop, in a simple manner, the formulae for the conformal conic projection of the ellipsoid. To do so, we can start from the spherical formulae, which we have developed in the first half-volume, sections 74 to 76, and replace the spherical quantities u_0 and λ by the spheroidal quantities φ_0 and l by means of equations (9). In addition, we must pass over from the spherical difference of latitude Δu to the spheroidal difference of latitude $\Delta \varphi$, for which equation (19), section 48, p. 251, is to be used.

Horvat has adopted this method in *Zeitschr. f. Verm.*, 1939, p. 622, and it becomes evident that the developments in series found hereby agree, in fact, with the formulae developed directly for the ellipsoid. In the above formulae (1)

to (11) we have generally proved the validity of this procedure.

Besides, we see that nothing changes in the whole foregoing consideration if we introduce arbitrary other values for the constants k and a instead of those determined in section 47; e.g., we also can use the values which follow from Gauss' theory of 1822 mentioned in the following paragraph and arrive then at the derivation of the conformal conic projection from Lagrange's projection, which we will discuss briefly at the end of this section.

The stereographic projection

We already have mentioned the stereographic projection of L. Krüger in section 45, p. 240, and referred to the fact that this projection is to be regarded likewise as a conformal double projection. For the projection of the ellipsoid on the sphere we do however not use here the theory treated in sections 46 to 52, but another theory originating likewise from Gauss, which was published in 1822 in the paper, "Auflösung der Aufgabe, die Teile einer gegebenen Fläche auf einer andern so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Teilen ähnlich wird." (Carl Friedrich Gauss' *Werke*, 4th volume, Göttingen, 1880, pp. 189-216.) The sphere is chosen here in such a way that it is tangent to the ellipsoid at a normal latitude, and its radius is assumed equal to the radius of curvature in the prime vertical for the normal latitude. It follows hence that the geographic longitudes are carried over to the sphere without change.

If we combine this projection with the stereographic projection of the sphere as one single mapping equation, then this agrees with the basic equation of Krüger's stereographic projection, as we have shown in *Zeitschr. f. Verm.*, 1936, p. 158.

Gauss describes the second form of the projection of the ellipsoid in 1844, which we have treated in detail in sections 46 to 52, as the more useful one, and this is obvious offhand, since the sphere approaches the ellipsoid better here than with the older method of 1822. One will therefore have to give the preference to the form of the double projection as it is applied in the Netherlands.

The projection of Lagrange

In the foregoing we have shown that by means of the conformal double projection we obtain the conformal conic projection as well as a stereographic projection of the terrestrial ellipsoid.

The projection of Lagrange, which can likewise be regarded as a conformal double projection, and which is distinguished by the property that it represents the meridians and parallels of the ellipsoid on the plane as circles, is of special significance. As special cases of Lagrange's projection we obtain again the conformal conic projection as well as the Mercator projection and also the stereographic projection treated above.

The projection of Lagrange is published in the paper, "Sur la construction des cartes géographiques," which has appeared in the *Memoirs of the Berlin Academy of Sciences* in 1781. Lehmann gives a detailed examination of this method of projection in *Zeitschr. f. Verm.*, 1939, pp. 329-344, 361-376, 424-432.

The work by Frank, "Beiträge zur winkeltreuen Abbildung des Erdellipsoids," *Zeitschr. f. Verm.*, 1940, pp. 97-112, 145-160, 193-204, contains a comprehensive treatment of the conformal projections, at the same time also a historical over-all view.

In many cases it is desirable to represent on the cadastral maps, maps of a district [Flurkarten], etc., the lines of the geographic net in addition to the net lines of the rectangular coordinates x, y , which form the foundation of the map.

The problem consists in computing, for definite even values of the geographic longitudes and latitudes L and φ , the rectangular coordinates in the system of projection on which the map is based and entering these values into the rectangular net of coordinates.

As an example for this, we take the map of the city of Linden near Hannover, whose survey Jordan carried out in 1887-1889 (cf., in this connection, vol. II, first half-volume, p. 528*). The rectangular coordinates are referred to the former Prussian coordinate system 27 Celle; therefore, the map is based on the Cassini-Soldner projection.

We have already developed the formulae for the computation of rectangular coordinates for practical use in the first half-volume, section 56; we gave additional formulae in section 27 of this second half-volume.

The boundary lies approximately between the geographic longitude $27^{\circ} 21'$ and $27^{\circ} 24'$ and between the latitudes $52^{\circ} 21'$ and $52^{\circ} 23'$; it thus comprises 6 sections with an extent of $1'$ longitude and latitude, and for the 12 corners of the corresponding minute net we have computed the rectangular coordinates y and x according to the pattern in the first half-volume, p. 142, as is indicated in the following summary:

	$\lambda = 27^{\circ} 21'$	$\lambda = 27^{\circ} 22'$	$\lambda = 27^{\circ} 23'$	$\lambda = 27^{\circ} 24'$
$\varphi = 52^{\circ} 23'$	$y = -27,135.04 \text{ m}$ $x = -26,896.62$	$y = -26,000.36 \text{ m}$ $x = -26,902.74$	$y = -24,865.68 \text{ m}$ $x = -26,908.61$	$y = -23,731.00 \text{ m}$ $x = -26,914.22$
$\varphi = 52^{\circ} 22'$	$y = -27,145.25 \text{ m}$ $x = -28,750.97$	$y = -26,010.14 \text{ m}$ $x = -28,757.09$	$y = -24,875.04 \text{ m}$ $x = -28,762.96$	$y = -23,739.94 \text{ m}$ $x = -28,768.56$
$\varphi = 52^{\circ} 21'$	$y = -27,155.47 \text{ m}$ $x = -30,605.32$	$y = -26,019.93 \text{ m}$ $x = -30,611.45$	$y = -24,884.40 \text{ m}$ $x = -30,617.32$	$y = -23,748.87 \text{ m}$ $x = -30,622.90$

We can check these coordinates in their differences by meridian arcs and parallel arcs by means of the auxiliary tables in the Appendix of the first half-volume; e.g., at the meridian of $\lambda = 27^{\circ} 21'$ we have from the above computation:

$\varphi = 52^{\circ} 23'$	$x = -26,896.62 \text{ m}$	
$\varphi = 52^{\circ} 22'$	$x = -28,750.97$	$\Delta x = 1854.35 \text{ m}$
$\varphi = 52^{\circ} 21'$	$x = -30,605.32$	$\Delta x = 1854.35$

According to p. [43] of the Appendix in the first half-volume, the meridian arc for 1 minute between $52^{\circ} 21'$ and $52^{\circ} 23'$ is $m = 1854.36 \text{ m}$. This agrees sufficiently with the values Δx . In order to obtain complete agreement, we would have to reduce the meridian arc m by means of the meridian convergence with the parallel line to the zero meridian and, besides, take into account the ordinate convergence according to (7) in the first half-volume, section 67, p. 192.

In order to check the ordinate differences, we can compute the minutes of longitude according to the first half-volume, section 42, pp. 77 and 78:

$$l' = \frac{N \cos \varphi}{e'} = \frac{60 \cos \varphi}{[2]},$$

where $\log [2]$ is to be taken from the auxiliary table on p. [37] of the Appendix to the first half-volume, e.g., for $\varphi = 52^{\circ} 23'$ we find $\log [2] = 8.508 8704$, and with this, according to the above formula $l' = 1134.69 \text{ m}$,

* Not translated.

[44] in the Appendix of the first half-volume by interpolation. By means of this latitude, the ordinates for even values of longitude are then to be computed.

For this computation we start from equation (20), section 33, p. 168, in which we neglect the terms of fifth order completely, and the term of third order in η_1^2 . The equation then reads:

$$l \cos \varphi_1 = \frac{y}{N_1} - \frac{y^3}{6 N_1^3} (1 + 2 t_1^2).$$

By setting in the last term $\frac{y}{N_1} = l \cos \varphi_1$, we obtain at once

$$\frac{y}{N_1} = l \cos \varphi_1 + \frac{1}{6} l^3 \cos^3 \varphi_1 (1 + 2 t_1^2)$$

or

$$y = N_1 \frac{l}{\varrho} \cos \varphi_1 \left(1 + \frac{l^2}{6 \varrho^2} \cos^2 \varphi_1 (1 + 2 t_1^2) \right),$$

and this is in logarithmic form

$$\log y = \log N_1 \frac{l}{\varrho} \cos \varphi_1 + \frac{\mu 10^7}{6 \varrho^2} l^2 \cos^2 \varphi_1 (1 + 2 t_1^2).$$

We introduce further the coefficient $[2]_1 = \frac{\rho}{N_1}$ and have then

$$\log y = \log \frac{l}{[2]_1} \cos \varphi_1 + \frac{\mu 10^7}{6 \varrho^2} l^2 \cos^2 \varphi_1 (1 + 2 t_1^2). \quad (1)$$

Here we have

$$\log \frac{\mu 10^7}{6 \varrho^2} = 5.2308,$$

and the value of $\log (1 + 2 t_1^2)$ can be taken directly from p. [62] in the Appendix of the first half-volume.

For the sheet boundaries at the right and the left, the ordinates are given, and we now have to compute the abscissae for even values of the latitude.

We use for this equation (7), section 33, p. 165, in which we omit the terms in y^6 and the term in $y^4 \eta^2$. Then we have

$$x - B = \frac{y^2}{2 N} t \left(1 + \frac{y^2}{12 N^2} (1 + 3 t^2) \right)$$

and in logarithmic form

$$\log (x - B) = \log \frac{y^2}{2 N} t + \frac{\mu 10^7}{12 N^2} (1 + 3 t^2).$$

If we use again the coefficient $[2] = \frac{\rho}{N}$, then we can write the equation in the form

$$\log (x - B) = \log \frac{[2]}{2 \varrho} t y^2 + \frac{\mu 10^7}{12 \varrho^2} [2]^2 (1 + 3 t^2) y^2. \quad (2)$$

The constant numerical values here are

$$\log \frac{1}{2\varrho} = 4.384\,5448\cdot7 \quad \log \frac{\mu\,10^7}{12\,\varrho^2} = 4\cdot9298.$$

Further we can take $\log(1 + 3t^2)$ from p. [62] and the meridional arc B from pp. [41] to [44] of the Appendix to the first half-volume.

As an example we will compute the sheets of the Deutsche Grundkarte which lie between the abscissae 5820 km and 5822 km.

We begin with the upper edge of this strip, i.e. with the abscissa $x = 5,822,000$ m, and find for this from p. [43] of the Appendix to the first half-volume by interpolation for the foot-point latitude φ_1 the value

$$\varphi_1 = 52^\circ 31' 57.115''.$$

With this, we then have the following numerical computation

$$\begin{array}{r|l} 1:[2]_1 & 1.491\,1332\cdot7 \\ \cos \varphi_1 & 9.784\,1255\cdot8 \\ \hline \cos \varphi_1 & 1.275\,2588\cdot5 \\ [2]_1 & \end{array} \quad \begin{array}{r|l} \frac{\mu\,10^7}{6\,\varrho^2} & 5\cdot2308 \\ \cos^2 \varphi_1 & 9.5683 \\ 1 + 2\,t_1^2 & 0.6439 \\ \hline & 5\cdot4430. \end{array}$$

Therefore, we have according to (1)

$$\left. \begin{array}{l} \log y_o = 1.275\,2588\cdot5 + \log l + K_o \\ \log K_o = 5\cdot4430 + \log l^2. \end{array} \right\} \quad (3)$$

For the lower edge of the same sheet strip we have the abscissa $x = 5,820,000$ m, and for this, we obtain

$$\varphi_1 = 52^\circ 30' 52.405'',$$

and for the computation of the ordinates there follows according to (3)

$$\left. \begin{array}{l} \log y_u = 1.275\,4361\cdot2 + \log l + K_u \\ \log K_u = 5\cdot4429 + \log l^2. \end{array} \right\} \quad (4)$$

The results of the computation according to the formulae (3) and (4) for a series of sheets of the strip between the two abscissae mentioned are shown in the following table.

l	l''	y_o	y_u
$-1^\circ 30''$	$-5400''$	$-101,796.65$ m	$-101,838.21$ m
$-1\ 29$	-5340	$-100,665.16$	$-100,706.26$
$-1\ 28$	-5280	$-99,533.69$	$-99,574.32$
$-1\ 27$	-5220	$-98,402.22$	$-98,442.39$
$-1\ 26$	-5160	$-97,270.78$	$-97,310.49$
$\dots\dots$	$\dots\dots$	$\dots\dots$	$\dots\dots$

There follows the computation of the points of intersection of the parallel $\varphi = 52^\circ 31'$ with the lateral edge of the sheets with the help of equation (2). For the latitude $\varphi = 52^\circ 31'$ we obtain from (2) the computation formula

$$\left. \begin{array}{l} \log(x - B) = 3.008\,6931 + \log y^2 + K_s \\ \log K_s = 2\cdot7329 + \log l^2. \end{array} \right\} \quad (5)$$

Here we have set consecutively $y = -102,000$ m, $-100,000$ m, $-98,000$ m, and so on. With the help of the meridional arcs B from p. [43] of the Appendix to the first half-volume we obtain the following values:

y	x
$-102,000$ m	5821,296.32 m
$-100,000$	5821,255.09
$-98,000$	5821,214.68
$-96,000$	5821,175.09

So far we have introduced only the differences of longitude l . Now let us assume that the zero meridian has the longitude 15° east of Greenwich, so that the eastings of the coordinates have the index 5, and with this, we will once again assemble the above results in the form of Gauss-Krüger coordinates.

L	Intersection Points of the Meridians		Intersection Points of the Parallel $52^\circ 31'$	
	Eastings (Upper Edge)	Eastings (Lower Edge)	Eastings	Northings
$13^\circ 30'$	5,398,203.35 m	5,398,161.79 m	5,398,000 m	5,821,296.32 m
$13^\circ 31'$	5,399,334.84	5,399,293.74	5,400,000	5,821,255.09
$13^\circ 32'$	5,400,466.31	5,400,425.68	5,402,000	5,821,214.68
$13^\circ 33'$	5,401,597.78	5,401,557.61	5,404,000	5,821,175.09
$13^\circ 34'$	5,402,729.22	5,402,689.51
...

If the computing formulae (3) to (5) are set up correctly, then no further computational check is required since the results can be easily checked by forming the differences. In the case of the points of intersection of the parallel, the second differences must be used for this.

Fig. 2 shows an illustration of the results found.

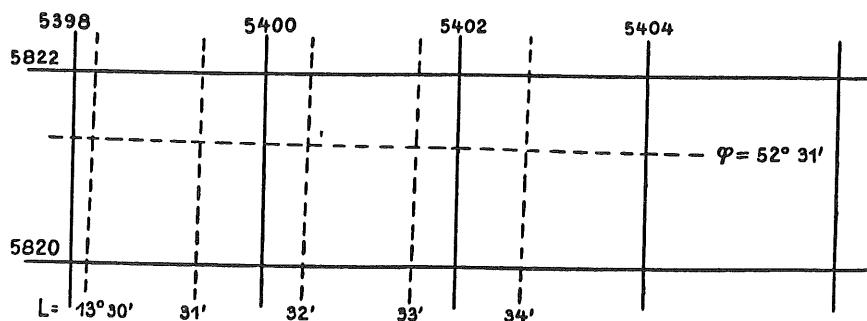


Fig. 2.

We have set up the formulae (3) to (4) for eight-place logarithmic computation; but, as a rule, seven-place computation in the case of formulae (3) and (4) and even six-place computation in the case of formula (5) may be sufficient.

We mention further in connection with the problem treated above:

M. Pehnack, "Einrechnung geographischer Netzlinien in ein konformes rechtwinkliges Koordinatennetz," *Mitt. d. Reichsamts f. Landesaufnahme*, 1934-35, pp. 119-121.

H. Wilsing, "Die Einrechnung geographischer Netzlinien in ein konformes rechtwinkliges Koordinatennetz," *Zeitschr. f. Verm.*, 1938, pp. 449-456.

E. Müller, "Über die vorbereitenden Berechnungen zur Herstellung der Katasterplankarte in Schleswig-Holstein," *Zeitschr. f. Verm.*, 1938, pp. 246-254, 261-272.

If we imagine on the ellipsoid the geographic net lines, longitude and latitude, plotted by whole degree numbers, then small trapezoids will hereby be formed between every two consecutive degrees of longitude and latitude, which we call a "graticule."

The individual sheets of the topographic map are subdivisions of these graticules, as is already pointed out in our vol. II, second half-volume, ninth edition, 1933, section 128, p. 591. Each such graticule is divided, for the map at the scale 1:25,000 (plane-table sheet), according to the latitude into ten parts, and according to the longitude into six parts, so that 60 sheets are formed, and for the map at 1:100,000 (Reichskarte [map of the Reich]), according to the latitude into four parts and according to the longitude into two, so that eight sheets are formed, while for the Reichsübersichtskarte [general map of the Reich] at 1:200,000 a division of the graticule into two sheets according to the latitude is made. The plane table, whose edge lines thus have a difference of longitude of 10' and a difference of latitude of 6', is used as survey map. Fig. 3 gives a diagram of the sheet division for the region of Hannover.

These small ellipsoidal trapezoids can now be regarded, each independently, as plane, and bounded rectilinearly, and can be drawn up directly according to their peripheral lines. By this method we replace, so to speak, the ellipsoid by a polyhedron bounded by plane surfaces, according to which the designation "polyhedric projection" is also used for this graticule projection.

The sheets of a latitude zone of 6' between the latitudes φ_1 and φ_2 lie on a conic surface and are bound, after the development of the cone, above and below by arcs of a circle, whose radii are equal to $N_1 \cot \varphi_1$ and $N_2 \cot \varphi_2$. For the plotting of a sheet we may be able to neglect at first the curvature of the upper and lower boundary line, so that each sheet can be regarded as a trapezoid.

In Fig. 4 such a plane-table sheet between the latitudes $52^\circ 24'$ and $52^\circ 30'$ is represented.

The side lengths of this trapezoid can be taken from p. [49] of the Appendix to the first half-volume. We have

$$\begin{aligned} \varphi &= 52^\circ 30' & AB &= 11,316.99 \text{ m} \\ \varphi &= 52^\circ 24' & CD &= 11,342.65 \end{aligned} \quad AC = BD = 11,126.31 \text{ m.}$$

For instance, if we do not have these dimensions available, but at least the radii of curvature M and N , then we can compute the sides of the trapezoid likewise. We have

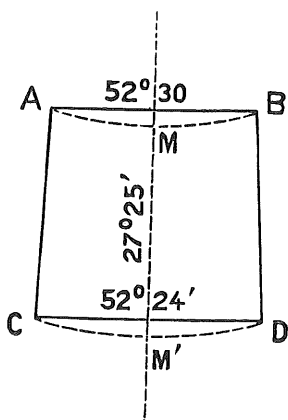


Fig. 4.

$$\begin{aligned} AC &= \frac{M}{\rho} 6' = \frac{360}{[1]} \\ AB &= \frac{N}{\rho} 10' \cos \varphi = \frac{600}{[2]} \cos \varphi, \end{aligned}$$

where M or [1] belongs to the mean latitude of A and C and N or [2] to the latitude of A and B itself.

The construction is based on the mean meridian $MM' = 27^\circ 25'$ as axis of abscissae, as Fig. 4 shows. Then, the length of the arc of meridian $AC = BD = 11,126.31 \text{ m}$ is set off at the scale 1:25,000 from M to M' , and at these points,

the lengths $\frac{1}{2} AB = \frac{1}{2} \cdot 11,316.99 \text{ m}$ or, as the case may be, $\frac{1}{2} CD = \frac{1}{2} \cdot 11,342.65 \text{ m}$

are set off, in both directions at right angles, and by so doing, the corner points A, B, C, D are constructed, whose connecting lines yield the edge lines. (At the topographic section of the Prussian Land Survey, the edge

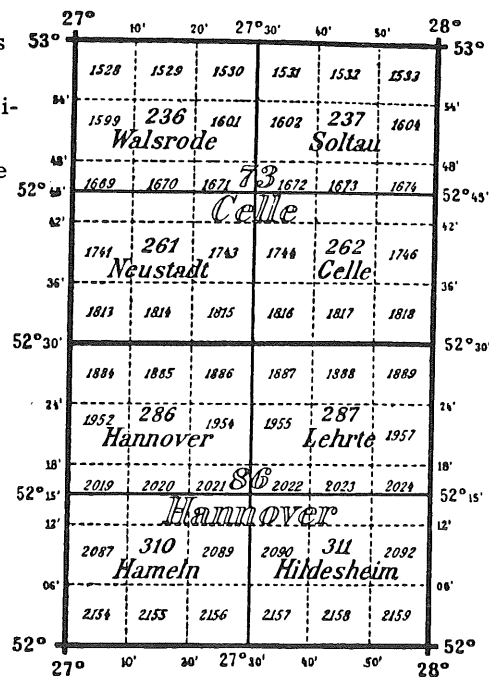


Fig. 3. Sheet division of the Prussian maps for the region of Hannover.

points are carried forward by metal stencils.) The division of the edge lines into minutes yields the minute net, in which the triangulation points are now entered by their geographic coordinates.

Now there only remains to be examined the allowance for the curvature of the parallels.

From the theory of the conic projection we use the basic equation (2), section 41, p. 221,

$$l' = l \sin \varphi_0.$$

The angle l' is the angle between the two meridians AC and BD in Fig. 4, and we will have $l' = 475.4''$. Further, the radius of the parallels, which we can regard as equally large for all latitudes within the plane-table sheet, is equal to 197 m for the scale 1:25,000 according to (1), section 41, p. 221, and with this, the distance of the center of the arc from the chord is found equal to 0.13 mm.

Likewise, we could also indicate the ordinates δ of the additional points of the arc for every second of the geographic longitude; we see that these values lie directly at the boundary of the graphic reproduction.

At the topographic section of the Reichsamt für Landesaufnahme, the curvature of the parallels is taken into account by adding to the geographic coordinates of the triangle points, for every minute of longitude of the sheet, a reduction δ , to be considered as equal, with sufficient accuracy, for all plane-table sheets, namely:

l :	0'	1'	2'	3'	4'	5'	6'	7'	8'	9'	10'
δ :	—0.00"	0.04"	0.07"	0.09"	0.10"	0.11"	0.10"	0.09"	0.07"	0.04"	0.00"

and then setting them off from the rectilinear minute net referred to CD and AB , Fig. 4, p. 275.

The thus computed dimensions of a plane-table sheet hold for the whole zone between the upper and the lower latitude; all sheets of this zone are equal and fit together precisely with the meridian edge line. Two neighboring zones can however not be put exactly together, since they originate from different conic surfaces and since, therefore, the angles l' between the meridians of the two sheet edges are not exactly equal. However, this is without any practical significance, with respect to the unavoidable paper shrinkage, if a few sheets are assembled as a general sheet.

In a corresponding manner, the plotting of the sheets of the map at 1:100,000 or, as the case may be, 1:200,000 is carried out.

Computation of rectangular coordinate lines into geographic net trapezoids

If the xy -net, i.e. the grid net of a rectangular system of coordinates, is drawn into a plane-table sheet or into another map sheet bound according to longitudes and latitudes, then we deal with the inversion of the problem solved above.

We will treat here only the special case in which the grid net of the Gauss-Krüger coordinates is entered on a plane-table sheet. In view of the scale 1:25,000 existing here it is sufficient to compute the longitudes L and the latitudes φ to two decimal places of a second.

The problem is solved again most appropriately in the form of determining the sheet corners of the grid net, i.e., we compute the geographic latitudes on the west and east edge of the sheet for even values of x and on the north and south edge for even values of y .

1. If we start with the west and east edge, then the problem will read: The latitude φ shall be computed for a given longitude L of the west and east edge and a given abscissa x .

With the help of the abscissa x we can determine the foot-point latitude φ_1 with the use of the table of meridional arcs on pp. [41] to [44] of the Appendix to the first half-volume by interpolation, and then we have according to (21), section 33, p. 168,

$$\varphi - \varphi_1 = -\frac{y^2 \varrho}{2 M_1 N_1} t_1 + \dots,$$

and since we have according to (22), section 33, p. 168,

$$y = \frac{l}{\varrho} N_1 \cos \varphi_1 + \dots \quad \text{and} \quad \frac{N_1}{M_1} = V_1^2$$

then we will have

$$\varphi - \varphi_1 = -\frac{V_1^2}{2\rho} l^2 \cos^2 \varphi_1 t_1 + \dots \quad (6)$$

For this, the constant $\log \frac{1}{2\rho} = 4.384\,545$ holds.

2. For the north or, as the case may be, the south edge, the ordinate y is given in addition to the latitude φ and the longitude L is required.

According to (6), section 33, p. 165, we have immediately

$$l = \frac{y\rho}{N \cos \varphi} - \frac{y^3 \rho}{6 N^3 \cos \varphi} (1 - t^2) + \dots = \frac{y\rho}{N \cos \varphi} \left(1 - \frac{y^2}{6 N^2} (1 - t^2) \right),$$

and we obtain hence

$$\log l = \log \frac{\rho y}{N \cos \varphi} - \frac{\mu 10^7 y^2}{6 N^2} (1 - t^2). \quad (7)$$

We introduce further the auxiliary quantity $[2] = \frac{\rho}{N}$ and with this, (7) has the final form

$$\log l = \log \frac{[2]}{\cos \varphi} y - \frac{\mu 10^7}{6 \rho^2} [2]^2 y^2 (1 - t^2). \quad (8)$$

We have here $\log \frac{\mu 10^7}{6 \rho^2} = 5.2308$. The computation of the factor $(1 - t^2)$ does not offer any difficulty

in view of the even values of φ ; we also can compute according to the formula $1 - t^2 = \frac{\cos 2\varphi}{\cos^2 \varphi}$.

In order to have a numerical example, we will apply the above formulae to plane-table sheet 1907, Teltow. The sheet has the following boundary:

$$\begin{array}{ll} \text{north } \varphi = 52^\circ 30', & \text{west } L = 13^\circ 10', \quad l = 1^\circ 10' = 4200'' \\ \text{south } \varphi = 52^\circ 24', & \text{east } L = 13^\circ 20', \quad l = 1^\circ 20' = 4800''. \end{array}$$

The following lines are computed into the net:

$$\begin{array}{ll} \text{northing} = 5812 \text{ km} & \text{or } x = 5,812,000 \text{ m} \\ \text{easting} = 4589 & \text{or } y = + 89,000 \end{array}$$

1. For the abscissa x we find on p. [43] of the Appendix to the first half-volume

$$\varphi_1 = 52^\circ 26' 33.56'',$$

and for this, we have

$$\begin{array}{r} \log V_1^2 = 0.001\,084 \\ \log \cos^2 \varphi_1 = 9.570\,026 \\ \log t_1 = 0.114\,120 \\ \log \frac{1}{2\rho} = 4.384\,545 \\ \hline 4.069\,775. \end{array}$$

With $l = 4200''$ in the west and $l = 4800''$ in the east we then obtain according to (6)

	West	East
$\varphi - \varphi_1$	$- 20.71''$	$- 27.06''$
φ_1	$52^\circ 26' 33.56''$	$52^\circ 26' 33.56''$
φ	$52^\circ 26' 12.85''$	$52^\circ 26' 6.50''$

(9)

2. For the ordinate

$$y = + 89,000 \text{ m}$$

we have, according to equation (8), the following computational procedure:

	North	South
	$\varphi = 52^\circ 30'$	$\varphi = 52^\circ 24'$
[2]	8.508 8675	8.508 8700
(1 : cos φ)	0.215 5529	0.214 5668
y	4.949 3900	4.949 3900
	3.673 8104	3.672 8268
	North	South
$\frac{\mu 10^7}{6 \varrho^2}$	5.2308	5.2308
(1 - t^2)	9.8441 n	9.8364 n
[2] ²	7.0177	7.0177
y^2	9.8988	9.8988
	1.9914 n	1.9837 n
	- 98	- 96
	3.673 8104	3.672 8268
	+ 98	+ 96
	3.673 8202	3.672 8364
$l = 4718.68''$	$l = 4708.00''$	
$= 1^\circ 18' 38.68''$	$= 1^\circ 18' 28.00''$	
L (North) = $13^\circ 18' 38.68''$	}	
L (South) = $13^\circ 18' 28.00''$		

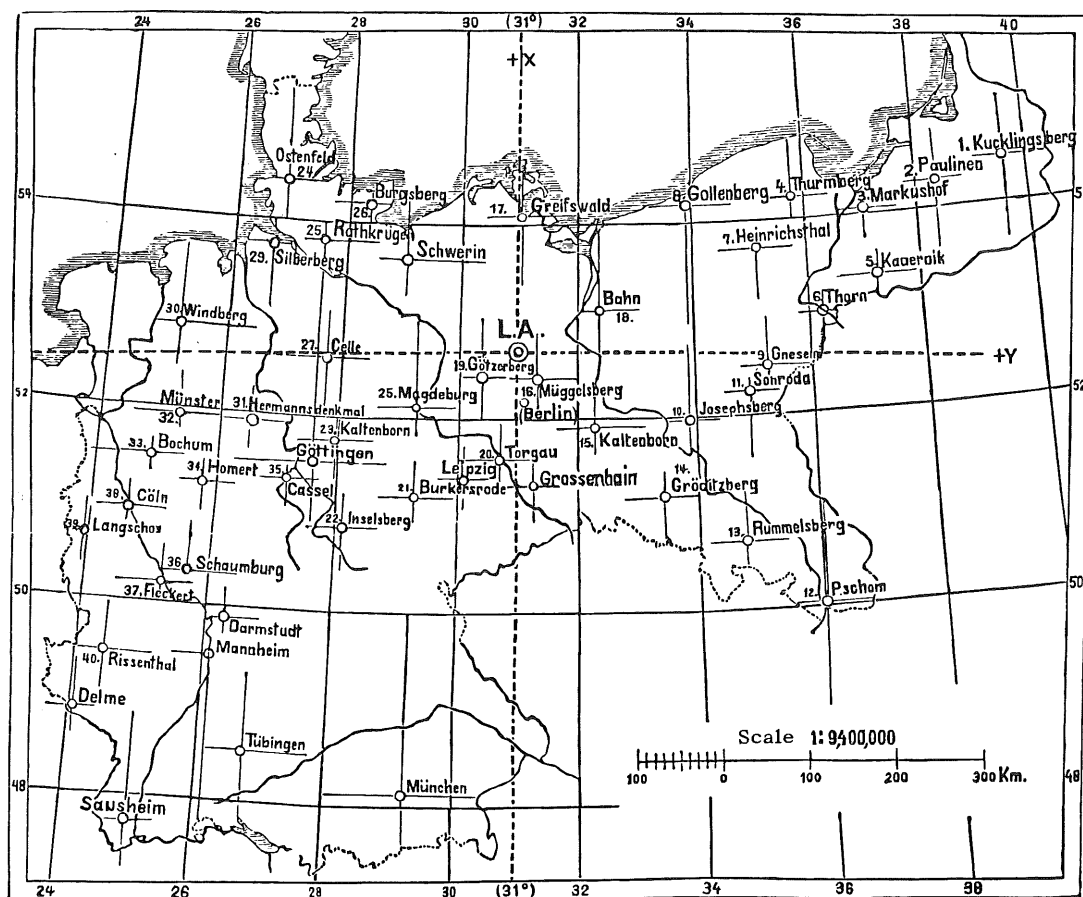
(10)

Even if the above computation does not offer any difficulties, we carry it out only for individual lines and, as for the remaining part, make use extensively of interpolations. The same also holds for the remaining computations of this section, especially for the computation of the geographic net lines to agree with the sheets of the Deutsche Grundkarte, to which reference is made also in the operations mentioned on p. 275.

Geographic net lines in the case of the conformal double projection

In the case of the maps which are based on a conformal double projection, the computation of the geographic net lines to agree with the net is carried out likewise in two stages. At first, we have to compute, for even values of the longitudes and latitudes of the ellipsoid, the corresponding values for the sphere and then, for the latter to determine the rectangular coordinates. We will however not discuss this problem more thoroughly, since it has at the present time no significance for German geodesy.

In Fig. 1 we have made a summary sketch of the German rectangular systems of coordinates which, at the same time, brings to mind a good piece of the history of the German surveys.



Frauenturm [tower of the Church of Our Lady] in Munich as origin of coordinates and the meridian of this point as axis of abscissae. Further details about this are given in the official work, *Die bayerische Landesvermessung in ihrer wissenschaftlichen Grundlage*, München, 1873, p. 253. Cf. also our Volume I, 8th edition, 1935, p. 547.

This system holds only for Bavaria proper on the right bank of the Rhine; for the Bavarian Palatinate, the same zero point, Mannheim, holds as for Baden.

A single Bavarian meridian x -axis which passes through the Frauenturm of Munich was sufficient at the time of the layout of the system and useful to make the whole system perspicuous, since only plane-table surveys at 1:5,000 were involved.

The largest ordinates of this system are in the east near Passau $y = 56,000$ rods = 163 km and north-west near Aschaffenburg $y = 64,000$ rods = 187 km, which gives a distortion of $\frac{y^2}{2r^2} = 0.00043$ or 0.43 m to 1 km. Since this distortion offers difficulties in a small survey, this has been remedied by introducing local systems with oblique x -axes, i.e. with such axes which are twisted with respect to the meridian of the zero point by the meridian convergence.

Cf., in this connection: *Technische Anleitung zu den trigonometrischen Netz- und Koordinatenrechnungen* von Dr. J. H. Franke, München, 1889, pp. 14 and 99. Further: "Transformation rechth.-sphär. Koordinaten," *Astr. Nachr.*, 126. Band 1890, p. 355, System I, and *Korrespondenzblatt des bayerischen Geometervereins*, Band IX, München, February, 1894, Nr. 1. "Betrachtungen über das Koordinaten- und Blattsystem der bayerischen Landesvermessung" von Dr. J. H. Franke, pp. 1 to 21. J. H. Franke, *Geodätische Punktkoordinierung in sphärischem Kleinsystem*, München, 1898; further *Zeitschr. f. Verm.*, 1896, p. 327; 1899, pp. 255-264. Finsterwalder, "Vorschlag zu einer Neugestaltung der bayerischen Koordinaten," *Zeitschr. d. V. d. höh. bayer. Verm.-Beamten*, 1914, pp. 75-82.

Württemberg

The Observatory of Tübingen as origin of a rectangular system of coordinates and the meridian of Tübingen as x -axis were assumed by Bohnenberger already in the eighteenth century for his map of Swabia; the azimuth Tübingen-Kornbühl oriented thereby was already measured in 1792 and is retained until the present day, although the measurement of 1819 resulted in a change of 15", so that the present-day system of Württemberg is thus twisted by 15" with respect to the meridian of Tübingen; cf., in this connection, our Vol. I, 8th edition, 1935, p. 551, with Fig. 1.

At the beginning of the past century, Bohnenberger computed in Württemberg rectangular geodetic coordinates piecewise as plane coordinates, which was sufficient in the case of the accuracy of the measurements then required to minutes. But Bohnenberger also solved splendidly, at once, the most important problem which follows, namely the conversion between rectangular and geographic coordinates and vice versa in his paper, *De computandis dimensionibus*, etc., Tubingae, 1826, where he says in section 16 about his formulae for rectangular coordinates: "Conveniunt cum iis, quibus usus est cel. Soldner in computandis dimensionibus bavaricis."

Cf. also Jordan-Steppes' *Deutsches Vermessungswesen*, 1882, I, pp. 244-259.

Baden

The topographic survey of the grand-duchy of Baden was referred, at an early date, to a single rectangular system of coordinates, with the observatory in Mannheim as zero point and with the meridian of Mannheim as x -axis. The azimuth Mannheim-Speyer used for orientation (cf. first half-volume, section 53, p. 123) was already measured in 1820 by Nicolai. The coordinates were formerly computed as plane coordinates; the newer spherical rectangular system of the cadastral survey of Baden, which was introduced by Obergeometer [first geometer] Rheiner, originates approximately from the time of 1840.

Hesse-Darmstadt

In the "law concerning the completion of the land cadaster" and instruction of 30 June 1824, there is decreed in Art. 3: "Spherical coordinates, and the meridian of Darmstadt shall hereby be assumed as the main

axis." In Jordan-Steppes' *Deutsches Vermessungswesen*, p. 289, there is reported on some characteristic features of the Hessian rectangular coordinates.

Hannover

For the land survey of Hannover, Gauss already ordered early a rectangular plane conformal system of coordinates with the origin Göttingen and the meridian of Göttingen as x -axis, whose theory we have treated, in the first approximation, in the first half-volume, section 68, and, in detail, in this half-volume, sections 32 to 38.

About the history of these coordinates, with reference to our Vol. I, 8th edition, 1935, pp. 522-526, as well as to Gauss' *Werke*, Bd. IX, Leipzig, 1903, and a report by Gäde about Gauss' practical geodetic work in *Zeitschr. f. Verm.*, 1885, pp. 113, 145, 161, 177, 193, 225, the following is said:

In connection with the Danish degree-measurement, which was begun by Schumacher in 1816, Gauss carried out the geodetic measurements of the arc between Göttingen and Altona in the years 1821-1823 (a picture of the triangulation net, concerning this, is given in our Vol. I, 8th edition, 1935, p. 523).

In 1824 and 1825, Gauss' triangles underwent a further extension toward the west for the purpose of a new attachment, which was not planned originally. In addition to the scientific interest of the degree-measurement, the purposes of the land survey were thereby kept in sight very early also. "It is now generally recognized that a precise land survey without a good triangulation is impossible" (Gauss, 1824). For instance, in 1823, Gauss expressly undertook angle measurements to topographic surveys on the Ägidius tower in Hannover, which did not belong to the points of the degree-measurement. From such auxiliary measurements, Gauss obtained, from 1821 to 1825, over 400 well-determined points; altogether, there were 2600. These points were computed according to coordinates and plotted on the plane tables. "The data of the position of an arbitrary starting point (the observatory of Göttingen) accurate to a few feet must be considered as the principal gain."

On 25 March 1828, the expansion of the triangulation over the whole kingdom was ordered; it was concluded in 1844. (The picture of the net of the main triangles with 89 points at the scale 1:1,000,000 is contained in Papens *Geogr. Karte des Königreichs Hannover und Herzogtums Braunschweig* and Gauss' *Werke*, Band IX, p. 434.)

In 1830 Gauss wrote: "Later it might be advisable to publish by printing the list of 2600 points, for the moment not yet, first because a scientific development of the numbers can be given only in connection with the development of my own mathematical theories, which I intend to furnish in about 3-4 treatises." Of the latter, the "Untersuchungen über Gegenstände der höheren Geodäsie" were published in 1845 and 1846, and the posthumous works on the theory of the conformal projection of the ellipsoid on the plane were examined later and published in Gauss' *Werke*, Band IX, 1903, pp. 143-218. The termination of the computational preparation of the land survey was delayed until 1848. In 1859, four years after Gauss' death, the Ministry desired the publication in print, which the General Staff refused, however, "because the coordinates not only have a very relative value but also many of them are unreliable and even wrong; of such, the list would first have to be cleaned."

The theory of these coordinates has at first been worked up by Schreiber in the work, *Theorie der Projektionsmethode der hannoverschen Landesvermessung* von Oskar Schreiber, Hauptmann im Kgl. Hannov. 1. Jägerbataillon Hannover, Hahnsche Hofbuchhandlung, 1866.

The preface of this work by Wittstein (May 1866) says: Even in Hannover, where on the basis of Gauss' projection, topographic surveys have taken place continuously, the knowledge of the fundamentals of this projection was as good as lost, and one worked only under the influence of a sort of tradition according to handed-down patterns. It was a matter of searching for existing hints and fragments; of tying with care to the latter, and thus venturing the experiment of calling to life again completely the analytical developments, which Gauss must already have possessed.

Soon afterwards there thus appeared: *Allgemeines Koordinatenverzeichnis als Ergebnis der hannoverschen Landesvermessung aus den Jahren 1821-1841, abgedruckt zum Zwecke der Benützung bei den Vermessungsarbeiten zur Vorbereitung der anderweitigen Regelung der Grundsteuer*, Hannover, 1868, printed by Wilh. Riemschneider, with an introduction by Wittstein, containing the most important formulae of coordinates with terms of the order $1:r^2$ inclusive.

Meanwhile Hannover had become Prussian, and it was thought of using Gauss' coordinates also for the cadastral survey, which had been made so far by districts with the chain and the compass and the like.

But it was thought, thereby, of cutting the coordinates up into 31 partial systems according to the political division of the country into districts. For the 31 new zero points Wittstein had to compute the meridian convergences and scale factors m , and according to these, the partial systems were converted. The city of Hannover received the new zero point of Osterwald, whose reduction formulae were given in our Vol. II, 3rd edition, 1888, pp. 196-197.

Cadastral secretary Clotten in Hannover, who had written several papers on the surveys in the former Kingdom of Hannover (*Zeitschr. f. Vermessung*, 1881, pp. 22, 292, 376, 425, 445; and 1882, pp. 22, 256), has told us earlier many

things about the period of transition after 1866. Since in the conformal projection one *can* compute a scale factor $m = 1 + \frac{y^2}{2r^2}$ at every point, it seems that it was believed that one *must* compute and use such reduction coefficients by districts, and this mistake may have been the reason for the splitting up of Gauss' system into 31 partial systems, since one went so far then with the dividing up until it was believed to be able to set, with sufficient accuracy, $1 + \frac{y^2}{2r^2} = 1$.

The 31 conformal partial systems were abolished again in 1879, and replaced by the Soldner systems generally introduced in Prussia (in the region of Hannover, see Fig. 1, 27. Celle, 28. Kaltenborn, 29. Silberberg, 30. Windberg, etc.).

It would have been desirable to retain (even from considerations of reverence toward Gauss) the old classical axis of Göttingen and also in order to make possible later a thorough critique of the old coordinates. (Cf. *Zeitschr. f. Verm.*, 1896, pp. 197-199.)

Electorate of Hesse

Originally, the triangulation by Gerling had no rectangular system of coordinates; however, the longitudes and latitudes of all 48 main points were computed in connection with Göttingen (Gerling, *Beiträge zur Geographie Kurhessens*, Kassel, 1839, pp. 200-204). Rectangular partial systems, with the church tower of the district in every case as origin and the meridian of the church tower in every case as x -axis, were then attached to these geographic coordinates by the cadastral authorities. Where the attachment to the main triangulation was missing, they measured a small base with measuring rods and an azimuth by corresponding solar altitudes, separately for every district.

When, about 1853, the general cadastral surveys were to be carried out in the provinces of Hanau and Fulda and extended to the rest of Hesse, the geographic coordinates of the trigonometric points usable for cadastral surveys were converted to rectangular spherical coordinates with the zero point Kassel, Martinsturm (with the earth's dimensions according to Walbeck; cf. first half-volume, p. 7).

(This is based on information by Landesvermessungsrat [Councillor for land surveying] Kaupert, as well as by Gehrmann in Jordan-Steppes' *Deutsches Vermessungswesen* II, p. 105.)

Thüringen-Gotha

In a treatise, *Über die Ergänzung der topographischen Aufnahme und Kartierung von Deutschland in bezug auf Thüringen*, von C. Frhrn. von Gross, Kammerherren, etc., Weimar, 1848, there is published on pp. 33-72 an "Instruktion für die Ausführung der Triangulation" written by the astronomer and geodesist Hansen in Gotha, which is interesting in some respects, and with respect to the coordinates, assumes a meridian x -axis from which the geographic longitudes are counted to the east $+ 10'$, $+ 20'$, etc., to the west $- 10'$, $- 20'$, etc. On this meridian, the latitude $50^\circ 36'$ is determined as the zero point for the abscissae x . The rectangular coordinates are at first computed approximately as *plane*, ξ , η , p. 51, whereupon corrections

of the order $\frac{s^3}{2r^2} \left(p. 53 \text{ and } \rho' + \frac{1}{2r^2} \right)$ are added further, by which x , y coordinates are obtained, "on the

curved surface of the earth, but in a somewhat different sense than we have formerly understood these coordinates" (p. 53). The theory of these coordinates is not given; the indicated formulae (p. 53) are symmetric with respect to x and y (which is not true in the case of Soldner's and Gauss' formulae) and can be converted to rectangular plane coordinates of the stereographic projection by the insertion of additional simple terms, like-

wise of the order $\frac{s^3}{r^2}$ (p. 72).

According to further information about the Thuringian surveys, these instructions by Hansen of 1848 with their characteristic coordinates x , y have remained only a proposal.

A new representation of the theoretical fundamentals of Hansen's rectangular coordinates is given by E. Anding in *Astr. Nachr.*, No. 5281, 1924.

About 1855 the duchy of Nassau accepted a rectangular system with the origin Schaumburg according to Soldner's theory. (For further details see *Zeitschrift für Vermessungswesen*, 1882, pp. 315-316, and Vol. I, 8th edition, 1935, p. 563.)

Prussia, Land Survey

In Prussia, in former times, the points were computed only according to geographic coordinates. For this, a lithographed *Instruktion für die topographischen Arbeiten des Königl. Preussischen Generalstabes* by the Chief, General Staff of the Army v. Müffling, Berlin, 15 January 1821, gives the necessary formulae for use. These formulae correspond to a treatise by Soldner "Über die kürzeste Linie auf dem Sphäroide" in the *Monthly Correspondence for the Promotion of Geography and Astronomy*, 1805, pp. 7-23. About the astronomic starting values of the older Prussian triangulations, especially the triangulation chain from Berlin to the Rhine measured by Müffling in 1817-1822, nothing has become known.

The newer geographic longitudes and latitudes, which the trigonometric section of the land survey publishes, are based on the point Rauenberg near Berlin, whose longitude and latitude were found by transfer from the observatory of Berlin. There was measured further the azimuth Rauenberg-Berlin, Marienkirche, by Baeyer in 1859. The starting values thus found are:

Rauenberg, longitude east of Ferro	31° 02' 04.928"
Rauenberg, latitude	52 27 12.021
Azimuth Rauenberg-Berlin	19 46 04.87.

Later measurements of the Geodetic Institute from 1886 and 1887 yielded the longitude to be smaller by 12.95", and the latitude and the azimuth larger by 0.17" and by 3.83", respectively. (*Veröffentl. des Kgl. Preuss. Geodätischen Instituts, astronomisch-geodätische Arbeiten I. Ordnung*, Berlin, 1889.)

In 1917-1918, a repeated astronomic orientation of the Prussian triangulation net took place at the Helmerturm of the Geodetic Institute, which was inserted likewise into the first-order net. According to this, the errors in the orientation of the net at the point Potsdam, Helmerturm, amount to

0.8492" in latitude
13.3940 in longitude
1.82" in azimuth.

Rectangular coordinates occur at the Prussian Land Survey for the first time in the work, *Die Königl. Preussische Landestriangulation*, Hauptdreiecke, erster Teil, zweite vermehrte Auflage, Berlin, 1870. However, these rectangular spheroidal coordinates introduced by Schreiber are used only for the unobjectionable insertion of triangle chains into systems of triangles already fixed.

Not until 1876, however, did the rectangular coordinates obtain fundamental significance when they were introduced by Schreiber for the adjustment of the points of second order according to indirect observations. The publication of this computation of coordinates took place in an autographed treatise: "Rechnungsvorschriften für die trigonometrische Abteilung der Landesaufnahme vom 8. September 1877," whose contents are represented also in the work, Jordan-Steppes, *Das deutsche Vermessungswesen*, Vol. I, 1882, pp. 151-164. As already indicated in section 53, p. 266, the Schreiber coordinates were based on a conformal double projection, namely first the conformal projection of the ellipsoid on the sphere according to Gauss' *Untersuchungen über Gegenstände der höheren Geodäsie*, erste Abhandlung, 1843, cf. sections 46-52, and then conformal projection of the sphere on the plane, which we have treated in the first half-volume, sections 68 to 70.

The system of coordinates had as x -axis the meridian of 31° longitude (Berlin); see Fig. 1, p. 279. The point with the latitude 52° 42' 2.53251" was used as the zero point, corresponding to the latitude 52° 40' on Gauss' conformal sphere; the mean radius of curvature A of this latitude is given by $\log A = 6.805\ 0274.003$. The ordinates go to the west as far as $y = 540$ km and to the east as far as $y = 620$ km. The conditions of distortion are therefore very considerable; as is seen from the auxiliary table,

p. [59], of the Appendix to the first half-volume, $\log m$ goes to 0.002 or m itself to 1.0046 or 4.6 mm per 1 m, so that the eccentricities in the case of eccentric placements of the theodolite and similar local measurements must already be reduced according to the ratio m .

Because of the large distortions, the Schreiber coordinates are not suited for immediate practical use in the land survey; they were used in the land survey, in the first place, for the adjustment of the triangulations of orders II and III, for which we have treated examples in Vol. I, 8th edition, 1935, section 71, pp. 245-250, section 103, pp. 423-429, section 106, pp. 437-446, and section 114, pp. 493-499.

Prussia, Cadastral Survey

In the Prussian cadastral system also, the rectangular systems of coordinates of greater extent have found acceptance at a rather late date.

In the treatise by General Baeyer, *Mein Entwurf zur Anfertigung einer guten Karte*, etc., Berlin, 1868, which is the best source for the historical development of the Prussian survey, the rectangular systems of coordinates, which, already then, had held good in South Germany for half a century, are not mentioned.

The *Anweisung vom 7. Mai 1868 für das Verfahren bei den Vermessungsarbeiten in den Provinzen Schleswig-Holstein, Hannover und Hessen-Nassau*, zweite Ausgabe, Berlin, 1870 [instructions of May 7, 1868, for the procedure in the survey operations in the provinces of Schleswig-Holstein, Hannover and Hesse-Nassau, second edition, Berlin, 1870], says in section 40, p. 35: "For the purpose of further use of the trigonometric measurements, the position of the triangle points with respect to each other is to be computed by rectangular coordinates, which are to be referred to the true meridian line of a suitable point, to be chosen for this according to the determination of the cadaster inspector."

About the rectangular system of coordinates in the Prussian Rhineland we have the following statement by F. G. Gauss in Jordan-Steppes' *Deutsches Vermessungswesen*, 1881, p. 165.

By the instructions of 12 March 1822, it was generally introduced that the detail nets were connected with each other by means of nets of higher order, as well as that their triangle sides were derived from sides of first order and oriented according to these.

For all points there should be computed rectangular coordinates which should be referred, for the points of first order, to the cathedral of Cologne and its meridian, for the points of second to fourth order, to a suitable first-order point, located in the district in question, and its meridian. This has been deviated from, since for the points of second to fourth order, not the meridian of the first-order point used as starting point for the coordinates was assumed as axis of abscissae, but the parallel line to the meridian of Cologne laid through it, and the coordinates which were computed without taking into account the earth's curvature were all referred, by their addition to those of the starting point, *nominally*, to the cathedral of Cologne. But, in fact, even further on, a rather large number of systems of coordinates existed therefore. Their extent depended on the division of the work districts and was very different from individual community districts to a few provinces. For the points which occurred in several systems, different distances from the meridian and vertical line of the cathedral of Cologne were computed in every system, since the computation of the coordinates, as lying on the plane, did not make the reciprocal agreement of the numbering possible.

In the eastern provinces of Brandenburg, Pomerania, Saxony, Silesia, Posen, Prussia, extensive, accurate surveys of parcels of land were not carried out before 1876; general systems of coordinates were not available.

The *Anweisung IX vom 25. Oktober 1881 für die trigonometrischen und polygonometrischen Arbeiten bei Erneuerung der Karten und Bücher des Grundsteuerkatasters*, Berlin, 1881 [instruction IX of 25 October 1881 for the trigonometric and polygonometric operations in the case of the renewal of the maps and books of the land-tax cadaster, Berlin, 1881], gave in the Appendix on pp. 337-351 the "Bestimmungen vom 29. Dezember 1879 über den Anschluss der Spezialvermessungen an die trigonometrische Landesvermessung" [regulations of 29 December 1879 about the attachment of the special surveys to the trigonometric land survey], issued by the central directorate of surveys in the Prussian state. By this, 40 coordinate zero points, on which in part older surveys were already based, were established; they are drawn up in our sketch map on p. 279.

These zero points are triangulation points of first or second order of the land survey, so that through their geographic coordinates the connection of the individual systems was given.

Oldenburg

As zero point of the rectangular system of coordinates there was used the castle tower of Oldenburg;

the meridian laid through this point is used as the axis of abscissae with $+x$ to the south, $-x$ to the north, and accordingly, $+y$ is counted to the west and $-y$ to the east.

We refer, in this connection, to our Vol. I, 8th edition, 1935, pp. 564-565, with the remark that the coordinates introduced from about 1835 to 1836 are spherical (according to Soldner or Bohnenberger, as the case may be). (About further details see 4th edition of the volume, 1896, pp. 333-335.)

Mecklenburg

Mecklenburg is the first State in Germany which has profited by the advantages of conformality up to the cadastral maps. The conformal principle has been retained, in practical geodesy, in the first to the third order of the triangulation by the geodesist Paschen, of Mecklenburg, who, as a direct pupil of Gauss, became acquainted with the geodetic ideas of the master in the university in Göttingen, and upon return to his homeland Mecklenburg, made use of the knowledge in a form, deviating from that of Hannover and adapted to a geographical extent from west to east. (Cf. Gauss' *Werke*, Band IX, pp. 135-140.)

It is the conformal conic projection with tangency at the central parallel of the state at the latitude $P = 53^\circ 45'$, and accordingly, the system of coordinates is set up in such a way that the x -axis lies at the meridian of the castle tower of Schwerin and the y -axis at right angles to it at the latitude $53^\circ 45'$. However, there was carried out, in addition, a displacement of the x 's by the constant amount of 13,919.812 m in order to shift the zero point to the castle tower of Schwerin itself, but this has only a formal significance: for in all cases of theoretical computation with the coordinates of Mecklenburg, it is necessary to use the original abscissae x computed from the latitude $53^\circ 45'$. The linear distortion agrees, in the first approximation, with that

of Gauss' conformal system, namely $m = 1 + \frac{x^2}{2r^2}$, because the x 's take the place of the y 's.

But in view of the distortion $m = 1 + \frac{x^2}{2r^2}$ Mecklenburg has reduced, by a little device, the maximum value

occurring with respect to its extent to half by introducing a token value equal to half of the maximum value, and since in the case of the extent of $0^\circ 45'$ in latitude or 82.5 km from the normal parallel to the south and to the north, the maximum value amounts to $\log m = 0.0000371$, which corresponds to 85.4 mm per 1 km, then the linear maximum distortion has been limited by that displacement to the maximum value of approximately 4 cm per 1 km in the whole of Mecklenburg.

We have treated the theory of the projection of Mecklenburg in sections 41 to 43; the official work about it is:

Grossherzoglich Mecklenburgische Landesvermessung, V. Teil, Die konforme Kegelprojektion und ihre Anwendung auf das trigonometrische Netz I. Ordnung, herausgegeben im Auftrage der Grossherzoglichen Ministerien des Innern und der Finanzen, Abteilung für Domänen und Forsten, von Dr. W. Jordan, Professor an der Technischen Hochschule in Hannover, Karl Mauck, Kammeringenieur in Schwerin, R. Vogeler, Kammeringenieur in Schwerin; mit einer lithographischen Netzkarte, Schwerin 1895. To be obtained through Stiller's booksellers to the court (J. Ritter).

Saxony

In the region of the former kingdom of Saxony, two points, Grossenhain and Leipzig, are entered on our sketch map and, in fact, according to information by Nagel of 5 May 1889, according to which the pillar for the intermediate base point, which carries the name of Grossenhain, holds as the actual zero point for Saxony. The pillar *B* Leipzig on the Pleissenburg in Leipzig holds only as the starting point of coordinates for the city survey of Leipzig.

In addition (according to a communication by Fuhrmann in *Zeitschr. f. Verm.*, 1894, pp. 266-270), in Saxony, a type of local systems has been assumed, in which we can compute, by districts, as on the plane, but with the sacrifice of having an attachment as a whole.

Alsace-Lorraine

For the cadastral survey of Alsace-Lorraine, two zero points of coordinates, Delme and Sausheim, were

assumed, about which a first communication was made by Rodenbusch in *Zeitschr. f. Verm.*, 1888, pp. 545-552. The official data about it are contained in the work, *Anweisung vom 30. Januar 1889 für das Verfahren bei der Stückvermessung von Gemarkungen zum Zwecke der Errichtung von Katasterurkunden*, Strassburg, 1889, p. 9.

Anhalt

For the duchy of Anhalt there was introduced, by the ducal Treasury in 1896, a conformal transverse-axis system of coordinates with the zero point: latitude = $51^{\circ} 50'$, longitude $29^{\circ} 18' 7.8117''$ (meridian of the Prussian system No. 23 Magdeburg), on the basis of the professional opinion of W. Jordan (27 November 1893). (Cf. Jordan, *Zeitschr. f. Verm.*, 1894, pp. 65-74; 1896, pp. 83-94.) Further details about the coordinates of Anhalt are contained in section 29 and sections 39 to 40.

Braunschweig

For the land survey of Braunschweig, there was assumed, in 1897, the Prussian cadastral coordinate system No. 28 Kaltenborn, whose zero point lies favorably to the various territorial parts of Braunschweig. Cf. C. Koppe, *Zeitschr. f. Verm.*, 1902, p. 402, "Die neue topographische Landeskarte des Herzogtums Braunschweig."

Ostmark

Rectangular coordinates were introduced in Austria in 1807, after a topographic section at the General Quartermaster Staff in Vienna had been installed in 1806; by it, the triangulation of the country was initiated at once. On the basis of an astronomic starting station the geographic coordinates of all triangle points were computed. Then there followed the computation of rectangular coordinates according to formulae which were indicated by Oriani in the ephemerides of Milan of 1807. These were mainly spherical formulae with a few spheroidal supplementary terms. These coordinates yielded, on the plane, a projection which, in substance, agrees with the Cassini-Soldner projection.

A new period began with the foundation of the Military Geographic Institute in 1839. In order to be able to use the coordinates computed on the terrestrial ellipsoid as plane coordinates for the cadastral surveys without too large distances, there were set up, for the territory of the Austrian state, seven coordinate systems, whose zero points coincided with main points of the land triangulation net. These coordinates were used for the Austrian survey until the World War. In 1917, negotiations were opened with the German Reich for the unification of survey in both states, and conformal plane coordinates were introduced in Austria on the basis of the Gauss-Krüger method of projection, for which the meridians 28° , 31° and 34° east of Ferro were assumed as zero meridians. Since the annexation of Austria to the German Reich in 1938, the adjustment of the Austrian triangulation net to the German net is in preparation, and with this, the Gauss-Krüger systems of coordinates of the German Reich referred to Greenwich will also be introduced for Ostmark.

Further details about the Austrian coordinates are given by Jordan, "Österreichische Geodäsie," *Zeitschr. f. Verm.*, 1899, pp. 52-60; Rohrer, "Zum neuen Projektionssystem Österreichs," *Österr. Zeitschr. f. Verm.*, 1934, pp. 89-97, 116-123; and Meyer, "Die amtlichen Kartenwerke des Landes Österreich," *Mitt. d. Reichsamts für Landesaufn.*, 1938, pp. 134-139.

The Gauss-Krüger Coordinates of the German Reich

The advisory council for surveying called into existence by the decree of 27 July 1921 for the advancement and gradual unification of surveying, on its first session in April 1922, already proposed to the states [Länder] the introduction of a uniform system of rectangular coordinates on the basis of a conformal projection of the terrestrial ellipsoid. This proposal was at first accepted by the Reichsamt für Landesaufnahme so that, since this time, in addition to geographic coordinates, the new conformal rectangular coordinates are used for the results of the triangulations.

Simultaneously with the introduction of the new systems of coordinates, the passage from Ferro to

Greenwich for the counting of geographic longitudes was decided upon and the relation

$$\text{Greenwich longitudes} = \text{Ferro longitudes} - 17^{\circ} 40'$$

was assumed for the conversion.

For the territory of the German Reich there exist 6 systems of coordinates whose axes of abscissae are the meridians 6° , 9° , 12° , 15° , 18° , 21° east of Greenwich; the region of validity of each system extends therefore to $1^{\circ} 30'$ west and east of the zero meridian. The position and boundary of the systems are illustrated in Fig. 2. These meridian strips are projected conformally on the plane, as represented thoroughly in sections 32 to 38.

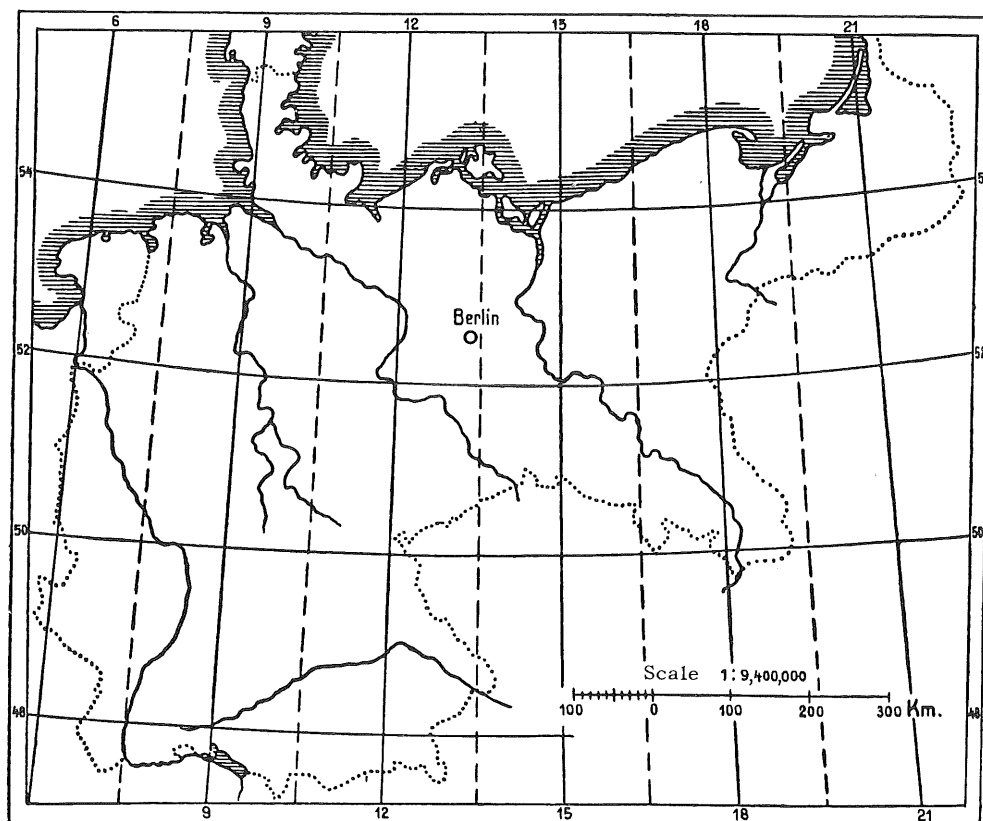


Fig. 2.
The Gauss-Krüger systems of coordinates.

The zero points for the counting of the abscissae lie at the equator for all systems. The numbering of the ordinates was based on a proposal by trigonometrist Baumgart in *Technik und Wehrmacht*, 1921, Heft 11-12, through which single values are obtained for all ordinates independently of the individual systems. For this, the value $+500,000$ m is added to the original ordinates so that, by so doing, also all negative ordinates receive positive values. In addition, each ordinate is preceded by an ordinal number, which is equal to the third part of the geographic longitude of the zero point.

According to this, e.g., the ordinate $y = -83,137.824$ m in the system 12° east of Greenwich has the value

$$-83,137.824 \text{ m} + 4,500,000 \text{ m} = +4,416,862.176 \text{ m}.$$

The ordinates thus converted are designated by *eastings* [Rechts] while the abscissae are designated accordingly by *northings* [Hoch].

The Gauss-Krüger coordinates constitute, in the form of northings and eastings, the foundation for all new surveys in the German Reich.

Concluding Observations

The general maps, Fig. 1, p. 279, and Fig. 2, p. 287, of the German systems of coordinates and the glance back at their gradual origin, both show a true picture of the unequal political development of the individual states of our homeland.

From the geodetic point of view, we do not have to regret this lack of uniformity in the past. From the 100 years' work of a *Bohnenberger*, *Soldner*, *Rheiner*, *Schleiermacher*, *Gauss*, *Paschen*, *Schreiber*, and whatever all their names are, such an abundance of experience has come forth that today we are in a position to choose and utilize in practice the most favorable method of projection for the introduction of rectangular plane coordinates for a problem under consideration.

The systems of coordinates of our land surveys are of a fundamental significance for the surveys themselves, for the mathematical determination, and for the graphical representation of the survey results, and for this reason, the value and the duration of a land survey depend, for the greatest part, on the more or less good choice of a coordinate system.

Chapter VI

DETERMINATION OF THE DIMENSIONS OF THE TERRESTRIAL ELLIPSOID BY DEGREE-MEASUREMENTS

Section 56. Determination of the Meridian Ellipse by Means of Two Latitude Measurements

The oldest means for the determination of the earth's dimensions are the so-called latitude measurements, whose history we have given in the introduction to the first half-volume, pp. 1 through 8.

By a measurement of latitude we understand the measurement of an arc of the meridian of the earth and of the polar altitude or geographic latitude of its end points.

If we know the measured results of such two degree-measurements at different latitudes, then we can compute the dimensions of the meridian ellipse thereby determined.

Before we occupy ourselves with this, a remark should be made about the measurement of the meridional arcs. To measure a line directly on the meridian was the endeavor of the first degree measurers (cf., e.g., first half-volume, pp. 2 and 3, Arabic degree-measurement, and pp. 5 and 6, American degree-measurement), and if the measured arc made a small angle α with the direction of the meridian, then one could easily carry out a reduction to the meridian which, in substance, consists in the multiplication of the measured arc by $\cos \alpha$. Even triangulation chains which, according to their main extent, lie near the direction of the meridian, can be reduced to the meridian, as we shall show more thoroughly in the following section 57.

According to the illustration of Fig. 1 we now assume that we have two degree-measurements on the same meridian, or, which is the same here, two degree-measurements whose elements are represented on a meridian ellipse. Let the first degree-measurement have the meridional arc m with the latitudes φ_1 and φ_2 of its end points, and the second degree-measurement, accordingly, the meridional arc m' with the latitudes φ_3 and φ_4 . For abbreviation, we will write for this:

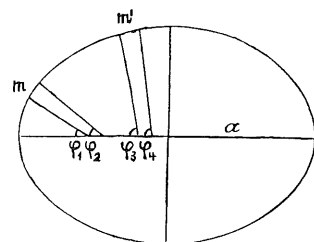


Fig. 1.
Two latitude measurements.

$$\varphi_2 - \varphi_1 = \Delta \varphi \qquad \varphi_4 - \varphi_3 = \Delta \varphi' \qquad (1)$$

$$\frac{\varphi_2 + \varphi_1}{2} = \varphi \qquad \frac{\varphi_4 + \varphi_3}{2} = \varphi' . \qquad (2)$$

Now we know from the first half-volume, section 41, p. 72, that we can compute the length m of a moderately large meridional arc as the arc of a circle whose radius is the radius of curvature in the meridian M for the mean latitude φ , and whose central angle is the difference of latitude $\Delta \varphi$; i.e., we have for the two degree-measurements:

$$m = \frac{\Delta \varphi}{\rho} M \qquad m' = \frac{\Delta \varphi'}{\rho} M' . \qquad (3)$$

In addition, we have according to (21) and (19) in the first half-volume, p. 49:

$$M = \frac{c}{V^3} \quad M' = \frac{c}{V'^3} \quad (4)$$

$$V^2 = 1 + e'^2 \cos^2 \varphi \quad V'^2 = 1 + e'^2 \cos^2 \varphi' \quad (5)$$

If we substitute these equations (5) and (4) in (3), and then divide the two equations (3), then we obtain:

$$\left(\frac{m}{m'} \frac{\Delta' \varphi}{\Delta \varphi} \right)^{2/3} = \frac{1 + e'^2 \cos^2 \varphi'}{1 + e'^2 \cos^2 \varphi} \quad (6)$$

For abbreviation, we write:

$$\left(\frac{m}{m'} \frac{\Delta \varphi'}{\Delta \varphi} \right)^{2/3} = q^2 \quad (6a)$$

Equation (6) is linear with respect to e'^2 , and therefore can be solved directly for e'^2 . If we use the abbreviation (6a) here, then we obtain:

$$e'^2 = \frac{1 - q^2}{q^2 \cos^2 \varphi - \cos^2 \varphi'} \quad (7)$$

If we have computed hence e'^2 , then we obtain with a check from (3), (4) and (5):

$$c = \frac{m}{\Delta \varphi} \rho V^3 \quad \text{or} \quad c = \frac{m'}{\Delta \varphi'} \rho V'^3 \quad (8)$$

We obtain the two semiaxes of the ellipse a and b from c and e'^2 according to (18), first half-volume, p. 49:

$$a = \frac{c}{\sqrt{1 + e'^2}} \quad b = \frac{c}{1 + e'^2} \quad (9)$$

where we can form once again as a check:

$$\frac{a^2 - b^2}{b^2} = e'^2$$

With this, we also have e^2 and the flattening α according to (5) and (7), first half-volume, pp. 41 and 42:

$$e^2 = \frac{e'^2}{1 + e'^2}, \quad \alpha = 1 - \sqrt{1 - e^2} \quad \text{or} \quad \alpha = 1 - \frac{1}{\sqrt{1 + e'^2}} \quad (10)$$

The length of the meridian quadrant Q also can be computed according to (25), first half-volume, p. 71:

$$Q = a \frac{\pi}{2} \left(1 - \frac{\alpha}{2} + \frac{\alpha^2}{16} \right) \quad (11)$$

For the application of the developed formulae we will use the well-known classic degree-measurements of Peru and Lapland.

According to Bessel's data in the fourteenth volume, 1837, of *Astronom. Nachr.*, pp. 334 and 337, the results of the degree-measurements in Peru and Lapland (Sweden) are the following:

Degree-measurement in Peru:

$$\left. \begin{aligned} m &= 176,875.5 \text{ toises} = 344,736.772 \text{ m} \\ \varphi_1 &= -3^\circ 4' 32.068'' \quad \varphi_2 = +0^\circ 2' 31.387''. \end{aligned} \right\} \quad (12)$$

Degree-measurement in Lapland:

$$\left. \begin{aligned} m' &= 92,777.981 \text{ toises} = 180,827.654 \text{ m} \\ \varphi_3 &= 65^\circ 31' 30.265'' \quad \varphi_4 = 67^\circ 8' 49.830''. \end{aligned} \right\} \quad (13)$$

We form hence the differences and the means:

$$\begin{aligned} \Delta \varphi &= 3^\circ 7' 3.455'' & \Delta \varphi' &= 1^\circ 37' 19.565'' \\ &= 11,223.455'' & &= 5839.565'' \\ \varphi &= -1^\circ 31' 00.3405'' & \varphi' &= 66^\circ 20' 10.0475''. \end{aligned}$$

We now compute according to the indicated formulae:

$$\begin{aligned} \log \frac{m}{\Delta \varphi} &= 1.487\,3610.4 & \log \frac{m'}{\Delta \varphi'} &= 1.490\,8843.5 & \log q^2 &= 9.997\,6511.3 \\ e'^2 &= \frac{1 - 0.994\,606\,119}{0.993\,901\,347 - 0.161\,098\,184} = 0.006\,476\,714 \\ \log V^2 &= 0.002\,8017.7 & \log V'^2 &= 0.000\,4529.0 \\ \log c &= 6.805\,9888.4 & \log a &= 6.804\,5869.7 & \log b &= 6.803\,1851.0 \\ \alpha &= 1 : 310.2977 & Q &= 10,000,156 \text{ m.} \end{aligned} \quad (14)$$

If we aim to have at first e^2 instead of e'^2 , then we can derive this from (7), for we have according to (5), first half-volume, section 37, p. 41, with application to (7):

$$e^2 = \frac{e'^2}{1 + e'^2} = \frac{1 - q^2}{\sin^2 \varphi' - q^2 \sin^2 \varphi}. \quad (15)$$

We are led to the same formula also directly by computing, from the outset, with the constants a , e^2 and W^2 instead of c and e'^2 and V^2 :

$$W^2 = 1 - e^2 \sin^2 \varphi \quad W'^2 = 1 - e^2 \sin^2 \varphi'. \quad (16)$$

If we treat this just as formerly in (3) to (6), then we are led to (15), after which there follows also a with a check from (17), first half-volume, section 38, p. 49.

Computations of such a kind played an important role at the time of the degree-measurements of the eighteenth century (cf. Introduction to first half-volume, p. 6); today, this is no longer the case, because the question of the earth's dimensions occurs now in a different form.

We now have to take up the problem mentioned at the beginning of the previous section 56, namely the computation of the meridional arc m , which corresponds to a measured arc s laid obliquely with respect to the meridian between the latitudes of the end points. Or, in connection with the following Fig. 1, we have the task of computing the meridional arc m which lies between the same latitudes φ_1 and φ_2 as an obliquely laid arc $AB = s$, whose direction is determined at least by *one* azimuth α .

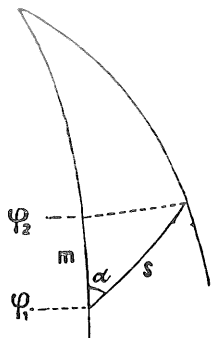


Fig. 1.

The arc s can be measured directly, but in general it is to be assumed that this arc s is computed as a long diagonal of a triangulation chain. Besides, it is assumed that the azimuth α could not be measured directly for the line s , but was referred to s by computation.

According to this, we can consider the result s with one azimuth α (or with two azimuths α_1, α_2 , according to Fig. 3, p. 294, as a geodetic line with geodetic azimuths, and treat it further as such.

According to Fig. 1, we now consider at first the simple case in which only *one* azimuth α is measured: but, on the other hand, we will then assume as a simplification that this α is rather *small*, i.e., that the measured arc s has nearly the direction of the meridian, which is desired, from the outset, in the case of pure latitude measurements.

Between $\varphi_2 - \varphi_1$, s and α there exists a relation which is expressed in the first approximation by the first two terms of (25), section 18, p. 78, namely:

$$\varphi_2 - \varphi_1 = V^2 \left(u - \frac{v^2}{2} t \right). \quad (1)$$

Here we have according to (22) and (23), p. 76:

$$V^2 = \frac{N}{M}, \quad u = \frac{s}{N} \cos \alpha, \quad v = \frac{s}{N} \sin \alpha, \quad t = \tan \varphi_1,$$

therefore:

$$\varphi_2 - \varphi_1 = \frac{s}{M} \cos \alpha - \frac{s^2}{2MN} \sin^2 \alpha \tan \varphi_1.$$

The same equation applied to the meridional arc m yields with $\alpha = 0$:

$$\varphi_2 - \varphi_1 = \frac{m}{M}.$$

This combined with the previous equation yields:

$$m = s \cos \alpha - \frac{s^2}{2N} \sin^2 \alpha \tan \varphi_1 + \dots \quad (2)$$

We can take a rounded-off approximate value for the radius of curvature in the prime vertical N , which occurs here as the denominator only in the second term. If the azimuth α is rather small, then the second term with $\sin^2 \alpha$ becomes very small, and then it is rather unimportant how the approximate value N needed here is assumed.

We could easily develop formula (2) also to higher terms by taking into account further terms of (25), p. 78, in the case of (1); however, we will not carry this out here, but take a simple numerical example in which the reduction formula (2) is fully sufficient.

As such an example, we shall use the degree-measurement of Pennsylvania, remarkable for its

simplicity, which was carried out in the eighteenth century, 1764-1768, by Mason and Dixon, not by triangulation but by direct measurement with rods, as we have already indicated in the Introduction to the first half-volume, p. 5.

The principal data about this noteworthy measurement were given by Prof. J. Howard Gore of Washington, in *Zeitschr. f. Verm.*, 1888, from which we take the following:

$$\varphi_2 = 39^\circ 56' 19''$$

$$m' = 32,010.24 \text{ m}$$

$$(39^\circ 39')$$

$$s = 132,327.16 \text{ m}$$

$$m = 132,042.98 \text{ m}$$

$$\alpha = 3^\circ 43' 30''$$

$$\varphi_1 = 38^\circ 27' 34''$$

$$\varphi_2 - \varphi_1 = 1^\circ 28' 45''$$

$$m' + m = 164,053.22 \text{ m}.$$

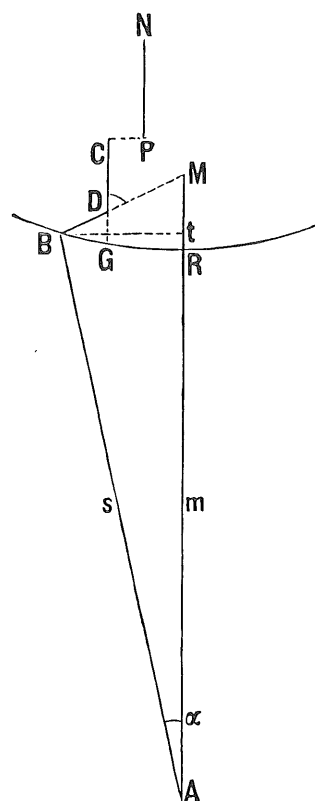


Fig. 2.
Pennsylvanian degree-measurement. (1764-1768.)

The principal measurement was carried out along the straight line AB , which extends from the southernmost point A with the astronomically measured latitude $\varphi_1 = 38^\circ 27' 34''$ with the azimuth $\alpha = 3^\circ 43' 30''$ to a point B whose latitude is not measured astronomically ($39^\circ 39'$ by supplementary interpolation). Then, in addition, there was measured a broken line $BD C P N$ to the northernmost point N , whose astronomically measured latitude is $\varphi_2 = 39^\circ 56' 19''$.

As measured lengths there are indicated: first, the oblique main length, $s = 434,011.64$ ft, and the sum of the two direct meridional arcs, $GC + PN = 104,988.4$ ft. (In Fig. 2 BGR represents the parallel of B and CP a small piece of the parallel of C .)

In addition, there is indicated that the English foot used here is $= \frac{107}{144}$ Paris foot, from which we compute $1 \text{ ft} = 0.304\,893\,06 \text{ m}$.

The present-day English foot is smaller, namely $= 0.304\,797\,27 \text{ m}$. The reduction $434,011.64 \text{ ft} = 132,286 \text{ m}$ indicated in our Introduction to the first half-volume, p. 5, is based on the new ratio $1 \text{ ft} = 0.304\,79727 \text{ m}$.

With this coefficient of proportion we convert the two given distances into meters, as is noted in Fig. 2:

$$m' = 104,988.4 \text{ ft} = 32,010.24 \text{ m} \quad \text{and} \quad s = 434,011.64 \text{ ft} = 132,327.16 \text{ m} . \quad (3)$$

Now comes the main problem which occupies us here, namely to reduce the oblique length s to the meridian length, for which we use formula (2). Here we substitute $s = 132,327.16 \text{ m}$, $\alpha = 3^\circ 43' 30''$, $\varphi_1 = 38^\circ 27' 34''$ and according to p. [20] of the Appendix to the first half-volume, approximately, $\log N = 6.80521$. The calculation yields:

$$m = 132,047.60 \text{ m} - 4.62 \text{ m} = 132,042.98 \text{ m} . \quad (4)$$

The two parts of this computation are illustrated from the geometric viewpoint in Fig. 2, p. 293, for we have:

$$A t = s \cos \alpha = 132,047.60 \text{ m} \quad \text{and} \quad t R = 4.62 \text{ m} , \quad \text{and hence} \quad m = A R = A t - t R . \quad (5)$$

We now have the whole measured meridional arc between the parallels of A and N according to (3) and (4):

$$m' + m = 164,053.22 \text{ m} . \quad (6)$$

In addition, the two astronomically measured latitudes:

$$\begin{aligned} \varphi_2 &= 39^\circ 56' 19'' \\ \varphi_2 - \varphi_1 &= 1^\circ 28' 45'' = 5325'' . \\ \varphi_1 &= 38 \ 27 \ 34 \end{aligned} \quad (7)$$

$$\text{Mean} \quad \frac{\varphi_2 + \varphi_1}{2} = \varphi = 39^\circ 11' 56.5'' . \quad (8)$$

From (6) and (8) we obtain the degree of the meridian for the mean latitude φ :

$$G = \frac{3600''}{5325''} 164,053.22 \text{ m} = 110,909.22 \text{ m} . \quad (9)$$

(This degree-measurement of Pennsylvania was used, along with others, by Laplace, Airy and Schubert for the computation of the earth's dimensions, but not by Bessel and Clarke.)

After having treated, by this historically interesting example, the reduction of a measured arc with *one* azimuth, we pass over to the case in which *two* azimuths are measured, namely α_1 and α_2 in Fig. 3.

For this, we have from section 21, (27), p. 98, the equation for the mean latitude φ :

$$\varphi_2 - \varphi_1 = \frac{s}{M} \cos \alpha \left(1 + \frac{l^2 \cos^2 \varphi}{24} (2 + 3 t^2 + 2 \eta^2) + \frac{b^2}{8 V^4} \eta^2 (t^2 - 1 - \eta^2 - 4 \eta^2 t^2) \right) . \quad (10)$$

The same formula also holds for the meridional arc m if the mean azimuth $\alpha = 0$ and also the difference of longitude $l = 0$ is set down, and hence:

$$\varphi_2 - \varphi_1 = \frac{m}{M} \left(1 + \dots + \frac{b^2}{8 V^4} \eta^2 (t^2 - 1 - \eta^2 - 4 \eta^2 t^2) \right) . \quad (11)$$

Now the division of (10) and (11) yields:

$$m = s \cos \alpha \left(1 + \frac{l^2 \cos^2 \varphi}{24} (2 + 3 t^2 + 2 \eta^2) \right) . \quad (12)$$

This formula can be applied directly if we substitute for the geographic difference of longitude l an approximate value and assume φ as the mean latitude (also in $\eta^2 = e'^2 \cos^2 \varphi$).

However, we also can set as a first approximation according to (26), section 21, p. 98:

$$l \cos \varphi = \frac{s}{N} \sin \alpha . \quad (13)$$

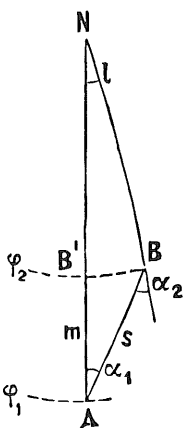


Fig. 3.

With this, (12) yields:

$$m = s \cos \alpha \left(1 + \frac{s^2 \sin^2 \alpha}{24 N^2} (2 + 3 l^2 + 2 \eta^2) \right) \quad (14)$$

Any approximate values easily procured are sufficient here for N^2 and η^2 .

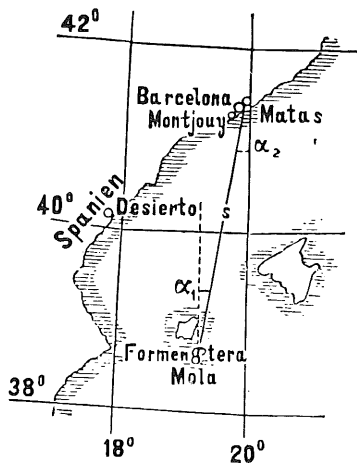
Formula (14) just now developed will be sufficient in most cases, and we could also carry it further in the way started; but we will give yet another form according to Bessel, where reduced latitudes are used.

If, in addition to Fig. 4 which illustrates our problem on the ellipsoid, we also undertake the transfer onto the sphere with reduced latitudes according to section 22, then we obtain, to Fig. 4, the corresponding Fig. 5, which contains the same azimuths as Fig. 4 and, as for the remaining part, spherical measurements, which we can treat according to section 26. We will, however, not repeat here the development for this, which was given in detail in our third edition of the third volume, 1890, in section 103, pp. 503-505, because such problems are applied less and less in modern times.

The final result of our development mentioned is:

$$m = s \frac{\cos \frac{\alpha_2 + \alpha_1}{2}}{\cos \frac{\alpha_2 - \alpha_1}{2}} \left\{ 1 + \frac{s^2 \sin^2 \alpha}{12 a^2} (1 + e'^2 \cos (\varphi_1 + \varphi_2)) - \frac{s^4 \sin^2 \alpha}{240 a^4} \left\{ -2 + 3 \cos^2 \alpha + 5 \cos^2 \alpha \tan^2 \frac{\varphi_1 + \varphi_2}{2} \right\} \right\} \quad (15)$$

This is, essentially, the formula which was indicated by Bessel in the fourteenth volume of *Astr. Nachr.*, 1837, p. 310; in *Gradmessung in Ostpreussen*, 1838, p. 446; and in General Baeyer's *Messen auf der sphäroidischen Oberfläche*, 1862, p. 48.



Matas $\varphi_2 = 41^\circ 30' 29.04''$
 $\alpha_2 = 11 \ 26 \ 1.71$

$$\frac{\varphi_2 + \varphi_1}{2} = 40^\circ 5' 12.575''$$

$$\frac{\alpha_2 + \alpha_1}{2} = 11 \ 11 \ 52.325$$

$\alpha_1 = 10^\circ 57' 42.94''$
Mola $\varphi_1 = 38 \ 39 \ 56.11$

$$\frac{\alpha_2 - \alpha_1}{2} = 0^\circ 14' 9.385''$$

Fig. 6.
South end of the French-Spanish
degree-measurement of 1792.
(Scale 1:10,000,000.)

As a numerical example for the use of formula (15) we take, in accordance with the previous Fig. 6, a communication by Bessel, *Astr. Nachr.*, nineteenth volume, 1841, pp. 112-114, about his new computation of the southern part of the old French-Spanish degree-measurement from Dunkirk to the Balearic Islands. The northern point Matas lies at the Spanish coast near Barcelona, and the southern point Mola is the southernmost

point of the degree-measurement on the island of Formentera.

From the triangulation, Bessel computed the geodetic line between Matas and Mola, $s = 165,108.586$ toises or $= 321,802.629$ m, as well as reduced the azimuths α_1 and α_2 , which, in addition to the latitudes φ_1 and φ_2 , are entered in Fig. 6, p. 295.

If we make, with this, the calculation according to formula (15), then we find:

$$m = 315,678.950 \text{ m} + 2.529 \text{ m} - 0.001 \text{ m} = 315,681.478 \text{ m}.$$

The last term of formula (15) thus contributes here only 1 mm; this term will usually be neglected.

Section 58. Adjustment of Several Latitude Measurements

In following the Introduction to the first half-volume, pp. 6 through 9, we come to the determination of the dimensions of the terrestrial ellipsoid from more than two latitude measurements or to the adjustment of several latitude measurements according to the method of least squares.

We start here from the idea that the astronomic measurement of the latitudes φ is relatively much more inaccurate than the geodetic measurement of the meridional arcs m , for an error of 1" in the polar altitude or latitude φ produces already a change of approximately 31 m in the meridional arc m , while the mean error of the geodetic measurement of a meridional arc is a much smaller one.

However, it was soon found out that even the errors in measurement of the latitudes φ were not sufficient for the explanation of the discrepancies in the various degree-measurements; yet the form of the adjustment computation, according to which the sum of the squares of all changes of latitude was brought to a minimum, was long retained, although it was known that the reason for the discrepancies of latitude, to a great extent, is not to be found in the errors of measurement at all but in the deflections of the vertical. In the case of such an application, the method of least squares has only the significance of an empirical adjustment of discrepant elements, and the errors v remaining here give the first indications at which points deflections of the vertical are to be found.

We will now undertake a part of such an adjustment of measurements of latitude and use for this the results of degree-measurements collected and sifted by Bessel from 1837 to 1841, from which Bessel in 1841 derived his well-known dimensions of the earth still in use today (cf. first half-volume, p. 44).

However we will not demonstrate here Bessel's computation itself but we will pick out only a few numerical values in order to show by them the computational procedure.

Of the French degree-measurement we shall use the following five stations:

Station	Latitude φ	$\Delta \varphi$	Meridional Arc	(1)
1. Formentera	$\varphi_1 = 38^\circ 39' 56.1''$.	.	
2. Barcelona	$\varphi_2 = 41 \ 22 \ 47.9$	$2^\circ 42' 51.8''$	301,354 m	
3. Carcassonne	$\varphi_3 = 43 \ 12 \ 54.3$	4 32 58.2	505,137	
4. Pantheon	$\varphi_4 = 48 \ 50 \ 49.4$	10 10 53.3	1,131,050	
5. D�nkirchen	$\varphi_5 = 51 \ 2 \ 8.8$	12 22 12.7	1,374,572	

In order to obtain the error equations for an adjustment according to indirect observations, we take Bessel's dimensions of the earth a and e^2 as a basis according to the first half-volume, section 37, p. 44, and determine such corrections of a and of e^2 , which make the sum of squares of all corrections to be applied to the latitudes φ a minimum.

For this, we must first determine relations between the differences of latitude $\Delta \varphi$ and the meridional arcs m belonging to them; and in order to have here a simple computation, we proceed according to the fundamental theorem of the first half-volume, section 41, p. 66, which says that we can compute a meridional arc m as an arc of a circle, with the radius of curvature in the meridian M of the mean latitude and with the central angle $\Delta \varphi$. Since we have rounded off the latitudes φ to 0.1", corresponding to 3 m, then the approximation indicated is admissible.

The first two measured latitudes φ_1 and φ_2 are connected with the meridional arcs m lying

between them by the relation:

$$\varphi_2 - \varphi_1 = \frac{m}{M} \rho,$$

where, according to (17) and (15), first half-volume, p. 49, the following formula holds for M :

$$\frac{1}{M} = \frac{(1 - e^2 \sin^2 \varphi)^{1/2}}{a(1 - e^2)} \quad \text{with} \quad \varphi = \frac{\varphi_1 + \varphi_2}{2}. \quad (2)$$

Since the earth's dimensions a and e^2 occurring here are the unknowns for our adjustment, we divide them into approximate values a_0 and e_0^2 with the corrections δa and δe^2 , belonging to them, i.e. we set down:

$$a = a_0 + \delta a \quad e^2 = e_0^2 + \delta e^2. \quad (3)$$

We also denote by M_0 the value of M , which arises from the approximate assumptions $a = a_0$ and $e^2 = e_0^2$; and accordingly, we develop according to Taylor's series:

$$\frac{1}{M} = \frac{1}{M_0} + \frac{\partial}{\partial a} \left(\frac{1}{M} \right) \delta a + \frac{\partial}{\partial e^2} \left(\frac{1}{M} \right) \delta e^2. \quad (4)$$

We now develop the two partial derivatives of the function $\frac{1}{M}$ needed here, in the first approximation, according to (2), neglecting all terms in e^2 :

$$\frac{\partial}{\partial a} \left(\frac{1}{M} \right) = -\frac{1}{a^2} \quad \text{and} \quad \frac{\partial}{\partial e^2} \left(\frac{1}{M} \right) = \frac{1}{a} \left(1 - \frac{3}{2} \sin^2 \varphi \right). \quad (5)$$

Now we can proceed to the formation of the error equations. Equation (1), which will in general not be satisfied because of the observational errors, is brought to agreement by adding the corrections v_1 and v_2 to the observed φ_1 's and φ_2 's, i.e.:

$$\varphi_2 - \varphi_1 + v_2 - v_1 = m \varrho \left(\frac{1}{M} \right). \quad (6)$$

If we substitute here (4) and (5), then we obtain:

$$\varphi_2 - \varphi_1 + v_2 - v_1 = m \varrho \left\{ \frac{1}{M_0} - \frac{\delta a}{a^2} + \left(1 - \frac{3}{2} \sin^2 \varphi \right) \frac{\delta e^2}{a} \right\}. \quad (7)$$

In the terms with δa and with δe^2 , instead of the unknowns a , we can set here their approximate value a_0 ; since all terms in e^2 are already neglected in the coefficients of δa and δe^2 anyhow, we will even set here $a = M$, and hence, according to (1) the product $\frac{m \rho}{a} = \frac{m \rho}{M} = \varphi_2 - \varphi_1$, and with this, (7) becomes:

$$v_2 - v_1 = -\frac{\varphi_2 - \varphi_1}{a_0} \delta a + \left(1 - \frac{3}{2} \sin^2 \varphi \right) (\varphi_2 - \varphi_1) \delta e^2 + \frac{m \varrho}{M_0} - (\varphi_2 - \varphi_1). \quad (8)$$

In order to obtain convenient numbers for the computation, we will not determine δa and δe^2 themselves, but the thousandth part of δa and a thousand times δe^2 ; i.e. we will introduce two new unknowns x and y by the equations:

$$x = \frac{\delta a}{1000} \quad y = 1000 \delta e^2. \quad (9)$$

This introduced in (8) will give:

$$v_2 - v_1 = a'x + b'y + l', \quad (10)$$

where a' , b' and l' have the following meanings:

$$a' = -1000 \frac{\varphi_2 - \varphi_1}{a_0} \quad b' = + \frac{\varphi_2 - \varphi_1}{1000} \left(1 - \frac{3}{2} \sin^2 \varphi \right) \quad (11)$$

$$l' = \frac{m \varrho}{M_0} - (\varphi_2 - \varphi_1). \quad (12)$$

As approximate values a_0 and e_0^2 we take the well-known Bessel values of 1841 according to section 37, first half-volume, p. 44, namely:

$$\left. \begin{array}{ll} a_0 = 6,377,397.155 \text{ m} & \log a_0 = 6.804\,6434\cdot6 \\ e_0^2 = 0.006\,674\,372 & \log e_0^2 = 7.824\,4104\cdot2 \end{array} \right\} \quad (13)$$

We will thereby have the convenience that our absolute terms l' in (12) immediately become equal to Bessel's final values $v_2 - v_1$, and so on; we will however not at first make use of them here, but show the application of formulae (10), (11), (12) by the first two values of table (1) of p. 296:

$$\left. \begin{array}{ll} \text{Formentera } \varphi_1 = 38^\circ 39' 56.1'' & m = 301,354 \text{ m} \\ \text{Barcelona } \varphi_2 = 41^\circ 22' 47.9'' & \\ \hline \varphi_2 - \varphi_1 = 2^\circ 42' 51.8'' = 9771.8'' & \\ \text{mean } \varphi = 40^\circ 1' 22.0'' & \end{array} \right\} \quad (14)$$

With this, we can compute at once according to formulae (11):

$$a' = -1.532 \quad b' = +3.709. \quad (15)$$

Even the computation of l according to (12) has no difficulty, since M_0 is here the value which corresponds to the values a_0 and e_0^2 of (13), i.e.:

$$M_0 = \frac{a_0 (1 - e_0^2)}{(1 - e_0^2 \sin^2 \varphi)^{3/2}}, \quad \log M_0 = 6.803\,5358. \quad (16)$$

However, we can also take advantage of the favorable circumstance that the a_0 's and e_0^2 's of (13) are the well-known Bessel values, on which our auxiliary tables in the first half-volume, pp. [12] to [33] of the Appendix are based; and therefore, instead of calculating $\log M_0$ according to (16), we can also take it from p. [22] of the Appendix or, better still, immediately from p. [23]:

$$\begin{array}{rcl}
\text{for } \varphi = 40^\circ 1' 22'' & \log [1] = \log \frac{\varrho}{M_0} & 8.510\ 8893 \\
\text{from (14) } \log m = \log 301\ 354 & & 5.479\ 0770 \\
\hline
& \log \frac{m \varrho}{M_0} & 3.989\ 9663 \qquad \frac{m \varrho}{M_0} = 9771.6'' \\
& & \text{from (14) } \varphi_2 - \varphi_1 = 9771.8'' \\
& \text{hence} & l' = -0.2''
\end{array} \tag{17}$$

If we combine this with (15), then we have the first equation of the form (10):

$$v_2 - v_1 = -1.53 x + 3.71 y - 0.2'' \tag{18}$$

Since we have thus shown in all thoroughness the setting up of an equation with the calculation of the coefficients and of the absolute term, we will briefly write down the result of the computation for the four remaining stations which belong to the French degree-measurements indicated in (1).

Equations of Error Differences

$$\begin{array}{ll}
\text{Formentera—Barcelona} & v_2 - v_1 = -1.53 x + 3.71 y - 0.2'' \\
\text{,, —Carcassonne} & v_3 - v_1 = -2.57 x + 5.83 y - 1.4 \\
\text{,, —Pantheon} & v_4 - v_1 = -5.75 x + 10.36 y - 2.1 \\
\text{,, —Dünkirchen} & v_5 - v_1 = -6.98 x + 11.31 y + 1.2
\end{array} \tag{19}$$

Similar groups of equations also arise for all other degree-measurements; Bessel's adjustment has a total of ten such groups of equations with altogether $38 - 10 = 28$ equations, as is seen from our summary of the first half-volume, section 1, p. 8; but we remark in connection with this that in group (1) there are already included only five stations, whereas according to p. 8, first half-volume, the number of the French stations is seven; we have omitted two.

Equations (19) are not error equations in the usual sense, because in each equation there occur two corrections v . The separation of the v 's is made by introducing for each degree-measurement itself a latitude correction v as an unknown. The case is entirely in correspondence with the adjustment of geodetic direction measurements, where we must also introduce a zero point correction as an unknown for each set of measurements.

In this manner, the following five actual error equations arise from the four equations (19):

$$\begin{array}{llll}
v_1 = v_1 & \dots & \dots & \dots \\
v_2 = v_1 & \dots & -1.53 x & + 3.71 y - 0.2'' \\
v_3 = v_1 & \dots & -2.57 x & + 5.83 y - 1.4 \\
v_4 = v_1 & \dots & -5.75 x & + 10.36 y - 2.1 \\
v_5 = v_1 & \dots & -6.98 x & + 11.31 y + 1.2
\end{array} \tag{20}$$

Each of the ten degree-measurements yields such a group of error equations, and since each degree-measurement brings in a special unknown v , as, e.g. v_1 in group (20), then we see at a glance that the number of all unknowns must be $= 10 + 2$, namely the ten special v 's and then the two actual unknowns x and y .

Nothing stands in the way now of forming the pertinent twelve normal equations from which we shall eliminate the ten auxiliary unknowns v as quickly as possible.

Since the procedure of the adjustment is made sufficiently clear by this, we will break off here. A more thorough computation extending over the whole adjustment with six degree-measurements in Europe and with a total of 20 stations was contained in our third edition of this volume, 1890, section 104, pp. 507-516.

If we establish a triangulation chain in the main extent from west to east and connect the two end points by an astronomic longitude determination, then we obtain a longitude measurement.

Whereas in earlier times, especially in the last century, this form of degree-measurement had little importance because of the great uncertainty of the astronomic longitude determinations, the relation has become a different one now, since they have nearly reached the accuracy of the latitude measurements, and the longitude measurements are now nearly on the same level as the latitude measurements.

In order to treat the theory of the longitude measurement from its fundamental principles, we only need to demonstrate equation (15), section 21, p. 95, for the mean latitude φ :

$$l \cos \varphi = S \sin \alpha \left\{ 1 + \frac{S^2}{24} (\sin^2 \alpha t^2 - \cos^2 \alpha (1 + \eta^2 - 9 \eta^2 t^2)) \right\}. \quad (1)$$

Since $N = c:V$, the meaning of S is here:

$$S = \frac{s}{c} V = \frac{s}{c} \sqrt{1 + e'^2 \cos^2 \varphi} \quad \text{and} \quad t = \tan \varphi, \quad (2)$$

in addition, also

$$\varphi = \frac{\varphi_1 + \varphi_2}{2} \quad \alpha = \frac{\alpha_1 + \alpha_2}{2}. \quad (3)$$

Besides the astronomic difference of longitude l and the geodetic arc s , also the latitudes φ_1, φ_2 and the azimuths α_1, α_2 are to be determined by measurement (cf. Fig. 1). If the measurement takes place in low latitudes (in the neighborhood of the equator), φ need not be very accurate, and if the arc s has essentially a west-east extent (α nearly $= 90^\circ$), α need not be very accurate.

The calculation according to formula (1) has the meaning of a reduction of the geodetic line s to the parallel of the mean latitude φ , and this reduction plays here the same role as the reduction of a latitude measurement to the meridian, which we have treated independently in detail in section 57.

We now imagine the reduction carried out according to (1) and set down for abbreviation:

$$s \sin \alpha \left\{ 1 + \frac{1}{24} \left(\frac{s}{c} V \right)^2 (\sin^2 \alpha t^2 - \cos^2 \alpha (1 + \eta^2 - 9 \eta^2 t^2)) \right\} = p. \quad (4)$$

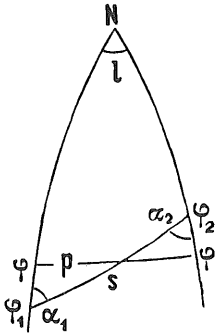


Fig. 1.

Then we have according to (1) and (2):

$$l \cos \varphi = \frac{p}{c} \sqrt{1 + e'^2 \cos^2 \varphi}. \quad (5)$$

The arc of parallel p appears here as a measured quantity in the same manner as the meridional arc m in the case of the latitude measurements.

Now there should exist two such measurements p , namely, in addition to (5), in the same correspondence:

$$l' \cos \varphi' = \frac{p'}{c} \sqrt{1 + e'^2 \cos^2 \varphi'}. \quad (6)$$

From equations (5) and (6) we can determine the two unknowns e'^2 and c ; we write here for abbreviation:

$$\frac{p l' \cos \varphi'}{p' l \cos \varphi} = q. \quad (7)$$

Then we will have:

$$e'^2 = \frac{1 - q^2}{q^2 \cos^2 \varphi - \cos^2 \varphi'}. \quad (8)$$

Then with a check from (5) and (6):

$$c = \frac{p}{l \cos \varphi} \sqrt{1 + e'^2 \cos^2 \varphi} = \frac{p'}{l' \cos \varphi'} \sqrt{1 + e'^2 \cos^2 \varphi'}. \quad (9)$$

These equations (7), (8), (9) entirely correspond to the previous equations found for two latitude measurements in section 56.

Section 60. Transfer of Azimuth

After we have seen that the eccentricity of the meridian ellipse can be determined by two latitude measurements, and that the same problem is also solved by two longitude measurements, there is, third, to be shown further that also two azimuth measurements with the latitudes corresponding to them and with a connection of triangulation lead to the determination of the eccentricity of the meridian ellipse.

Azimuth measurements have already been used in the case of latitude measurements and in the case of longitude measurements, but more only as *auxiliary* measurements, for the reduction of the measured arcs to the meridian or perpendicularly to the meridian; but in the case of the third problem in which we are now engaged, the azimuths are indeed the principal values of the measurement.

If we have measured the two latitudes φ, φ' and the two azimuths α, α' according to the indication of Fig. 1, then we can establish a relation between these four quantities, on the one hand, and the eccentricity of the meridian ellipse, on the other, by means of the reduced latitudes.

If we denote the reduced latitudes by ψ and ψ' , then we have according to (8), section 22, p. 105:

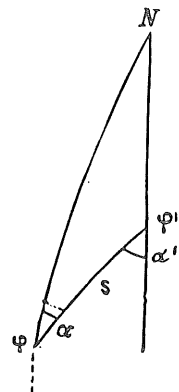


Fig. 1.

$$\cos \psi = \frac{\cos \varphi}{V \sqrt{1 - e^2}} \quad \cos \psi' = \frac{\cos \varphi'}{V' \sqrt{1 - e'^2}}, \quad (1)$$

or

$$\cos \psi = \frac{\cos \varphi \sqrt{1 + e'^2}}{V} \quad \cos \psi' = \frac{\cos \varphi' \sqrt{1 + e'^2}}{V'} \quad (1a)$$

The reduced latitudes ψ and ψ' yield the following equation with the azimuths α and α' according to (1), section 23, p. 109:

$$\cos \psi \sin \alpha = \cos \psi' \sin \alpha'. \quad (2)$$

This in conjunction with (1a) yields:

$$\left(\frac{\cos \varphi' \sin \alpha'}{\cos \varphi \sin \alpha}\right)^2 = \frac{V'^2}{V^2} = \frac{1 + e'^2 \cos^2 \varphi'}{1 + e'^2 \cos^2 \varphi}.$$

If we set for abbreviation:

$$\frac{\cos \varphi' \sin \alpha'}{\cos \varphi \sin \alpha} = q, \quad (3)$$

then we will have:

$$e'^2 = \frac{1 - q^2}{q^2 \cos^2 \varphi - \cos^2 \varphi'}. \quad (4)$$

With this, we also have:

$$1 + e'^2 = \frac{\sin^2 \varphi' - q^2 \sin^2 \varphi}{q^2 \cos^2 \varphi - \cos^2 \varphi'} = \frac{1}{1 - e^2}. \quad (5)$$

In (3) to (5) we thus have once more a system of equations of the same form as in the case of two latitude measurements in section 56 and in the case of two longitude measurements in section 59.

As a numerical example we take:

$$\begin{array}{lll} \text{Trunz} & \varphi' = 54^\circ 13' 11.466'' & \alpha' = 67^\circ 26' 56.152'' \\ \text{Berlin} & \varphi = 52^\circ 30' 16.680'' & \alpha = 62^\circ 31' 15.416'' \end{array} \quad \left. \vphantom{\begin{array}{l} \text{Trunz} \\ \text{Berlin} \end{array}} \right\} \quad (6)$$

$$\log s = 5.654\,2046\,3.$$

If we substitute these equations (6) in formulae (3), (4) and (5), then we obtain:

$$\log q^2 = 9.999\,9152\,63, \quad e'^2 = \frac{0.000\,195\,095}{0.370\,440\,0839 - 0.341\,846\,755}. \quad (7)$$

$$\log e'^2 = 7.833\,981, \quad \log(1 + e'^2) = \log \frac{1}{1 - e^2} = 9.997\,0468. \quad (8)$$

With this, the eccentricity of the meridian ellipse is determined. We see from equations (3) and (4) immediately that the whole procedure becomes useless if the two points at which the latitudes φ , φ' and the reciprocal azimuths α , α' are measured lie either on the same meridian or on the same latitude, for at the meridian we have $\alpha = \alpha' = 0$, therefore $q = \frac{0}{0}$; and if $\varphi' = \varphi$, then because of (1) and (2), we must also have

$\alpha' = \alpha$, and hence, again $q = \frac{0}{0}$, i.e. indeterminate. Even if both φ and φ' are small, i.e. the measurement takes place in the neighborhood of the equator, the method fails to work, because α and α' are then very little different, and hence, q nearly = 1 and e'^2 nearly = $\frac{0}{0}$.

According to this, the method is applicable at higher latitudes with an extent oblique to the meridian.

If, in addition to the astronomic measured results φ , φ' , α , α' , the length s of the connecting geodetic line is also known, then we can also determine the equatorial radius.

For this, we can use equation (17), section 26, p. 121, for with the substitution of S according to (10), p. 120:

$$a \sqrt{1 - e^2} = \frac{s}{c} V \left\{ 1 - \frac{\eta^2}{24} \left(2 \left(\frac{s}{c} V \sin \alpha \right)^2 l^2 + \left(\frac{s}{c} V \cos \alpha \right)^2 (1 - l^2 + \eta^2 + 6 \eta^2 l^2) \right) \right\}.$$

Here we have according to (9), first half-volume, p. 42, $c \sqrt{1 - e^2} = a$, and hence, the above equation yields with the insertion of the necessary ρ 's:

$$a = s V \frac{\rho}{\sigma} \left\{ 1 - \frac{\eta^2}{24} \left(2 \left(\frac{s}{c} V \sin \alpha \right)^2 t^2 + \left(\frac{s}{c} V \cos \alpha \right)^2 (1 - t^2 + \eta^2 + 6 \eta^2 t^2) \right) \right\}. \quad (9)$$

Therefore, there only remains to compute σ , and this is a purely spherical problem, which is solved with the help of Fig. 2 or with the more detailed Fig. 2 in section 24, p. 111.

In preparation for this, we compute the two reduced latitudes ψ and ψ' , where the value of eccentricity e or, as the case may be e' , previously found in (5) and (7), is to be used.

$$\tan \psi = \sqrt{1 - e^2} \tan \varphi, \quad \tan \psi' = \sqrt{1 - e'^2} \tan \varphi'. \quad (10)$$

Now there are given four elements in the spherical triangle of Fig. 2, namely $\psi, \psi', \alpha, \alpha'$; the computation of σ is thus not only determined, but even overdetermined, whereby a computational check arises, for the reduced latitudes ψ and ψ' according to (10) are based on that eccentricity e or, as the case may be e' , which has been derived in (3) to (5) from the four quantities $\varphi, \varphi', \alpha, \alpha'$ themselves. Should thus the computational check in the case of the double determination of σ not agree, then the reason could lie either in the spherical computation according to Fig. 2 or also in the previous computation of the reduced latitudes according to (10).

We can take the formulae needed for the mentioned spherical computation of σ according to Fig. 2 from section 24, p. 111; we will introduce here two values M , the first, belonging to ψ and α , the second, belonging to ψ' and α' ; then we have

$$\sigma = M' - M. \quad (11)$$

For M and M' we have according to (2), p. 111:

$$\tan M = \frac{\tan \psi}{\cos \alpha}, \quad \tan M' = \frac{\tan \psi'}{\cos \alpha'}. \quad (12)$$

From (12) and (11) we have already the required σ . The pertinent computational check mentioned above can be achieved in different ways, e.g. by means of the arc m , which is the same for M and M' . We have according to (2) and (3), p. 111:

$$\sin m = \cos \psi \sin \alpha = \cos \psi' \sin \alpha';$$

then

$$\sin M = \frac{\sin \psi}{\cos m}, \quad \sin M' = \frac{\sin \psi'}{\cos m}. \quad (13)$$

With this, we have the second determination of M and M' as insurance for equations (12). But if M and M' are considerably larger than 45° , then the determinations (13) are not favorable; then we compute better:

$$\begin{aligned} \cot \lambda_1 &= \tan M \sin m, & \cot \lambda_2 &= \tan M' \sin m \\ \lambda_2 - \lambda_1 &= \lambda \\ \sin \sigma &= \frac{\sin \lambda \cos \psi'}{\sin \alpha} = \frac{\sin \lambda \cos \psi}{\sin \alpha'}. \end{aligned} \quad (14)$$

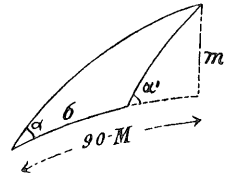


Fig. 2.
(Corresponding to
Fig. 2, p. 111.)

The application of these formulae to our example (6), p. 302, yields:

Trunz	$\psi' = 54^\circ 7' 38.6482''$	$\alpha' = 67^\circ 26' 56.152''$	
Berlin	$\psi = 52^\circ 24' 37.8514''$	$\alpha = 62^\circ 31' 15.416''$	
	$M = 70^\circ 26' 40.0950''$	$M' = 74^\circ 29' 58.3496''$	
	$M' - M = \sigma = 4^\circ 3' 18.2546''$,	$m = 32^\circ 45' 50.2488''$	(15)
	$\lambda_2 - \lambda_1 = \lambda = 6^\circ 8' 45.6806''$,	$\sigma = 4^\circ 3' 18.2548''$.	(16)

Both values of σ according to (15) and (16) agree with each other sufficiently; we have computed further with the mean $\sigma = 4^\circ 3' 18.2547''$ and obtained with this from (9):

$$\log \eta^2 = 7.385\,5669, \quad \log V^2 = 0.001\,0593\cdot6$$

$$a = 6,380,516.074\text{ m} - 9.543\text{ m} + 0.448\text{ m} = 6,380,506.979\text{ m}, \quad \log a = 6.804\,8551\cdot88. \quad (17)$$

The correction terms of (9) thus amount here to only 9.5 m and 0.4 m, whence it is seen at the same time that a repetition of the computation is not necessary because of $c = a \sqrt{1 + e'^2}$ used preliminarily in (9).

The fundamental idea to determine the eccentricity of the meridian ellipse from a degree-measurement diagonally to the meridian was first considered by J. Tobias Mayer, as follows from *Astr. Nachr.*, vol. 13, 1836, p. 353. This idea was first set forth in *Gradmessung in Ostpreussen* by Bessel, who in the preface on pp. V-VI of his work about this degree-measurement mentions Tobias Mayer.

Degree-measurement diagonally to the meridian

If we lay a geodetic line s (or, as the case may be, a triangulation chain) diagonally to the meridian according to Fig. 3, measure at the starting point and at the end point the azimuths α_1, α_2 and the latitudes φ_1, φ_2 and finally the difference of longitude l astronomically, then we have combined now everything which we have so far treated separately as latitude measurement, longitude measurement and transfer of azimuth; and since a diagonal geodetic line with azimuths and latitudes at the end points is sufficient, according to the above, for the determination of the dimensions of the ellipsoid, then we already have in the combination of the six measurements mentioned a determination of the earth's dimensions exceeding the immediate requirement.

We can understand this also thusly: A spherical triangle of the form of Fig. 3 is determined with respect to its form by three elements, e.g., by φ_1, φ_2, l ; in order to determine also the radius on which the spherical triangle shall lie, we need a fourth element, s measured linearly. If we change to an ellipsoid on which the triangle of Fig. 3 shall lie, then a further unknown occurs in the eccentricity, so that five measuring elements become now necessary. If, in Fig. 3, six elements are measured in all, then *one* measurement is in excess even for the ellipsoid, or we have to deal with an adjustment.

We also see at a glance immediately that we can combine as a total adjustment several such systems, as is represented in Fig. 3.

This is the fundamental idea of the international measurement of the earth of today. However, since the deflections of the vertical play an important role here, we will not pursue this problem further here, but return to it in the later Chapter IX.

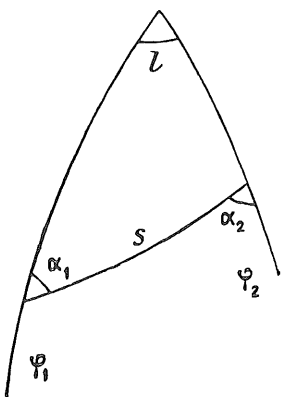


Fig. 3.

THE MATHEMATICAL SHAPE OF THE EARTH AND THE FORCE OF GRAVITY

Section 61. Attraction of a Particle by the Terrestrial Body

For the degree-measurements treated in the preceding chapter, we have taken a flattened ellipsoid of rotation as the immediate basis for the mathematical shape of the earth, and the correctness of this assumption is confirmed by the nearly agreeing results of the various degree-measurements. But before we can pass over to the theory of modern degree-measurements, we must examine more closely the question of the theoretical shape of the earth's surface.

The theoretical surface of the earth, the *geoid*, is represented according to the first half-volume, p. 12, by the surface of the sea at rest and its continuation beneath the continents. Since the surface of the sea at rest is in the equilibrium, then it must lie everywhere perpendicularly to the direction of the force of gravity; therefore, it must be a *level surface*. The force of gravity is the resultant from the force of attraction of the terrestrial body and the centrifugal force generated by the rotation of the earth; therefore, we have to occupy ourselves at first with these two forces and begin with the examination of the attraction of the terrestrial body on a particle lying on the outside.

Let a particle dm of the terrestrial body have the coordinates a, b, c , with respect to a coordinate system rigidly connected with the latter. Let a particle P , whose mass we will assume equal to unity, have the coordinates x, y, z in the same system. The following equation then exists for the distance u between the two points according to Fig. 1:

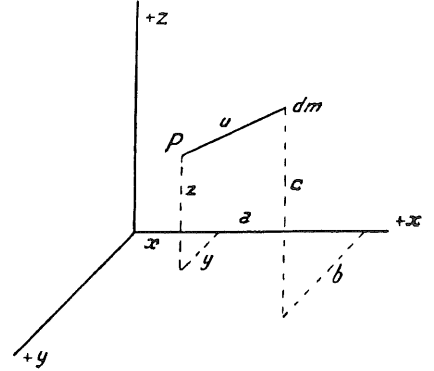


Fig. 1.

$$u^2 = (a - x)^2 + (b - y)^2 + (c - z)^2. \quad (1)$$

The particle dm will now exert on the unit particle at the point P a force of attraction dK , which is represented, according to Newton's law of gravitation, by the expression:

$$dK = f \frac{dm}{u^2} \quad (2)$$

in which f designates the constant of attraction. We divide the force dK into three components by projecting it on the axes of coordinates. If the direction of the force makes with the axes the angles α, β and γ , then we have:

$$\cos \alpha = \frac{a - x}{u} \quad \cos \beta = \frac{b - y}{u} \quad \cos \gamma = \frac{c - z}{u},$$

and the components of the force are:

$$\left. \begin{aligned} dX &= f \frac{dm}{u^2} \frac{a-x}{u} \\ dY &= f \frac{dm}{u^2} \frac{b-y}{u} \\ dZ &= f \frac{dm}{u^2} \frac{c-z}{u} \end{aligned} \right\} \quad (3)$$

We can set these components up for every particle of the earth and the components of the force exerted by the whole terrestrial body on the unit particle at P are then:

$$\left. \begin{aligned} X &= f \int \frac{dm}{u^2} \frac{a-x}{u} \\ Y &= f \int \frac{dm}{u^2} \frac{b-y}{u} \\ Z &= f \int \frac{dm}{u^2} \frac{c-z}{u} \end{aligned} \right\} \quad (4)$$

From equation (1) there follows:

$$u \frac{\partial u}{\partial x} = -(a-x) \quad u \frac{\partial u}{\partial y} = -(b-y) \quad u \frac{\partial u}{\partial z} = -(c-z). \quad (5)$$

We have further:

$$\frac{\partial \frac{1}{u}}{\partial x} = -\frac{1}{u^2} \frac{\partial u}{\partial x},$$

and hence, according to (5):

$$\frac{\partial \frac{1}{u}}{\partial x} = +\frac{1}{u^3} (a-x).$$

Likewise, we also have:

$$\frac{\partial \frac{1}{u}}{\partial y} = \frac{1}{u^3} (b-y).$$

$$\frac{\partial \frac{1}{u}}{\partial z} = \frac{1}{u^3} (c-z).$$

With this, we obtain for the three components of the force of attraction according to (4):

$$X = f \int \frac{\partial \frac{1}{u}}{\partial x} dm \quad Y = f \int \frac{\partial \frac{1}{u}}{\partial y} dm \quad Z = f \int \frac{\partial \frac{1}{u}}{\partial z} dm. \quad (6)$$

Now we introduce a new function by setting:

$$V = f \int \frac{dm}{u}, \quad (7)$$

and call V the *potential* of the attraction of the terrestrial body with respect to point P . From (7) we obtain by differentiation with respect to x , if we take into account (6) at the same time:

$$\frac{\partial V}{\partial x} = f \int \frac{\partial \frac{1}{u}}{\partial x} dm = X.$$

If we form likewise the differential quotients of (7) with respect to y and x , then we obtain together:

$$X = \frac{\partial V}{\partial x} \quad Y = \frac{\partial V}{\partial y} \quad Z = \frac{\partial V}{\partial z}. \quad (8)$$

The centrifugal force. Since the particle P takes part in the rotation of the terrestrial body around its axis, then we still have to take into account the centrifugal force which follows from this rotation. For this, we imagine the system of coordinates, which so far was assumed quite arbitrarily, laid in such a way that the z -axis coincides with the axis of rotation. If we denote the angular velocity of the rotation by ω , the distance of point P from the axis by r , then the centrifugal acceleration for the point P is equal to $r \omega^2$. The projections of this quantity on the axes yield the components of the centrifugal acceleration, which, in the present case, are at the same time the components of the centrifugal force, since the mass imagined at point P is taken as equal to unity. We thus have as components of the centrifugal force the three quantities:

$$x \omega^2 \quad y \omega^2 \quad 0. \quad (9)$$

The resultant of the force of attraction and of the centrifugal force yields the force of gravity g at point P , whose components are therefore:

$$\left. \begin{aligned} g_x &= \frac{\partial V}{\partial x} + \omega^2 x \\ g_y &= \frac{\partial V}{\partial y} + \omega^2 y \\ g_z &= \frac{\partial V}{\partial z} \end{aligned} \right\} \quad (10)$$

Just as we previously obtained a simplification, with equation (7), by application of the function V , we can now introduce a new function by setting:

$$W = V + \frac{1}{2} (x^2 + y^2) \omega^2. \quad (11)$$

Then we have, as we see at once:

$$g_x = \frac{\partial W}{\partial x} \quad g_y = \frac{\partial W}{\partial y} \quad g_z = \frac{\partial W}{\partial z}, \quad (12)$$

and therefore, we can call the function W the potential of the force of gravity, by analogy with equation (7).

All this holds for an arbitrary point P with the coordinates x, y, z . We now imagine, starting from this point, a surface constructed, which is perpendicular everywhere to the direction of the force of gravity, and hence, a *level surface*, and we can then indicate the equation of this surface.

If a linear element ds lying on the surface starts from P , then it must also be perpendicular to the direction of the force of gravity. Let the projections of ds on the three axes be dx, dy, dz , with which ds makes the angles ξ, η, ζ . Then we have:

$$\cos \xi = \frac{dx}{ds} \quad \cos \eta = \frac{dy}{ds} \quad \cos \zeta = \frac{dz}{ds}. \quad (13)$$

On the other hand, if the angles between the direction of the force of gravity and the three axes are denoted by ξ', η', ζ' , then we have:

$$\cos \xi' = \frac{g_x}{g} \quad \cos \eta' = \frac{g_y}{g} \quad \cos \zeta' = \frac{g_z}{g}. \quad (14)$$

Since the angle φ between ds and the direction of the vertical is equal to 90° , then we must have:

$$\cos \varphi = \cos \xi \cos \xi' + \cos \eta \cos \eta' + \cos \zeta \cos \zeta' = 0$$

or, according to (13) and (14):

$$g_x dx + g_y dy + g_z dz = 0$$

With the help of equations (10), this changes to:

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz + (x dx + y dy) \omega^2 = 0. \quad (15)$$

The first three terms of (15) are the total differential dV , while the last term represents the total differential $d \frac{x^2 + y^2}{2} \omega^2$. Instead of (15) we thus can write:

$$dV + d \frac{x^2 + y^2}{2} \omega^2 = 0,$$

or integrating:

$$V + \frac{1}{2} (x^2 + y^2) \omega^2 = \text{Const.} \quad (16)$$

This is the equation of the level surface passing through point P , which, according to (11), we can also write in the form of

$$\mathbb{W} = \text{Const.} \quad (17)$$

It follows hence that on every level surface the potential of gravity has a definite constant value. Of special interest is for us that level surface which coincides with the surface of the sea at rest and which we have designated as the theoretical surface of the earth.

Without examining the shape of the level surface more closely, we can already now say a few things about the properties of this surface.

According to equations (12), the differential quotient of \mathbb{W} in any direction indicates the component of gravity for this direction. Since the force of gravity itself is directed perpendicularly to the level surface, then we have for any arbitrary point of a level surface:

$$\frac{\partial \mathbb{W}}{\partial h} = -g, \quad (18)$$

where ∂h designates a linear element perpendicular to the level surface. The negative sign is introduced here, because ∂h shall be understood in the sense of a change of elevation and \mathbb{W} decreases with the elevation. Of the earth's surface we know from pendulum measurements that on it, gravity is different from place to place; therefore, equation (18) gives us information on the reciprocal position of two neighboring level surfaces. Since for each of these two surfaces the potential \mathbb{W} is equal to a constant, then the difference $\mathbb{W}_2 - \mathbb{W}_1 = d\mathbb{W}$ also is a constant. But, on the other hand, the force of gravity g is variable within each level surface, and therefore, the distance dh of the two surfaces is likewise variable according to (18); the two surfaces thus are not parallel to each other.

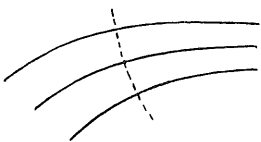


Fig. 2.

If Fig. 2 represents several level surfaces lying above each other, then we see that each plumb line is a curved line whose tangent indicates the direction of the plumb line [direction of the vertical] for any arbitrary point.

Before we continue the development, we will insert a section from the mechanics of rigid bodies, in order to explain some theorems to be applied in the next section.

If in Fig. 1 a force P acts on a point A of a rigid system, then it can be shifted to another point of attack O by applying here two equal and oppositely directed forces P' and $-P'$ parallel to the original force P . The two forces P and $-P'$ together are called a *couple* with the arm e , and the product eP is called the momentum of the pair of forces. We can then replace the original force P by the force P' and the pair of forces eP . While the force P' aims at a displacement of the point system, the pair of forces seeks to bring about a rotation. The couple can be shifted arbitrarily in a parallel way, since nothing changes hereby in its effect.

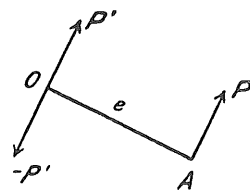


Fig. 1.

We will now assume that point A has the coordinates x, y, z in a rectangular system of coordinates (Fig. 2), and that the forces acting upon it jointly yield the three components X, Y, Z . In order to be able to learn the total effect of all forces on the whole system, it is necessary to shift all forces

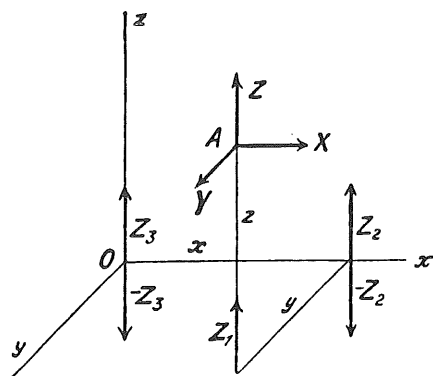


Fig. 2.

to a common point of attack, e.g. the zero point of coordinates, whereby, however, couples are still to be added to the forces. By treating now at first the component Z for the point A further, we shift it to Z_1 , and since a pair of forces does not originate hereby, then Z is replaced completely by Z_1 . But according to the above, we can substitute for Z_1 the force Z_2 as well as the couple $Z_1, -Z_2$ with the moment $yZ_1 = yZ$. If we shift Z_2 again to Z_3 , then a new couple $xZ_2 = xZ$ originates. We thus have found as substitute for the force Z the force Z_3 and the two couples yZ and xZ .

If we treat the components X and Y , likewise, then we obtain two new components X_3 and Y_3 acting on the starting point O , which have the same direction and magnitude as X and Y and further four couples with the moments yX and xX as well as xY and yY .

If we bring the two couples xZ and xX onto the xy -plane by parallel displacement, then we see that they bring about two oppositely directed rotations; their sum is therefore $xZ - xX$. Likewise, we can also combine the couples on the xy -plane and on the yz -plane and then obtain the following result:

The three force components X, Y, Z acting on point A of a rigid system can be replaced by three components equal to them and equally directed, acting on the starting point, and by the three couples:

$$yZ - xY \quad xY - yX \quad xZ - xX.$$

The same consideration holds for all points of the rigid system, and we obtain the total effect of the force on the whole system by forming the sum of the corresponding force components and couples.

If all forces acting on the system shall be in equilibrium, then these six sums must be set equal to zero. The conditions of equilibrium thus are:

$$\left. \begin{aligned} \sum X &= 0 & \sum (yZ - xY) &= 0 \\ \sum Y &= 0 & \sum (xZ - xX) &= 0 \\ \sum Z &= 0 & \sum (xY - yX) &= 0. \end{aligned} \right\} \quad (1)$$

We will apply the above on a body which rotates around an axis. Hereby originate centrifugal forces which act on the individual points of the body. If we lay the x -axis on the axis of rotation, then, according to equation (9), section 61, p. 307, the components of the centrifugal force for a particle are $\omega^2 x$, $\omega^2 y$ and 0; consequently, the following forces act on a particle dm :

$$\left. \begin{aligned} X &= \omega^2 x dm \\ Y &= \omega^2 y dm \\ Z &= 0. \end{aligned} \right\} \quad (2)$$

According to this, the rotating body will then be in equilibrium, if the following equations exist according to (1):

$$\left. \begin{aligned} \omega^2 \int x \, dm &= 0 & \omega^2 \int xy \, dm - \omega^2 \int yx \, dm &= 0 \\ \omega^2 \int y \, dm &= 0 & \omega^2 \int xz \, dm &= 0 \\ & & \omega^2 \int yz \, dm &= 0 \end{aligned} \right\} \quad (3)$$

We have so far assumed the position of the x -axis completely arbitrarily. We will now give this axis such a direction that $\int xy \, dm = 0$. For if we at first assume an arbitrary position x_0, y_0 and rotate these axes by the angle α , then we obtain a system x, y , and we have according to Vol. II, first half-volume, 9th edition, 1931,* p. 156:

$$\left. \begin{aligned} x &= x_0 \cos \alpha - y_0 \sin \alpha \\ y &= x_0 \sin \alpha + y_0 \cos \alpha \end{aligned} \right\} \quad (4)$$

There follows hence:

$$xy = (x_0^2 - y_0^2) \sin \alpha \cos \alpha + x_0 y_0 (\cos^2 \alpha - \sin^2 \alpha)$$

or

$$xy = (x_0^2 - y_0^2) \frac{1}{2} \sin 2\alpha + x_0 y_0 \cos 2\alpha.$$

Consequently,

$$\int xy \, dm = \frac{1}{2} \sin 2\alpha \int (x_0^2 - y_0^2) \, dm + \cos 2\alpha \int x_0 y_0 \, dm.$$

From this equation, α can always be determined in such a way that $\int xy \, dm = 0$.

Further, we can also shift the zero point of coordinates in the z -axis so as to coincide with the center of gravity. The conditions for this are:

$$\int x \, dm = 0 \quad \int y \, dm = 0 \quad \int z \, dm = 0. \quad (5)$$

Since the first two conditions are already satisfied by (3), then we can also satisfy the third equation (5) by a suitable assumption of the zero point on the axis of rotation.

In this connection, we thus have found for a rotating body in equilibrium:

$$\left. \begin{aligned} \int x \, dm &= 0 & \int yx \, dm &= 0 \\ \int y \, dm &= 0 & \int xz \, dm &= 0 \\ \int z \, dm &= 0 & \int xy \, dm &= 0 \end{aligned} \right\} \quad (6)$$

If the three axes of coordinates are laid, as in the present case, in such a way that they satisfy the last three conditions (6), then they coincide with the so-called three *main axes* of the body.

Moment of inertia

By the moment of inertia of a rigid point system with respect to a straight line, we understand the sum of the product of all particles by the square of their distance from the straight line. If the axes of coordinates coincide with the main axes of the body and if we denote the moments of inertia with respect to the three main axes by A, B and C , then we have:

$$\left. \begin{aligned} A &= \int (y^2 + z^2) \, dm \\ B &= \int (x^2 + z^2) \, dm \\ C &= \int (x^2 + y^2) \, dm \end{aligned} \right\} \quad (7)$$

* Not translated.

where the integration refers to all particles of the whole point system, as up to now.

In the following section 63 we shall make use of the above fundamental theorems. It may be mentioned further that in section 61 we have used for the coordinates of the particle dm of the terrestrial body the denotations a, b, c instead of x, y, z of the above formulae, while the coordinates of the attracted point P were denoted by x, y, z . These denotations shall also be retained further.

Section 63. Development in Series for the Potential of Gravity

For the further examination of the level surfaces we now pass to a development in series of the function:

$$W = V + \frac{1}{2} (x^2 + y^2) \omega^2, \quad (1)$$

which we have found in (16), section 61, p. 308, where we will assume that the system of coordinates corresponds to the one introduced in the previous section 62. According to (7), section 61, p. 306, and (1), section 61, p. 305, we have:

$$V = \int \frac{dm}{u} \quad u = \sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2}; \quad (2)$$

therefore, we have to develop the expression:

$$\frac{1}{u} = \frac{1}{(x^2 + y^2 + z^2 + a^2 + b^2 + c^2 - 2ax - 2by - 2cz)^{1/2}}$$

in a series.

If we denote the distance of the particle dm from the zero point by e , and that of point P by r , then we have

$$\frac{1}{u} = \frac{1}{(r^2 + e^2 - 2(ax + by + cz))^{1/2}}, \quad (3)$$

or

$$\frac{1}{u} = \frac{1}{r} \frac{1}{\left(1 + \frac{e^2}{r^2} - \frac{2}{r^2} (ax + by + cz)\right)^{1/2}}. \quad (4)$$

If we apply the development in series to this expression, then the series will converge only as long as $\frac{e}{r} < 1$ or as long as point P lies outside the terrestrial body.

If only the first terms of the development in series are set up, then the neglected higher terms will thus be insignificant only when level surfaces outside the earth are involved; for the surface of the sea itself, from which already considerable masses of the earth project, the neglect of the higher terms, strictly speaking, is no longer admissible.

The coefficients of the development in series are known by the name of *spherical functions*. However, we need not apply these functions here, since we will limit ourselves to the first terms of the series. The following is still to be considered here. If γ is the angle between the two lengths e and r , then we have:

$$u^2 = r^2 + e^2 - 2re \cos \gamma,$$

but on the other hand, according to (3):

$$u^2 = r^2 + e^2 - 2(ax + by + cz);$$

consequently, we have:

$$ax + by + cz = r \cos \gamma, \quad (5)$$

This is to be taken into account in the case of the development in series, in order to recognize the order of the terms. If we limit ourselves to terms in $\frac{e^2}{r^2}$, then we obtain from (4):

$$\frac{1}{u} = \frac{1}{r} \left\{ 1 - \frac{e^2}{2r^2} + \frac{ax + by + cz}{r^2} + \frac{3(ax + by + cz)^2}{2r^4} + \dots \right\},$$

and with this we will have according to (2):

$$V = \frac{f}{r} \int dm + \frac{f}{r^3} \int (ax + by + cz) dm - \frac{f}{2r^3} \int e^2 dm + \frac{3f}{2r^5} \int (ax + by + cz)^2 dm + \dots \quad (6)$$

The value of the individual integrals can be easily indicated, where we must bear in mind that for the integration the quantities x, y, z are constant. If we denote the mass of the whole earth by M , then we have for the first term

$$\int dm = M. \quad (7)$$

We can dissolve the second term into three individual integrals, each of which becomes equal to zero according to (5), section 62, p. 310.

If we replace in the third term e^2 by $a^2 + b^2 + c^2$, then this term changes into three integrals, which we must determine individually. With the help of the three identities:

$$\begin{aligned} a^2 &= \frac{a^2 + b^2 - ((b^2 + c^2) - (a^2 + c^2))}{2} \\ b^2 &= \frac{a^2 + b^2 + ((b^2 + c^2) - (a^2 + c^2))}{2} \\ c^2 &= \frac{-(a^2 + b^2) + ((b^2 + c^2) + (a^2 + c^2))}{2} \end{aligned}$$

and equations (7), section 62, p. 310, in which we must replace everywhere x, y, z by a, b, c , we obtain:

$$\left. \begin{aligned} \int a^2 dm &= \frac{C - (A - B)}{2} \\ \int b^2 dm &= \frac{C + (A - B)}{2} \\ \int c^2 dm &= \frac{-C + (A + B)}{2} \end{aligned} \right\} \quad (8)$$

In the fourth term of (6) we have:

$$\int (ax + by + cz)^2 dm = \int (a^2 x^2 + b^2 y^2 + c^2 z^2 + 2abxy + 2acxz + 2bcyz) dm.$$

In this, the last three terms become equal to zero according to (6), section 62, p. 310, while the first three terms are determined by equation (8).

If we introduce all these in equation (6), then we find:

$$V = f \frac{M}{r} + f \frac{C}{2} \left\{ \left(\frac{3x^2}{2r^5} - \frac{1}{2r^3} \right) + \left(\frac{3y^2}{2r^5} - \frac{1}{2r^3} \right) - \left(\frac{3z^2}{2r^5} - \frac{1}{2r^3} \right) \right\} \\ + f \frac{A-B}{2} \left\{ - \left(\frac{3x^2}{2r^5} - \frac{1}{2r^3} \right) + \left(\frac{3y^2}{2r^5} - \frac{1}{2r^3} \right) \right\} \\ + f \frac{A+B}{2} \left(\frac{3z^2}{2r^5} - \frac{1}{2r^3} \right) + \dots$$

or

$$V = f \frac{M}{r} + f \frac{C}{2} \left\{ \frac{3}{2r^5} (x^2 + y^2 - z^2) - \frac{1}{2r^3} \right\} + f \frac{A-B}{2} \left\{ \frac{3}{2r^5} (y^2 - x^2) \right\} \\ + f \frac{A+B}{2} \left\{ \frac{3z^2}{2r^5} - \frac{1}{2r^3} \right\} + \dots \quad (9)$$

If we take into account further that

$$x^2 + y^2 - z^2 = r^2 - 2z^2,$$

and if we collect the terms with C and $A+B$ and also insert further the term $\frac{1}{2} (x^2 + y^2) \omega^2$, then we have

$$W = f \left\{ \frac{M}{r} + \frac{2C - (A+B)}{2r} \left(\frac{1}{2r^2} - \frac{3z^2}{2r^4} \right) + \frac{3(A-B)}{4r^5} (y^2 - x^2) \right\} \\ + \frac{1}{2} (x^2 + y^2) \omega^2 + \dots \quad (10)$$

In this, we could substitute further for r the expression $\sqrt{x^2 + y^2 + z^2}$ and would then obtain an equation between the constants M, f, A, B and C on the one hand and the variables x, y and z on the other, and hence, the equation of the level surfaces with respect to a rectangular system of coordinates. It is more convenient, however, to change to polar coordinates by determining point P by its geographic longitude and geocentric latitude. As is represented in Fig. 1, the geographic longitude λ is counted in this system from the x -plane so far used. For the change to the system of polar coordinates, we have according to Fig. 1:

$$x = r \cos \psi \cos \lambda \\ y = r \cos \psi \sin \lambda \\ z = r \sin \psi.$$

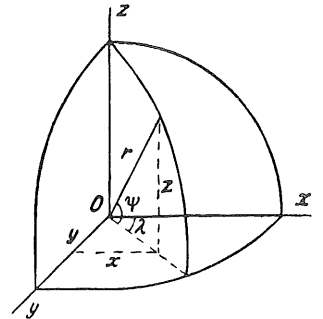


Fig. 1.

If we set further for abbreviation:

$$C - \frac{A+B}{2} = KM \quad B - A = K'M, \quad (11)$$

then equation (10) changes to:

$$W = f \frac{M}{r} \left\{ 1 + \frac{K}{2r^2} (1 - 3 \sin^2 \psi) + \frac{3}{4} \frac{K'}{r^2} \cos^2 \psi \cos 2\lambda \right\} + \frac{\omega^2}{2} r^2 \cos^2 \psi + \dots \quad (12)$$

The development in series of the previous section 63 converges, as we have already noticed on p. 311, only for such level surfaces as are located outside a sphere completely enclosing the physical surface of the earth. Therefore, strictly speaking, only for these level surfaces can the first terms of the development in series, to which we have limited ourselves, lead to a usable result.

Meanwhile, experience teaches that also in the neighborhood of the surface of the sea the terms neglected in our development in series are not very important, so that the terms indicated above in equation (12) of the previous section 63 will be approximately sufficient also for the level surfaces located in the neighborhood of the geoid and for the geoid itself. The above equation (12), section 63, however, will then no longer be able to represent these surfaces accurately in all details, but only in their general form, and we will designate the surfaces corresponding to equation (12), section 63, as *spheroids*. We shall therefore consider the ideal surface of the sea as the *terrestrial spheroid*, in contrast to the geoid. The following examination will be limited to these surfaces, and the value of W , which results from the few terms indicated in (12), section 63, shall be denoted by U . The equation of the spheroid is then:

$$U = f \frac{M}{r} \left\{ 1 + \frac{K}{2r^2} (1 - 3 \sin^2 \varphi) + \frac{3}{4} \frac{K'}{r^2} \cos^2 \varphi \cos 2\lambda \right\} + \frac{\omega^2}{2} r^2 \cos^2 \varphi \quad (1)$$

or in rectangular coordinates according to (10), section 63, p. 313, with the notation of (11), section 63:

$$U = f \frac{M}{r} \left\{ 1 + \frac{K}{2r^4} (r^2 - 3z^2) + \frac{3}{4} \frac{K'}{r^4} (x^2 - y^2) \right\} + \frac{\omega^2}{2} (x^2 + y^2).$$

If we consider the spheroids at first as purely algebraic surfaces, then we obtain from the last equation:

$$\left(U - \frac{\omega^2}{2} (x^2 + y^2) \right) r^5 = f M \left\{ r^4 + \frac{K}{2} (r^2 - 3z^2) + \frac{3}{4} \frac{K'}{4} (x^2 - y^2) \right\},$$

and since

$$r = \sqrt{x^2 + y^2 + z^2}$$

then we must write:

$$\left(U - \frac{\omega^2}{2} (x^2 + y^2) \right)^2 r^{10} = f^2 M^2 \left\{ r^4 + \frac{K}{2} (r^2 - 3z^2) + \frac{3}{4} \frac{K'}{4} (x^2 - y^2) \right\}^2.$$

We see hence that the spheroids are algebraic surfaces of the fourteenth degree, and we will find later on p. 323 that these surfaces deviate very little from ellipsoids of rotation.

The gravity measurements give us information about the various coefficients which occur in the equation of the spheroid. The force of gravity at sea level is dependent on the position of the point on the earth's surface; it can therefore be regarded as a function of the geographic longitude and latitude. But now every such function can be developed, according to spherical functions, in a series whose individual terms, in addition to constant coefficients, contain only the *sine* and *cosine* of the geographic longitude and latitude and of multiples of the geographic latitude and longitude. The coefficients of this development in series can be computed from the measured values of gravity, and it has been found that only a few terms have to be determined with certainty, while the coefficients of the remaining terms can be assumed equal to zero. We shall return to these calculations at the end of this section on p. 319; for our immediate purpose, the statement is sufficient that the force of gravity for any arbitrary point of the sea level can be rendered, to a good approximation, by the equation:

$$g = g_1 (1 + \beta \sin^2 \varphi) \quad (2)$$

in which both coefficients g_1 (the force of gravity at the equator) and β are known from the measurements. We will call the value represented by this equation the *normal* value of gravity, from which the real value will deviate by a small amount dependent on the local position of the point in question on the earth's surface.

According to equation (18), section 61, p. 308, we have:

$$g = -\frac{\partial U}{\partial h},$$

where ∂h lies in the direction of the vertical. The direction of the radius vector r in (1) refers to the geocentric latitude, while the direction of the vertical corresponds to the geographic latitude φ . According to equation (g), first half-volume, section 38, p. 53, we have:

$$\varphi - \psi = \frac{1}{2} e^2 \sin 2 \varphi,$$

and since we have, according to (7), first half-volume, section 37, p. 42:

$$e^2 = 2 a \left(1 - \frac{1}{2} a \right)$$

then we have

$$\varphi - \psi = a \sin 2 \varphi + \dots \quad (2a)$$

A displacement $\frac{\partial r}{\cos(\varphi - \psi)} = \partial r (1 + a^2 \sin^2 2 \varphi + \dots)$ of the point in the direction of the vertical thus corresponds to a displacement ∂r in the direction of r . If we, therefore, neglect terms in a^2 , then we will have:

$$g = -\frac{\partial U}{\partial r}.$$

From (1) we then obtain, if we replace in the last term, $\cos^2 \psi$, by $1 - \sin^2 \psi$:

$$g = \frac{f M}{r^2} \left\{ 1 + \frac{3 K}{2 r^2} - \frac{\omega^2 r^3}{f M} - \left(\frac{9 K}{2 r^2} - \frac{\omega^2 r^3}{f M} \right) \sin^2 \psi + \frac{9 K}{4 r^2} \cos^2 \psi \cos 2 \lambda \right\}. \quad (3)$$

The gravity measurements on the earth's surface have shown that a dependence of gravity on the geographic longitude probably exists; this dependence, however, can so far be determined with very little certainty. Therefore, we will omit the term including λ in g and also in U , whence it follows that we must assume, according to equation (11), section 63, p. 313, that:

$$B = A. \quad (4)$$

The expressions for U and g then read in simplified form:

$$U = \frac{f M}{r} \left\{ 1 + \frac{K}{2 r^2} (1 - 3 \sin^2 \psi) + \frac{\omega^2 r^3}{2 f M} \cos^2 \psi \right\} \quad (5)$$

$$g = \frac{f M}{r^2} \left\{ 1 + \frac{3 K}{2 r^2} - \frac{\omega^2 r^3}{f M} + \left(\frac{\omega^2 r^3}{f M} - \frac{9 K}{2 r^2} \right) \sin^2 \psi \right\}. \quad (6)$$

We will now apply these two equations to the two principal cases $\psi = 0^\circ$ and $\psi = 90^\circ$; for this, r passes into the major and minor semiaxis of the terrestrial spheroid. If we denote these by a and b and the

two corresponding values of gravity by g_1 and g_2 , then we have

$$1. \text{ For } \psi = 0^\circ: \quad U = \frac{f M}{a} \left(1 + \frac{3 K}{2 a^2} + \frac{\omega^2 a^3}{2 f M} \right) \quad (7)$$

$$g_1 = \frac{f M}{a^2} \left(1 + \frac{3 K}{2 a^2} - \frac{\omega^2 a^3}{M f} \right), \quad (8)$$

$$2. \text{ For } \psi = 90^\circ: \quad U = \frac{f M}{b} \left(1 - \frac{K}{b^2} \right) \quad (9)$$

$$g_2 = \frac{f M}{b^2} \left(1 - \frac{3 K}{b^2} \right). \quad (10)$$

If we now introduce the flattening in the ellipsoid of rotation, then we set down, according to equation (2), first half-volume, section 37, p. 41:

$$a = \frac{a-b}{a} \quad \text{or} \quad b = a(1-a).$$

With this, (9) changes to

$$U = \frac{f M}{a(1-a)} \left(1 - \frac{K}{a^2(1-a)^2} \right), \quad (11)$$

and the difference of equations (7) and (9) yields:

$$0 = -\frac{f M}{a} \frac{a}{1-a} + \frac{f M K}{a^3} \left(\frac{1}{2} + \frac{1}{(1-a)^3} \right) + \frac{\omega^2}{2} a^2.$$

This can easily be transformed to:

$$a(1+a) = \frac{3 K}{2 a^2} (1+2a) + \frac{\omega^2 a^3}{2 f M}, \quad (12)$$

for which we can set in the first approximation:

$$a = \frac{3 K}{2 a^2} + \frac{\omega^2 a^3}{2 f M} + \dots \quad (12a)$$

In the same manner, we introduce the flattening in equation (10) and obtain:

$$g_2 = \frac{f M}{a^2} \left\{ 1 + a(2-a) + 4a^2 \right\} \left\{ 1 - \frac{3 K}{a^2} (1+2a) \right\},$$

and if we get rid of the parentheses and limit ourselves to terms in a^2 :

$$g_2 = \frac{f M}{a^2} \left\{ 1 + 2a + 3a^2 - \frac{3 K}{a^2} (1+4a) \right\}. \quad (13)$$

From (8) and (13) we form the expression $\frac{g_1 - g_2}{g_1}$ and find:

$$\frac{g_1 - g_2}{g_1} = \left\{ \frac{9K}{2a^2} + a \frac{12K}{a^2} - a(2 + 3a) - \frac{\omega^2 a^3}{Mf} \right\} \left(1 - \frac{3K}{2a^2} + \frac{\omega^2 a^3}{Mf} \right), \quad (14)$$

For the term $\frac{9K}{2a^2}$ we can set, according to (12):

$$\frac{9K}{2a^2} = 3a(1 + a) - \frac{9K}{a^2}a - \frac{3}{2} \frac{\omega^2 a^3}{fM}$$

and then we obtain from (14):

$$\frac{g_1 - g_2}{g_1} = \left\{ a + a \frac{3K}{a^2} - \frac{5}{2} \frac{\omega^2 a^3}{fM} \right\} \left(1 - \frac{3K}{2a^2} + \frac{\omega^2 a^3}{Mf} \right) \quad (15)$$

or, by neglecting higher terms:

$$\frac{g_1 - g_2}{g_1} = a + a \left(\frac{3K}{2a^2} + \frac{\omega^2 a^3}{fM} \right) - \frac{5}{2} \frac{\omega^2 a^3}{fM}. \quad (16)$$

We can likewise eliminate the term $\frac{3K}{2a^2}$; however, since the term is multiplied by the factor a , the approximate expression (12a) is sufficient here. According to this, we have:

$$\frac{3K}{2a^2} = a - \frac{\omega^2 a^3}{2fM},$$

and hence

$$\frac{g_1 - g_2}{g_1} = a + a^2 + a \frac{\omega^2 a^3}{2fM} - \frac{5}{2} \frac{\omega^2 a^3}{fM}. \quad (17)$$

A further simplification results in the last two terms if we introduce according to (8):

$$g_1 = \frac{fM}{a^2} (1 + \dots) \quad \text{or} \quad \frac{1}{fM} = \frac{1}{g_1 a^2} (1 + \dots) \quad (17a)$$

With this, we then obtain

$$a + \frac{g_2 - g_1}{g_1} = \frac{5}{2} \frac{\omega^2 a}{g_1} - a \left(a + \frac{\omega^2 a}{2g_1} \right). \quad (18)$$

The above simple relation between the flattening of the spheroid, the angular velocity of the earth's rotation and the force of gravity forms Clairaut's extended theorem. Since the angular velocity is known, then we can determine the flattening of the terrestrial spheroid from gravity measurements, with the help of this theorem, if the major semiaxis a is given by other measurements.

Equation (18) was derived in a simpler form, namely without the right-hand term multiplied by a , for the first time by Clairaut in his classical work, *La Figure de la Terre* in 1743, where he starts from the assumption of homogeneous stratification of the earth's mass and delimitation of the layers by coaxial ellipsoids of rotation. Another representation was given by Stokes in 1849 (cf. G. G. Stokes, *Mathematical and physical papers*, Vol. II, Cambridge, 1883, pp. 131-171: On the Variation of Gravity at the Surface of the Earth). In this treatise, Stokes excludes the assumption of a definite surface and merely regards the surface of the earth as a level surface which is approximately an ellipsoid of rotation with a small flattening. In the same manner, the theorem was treated by H. Bruns in *Die Figur der Erde*, Berlin, 1878. Finally,

Clairaut's theorem was extended, by including higher terms, by F. R. Helmert in *Die math. u. phys. Theorien der höheren Geodäsie*, Vol. II, Leipzig, 1884, p. 78. Cf. also F. R. Helmert, "Die Schwerkraft und die Massenverteilung der Erde," *Enzykl. d. math. Wiss.*, Vol. VI, 1 B. Leipzig, 1910.

If the flattening of the terrestrial spheroid is determined, with the help of Clairaut's theorem, then the constants of equation (5), of the surface of the spheroid can also be indicated.

At first, we set up the following term from equation (8), p. 316:

$$\frac{\omega^2 a^3}{Mf} = \frac{\omega^2 a}{g_1} \frac{g_1 a^2}{Mf} = \frac{\omega^2 a}{g_1} \left(1 + \frac{3K}{2a^2} - \frac{\omega^2 a^3}{Mf} \right) \quad (19)$$

or, if in the last term $\frac{\omega^2 a^3}{Mf}$ is replaced by $\frac{\omega^2 a}{g_1}$:

$$\frac{\omega^2 a^3}{Mf} = \frac{\omega^2 a}{g_1} \left(1 + \frac{3K}{2a^2} - \frac{\omega^2 a}{g_1} \right). \quad (20)$$

With this, equation (12) changes to:

$$a(1+a) = \frac{3K}{2a^2} \left(1 + 2a + \frac{\omega^2 a}{2g_1} \right) + \frac{\omega^2 a}{2g_1} \left(1 - \frac{\omega^2 a}{g_1} \right). \quad (21)$$

On the other hand, (18) yields:

$$a(1+a) = \frac{5}{2} \frac{\omega^2 a}{g_1} - a \frac{\omega^2 a}{2g_1} - \frac{g_2 - g_1}{g_1}, \quad (22)$$

and from (21) and (22) we obtain for $\frac{3K}{2a^2}$ the expression:

$$\frac{3K}{2a^2} = \frac{2\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1} - \frac{9}{2} a \frac{\omega^2 a}{g_1} - \frac{1}{2} \left(\frac{\omega^2 a}{g_1} \right)^2 + \frac{g_2 - g_1}{g_1} \left(2a + \frac{\omega^2 a}{2g_1} \right).$$

There follows another simplification for the terms of second order, if we set in them, according to (18),

$$\frac{g_2 - g_1}{g_1} = -a + \frac{5}{2} \frac{\omega^2 a}{g_1} + \dots$$

Then we find:

$$\frac{3K}{2a^2} = \frac{2\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1} - 2a^2 + \frac{3}{4} \left(\frac{\omega^2 a}{g_1} \right)^2. \quad (23)$$

Further, we can at once derive a formula for the computation of Mf from (20): For if we substitute in (20) the value of $\frac{3K}{2a^2}$ from (23), then there follows:

$$Mf = \frac{g_1 a^2}{1 + \frac{\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1} - 2a^2 + \frac{3}{4} \left(\frac{\omega^2 a}{g_1} \right)^2}. \quad (24)$$

Finally, we find from (7) the value of U , so that, with this, all coefficients for equation (5) of the terrestrial spheroid are known, if the flattening α has been determined by gravity measurements and the semiaxis a is given.

Although the methods for the determination of gravity shall be treated only in the following Chapter VIII, the results of gravity measurements may already be applied to the above theories here. We draw from the computations indicated at the end of the section, as well as from the later section 83, the following formula for the normal value of gravity at the surface of the sea, by limiting ourselves to the principal terms:

$$g = 978.052 (1 + 0.005285 \sin^2 \varphi), \quad (25)$$

Thus we will have:

$$g_1 = 978.052 \quad g_2 = 978.052 (1 + 0.005285),$$

and hence,

$$\frac{g_2 - g_1}{g_1} = 0.005285.$$

The numerical values of g or, as the case may be, g_1 and g_2 indicate here the acceleration of the velocity of descent in centimeters for 1 second.

The calculation of the angular velocity ω of the motion of rotation of the earth results from the duration of rotation of the earth, which is equal to 86,164.09 seconds of mean time. Consequently,

$$\omega = \frac{2\pi}{86,164.09}.$$

If we apply to this further the rounded-off value $a = 6,378,400$ m for the major semiaxis of the terrestrial spheroid (cf. section 97), then we obtain:

$$\frac{\omega^2 a}{g_1} = 0.0034 \ 679.$$

With these numerical values, the terms of first order in equation (18), p. 317, yield for α a first approximate value of 0.003385, and with this, we find for the last term in (18) the value of 0.000017.

The second approximation for the flattening, taking into account the last term, is then:

$$\alpha = 0.003368. \quad (26)$$

A further repetition of the calculation no longer gives a variation of this numerical value, which we can also write in the form of

$$\alpha = \frac{1}{296.9}. \quad (27)$$

Helmert carried yet an additional term for the flattening, with which there resulted the value of:

$$\alpha = \frac{1}{296.7} \quad (28)$$

which can be regarded at the present time as the most accurate determination of the flattening from gravity measurements.

In order to form an opinion about the shape of the spheroids, it is useful to set up their equation with respect to the geocentric system of polar coordinates.

From equations (5) and (7), section 64, p. 315 and p. 316, we obtain by division:

$$r = a \left\{ 1 + \frac{K}{2r^2} (1 - 3 \sin^2 \psi) + \frac{\omega^2 r^3}{2fM} (1 - \sin^2 \psi) \right\} \left(1 - \frac{K}{2a^2} - \frac{\omega^2 a^3}{2fM} \right). \quad (1)$$

In this, we have to eliminate on the right-hand side the quantity r , which we will do by means of successive approximation.

In the first approximation there follows from equation (1):

$$r = a + \dots, \quad (2)$$

and if we introduce this in (1) on the right-hand side:

$$r = a \left\{ 1 - \left(\frac{3K}{2a^2} - \frac{\omega^2 a^3}{2fM} \right) \sin^2 \psi \right\} + \dots$$

But according to equation (12a), section 64, p. 316, this is:

$$r = a (1 - a \sin^2 \psi) + \dots, \quad (3)$$

with which we have obtained a second approximation for r . If this is introduced again in (1), then there follows by neglecting terms in a^2 :

$$r = a \left\{ 1 + \frac{K}{2a^2} (1 - 3 \sin^2 \psi) (1 + 2a \sin^2 \psi) + \frac{\omega^2 a^3}{2fM} (1 - \sin^2 \psi) (1 - 3a \sin^2 \psi) \right\} \left(1 - \frac{K}{2a^2} - \frac{\omega^2 a^3}{2fM} \right)$$

or in another arrangement:

$$r = a \left\{ 1 + \frac{K}{2a^2} + \frac{\omega^2 a^3}{2fM} - \left(\frac{3K}{2a^2} + \frac{\omega^2 a^3}{2fM} - \frac{K}{a^2} a + \frac{3\omega^2 a^3}{2fM} a \right) \sin^2 \psi + a \left(\frac{3\omega^2 a^3}{2fM} - \frac{3K}{a^2} \right) \sin^4 \psi \right\} \left(1 - \frac{K}{2a^2} - \frac{\omega^2 a^3}{2fM} \right). \quad (4)$$

Here we recall that the last factor in (4) was substituted for the expression $\frac{1}{1 + \frac{K}{2a^2} + \frac{\omega^2 a^3}{2fM}}$ in the

case of the above division of equations (5) and (7) from section 64, p. 315 and p. 316. If we bear this in mind with respect to the terms free of ψ in (4), then we will have:

$$r = a \left\{ 1 - \left(\frac{3K}{2a^2} + \frac{\omega^2 a^3}{2fM} - \frac{K}{a^2} a + \frac{3\omega^2 a^3}{2fM} a \right) \left(1 - \frac{K}{2a^2} - \frac{\omega^2 a^3}{2fM} \right) \sin^2 \psi + a \left(\frac{3\omega^2 a^3}{2fM} - \frac{3K}{a^2} \right) \sin^4 \psi \right\}. \quad (5)$$

In the last term, the factor $\left(1 - \frac{K}{2a^2} - \frac{\omega^2 a^3}{2fM}\right)$ could at the same time be omitted, whereby we would not exceed the omissions so far permitted.

For further simplification, we will write equation (5) with simplified notation:

$$r = a (1 - A_2 \sin^2 \psi + A_4 \sin^4 \psi), \quad (6)$$

and treat now the coefficients A_2 and A_4 individually. For A_2 we obtain, if we keep the accuracy so far observed,

$$A_2 = \left(\frac{3K}{2a^2} + \frac{\omega^2 a^3}{2fM}\right) \left(1 - \frac{K}{2a^2} - \frac{\omega^2 a^3}{2fM}\right) - a \left(\frac{K}{a^2} - \frac{3\omega^2 a^3}{2fM}\right). \quad (7)$$

According to equation (12), section 64, p. 316, we have for the first factor:

$$\frac{3K}{2a^2} + \frac{\omega^2 a^3}{2fM} = a(1 + a) - \frac{3K}{a^2} a.$$

We can substitute further in the second factor according to (12), section 64, p. 316:

$$-\frac{\omega^2 a^3}{2fM} = \frac{3K}{2a^2} - a + \dots$$

This yields in (7):

$$A_2 = \left(a(1 + a) - \frac{3K}{a^2} a\right) \left(1 + \frac{K}{a^2} - a\right) - a \left(\frac{K}{a^2} - \frac{3\omega^2 a^3}{2fM}\right)$$

or contracting,

$$A_2 = a \left(1 - \frac{3K}{a^2} + \frac{3\omega^2 a^3}{2fM}\right). \quad (8)$$

If we eliminate hence $\frac{3K}{a^2}$ according to (12) or (12a), section 64, p. 316, then we will have:

$$A_2 = a \left(1 - 2a + \frac{5\omega^2 a^3}{2fM}\right),$$

and this is according to Clairaut's theorem (18), section 64, p. 317:

$$A_2 = a \left(1 - a + \frac{g_2 - g_1}{g_1}\right). \quad (9)$$

For the coefficient A_4 of $\sin^4 \psi$ in (6) or (5), as the case may be, we obtain with the help of (8) the equation

$$A_2 = a + A_4;$$

therefore, according to (9):

$$A_4 = a \left(\frac{g_2 - g_1}{g_1} - a\right). \quad (10)$$

The equation of the spheroids thus is:

$$r = a \left\{ 1 - a \left(1 - a + \frac{g_2 - g_1}{g_1} \right) \sin^2 \psi + a \left(\frac{g_2 - g_1}{g_1} - a \right) \sin^4 \psi \right\}. \quad (11)$$

In addition, we will now compare the spheroid with an ellipsoid of rotation, whose semiaxes a and b agree with those of the former. If we denote the radius vector of the ellipsoid by r' , then we have according to (f), first half-volume, section 38, p. 52:

$$r' = a \left\{ 1 - \frac{1}{2} e^2 (1 - e^2) \sin^2 \varphi - \frac{5}{8} e^4 \sin^4 \varphi \right\}.$$

For the introduction of the flattening we take from (7), first half-volume, section 37, p. 42:

$$e^2 = a (2 - a)$$

and obtain:

$$r' = a \left(1 - a \sin^2 \varphi + \frac{5}{2} a^2 \sin^2 \varphi - \frac{5}{2} a^2 \sin^4 \varphi \right) \quad (12)$$

Further, we can set down for the elimination of φ :

$$\sin^2 \varphi = \sin^2 (\psi + (\varphi - \psi)) = \sin^2 \psi + 2 (\varphi - \psi) \sin \psi \cos \psi,$$

and according to (2a), section 64, p. 315, where we can replace forthwith φ by ψ :

$$\sin^2 \varphi = \sin^2 \psi + 4 a \sin^2 \psi \cos^2 \psi$$

or

$$\sin^2 \varphi = \sin^2 \psi + 4 a \sin^2 \psi (1 - \sin^2 \psi).$$

With this, (12) changes to:

$$r' = a \left\{ 1 - a \left(1 + \frac{3}{2} a \right) \sin^2 \psi + \frac{3}{2} a^2 \sin^4 \psi \right\}. \quad (13)$$

The difference $r - r'$ of (11) and (13) gives us the elevation of the spheroid above the ellipsoid of rotation, and we obtain:

$$r - r' = a \left\{ \left(\frac{5}{2} a^2 - \frac{g_2 - g_1}{g_1} a \right) \sin^2 \psi - \left(\frac{5}{2} a^2 - \frac{g_2 - g_1}{g_1} a \right) \sin^4 \psi \right\}$$

or also:

$$r - r' = \frac{1}{4} a \left(\frac{5}{2} a - \frac{g_2 - g_1}{g_1} \right) a \sin^2 2 \psi. \quad (14)$$

In order to be able to calculate the coefficient of $\sin^2 2 \psi$ for the terrestrial spheroid numerically, we take from section 64, p. 319:

$$a = \frac{1}{296.7} \quad \text{and} \quad \frac{g_2 - g_1}{g_1} = 0.00528.$$

If we assume further $a = 6378$ km, then there follows:

$$r - r' = 16.8 \sin^2 2\psi. \quad (15)$$

We see hence that the spheroid encloses the surface of the ellipsoid; the maximum elevation of the spheroid above the ellipsoid amounts at 45° latitude to approximately 16.8 m.

We thus can summarize the developments represented in this chapter to the effect that the terrestrial spheroid or the ideal surface of the sea can be regarded to within quite minor deviations as an ellipsoid of rotation.

Section 66. Determination of the Major Semiaxis of the Terrestrial Ellipsoid from the Parallax of the Moon and from Gravity

In connection with the above theories we treat further a few methods for the determination of the shape of the earth, which result from the astronomical observations of the lunar motion. For the determination of the lunar parallax, i.e. the angle at which the major semiaxis of the terrestrial ellipsoid appears from the center of the moon, two methods are used. One is based on a purely trigonometric foundation, while the other results from the theory of gravitation in connection with the measurements of gravity on the earth's surface. The combination of these two methods was used by Helmert to determine the major semiaxis of the terrestrial ellipsoid.

For the representation of this method we start from Kepler's third law by assuming at first that only the earth and the moon influence each other, and hence, neglecting the attraction of the remaining bodies of the solar system, especially of the sun itself. Kepler's third law, for the derivation of which we refer to the textbooks of theoretical astronomy and celestial mechanics, reads:

$$T^2 = \frac{4\pi^2 R^3}{f(M + m)}. \quad (1)$$

In this, T designates the time of revolution, m the mass of the moon and R the major semiaxis of the moon's orbit, further, M the mass of the earth and f the constant of gravitation.

In expression (1), all quantities are to be regarded as known, with the exception of the constant of gravitation f , which we can express by values of gravity however. According to equation (24), section 64, p. 318, we have, if we omit terms of second order,

$$f = \frac{g_1 a^2}{M \left(1 + \frac{\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1}\right)}. \quad (2)$$

With this, we find for (1) the expression:

$$T^2 = \frac{4\pi^2 R^3 \left(1 + \frac{\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1}\right)}{g_1 a^2 \left(1 + \frac{m}{M}\right)}$$

$$a = \frac{a^3 g_1 \left(1 + \frac{m}{M}\right) T^2}{R^3 4\pi^2 \left(1 + \frac{\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1}\right)} \quad (3)$$

and

But for the lunar parallax p we have the equation

$$\sin p = \frac{a}{R};$$

consequently, we obtain:

$$a = \frac{g_1 \left(1 + \frac{m}{M}\right) T^2 \sin^3 p}{4 \pi^2 \left(1 + \frac{\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1}\right)}. \quad (4)$$

If this equation is to be used for the computation of a , then we can substitute for a an approximate value in the middle term of the denominator.

For the development of this equation we have started from the assumption that the motion of the moon around the earth takes place only under the influence of the mutual attraction. Since this simple motion, however, is disturbed by the remaining celestial bodies, in the first place by the sun, then equation (4) found before is not rigorously correct. Without discussing the theory of the lunar motion more thoroughly, we take the correction to equation (4) from a treatise by Hansen in *Astronomische Nachrichten*, Vol. 17, 1840, No. 403, in which the terms of disturbance are indicated as far as they occur in the lunar parallax. Hansen found that, if we wish to compute the lunar parallax from equation (4), a being known, there is to be inserted further a constant factor 1.006537, apart from small periodic terms, so that the expression for the lunar parallax reads:

$$\sin p = 1.006537 \left\{ \frac{4 a \pi^2 \left(1 + \frac{\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1}\right)}{g_1 \left(1 + \frac{m}{M}\right) T_a^2} \right\}^{\frac{1}{3}}. \quad (5)$$

We have set here the symbol T_a instead of T , because Hansen uses the anomalistic time of revolution of the moon instead of the sidereal time of revolution. If we wish to introduce in (5) the T used so far again, then we must write:

$$\sin p = 1.006537 \left(\frac{T}{T_a}\right)^{\frac{2}{3}} \left\{ \frac{4 a \pi^2 \left(1 + \frac{\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1}\right)}{g_1 \left(1 + \frac{m}{M}\right) T^2} \right\}^{\frac{1}{3}}. \quad (6)$$

Now we have

$$T = 2,360,592 \text{ sec.}$$

$$T_a = 2,380,713 \text{ sec.,}$$

and hence, we will have

$$1.006537 \left(\frac{T}{T_a}\right)^{\frac{2}{3}} = 1.000864.$$

After the determination of this numerical coefficient we return again to equation (3) and obtain with the above coefficient:

$$a = 0.997412 \frac{g_1 \left(1 + \frac{m}{M}\right) T^2 \sin^3 p}{4 \pi^2 \left(1 + \frac{\omega^2 a}{g_1} - \frac{g_2 - g_1}{g_1}\right)}. \quad (7)$$

If the parallax of the moon is determined by the trigonometric method, then, with the help of the above

equation, the equatorial radius of the earth can be computed.

In order to be able to demonstrate such a computation numerically, we put together at first the various quantities occurring in (7).

For the gravity g_1 and the coefficient $\frac{g_2 - g_1}{g_1}$ we use the values found by Helmert and already indicated on p. 319:

$$g_1 = 9.78052 \text{ m} \quad \frac{g_2 - g_1}{g_1} = + 0.005285.$$

The ratio of the masses of the moon and the earth is according to Bauschinger, *Die Bahnbestimmung der Himmelskörper*, Leipzig, 1906, p. 83:

$$\frac{m}{M} = \frac{1}{81.5}; \quad \text{we thus have} \quad \log \left(1 + \frac{m}{M} \right) = 0.005296.$$

For the time of revolution of the moon we use the value already indicated above:

$$T = 2,360,592 \text{ seconds of mean time.}$$

We take further from p. 319:

$$\frac{\alpha \omega^2}{g_1} = 0.0034679. \quad (8)$$

Now we still lack the most important quantity in (7), the parallax of the moon p . Trigonometric determinations of the lunar parallax were for the first time carried out from 1751 to 1753 by Lacaille at the Cape of Good Hope in connection with a series of European observatories and published in the *Memoirs of the Paris Academy* in 1753-1761. (Cf. R. Wolf, *Handb. d. Astr.*, Vol. II, Zürich, 1892, p. 241.) A new working up of these observations was carried out in 1837 by Olufsen in Copenhagen in *Astr. Nachr.*, No. 326. Helmert reports about a further working up of these measurements by Breen and Stone in *Die math. u. phys. Theorien d. höh. Geodäsie*, Leipzig, 1884, p. 464.

Meanwhile, further measurements were taken in 1832-1833 by Henderson at the Cape, which were used, in connection with Greenwich measurements, for the determination of the lunar parallax. The results are given in *Astr. Nachr.*, No. 338.

For the application of the above developed formulae we use new measurements from the beginning of the twentieth century, for which the following publication is available:

"Determination of the Moon's Parallax from Meridian Observations of the Crater Mösting A at the Royal Observatories of Greenwich and the Cape of Good Hope in the Years 1906-1910," *Monthly Notices of the Royal Astronomical Society*, Vol. 71, London, 1911, pp. 526-540.

At the two observatories there took place simultaneous meridian observations of the moon on a total of 100 days. The computations of the moon were based on the lunar places of the *Berliner Astronomisches Jahrbuch*, which rest on Hansen's value of the moon's parallax, and there followed hence the corrections of Hansen's lunar parallax according to the measurements.

The flattening α of the terrestrial ellipsoid plays an important role in these computations, and therefore, the correction of Hansen's lunar parallax is indicated in the result for different values of flattening. From this, we will use two values, namely $1:\alpha = 296.7$ according to Helmert from gravity measurements and a rounded-off comparison value $1:\alpha = 298.0$. Then we have:

$$\begin{aligned} \text{for } 1:\alpha = 296.7 & \quad \text{correction of the lunar parallax} + 0.31'' \\ \text{for } 1:\alpha = 298.0 & \quad \text{correction of the lunar parallax} + 0.24 . \end{aligned}$$

According to H. Battermann (*Beob.-Erg. d. Kgl. Sternw. Berlin*, No. 13, Berlin, 1910, p. 11) the value 3422.28" is to be assumed for Hansen's lunar parallax; therefore, we have the two values 3422.59" and 3422.52" for the lunar parallax according to the above measurements.

If all numerical values are substituted in equation (7), then there follows:

$$\begin{array}{ll} \text{for } 1:a = 296.7 & a = 6,378,743 \text{ m,} \\ \text{for } 1:a = 298.0 & a = 6,378,343 \text{ .} \end{array} \quad (9)$$

Helmert (cf. *Zeitschr. f. Verm.*, 1913, p. 753) finds from a combination of the newer degree-measurements in Europe, Asia, Africa and America, taking into account the different accuracies, a mean value of:

$$a = 6,378,350 \text{ m,}$$

with which the two results (9) agree well.

Section 67. Determination of the Flattening of the Earth from the Irregularities of the Moon's Motion

The observations of the moon's motion furnish a further contribution for the determination of the shape of the earth. The flattening of the terrestrial body causes certain irregularities in the motion of the moon, whose measurement, conversely, renders possible a conclusion as to the flattening itself. For this, we have to consider the attracting effect of the earth on the moon.

For the potential of the attraction of the terrestrial body on a point lying outside the earth, we have found in section 63, p. 313, the expression:

$$V = f \frac{M}{r} \left\{ 1 + \frac{1}{2 r^2} \left(\frac{C - \frac{A+B}{2}}{M} \right) (1 - 3 \sin^2 \psi) + \frac{3}{4 r^2} \frac{B - A}{M} \cos^2 \psi \cos 2 \lambda \right\}. \quad (1)$$

As remarked on p. 315, there follows from the gravity measurements that the last term in (1) is to be neglected, whence there follows at the same time that it is also permitted to set $A = B$. Consequently, (1) changes to the simplified form

$$V = f \frac{M}{r} \left\{ 1 + \frac{1}{2 r^2} \frac{C - A}{M} (1 - 3 \sin^2 \psi) \right\} \quad (2)$$

This potential V also holds for the attraction which the earth exerts on the celestial bodies, where by ψ the declination of the latter is to be understood. But since the difference $C - A$ is small, then the second term in (2) is no longer considered for the planets because of the large distances r . On the other hand, the influence of this second term on the moon's motion is so great that it causes small irregularities which can be determined by observations, and conversely, we can now conclude from the size of the disturbances in the moon's motion as to the amount of the second term in (2). If the quotient $\frac{C - A}{M}$ has thereby become known, then the flattening, on which the difference $C - A$ depends, can also be computed.

We cannot treat in detail the theories of celestial mechanics belonging here and will therefore indicate immediately the terms of disturbance of the moon's motion. According to Tisserand, *Traité de Mécanique Céleste*, Tome II, Paris, 1891, p. 367, the influence of the earth on the longitude of the moon is:

$$\Delta \lambda = + 6540'' \frac{C - A}{M a^2} \sin l \quad (3)$$

and on the latitude of the moon it is

$$\Delta \beta = -7439'' \frac{C-A}{M a^2} \sin l', \quad (4)$$

where l is the mean longitude of the ascending node of the moon's orbit, l' the mean longitude of the moon and a the equatorial radius of the earth.

The latter of these two magnitudes, the disturbance $\Delta \beta$, in the latitude of the moon is to be determined more accurately than the former, and therefore we will now consider $\Delta \beta$ more closely. From the observations by Hansen, the coefficient of $\sin l'$ in (4) is to be assumed equal to $-8.382''$ according to Tissérand; from this, however, there is to be deducted an amount of $0.240''$ which originates from the effect of attraction of the planets. Accordingly, we have:

$$7439'' \frac{C-A}{M a^2} = 8.142''$$

or
$$\frac{C-A}{M a^2} = 0.001\ 0945. \quad (5)$$

For the determination of the flattening from this value taken from the astronomical observations we recall at first that according to (11), section 63, p. 313:

$$\frac{C-A}{M a^2} = \frac{K}{a^2}$$

so that the quotient $\frac{K}{a^2}$ is known from (5).

Further, we have derived from Clairaut's theorem the expression (21), section 64, p. 318:

$$\alpha(1+\alpha) = \frac{3}{2} \frac{K}{a^2} \left(1 + 2\alpha + \frac{\omega^2 a}{2g_1} \right) + \frac{\omega^2 a}{2g_1} \left(1 - \frac{\omega^2 a}{g_1} \right), \quad (6)$$

from which α can be determined.

For the numerical computation we have from (5):

$$\frac{K}{a^2} = 0.001\ 0945,$$

and from section 64, p. 319:

$$\frac{\omega^2 a}{g_1} = 0.003\ 4679,$$

with which we obtain:

$$\alpha(1+\alpha) = 0.003\ 284\ \alpha + 0.003\ 372.$$

By a stepwise solution of this equation we find easily:

$$\alpha = 0.003\ 372 \quad \text{or} \quad \alpha = \frac{1}{296.6}. \quad (7)$$

The certainty of this determination of the flattening depends mainly on the accuracy with which the above indicated constant $8.142''$ has been found. In order to be able to indicate the influence of an

uncertainty of this constant, we have carried out the computation of a once more with the constant $8.242''$ increased by $0.1''$ and found hence the value $a = \frac{1}{294.8}$. Therefore, even if the constant should be uncertain by several tenths of a second, there results hence in the reciprocal value of the flattening an uncertainty of only a few units, whence it becomes clear that this method forms a valuable control of the remaining methods for the determination of the flattening.

Utilization of the constants of precession and nutation. The secular and periodic disturbances in the position of the equator and the ecliptic, designated by the name of precession and nutation, are likewise produced, for the greatest part, by the attracting effect of the sun and the moon on the terrestrial ellipsoid. From the constants of precession and nutation, which are known from astronomical observations, the quotient:

$$\frac{C - A}{C} \quad \text{or} \quad \frac{C - A}{A}$$

can be computed numerically. If we wished to make use of this for the determination of the flattening, then we would have to try at first to compute C or A by introducing plausible hypotheses for the density of the interior of the earth. In this manner, the mass M of the terrestrial body would also have to be determined, with which we would then obtain a value for the magnitude $\frac{K}{a^2}$ which results immediately from the above-mentioned observations of the lunar motion.

Chapter VIII

THE MEASUREMENT OF GRAVITY

Section 68. The Mathematical Pendulum

Since we have recognized in the preceding chapter the importance of the knowledge of gravity at the surface of the earth for the determination of the shape of the earth, we will now consider the means for the measurement of gravity.

The most important method for the measurement of gravity is the observation of the duration of oscillation of a pendulum, and therefore we must first deal with the theory of the pendulum.

By a mathematical pendulum we understand a particle which is suspended on a weightless thread and merely oscillates about a horizontal axis under the influence of gravity.

By means of the force of gravity, a vertically directed acceleration g is assigned to the particle at P in Fig. 1. If ds is the path covered by the particle in the time dt , and v its velocity, then we have

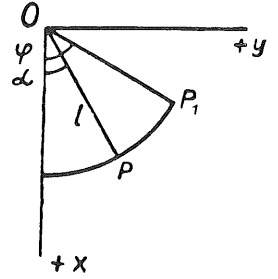


Fig. 1.

$$v = \frac{ds}{dt}. \quad (1)$$

If we denote by w the angular velocity and by l the length of the pendulum, then we also have

$$v = l w. \quad (2)$$

If α is the angle of deflection of the pendulum, the components of the velocity in the direction of the two axes of coordinates then are

$$\left. \begin{aligned} \frac{dx}{dt} &= \frac{ds}{dt} \sin \alpha = v \frac{y}{l} = y w \\ \frac{dy}{dt} &= -\frac{ds}{dt} \cos \alpha = -v \frac{x}{l} = -x w \end{aligned} \right\} \quad (3)$$

There follows hence:

$$y \frac{dx}{dt} - x \frac{dy}{dt} = y^2 w + x^2 w = l^2 w. \quad (4)$$

If we differentiate this equation, then we obtain:

$$\begin{aligned} y \frac{d^2 x}{dt^2} + \frac{dy}{dt} \frac{dx}{dt} - x \frac{d^2 y}{dt^2} - \frac{dx}{dt} \frac{dy}{dt} &= l^2 \frac{dw}{dt} \\ y \frac{d^2 x}{dt^2} - x \frac{d^2 y}{dt^2} &= l^2 \frac{dw}{dt}. \end{aligned} \quad (5)$$

or

In this, $\frac{d^2 x}{dt^2}$ and $\frac{d^2 y}{dt^2}$ are the components of the acceleration in the direction of the two axes. But we have

$$\frac{d^2 x}{dt^2} = g \quad \frac{d^2 y}{dt^2} = 0.$$

Therefore, (5) changes to

$$g y = l^2 \frac{dw}{dt} \quad (6)$$

and since $y = l \sin \alpha$, then we have:

$$g \sin \alpha = l \frac{dw}{dt}. \quad (7)$$

The angular velocity can also be expressed by the angle of deflection α , for we have

$$w = -\frac{d\alpha}{dt} \quad \text{and hence} \quad \frac{dw}{dt} = -\frac{d^2\alpha}{dt^2}.$$

Consequently, we obtain from (7)

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l} \sin \alpha = 0 \quad (8)$$

as the differential equation of motion of the mathematical pendulum.

For the integration we form hence:

$$2 \frac{d^2\alpha}{dt^2} \frac{d\alpha}{dt} dt = -2 \frac{g}{l} \sin \alpha d\alpha,$$

and this integrated yields

$$\left(\frac{d\alpha}{dt}\right)^2 = 2 \frac{g}{l} \cos \alpha + C.$$

If the angle of deflection α reaches its maximum value, the amplitude φ , then the angular velocity is $w = -\frac{d\alpha}{dt} = 0$, and hence, we have in this case

$$0 = 2 \frac{g}{l} \cos \varphi + C,$$

and we obtain

$$\left(\frac{d\alpha}{dt}\right)^2 = 2 \frac{g}{l} (\cos \alpha - \cos \varphi); \quad (8a)$$

therefore,

$$dt = \pm \sqrt{\frac{l}{2g}} \frac{d\alpha}{\sqrt{\cos \alpha - \cos \varphi}}. \quad (9)$$

Since

$$\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2}$$

$$\cos \varphi = 1 - 2 \sin^2 \frac{\varphi}{2}$$

we can bring (9) into the following form:

$$dt = \pm \frac{1}{2} \sqrt{\frac{l}{g}} \frac{d\alpha}{\sqrt{\sin^2 \frac{\varphi}{2} - \sin^2 \frac{\alpha}{2}}} . \quad (10)$$

The integration of this equation between the limits φ and 0 or, as the case may be, 0 and φ gives us the time which the pendulum needs to arrive from the deflection φ to the vertical position or from the vertical position to the deflection φ .

In order to set up the integration, we introduce a new variable by setting:

$$\sin \frac{\alpha}{2} = \sin \frac{\varphi}{2} \sin \rho , \quad (11)$$

then we have

$$\frac{1}{2} \cos \frac{\alpha}{2} d\alpha = \sin \frac{\varphi}{2} \cos \rho d\rho ,$$

and hence

$$d\alpha = \frac{2 \sin \frac{\varphi}{2}}{\cos \frac{\alpha}{2}} \cos \rho d\rho .$$

With this, equation (10) changes to

$$dt = \pm \sqrt{\frac{l}{g}} \frac{\sin \frac{\varphi}{2} \cos \rho d\rho}{\cos \frac{\alpha}{2} \sqrt{\sin^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2} \sin^2 \rho}} = \pm \sqrt{\frac{l}{g}} \frac{d\rho}{\cos \frac{\alpha}{2}}$$

or

$$dt = \pm \sqrt{\frac{l}{g}} \frac{d\rho}{\sqrt{1 - \sin^2 \frac{\varphi}{2} \sin^2 \rho}} . \quad (12)$$

The integral is to be formed within the limits φ and 0. If $\alpha = \varphi$, then we must have according to (11) $\sin \rho = 1$, and hence $\rho = \frac{\pi}{2}$; therefore:

$$t = \pm \sqrt{\frac{l}{g}} \int_{\frac{\pi}{2}}^0 \frac{d\rho}{\sqrt{1 - \sin^2 \frac{\varphi}{2} \sin^2 \rho}} = \mp \sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\rho}{\sqrt{1 - \sin^2 \frac{\varphi}{2} \sin^2 \rho}} . \quad (13)$$

For the computation of the integral we will limit ourselves to a small amplitude φ and then can use a development in series. If we set

$$\sin^2 \frac{\varphi}{2} = e^2 ,$$

then we have according to the first half-volume, section 34, p. 20:

$$\frac{1}{\sqrt{1-e^2 \sin^2 \varrho}} = 1 + \frac{1}{2} e^2 \sin^2 \varrho + \frac{3}{8} e^4 \sin^4 \varrho + \dots$$

But according to the first half-volume, section 35, p. 28, we have:

$$\begin{aligned} \sin^2 \varrho &= \frac{1}{2} - \frac{1}{2} \cos 2\varrho \\ \sin^4 \varrho &= \frac{3}{8} - \frac{1}{2} \cos 2\varrho + \frac{1}{8} \cos 4\varrho \\ &\dots \end{aligned}$$

then,

$$\frac{1}{\sqrt{1-e^2 \sin^2 \varrho}} = 1 + \frac{1}{4} e^2 + \frac{9}{64} e^4 - \left(\frac{1}{4} e^2 + \frac{3}{16} e^4 \right) \cos 2\varrho + \frac{3}{64} e^4 \cos 4\varrho + \dots$$

and the integral from (13) changes to:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{d\varrho}{\sqrt{1-e^2 \sin^2 \varrho}} &= \left(1 + \frac{1}{4} e^2 + \frac{9}{64} e^4 \right) \int_0^{\frac{\pi}{2}} d\varrho - \left(\frac{1}{4} e^2 + \frac{3}{16} e^4 \right) \int_0^{\frac{\pi}{2}} \cos 2\varrho d\varrho \\ &\quad + \frac{3}{64} e^4 \int_0^{\frac{\pi}{2}} \cos 4\varrho d\varrho + \dots \end{aligned} \quad (14)$$

But we have

$$\int \cos 2\varrho d\varrho = \frac{1}{2} \sin 2\varrho \quad \int \cos 4\varrho d\varrho = \frac{1}{4} \sin 4\varrho$$

and

$$\int_0^{\frac{\pi}{2}} \cos 2\varrho d\varrho = 0 \quad \int_0^{\frac{\pi}{2}} \cos 4\varrho d\varrho = 0.$$

Since we have further

$$\int_0^{\frac{\pi}{2}} d\varrho = \frac{\pi}{2}$$

then we obtain:

$$\int_0^{\frac{\pi}{2}} \frac{d\varrho}{\sqrt{1-e^2 \sin^2 \varrho}} = \left(1 + \frac{1}{4} e^2 + \frac{9}{64} e^4 \right) \frac{\pi}{2}.$$

According to (13) we thus will have

$$t = \frac{\pi}{2} \sqrt{\frac{l}{g}} \left(1 + \frac{1}{4} \sin^2 \frac{\varphi}{2} + \frac{9}{64} \sin^4 \frac{\varphi}{2} + \dots \right).$$

The whole oscillation T is equal to $2t$, and hence, we have:

$$T = \pi \sqrt{\frac{l}{g}} \left(1 + \frac{1}{4} \sin^2 \frac{\varphi}{2} + \frac{9}{64} \sin^4 \frac{\varphi}{2} + \dots \right). \quad (15)$$

This equation, in which in practice the term of fourth order can further be neglected, forms the basis of gravity measurements.

For infinitely small amplitudes we also can set:

$$T = \pi \sqrt{\frac{l}{g}}. \quad (16)$$

The measurement of the duration of oscillation always refers, it is true, to finite, though small oscillations; in equation (15), however, we have a means of reducing the duration of oscillation to infinitely small oscillations, so that equation (16) can be used for the computation of gravity.

From (16) we obtain for the gravity the value

$$g = \pi^2 \frac{l}{T^2}. \quad (17)$$

We see from this at the same time what dimension gravity or, more correctly, the acceleration of gravity is. If we assume as unit of length the centimeter, as unit of time the second, then the dimension of g is equal to $\frac{\text{cm}}{\text{sec}^2}$. The unit of gravity is designated as *gal* in honor of Galileo; a thousandth of it is called a milligal or, in abbreviated manner of writing, *mgal*.

Section 69. The Physical Pendulum

Let a rigid body oscillate under the influence of gravity around a horizontal axis passing through O in Fig. 1. We consider a mass particle dm of the body which has the distance r from the axis of rotation and the coordinates x and y . We can apply the developments of the previous section 68 to this mass particle and obtain equation (6) found on p. 330. If we multiply this equation by dm , then we have

$$g y dm = r^2 \frac{dw}{dt} dm. \quad (1)$$

If we wish to integrate this, then we must consider that $\frac{dw}{dt}$ is independent of dm , since it has the same value for all mass particles of the body. Consequently, the integration of equation (1) yields:

$$\frac{dw}{dt} \int r^2 dm = g \int y dm. \quad (2)$$

According to section 62, p. 310, the integral $\int r^2 dm$ is hereby the moment of inertia of the body with regard to the axis O .

We now introduce further the center of gravity S of the body, for which we assume the coordinates x_s and y_s , whereby S however need not lie in the xy -plane. If M is the mass of the body, then we have according to the definition of the center of gravity:

$$\int y dm = y_s M, \quad (3)$$

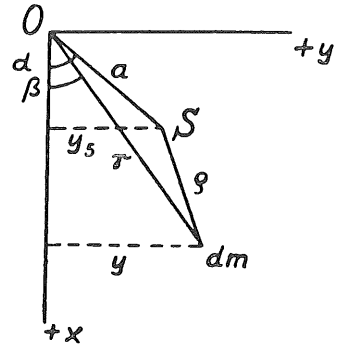


Fig. 1.

therefore, (2) becomes:

$$\frac{dw}{dt} \int r^2 dm = g y_s M. \quad (4)$$

If a is the distance of the center of gravity from the axis of rotation, then we have according to Fig. 1:

$$y_s = a \sin \alpha;$$

consequently:

$$\frac{dw}{dt} \int r^2 dm = g M a \sin \alpha. \quad (5)$$

For the further treatment of the problem we imagine a line parallel to the axis of rotation laid through S . If ρ is the distance of the mass particle dm from this parallel line, then we have

$$\rho^2 = (x - x_s)^2 + (y - y_s)^2.$$

Since we can set further:

$$\begin{aligned} x &= x_s + (x - x_s) \\ y &= y_s + (y - y_s), \end{aligned}$$

then we obtain:

$$\begin{aligned} r^2 &= x^2 + y^2 = x_s^2 + y_s^2 + (x - x_s)^2 + (y - y_s)^2 + 2x_s(x - x_s) + 2y_s(y - y_s) \\ \text{or:} \quad r^2 &= a^2 + \rho^2 + 2x_s(x - x_s) + 2y_s(y - y_s). \end{aligned}$$

With this, the integral becomes:

$$\int r^2 dm = a^2 \int dm + \int \rho^2 dm + 2x_s \int (x - x_s) dm + 2y_s \int (y - y_s) dm. \quad (6)$$

Since $x - x_s$ and $y - y_s$ are the coordinates of dm with respect to the center of gravity S as zero point, then the last two integrals of the previous equation are equal to zero according to the definition of the center of gravity.

The integral

$$\int \rho^2 dm$$

according to section 62, p. 310, is the moment of inertia of the body with respect to the axis assumed through S . We will denote this moment of inertia by $k^2 M$, hence set:

$$\int \rho^2 dm = k^2 M \quad (7)$$

and then obtain from equation (6):

$$\int r^2 dm = (a^2 + k^2) M. \quad (8)$$

Before we introduce this in equation (5), we will further express the angular velocity w by the angle α . We have

$$w = -\frac{d\alpha}{dt}, \quad \text{and hence} \quad \frac{dw}{dt} = -\frac{d^2\alpha}{dt^2}. \quad (9)$$

Now we set (8) and (9) into (5) and obtain after omitting M

$$\frac{d^2\alpha}{dt^2}(a^2 + k^2) = -g a \sin \alpha \quad (10)$$

as the differential equation for the motion of the physical pendulum.

If we set $\frac{a^2 + k^2}{a} = l$, then this equation changes to:

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l} \sin \alpha = 0. \quad (11)$$

This equation is identical with equation (8), section 68, p. 330, which we have found for a mathematical pendulum of the length l .

It follows hence that the physical pendulum considered by us has the same duration of oscillation as a mathematical pendulum whose length is

$$l = \frac{a^2 + k^2}{a} = a + \frac{k^2}{a}. \quad (12)$$

We call l the *reduced length* of the physical pendulum.

If we lay through the center of gravity S a line perpendicular to the axis of rotation, then that point of the perpendicular line which has the distance l from the axis is the *center of oscillation* of the pendulum.

Influence of air resistance

In the foregoing, we have assumed implicitly that the pendulum moves in a vacuum. A body in a fluid has a perpendicularly directed buoyancy which acts on the body at the center of gravity of the displaced fluid mass. This buoyancy causes a loss of weight of the body, which is equal to the weight of the displaced fluid.

Since we have to regard the air surrounding the pendulum likewise as a fluid, we must replace the product $g dm$ in the previous equation (1), p. 333, by $g(dm - dm')$ where dm' designates the mass of the air volume corresponding to the element dm . By analogy to equation (3) we now introduce also the center of gravity S' of the volume of air corresponding to the pendulum. If we denote its ordinate by $y_{s'}$, then we have

$$\int y dm' = y_{s'} M',$$

and we obtain instead of equation (4), p. 334:

$$\frac{dw}{dt} \int r^2 dm = g(y_s M - y_{s'} M'). \quad (13)$$

We will now make the assumption, which always proves correct in the case of pendulum measurements, that the figure of the pendulum is symmetric to the extent that the center of gravity S' of the displaced air

mass lies in the plane which we can pass through the axis of rotation and S . If the distance of the center of gravity S' from the axis is equal to a' , then a' likewise makes the angle α with the direction of the vertical, and we have

$$y_{S'} = a' \sin \alpha,$$

and hence (5), p. 334, changes to

$$\frac{d}{dt} \int r^2 dm = g (a M - a' M') \sin \alpha. \quad (14)$$

The further previous developments remain unchanged, so that we finally obtain instead of (10):

$$\frac{d^2 \alpha}{dt^2} (a^2 + k^2) M = -g (a M - a' M') \sin \alpha. \quad (15)$$

If we now set

$$l = \frac{M (a^2 + k^2)}{a M - a' M'} = \frac{a^2 + k^2}{a \left(1 - \frac{a' M'}{a M}\right)}, \quad (16)$$

then l is the length of the mathematical pendulum oscillating in a vacuum, which has the same duration of oscillation as the physical pendulum oscillating in the air.

In the above equation (16), the influence of the air resistance is taken into account only in part. For it is to be considered that by the motion of the pendulum the air also is put in motion, so that in addition to the mass of the pendulum, the mass of the stirred air is also to be considered. The rigorous taking into account of this influence meets with great difficulties. We can take into account the motion of the air approximately by adding, according to Bessel's suggestion, to the moment of inertia a term which must be determined in an experimental way. If we denote this term by $\kappa^2 M'$, where κ is a quantity dependent on the shape of the pendulum, then we have to replace k^2 by $k^2 + \kappa^2 \frac{M'}{M}$ in equation (16) and then obtain:

$$l = \frac{a^2 + k^2 + \kappa^2 \frac{M'}{M}}{a \left(1 - \frac{a' M'}{a M}\right)}. \quad (17)$$

If this equation is applied to two pendulums of the same figure, but of a different mass, then the coefficient κ^2 can be determined. This method is not rigorously correct; however, experience has taught that it is sufficient in practice.

Stokes (1856) gave a treatment of the problem for definite figures of the pendulum, approaching the actual conditions, which we will however not discuss in greater detail. For the literature concerning this subject we refer to Kühnen und Furtwängler, "Bestimmung der absoluten Grösse der Schwerkraft zu Potsdam mit Reversionspendeln," *Veröff. d. Kgl. Preuss. Geod. Inst.*, Berlin, 1906, p. XII.

Influence of air resistance on the amplitude of the pendulum

A further consequence of the air resistance is the damping effect which the air exerts on the swinging pendulum, and because of which the amplitude of the oscillation gradually decreases. We will base the following treatment of the subject on the assumption of very small amplitudes, which corresponds to the actual conditions in the case of pendulum measurements.

If a mass particle dm turnable around an axis is deflected from its position of rest, then there arises a moment of rotation which tends to bring the mass particle back into the position of rest. There acts counter to it the air resistance, which is a function of the angular velocity and, in the case of a small angle of deflection of the angular velocity, can be assumed proportional. We will therefore express the air resistance by a term $-akw dm$ in which a has the meaning so far used while k is a constant coefficient, which must however not be mistaken for the constant k^2 of the moment of inertia. If we insert this term $-akw dm$ on the left-hand side of equation (1), p. 333, then, finally, the term $+akw$ or $-ak \frac{d\alpha}{dt}$ is added on the right-hand side in (10), p. 335.

Consequently, the complete differential equation for the motion of the physical pendulum reads now for small deflections according to (10) and (12), p. 335:

$$\frac{d^2 \alpha}{dt^2} + \frac{k}{l} \frac{d\alpha}{dt} + \frac{g}{l} \alpha = 0. \quad (18)$$

For the integration of this homogeneous differential equation of second order we set experimentally

$$\alpha = e^{rt}, \quad (19)$$

where r is a still unknown constant. Then we have

$$\frac{d\alpha}{dt} = r e^{rt} \quad \text{and} \quad \frac{d^2 \alpha}{dt^2} = r^2 e^{rt}.$$

This substituted in (18) yields:

$$r^2 e^{rt} + \frac{k}{l} r e^{rt} + \frac{g}{l} e^{rt} = 0.$$

Therefore, if the value (19) is to satisfy equation (18), then r must be determined in such a way that

$$r^2 + \frac{k}{l} r + \frac{g}{l} = 0$$

There follows hence

$$r = -\frac{k}{2l} \pm \frac{1}{2l} \sqrt{k^2 - 4gl}.$$

Since k is a very small quantity, then r will become imaginary, and we write therefore:

$$r = -\frac{k}{2l} \pm i \sqrt{\frac{g}{l} - \frac{k^2}{4l^2}}, \quad \text{where } i = \sqrt{-1}.$$

If we set further for abbreviation:

$$\frac{k}{2l} = z \quad \sqrt{\frac{g}{l} - z^2} = u, \quad (20)$$

then we will have

$$r = -z \pm iu.$$

According to (19), the two values:

$$e^{(-z + i u) t} \quad \text{and} \quad e^{(-z - i u) t} \quad (21)$$

will thus satisfy equation (18); consequently, they will be two particular integrals of this equation.

By multiplication with arbitrary factors and addition, an arbitrary number of other particular integrals can be derived therefrom, e.g.:

$$\begin{aligned} \frac{1}{2} (e^{(-z + i u) t} + e^{(-z - i u) t}) &= e^{-z t} \cos u t \\ \frac{1}{2i} (e^{(-z + i u) t} - e^{(-z - i u) t}) &= e^{-z t} \sin u t. \end{aligned}$$

Let these two new particular integrals of equation (18) be denoted by $\varphi(t)$ and $\psi(t)$, so that we have thusly:

$$\varphi(t) = e^{-z t} \cos u t \quad \psi(t) = e^{-z t} \sin u t. \quad (22)$$

If we denote by c_1 and c_2 two arbitrary constants, then the general integral of (18) is:

$$\alpha = e^{-z t} (c_1 \cos u t + c_2 \sin u t). \quad (23)$$

The two constants c_1 and c_2 can easily be indicated. For if the pendulum has at the beginning of the motion, and hence for $t = 0$, the deflection $\alpha = \varphi_0$, then there follows $c_1 = \varphi_0$. If the starting velocity for $t = 0$ is likewise equal to zero, and therefore $\left(\frac{d\alpha}{dt}\right)_{t=0} = 0$, then we find by differentiation of (23) $c_2 = 0$; consequently, we have, if the value of u from (20) is substituted at the same time:

$$\alpha = \varphi_0 e^{-z t} \cos \sqrt{\frac{g}{l} - z^2} t. \quad (24)$$

The deflection reaches its largest value, the amplitude of the oscillation, if

$$\cos \sqrt{\frac{g}{l} - z^2} t = 1$$

the amplitude φ after the time t thus becomes:

$$\varphi = \varphi_0 e^{-z t}, \quad (25)$$

if φ_0 is the starting amplitude.

The factor z occurring in (25) is designated as the damping factor.

The reversion pendulum

If, through the center of oscillation of a physical pendulum defined on p. 335, we imagine a straight line laid which is parallel to the axis of rotation, then we can consider this straight line as a new axis of rotation. Since the distance of the center of gravity from the new axis is equal to $l - a$, then the length of

the mathematical pendulum, which corresponds to the physical pendulum swinging around the new axis, is according to (12), p. 335:

$$l' = l - a + \frac{k^2}{l - a}.$$

But since we have likewise according to (12):

$$l - a = \frac{k^2}{a} \quad \text{or} \quad \frac{k^2}{l - a} = a,$$

then we will have

$$l' = l - a + a = l. \quad (26)$$

Conversely, if we can thus find, in the case of a physical pendulum, two parallel axes, which lie in the same plane as the center of gravity, and around which the pendulum describes oscillations of equal duration, then the interval of the two axes is equal to the length of the corresponding mathematical pendulum. We designate a pendulum with two axes of oscillation, constructed according to this principle, as a reversion pendulum.

Since it is technically difficult to make the distance between the two axes precisely equal to l , then this condition will in practice be satisfied only approximately, a case which we will now consider further.

Let there be present in a physical pendulum two parallel axes, lying in the same plane as the center of gravity, and whose distances from the center of gravity are equal to a_1 and a_2 . If the pendulum moves around the first axis, then the length of the corresponding mathematical pendulum is according to (12), p. 335:

$$l_1 = a_1 + \frac{k^2}{a_1}, \quad (27)$$

where the duration of oscillation is equal to T_1 .

For the oscillations around the second axis we have accordingly:

$$l_2 = a_2 + \frac{k^2}{a_2} \quad (28)$$

and the duration of oscillation T_2 .

We can now compute from this the duration of oscillation T for a mathematical pendulum whose length is equal to the distance $a_1 + a_2$ between the two axes. Since according to (15), section 68, p. 333, the lengths of two mathematical pendulums in the case of equal amplitude are proportional to the squares of the times of oscillation, then we have:

$$\frac{a_1 + a_2}{l_1} = \frac{T^2}{T_1^2} \quad \text{and} \quad \frac{a_1 + a_2}{l_2} = \frac{T^2}{T_2^2}. \quad (29)$$

If we substitute here the above values of l_1 and l_2 , then there follows:

$$\begin{aligned} (a_1 + a_2) T_1^2 &= a_1 T^2 + \frac{k^2 T^2}{a_1} \\ (a_1 + a_2) T_2^2 &= a_2 T^2 + \frac{k^2 T^2}{a_2} \\ \text{or:} \quad (a_1 + a_2) (a_1 T_1^2 - a_2 T_2^2) &= (a_1^2 - a_2^2) T^2, \\ \text{and we find hence:} \quad T^2 &= \frac{a_1 T_1^2 - a_2 T_2^2}{a_1 - a_2}. \end{aligned} \quad (30)$$

For the computation of T , the knowledge of the distances of the center of gravity from the axes of oscillation is therefore still needed.

Influence of air resistance in the case of the reversion pendulum

Equation (17) found above puts us in a position to determine the influence of the air resistance also for the reversion pendulum. Instead of the two equations (27) and (28) we have then according to (17):

$$\text{and} \quad \left. \begin{aligned} l_1 (a_1 M - a_1' M') &= k^2 M + \kappa^2 M' + a_1^2 M \\ l_2 (a_2 M - a_2' M') &= k^2 M + \kappa^2 M' + a_2^2 M \end{aligned} \right\} \quad (31)$$

If we now limit ourselves to such reversion pendulums, which from their outward form are exactly symmetrical, then the center of gravity of the displaced air mass is at the same distance from the two axes, so that:

$$a_1' = a_2' = a'.$$

With this, we obtain from the two equations (31) by subtraction:

$$(l_1 a_1 - l_2 a_2) M - (l_1 - l_2) a' M' = (a_1^2 - a_2^2) M.$$

But according to the two equations (29) we have:

$$l_1 = (a_1 + a_2) \frac{T_1^2}{T^2} \quad \text{and} \quad l_2 = (a_1 + a_2) \frac{T_2^2}{T^2},$$

and hence, we obtain:

$$\frac{a_1 + a_2}{T^2} (a_1 T_1^2 - a_2 T_2^2) M - \frac{a_1 + a_2}{T^2} (T_1^2 - T_2^2) a' M' = (a_1^2 - a_2^2) M,$$

and therefrom:

$$T^2 = \frac{a_1 T_1^2 - a_2 T_2^2}{a_1 - a_2} - \frac{(T_1^2 - T_2^2) a' M'}{(a_1 - a_2) M}. \quad (32)$$

In the case of the symmetrical reversion pendulum, the knowledge of the factor k^2 and of the two quantities a_1' and a_2' is therefore not necessary.

With regard to the practical execution of the pendulum measurements, we will in addition consider the case in which a physical pendulum does not swing around a straight line, but rolls with a cylindrical axis on a plane in the case of the oscillations. Let the arc of a circle BB in Fig. 1 represent a part of the cylinder with which the pendulum is rigidly connected. Let S be the center of gravity of the pendulum, so that the latter at rest lies on the axis passing through O .

If the pendulum has a deflection α , then it touches the plane with the axis passing through C , and if it carries out from here an infinitely small oscillation $d\alpha$, then this axis can be regarded as the instantaneous axis of oscillation of the pendulum. To the infinitely small oscillation $d\alpha$ we can then apply the differential equation found in equation (10), section 69, p. 335, for the physical pendulum and have thus:

$$\frac{d^2 \alpha'}{dt^2} (a'^2 + k^2) M = -g M a' \sin \alpha'. \quad (1)$$

In this, we must replace a' and α' by a and α .

At first, we can substitute immediately $d\alpha$ for $d\alpha'$ since for the infinitely small oscillation $d\alpha$ the points O , C and S can be considered as rigidly connected.

We have further according to Fig. 1:

$$\begin{aligned} a' \cos \alpha' &= a \cos \alpha - p \cos \gamma \\ a' \sin \alpha' &= a \sin \alpha + p \sin \gamma, \end{aligned}$$

and there follows hence:

$$a'^2 = a^2 + p^2 - 2ap \cos (\alpha + \gamma).$$

For equation (1) we thus obtain:

$$\frac{d^2 \alpha}{dt^2} (a^2 + k^2 + p^2 - 2ap \cos (\alpha + \gamma)) M = -g M (a \sin \alpha + p \sin \gamma). \quad (2)$$

The two quantities p and γ , which still appear in (2), can be expressed by the constants of the pendulum and the deflection α . In order to be able to set up the necessary relations, the points O and C are

once more represented in Fig. 2. At the same time, we introduce the two systems of coordinates x, y and ξ, η with the zero points K and O , where K is assumed arbitrarily. If we denote the coordinates of the two points O and C by x and y or, as the case may be, ξ and η , then we have

$$\left. \begin{aligned} x &= \eta \sin \alpha - \xi \cos \alpha \\ y &= s - \eta \cos \alpha - \xi \sin \alpha, \end{aligned} \right\} \quad (3)$$

where $s = CK$.

Further we take from Fig. 2 immediately:

$$\left. \begin{aligned} p \sin \gamma &= s - \gamma & p \cos \gamma &= x \\ p \cos (\alpha + \gamma) &= -\xi, \end{aligned} \right\} \quad (4)$$

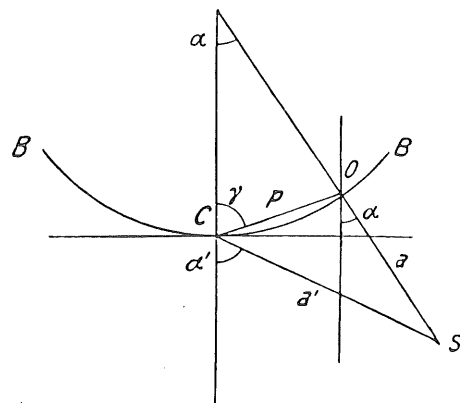


Fig. 1.

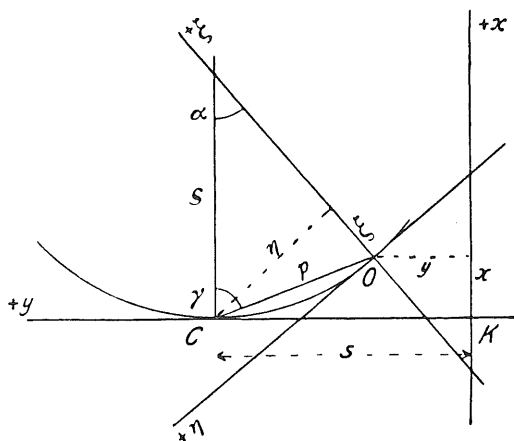


Fig. 2.

or with the help of (3):

$$\left. \begin{aligned} p \sin \gamma &= \eta \cos \alpha + \xi \sin \alpha \\ p \cos \gamma &= \eta \sin \alpha - \xi \cos \alpha \\ p \cos (\alpha + \gamma) &= -\xi \end{aligned} \right\} \quad (5)$$

From Fig. 2 there also follows the relation:

$$p^2 = \xi^2 + \eta^2. \quad (6)$$

We introduce this in (2), whereby we find:

$$\frac{d^2 \alpha}{d t^2} (a^2 + k^2 + \xi^2 + \eta^2 + 2 a \xi) M = -g M (a \sin \alpha + \eta \cos \alpha + \xi \sin \alpha). \quad (7)$$

In addition, if we denote the radius of the cylinder by ρ , then the equation of the circle in the system ξ, η is:

$$\xi^2 + \eta^2 = 2 \rho \xi,$$

and since we also have

$$\eta = \rho \sin \alpha \quad \xi = \rho - \rho \cos \alpha$$

then (7) changes to

$$\frac{d^2 \alpha}{d t^2} (a^2 + k^2 + 2 \rho (a + \rho) (1 - \cos \alpha)) M = -g M (a + \rho) \sin \alpha. \quad (8)$$

This is the differential equation for the oscillation of the pendulum around a cylindrical axis.

This equation is not convenient for integrating. But since only small amplitudes α always occur in the case of pendulum measurements, then we can substitute for $\cos \alpha$ the development in series and obtain:

$$1 - \cos \alpha = \frac{\alpha^2}{2} + \dots$$

Besides, if we limit ourselves to very small values of ρ , which is all that occurs in practice, then the last term on the left-hand side in (8) is very small in proportion to the first two terms, for which reason we can neglect it without a noticeable error. With this omission, equation (8) assumes the form:

$$\frac{d^2 \alpha}{d t^2} (a^2 + k^2) M = -g M (a + \rho) \sin \alpha. \quad (9)$$

This equation differs from the previous equation (10) in section 69, p. 335, only by the quantity ρ . If we pursue the previous further development by inserting ρ , then there follows for the length l of the corresponding mathematical pendulum the equation:

$$\frac{a + \rho}{a^2 + k^2} = \frac{1}{l}$$

$$l = \frac{a^2 + k^2}{a \left(1 + \frac{\rho}{a} \right)},$$

or

for which we can also set:

$$l = \left(a + \frac{k^2}{a} \right) \left(1 - \frac{\rho}{a} \right), \quad (10)$$

since ρ is assumed as very small.

Application to the reversion pendulum

With the help of the above equation (10) we can also treat the theory of the reversion pendulum swinging around two cylindrical axes. We denote the radii of the two cylinders by ρ_1 and ρ_2 ; besides, we retain the previous designations, so that the distances of the cylinders from the center of gravity are equal to a_1 and a_2 .

To the oscillations around the two cylinders there correspond then the two mathematical pendulums:

$$\left. \begin{aligned} l_1 &= \left(a_1 + \frac{k^2}{a_1} \right) \left(1 - \frac{\rho_1}{a_1} \right) \\ l_2 &= \left(a_2 + \frac{k^2}{a_2} \right) \left(1 - \frac{\rho_2}{a_2} \right) \end{aligned} \right\} \quad (11)$$

and since we have for infinitely small oscillations quite generally:

$$T^2 = \frac{\pi^2}{g} l,$$

then

$$\left. \begin{aligned} T_1^2 &= \frac{\pi^2}{g} \left(a_1 + \frac{k^2}{a_1} \right) \left(1 - \frac{\rho_1}{a_1} \right) \\ T_2^2 &= \frac{\pi^2}{g} \left(a_2 + \frac{k^2}{a_2} \right) \left(1 - \frac{\rho_2}{a_2} \right) \end{aligned} \right\} \quad (12)$$

If we compute hence the quantity T^2 according to equation (30), section 69, p. 339, then we have:

$$T^2 = \frac{a_1 T_1^2 - a_2 T_2^2}{a_1 - a_2} = \frac{\pi^2}{g(a_1 - a_2)} \left(a_1^2 - a_2^2 - \frac{a_1^2 + k^2}{a_1} \rho_1 + \frac{a_2^2 + k^2}{a_2} \rho_2 \right).$$

In the two terms multiplied by the small quantities ρ_1 and ρ_2 we can substitute

$$l_1 = \frac{a_1^2 + k^2}{a_1} + \dots \quad l_2 = \frac{a_2^2 + k^2}{a_2} + \dots$$

and obtain then:

$$T^2 = \frac{\pi^2}{g} \left(a_1 + a_2 - \frac{l_1 \rho_1 - l_2 \rho_2}{a_1 - a_2} \right).$$

Now since the reversion pendulum will always be constructed in such a way that the times of oscillation T_1 and T_2 nearly agree, then we also have approximately:

$$l_1 = l_2 = a_1 + a_2,$$

and there follows:

$$T^2 = \frac{\pi^2}{g} (a_1 + a_2) \left(1 - \frac{\rho_1 - \rho_2}{a_1 - a_2} \right). \quad (13)$$

To the time of oscillation T there thus corresponds a mathematical pendulum of length:

$$(a_1 + a_2) \left(1 - \frac{\rho_1 - \rho_2}{a_1 - a_2} \right).$$

We now imagine the reversion pendulum arranged in such a way that the two cylinders of oscillation can be exchanged with one another. If the measurement is repeated after the exchange, then there follow the two values T_1' and T_2' and therefrom:

$$T_1'^2 = \frac{a_1 T_1'^2 - a_2 T_2'^2}{a_1 - a_2}.$$

We have further in correspondence with (13):

$$T'^2 = \frac{\pi^2}{g} (a_1 + a_2) \left(1 - \frac{\rho_2 - \rho_1}{a_1 - a_2} \right). \quad (14)$$

From the two equations (13) and (14) we find:

$$\frac{T^2 + T'^2}{2} = \frac{\pi^2}{g} (a_1 + a_2). \quad (15)$$

To the mathematical pendulum of the length $a_1 + a_2$ there thus corresponds the time of oscillation

$$V \sqrt{\frac{T^2 + T'^2}{2}}.$$

The above examination was carried out for the first time by F. W. Bessel in his fundamental treatise, *Untersuchungen über die Länge des einfachen Sekundenpendels*, Berlin, 1828. Bessel treats the subject in a general form by substituting for the cross section of the knife-edge a conic section. Then the above important result is found that in the case of the reversion pendulum we can eliminate the influence of the form of the knife-edge by exchanging the knife-edges.

Although we take care from the outset, in the case of pendulum measurements, to make the suspension of the pendulum as stable as possible, a slight elastic co-vibration of the suspension console or of the support will be unavoidable through the oscillatory motion of the pendulum. The co-vibration causes a change of the duration of oscillation, so that the length of the mathematical pendulum computed from the observed duration of oscillation does not correspond to the physical pendulum.

While the mass particle dm in Fig. 1, section 69, p. 333, covers the path ds , we assume that the axis O moves in the direction of the y -axis by the length $-d\sigma$. Consequently, to the velocity component $\frac{dy}{dt}$ there is to be added a further term $-\frac{d\sigma}{dt}$, and the second equation (3), section 68, p. 329, thus reads more completely:

$$\frac{dy}{dt} = -v \frac{x}{r} - \frac{d\sigma}{dt} = -xw - \frac{d\sigma}{dt}. \quad (1)$$

If we pursue, with this, the development of p. 329 further, then the following equation takes now the place of equation (2), section 69, p. 333:

$$\frac{dw}{dt} \int r^2 dm = g \int y dm - \frac{d^2\sigma}{dt^2} \int x dm, \quad (2)$$

and integrating:

$$\frac{dw}{dt} \int r^2 dm = g y_s M - \frac{d^2\sigma}{dt^2} x_s M. \quad (3)$$

According to Fig. 1, p. 333, we have

$$y_s = a \sin \alpha \quad x_s = a \cos \alpha, \quad (4)$$

and obtain:

$$\frac{dw}{dt} \int r^2 dm = g M a \sin \alpha - \frac{d^2\sigma}{dt^2} M a \cos \alpha. \quad (5)$$

The further development, in correspondence with equation (10), p. 335, leads to the equation:

$$\frac{d^2\alpha}{dt^2} (a^2 + k^2) M = -g M a \sin \alpha + \frac{d^2\sigma}{dt^2} M a \cos \alpha, \quad (6)$$

which represents the differential equation of the motion of the pendulum taking into account the co-vibration of the support.

If l is the length of the mathematical pendulum corresponding to the physical pendulum under consideration, without the co-vibration of the support, then we have according to (12), p. 335:

$$l = \frac{a^2 + k^2}{a},$$

and hence, we have

$$\frac{d^2\alpha}{dt^2} l = -g \sin \alpha + \frac{d^2\sigma}{dt^2} \cos \alpha. \quad (7)$$

Now we pass over to the determination of the dynamic effect of the swinging pendulum on the support. According to equation (1), the acceleration of the mass particle dm in the direction of the y -axis results in:

$$\frac{d^2 y}{d t^2} = -x \frac{d w}{d t} - w \frac{d x}{d t} - \frac{d^2 \sigma}{d t^2},$$

or according to (3), section 68, p. 329:

$$\frac{d^2 y}{d t^2} = -x \frac{d w}{d t} - y w^2 - \frac{d^2 \sigma}{d t^2},$$

and the force which the mass particle $d m$ exerts in the horizontal direction is therefore:

$$\frac{d^2 y}{d t^2} d m = -x \frac{d w}{d t} d m - y w^2 d m - \frac{d^2 \sigma}{d t^2} d m. \quad (8)$$

By integration we obtain therefrom the force component Y of the pendulum:

$$Y = \int \frac{d^2 y}{d t^2} d m = -\frac{d w}{d t} x_s M - w^2 y_s M - M \frac{d^2 \sigma}{d t^2}.$$

Since the last term on the right-hand side is very small here in proportion to the two preceding terms, and we only deal with an approximate consideration of the co-vibration, then we can neglect the term $M \frac{d^2 \sigma}{d t^2}$. At the same time, we will substitute further according to Fig. 1, section 69, p. 333:

$$x_s = a \cos \alpha \quad \text{and} \quad y_s = a \sin \alpha,$$

as well as according to (9), p. 335:

$$w = -\frac{d \alpha}{d t} \quad \text{and} \quad \frac{d w}{d t} = -\frac{d^2 \alpha}{d t^2},$$

whereby we obtain:

$$Y = a \cos \alpha M \frac{d^2 \alpha}{d t^2} + a \sin \alpha M \left(\frac{d \alpha}{d t} \right)^2. \quad (9)$$

But there follows from (10), p. 335:

$$\frac{d^2 \alpha}{d t^2} = -g \frac{a \sin \alpha}{a^2 + k^2} = -\frac{g}{l} \sin \alpha \quad (10)$$

and from (8a), p. 330:

$$\left(\frac{d \alpha}{d t} \right)^2 = \frac{2 g a}{a^2 + k^2} (\cos \alpha - \cos \varphi) = \frac{2 g}{l} (\cos \alpha - \cos \varphi). \quad (11)$$

Since α and φ are small angles, then we can in addition simplify equation (11) by development in series by substituting:

$$\begin{aligned} \cos \alpha &= \sqrt{1 - \sin^2 \alpha} = 1 - \frac{1}{2} \sin^2 \alpha + \dots \\ \cos \varphi &= \sqrt{1 - \sin^2 \varphi} = 1 - \frac{1}{2} \sin^2 \varphi + \dots \end{aligned}$$

Then we have

$$\cos \alpha - \cos \varphi = \frac{1}{2} (\sin^2 \varphi - \sin^2 \alpha),$$

therefore,

$$\left(\frac{d\alpha}{dt} \right)^2 = \frac{g}{l} (\sin^2 \varphi - \sin^2 \alpha). \quad (12)$$

By introducing (10) and (12) in (9) we obtain:

$$Y = \frac{ag}{l} M (-\sin \alpha \cos \alpha + \sin \alpha \sin^2 \varphi - \sin^3 \alpha),$$

and if we neglect terms of third order with respect to α or φ , then we have

$$Y = -\frac{ag}{l} M \alpha. \quad (13)$$

The elasticity of the support will act counter to the horizontal pressure of the pendulum, and in each moment the tension produced by the elasticity will be equal to the pressure. If the axis of oscillation of the pendulum is displaced by the amount $-\sigma$ at the time t , and if ε denotes a constant depending on the elasticity of the support, then the tension is to be assumed approximately equal to $-\varepsilon\sigma$, and if equilibrium exists, then we must have:

$$\varepsilon\sigma + \frac{ag}{l} M \alpha = 0$$

or

$$\sigma = -\frac{ag}{\varepsilon l} M \alpha \quad (14)$$

for which we will write in simplified notation:

$$\sigma = -\gamma \alpha \quad \text{where} \quad \gamma = \frac{agM}{\varepsilon l}. \quad (14a)$$

We can substitute this expression in the differential equation (7), p. 345, and obtain:

$$\frac{d^2 \alpha}{dt^2} l = -g \sin \alpha - \frac{ag}{\varepsilon l} M \cos \alpha \frac{d^2 \alpha}{dt^2}$$

or

$$\frac{d^2 \alpha}{dt^2} l \left(1 + \frac{ag}{\varepsilon l^2} M \cos \alpha \right) = -g \sin \alpha.$$

We can also set here $\cos \alpha = 1 - \dots$ and write in simplified form:

$$\frac{d^2 \alpha}{dt^2} l \left(1 + \frac{ag}{\varepsilon l^2} M \right) = -g \sin \alpha. \quad (15)$$

If we compare this expression with (10), p. 335, where we must also imagine l substituted for $\frac{a^2 + k^2}{a}$, then we find for the length of the mathematical pendulum, which has the same duration of oscillation as the physical pendulum with co-vibrating support, the value:

$$l' = l \left(1 + \frac{ag}{\varepsilon l^2} M \right) = l + \gamma. \quad (16)$$

The length of the mathematical pendulum thus will always be greater for the pendulum disturbed by the co-vibration than for the pendulum swinging around a fixed axis.

Application to the reversion pendulum

According to the above equation (16), the length of the mathematical pendulum is changed by the co-vibration of the support by the amount of

$$\Delta l = \frac{M g a}{\varepsilon l}.$$

Since for the duration of oscillation of the mathematical pendulum there holds the equation:

$$T^2 = \frac{\pi^2}{g} l,$$

then by the co-vibration there follows for T^2 a change of:

$$\Delta(T^2) = \frac{\pi^2}{g} \Delta l = \pi^2 \frac{M a}{\varepsilon l}. \quad (17)$$

If for the reversion pendulum the times of oscillation for the two knife-edges are equal to T_1 and T_2 , then, according to (30), section 69, p. 339, the duration of oscillation for a mathematical pendulum, whose length is equal to the distance between the knife-edges $a_1 + a_2$, is:

$$T^2 = \frac{a_1 T_1^2 - a_2 T_2^2}{a_1 - a_2}.$$

Since the two times of oscillation T_1 and T_2 are found too large because of the co-vibration, then the corrected times of oscillation are $T_1 - \Delta T_1$ and $T_2 - \Delta T_2$, and therefore, $T - \Delta T$ is the corrected duration of oscillation for the mathematical pendulum of length $a_1 + a_2$. We have

$$T^2 - \Delta(T^2) = \frac{a_1 T_1^2 - a_2 T_2^2 - a_1 \Delta(T_1^2) + a_2 \Delta(T_2^2)}{a_1 - a_2},$$

and if we substitute here $\Delta(T_1^2)$ and $\Delta(T_2^2)$ according to (17), then we obtain:

$$\Delta(T^2) = \frac{\pi^2 M}{\varepsilon} \left(\frac{a_1^2}{l_1} - \frac{a_2^2}{l_2} \right) \frac{1}{a_1 - a_2}. \quad (18)$$

But since we have approximately:

$$l_1 = l_2 = a_1 + a_2,$$

then (18) changes to

$$\Delta(T^2) = \frac{\pi^2 M}{\varepsilon}. \quad (19)$$

In addition, we will now determine the change in the length of the second pendulum resulting therefrom. If we denote the latter by L and set $a_1 + a_2 = l$, then we have

$$L = \frac{l}{T^2 - \Delta(T^2)},$$

$$L = \frac{l}{T^2} + \frac{l}{T^4} \Delta(T^2),$$

therefore,

for which we obtain according to (19):

$$L = \frac{l}{T^2} + \frac{l}{T^4} \frac{\pi^2 M}{\varepsilon}. \quad (20)$$

Now since we can set in the second term:

$$\frac{1}{T^2} = \frac{L}{l} \quad \text{and also} \quad \frac{1}{T^2} = \frac{g}{\pi^2 l}$$

then we obtain:

$$L = \frac{l}{T^2} + \frac{L}{l} \frac{M g}{\varepsilon} \quad (21)$$

as the true length of the second pendulum.

For the elimination of the unknown ε Defforges suggested using two pendulums of different length, but of equal weight, which swing with the same knife-edges on the same support. If the distances between the knife-edges of these two pendulums are equal to l_1 and l_2 , and if there follow two preliminary values L_1 and L_2 , from the oscillations for the second pendulum, then we have according to (21):

$$\left. \begin{aligned} L &= L_1 - \frac{L_1 M g}{l_1 \varepsilon} \\ L &= L_2 - \frac{L_2 M g}{l_2 \varepsilon} \end{aligned} \right\} \quad (22)$$

whence there follows easily:

$$L = \frac{L_1 L_2 (l_1 - l_2)}{l_1 L_2 - l_2 L_1}. \quad (23)$$

This equation can be simplified for practical use, since L_1 and L_2 differ only little from one another. Therefore, we set:

$$L_1 = L_2 + (L_1 - L_2) \quad (24)$$

and then obtain from (23):

$$L = \frac{L_1 L_2 (l_1 - l_2)}{L_2 (l_1 - l_2) - l_2 (L_1 - L_2)}.$$

If this expression is developed in a series according to Taylor's theorem, then, neglecting the terms of second and higher order, there follows:

or

$$L = \frac{L_1 L_2 (l_1 - l_2)}{L_2 (l_1 - l_2)} + \frac{L_1 L_2 (l_1 - l_2)}{L_2^2 (l_1 - l_2)^2} l_2 (L_1 - L_2) + \dots$$

$$L = L_1 + \frac{L_1 (L_1 - L_2)}{L_2 (l_1 - l_2)} l_2. \quad (25)$$

Since we have according to (24):

$$\frac{L_1}{L_2} = 1 + \frac{L_1 - L_2}{L_2}$$

then we can also replace equation (25) by:

$$L = L_1 + \frac{L_1 - L_2}{l_1 - l_2} l_2 \quad (26)$$

since terms of second order shall remain neglected. We can at once conclude therefrom that in order to apply the method of Defforges, two pendulums are necessary of length different as much as possible, where the knife-edge distance of the shorter pendulum must at the same time be as small as possible.

The two equations (16) and (20) were set up for the first time in 1875 by Peirce whose attention had been drawn by Baeyer to the co-vibration of the support and who, on the occasion of his pendulum measurements for the Coast Survey in America, examined theoretically the influence of the co-vibration. The development of the above basic equations is printed in *Bericht über die Verhandlungen der 5. allgemeinen Konferenz der Europäischen Gradmessung in Stuttgart 1877*, pp. 171-187.

For the theory of the pendulum treated in the preceding sections 68-71 we have to mention as the most important work: F. R. Helmert, *Beiträge zur Theorie des Reversionspendels*, Potsdam, 1898. Of additional literature we add:

Giuseppe Lorenzoni, *Relazione sulle esperienze istituite nel R. Osservatorio Astronomico di Padova per determinare la lunghezza del pendolo semplice a secondi*. Roma, 1888. — Defforges, "Observations du pendule," *Mémoires du Dépôt Général de la Guerre*, Tome XV, Paris, 1894. — F. Kühnen und Ph. Furtwängler, *Bestimmung der absoluten Grösse der Schwerkraft zu Potsdam mit Reversionspendeln*, Berlin, 1906.

We shall treat the determination of the constants of co-vibration in the later section 74.

Section 72. Older Pendulum Measurements

For infinitesimal amplitudes of oscillation the duration of oscillation of a mathematical pendulum is represented by the equation

$$T = \pi \sqrt{\frac{l}{g}}.$$

If the length l of the pendulum is known and the duration of oscillation T is observed, then a simple means for the determination of the gravity g follows from the above equation.

The first measurements of this kind were carried out in 1644 by Mersenne, in 1647 by Riccioli and with greater accuracy by Picard on the occasion of his degree-measurement in the years 1669 and 1670. The latter measurements had at the same time the purpose of determining the unit of length through the length of the second pendulum considered as the natural measure.

The apparatuses used in these experiments were very simple; the pendulum consisted of a lead or copper sphere of a diameter of about 1 in., which was suspended from a thread. The upper end of the thread was held firm by metal tongs fastened to a wall. The length of the pendulum was considered to be the distance of the center of the sphere from the lower surface of the tongs. In the case of Picard, the determination of the

duration of oscillation was carried out by regulating the length of the pendulum by shortening or lengthening the thread so long until the oscillations agreed precisely with those of the pendulum of a clock going according to mean time.

Further measurements of the seventeenth and eighteenth century, e.g. by Bouguer and Condamine in connection with the degree-measurement in Peru in 1735, and by Boscovich in 1785 among others, led gradually to the perfecting of the pendulum apparatus and the methods of observation. For the measurement of the duration of oscillation there were observed those moments at which the pendulum and the clock pendulum reached simultaneously the greatest deflection. From the time elapsed between two such observations there resulted the duration of oscillation of the pendulum. Another method used by Bradley in Greenwich in the years 1743-1749 consisted in the determination of the moment at which the passage of the pendulum thread through the position of rest coincided with a second beat of the watch.

We find all developments in the field of pendulum measurements until the end of the eighteenth century utilized in the papers by Borda and Cassini, which are regarded as the first exact determinations of the length of the second pendulum, and therefore, we will discuss in greater detail the methods used by these two French scholars.

The pendulum apparatus of Borda and Cassini

In connection with the great French degree-measurement by Delambre and Méchain there also was decided upon the determination of the length of the second pendulum, operations which Borda and Cassini carried out. The description of these measurements is contained in the work: *Base du Système métrique décimal*, Tome III, Paris, 1810, pp. 337-401, *Expériences pour connaître le longueur du pendule, qui bat les secondes à Paris par M. M. Borda et Cassini*.

The observations took place in a room of the Paris observatory, where the pendulum apparatus was fastened to a strong isolated wall. Onto the wall there was put a stone block hewn out in the shape of a fork to receive the suspension device. For the suspension of the pendulum there was used a copper plate (Fig. 1), which was fastened onto the stone block by three screws; with the latter, the smoothly ground-off surface of the steel frame MN on the copper plate could at the same time be set exactly horizontal. Over the center of the frame there rested the prism OP carrying the pendulum during the oscillations. For the measurement of the length of the pendulum, the knife-edge was shifted to QR while at OP the auxiliary scale, yet to be mentioned, was suspended.

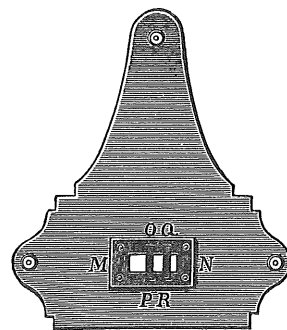


Fig. 1.

The pendulum itself is illustrated in Fig. 2. The lower edge of the steel prism AB forms the axis of oscillation of the pendulum. The thread is fastened to a stem in whose extension a rod with a movable weight G projects above from the prism. This suspension device forms by itself a small physical pendulum, and this can be regulated by moving of the sliding weight G , in such a manner that it possesses the same duration of oscillation as the thread pendulum. It can easily be proved that in this case the oscillations of the thread are quite independent of the suspension device.

For the suspension of the body of the pendulum, a thin iron wire of 12-ft length was used so that the pendulum had a duration of oscillation of two seconds. As the pendulum body a platinum ball with diameter 36.5 mm and weight 526 g was used. To the thread a small spherical cap [calotte] was fastened in whose spherical surface a thin layer of tallow was inserted, which was sufficient in order to hold the ball fast. In this manner, the ball could be suspended in every possible way in order to adjust irregularities of density in the interior.

Fig. 2 on p. 352 also shows an auxiliary arrangement for the measurement of the length of the pendulum. For this, under the pendulum there was fastened a console to the wall, and on it there was a stand with a small table, whose height was adjustable; it was adjusted in such a way that it just touched the ball at the bottom. The measurement of the distance of the knife-edge from the surface of the table took place by means of a platinum scale, Fig. 3, which, in conjunction with a copper scale, formed a metal thermometer. With the help of the T-shaped head whose lower surface was smoothly ground off, the scale instead of the pendulum could be hung on the steel frame. From the lower end of the scale a slide was drawn down as far as the

contact with the table surface and its position read off at a Vernier scale. A second Vernier scale was used for reading off the metal thermometer.

The duration of oscillation was measured by Borda and Cassini for the first time according to the so-called "coincidence method." Behind the pendulum there was attached to the wall a pendulum clock; as

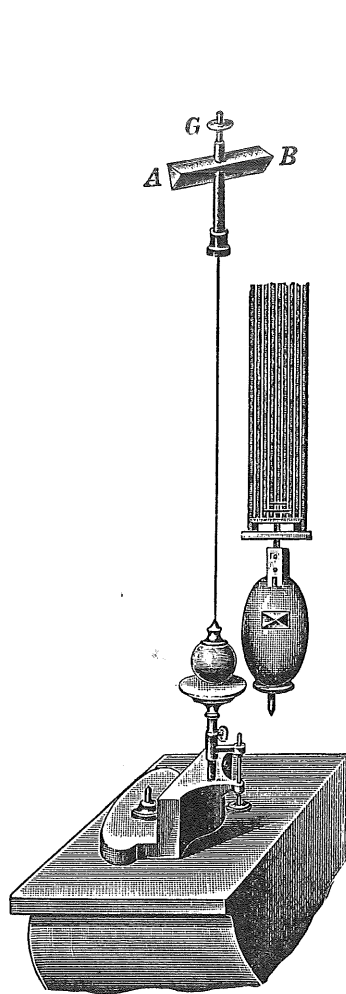


Fig. 2.

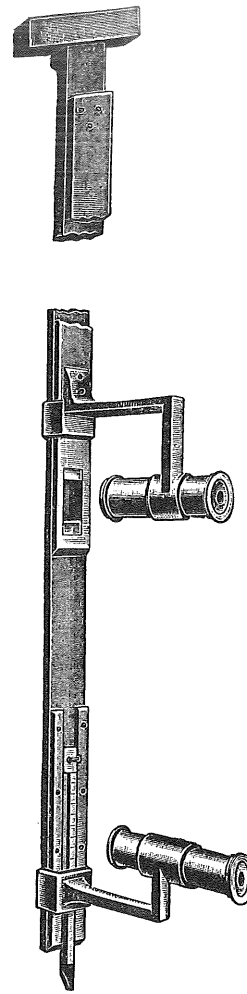


Fig. 3.

Fig. 2, above, shows the pendulum lens of the clock carried on a small sheet of paper a white diagonal cross on a dark background. At some distance before the apparatus there stood an observation telescope in whose field of vision the thread of the pendulum covered exactly the center of the cross when the two pendulums were in the position of rest. Besides, there was set up directly before the apparatus a screen with a perpendicular edge which, in case the pendulums were at rest, passed in the telescope through the center of the thread and the cross. In the telescope there now was observed the moment at which the thread and the cross center vanished at the same time behind the screen. Since between two such coincidences the pendulum clock carried out exactly two oscillations more or less than the thread pendulum, then the ratio of the two times of oscillation could be computed therefrom.

Since that time this coincidence method has been applied in the case of all pendulum measurements; later in section 73 we shall discuss it more thoroughly.

With this apparatus, 20 determinations of the length of the second pendulum were carried out by Borda and Cassini in Paris during the time from 15 June to 4 August 1792. Later Biot and Arago used the same apparatus for pendulum measurements at several points of the meridional arc extending from the Shetland Islands to Fermentera, whereby the steel plate *MN* in Fig. 1, p. 351, was replaced however by an agate plate.

These latter measurements were reported in Biot and Arago, *Recueil d'observations géodésiques, astronomiques et physiques*, Paris, 1821, pp. 441-588.

The pendulum measurements of Kater

Already in 1792 when the just described measurements by Borda were at issue, the French physicist de Prony proposed instead of the thread pendulum a rigid pendulum of rather large weight which could swing around several solid parallel axes lying in the same plane as the center of gravity. De Prony proved that the measurement of the duration of oscillation around three such axes was sufficient to compute the moment of inertia of the pendulum with respect to the center of gravity and, therefore, also the lengths of the three corresponding mathematical pendulums. The construction, however, was considered too complicated, and the proposal did not meet with approval. Later de Prony remodeled his apparatus so that it agreed with the actual reversion pendulum.

In 1811 the idea of the reversion pendulum was proposed likewise by Bohnenberger, astronomer of Württemberg, without being materialized, however.

Finally, in 1817, the English physicist Kater repeated this proposal once again without knowing anything about his predecessors. The construction was carried out by Kater, so that he determined for the first time the length of the second pendulum with a reversion pendulum.

The measurements are described in the treatise, "An Account of Experiments for Determining the Length of the Pendulum Vibrating Seconds in the Latitude of London," by Capt. Henry Kater, F. R. S., in *Philosophical Transactions of the Royal Society of London for the Year 1818*, Part I, pp. 33-102.

Kater's pendulum, which is illustrated in Fig. 4, consists in its principal part of a brass bar, about 135 cm long, with a rectangular cross section, approximately 4 cm wide and 3 mm thick, in which two rectangular sectors with an interval of about 1 m are provided. At the ends, at both sides of the bar, lashes of hammered brass are screwed on, which begin at the sectors and project about 5 cm over the ends. These lashes have the same width as the bar and are about 2 cm strong; at the sectors they are bent outwardly into right angles and ground off in such a way that they offer a plane surface for placing the prisms, which carry the knife-edges. As a continuation of the brass bar there is attached at each end a wooden bar, about 43 cm long, between the projecting lashes; the wooden bars are half as wide as the brass bar and painted black. At the extreme ends each carries a point of fishbone with which the amplitude of the oscillation can be read off on a graduated arc.

The prisms with the knife-edges are made of steel and are about 4.5 cm long; they lean with their backs against the angles, formed by the lashes, at which they are attached by screws.

There are three weights on the pendulum. A cylindrical brass weight of about 950 g is perforated according to the cross section of the pendulum and connected at one end on the two lashes. A second smaller weight of 230 g is screwed onto the brass bar itself between the knife-edges. The third weight of about 125 g is placed as a slide on the bar approximately at its center and can be adjusted by means of a fine movement. To facilitate the adjustment, the bar is provided at this place with a graduation which becomes visible by a sector in the slide.

In the observation room, the pendulum was suspended in a niche directly before the pendulum clock. For this purpose, above the clock there was attached a horizontal strong wooden board in whose sector the pendulum could be suspended. This board carried a metal piece cut out in the shape of a fork, which was covered at both sides of the sector with agate plates for carrying the knife-edge. A special suspension device made it possible to lay the knife-edge always at the same place on the agate plates.

The measurement of the duration of oscillation was carried out likewise by observing the coincidences of the oscillating pendulum and of the clock pendulum. At the lens of the latter, there was attached on a dark background a white circular disc whose diameter agreed with the width of the wooden bars fastened to the pendulum. At some distance from the apparatus, an observation telescope was set up in such a way that in the case of resting pendulums, the white circle on the pendulum lens was completely covered by the wooden bar. Besides, there was placed in the image plane a screen which left free only a perpendicular slit of the apparent width of the



Fig. 4.

wooden bar and of the white circle; the oscillating pendulums could thus be observed only upon their passage through the position of rest. The duration of oscillation resulted then from the observation of those times at which both pendulums passed through the position of rest at the same time.

The measuring procedure now consisted in determining at first the duration of oscillation for the oscillations around one knife-edge and then for the oscillations around the other knife-edge. To eliminate the difference hereby showing, the slide was shifted with the help of the fine movement and the measurement repeated. By continuation of this procedure there was finally obtained exactly the same duration of oscillation for both knife-edges. The thus determined, final, duration of oscillation, according to section 69, p. 339, corresponds then to the mathematical pendulum, whose length is equal to the interval between the knife-edges.

For the measurement of the interval between the knife-edges, the pendulum was compared with a length scale on a comparator device. A peculiar phenomenon was hereby observed. For, if the background of the knife-edges was bright, then the distance between the knife-edges was found larger with the screw-microscopes than in the case of a dark background. Since the cause of this phenomenon could not be detected, then one accepted the arithmetic mean from both measurements as the true distance between the knife-edges.

Bessel's pendulum apparatus

The activities of Bessel in Königsberg from the years 1826 and 1827, in which he sought to overcome by new means the difficulties found in the case of the measurements hitherto carried out, meant a further important step in the development of pendulum measurements.

The suspension of the pendulum with the help of knife-edges is not so unobjectionable as it appears at first sight, since the mechanic will never succeed in producing the knife-edges as true mathematical straight lines, and the deviations exert a considerable influence. The effect of these irregularities is the same as if instead of the knife-edge there were a cylinder which rolls on the bearing plane in the case of the oscillations of the pendulum (cf. section 70, p. 343).

A second source of error lies in the precise measurement of the distance between the knife-edges in the case of the reversion pendulum, as was already indicated above.

Bessel therefore returned again to the thread pendulum. He avoided the difficulty of the suspension of the pendulum by determining the periods of oscillation of two homogeneous pendulums of a known difference of length. For, if there hold for the two pendulums the equations

$$T_1 = \pi \sqrt{\frac{l_1}{g}} \quad \text{and} \quad T_2 = \pi \sqrt{\frac{l_2}{g}},$$

then we have

$$T_1^2 - T_2^2 = \frac{\pi^2}{g} (l_1 - l_2),$$

and hence,

$$g = \pi^2 \frac{l_1 - l_2}{T_1^2 - T_2^2}.$$

Therefore, one can only determine the gravity from the difference of the lengths of the pendulums and the observed periods of oscillation T_1 and T_2 .

Bessel eliminated the errors in the determination of the difference of length $l_1 - l_2$ by making this quantity exactly equal to the length of a toise scale.

The apparatus used by Bessel, which is still found today at its original place in the observatory of Königsberg, is illustrated in Fig. 5. As bearer of the whole apparatus there is used a framework prepared of mahogany wood, which is fastened to the wall above and below by iron bolts; in Fig. 5, only the upper fastening is visible. In the framework there is suspended above an iron rod, about 3.20 m long, 10 cm wide and 9 mm thick, which is adjusted exactly vertically with the help of a plumb and held fast by screws at the lower end. On the iron rod, at about a third of its height, there is placed a small steel cylinder, which rests below on a small console and above has the form of a truncated cone.

Since the apparatus is used for two pendulums of different lengths, then the suspension device of the

pendulum is placed either directly on the small cylinder, or, as Fig. 5 shows, on the upper end of a toise-bar which is borne by the cylinder. If the pendulum ball remains thereby in both cases at the same height, then the difference of length of the two pendulums is exactly equal to the length of the toise.

The toise is held upright by weak lateral springs. In its center there is clamped fast a hull which is borne by two levers. The counterweights are adjusted in such a way that they carry the toise straight; in this manner, a shortening of the rod, which would be likely to happen, is avoided by the pressure of its weight, since the lower half is just as much lengthened as the upper is shortened.

The suspension device which carries the thread pendulum rests on one of the two bearings provided for this purpose and on the upper end of the toise or directly on the steel cylinder. For the explanation of details, the suspension device is reproduced once again at a larger scale in Fig. 6. It consists in its principal part *A* of a *T*-shaped frame whose crosspiece carries below two sectors for the mounting on a cylinder *B*

lying in the bearing *Q*. By means of the screw *S*, with the help of a striding level, the cylinder *B* can be set horizontal. Under the other end of the frame there is attached a second cylinder *C* which carries in the front a cylindrical apex and in the back a spherical head, with which it rests on the toise *K*. There is attached to the frame an arm directed obliquely above, in which the thread of the pendulum is clamped fast, whereby the clamp can be shifted a little by a finely moving screw. For the regulation of the pressure on the toise there is further a counterweight *G*.

The steel thread of the pendulum runs around the cylindrical apex of the cylinder *C*; in its oscillations, the pendulum therefore does not describe a circle but the unwinding of a circle. The ball of the pendulum has a diameter of approximately 52 mm and is made of brass; besides, there was used by Bessel an ivory ball of the same size for the determination of the atmospheric influence.

In order to be able to examine the position of elevation of the ball, there is placed under the latter, in a box (Fig. 5), a steel cylinder, which can be moved by a screw arrangement from the outside; the revolutions of the screw are used at the same time for the measuring of the movement of the steel cylinder. The latter, however, is not moved upward as far as the contact with the ball; there is rather inserted further a sensitive nipper-lever,

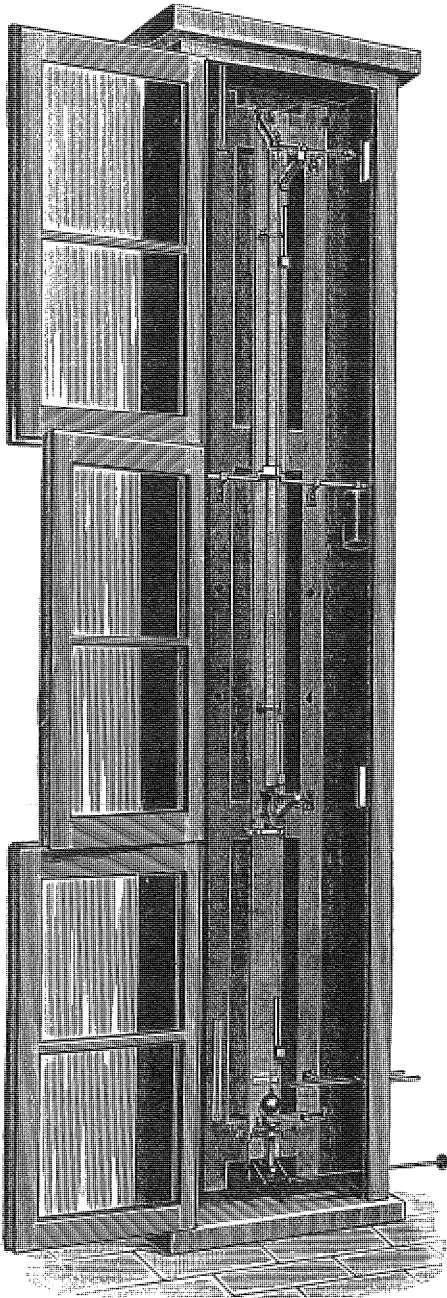


Fig. 5.

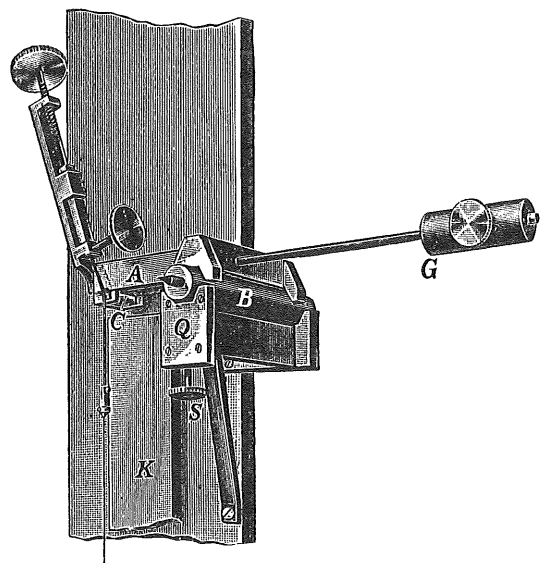


Fig. 6.

whereby the adjustment of the cylinder becomes much more rigorous.

Above the ball there is placed a device with which the pendulum can be put in motion, while a scale is used for reading off the amplitudes of oscillation. Finally, there are still to be mentioned several thermometers, which indicate the temperature of the rod as well as the temperature of the air.

The whole apparatus is closed by a casing with mirror-glass windows, from which only the handles project.

For the measurement of the duration of oscillation, Bessel likewise made use of the coincidence method. In order to avoid, however, any reciprocal influence of the clock pendulum and the thread pendulum, the clock was attached about 2.5 m in front of the pendulum apparatus. Between the two, a telescope was set up from which the ocular was removed, so that the image of the thread pendulum projected by the objective lens fell exactly on a disc of paper placed below at the clock pendulum and provided with an opening. With the help of an observation telescope set up before the clock, the paper disc could be seen at the same time as the thread of the pendulum when the pendulums were at rest.

As to the rest, the observation of the coincidences took place similarly as in the case of the measurements already described.

Bessel's measurements are less significant for their results than rather for the admirable theoretical and practical investigations which are connected with this work and which guided the problem of the pendulum measurement into new paths. The whole work is published under the title: "Untersuchungen über die Länge des einfachen Sekundenpendels" in *Abhandlungen der Kgl. Ak. d. Wiss. in Berlin für das Jahr 1826*, Berlin, 1828.

The main difficulties which Bessel met with in the course of his investigations consist in taking into account the influence of the air, in which he succeeded only imperfectly, and in determining the position of the center of oscillation, for which doubts remained even in the case of the well-considered arrangement of his thread pendulum. On the basis of his theoretical discussions, Bessel arrived at the result that these error sources can be made harmless more easily in the case of the reversion pendulum.

On the basis of Bessel's theory, we already have proved on p. 340 that in the case of a reversion pendulum, symmetrical with respect to the knife-edges, the influence of the air resistance is eliminated. The determination of the centers of oscillation does not offer any difficulty in the case of the reversion pendulum. On the other hand, if a new error source is added by the imperfection of the knife-edges, then it has been shown on p. 344 according to Bessel's procedure that this error source can be eliminated by exchanging the knife-edges.

Consequently, at the end of his treatise, Bessel indicates the conditions for the construction of an unobjectionable reversion pendulum. In a later paper on the "Konstruktion eines symmetrisch geformten Pendels mit reziproken Achsen," which was published after his death in *Astronomische Nachrichten*, 30. Band, 1849, Nr. 697, Bessel returns to the same subject and gives now thorough data about the pendulum proposed by him. The most important principles are the following:

The pendulum must be constructed symmetrically so far as the outward figure is concerned, and hence, since it cannot be symmetrical in mass, it must possess two equally large lenses, equally disposed against the knife-edges, on a rod, of which one is hollow.

Furthermore the knife-edges must be arranged in such a way that they can be exchanged with one another.

Finally, Bessel proposes to leave off the movable weight of Kater's pendulum and to shorten the rod gradually, exactly symmetrically at both ends, until the periods of oscillation are nearly equal.

Bessel's proposals were realized for the first time by G. Neumayer, who, about 1861, gave an order for the manufacture of a symmetrical reversion pendulum by the mechanic W. J. Lohmeier in Hamburg for the determination of gravity in Melbourne. The outward form was quite similar to that of Kater's pendulum; the weights, however, were distributed symmetrically, while the movable weight was left out entirely. By symmetrical shortening of the two ends of the pendulum rod, made originally a little too long, the periods of oscillation were made exactly equal, as Bessel had indicated.

In contrast to Kater's measurements, the determination of the distance between the knife-edges was carried out for the first time with a suspended pendulum, which is to be regarded as a progress. The measurements took place in the fall of 1863. Upon reduction of the observations, however, there appeared an

uncertainty in the length of the scale used in the measurements, and the clarification of the differences found was delayed for several decades without leading to a perfectly satisfying result. Therefore, the whole material was not published until 1902 in *Abhandlungen der Kgl. Bayer, Akademie der Wissenschaften*, 21. Band, 3. Abt., pp. 479-556.

In addition to the references to literature communicated already in the foregoing paragraphs we mention further the work, *Collection de mémoires relatifs à la Physique*, publiés par la Société française de Physique, Tomes IV et V: C. Wolf, "Mémoires sur le Pendule," Paris, 1889-1891. In this work there also is contained the very extensive literature about pendulum measurements up to the year 1886.

Section 73. The Repsold Pendulum Apparatus

The Swiss Commission of the European Degree Measurement decided in one of its first sessions in 1862 to include also gravity measurements in the program of its activities, and, for this purpose, in accordance with Bessel's proposals, ordered the construction of a reversion pendulum by the mechanics A. Repsold and Sons in Hamburg; with it, Plantamour determined the length of the second pendulum in Geneva. These activities are described in E. Plantamour's *Expériences faites à Genève avec le pendule à réversion*, Genève et Bâle 1866, and *Nouvelles expériences faites avec le pendule à réversion et détermination de la pesanteur à Genève et au Righi-Kulm*, Genève et Bâle 1872.

A similar instrument was made in 1869 for the Prussian Geodetic Institute (cf. *Publ. d. Kgl. Preuss. Geod. Inst., Astr.-geod. Arb. im Jahre 1870*, Leipzig, 1871, pp. 107 and following), and further instruments of the same construction came gradually into use in many other states.

In Fig. 1 we give an illustration of the Repsold pendulum apparatus. It consists of the stand, the pendulum, the comparator and the scale; in Fig. 1, the latter is only outlined on the stand and separately illustrated on the side. The comparator and scale are used for the measurement of the distance between the knife-edges.

The stand consists of three arms which rest on a baseplate provided with foot screws. The head connecting the arms above is shaped in the form of a horseshoe so that the pendulum can be hung in from the side.

The pendulum is formed by a hollow brass cylinder with an outer diameter of about 4 cm and a length of 1.25 m. Of the two weights, which are placed near the ends, one is hollow, the other is filled. Between the weights the rod contains two notches to receive the knife-edges; two oblong brass frames, which expand at the center in the form of an arc, embrace the rod and carry at their ends two bows, provided with clamp screws, in which the knife-edges are attached. At the top of the knife-edges, the wall of the tube is bored through on both sides, so that one can see the knife-edges through the bore opening; this is made use of in measuring the distance between the knife-edges. These details are clearly visible in the upper rod end, illustrated separately in Fig. 1 on the right-hand side. The knife-edges themselves are made of steel in the case of the first apparatuses, later of agate; they can be reversed as well as exchanged with one another.

For the suspension of the pendulum, on the headplate of the stand there is attached a horizontal arm with a steel (later agate) plate, precisely evenly ground off, which goes through the cutout of the pendulum rod and serves as the bearing for the knife-edges. In addition, there is also provided a lever arrangement, into which the pendulum is hung at first, and by means of which it can then carefully be lowered down to the bearing.

Below the pendulum, the stand plate carries a scale, standing edgewise, for the measurement of the amplitude of oscillation.

The comparator consists of a cylindrical brass tube, which is attached at the lower and the upper stand plate and can be turned around its longitudinal axis. Its lower end can be moved a little by four adjusting screws; in addition, a small vertical movement is provided for. At an interval of nearly 1 m, the tube carries two screw microscopes, which can be set exactly horizontal by means of a mounting level.

The scale, illustrated separately in Fig. 1, is likewise a brass tube of the dimensions of the pendulum rod. In the interior, at an interval of about 1 m there are placed two small metal blocks which reach as far as the center of the tube and on which the length of the scale is designated by scratch marks; by cuts in the wall of the tube, these marks are visible from the outside. The upper small block carries three subdividing marks at an interval of 0.1 mm, the central one being the zero mark, while the lower small block is divided from

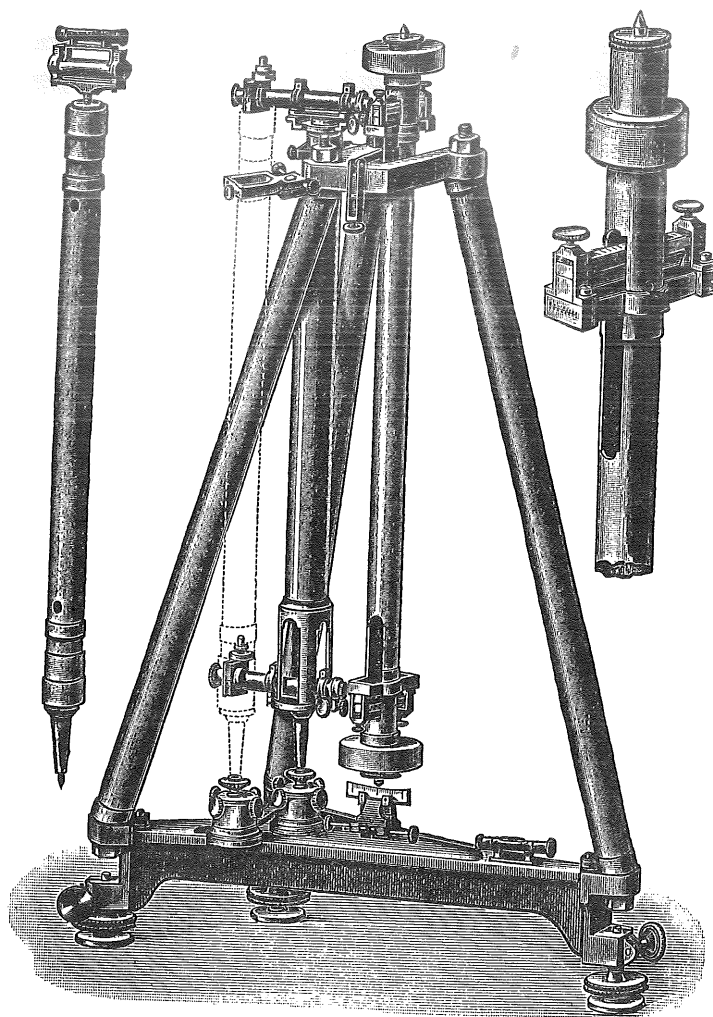


Fig. 1.

999 mm to 1001 mm into tenths of a millimeter. In the tube, in addition, there is a metal thermometer which consists of a steel tube and a zinc tube lying in the latter; the first is fastened at the brass tube above and can be moved freely downward. At the lower end the two tubes of the metal thermometer are connected with one another. The different change in length of the two metals, steel and zinc, will therefore express itself in the fact that the upper end of the zinc tube changes its position; for the measurement of this displacement there is placed at the zinc tube a small brass plate with a tenths-of-a-millimeter division, which can be read at the same time with the upper scale.

The scale can also be put in a reverse position into the stand, i.e. with the zero point below.

Determination of the center of gravity of the pendulum

To the pendulum apparatus there also belongs an auxiliary device in order to measure the distance of the center of gravity from the two knife-edges, i.e. the quantities a and a' . For this purpose, there is placed over a horizontal scale and lying crosswise to it a double cone, on which the pendulum is laid in such a way that it is horizontally in free suspension. Then a slide gliding on the scale is brought up to one of the two knife-edges and its position read. If we reverse the pendulum and repeat the measurement, then the difference of the two scale readings yields the difference $a - a'$; if the distance between the knife-edges $a + a'$ is taken, then a and a' are known.

The Repsold pendulum apparatus has been used in the case of most of the absolute gravity measurements carried out in the last decades. The arrangement has thereby remained in substance unchanged; in the case of newer measurements, however, above all, the stand, whose solidity did not suffice for higher demands, has been replaced by a pair of solidly founded pillars standing near one another. Without entering into details, we add further a few data about the arrangements which were used in the case of the pendulum measurements of the Geodetic Institute in Potsdam in the years 1898 to 1904 and described in detail in F. Kühnen and Ph. Furtwängler's *Bestimmung der absoluten Grösse der Schwerkraft zu Potsdam mit Reversionspendeln*, Berlin, 1906.

The measurements took place in the Pendulum Room of the Institute, a room situated in the interior of the building, which can be kept on a temperature as constant as possible by special heating and ventilating installations. Two heavy sandstone pillars founded independently of the floor, between which a space of a cross section of 60 by 60 cm exists, are used as a pendulum stand. This space is enclosed by wooden frames which are covered with linen and, in addition, on both sides with tinfoil. A cap made in the same way and provided with glass windows is put on the pillars, so that a closed pendulum cupboard, very well protected against heat radiation, exists. On the two pillars there rests as a bridge a heavy triangular cast piece which contains in the center a circular notch for the pendulum. The actual pendulum console with the agate bearing, on which the pendulum swings, is set on the cast piece; the fastening of the console can be carried out in three positions, for oscillations in the north-south, in the east-west and in the northeast-southwest direction. For this purpose, there are several levels with which the horizontal adjustment of the agate bearing is carried out.

The bridge contains further, on the upper and on the lower side, a surface ground off exactly evenly, at which the two parts of a vacuum cylinder, made of sheet copper, can be connected as soon as the oscillations are to take place in the space in which the air has been rarefied. The diameter of the cylinder is dimensioned so large that the pendulum console can be conveniently placed in the interior. Through glass windows the thermometer can be read off as well as the oscillations observed. Further there are faucets for the connection of an air pump and a pressure gauge, as well as a lever device with which the pendulum can be put in motion.

The reversion pendulum of Defforges

Ch. Defforges indicated another form of the reversion pendulum and had it constructed by the mechanic Brunner in Paris. The pendulum consists of a brass tube, about 3 cm in external diameter, at which the knife-edges are connected similarly as in the case of Repsold's. In the interior there are placed lead weights, which correspond to the weights of Repsold's pendulum. The cylindrical shape is of special advantage because the influence of the air resistance can theoretically be determined more easily for this form.

Defforges uses two pendulums of different lengths, but of equal weight, in order to be able to eliminate the co-vibration of the stand (cf. p. 349).

The pendulum apparatus of Defforges is described in *Mémoires du Dépôt général de la guerre*, Tome XV, 1^{er} fasc., Paris, 1894.

The measurement of the duration of oscillation

In the case of the first measurements with the Repsold pendulum apparatus, the determination of the duration of oscillation was carried out by observing the transits of the pendulum through the position of rest, whereby the transits in *one* direction only were considered, however. For instance, every 50 such transits were registered by pressing on the key of a chronograph; after this, 900 oscillations were not observed, and finally, 50 transits were again registered. The mean value of the first 50 observations, subtracted from the mean value of the last 50 observations, yielded the period of time for 500 oscillations.

Later, one returned from this simple method to the observation of coincidences, which were already used in the measurements of the seventeenth and eighteenth centuries and reached a high degree of accuracy especially through Borda and Cassini (cf. section 72, p. 352).

For this method there was constructed by the Austrian Major v. Sterneck (died in 1910 as a Major General of the Reserve), in 1887, an especially convenient apparatus, which is illustrated in Figs. 2 and 3.

(A description is contained in *Mitt. d. k. k. Mil.-geogr. Inst.*, Vol. 7, 1887.)

In a small box, resting on three foot screws, with the dimensions 20, 10 and 14 cm there is placed an electro-magnet, which is connected with the contact mechanism of a clock and, upon each closing of the current, attracts the armature once, while upon opening of the current, a spring lifts the armature again. At the front end, the armature carried a small metal plate with a horizontal slit. Another small plate with a corresponding slit is attached in the interior of the small box, so that upon closing as well as upon opening of the current the two slits are opposite one another for a moment. At this moment, the light of a lamp outside the small box, which comes through a glass window and through an obliquely placed mirror onto the fixed small plate, penetrates the two slits and becomes visible through a small opening in the end face of the small box. In addition,

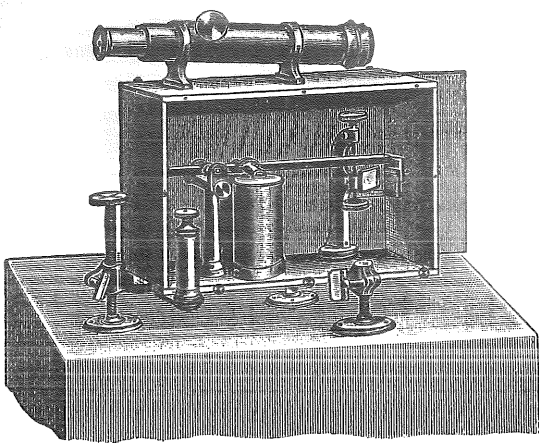


Fig. 2.

this surface carried a scale whose zero mark at the center passes through the just-mentioned opening. On the small box there is placed a small observation telescope with a horizontal thread.

For the measurement of the duration of oscillation, there is further necessary a small mirror, which is connected at the pendulum near the axis of oscillation.

The coincidence apparatus is set up at a distance of about 3 m from the pendulum in such a manner that in the resting position of the pendulum the zero mark of the scale is reflected, by means of the pendulum mirror, into the telescope and coincides with the horizontal thread of the latter. The image of the illuminated slit is then, at the same time, congruent with the thread. If the pendulum is set into vibration, then the image of the scale also swings in the telescope up and down, and by reading off the thread, the amplitude of the oscillations of the pendulum can be determined.

As soon as the electromagnet of the coincidence apparatus is also connected with the contact of the clock, the light penetrating through the slit can coincide with the thread in the telescope only when the slit is opened at the moment at which the pendulum passes through the position of rest. If this happens in the case of a definite oscillation of the pendulum, then, since the latter does not swing exactly evenly with the clock pendulum, in the case of the next oscillation the light flash appears a little over or under the thread. Not until the pendulum has executed a whole oscillation more or less than the clock pendulum does a coincidence occur again.

Of the two light flashes which occur upon closing and upon opening of the current, the first is not quite regular and therefore we consider only the second light flash. A device by which the light flash is screened off upon closing of the current is useful.

Therefore, if everything is set in motion, we see in the field of vision of the telescope the individual light flashes, which from second to second progress upward or, as the case may be, downward, and now we have to interpolate the time at which the light flash passes through the horizontal thread.

If the number of the oscillations of the clock pendulum between two coincidences is equal to c , then the pendulum has carried out, during this time, $c \pm 1$ oscillations; consequently, the duration of one oscillation in seconds of the clock is:

$$T = \frac{c}{c+1} = 1 - \frac{1}{c+1} \quad (1)$$

or, as the case may be,

$$T = \frac{c}{c-1} = 1 + \frac{1}{c-1} \quad (2)$$

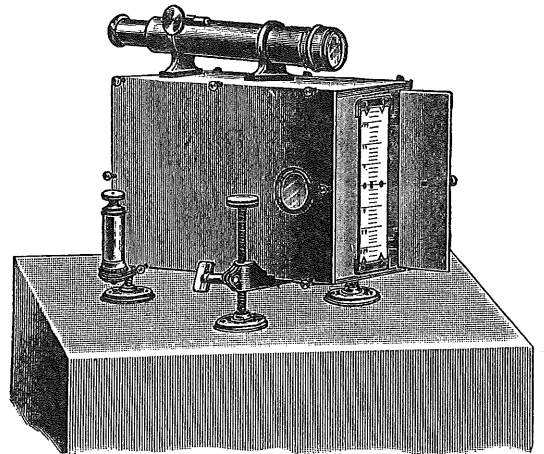


Fig. 3.

Another simple method for the observation of the coincidences was used by Kühnen and Furtwängler in the case of a part of the measurements mentioned on p. 359. Fig. 4 shows the arrangement necessary for it. At the clock pendulum, near the suspension, there is fastened a small mirror U and at the same height in the clock case there is provided an opening which is closed by a lens of a focal length of about 3 m. The rays sent out by a small lamp L are thrown through the lens on the mirror of the clock, arrive, after the reflection, at a second mirror set up at K , the coincidence mirror, and, after a third reflection at the pendulum mirror P , become visible in the telescope.

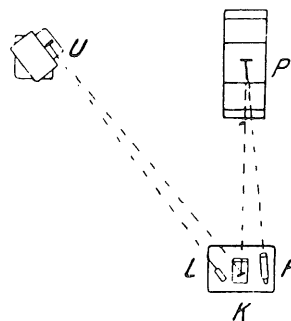


Fig. 4.

An optical coincidence apparatus used in the gravity measurements of the Finnish Geodetic Institute is described in *Veröff. d. Finn. Geod. Inst.*, No. 13, Helsinki, 1930, pp. 164-168 and No. 23, 1937, pp. 6-7.

In order to determine eventual disturbances of the motions of oscillation, which can arise from no evident causes, there is constructed, for the complete representation of the oscillation, an apparatus for the photographic registration of the motion of the two pendulums by General Madsen, the former director of the Danish degree-measurement; the apparatus is described in detail in *Den Danske Gradmaaling*, Ny Raekke, Hefte Nr. 14, Registreringsapparat til Tyngdemaalingspenduler, udgivet af Generalmajor V. H. O. Madsen, Direktor for den Danske Gradmaaling, Beskrevet af Ingenior Aage Petersen, Kjøbenhavn 1915. On a strip of sensitized paper, moving by a clock mechanism, one obtains the sine curves corresponding to the oscillations of the two pendulums, which intersect from time to time, and by whose measurement one finds the ratio of the two periods of oscillation. (Cf. also *Zeitschrift für Vermessungswesen*, 1915, p. 444.)

We shall describe a new registering apparatus of the Geodetic Institute in Potsdam in section 76, p. 378.

Correction because of the rate of the clock

The duration of oscillation of the pendulum is at first determined in seconds of clock time. If Δu is the daily rate of the clock, then a second of the clock time needs a correction of $\frac{\Delta u}{86,400}$ seconds; therefore, the duration of oscillation T is to be corrected by the amount of:

$$\Delta T = \frac{\Delta u}{86,400} T. \quad (3)$$

Reduction to an infinitely small amplitude

The duration of oscillation round thus far refers to the amplitude of the pendulum during the measurement; therefore, a reduction to an infinitely small amplitude is needed further. If we designate for the latter the duration of oscillation by T_0 , then we have according to (15) and (16), section 68, p. 333, with sufficient accuracy:

$$T_0 = T \left(1 - \frac{\varphi^2}{16} \right) \quad (4)$$

where φ is the mean amplitude during the measurement.

In the previous section 71, p. 345, we have established the influence of the co-vibration of the support of the pendulum, and now we will discuss the auxiliary means for the determination of the constant ε thereby occurring.

One of the most important methods consists in the observation of a thread pendulum, which is suspended at the support head and, for protection against air currents, is surrounded by a tube set up perpendicularly. As soon as the reversion pendulum is set in motion, the thread pendulum being at rest at first is moved likewise by the co-vibration of the support, and we can compute, in theory, the co-vibration of the support from the oscillations of the thread pendulum. This method was applied by C. von Orff in his pendulum measurements carried out at the observatory in Bogenhausen near Munich in 1877, about which an account is given in *Abhandlungen der mathematisch-physikalischen Klasse der Kgl. Bayer. Akademie der Wissenschaften*, 14. Band, 1883, pp. 161-294. With the use of this communication, in the following we develop at first the theory of the thread pendulum.

In equation (7), section 71, p. 345, we have found the general differential equation for the motion of oscillation of a physical pendulum in the case of co-vibrating support, which we can apply also to the thread pendulum. To distinguish this from the reversion pendulum, we will denote, for the thread pendulum, the angle of deflection by α_2 and the length of the mathematical pendulum corresponding to it by l_2 , while for the reversion pendulum α_1 and l_1 shall now hold. If we set, because of the small amount of the deflection, $\sin \alpha_2 = \alpha_2$ and $\cos \alpha_2 = 1$, then equation (7), section 71, p. 345, changes to:

$$\frac{d^2 \alpha_2}{dt^2} l_2 = -g \alpha_2 + \frac{d^2 \sigma}{dt^2}. \quad (1)$$

The displacement σ of the support head is a function of the deflection α_1 of the reversion pendulum, and we have already found in (14a), section 71, p. 347, that we can assume approximately:

$$\sigma = -\gamma_1 \alpha_1 \text{ with } \gamma_1 = \frac{a g M_1}{\varepsilon l_1}$$

and herewith

$$\frac{d^2 \sigma}{dt^2} = -\gamma_1 \frac{d^2 \alpha_1}{dt^2} \quad (2)$$

If we introduce this in the differential equation (7), section 71, p. 345, then we obtain for the reversion pendulum:

$$\frac{d^2 \alpha_1}{dt^2} l_1 = -g \alpha_1 - \gamma_1 \frac{d^2 \alpha_1}{dt^2} \quad \frac{d^2 \alpha_1}{dt^2} (l_1 + \gamma_1) = -g \alpha_1$$

or

$$\frac{d^2 \alpha_1}{dt^2} \gamma_1 = -\frac{g \gamma_1}{l_1 + \gamma_1} \alpha_1 = -\frac{g \gamma_1}{l_1} \alpha_1, \quad (3)$$

since the small quantity γ_1 can be neglected without hesitation in the denominator. If we substitute (2) and (3) in (1), then there follows:

$$\frac{d^2 \alpha_2}{dt^2} l_2 = -g \alpha_2 + \frac{g \gamma_1}{l_1} \alpha_1. \quad (4)$$

Now we still have to take into account the influence of the air resistance on the angle of deflection α_1 , for which, according to section 69, p. 337, the term $-k \frac{d \alpha_2}{dt}$ must be added on the right-hand side.

Therefore, we obtain for the thread pendulum, taking into account the co-vibration of the support and the air resistance, the differential equation:

$$\frac{d^2 \alpha_2}{dt^2} l_2 = -g \alpha_2 - k_2 \frac{d \alpha_2}{dt} + \frac{g \gamma_1}{l_1} \alpha_1. \quad (5)$$

We can likewise easily express the value of α_1 still occurring here as a function of t by making use of equation (24), section 69, p. 338. Accordingly, we have:

$$\alpha_1 = \varphi_1^0 e^{-z_1 t} \cos \sqrt{\frac{g}{l_1} - z_1^2} t,$$

but since, with respect to the large weight of the reversion pendulum, the damping factor z_1 is very small, then we can also write for this:

$$\alpha_1 = \varphi_1^0 e^{-z_1 t} \cos \sqrt{\frac{g}{l_1}} t. \quad (6)$$

Then we obtain for (5):

$$\frac{d^2 \alpha_2}{dt^2} + 2 z_2 \frac{d \alpha_2}{dt} + \frac{g}{l_2} \alpha_2 = \frac{g \gamma_1}{l_1 l_2} \varphi_1^0 e^{-z_1 t} \cos \sqrt{\frac{g}{l_1}} t. \quad (7)$$

This differential equation has essentially the same form as equation (18), section 69, p. 337; it differs from the latter only by the term on the right-hand side.

Therefore, we will now use the two particular integrals (22), section 69, p. 338:

$$\varphi(t) = e^{-z_1 t} \cos u t \quad \psi(t) = e^{-z_1 t} \sin u t \quad (8)$$

and, by analogy to equation (23), section 69, p. 338, form from them the general integral of the above equation (7), where the two constants c_1 and c_2 can no longer be assumed arbitrarily, however. Therefore, we set tentatively:

$$\alpha_2 = f_1(t) \varphi(t) + f_2(t) \psi(t) \quad (9)$$

and try to determine the two functions $f_1(t)$ and $f_2(t)$ in such a way that the value (9) satisfied the differential equation (7). Since we can introduce also a second condition for the two functions f_1 and f_2 , then we assume further for simplification:

$$\frac{d f_1(t)}{dt} \varphi(t) + \frac{d f_2(t)}{dt} \psi(t) = 0. \quad (10)$$

Then we will have:

$$\frac{d \alpha_2}{dt} = f_1(t) \varphi'(t) + f_2(t) \psi'(t) \quad (11)$$

$$\frac{d^2 \alpha_2}{dt^2} = f_1(t) \varphi''(t) + f_2(t) \psi''(t) + f_1'(t) \varphi'(t) + f_2'(t) \psi'(t). \quad (12)$$

If the values (9), (11) and (12) are substituted in (7), then we obtain:

$$f_1(t) \left\{ \varphi''(t) + 2z_2 \varphi'(t) + \frac{g}{l_2} \varphi(t) \right\} + f_2(t) \left\{ \psi''(t) + 2z_2 \psi'(t) + \frac{g}{l_2} \psi(t) \right\} \\ + f_1'(t) \varphi'(t) + f_2'(t) \psi'(t) = \frac{g \gamma_1}{l_1 l_2} \varphi_1^0 e^{-z_1 t} \cos \sqrt{\frac{g}{l_1}} t. \quad (13)$$

But since $\varphi(t)$ and $\psi(t)$ satisfy equation (6), then the two expressions in brackets in (13) are equal to zero, and there remains then:

$$f_1'(t) \varphi'(t) + f_2'(t) \psi'(t) = \frac{g \gamma_1}{l_1 l_2} \varphi_1^0 e^{-z_1 t} \cos \sqrt{\frac{g}{l_1}} t. \quad (14)$$

If we take to this, in addition, equation (10):

$$f_1'(t) \varphi(t) + f_2'(t) \psi(t) = 0,$$

then we have two equations for the determination of the two auxiliary functions $f_1(t)$ and $f_2(t)$.

We set for the moment for simplification:

$$\sqrt{\frac{g}{l_2}} = v \quad \frac{g \gamma_1}{l_1 l_2} \varphi_1^0 = a, \quad (15)$$

and have then according to (8):

$$\left. \begin{aligned} f_1'(t) \frac{d e^{-z_1 t} \cos u t}{d t} + f_2'(t) \frac{d e^{-z_1 t} \sin u t}{d t} &= a e^{-z_1 t} \cos v t \\ f_1'(t) e^{-z_1 t} \cos u t + f_2'(t) e^{-z_1 t} \sin u t &= 0. \end{aligned} \right\} \quad (16)$$

If the differentiation is carried out here in the first equation (16) and thereby the second equation (16) is taken into account, then we obtain:

$$\left. \begin{aligned} -f_1'(t) \sin u t + f_2'(t) \cos u t &= \frac{a}{u} e^{(z_1 - z_1) t} \cos v t \\ + f_1'(t) \cos u t + f_2'(t) \sin u t &= 0, \end{aligned} \right\} \quad (17)$$

and there follow hence for $f_1'(t)$ and $f_2'(t)$ the values:

$$\left. \begin{aligned} f_1'(t) &= -\frac{a}{u} e^{(z_1 - z_1) t} \sin u t \cos v t \\ f_2'(t) &= +\frac{a}{u} e^{(z_1 - z_1) t} \cos u t \cos v t. \end{aligned} \right\} \quad (18)$$

The integration of these two equations is simplified if we introduce, on the right-hand side, the transformations:

$$\begin{aligned} \sin u t \cos v t &= \frac{1}{2} \sin(u - v) t + \frac{1}{2} \sin(u + v) t \\ \cos u t \cos v t &= \frac{1}{2} \cos(u - v) t + \frac{1}{2} \cos(u + v) t. \end{aligned}$$

If we use then the integrals

$$\int e^{ax} \sin bx dx = e^{ax} \frac{a \sin bx - b \cos bx}{a^2 + b^2} + \text{const.}$$

$$\int e^{ax} \cos bx dx = e^{ax} \frac{a \cos bx + b \sin bx}{a^2 + b^2} + \text{const.},$$

then we find easily:

$$\left. \begin{aligned} f_1(t) &= -\frac{a}{2u} e^{(z_2 - z_1)t} \left\{ \frac{(z_2 - z_1) \sin(u - v)t - (u - v) \cos(u - v)t}{(z_2 - z_1)^2 + (u - v)^2} \right. \\ &\quad \left. + \frac{(z_2 - z_1) \sin(u + v)t - (u + v) \cos(u + v)t}{(z_2 - z_1)^2 + (u + v)^2} \right\} + c_1 \\ f_2(t) &= +\frac{a}{2u} e^{(z_2 - z_1)t} \left\{ \frac{(z_2 - z_1) \cos(u - v)t + (u - v) \sin(u - v)t}{(z_2 - z_1)^2 + (u - v)^2} \right. \\ &\quad \left. + \frac{(z_2 - z_1) \cos(u + v)t + (u + v) \sin(u + v)t}{(z_2 - z_1)^2 + (u + v)^2} \right\} + c_2. \end{aligned} \right\} \quad (19)$$

With these, we have found for the functions $f_1(t)$ and $f_2(t)$ those values with which equation (9) satisfies the differential equation (7), and if we now substitute (19) and (8) in equation (9), then we obtain after a simple transformation:

$$\alpha_2 = \frac{a}{2u} e^{-z_1 t} \left\{ \frac{(z_2 - z_1) \sin vt + (u - v) \cos vt}{(z_2 - z_1)^2 + (u - v)^2} - \frac{(z_2 - z_1) \sin vt - (u + v) \cos vt}{(z_2 - z_1)^2 + (u + v)^2} \right\} + c_1 \varphi(t) + c_2 \psi(t)$$

or

$$\alpha_2 = \frac{a}{2u} e^{-z_1 t} \left\{ \left(\frac{u - v}{(z_2 - z_1)^2 + (u - v)^2} + \frac{u + v}{(z_2 - z_1)^2 + (u + v)^2} \right) \cos vt \right. \\ \left. + \left(\frac{z_2 - z_1}{(z_2 - z_1)^2 + (u - v)^2} - \frac{z_2 - z_1}{(z_2 - z_1)^2 + (u + v)^2} \right) \sin vt \right\} + c_1 \varphi(t) + c_2 \psi(t). \quad (20)$$

This expression can be considerably simplified after introducing the values for u and v from (15), p. 364, and (20), section 69, p. 337. We will not cite this transformation, which does not offer any difficulties, in detail here, but reproduce only the result. For if we introduce the following notation:

$$p = g \frac{l_1 - l_2}{l_1} - 2l_2 z_1 z_2 + l_2 z_1^2 \quad q = 2l_2 (z_2 - z_1) \sqrt{\frac{g}{l_1}}, \quad (21)$$

then we have, if at the same time the values for $\varphi(t)$ and $\psi(t)$ from (8) are also substituted in the last two terms of (20):

$$\alpha_2 = e^{-z_1 t} \left(c_1 \cos \sqrt{\frac{g}{l_2} - z_2^2} t + c_2 \sin \sqrt{\frac{g}{l_2} - z_2^2} t \right) \\ + \frac{g \gamma_1}{l_1 (p^2 + q^2)} \varphi_1^0 e^{-z_1 t} \left(p \cos \sqrt{\frac{g}{l_1}} t + q \sin \sqrt{\frac{g}{l_1}} t \right). \quad (22)$$

The determination of the constants of the co-vibration turns out simplest, if the weight of the thread pendulum is very small, since the air-damping acts most strongly, and hence z_2 becomes very large. In this case, the influence of the first part of (22) is infinitesimal and can be neglected if we wish to limit ourselves

to an approximate computation. Then we set for further simplification

$$p = r \sin \omega \quad q = r \cos \omega ;$$

therefore

$$p^2 + q^2 = r^2 \quad \text{and} \quad \tan \omega = \frac{p}{q},$$

and have then:

$$p \cos \sqrt{\frac{g}{l_1}} t + q \sin \sqrt{\frac{g}{l_1}} t = \sqrt{p^2 + q^2} \sin \left(\sqrt{\frac{g}{l_1}} t + \omega \right).$$

With this, (22) changes to

$$\alpha_2 = \frac{g \gamma_1}{l_1 \sqrt{p^2 + q^2}} \varphi_1^0 e^{-z_1 t} \sin \left(\sqrt{\frac{g}{l_1}} t + \omega \right). \quad (23)$$

The angle of deflection α_2 reaches its maximum φ_2 in the case of each oscillation when

$$\sin \left(\sqrt{\frac{g}{l_1}} t + \omega \right) = 1; \quad \text{therefore,}$$

$$\varphi_2 = \frac{g \gamma_1}{l_1 \sqrt{p^2 + q^2}} \varphi_1^0 e^{-z_1 t} \quad (24)$$

is the amplitude of the thread pendulum.

Accordingly, there follows from (6), p. 363, the following value for the reversion pendulum

$$\varphi_1 = \varphi_1^0 e^{-z_1 t}. \quad (25)$$

We thus have from (24) and (25) for the constant γ_1 of the co-vibration the expression

$$\gamma_1 = \frac{\varphi_2 l_1}{\varphi_1 g} \sqrt{p^2 + q^2}. \quad (26)$$

Now we can determine hence also the elasticity constant ε of the pillar, or better still the magnitude $\frac{M}{\varepsilon} g$, which we need for the computation of the correction of the length of the second pendulum according to equation (21), section 71, p. 349. For, if a denotes again the distance of the center of gravity from the axis of oscillation in the case of the measurements executed for the determination of γ_1 , then we have according to (14a), section 71, p. 347:

$$\frac{M}{\varepsilon} g = \frac{\gamma_1}{a} l_1, \quad (27)$$

and with this, the length of the second pendulum can then be determined.

In the foregoing, we have treated the theory of the co-vibration only for the case in which a thread pendulum with a great damping is used. If both pendulums are at rest at the beginning, and if the reversion pendulum is then set in motion, then the thread pendulum also begins to swing, and oscillations of the latter grow larger until, after some time, a maximum of the amplitude is reached. The two pendulums assume then the same duration of oscillation, while the amplitudes slowly decrease and their ratio becomes nearly

constant. If the amplitudes are then measured, then the above simple formulae can be used.

As the thread pendulum we use a small brass cylinder with a weight of 20 to 40 g which is suspended on a brass wire, about 0.05 mm thick. The point of suspension should lie as accurately as possible at the same height as the knife-edge of the reversion pendulum, while the thread length must be regulated so that the pendulum, upon free oscillation, has a duration of oscillation of 1 second. The pendulum is placed in a tube standing upright which, above the measuring cylinder, carries a window for the observation of the thread. The measurement of the amplitude is carried out by means of the movable thread of a screw microscope fastened before the window.

In addition to the literature communicated already at the beginning of this section, we mention further the following papers on the theory of the simultaneous oscillations of two pendulums on the same support:

Ph. Furtwängler, "Über die Schwingungen zweier Pendel mit annähernd gleicher Schwingungsdauer auf gemeinsamer Unterlage," *Sitzungsbericht der Kgl. Preuss. Akademie der Wissenschaften zu Berlin*, 1902, pp. 245-253.

F. Kühnen and Ph. Furtwängler, "Bestimmung der absoluten Grösse der Schwerkraft zu Potsdam mit Reversionspendeln," *Veröffentlichung des Geodätischen Instituts in Potsdam*, N. F. Nr. 27, Berlin, 1906, pp. 49-69.

A. Berroth, "Schweremessungen," *Handbuch der Physik*, Band II, Berlin, 1926, pp. 450-455.

H. Schmehl and K. Jung, "Figur, Schwere und Massenverteilung der Erde," *Handbuch der Experimentalphysik*, Band 25, II. Teil, Leipzig, 1931, pp. 224-238.

Concerning the use of an interferometer for the determination of the co-vibration, an account is given in *United States Coast and Geodetic Survey, Report for 1910*, App. 6.

Section 75. Numerical Example

In order to be able to apply the theories developed in the preceding sections to a numerical example, we use, in extract form, a few results from the basic measurements which were carried out by Kühnen and Furtwängler in Potsdam in the years 1898-1904, and which are described with all details in the just-mentioned work, *Bestimmung der absoluten Grösse der Schwerkraft zu Potsdam mit Reversionspendeln*, by F. Kühnen and Ph. Furtwängler, Berlin, 1906.

We have already communicated a few things about these measurements in section 73, p. 359, to which we now add, supplementarily, some further data.

There were used in all five reversion pendulums, which differed from each other especially with respect to weight.

With the pendulum belonging to the observatory in Padova there was carried out, at the end of May 1899, a complete series of oscillation observations according to the following scheme:

- 1st day: Heavy weight below, heavy weight above. Turning of the pendulum around the vertical axis. Heavy weight above, heavy weight below.
- 2nd day: Measurement of the distance between knife-edges in the above four positions. Determination of the center of gravity. Exchanging of the knife-edges.
- 3rd day: Repeated measurement of the distance between knife-edges and determination of the center of gravity as on the second day.
- 4th day: Observations as on the first day.

From the observations of the first day, with the heavy weight below, we indicate the following coincidences, at which the clock ran on sidereal time.

Coincidences				Amplitudes		
No.		No.			Above	Below
1	1 ^h 54 ^m 31.6 ^s	17	2 ^h 41 ^m 01.8 ^s	1 ^h 53.1 ^m	11.1 ^p	11.2 ^p
2	57 25.4	18	43 56.0	56.0	11.0	11.0
3	2 00 20.2	19	46 50.4	2 13.4	9.0	9.0
4	3 14.6	20	49 44.8	16.3	8.8	8.8
5	6 09.4	21	52 39.6	39.6	6.7	6.8
6	9 03.2	22	55 34.0	42.5	6.3	6.4
7	11 58.2	23	58 28.2	3 00.0	5.2	5.3
8	14 52.2	24	3 01 23.4	3 02.8	5.0	5.1

The scale at which the amplitudes were read off was subdivided in 3 mm and had its zero mark at the center. The distance of the scale from the pendulum mirror amounted to 3.28 m.

Before the use of the coincidences, there is to be taken into special consideration an eventual error in the position of the thread in the observation telescope, which, according to p. 360, must be arranged so that it coincides with a light flash occurring upon the passing of the pendulum through the position of rest. If the thread lies incorrectly, then one coincidence is found a little too early and the next one by the same amount too late, as is easily understood. Therefore, the mean value is obtained from every two of the consecutive coincidences, from the outset. If we then make the assumption that the duration of oscillation decreases in proportion to the time, then the computation of the interval between coincidences c is carried out in the simplest manner by forming the difference of the first and the last, of the second and next-to-last observation, etc. For the above assumed 16 coincidences we thus obtain the equations:

$$\begin{aligned} 23/24 - 1/2 &= 63^m 57.3^s = 22 c \\ 21/22 - 3/4 &= 52 \quad 19.4 = 18 c \\ 19/20 - 5/6 &= 40 \quad 41.3 = 14 c \\ 17/18 - 7/8 &= 29 \quad 03.7 = 10 c . \end{aligned}$$

If we introduce, for simplification, the value:

$$c = 174.4^s + \zeta ,$$

then the foregoing equations yield

$$\begin{aligned} +0.5 &= 22 \zeta \\ +0.2 &= 18 \zeta \\ -0.3 &= 14 \zeta \\ -0.3 &= 10 \zeta . \end{aligned}$$

We find hence the normal equations:

$$1104 \zeta - 7.4 = 0 \quad \text{and} \quad \zeta = +0.007^s ,$$

with which we thus will have

$$c = 174.407^s .$$

According to equation (2), section 73, p. 360, we then obtain the preliminary value of the duration of oscillation:

$$T = 1.005 \, 7668^s .$$

For the amplitude correction we first have to determine the mean amplitude φ_m which best results from a graphical recording of the measured amplitudes. For the mean from the first and last coincidence $t_m = 2^h 28.0^m$ we then obtain

$$\varphi_m = 7.5 p .$$

This refers to the divisions of the scale. Since the scale intervals are equal to 3 mm, and the distance of the scale from the mirror is equal to 3280 mm, then we have:

$$\varphi_m = \frac{7.5 \times 3}{6560} 3438 = 11.8' .$$

With this, we can determine the amplitude reduction according to p. 361. We have in units of the seventh place:

$$\frac{\varphi_m^2}{16} T = 1.005\,7668 \frac{11.8^2 \times 10^7}{3438^2 \times 16} = 8 ,$$

and with this, the duration of oscillation reduced to an infinitely small amplitude is:

$$T_0 = 1.005\,7660^s .$$

In the above-mentioned publication, there has been used a more exact form of the reduction, which we will not discuss more thoroughly, however, since the demonstration of an example as simple as possible is only involved here.

Consideration of the pendulum temperature. For the simplification of the computation, it is advisable to reduce the individual determinations of the duration of oscillation to a uniform mean temperature of the pendulum. If l is the pendulum length, Θ the measured temperature of the pendulum, and Θ_m the mean temperature, and β denotes the coefficient of expansion of the pendulum, then there corresponds to the reduction from Θ to Θ_m a change in length $-l\beta(\Theta - \Theta_m)$ of the pendulum.

From the equation:

$$T^2 = \pi^2 \frac{l}{g}$$

there follows easily:

$$\Delta T_\Theta = \frac{\pi^2}{2gT} \Delta l \quad \text{or} \quad \Delta T_\Theta = \frac{T}{2l} \Delta l ,$$

therefore, the correction of the duration of oscillation for the reduction to the temperature Θ_m is:

$$\Delta T_\Theta = -T \frac{\beta}{2} (\Theta - \Theta_m) .$$

In the present case, the coefficient of expansion β of the pendulum was assumed equal to that of the scale, the latter of which gave a slightly different result for different temperatures. For about 10° there can be set $\beta = 0.000\,018\,666$ so that according to the foregoing equation the reduction of the duration of oscillation in units of the seventh place is:

$$\Delta T_\Theta = -93.3 (\Theta - \Theta_m) T .$$

The temperature of the pendulum, as the mean of the readings of different thermometers, gave a result of $\Theta = 14.525^\circ$, and if $\Theta_m = 14.5^\circ$ is assumed, then we have

$$\Delta T_\Theta = -93.3 \times 0.025 T = 2.3 T$$

or $\Delta T_\Theta = -2$ units of the seventh place of T .

This number still needs a small correction, since a base correction is to be added to the thermometer readings. Consequently, ΔT_Θ is to be increased by $+4$ units of the seventh place so that we must assume as final value:

$$\Delta T_\Theta = +2 \text{ units of the seventh place.}$$

A second, very small correction because of vertical temperature stratification is neglected here.

For the taking into account of the rate of the clock, there was found from the time determinations as the daily rate of the clock:

$$\Delta u = -0.54^s$$

with which, according to p. 361, the correction of the duration of oscillation is:

$$\Delta T_u = -63 \text{ units of the seventh place of } T.$$

The final value of the duration of oscillation, with the heavy weight below, is therefore:

$$T_1 = 1.005\,7599^s.$$

Thus far we only have treated the observations with the heavy weight below. The measurements with the heavy weight above are to be reduced in the same manner; but we will not demonstrate this, but take from the publication by Kühnen and Furtwängler, on p. 173,

$$T_2 = 1.005\,7874^s.$$

Duration of oscillation of the mathematical pendulum. The length measurements yielded that the distance between knife-edges was 1.9μ smaller than the scale length. For a temperature of 0° the length of the scale amounted to 999.9431 mm; therefore, at 0° the distance between knife-edges was equal to 999.9431 mm and at 14.5° :

$$\text{distance between knife-edges } l = 1000.2130 \text{ mm}.$$

For the computation of the duration of oscillation T of the mathematical pendulum of the length of the distance between knife-edges according to equation (32), section 69, p. 340, there is further to be determined the position of the center of gravity by means of the method indicated on p. 358. For the pendulum used here there had been found for the temperature 0° :

$$a_1 - a_2 = 381.78 \text{ mm},$$

and since according to the above:

$$a_1 + a_2 = 999.94 \text{ mm},$$

then we have

$$a_1 = 690.86 \text{ mm}, \quad a_2 = 309.08 \text{ mm} \quad \text{and} \quad \frac{a_2}{a_1 - a_2} = 0.80958.$$

Further we are to set for the pendulum used:

$$\frac{M'}{M} = 0.000\,152 \quad a' = 0.5\,l$$

and if all these numerical values are substituted in equation (32), section 69, p. 340, then we obtain for the duration of oscillation T of the mathematical pendulum of length l the value:

$$T = 1.005\,7376^s \text{ in sidereal time}.$$

We can now pass over to the computation of the length of the second pendulum. But since this shall be determined for a second of mean time, then the value of T is still to be converted into mean time, which yields:

$$T = 1.002\,9915^s \text{ in mean time}.$$

If we denote the value for the length of the second pendulum resulting from T and l by L_0 , then we have:

$$L_0 = \frac{l}{T^2} = \frac{1000.2130}{1.002\,9915^2} = 994.256 \text{ mm}.$$

Correction due to the co-vibration of the support. The determination of the constants ε for the co-vibration of the support was carried out with the help of a thread pendulum according to the method indicated in section 74, p. 366. The thread pendulum was at first quieted as much as possible, then the reversion pendulum pushed and then from time to time the amplitude of the two pendulums measured. This was continued until the ratio of the two amplitudes reached a constant value, which occurred in the case of the small damping of the thread pendulum only after about $1\frac{1}{2}$ hours.

Of the different measurements which were carried out for the determination of the co-vibration, we select the one of 17 December 1898 (observer K.), and take therefrom by interpolation for the time $7^h 06.8^m$ computed in arc measure:

$$\begin{aligned} \text{Amplitude of the reversion pendulum} &= 7.03 \cdot 10^{-3}, \\ \text{Amplitude of the thread pendulum} &= 0.24 \cdot 10^{-4}, \end{aligned}$$

whereby it is noted that the reversion pendulum swung with the "heavy weight below."

Now there are further to be determined the damping coefficients z_1 and z_2 for the two pendulums.

Equation (25) found in section 69, p. 338, holds for every physical pendulum, and hence also for a thread pendulum swinging at the immovable support. We therefore have for the two pendulums for the amplitude φ the expression:

$$\varphi = \varphi_0 e^{-z t}.$$

If for two different times t and t' the amplitudes φ and φ' are measured, then we obtain:

$$\frac{\varphi}{\varphi'} = e^{z(t' - t)},$$

whence z can be computed.

For the reversion pendulum, the amplitude measurements mentioned already above could directly be used, since the reaction of the thread pendulum on the reversion pendulum is not involved. For the thread pendulum, however, special observation series with the main pendulum at rest were measured. In this way there resulted:

$$\begin{aligned} \text{for the reversion pendulum} \quad z_1 &= 0.000\,214, \\ \text{for the thread pendulum} \quad z_2 &= 0.000\,675. \end{aligned}$$

The following lengths of the mathematical pendulum corresponded to these measurements:

$$\begin{aligned} \text{for the reversion pendulum} \quad l_1 &= 1005.15 \text{ mm}, \\ \text{for the thread pendulum} \quad l_2 &= 1004.64 \text{ mm}. \end{aligned}$$

With the numerical values thus obtained we compute the auxiliary quantities p and q of section 74, p. 365, and obtain:

$$p = 4.49 \quad q = 2.89 \quad \sqrt{p^2 + q^2} = +5.33.$$

Only the positive sign is usable for the latter root, since the coefficient γ must be positive. According to equation (26), section 74, p. 366, we have then:

$$\gamma_1 = \frac{\varphi_2}{\varphi_1} \frac{l_1}{g} \sqrt{p^2 + q^2} \quad \text{or} \quad \gamma_1 = 0.0019 \text{ mm}.$$

Since with the heavy weight below, the distance of the center of gravity from the axis of oscillation is $a = a_1 = 691 \text{ mm}$, then according to equation (27), section 74, p. 366, there follows hence further:

$$\frac{M g}{\varepsilon} = \frac{\gamma_1}{a} l_1 = 0.00276 \text{ mm}.$$

With this, the correction of the length of the second pendulum will be according to (21), section 71, p. 349:

$$\frac{L}{l_1} \frac{M g}{\varepsilon} = 0.00273 \text{ mm},$$

and the corrected length of the second pendulum is:

$$L = 994.259 \text{ mm}.$$

Influence of the air resistance. As we have recognized on p. 340, the air resistance does not exert an influence on the oscillations of a reversion pendulum if the pendulum in its outward form is constructed perfectly symmetrically. But since a small lack of symmetry can hardly be avoided, then the above length of the second pendulum still needs a correction because of the air density.

The influence of the air density is determined by observations of oscillations in the vacuum cylinder (cf. p. 359) with different air pressure. Since the variations of the air density during the actual pendulum measurements are only small, then it is sufficient to assume the correction in proportion to the air density; therefore, if L denotes the length of the second pendulum in the vacuum and L' that with an air density D , then we have:

$$L = L' + k \frac{D}{D_0},$$

where k is the reduction factor to be determined from experimental measurements, and $D_0 = 0.0012928$ is the air density with an air pressure of 760 mm and the temperature 0° . For the pendulum used here there was found $k = -0.0133 \text{ mm}$.

For this, we take from the observation tables:

Air pressure	$B = 752 \text{ mm}$,
Air temperature	$t = +14.6^\circ$,
Air humidity	$f = 79\%$.

Since for the temperature $+14.6^\circ$ the pressure of saturation of the water vapor is equal to 12.4 mm, then we have in the case of a relative air humidity of 79% an absolute pressure of water vapor $e = 0.79 \times 12.4 \text{ mm}$ or $e = 9.8 \text{ mm}$. Then we have (cf. Vol. II, 2nd half-volume, 1933, sections 41 and 52):

$$\frac{D}{D_0} = \frac{B - 0.377 e}{760 (1 + 0.003665 t)} = 0.93 \quad k \frac{D}{D_0} = -0.012 \text{ mm},$$

and there follows hence the value:

$$L = 994.247 \text{ mm}$$

for the length of the second pendulum free from the air resistance.

This result has followed from two observations of oscillations with the heavy weight above and the heavy weight below.

According to p. 344, two further observations are to be combined with this, after an exchange of knife-edges was undertaken in order to eliminate a difference in the radii of curvature of the knife-edges. For our example, for this there are to be taken the observations of 29 May from which, after taking into account all reductions, there follows the value $L = 994.220$ mm. We therefore have as the final length of the second pendulum:

$$L = 994.234 \text{ mm} .$$

As follows from the observation program indicated on p. 367, a repetition of the whole series of measurements after turning the pendulum around the vertical axis took place again in order to eliminate still further systematical errors; in general, only very small deviations resulted in the length of the second pendulum in the case of these repetitions.

In the foregoing, we have not taken into account two small corrections, which result from a bending and expanding of the pendulum in the case of the oscillations, and which in the case of the measurements in Potsdam together amounted to 3 to 4 thousandths of a millimeter at the most.

In all there were measured 40 series with five different pendulums among which there also was a half-second pendulum. The circumstance that the masses of these five pendulums were different made it possible that a few further error sources were taken into account, e.g., a constant error in the measurement of the distance between the knife-edges, since the subdividing marks of the scale and the knife-edges formed objects of very different kinds to be adjusted. Further, there are involved minor temperature changes during the measurement, gliding of the knife-edge on the bearing and others. Since these error sources had to manifest themselves in the case of the individual pendulums in a different manner, error equations could be set up, from which the unknowns could be computed according to the method of least squares.

We pass over this adjustment and give only the final result:

For the Pendulum Room of the Geodetic Institute in Potsdam, $52^{\circ}22.86'$ north latitude, $13^{\circ}4.06'$ longitude east of Greenwich, 87 m above sea level we have:

$$\begin{aligned} \text{the length of the simple second pendulum} &= 994.239 \pm 0.003 \text{ mm} \\ \text{and the acceleration of gravity} &= 981.274 \pm 0.003 \text{ gal} . \end{aligned}$$

More recent absolute gravity measurements

Washington. During the time from September 1934 to July 1935, a new determination of the absolute gravity was carried out by the National Bureau of Standards in Washington. A detailed report has been published about these measurements: "The value of gravity at Washington," by Paul R. Heyl and Guy S. Cook, *Journal of Research of the National Bureau of Standards*, vol. 17, 1937, pp. 805-839.

There were used four reversion pendulums of quartz in the form of cylindrical tubes with the following dimensions:

Outer diameter	4	4.5	5	7	cm
Inner diameter	3.4	3.4	3.6	5.8	

For the reception of the suspension device, the pendulums had at two places openings into which a plate with a rectangular cross section could be set, whose undersurface was smoothed completely, and which projected a little from the body of the pendulum on both sides. The side faces of the plate were provided with a mirror covering for the coincidence observations.

As the bearer of the pendulum apparatus there was used a heavy console, fastened to the wall of the observation room, on which, over a circular sector, the knife-edge was set, after it was pushed through the opening of the pendulum body. Different knife-edges of quartz, agate and other materials were used, whereby for the knife-edge an angle of 135° was found as the most favorable angle.

There is further to be mentioned the vacuum container in the form of a brass cylinder with an inner

diameter of 16 cm, which was provided with the necessary windows for the observation of the coincidences and for the measurement of the amplitude. Further there were provided openings for the connection of the air pump and the pressure gauge. Finally, a stopcock placed at the lower end served as pneumatic starter of the pendulum. At the beginning of the evacuation, by means of this cock, there was admitted the entry of air blasts which set the pendulum in motion until the desired amplitude was reached. Then further evacuation took place until less than 0.1 mm air pressure remained.

We pass over the further details of the apparatus and of the measurement, for which we refer to the above-mentioned publication, and only give the results.

From 70 individual measurements there was found:

$$g = 980.080 \pm 0.003 \text{ gal.}$$

The following values hold here for the position of the place of observation:

latitude	= 38°56' 30.143"
longitude	= 77 03 56.893
elevation above sea-level	= 94.75 m.

We give a few additional values for gravity in Washington, which were found in previous years by relative measurements (cf. section 76) in connection with European stations.

G. R. Putnam, 1900	$g = 980.112 \text{ gal}$
through Ottawa	980.117
Vening Meinesz	980.121
Brown, 1933	980.118

The differences occurring between these different determinations, which reach nearly 10 mgal, gave rise to the above-mentioned direct measurement of gravity in Washington.

London. A further determination of absolute gravity took place in the years 1936-1938 in the National Physical Laboratory in Teddington near London, about which there is a report in "An absolute determination of the acceleration due to gravity," by J. S. Clark, *Phil. Trans. of the Royal Society of London*, Series A. Math. and Phys. Sciences, No. 787, vol. 238, pp. 65-123, 17 February 1939, London, Cambridge University Press.

In these fundamental measurements, also, the reversion pendulum with level parallel surfaces which rested on the fixed knife-edges was used; another form of the pendulum rod was chosen, however. As shown in Fig. 1, the latter had an I-shaped cross section with dimensions 4.5 and 10 cm and a wall thickness of about 7 mm. In Fig. 1, the upper end of the pendulum is represented in the front and side view, as well as in the cross section. At the end of the I-shaped rod there is a cut for inserting the knife-edge *A*. On the rod there is set, by means of screw bolts, a plate *B* with level parallel end faces, whose lower surface *aa* serves as the supporting plane for the knife-edge, while the upper surface *bb* is used as the mirror for the measurement of the amplitude of oscillation.

The lower end of the rod carries a similar plate, to which two further plates are connected, however, so that these three plates form the heavier weight of the reversion pendulum. The undermost surface serves again as the mirror in the second position of the pendulum. The thickness of the centerplate is corrected by grinding off in such a way that the same duration of oscillation results for both positions of the pendulum.

The knife-edges are made of tempered steel and have an angle of 120°; thereby the sharpness of the knife-edges was such that their radius of curvature could be assumed smaller than 20 μ . Experiments with

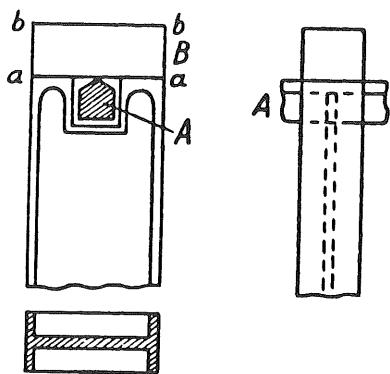


Fig. 1.

larger radii of curvature of 78, 173 and 282 μ showed a clear increase of the duration of oscillation with increasing radius.

For the measurements there was used a support of cast iron, which carries on top a heavy bronze plate for receiving the knife-edge. The two parts of the vacuum cylinder are connected to the plate, downward and upward.

The arrangement of the measurements was such that, after the pendulum had reached the desired amplitude, there was inserted a contact device with which the oscillations were registered during half a minute. After this, one let the pendulum swing freely for three hours, and finally a renewed registration for half a minute was carried out as the termination of the measurements. By taking into account the necessary corrections, which we have already learned in the foregoing on pp. 369 and 370, there followed hence the duration of oscillation.

For the determination of the pendulum length as given by the distance of the surface aa in Fig. 1 from the corresponding surface at the other end, the comparison of this length took place with a normal meter *à bouts* on an interference comparator. The pendulum length was hereby found with a mean error of $\pm 0.25 \mu$.

The measurements with the reversion pendulum took place in nine series during the period from July 1936 to February 1938. As the final result there followed as the mean value of all measurements:

$$g = 981.1815 \text{ gal} \pm 1.4 \text{ mgal}.$$

The geographic values for the place of observation are hereby:

$$\varphi = 51^\circ 25' 14''$$

$$L = 0^\circ 20' 21'' \text{ w. Gr.}$$

$$H = 10 \text{ m}.$$

From the relative connecting measurements of Potsdam there follows the value $g = 981.193 \text{ gal}$. Conversely, if we compute the gravity at Potsdam from the value of g at Teddington and, also, from the value at Washington with the help of the connecting measurements, then we obtain a value which, in the mean, is about 20 mgal smaller than the value found by Kühnen and Furtwängler in 1906.

Section 76. The Pendulum Apparatus of Sterneck

The measurement of the absolute value of gravity is connected with great difficulties, as is seen from the preceding sections, and can therefore be carried out only at a few points of the earth's surface.

The ratio of the values of gravity for two points can be determined with far less difficulty. For, if the duration of oscillation of the same pendulum is measured at two places, then we have:

$$\frac{T_1^2}{T_2^2} = \frac{g_2}{g_1} \quad \text{or} \quad g_2 = g_1 \frac{T_1^2}{T_2^2},$$

i.e. the ratio of g_2 and g_1 can be determined from the two periods of oscillation without the knowledge of the length of the mathematical pendulum. Hence, a pendulum of any arbitrary form can be used for this.

In 1887 the Austrian Major von Sterneck constructed a pendulum apparatus, especially suited for relative gravity measurements. Of this apparatus, which with some improvements is used at present in all civilized countries, we give at first a description of the original form, as is represented in the *Mitteilungen des k. k. Militär-geographischen Instituts*, Band VII, 1887.

In order to be able to carry out the determinations of gravity also at points difficult of access, for instance on high mountains, it is important that the size of the apparatus be as small as possible, and therefore Sterneck introduced a pendulum with a duration of oscillation of half a second.

The whole apparatus, which Fig. 1, p. 376, shows, consists of the support, the pendulum and the thermometers.

The pendulum has a length of approximately 25 cm and consists of a brass rod as well as a lens made likewise of brass in the form of a truncated double cone; both parts are heavily gilded. For the reception of the knife-edges of agate, a holder, which carries at the same time a small mirror *s* of 10×15 mm face, is screwed on at the upper end of the rod.

Below at both sides of the rod there are ground to the agate two knife-edges each. The knife-edges at the extreme end, the auxiliary knife-edges, are used for suspending the pendulum, while the two knife-edges near the pendulum rod lie on the agate plate of the support after the suspension device has been let down. Underneath the auxiliary knife-edges, depressions are provided in the agate plate so that they do not touch the latter when swinging.

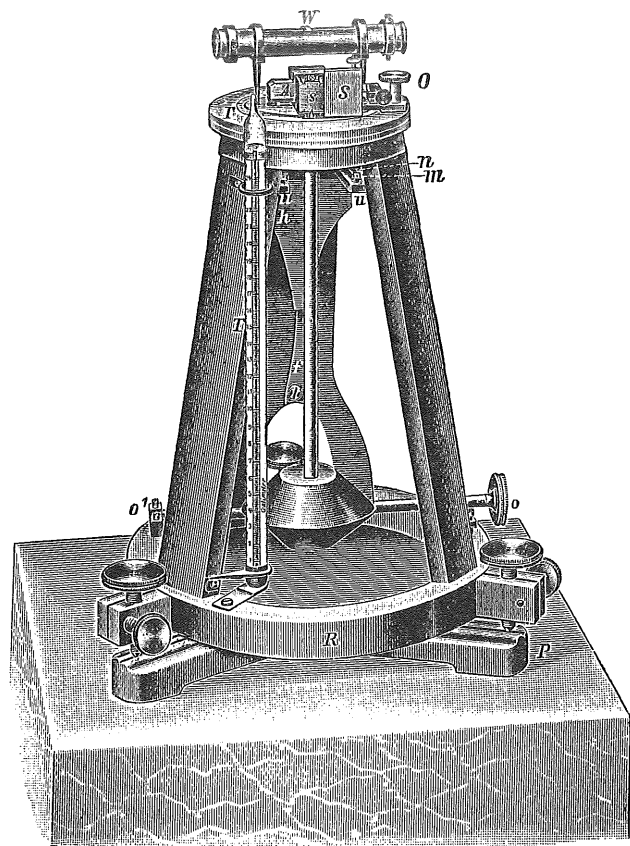


Fig. 1.

be set parallel to the pendulum mirror *s* by two small adjusting screws when the pendulum is at rest.

Finally, there is, in addition, a set level *W* of 6" of arc [per subdividing mark] for the horizontal setting of the agate plate.

At the lower part of the support there is visible a special device, with which the pendulum can be set in motion or stopped. For this, there is placed at a rod *oo'*, which is movable by means of a button, an arm with which the pendulum lens can be brought out of the rest position as far as desired. After this, if the arm is quickly turned back, then the pendulum becomes free and can now swing around the knife-edge. In addition, the rod *oo'* is provided with two adjustable attachments, of which one prevents the further turning as soon as a definite deflection is given the pendulum, while the second indicates the rest position of the rod in which the arm does not offer a hindrance to the oscillations of the pendulum.

Since the temperature of the pendulum must be taken into account in the case of the observations of oscillations, a very carefully tested mercury thermometer *T* is fastened at the pendulum support.

After the correct mounting, the whole pendulum apparatus is covered with a glass case, which keeps off air currents, dust and so on and also prevents quick temperature changes. Through small openings, provided with closing flaps, the raising device and the rod *oo'* can be used from the outside.

For the setting up of the pendulum apparatus in the field, there was used a stone pillar consisting of

As a rule, there are several pendulums of the same make, which are used one after the other in order to render various error sources as harmless as possible.

The support is made of red cast metal in one piece and consists mainly of two strong rings which are connected with one another by three stays. The lower ring *R* is provided with three lugs through which three foot screws go. With the latter, the support rests on a three-arm cast-iron plate *P*, which is provided above with deep grooves and rests only on the pillar underneath the support base.

On the headplate *r* of the support there rests a round agate plate, evenly ground off, with a thickness of 5 mm, which is held firm by a brass ring. For the suspension of the pendulum, the agate plate as well as the support head is cut in an oval opening; in addition, there are two small borings for the pins of the suspension device. For the raising and lowering of the pins, a horseshoe-shaped fork *uu* is used; the latter carries two flat springs *n*, which can be adjusted by means of the small screws *m*. The fork *uu* forms the horizontal arm of a knee-jointed lever, whose other arm *h*, directed downward, is moved by a screw. The spring *f* presses the arm *h* continually against the screw. Through the fact that the raising pins do not rest directly on the lever, but on the springs *n*, a very smooth raising and lowering of the pendulum is possible.

With the help of the screw *O*, there is fastened on the support head a rather large mirror *S*, which can

four parts and therefore easily transportable.

To the pendulum apparatus there belongs, in addition, the coincidence apparatus likewise constructed by von Sterneck, which we have already described in section 73, pp. 359 and 360.

Newer pendulum apparatuses

Sterneck's pendulum apparatus turned out to be a very valuable auxiliary means for relative gravity measurements and, therefore, found reception in all countries which took part in geodetic operations. On the basis of experience thereby collected, improvements for the removal of some error sources were introduced.

Increased care was given the measurement of the temperature of the pendulum by introducing a special pendulum thermometer, which in its outward form resembles the pendulum perfectly and is made also of the same material. The rod carries through its whole length a boring which is filled by the thermometer tube. In the newer apparatuses, the pendulums are, as a rule, made of invar, a fact by which the influence of the temperature is considerably lessened. For pendulum measurements, finally, one chooses rooms which are subject only to small variations of temperature.

In the first measurements with the Sterneck apparatus it was assumed that with respect to the small length of the pendulum the co-vibration of the support would be meaningless, which did not prove to be true, however. An improvement was achieved by wall supports, which were fastened at the masonry of the observation room. By these, the co-vibration was not entirely cut out, however, and for the determination of the co-vibration there was used the thread pendulum, of which we have already learned in section 74, p. 362. It is more convenient, however, to determine the co-vibration by two equal pendulums which swing on the same support and on the same plane. This method has proved best and is used exclusively at the present time. The oscillations of the two pendulums can also be arranged so that the co-vibration of the support is almost entirely cut out.

The use of several pendulums brings the additional advantage that irregular small changes of the pendulums can thereby be determined. Therefore, one uses now mostly four-pendulum apparatuses, in which the four pendulums swing in pairs on two planes perpendicular to one another.

In the newer apparatuses, finally, in order to make the influence of the air resistance as harmless as possible, the pendulums are placed in a vacuum in which the air pressure can be lowered to a few millimeters.

The four-pendulum apparatus of the Geodetic Institute in Potsdam

As an example of a new pendulum apparatus, in the following we describe briefly the four-pendulum apparatus of the Geodetic Institute in Potsdam, which is constructed in the workshop of the Institute by M. Fechner. There are two different forms of the apparatus in use, the hood apparatus [Haubenapparat] and the pot apparatus [Topfapparat].

In the case of the *hood apparatus*, at the center of a ring-shaped baseplate provided with three foot screws there rises a column whose head carries the bearings of the four pendulums. The planes of oscillation of the latter pass through the axis of the column, and the oscillation chambers are separated from one another by partitions in order to prevent disturbances by the air motion. The pendulums made of invar correspond to the original construction of Sterneck, Fig. 1, p. 376; like there, they also carry small mirrors for the observation of the recording of the oscillations. Since it is necessary, however, to be able to observe the oscillations of all four pendulums from the same direction, there is in addition required a special optical system, which is represented in Fig. 2 according to the specification of H. Schmehl. Let P_1 , P_2 , P_3 and P_4 denote the four pendulum mirrors, to which two reflecting prisms and two mirrors are connected. The method of operation of this *optical bridge* is evident in Fig. 2. Over the whole there is mounted a hood, shaped slightly conically, whose lower edge rests on the baseplate and gives an airtight closure by greasing.

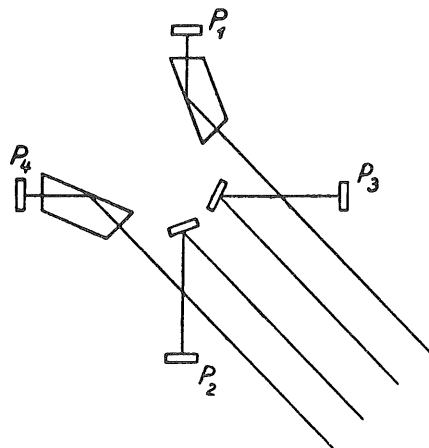


Fig. 2.

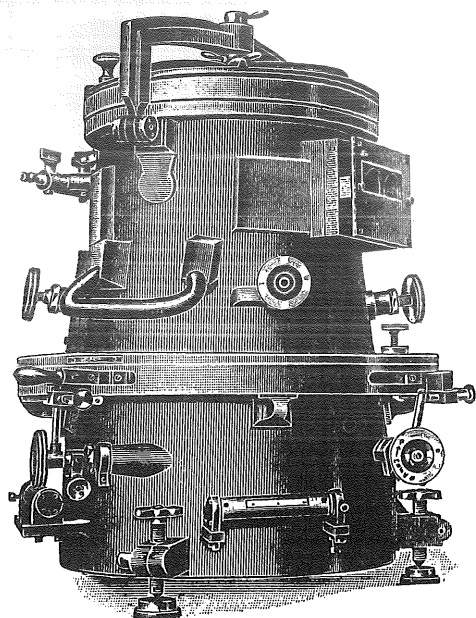


Fig. 3.

Fig. 3 represents a *pot apparatus* constructed likewise in the workshop of the Institute, in the case of which the pendulum bearings rest on four small consoles fastened at the pot wall. The closing of the apparatus is effected here by a cover, which is pressed on the edge of the pot by means of a bow and a screw. The pot apparatus guarantees a greater stability of mounting than the hood apparatus.

In addition, both apparatuses carry, in the interior, a shortened mercury barometer as well as several thermometers, which can be read through windows in the wall of the pot. The turnable discs placed at the lower part of the pot serve, in connection with the turnable ring above it, which is laid around the pot, to set the pendulums in motion, in pairs or both pairs together at the same time. Four additional handwheels at the upper part of the pot effect the arresting of the pendulums. Additional devices are provided for the connection of the air pump; besides, there are several tube levels for the setting up of the apparatus.

The registering apparatus. The measurement of the duration of oscillation occurs simultaneously for all four pendulums by photographic recording with the help of international time signals, of which we have already learned in the first half-volume, section 111, p. 586.* The apparatus used here is represented dia-

grammatically in Fig. 4. The bearer of the sensitized film is a drum T , which is turned by a clockwork on a screw spindle so that the drum hereby moves uniformly on longitudinally. The drum is in a small box, in whose front face there is a small opening O . At P there is indicated one of the four pendulum mirrors, throws from the lamp L_1 , by means of a lens system, a light flash on the film, which is registered there in the form of a point. A second mirror z is connected with the receiver of the time signals, and the impulses which the receiver receives are amplified in such a way that each signal causes a small turn of the mirror and, with this, throws a light flash on the film from the lamp L_2 .

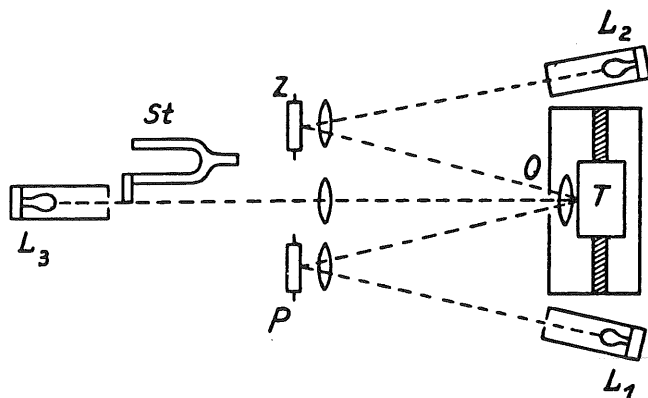


Fig. 4.

In this manner, the oscillations of the four pendulums and the time signals are registered on the film.

The accuracy of the recording depends on the clockwork moving the drum completely evenly, which will never be achieved. This inconvenience is eliminated with the help of a tuning fork St . At one of the two arms, the tuning fork carries a small screen which, in the position of rest, just reaches into the light path of a lamp L_3 . The tuning fork is set in motion by an electromagnet and in the case of each oscillation, a light flash likewise falls on the film from the lamp L_3 . A dashed line is hereby produced by the lamp L_3 , and since the oscillations of the tuning fork are completely uniform, a satisfactory subdivision of the time intervals between the time signals is obtained.

The accuracy of the determination of the duration of oscillation depends naturally on the time signals, and in order to eliminate the small inaccuracies with which the time signals are affected, there are used the quartz clocks of the Geodetic Institute whose rate changes are so minor that they can be neglected. The time signals, which are to be used according to a program set up before, are transferred also to the quartz clocks, so that the latter form the foundation for the evaluation of the observations.

* Not translated.

On the film one sees the time signals in seconds at intervals of 5 cm, further the dashed line produced by the tuning fork, whereby 40 dashes correspond to a second; finally, one sees the points which designate the oscillations of the four pendulums.

Of the five minutes during which the time signals of the international radio stations are transmitted, only the signals of the first and the last minute are received. Then one lets the pendulums swing for two hours until the next time signal reception, and from the oscillations of the two hours the duration of oscillation is computed.

We find the first description of a four-pendulum apparatus in *Zeitschr. f. Instr.*, 1896, pp. 193-196. The apparatus was constructed by C. Bamberg according to specifications of Haid and is based in its outer form completely on the Sterneck apparatus. We have described in detail this Haid four-pendulum apparatus in the 7th edition of the 3rd volume, 1923, pp. 685-686.

A three-pendulum apparatus of the Geodetic Institute in Potsdam is described in the *Veröffentlichung der Internationalen Erdmessung*, Neue Folge, Nr. 16, "Bestimmung der Schwerkraft auf dem Indischen und Grossen Ozean und an deren Küsten," by O. Hecker, Berlin, 1908. Since 1910, a four-pendulum apparatus has been used for the measurements of the Geodetic Institute; cf. *Veröffentlichung des Geodätischen Instituts*, Neue Folge, Nr. 71, "Bestimmung der Intens. d. Schwerkraft auf 35 Stationen, etc.," by L. Haasemann, Berlin, 1916, p. 29. Additional information, in this connection, is given by E. Kohlschütter, "Der neue Pendelapparat des Preuss. Geod. Inst.," *Verhandlungen der Baltischen Geodätischen Kommission*, Helsinki, 1928, p. 95. Finally, we mention in addition: O. Meisser, "Ein neuer Vierpendelapparat für relative Schweremessungen," *Zeitschr. f. Geophysik*, 1930, pp. 1-12.

A new form of pendulum for a quarter-of-a-second pendulum was proposed by J. Wilsing in 1897. The pendulum consists of a circular glass disc with an eccentric sector, through which the knife-edge passes. Cf.: "Über eine neue Form invariabler Pendel," *Zeitschr. f. Instr.*, 1897, pp. 109-114. The theoretical advantage of the invariable pendulums consists in the fact that small changes of the distance of the center of gravity from the knife-edge are almost without influence on the duration of oscillation. Experiments, which were undertaken with such a pendulum by the Geodetic Institute in 1899, did not lead to satisfactory results. (*Veröffentl. d. Geod. Inst.*, N.F., Nr. 19, "Bestimmung der Intensität der Schwerkraft auf 66 Stationen im Harz, etc.," by L. Haasemann, Berlin, 1905, p. 1.) A further invariable pendulum was proposed by E. Kohlschütter, "Über Pendelformen," *Verh. d. Balt. Geod. Komm.*, Helsinki, 1928, pp. 83-90.

L. Haasemann reports about other experiments with a quarter-of-a-second apparatus, which is equipped with four pendulums of the usual form, in the *Veröffentlichung des Geodätischen Instituts*, Neue Folge, Nr. 41, "Bestimmung der Intensität der Schwerkraft auf 42 Stationen im nördlichen Teil von Hannover, etc.," Berlin, 1909.

Section 77. The Co-vibration of the Support of the Sterneck Pendulum

Formerly it has been assumed that in view of the small size and the stable construction of the Sterneck pendulum apparatus the co-vibration of the support could be neglected. This assumption, however, has proven untenable, and the co-vibration must therefore be taken into account each time even in the case of relative gravity measurements.

For the determination of the constants of the co-vibration, there was used at first the thread pendulum, which had proved itself to be reliable in the case of the absolute gravity measurements. Such measurements, for instance, are described by Kühnen in the *Veröffentlichung des Geodätischen Instituts in Potsdam*, "Bestimmung der Polhöhe und der Intensität der Schwerkraft auf 22 Stationen von der Ostsee bis zur Schneekoppe," Berlin, 1896, pp. 249-258.

The difficulties of the measurements with the thread pendulum, however, are not reasonably proportional to the measuring procedure with the Sterneck pendulum apparatus, which is, after all, fairly simple.

A very simple method for the approximate determination of the constants of co-vibration was indicated by R. Schumann in *Astronomische Nachrichten*, Band 140, 1896, pp. 257-262. This "whip method" [Wippmethode] by Schumann consists in setting the pillar artificially in motion, by which the pendulum starts likewise to swing. The equipment required for this is indicated diagrammatically in Fig. 1 according to the above-mentioned publication of the Geodetic Institute. In Fig. 1, p. 380, we see at the left-hand side the pendulum pillar with the mounted support, while at the right-hand side there is placed, on a special support, a knee-jointed lever, by whose regular up and down movement the pillar is set in motion. The force thereby acting on the pillar is measured by an inserted dynamometer, and the knee-jointed lever can be adjusted in such a way that at each move the same definite force is applied.

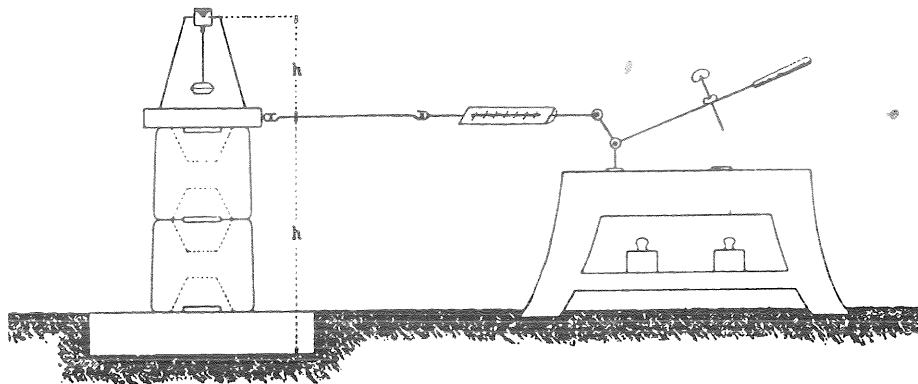


Fig. 1.

After the pendulum has started to swing through a number of moves, the amplitude is measured. By theoretical investigations, which were confirmed by experience, it can be proved that the deflection of the pendulum is nearly proportional to the number of moves and to the force thereby applied, and the measured deflection is therefore reduced to a single move and to the unit of force. If the reduced deflection is denoted by ϑ , then the following expression can easily be derived from the differential equation of the movement of the pendulum:

$$\frac{1}{\varepsilon} = \frac{2l}{\pi} \vartheta,$$

where ε denotes the constant of elasticity and l the length of the mathematical pendulum. From ε , the increase of the duration of oscillation caused by the co-vibration can then be determined (cf. section 71, p. 348). For this as well as for the development of the foregoing equation we refer to the communications by E. Borrass in *Bestimmung der Polhöhe und der Intensität der Schwerkraft in der Nähe des Berliner Meridians von Arkona bis Elsterwerda*, Berlin, 1902, pp. 95-99.

In the case of the newer pendulum apparatuses, in which there are two pendulums swinging in the same plane, there follows a simple method of determining the co-vibration with these two pendulums. If one pendulum is set in motion while the other one is at rest, then the latter begins likewise to swing due to the co-vibration of the support, and from the amplitudes of the two pendulums we can determine the amount of the co-vibration.

In the following, we will designate the driving pendulum as the first pendulum and the driven as the second pendulum. For the theory of these oscillations we can start with equation (22), section 74, p. 365, which was developed for the thread pendulum. The first and the second pendulums take now the place of the reversion and the thread pendulum. If we introduce the indices 1 and 2 for the two pendulums, then (22), section 74, p. 365, changes to

$$\alpha_2 = e^{-z_1 t} \left(c_1 \cos \sqrt{\frac{g}{l_2} - z_2^2} t + c_2 \sin \sqrt{\frac{g}{l_2} - z_2^2} t \right) + \frac{g \gamma_1}{l_1 (p^2 + q^2)} \varphi_1 e^{-z_1 t} \left(p \cos \sqrt{\frac{g}{l_1}} t + q \sin \sqrt{\frac{g}{l_1}} t \right). \quad (1)$$

The assumption that z_2 is very large in comparison to z_1 , which was admissible in section 74, p. 363, does now no longer prove correct; and hence, the first part of the foregoing equation, too, cannot be neglected.

We can determine the two constants of integration c_1 and c_2 by substituting $\alpha_2 = 0$ for $t = 0$ as well as $\frac{d\alpha}{dt} = 0$.

For $(\alpha_2)_{t=0} = 0$ we have $c_1 = -\frac{g \gamma_1 p}{l_1 (p^2 + q^2)} \varphi_1^\circ$ (2)

and for $\left[\frac{d\alpha_2}{dt}\right]_{t=0} = 0$ we will have

$$z_2 c_1 = c_2 \sqrt{\frac{g}{l_2} - z_2^2} - z_1 \frac{g \gamma_1 p}{l_1 (p^2 + q^2)} \varphi_1^\circ + \frac{g \gamma_1 q}{l_1 (p^2 + q^2)} \sqrt{\frac{g}{l_1}} \varphi_1^\circ. \quad (3)$$

If (2) is introduced in here, then we have

$$c_2 \sqrt{\frac{g}{l_2} - z_2^2} = -\frac{g \gamma_1}{l_1 (p^2 + q^2)} \varphi_1^\circ \left\{ (z_2 - z_1) p + \sqrt{\frac{g}{l_1}} q \right\},$$

and if the value from (21), section 74, p. 365, is used for q ,

$$c_2 \sqrt{\frac{g}{l_2} - z_2^2} = -\frac{g \gamma_1 (z_2 - z_1)}{l_1 (p^2 + q^2)} \varphi_1^\circ \left[p + 2 l_2 \frac{g}{l_1} \right]. \quad (4)$$

The two pendulums shall now be constructed in such a manner that the damping is the same in both cases, so that we can set $z_1 = z_2 = z$. Then there follows from (4) that $c_2 = 0$, and from (21), section 74, p. 365, that $q = 0$. Besides, since with respect to the large weight of the pendulums z is very small, then we can set $\sqrt{\frac{g}{l_2}}$ instead of $\sqrt{\frac{g}{l_2} - z^2}$. With these values, (1) changes to

$$\alpha_2 = -e^{-z t} \frac{g \gamma_1}{l_1 p} \varphi_1^\circ \cos \sqrt{\frac{g}{l_2}} t + e^{-z t} \frac{g \gamma_1}{l_1 p} \varphi_1^\circ \cos \sqrt{\frac{g}{l_1}} t. \quad (5)$$

According to (25), section 69, p. 338, $\varphi_1^\circ e^{-z t}$ is the amplitude of the first pendulum after the time t , which we will denote by φ_1 . Therefore, we have

$$\alpha_2 = -\frac{g \gamma_1}{l_1 p} \varphi_1 \left(\cos \sqrt{\frac{g}{l_2}} t - \cos \sqrt{\frac{g}{l_1}} t \right). \quad (6)$$

With the omissions hitherto used we have $p = g \frac{l_1 - l_2}{l_1}$. If we call further the duration of oscillation of the pendulums T_1 and T_2 , then we have

$$\frac{g}{l_1} = \frac{\pi^2}{T_1^2} \quad \text{and} \quad \frac{g}{l_2} = \frac{\pi^2}{T_2^2}, \quad (7)$$

therefore we will have

$$\begin{aligned} \alpha_2 &= -\varphi_1 \frac{\gamma_1}{l_1 - l_2} \left[\cos \pi \frac{t}{T_2} - \cos \pi \frac{t}{T_1} \right], \\ \text{or} \quad \alpha_2 &= \varphi_1 \frac{2 \gamma_1}{l_1 - l_2} \sin \frac{\pi t}{2} \frac{T_1 - T_2}{T_1 T_2} \sin \frac{\pi t}{2} \frac{T_1 + T_2}{T_1 T_2}. \end{aligned} \quad (8)$$

This equation tells us that the amplitude of the second pendulum, which is determined essentially by the product $\varphi_1 \sin \frac{\pi t}{2} \frac{T_1 + T_2}{T_1 T_2}$, first gradually decreases because of the damping of φ_1 , second, however, has a period because of the factor $\sin \frac{\pi t}{2} \frac{T_1 + T_2}{T_1 T_2}$. The pendulum reaches the amplitude every time when

$\sin \frac{\pi t}{2} \frac{T_1 + T_2}{T_1 T_2} = 1$; if we denote this amplitude by φ_2 , then we have accordingly

$$\begin{aligned} \varphi_2 &= \varphi_1 \frac{2\gamma_1}{l_1 - l_2} \sin \frac{\pi t}{2} \frac{T_1 - T_2}{T_1 T_2}, \\ \text{or according to (7)} \quad \varphi_2 &= \varphi_1 \frac{2\gamma_1 \pi^2}{g(T_1^2 - T_2^2)} \sin \frac{\pi t}{2} \frac{T_1 - T_2}{T_1 T_2}, \end{aligned} \quad (9)$$

and we find hence:

$$\gamma_1 = \frac{\varphi_2 (T_1^2 - T_2^2)}{\varphi_1} \frac{g}{2\pi^2} \operatorname{cosec} \frac{\pi t}{2} \frac{T_1 - T_2}{T_1 T_2}. \quad (10)$$

In the case of relative gravity measurements, however, only the duration of oscillation is of interest to us, and we will therefore determine, in addition, the influence of the co-vibration on the duration of oscillation of the first pendulum. If we denote this influence by ΔT_1 , then we have according to (17), section 71, p. 348,

$$\Delta(T_1^2) = \pi^2 \frac{a_1 M_1}{\varepsilon l_1},$$

and according to (14a), section 71, p. 347,

$$\Delta(T_1^2) = -\pi^2 \frac{\sigma}{g a_1} = +\pi^2 \frac{\gamma_1}{g}. \quad (11)$$

But since

$$\Delta(T_1^2) = (T_1 + \Delta T_1)^2 - T_1^2 = T_1^2 + 2 T_1 \Delta T_1 + \dots - T_1^2$$

or

$$\Delta(T_1^2) = 2 T_1 \Delta T_1$$

then we will have

$$\Delta T_1 = \frac{\Delta(T_1^2)}{2 T_1} = \frac{\pi^2}{2 T_1} \frac{\gamma}{g}$$

and with (10)

$$\Delta T_1 = \frac{\varphi_2}{\varphi_1} \frac{T_1^2 - T_2^2}{4 T_1} \operatorname{cosec} \frac{\pi t}{2} \frac{T_1 - T_2}{T_1 T_2}.$$

This expression can still be simplified a little since T_1 and T_2 are nearly equal to one another. Therefore, we can set

$$T_1^2 - T_2^2 = (T_1 - T_2)(T_1 + T_2) = (T_1 - T_2) 2 T_1 + \dots$$

and obtain with this

$$\Delta T_1 = \frac{\varphi_2}{\varphi_1} \frac{T_1 - T_2}{2} \operatorname{cosec} \frac{\pi t}{2} \frac{T_1 - T_2}{T_1 T_2}. \quad (12)$$

If T_1 is the observed duration of oscillation and T_1° the reduced duration of oscillation, then we have

$$T_1^\circ = T_1 - \Delta T_1. \quad (13)$$

For a more detailed explanation of this method, we extract an example which Hecker communicates in *Bestimmung der Schwerkraft auf dem Atlantischen Ozean sowie in Rio de Janeiro, Lissabon und Madrid*, Berlin, 1903, pp. 98-99.

The two pendulums Nos. 21 and 28 are used. Pendulum No. 28 is set at rest while pendulum No. 21 receives a push at the clock time $3^h 43^m 30^s$. The deflection of the driving pendulum No. 21 is read at the scale of the Sterneck coincidence apparatus of section 73, p. 360, while for the measurement of the deflection of the driven pendulum No. 28 there is used the micrometer screw at the telescope of the coincidence apparatus, whose movable thread is adjusted to the two reversing points of the oscillation of the pendulum.

Pendulum No.	Time of Observation	Deflection		Double Amplitude
		Above	Below	
21	$3^h 47^m$	6.5	4.9	11.4 <i>p</i>
28	48	1.16	1.26	0.10 <i>rev</i>
28	49	1.16	1.28	0.12 <i>rev</i>
21	50	6.3	4.9	11.2 <i>p</i>
21	57	8.0	2.6	10.6 <i>p</i>
28	58	0.91	1.17	0.26 <i>rev</i>
28	59	0.91	1.18	0.27 <i>rev</i>
21	4 00	4.8	5.7	10.5 <i>p</i>
21	7	4.5	5.3	9.8 <i>p</i>
28	8	0.88	1.20	0.32 <i>rev</i>
28	9	0.88	1.19	0.31 <i>rev</i>
21	10	4.3	5.2	9.5 <i>p</i>

The magnitude t in (12) is to be counted from the time of starting. The duration of oscillation T_1 and T_2 is known for the two pendulums from the actual pendulum measurements; we have

$$T_1 - T_2 = 0.000172^s \quad T_1 T_2 = 0.26005 .$$

If we form mean values for each of the three above measuring groups, and take thereby into account that $1p$ of the scale is equal to 0.915 rev of the micrometer screw, then we obtain:

t	φ_1	φ_2	$\frac{T_1 - T_2}{2} \operatorname{cosec} \frac{\pi t}{2} \frac{T_1 - T_2}{T_1 T_2}$	ΔT
5^m	11.30'	0.12'	0.000 280	$30 \cdot 10^{-7}$
15	10.55	0.29	0.000 107	29
25	9.65	0.34	0.000 086	31

The mean value of the correction because of the co-vibration of the support for pendulum No. 21 and for the listing used here thus is:

$$\Delta T = 0.000 0030^s .$$

In the case of the above measurements, the oscillations occurred in the north-south direction; for every other direction of oscillation, the correction has to be determined anew.

The foregoing formula (12) for the computation of the co-vibration was first derived by E. Borrass in *Bestimmung der Polhöhe und der Intensität der Schwerkraft in der Nähe des Berliner Meridians von Arkona bis Elsterwerda*, Berlin, 1902, p. 90.

A thorough theoretical examination "Über die Verwendung zweier Pendel auf gemeinsamer Unterlage zur Bestimmung der Mitschwingung" was communicated by R. Schumann in *Zeitschrift für Mathematik und Physik*, 1899, pp. 102-138. M. Haid gave another representation in *Astronomische Nachrichten*, Band 143, 1897, pp. 145-152. The treatise by Ph. Furtwängler mentioned already in section 74, p. 367, is of fundamental significance.

The first determination of the co-vibration was carried out by Peirce (cf. p. 350) in the manner that on the pendulum bearing there was exerted a horizontal motion of a known quantity, whose effect was read off immediately on a scale with a microscope at the pendulum bearing. This method was improved by Hirsch to the effect that the motions of the pendulum bearing were converted to rotations of a small mirror around a horizontal axis. (*Verhandlungen der 5. allgemeinen Konferenz der europäischen Gradmessung*, Berlin, 1878.) This method was used also in the Geodetic Institute in Potsdam by L. Haasemann, about which there is a report in the *Veröffentlichung des Geodätischen Instituts*, N.F. Nr. 41, "Bestimmung der Intensität der Schwerkraft auf 42 Stationen im nördlichen Teil von Hannover, etc.," Berlin, 1909. The experiments led to good results, but did not show any advantages compared with the two-pendulum method.

Section 78. Reduction of the Pendulum Measurements.

Influence of the Air Density and the Pendulum Temperature

With the use of the Sterneck coincidence apparatus (section 73, p. 360), the computation of the duration of oscillation from the coincidences is carried out in the same manner as in the case of the absolute measurements (section 73, p. 361). If, as on p. 360, c denotes again the number of the oscillations of the clock pendulum between two coincidences, then the half-second pendulum has carried out $2c \pm 1$ oscillations during this time, and the duration of oscillation in seconds of the clock is

$$T = \frac{c}{2c \pm 1} \quad (1)$$

or
$$T = \frac{1}{2} - \frac{1}{4c + 2} \quad \text{or, as the case may be,} \quad T = \frac{1}{2} + \frac{1}{4c - 2} \quad (2)$$

If Δu is the daily rate of the clock, then according to (3), section 73, p. 361, the correction because of the rate of the clock is

$$\Delta T_u = \frac{\Delta u}{86,400} T. \quad (3)$$

Since for precise time determinations only fixed stars are considered, then a clock going by sidereal time is used, and hence, the duration of oscillation in seconds of sidereal time is also obtained. A conversion to mean time is not required, because only the ratio of the periods of oscillation of different places is to be determined. If sidereal time is not used everywhere, then all measurements must be reduced to the same measure of time.

For the reduction of mean time to sidereal time, the measured duration of oscillation is to be multiplied by 1.002 73791.

For the reduction of sidereal time to mean time, the measured duration of oscillation is to be multiplied by 0.997 26957.

In the case of photographic recording according to section 76, p. 378, the duration of oscillation is obtained by measurement of the film.

The amplitude reduction is carried out according to the method indicated previously in section 73, p. 361. If φ is the mean amplitude, then we have according to (4), section 73, p. 361,

$$\Delta T_\varphi = -\frac{\varphi^2}{16} T, \quad (4)$$

where φ (as an *arc*) is determined from the scale readings at the coincidence apparatus and the distance of the latter from the pendulum.

The correction because of the co-vibration of the support is according to (12) and (13), section 77, p. 382:

$$\Delta T_m = - \frac{\varphi'}{\varphi} \frac{T - T'}{2} \operatorname{cosec} \frac{\pi t}{2} \frac{T - T'}{T T'},$$

where φ' and T' denote the amplitude and the duration of oscillation of the auxiliary pendulum driven by the observation pendulum after a time t , which is counted from the beginning of the oscillation.

Influence of the air resistance

For the determination of the influence of the air resistance on the duration of oscillation, it has proved sufficient to assume the influence of the air resistance proportional to the relative air density, so that the

$$\text{correction because of air density is } \Delta T_l = - k_l \frac{D}{D_0}. \quad (5)$$

In this, k_l denotes the constant of air density, D the air density and D_0 the air density for an air pressure of 760 mm and an air temperature of 0° . According to section 75, p. 372, we have

$$\frac{D}{D_0} = \frac{B - 0.377 e}{760 (1 + 0.003665 t)}, \quad (6)$$

where the following designations hold:

B = air pressure

e = water vapor pressure

t = air temperature .

The air density constant k_l can therefore be found if the periods of oscillation in the case of different values of the air pressure B are measured. As an example of such a determination, we take a few observation data from O. Hecker, *Bestimmung der Schwerkraft auf dem Schwarzen Meere und an dessen Küste*, etc., Berlin, 1910, p. 4. For each pendulum there took place six oscillation observations with a normal air pressure as well as with an air pressure of about 320 mm, of which we indicate in table (7), in the form of an extract, three observations in each case for pendulum No. 6.

P e n d u l u m M e a s u r e m e n t s f o r D i f f e r e n t
A i r P r e s s u r e s (7)

Air Press. B	Vapor Press. e	Pend. Temp. t	Deflec- tion α	Daily Clock Rate u	Period of Coinci- dences c
750.3 mm	8.2 mm	+ 11.86°	18.8'	—0.26 ^s	100.2283 ^s
754.4	8.4	11.83	20.1	—0.17	100.2275
763.6	8.6	12.04	20.7	—0.07	100.1762
318.9	3.6	12.15	21.8	—0.05	101.5862
321.5	3.6	12.27	21.4	—0.16	101.5604
322.0	3.7	12.35	21.4	—0.23	101.4912

In the case of the reduction of these observations, there is also to be taken into account the influence of temperature, which we will treat only in the following. For this, we have according to (10), p. 387,

$$\Delta T_t = -k_t t,$$

where k_t denotes the temperature constant. For the pendulum used here, there has been found

$$k_t = 0.000\ 0048.02.$$

The complete reduction of the above measurements is assembled in table (8).

Reduction of the Observations

(8)

Observed Duration of Oscillation T	Correction Because of			Reduced Duration of Oscillation T_0	Relative Air Density $\frac{D}{D_0}$
	Deflec- tion ΔT_α	Tempera- ture ΔT_t	Clock Rate ΔT_u		
0.502 5068 ^s	— 10	— 571	— 15	0.502 4472 ^s	0.9419
0.502 5068	— 11	— 566	— 10	0.502 4481	0.9475
0.502 5081	— 12	— 576	— 4	0.502 4489	0.9585
0.502 4731	— 13	— 586	— 3	0.502 4129	0.4004
0.502 4738	— 12	— 590	— 9	0.502 4127	0.4035
0.502 4754	— 12	— 595	— 13	0.502 4134	0.4040

The corrections are computed here in units of the seventh decimal of the duration of oscillation. A correction because of the co-vibration of the support was not required, because the influence of the co-vibration remained constant for all measurements.

For the computation of k_t we denote the duration of oscillation freed from the influence of the air density by x and set $k_t = y$, with which we can then set up the error equation

$$T_0 + v = x + \frac{D}{D_0} y.$$

With the numerical values from (8) we obtain:

$$\left. \begin{aligned} 0.502\ 4472 + v_1 &= x + 0.9419\ y \\ 0.502\ 4481 + v_2 &= x + 0.9475\ y \\ 0.502\ 4489 + v_3 &= x + 0.9585\ y \\ 0.502\ 4129 + v_4 &= x + 0.4004\ y \\ 0.502\ 4127 + v_5 &= x + 0.4035\ y \\ 0.502\ 4134 + v_6 &= x + 0.4040\ y. \end{aligned} \right\} \quad (9)$$

We will no longer carry out the further treatment of these error equations with all details, but refer to our first volume, 8th edition, 1935, chapter I.

After setting up the normal equations, we obtain finally

$$k_t = 0.000\ 0642^s$$

as the final value of the air density constant.

For the determination of the influence of the air temperature also, the pendulum apparatus must be set up in a container, in which the air can be brought to different temperatures by the use of ice or hot water.

The change of the duration of oscillation occurs likewise nearly proportional to the air temperature, and therefore we can set:

$$\Delta T_t = -k_t t. \quad (10)$$

In order to be able to show this determination also by an example, in table (11) we assemble a few measurements, which were carried out in the Geodetic Institute in Potsdam for the same pendulum No. 6. According to O. Hecker, *Bestimmung der Schwerkraft auf dem Indischen und Grossen Ozean und an deren Küsten*, Berlin, 1908, p. 11, we have

Reduced Duration of Oscillation T_0	Pendulum Temperature t
0.508 3788 ^s	5.14°
0.508 3787	4.97
0.508 3785	4.87
0.508 5367	40.04
0.508 5370	39.94
0.508 5380	40.49

(11)

For the computation of k_t , error equations would have to be set up therefrom, which we have just shown in all thoroughness. But since the measurements are limited essentially to two nearly constant temperatures, then we will form the mean values for these two temperatures and obtain

T_0	t
0.508 3787 ^s	4.99°
0.508 5372	40.16
0.000 1585 ^s	35.17°

therefore, we will have

$$k_t = \frac{0.000\ 1585}{35.17} = 0.000\ 0045 \cdot 1^s.$$

The adjustment by the method of least squares led to the same result.

The numerical data in (11) form only a small part of the measurements, communicated elsewhere, from which the more accurate value of k_t used already on p. 386, was computed.

Example of a pendulum observation

The measurement of the duration of oscillation by means of the coincidence apparatus (section 73, p. 360), is carried out as a rule in such a manner that about ten coincidences are observed, whereupon a pause occurs after which the same amount of coincidences is observed again. According to the scheme introduced in the Geodetic Institute, let us communicate a complete example from Hecker, *Bestimmung der Schwerkraft auf dem Atlantischen Ozean*, p. 115.

Coincidences

No.	Clock Time	No.	Clock Time	72 c	Mean: 72 c = 31 ^m 29.78 ^s c = 26.2469 ^s T = 0.509 7099 ^s
1	19 ^h 46 ^m 10.6 ^s	73	20 ^h 17 ^m 40.3 ^s	31 ^m 29.7 ^s	
2	37.5	74	18 07.4	29.9	
3	47 03.1	75	32.8	29.7	
4	30.0	76	59.8	29.8	
5	55.6	77	19 25.3	29.7	
6	48 22.4	78	52.2	29.8	
7	48.1	79	20 17.8	29.7	
8	49 14.9	80	44.7	29.8	
9	40.6	81	21 10.4	29.8	
10	50 07.4	82	37.3	29.9	

	Barom.	Temp.	Hygrom.	Pendulum Temperature	Deflection	
					Above	Below
Beginning .	709.3 mm	+ 15.5°	52 %	+ 14.90°	5.3 p	5.3 p
End	709.4	+ 15.5	52	+ 14.98	4.1	4.1
Mean	709.35	+ 15.5	52	+ 14.94	4.7	4.7
Correction.	707.10		or 6.5 mm	+ 14.90		

Distance: Mirror-scale: 1.72 m. Division of the scale: 1 p = 3 mm.

Deflection: $\alpha = 4.7 p = 14.1'$.

Direction of oscillation: East-west. Daily rate of clock: -0.65^s .

Observed duration of oscillation	0.509 7099 ^s
Correction because of deflection	— 5
Correction because of air pressure	— 571
Correction because of temperature	— 654
Correction because of rate of clock	— 38
Correction because of co-vibration	— 37
Reduced duration of oscillation	0.509 5794 ^s .

At the end of our representations about gravity measurements with the help of the pendulum, we treat further in this section two special methods, which likewise make use of the pendulum, but, otherwise, vary substantially from the methods thus far described.

1. *Gravity measurements by F. A. Vening Meinesz in a submarine*

The use of two pendulums swinging in the same plane offers the possibility of determining the co-vibration of the support in a simple manner, as we have seen in section 77. In 1923 it was proved by F. A. Vening Meinesz that such a two-pendulum apparatus is suited also for gravity measurements on unsteady ground, and that this method can also be utilized for measurements on the ocean in the submarine. It turned out that at a depth of about 30 m below sea level the motions of the water are already damped in such a way that a submarine is subject only to small fluctuations. There is a report about the first measurements by this method in the publication: F. A. Vening Meinesz, "Observations de pendule sur la mer pendant un voyage en sous-marin de Hollande à Java 1923," *Publ. Comm. Géod. Néerl.*, Delft, 1924.

Without entering into details, in the following we give a brief representation of the theory of these measurements.

We start from the differential equation of motion of the pendulum, in which we disregard, for the sake of simplicity, the damping by the air resistance, and have then according to (11), section 69, p. 335, for small angles of deflection α

$$\frac{d^2 \alpha}{dt^2} + \frac{g}{l} \alpha = 0 \quad (1)$$

or if we set
$$\frac{g}{l} = n^2 \quad (2)$$

$$\frac{d^2 \alpha}{dt^2} + n^2 \alpha = 0. \quad (3)$$

We bring this equation into another form by introducing a complex quantity q for which we set

$$q = \alpha - \frac{i}{n} \frac{d\alpha}{dt}. \quad (4)$$

Then we have

$$\begin{aligned} \frac{d\alpha}{dt} &= -in\alpha + inq \\ \text{and} \quad \frac{d^2 \alpha}{dt^2} &= -in \frac{d\alpha}{dt} + in \frac{dq}{dt} \\ \text{or} \quad \frac{d^2 \alpha}{dt^2} &= -n^2 \alpha + n^2 q + in \frac{dq}{dt}. \end{aligned} \quad (5)$$

If we introduce this into (3), then we obtain

$$n^2 q + in \frac{dq}{dt} \quad \text{or} \quad \frac{dq}{dt} - in q = 0. \quad (6)$$

The quantity q , which proves very useful in the theory of the oscillations of two pendulums on a common base, can be represented as a plane vector and therefore is designated as a *pendulum vector*. In this

connection, we refer to the paper by Ph. Furtwängler, mentioned already at the end of section 74, p. 367.

Thus far, the base of the pendulum support was regarded as immovable. If the axis of oscillation of the pendulum describes the length $d\sigma$ during the element of oscillation $d\alpha$, then we have according to (7), section 71, p. 345, for small deflections α

$$\frac{d^2 \alpha}{dt^2} = -\frac{g}{l} \alpha + \frac{1}{l} \frac{d^2 \sigma}{dt^2} \quad (7)$$

or with (2)

$$\frac{d^2 \alpha}{dt^2} = -n^2 \alpha + \frac{n^2}{g} \frac{d^2 \sigma}{dt^2}. \quad (8)$$

Now let two pendulums, whose reduced lengths are equal to l_1 and l_2 , swing on the same support. Then there exist the two equations

$$\left. \begin{aligned} \frac{d^2 \alpha_1}{dt^2} &= -n_1^2 \alpha_1 + \frac{n_1^2}{g} \frac{d^2 \sigma}{dt^2} \\ \frac{d^2 \alpha_2}{dt^2} &= -n_2^2 \alpha_2 + \frac{n_2^2}{g} \frac{d^2 \sigma}{dt^2} \end{aligned} \right\} \quad (9)$$

If we introduce here the auxiliary quantity q again, then we obtain in correspondence with the above equation (6)

$$\left. \begin{aligned} \frac{dq_1}{dt} &= i n_1 q_1 - i \frac{n_1}{g} \frac{d^2 \sigma}{dt^2} \\ \frac{dq_2}{dt} &= i n_2 q_2 - i \frac{n_2}{g} \frac{d^2 \sigma}{dt^2} \end{aligned} \right\} \quad (10)$$

If we set $n = \frac{n_1 + n_2}{2}$ and multiply the two equations (10) by $\frac{n}{n_1}$ and $\frac{n}{n_2}$, then we will have

$$\begin{aligned} \frac{n}{n_1} \frac{dq_1}{dt} &= i n q_1 - i \frac{n}{g} \frac{d^2 \sigma}{dt^2} \\ \frac{n}{n_2} \frac{dq_2}{dt} &= i n q_2 - i \frac{n}{g} \frac{d^2 \sigma}{dt^2} \end{aligned}$$

Finally, if we set further

$$q = \frac{n}{n_1} q_1 - \frac{n}{n_2} q_2, \quad (11)$$

then there follows from these two equations

$$\frac{dq}{dt} - i n (q_1 - q_2) = 0. \quad (12)$$

If we have the simple case in which $l_1 = l_2$, then we have also $n_1 = n_2 = n$ and $q = q_1 - q_2$. With this, (12) changes to

$$\frac{dq}{dt} - i n q = 0. \quad (13)$$

The acceleration $\frac{d^2\sigma}{dt^2}$ of the horizontal motion of the pendulum support no longer occurs here, and equation (13) has the same form as equation (6) and represents the equation of motion of a *hypothetical pendulum* corresponding to the concurrence of the two pendulums, which is free from the motion of the pendulum support.

For the utilization of this theory, the oscillation curves of the two pendulums are recorded photographically. Then we can determine the two individual vectors q_1 and q_2 from the recordings and derive the oscillations of the hypothetical pendulum therefrom.

For the details of this method we refer to the above-mentioned publication.

In this way, Vening Meinesz determined gravity on the ocean in a series of journeys with submarines and reached thereby an accuracy of approximately ± 4 mgal. Such measurements were also carried out in other countries during the last years, whereby valuable contributions to the knowledge of the course of gravity on the earth's surface were obtained.

More detailed information about the measurements of Vening Meinesz is contained in:

"Observations de l'intensité relative de la pesanteur par M. Vening Meinesz, à bord d'un sous-marin des États-Unis," *Bull. Géod.*, No. 21, 1929, pp. 42-49.

F. A. Vening Meinesz, "Ergebnisse der Schwerkraftbeobachtungen auf dem Meere in den Jahren 1923 bis 1932," *Ergebnisse der kosmischen Physik mit Einschluss der Geophysik*, 2. Band, pp. 153-184.

2. The pendulum of Holweck-Lejay

A new form of the pendulum indicated by F. Holweck and P. Lejay for the determination of gravity was treated in 1930 for the first time in the *Bulletin géodésique*, No. 25, p. 16, and in No. 28, p. 577, by R. P. Lejay.

As the diagram in Fig. 1 shows, the pendulum consists of a rigid rod which is fastened to a base by means of an elastic joint O and which carries at the upper end the pendulum body P .

If the pendulum is taken out of the position of equilibrium by the angle α , then, on the one hand, it is subject to the effect of gravity, and on the other hand, to the elasticity of the joint, whereby the two forces act in opposite direction.

The effect of gravity is given by equation (11), section 69, p. 335, whereby gravity tends here toward an increase of α , however; therefore, the equation in the present case reads:

$$l \frac{d^2\alpha}{dt^2} - g \sin \alpha = 0.$$

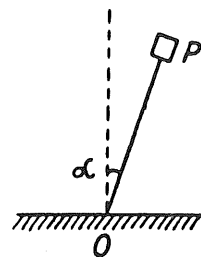


Fig. 1.

The effect of the elasticity of the joint can in the case of a small angle of deflection α be assumed proportional to the latter, and if we denote the factor of proportionality by k , then we have as the equation of motion of the pendulum the equation

$$l \frac{d^2\alpha}{dt^2} + k\alpha - g \sin \alpha = 0. \quad (14)$$

We write this equation in the form

$$2 \frac{d^2\alpha}{dt^2} \frac{d\alpha}{dt} dt = -2 \frac{k}{l} \alpha d\alpha + 2 \frac{g}{l} \sin \alpha d\alpha$$

and obtain hence by integration:

$$\left(\frac{d\alpha}{dt}\right)^2 = -\frac{k}{l}\alpha^2 - 2\frac{g}{l}\cos\alpha + C. \quad (15)$$

In the case of the largest deflection of the pendulum, the amplitude φ , we have $\frac{d\alpha}{dt} = 0$; for this, we thus obtain

$$0 = -\frac{k}{l}\varphi^2 - 2\frac{g}{l}\cos\varphi + C,$$

whereby the constant of integration C is determined. With this, (15) yields

$$\left(\frac{d\alpha}{dt}\right)^2 = -\frac{k}{l}(\alpha^2 - \varphi^2) - 2\frac{g}{l}(\cos\alpha - \cos\varphi).$$

If we introduce a development in series for $\cos\alpha$ and $\cos\varphi$, limiting ourselves thereby to the terms with α^4 and φ^4 , then we obtain

$$\left(\frac{d\alpha}{dt}\right)^2 = -\frac{k}{l}(\alpha^2 - \varphi^2) + 2\frac{g}{l}\left(\frac{\alpha^2}{2} - \frac{\varphi^2}{2}\right) - 2\frac{g}{l}\left(\frac{\alpha^4}{24} - \frac{\varphi^4}{24}\right)$$

or

$$\left(\frac{d\alpha}{dt}\right)^2 = -\frac{k-g}{l}(\alpha^2 - \varphi^2) - \frac{g}{12l}(\alpha^4 - \varphi^4).$$

This can also be written in the form:

$$\left(\frac{d\alpha}{dt}\right)^2 = \frac{k-g}{l}(\varphi^2 - \alpha^2)\left(1 + \frac{g}{12(k-g)}(\varphi^2 + \alpha^2)\right)$$

or

$$\frac{d\alpha}{dt} = \sqrt{\frac{k-g}{l}} \sqrt{\varphi^2 - \alpha^2} \left(1 + \frac{g}{24(k-g)}(\varphi^2 + \alpha^2)\right).$$

For the repeated integration we bring this equation into the following form:

$$dt = \sqrt{\frac{l}{k-g}} \frac{d\alpha}{\sqrt{\varphi^2 - \alpha^2}} - \sqrt{\frac{l}{k-g}} \frac{g\varphi^2}{24(k-g)} \frac{d\alpha}{\sqrt{\varphi^2 - \alpha^2}} - \sqrt{\frac{l}{k-g}} \frac{g}{24(k-g)} \frac{\alpha^2}{\sqrt{\varphi^2 - \alpha^2}} d\alpha \quad (16)$$

We have here the general integrals

$$\int \frac{d\alpha}{\sqrt{\varphi^2 - \alpha^2}} = \arcsin\left(\frac{\alpha}{\varphi}\right) \quad \int \frac{\alpha^2 d\alpha}{\sqrt{\varphi^2 - \alpha^2}} = -\frac{\alpha}{2} \sqrt{\varphi^2 - \alpha^2} + \frac{\varphi^2}{2} \arcsin\left(\frac{\alpha}{\varphi}\right)$$

and the definite integrals will be:

$$\int_0^{\varphi} \frac{d\alpha}{\sqrt{\varphi^2 - \alpha^2}} = \frac{\pi}{2} \quad \int_0^{\varphi} \frac{\alpha^2 d\alpha}{\sqrt{\varphi^2 - \alpha^2}} = \frac{\varphi^2 \pi}{4}.$$

The integral of dt in (16) yields for the limits $\alpha = 0$ and $\alpha = \varphi$ half the duration of oscillation $\frac{T}{2}$. Therefore, we will have

$$T = \pi \sqrt{\frac{l}{k-g}} \left(1 - \frac{g}{16(k-g)} \varphi^2 \right). \quad (17)$$

For the use of the pendulum, the two constants l and k must be determined. This determination can be done by measuring the duration of oscillation T at two places at which the value of gravity is already known.

In the execution of the instrument, the elastic joint indicated in Fig. 1 consists of a lamina of invar, about 0.02 mm thick, which represents the connection between the pendulum rod and the baseplate. The pendulum rod is a quartz bar with a length of 6 cm and a thickness of 4 mm, on which the pendulum body made of platinum is movably connected.

Since in view of the small weight of the pendulum, amounting to only a few grams, the influence of the air resistance would be very great, the whole is placed in a vacuum. There is a thermometer in the interior of the vacuum in connection with the lamina in order to be able to measure the temperature of the latter.

As shown in Fig. 2, the pendulum is fastened, together with the vacuum cylinder, on a baseplate, provided with three foot screws, on which there is a tube level for setting the plane of oscillation perpendicular. In addition, the baseplate carries, to the right, a microscope with which we can observe the oscillations of the pendulum, and to the left, a device for the illumination and reading of the thermometer.

After the setting up of the instrument, the measuring procedure is that the time interval for a definite number of oscillations, e.g., for 20 oscillations or for 100 oscillations, is measured by means of a chronograph.

In the publication: F. Holweck, "Description du nouveau gravimètre Holweck-Lejay. Résultats obtenus," *Bulletin géodésique*, No. 46, 1935, pp. 295-303, which contains a description of the instrument, there are indicated also a few experimental measurements at stations for which accurate gravity determinations are already available. It turns out here that the measurements deviate from the given gravity values on an average by ± 1 mgal.

N. E. Nörlund gives further accuracy data in: "Untersuchungen über die Genauigkeit relativer Schweremessungen mit dem Holweck-Lejay-Pendel," *Verhandlungen der 7. Tagung der Baltischen Geodätischen Kommission 1934 in Moskau*, II. Teil, Helsinki, 1935, pp. 224-231. For two pendulums there followed the result that gravity can be determined from five observations with a mean error of ± 4 or, as the case may be, ± 5 mgal. The measurements are set forth in detail in *Institut Géodésique de Danemark, Mémoires, Troisième série, Tome premier*, "Observations de l'intensité de la pesanteur avec le nouveau modèle de pendule de Holweck-Lejay," publiées par N. E. Nörlund avec la collaboration de A. Schneider, Copenhagen, 1934.

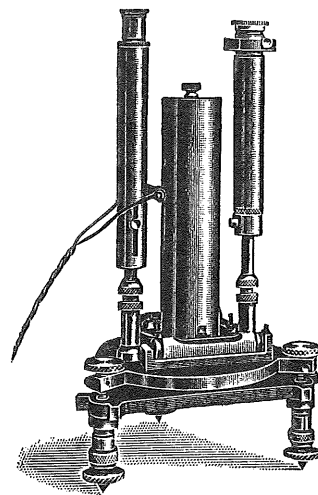


Fig. 2.

In the last-mentioned two papers there are indicated further references about publications in regard to the pendulum of Holweck-Lejay. We mention in addition:

A. Graf, "Zur Theorie elastischer Pendel mit besonderer Berücksichtigung des Holweck-Lejayschen Stabpendels," *Zeitschrift für Geophysik*, 1934, pp. 73-84.

F. Holweck, "Description du nouveau gravimètre Holweck-Lejay. Résultats obtenus," *Bull. géod.*, No. 46, 1935, pp. 295-303.

F. Baeschlin, "Das astasierte elastische Pendel als Schweremesser," *Schweiz. Zeitschr. f. Verm. u. Kult.*, 1938, pp. 30-32, 41-47, 65-70, 97-109.

C. F. Baeschlin et A. J. Corpaci, "Analyse mathématique des oscillations d'un pendule astatique élastique," *Bull. géod.*, No. 58, 1938, pp. 133-147.

Pierre Lejay, "Sur la précision des mesures effectuées avec le gravimètre Holweck-Lejay. Quelques remarques à propos de la liaison gravimétrique de Varsovie à Potsdam," *Bull. géod.*, No. 63, 1939, pp. 699-710.

Gustaf Ising, "Relative Schweremessungen mit Hilfe astasierter Pendel," *Bull. géod.*, No. 28, 1930, pp. 556-576.

Section 80. The Static Gravimeters

Gravity is determined from the pendulum measurements according to the principles of dynamics by measuring the velocity with which a body changes its position under the influence of gravity.

In addition, we will now treat the methods of gravity measurement which are based on the principles of statics.

The hypsometrical method

A method successfully used by O. Hecker several times on sea journeys during the first years of the twentieth century consists of a connection of measurements with the mercury barometer and a thermometer showing the boiling point of water.

In order to make the readings of a mercury barometer comparable with those of a boiling-point thermometer, we have to add, to the first, the gravity correction according to volume II, second half-volume, section 44, p. 191.* Of the two parts of the gravity correction, i.e. the reduction to sea level and the reduction to the geographic latitude of 45° , we will only consider the latter here, as we assume that the reading of the mercury barometer is already reduced to sea level.

If we denote by B and g the barometer reading and gravity at sea level of the place of observation and by B_0^{45} and g_0^{45} the same quantities for the geographic latitude of 45° , then we have according to volume II, second half-volume, section 44, p. 191:*

$$Bg = B_0^{45} g_0^{45}.$$

The value of B_0^{45} is read off directly from the boiling-point thermometer. If we call this reading S , then we have accordingly

$$g = \frac{S}{B} g_0^{45}. \quad (1)$$

If measurements are carried out with the mercury-barometer and with the boiling-point thermometer simultaneously at various points, then we can set up an equation (1) for each point and determine, with this, the ratio of the different values of g .

Equation (1) forms the basis for the relative gravity measurement with the help of the boiling-point thermometer.

For practical use it is more convenient to introduce the gravity correction

$$s = S - B. \quad (2)$$

If we write (1) in the form

$$g = \frac{S}{B} g_0^{45} - \frac{B}{B} g_0^{45} + g_0^{45},$$

then we have

$$g - g_0^{45} = \frac{s}{B} g_0^{45}. \quad (3)$$

* Not translated.

We will now use equation (3) for determining the deviation of the actual gravity g from its normal value g_n .

According to volume II, second half-volume,* section 44, p. 191, equation (3), there corresponds to the normal gravity g_n the normal gravity correction

$$s_n = -B \beta \cos 2 \varphi, \quad (4)$$

where $\beta = 0.00264$. Equation (3) yields then for g_n the expression

$$g_n - g_0^{45} = \frac{s_n}{B} g_0^{45},$$

and therefore, the gravity anomaly is

$$g - g_n = \Delta g = (s - s_n) \frac{g_0^{45}}{B}. \quad (5)$$

According to this basic idea, gravity measurements on the sea have been carried out with great success by O. Hecker by order of the Geodetic Institute in Potsdam and the Zentralbureau der Internationalen Erdmessung since the beginning of this century; for these gravity determinations, the use of a pendulum apparatus is not possible.

The following reports have been published about these measurements: O. Hecker, *Bestimmung der Schwerkraft auf dem Atlantischen Ozean, sowie in Rio de Janeiro, Lissabon und Madrid*, Berlin, 1903. O. Hecker, *Bestimmung der Schwerkraft auf dem Indischen und Grossen Ozean und an deren Küsten, sowie erdmagnetische Messungen*, Berlin, 1908. O. Hecker, *Bestimmung der Schwerkraft auf dem Schwarzen Meere und an dessen Küste, sowie neue Ausgleichung der Schwerkraftmessungen auf dem Atlantischen, Indischen und Grossen Ozean*, Berlin, 1910.

The idea of determining gravity from the gravity correction of the mercury barometer was expressed for the first time in 1866 by Freiherr von Wüllerstorff-Urbair in the treatise, *Das Aneroid als Instrument zur Messung der Änderung der Schwere*; this proposal, however, could not have any practical significance due to the limited reliability of the aneroid. In 1894, the replacement of the aneroid by the boiling-point thermometer was suggested by Guillaume, without pursuing the idea further, however. Only by Mohn were there undertaken thorough investigations about the usability of this new method, which led to a satisfactory result and induced the Geodetic Institute in Potsdam to study the applicability of the method for gravity measurements on the ocean.

We have taken the foregoing historical notes from the first of the above-mentioned publications by O. Hecker, in which there are given also the succeeding experiments, hitherto made, for the measurement of gravity at sea.

In the former seventh edition of the third volume, Stuttgart, 1923, in section 133, pp. 697-701, we have reported in detail about the apparatus used by Hecker and about the measuring results. We add to this that a measuring accuracy of about ± 40 mgal was thereby reached. Since at the present time the method has only a historical interest, we limit ourselves to this brief reference and turn to the more recent static gravimeters, which have found extensive application since about ten years ago.

The static gravimeter of Fr. Haalck

Since 1931, Prof. Haalck, in the Geodetic Institute in Potsdam, has developed a gravimeter which rests likewise on the barometric principle. A mercury column presses here on an elastic gas mass, and if the temperature remains unchanged, in the case of changing gravity, a change in the reading of the mercury column will occur. If the latter is measured, then the change of gravity can be determined therefrom.

The difficulty in the practical use of this simple basic idea consists in the precise measurement of the variation of height of the mercury column, and there have been tried various ways for it during the last few

* Not translated.

years. In Fig. 1, the present execution of Haalck's gravimeter is illustrated in diagrammatical form.

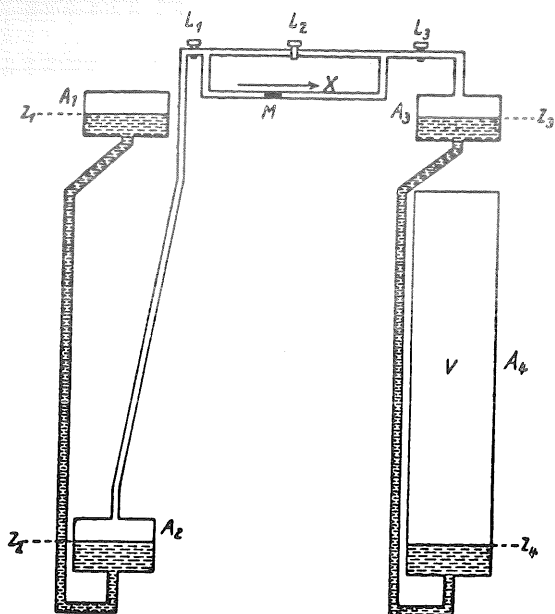


Fig. 1.

A U-shaped tube, which carries at both ends vascular enlargements A_1 and A_2 , is filled with mercury. The upper vessel is closed and contains, above the mercury surface, a vacuum, while the lower vessel is connected with the vessel A_3 by a narrow tube filled with a gas. To the latter vessel, there is connected again a U-shaped tube, filled with mercury, with a larger gas vessel A_4 . If we consider, of the two horizontal tubes indicated in Fig. 1 above, only the lower one, which is inserted through the open valves L_1 and L_3 , while L_2 is supposed to be closed, then in the vessels A_2 and A_3 above the mercury surface, the pressure of the left mercury column will be predominant, while the mercury surface in A_4 stays under the pressure of both mercury columns. We shall disregard here the small gas mass in the connecting tube.

If we denote by z_1, z_2, z_3 and z_4 , the heights of the four mercury surfaces with respect to an arbitrary starting surface, by σ the density of the mercury and by p the pressure in the vessel A_4 , then there exists the condition of equilibrium

$$(z_1 - z_2) \sigma g + (z_3 - z_4) \sigma g = p, \quad (6)$$

whereby it shall be assumed that the four vessels have the same cross sections.

If gravity changes for a constant temperature, then there follows hence a change of the weight of the mercury columns, and consequently, the heights of the mercury surfaces will change by the amounts dz_1, dz_2, dz_3 and dz_4 . We have here

$$dz_1 = -dz_2 = dz_3 = -dz_4.$$

For the measurement of these magnitudes, in the horizontal capillary there is a drop of liquid M , whose position is determined by the reading x on a horizontal scale. The change of height of the mercury columns effects a displacement dx of the drop, and if the size of the mercury surfaces is denoted by F , the cross section of the capillary by q , then we have

$$dz_1 = -dz_2 = dz_3 = -dz_4 = -\frac{q}{F} dx. \quad (7)$$

From (6) we obtain by differentiation

$$(dz_1 - dz_2 + dz_3 - dz_4) \sigma g + (z_1 - z_2 + z_3 - z_4) \sigma dg = dp$$

or with the help of (7), if we set at the same time

$$z_1 - z_2 + z_3 - z_4 = 2s,$$

$$-\frac{4q}{F} \sigma g dx + 2s \sigma g = dp. \quad (8)$$

But we have according to Mariotte's law (Vol. II, second half-volume, 1933, section 41, p. 176)

$$(v + dv)(p + dp) = vp$$

and hence we obtain

$$p dv = -v dp \quad \text{or} \quad dp = -\frac{p}{v} dv.$$

Consequently, (8) changes to

$$-\frac{4q}{F} \sigma g dx + 2s \sigma dg = \frac{p}{v} dv.$$

But we have

$$2s \sigma g = p \quad \text{and} \quad dv = -q dx,$$

and hence we will have

$$q \left(\frac{2q}{sF} + \frac{1}{v} \right) dx = \frac{1}{g} dg \quad \text{or} \quad dg = qg \left(\frac{2}{sF} + \frac{1}{v} \right) dx. \quad (9)$$

In addition, we set

$$qg \left(\frac{2}{sF} + \frac{1}{v} \right) = c, \quad (10)$$

so that we have

$$dg = c dx. \quad (11)$$

The dimensions of the instrument are chosen in such a way that we will have the constant $c = 0.01$. Therefore, to a change of reading of 1 mm there corresponds a change of gravity of 1 mgal.

The valves L_1 , L_2 and L_3 indicated in Fig. 1, p. 396, are used for locking the instrument in transport.

The reading depends to a great extent on the temperature. At the apparatus there is therefore placed a compensation device which makes the influence of temperature harmless for the most part. In order to eliminate the remainder, the whole apparatus is surrounded with a casing, which is filled with ice.

Determination of the constant c . We can determine the constant c by bringing the instrument out of the vertical position, by a small angle δ , perpendicular to the direction of the reading capillary. Instead of the gravity g there acts then only the component $g \cos \delta$, and the readings yield then the variation of gravity

$$dg = g \cos \delta - g = c dx.$$

For a small angle δ we have with sufficient accuracy

$$g \cos \delta - g = -g \frac{\delta^2}{2}$$

and then we will have

$$c = -g \frac{\delta^2}{2} \quad (12)$$

At the same time, we see hence that great importance is to be attached to the vertical position of the instrument in the case of the measurement, i.e. care has to be taken that the attachment of the levels at the instrument remains unchanged.

A static gravimeter of the Geodetic Institute in Potsdam, constructed according to the above-mentioned principles by Askania-Werke in Berlin-Friedenau, has been used for the geophysical survey of the Reich since

1935. The instrument consists of four apparatuses independent of one another, which are enclosed by a common covering. The latter has a height of 1.25 m and a diameter of 0.62 m. To keep the temperature constant, the apparatuses are kept under ice during the measurement, and the whole instrument together with the ice filling has a weight of about 7 cwt [German: Zentner = 50 kg]. For the measurements, the instrument is placed in a power vehicle in cardanic suspension.

The measuring procedure at a station is very simple. After the instrument has been adjusted with the help of the levels, the arresting faucets are opened and the two scales are read four or five times. After this, the faucets are closed again, the instrument fixed, and the vehicle can proceed to the next point. From the stopping to the proceeding of the vehicle, a time of 3 to 4 minutes is required.

In this form, more than 2000 points have been surveyed during the last years, whereby approximately ± 1 mgal was obtained as the average measuring accuracy in an average monthly work performance of about 100 points.

After the measurements for the geophysical survey of the Reich had proved to a great extent the usability of Haalck's gravimeter, experiments were begun to utilize the instrument for gravity measurements on moving ships. Experimental measurements, which proved very promising, have repeatedly been carried out since 1934. On the basis of experiences thereby obtained, there was constructed a smaller instrument, with which, however, only a few experimental trips from Hamburg through the North Sea Bay to Bremen, in the fall of 1938, could be carried out thus far.

About the development of the static gravimeter, H. Haalck has published the following reports in *Zeitschrift für Geophysik*:

"Ein statischer Schweremesser" (Vorläufige Mitteilung [preliminary communication]), 1931, pp. 95-103.

"Ein statischer Schweremesser" (Zweite Mitteilung [second communication]), 1932, pp. 17-30, 197-204.

"Neue Messungsergebnisse mit dem statischen Schweremesser," 1933, pp. 285-295.

"Messungsergebnisse mit dem statischen Schweremesser auf der Nord- und Ostsee und in Norddeutschland," 1935, pp. 55-74.

"Der neue statische Schweremesser des Geodätischen Instituts in Potsdam," 1936, pp. 1-21.

"Bericht über den gegenwärtigen Stand der Entwicklung des statischen Schweremessers," 1936, pp. 356-360.

A detailed description of the instrument and its practical handling, the theory and the influences of errors is contained in the treatise: H. Haalck, "Der statische (barometrische) Schweremesser für Messungen auf festem Lande und auf See," *Beiträge zur angewandten Geophysik*, 1939, pp. 285-316, 392-447.

The Askania gravimeter of A. Graf

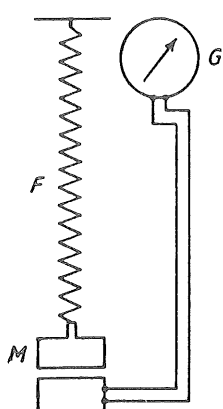


Fig. 2.

A gravimeter, which was constructed in 1939 according to specifications by A. Graf in Askania-Werke in Berlin-Friedenau, is based on another principle. The basic idea of the apparatus is represented in Fig. 2. To a support there is fastened a helical spring F from which there is suspended a mass M which can be moved freely in all directions. As gravity changes, the mass moves in the vertical direction, and the amount of the displacement gives a measure for the variation of gravity.

The measurement of the displacement is carried out electrically by one of the resistance methods. The method is developed so that a length displacement of 0.002μ could be made visible on a galvanometer G .

Fig. 3 shows the design of the gravimeter. The spring and the electrical measuring equipment are placed in a cylindrical, thick-walled, cast casing. The mechanisms, which are required for the arresting and setting in motion of the measuring device, penetrate the casing in an airtight manner, and for more convenient handling, all are placed in the upper part of the instrument.

The baseplate carries a calotte [segment of a sphere], on which the instrument can be adjusted approximately by means of a rough box level. The fine adjustment is then carried out by means of the foot screws and two tube levels. The foot screws are thereby carried likewise to the top, so that they can be used from above like all the other operating devices.

The gravimeter is built into a measuring carriage, in which it is suspended elastically on ropes for transport. The three spring rods are enclosed in tubes and in Fig. 4 likewise visible. For the measurement, the apparatus, which has a weight of 57 kg, is let down to the ground by means of a small rope winch through an opening in the carriage.

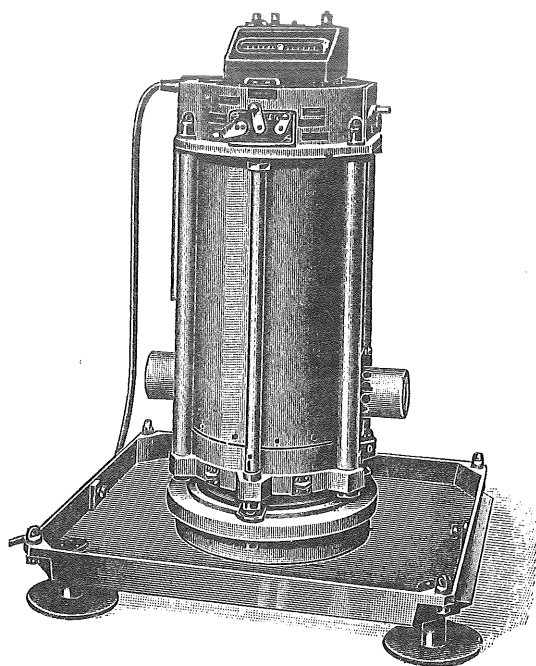


Fig. 3.

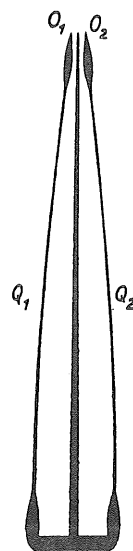


Fig. 4.

In the case of this gravimeter, also, a time of 4 to 5 minutes is needed for each station from the stopping of the carriage to the continuation of the trip, so that about 20 points at intervals of 3 km per day can be taken care of.

For the examination of the accuracy of the instrument, there were measured about 300 to 400 stations at intervals of 1 to 3 km in the field, whence a mean error of $\pm 0.1-0.2$ mgal resulted.

Further data about the Askania gravimeter are contained in: A. Graf, "Ein neuer statischer Schweremesser zur Messung und Registrierung lokaler und zeitlicher Schwereänderungen," *Zeitschrift für Geophysik*, 1938, pp. 152-172. In this treatise on pp. 154-156, there is also given a historical sketch about the many static gravimeters constructed during the last 10 to 12 years.

The static gravimeter of the Danish Geodetic Institute

In connection with Fig. 4 we mention further, briefly, a static gravimeter constructed in the Danish Geodetic Institute in Copenhagen. It consists of two quartz springs Q_1 and Q_2 , which, being inclined a little toward one another, are fastened on a base made likewise of quartz. The springs are loaded at the upper ends by thickenings. An increase of gravity will decrease the distance between the two points O_1 and O_2 of the quartz springs, and this decrease, which is measured with a micrometer arrangement, yields a measure for the variation of gravity.

For the determination of the vertical position of the instrument there is used a quartz column, fastened likewise on the baseplate, and the instrument must be set up in such a way that the two springs have equal distances from the point of the column.

The first experiments with the instrument were carried out in the years 1928-1930; about further successful experiments during the last years there is a report in:

Geodaetisk Institut, Meddelelse Nr. 10, "Ein statischer Quarzschweremesser und Schweremessungen" von G. Noergaard, Kopenhagen 1939.

In addition, experiments with horizontal springs are also in progress; they appear successful even for gravity measurements at sea.

The values of gravity measured on the physical surface of the earth are not comparable with one another, since they belong to different elevations above sea level and also are influenced, in a different manner, by the masses located above the sea level. Therefore, there arises the problem of computing, from the measured values of gravity, the values valid for sea level.

1. The free-air reduction to sea level

A first correction of the measured gravity g is used for taking into account the height H above sea level of the measurement station. In equation (3), section 64, p. 315, we have found the following expression for gravity at sea level, if we only take into account the main term:

$$g_0 = f \frac{M}{r^2}.$$

Gravity at the height H above sea level is then

$$g = f \frac{M}{(r + H)^2}$$

and we obtain hence with sufficient accuracy

$$g = f \frac{M}{r^2} \left(1 - \frac{2H}{r}\right) = g_0 \left(1 - \frac{2H}{r}\right).$$

Therefore, we have

$$g = g_0 - g_0 \frac{2H}{r},$$

or if we replace in the last term g_0 by g

$$g_0 - g = \Delta g = \frac{2g}{r} H. \quad (1)$$

The assumption is hereby made that the measurement station is situated in free air at the height H above sea level.

2. The method of reduction by Bouguer

In contrast to the free-air reduction, Bouguer's method is based on the idea of deducting, from the measured gravity, the vertical attraction of the masses located between sea level and the station in the neighborhood of the masses. If we assume for this that the terrain around the station is flat, and if we disregard the curvature of the earth's surface, then we have to determine the attraction of the plate bounded by the surface of the terrain and the sea level.

We imagine this plate at first in the form of a vertical circular cylinder with the arbitrary radius a , whose axis passes through the station. In Fig. 1 there is represented an element of the thickness dz of the cylindrical

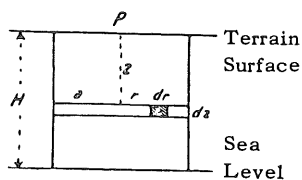


Fig. 1.

plate located below the terrain at the depth z . If we consider, of this, a ring with the inner radius r and the width dr , then the volume of this ring is equal to $2\pi r dr dz$. If Θ is the density of the terrestrial body below the surface of the terrain, then the mass of the ring is

$$dm = 2\pi r \Theta dr dz. \quad (2)$$

According to section 61, p. 306, we then obtain for the potential of the attraction of the element of the mass dm at the point P the expression

$$dV = 2\pi \Theta f dz \frac{r}{\sqrt{r^2 + z^2}} dr, \quad (3)$$

where f denotes the constant of attraction.

By integration we obtain hence the potential of the whole circular disc, namely:

$$V = 2\pi \Theta f dz \int_0^a \frac{r}{\sqrt{r^2 + z^2}} dr = 2\pi \Theta f dz (\sqrt{a^2 + z^2} - z)$$

and the attraction of the disc at the point P is

$$-\frac{\partial V}{\partial z} = 2\pi \Theta f \left(dz - \frac{z}{\sqrt{a^2 + z^2}} dz \right). \quad (4)$$

The negative sign has been introduced here, because the attraction decreases as z increases.

If we denote the elevation above sea level of the point P by H , then we find the total attraction of the plate located below P by integration of (4) between the limits 0 and H . This attraction will then be equal to

$$2\pi \Theta f \left(\int_0^H dz - \int_0^H \frac{z}{\sqrt{a^2 + z^2}} dz \right)$$

or equal to

$$2\pi \Theta f (H - \sqrt{a^2 + H^2} + a). \quad (5)$$

If we assume the radius a very large in proportion to the elevation above sea level H , then we can write the expression (5) in the form:

$$2\pi \Theta f \left(H - a \sqrt{1 + \frac{H^2}{a^2}} + a \right),$$

and if we set

$$\sqrt{1 + \frac{H^2}{a^2}} = 1 + \frac{H^2}{2a^2} + \dots$$

then the expression (5) will change to

$$2\pi \Theta f H \left(1 - \frac{H}{2a} \right). \quad (6)$$

For the elimination of the constant of attraction f we can introduce gravity and have according to section 64, equation (3), p. 315, with sufficient accuracy:

$$g = \frac{fM}{r^2} + \dots$$

If Θ_m is the mean density of the terrestrial body, then we have $M = \frac{4}{3} r^3 \pi \Theta_m$, and hence

$$g = \frac{4}{3} r \pi \Theta_m f, \quad (7)$$

and if this is introduced in (6), then the attraction is:

$$\frac{3}{2} \frac{\Theta}{\Theta_m} \frac{H}{r} g \left(1 - \frac{H}{2a}\right).$$

Under the assumption that a can always be assumed sufficiently large in proportion to H , we can, in addition, also neglect the expression within parentheses on the right-hand side, so that the reduction has the value:

$$\Delta g' = -\frac{3}{2} \frac{\Theta}{\Theta_m} \frac{H}{r} g. \quad (8)$$

If the amount computed according to (8) is deducted from the measured gravity, then the effect of the plate below the station is eliminated with this.

For the mean density of the earth we can introduce the value $\Theta_m = 5.52$, while the density Θ on the earth's surface must be estimated from case to case. As a rule, Θ may be assumed nearly equal to half of Θ_m .

The reduction method treated in the foregoing was used by Bouguer in 1749 in his treatise *La Figure de la Terre*, in the working up of the gravity measurements carried out in Peru on the occasion of the degree-measurement.

3. The topographic correction

In the foregoing we have limited ourselves to the assumption of a level terrain in the neighborhood of the station. If the station is located higher than the surrounding terrain, then the mass expressed in equation (5) is too large, and a positive correction is therefore to be added further to (8). On the other hand, if the terrain around the station is higher than the latter, then gravity at the station is measured too small due to the attraction of the parts of the terrain located higher and requires likewise a positive correction. Therefore, the deviations of the terrain from the horizontal plane passing through the station will in any case make a positive correction term necessary.

For the computation of this topographic correction, a convenient method has been given by Helmert in *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, vol. II, Leipzig, 1884, p. 169. For this, the terrain around the station is divided on a topographical map provided with contour curves, by means of concentric circles, into ring-shaped segments, which are divided again by radial rays into a number of equal sections. Since the influence of the masses very near the station is greater than at a greater distance, the first rings must be assumed very small, while they can become increasingly wider with increasing radius. Moreover, the individual sections are to be chosen so small that within their boundaries the terrain surface can be regarded nearly as plane, and for each section there is taken, from the plan, the mean elevation above sea level, with which the difference of elevation h with respect to the elevation of the station is then also known.

To each section there corresponds then a column whose base is formed by the terrain while the surface lies at the level surface of the point P . The attraction of all these columns around the point P is to be deducted from the attraction of the horizontal plate given by (8).

From (5) we obtain for the attraction of a ring-shaped plate with the two radii a_1 and a_2 and the height h the expression

$$2\pi \Theta f (\sqrt{a_1^2 + h^2} - \sqrt{a_2^2 + h^2} + a_2 - a_1),$$

and if the ring is divided into n equal sections,

$$\frac{2\pi}{n} \Theta f (\sqrt{a_1^2 + h^2} - \sqrt{a_2^2 + h^2} + a_2 - a_1). \quad (9)$$

The constant of attraction f can again be eliminated with the help of (7).

The value of this expression (9) is to be computed for all sections, and the sum yields the topographic correction $\Delta g''$.

Examples for the computation of the topographic reduction according to the above method are contained, among other things, in: *Veröffentlichung der Kgl. Bayer. Kommission für die International Erdmessung. Astronomisch-Geodätische Arbeiten*, Heft 6, "Relative Schweremessungen in Bayern," München 1904, pp. 151-155.

E. Anding has developed another method in *Astronomische Nachrichten*, vol. 159, pp. 65-82, for which examples are likewise given in the aforementioned publication.

If we collect the different reductions treated in the foregoing and denote the gravity hereafter reduced to sea level by g_0'' , then we have

$$g_0'' = g + \Delta g + \Delta g' + \Delta g'' \quad (10)$$

or

$$g_0'' = g + \frac{2H}{r}g - \frac{3}{2} \frac{\Theta}{\Theta_m} \frac{H}{r}g + \text{Top. Corr.} \quad (11)$$

If the question is to find, from the gravity measurements, a formula for the normal gravity at sea level and to determine also the flattening of the terrestrial spheroid, then Bouguer's reduction method is not suited for this. The decrease of gravity by the attraction of the masses located between the station and the sea level is synonymous with the theory of imagining these masses completely removed, and it is evident that very considerable displacements of the sea level itself can be caused by this. Therefore, Helmert has proposed another form of reduction which corresponds better to the purpose in question. For this, let us imagine the outer masses displaced in the vertical direction as far as the sea level and intensified here as a surface layer. By this condensation of the masses, it is true, the sea level will also suffer a small disturbance; Helmert, however, proved that the displacement of the sea level does not reach anywhere the amount of 3 m, and hence, can be neglected completely.

Although Bouguer's reduction thus is not suited for the purpose in question, this method has great significance in another field, since it can be used to conclude from gravity measurements to disturbances of the masses under the earth's surface as well as to disturbances of the height of the geoid with respect to the terrestrial spheroid, as Helmert has shown in 1884 in his *Mathematische und physikalische Theorien der höheren Geodäsie*, vol. II, p. 259 and following. It would lead too far to represent these theories here and we limit ourselves, therefore, to indicate, in this connection, additional literature.

After the first work, which we have already mentioned above, Helmert has pursued the foregoing theories further in *Veröffentlichung des Geodätischen Instituts und des Zentralbureaus der Internationalen Erdmessung*, "Die Schwerkraft im Hochgebirge, insbesondere in den Tiroler Alpen in geodätischer und geologischer Beziehung," Berlin, 1890. Further

there are referred to this two communications by Helmert in *Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften*, "Über die Reduktion der auf der physischen Erdoberfläche beobachteten Schwerebeschleunigungen auf ein gemeinsames Niveau," erste Mitteilung 1902, pp. 843-855; zweite Mitteilung 1903, pp. 650-667. Finally, there are additional supplements to this in the chapter of *Enzykl. d. math. Wiss.*, Band VI, 1, prepared by Helmert: "Die Schwerkraft und die Massenverteilung der Erde," 1910, pp. 99-111.

Section 82. The Isostatic Reduction of Gravity Measurements

In the middle of the previous century, Pratt, in a comparison of the attracting effect of the Himalayan mountains and the deflections of the vertical found from the astronomic-geodetic surveys in India, discovered that the mountain masses of the Himalayas would have to be compensated by masses of lesser density below the earth's surface. This was also confirmed after a short time by gravity measurements on the Himalayas, and Pratt now believed that he could assert that at a certain depth below the earth's surface there exists a surface of compensation for which the pressure on the surface unit is the same at every point, while the masses lying above it are distributed unequally through height disturbances.

Almost at the same time, an investigation which gives another interpretation for the observational results on the Himalayan mountains was published by Airy. He assumes that the mountain mass floats on the flexible layer below the earth's crust, and hence that the highest parts of the mountains penetrate most deeply.

The hypotheses of Pratt and Airy are published in:

J. H. Pratt, "On the attraction of the Himalaya Mountains and of the elevated regions beyond upon the plumb-line in India," *Philos. Transactions of the Royal Soc. of London*, 1855, Vol. 145, p. 53.

G. B. Airy, "On the computation of the effect of the attraction of mountain masses as disturbing the apparent astronomical latitude of stations in geodetic surveys," *Phil. Trans. of the R. Soc. of London*, 1855, Vol. 145, p. 101.

We will leave undiscussed the question which of the two hypotheses of the compensation of the masses or of *isostasy* has the greater justification for itself, especially because it has been shown that for the problems of geodesy, both hypotheses lead to nearly the same results. In the following, we shall therefore take as a basis mainly Pratt's theory, which is easily accessible to computational treatment.

Let us consider at first a vertical column starting from the surface of compensation, whose cross section is equal to the surface unit and whose density is equal to Θ . Let this column be located at the sea coast, and if we denote the depth of the surface of compensation by T , then the mass of this column is equal to $T\Theta$. This mass must be the same for all columns between the surface of compensation and the surface of the terrain. In the case of a terrain elevation H , the mass of that part of a column lying above sea level is equal to $H\Theta$. The part of this column lying below it must then have a lesser density $\Theta - \Delta\Theta$ so that its mass is equal to $T(\Theta - \Delta\Theta)$. For the whole column we have therefore the mass $T(\Theta - \Delta\Theta) + H\Theta$, and this must be equal to $T\Theta$ due to isostasy. We have therefore the equation

$$T(\Theta - \Delta\Theta) + H\Theta = T\Theta \quad (1)$$

and there follows hence

$$T\Delta\Theta = H\Theta \quad \text{or} \quad \frac{\Delta\Theta}{H} = \frac{\Theta}{T}. \quad (2)$$

Since Θ and T are constant magnitudes, then we must have for all points above sea level

$$\frac{\Delta\Theta}{H} = \text{const.} \quad (3)$$

For a point on the bottom of the sea H is negative, and hence also $\Delta\Theta$ is negative; i.e. below the bottom of the sea we have an increased density $\Theta + \Delta\Theta$. Here we are to take into account, however, also the water mass reaching to sea level, whose density is to be assumed equal to 0.385Θ . Therefore, the condition of equilibrium now reads

$$0.385 H\Theta + (T - H)(\Theta + \Delta\Theta) = T\Theta$$

or

$$-0.615 H\Theta + (T - H)\Delta\Theta = 0$$

and there follows hence

$$\Delta \Theta = 0.615 \frac{H \Theta}{T - H}. \quad (4)$$

Here we can also neglect in the denominator the depth below sea level H , which is very small in proportion to T , so that we have

$$\Delta \Theta = 0.615 \frac{H \Theta}{T}. \quad (5)$$

To carry out the isostatic reduction we compute at first the attraction of the masses located below the terrain to sea level. For this, we imagine again, at first, the terrain around the station P divided, by means of concentric circles, into rings and the latter divided into sections in a radial direction. The innermost part of this system of rings is a circular cylinder whose height is equal to the elevation above sea level H of station P . If we denote the radius of this cylinder by a , then we have for the attraction, according to (5), section 81, p. 401, the expression

$$2\pi \Theta f(H - \sqrt{a^2 + H^2} + a). \quad (6)$$

Let a_1 and a_2 be the two radii for an arbitrary ring. Then, if the ring is divided into n equal sections, the attraction of a column of a section, reaching from sea level to the height H , according to (5), section 81, p. 401, is

$$\frac{2\pi}{n} \Theta f(\sqrt{a_1^2 + H^2} - \sqrt{a_2^2 + H^2} + a_2 - a_1). \quad (7)$$

If the height of the terrain of this section is equal to H' , and if we set $H - H' = h$, then the attraction of the column lying between the heights H' and H is to be deducted from the expression (7). We thus obtain for the attraction of the section the value

$$\frac{2\pi}{n} \Theta f(\sqrt{a_1^2 + H^2} - \sqrt{a_2^2 + H^2} - \sqrt{a_1^2 + h^2} + \sqrt{a_2^2 + h^2}). \quad (8)$$

For the computation of the isostatic compensation we determine in the same manner the attraction of the mass column of the same section which reaches from sea level to the surface of compensation and whose density is to be assumed equal to the density defect according to (2), and hence equal to $\frac{H}{T} \Theta$. In equation (8), we are to replace further the quantity h by the height H of the station. Therefore, we have the following expression for the effect of isostasy:

$$-\frac{2\pi H}{n T} \Theta f(\sqrt{a_1^2 + (H + T)^2} - \sqrt{a_2^2 + (H + T)^2} - \sqrt{a_1^2 + H^2} + \sqrt{a_2^2 + H^2}). \quad (9)$$

The two expressions (8) and (9) together yield the isostatic reduction of the measured gravity.

The foregoing representations are referred to the case in which the station as well as all sections of the individual rings lie on land. For such sections which fall on the oceans there is to be borne in mind that the mass defect in the sea is to be computed in the negative sense and the compensating mass in the positive sense, whereby the change of density is given by (4) or, as the case may be, (5).

Equations (8) and (9) are usable only in the nearer vicinity of the station, say, up to a distance of 200 km; for larger distances, which need to be taken into account for the isostatic reduction in every case, the curvature of the earth's surface must be taken into account. For this, as well as for the different cases which occur in the case of the reduction, we refer to the work: *Coast and Geodetic Survey*, "The effect of topography and isostatic compensation upon the intensity of gravity," by John F. Hayford and William Bowie, *Spec. publ.* No. 10, Washington 1912.

The scheme indicated in the following table is used in this basic work for the division of the terrain around a station into zones and sections. Zone A starts at the station and zone 1 ends at the antipodal point, so that the zones cover the whole surface of the earth. In addition to the computing formulae, in the above publication there are also given auxiliary tables for the computation of the reduction for the individual zones.

Division into Zones for the Computation of the Isostatic Reduction

Designation of Zone	Inner Radius of Zone	Outer Radius of Zone	Number of Sections
A	0 m	2 m	1
B	2	68	4
C	68	230	4
D	230	590	6
E	590	1 280	8
F	1 280	2 290	10
G	2 290	3 520	12
H	3 520	5 240	16
I	5 240	8 440	20
J	8 440	12 400	16
K	12 400	18 800	20
L	18 800	28 800	24
M	28 800	58 800	14
N	58 800	99 000	16
O	99 000	166 700	28
18	1° 29' 58"	1° 41' 13"	1
17	1 41 13	1 54 52	1
16	1 54 52	2 11 53	1
15	2 11 53	2 33 46	1
14	2 33 46	3 3 5	1
13	3 3 5	4 19 13	16
12	4 19 13	5 46 34	10
11	5 46 34	7 51 30	8
10	7 51 30	10 44	6
9	10 44	14 9	4
8	14 9	20 41	4
7	20 41	26 41	2
6	26 41	35 58	18
5	35 58	51 4	16
4	51 4	72 13	12
3	72 13	105 48	10
2	105 48	150 56	6
1	150 56	180 0	1

Further representations about the isostatic reduction are given in the following papers, which appear likewise as publications of the Coast and Geodetic Survey:

"Effect of topography and isostatic compensation upon the intensity of gravity" (second paper) by William Bowie, *Spec. publ.* No. 12, Washington, 1912.

"Investigations of gravity and isostasy," by William Bowie, *Spec. publ.* No. 40, Washington, 1917.

In this connection, we mention a few additional treatises of the same period:

Erich Hübner, "Beitrag zur Theorie der isostatischen Reduktion der Schwerebeschleunigung," *Gerlands Beiträge zur Geophysik*, XII. Band, 1913, pp. 588-638.

O. Meissner, "Tabellen zur isostatischen Reduktion der Schwerkraft," *Astr. Nachr.*, No. 4924-25, Band 206, 1918.

O. Meissner, "Isostatische Reduktion von 34 Stationen, ausgeführt im Geodatischen Institut von Dr. E. Hübner und O. Meissner," *Astr. Nachr.*, No. 4967, Band 207, 1918.

O. Meissner, "Neue Tabellen zur isostatischen Reduktion der Schwerkraft," *Astr. Nachr.*, No. 5125, Band 214, 1921.

After the publication of the basic papers by Hayford and Bowie, the theory of isostasy has stepped more and more into the foreground of interest in physical geodesy because of its great practical significance. In more recent time, a special institute for isostasy under the direction of W. Heiskanen was established in Helsinki by the International Association for Geodesy.

Of newer tables for the computation of the isostatic reduction we give the following:

U. S. Coast and Geodetic Survey, "Tables for determining the form of the geoid and its indirect effect on gravity," by Walter D. Lambert and F. W. Darling, *Spec. publ.* No. 199, Washington, 1936. In this, the third part contains Bowie isostatic reduction tables.

R. Politecnico, Milano, *Pubbl. dell'Istituto di Top. e. Geod.*, G. Cassinis, P. Dore, S. Ballarin, "Tavole fondamentali per la riduzione dei valori osservati della gravita," Milano 1937. The tables are usable for the determination of the topographic correction, for the isostatic reduction according to Hayford, according to Airy-Heiskanen, for the condensation method by Helmert and also for the free-air reduction.

Publ. of the Isostatic Institute of the International Association of Geodesy, No. 2, "New isostatic tables for the reduction of gravity values calculated on the basis of Airy's hypothesis," by Dr. W. Heiskanen, Helsinki, 1938.

Conclusively, we list, in addition, some publications from the last ten years on reduction of gravity measurements:

Walter D. Lambert, "The reduction of observed values of gravity to sea level," *Bull. géod.* No. 26, 1930, pp. 107-181.

F. A. Vening Meinesz, "Une nouvelle méthode pour la réduction isostatique régionale de l'intensité de la pesanteur," *Bull. géod.* No. 29, 1931, pp. 33-51.

W. Heiskanen, "Tables isostatiques pour la réduction, dans l'hypothèse de Airy, des intensités de la pesanteur observées," *Bull. géod.* No. 30, 1931, pp. 87-153.

W. Heiskanen, "Der heutige Stand der Isostasiefrage," *Gerl. Beitr. z. Geophysik*, Band 36, 1932, pp. 177-205.

W. Heiskanen, "Rapport sur l'isostasie," *Trav. de l'assoc. de géod. Rapp. généraux établis à l'occasion de la cinqu. assemblée gén. Lisbonne, 14-25 Sept. 1933.*

W. Heiskanen, "Rapport sur l'isostasie," *Trav. de l'assoc. de géod. Rapp. généraux établis à l'occasion de la six. assemblée gén. Edimbourg, 14-26 Sept. 1936.*

W. Heiskanen and U. Nuotio, "Topographic-isostatic world maps of the effect of the Hayford zones 10, 9, 8 and 7 to 1," *Publ. of the isost. Inst. of the Int. Assoc. of Geodesy*, No. 3, Helsinki, 1938.

W. Heiskanen, "Catalogue of the isostatically reduced gravity stations," *Publ. of the isost. Inst. of the Int. Assoc. of Geodesy*, No. 5, Helsinki, 1939.

F. A. Vening Meinesz, "Tables fondamentales pour la réduction isostatique régionale," *Bull. géod.* No. 63, 1939, pp. 711-776.

In 1884 the results of the gravity measurements carried out until then were collected by Helmert for setting up a formula for the normal part of gravity at sea level. Hence there followed according to Helmert, *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, Band II, Leipzig 1884, p. 241:

$$\gamma_0 = 978.000 (1 + 0.005310 \sin^2 \varphi). \quad (1)$$

Since that time, the significance of gravity measurements has more and more been recognized, and since a simple and very efficient means for the determination of gravity had been found in the Sterneck pendulum apparatus, such measurements were carried out in great number in all countries. In 1900 the results of 1400 stations were already available, and this material was utilized by Helmert for a new computation of the formula for the normal gravity at sea level. This computation is published in *Sitzungsber. d. Preuss. Ak. d. Wiss. zu Berlin*, math.-phys. Klasse, 1901, pp. 328-336.

Only the gravity values found on continental and coastal stations were used, since the insular stations yield, by experience, too large gravity values; stations on high mountain peaks, which likewise show systematic deviations, were also excluded. The reduction because of the elevation above sea level was carried out according to the simple method of p. 403, where the topographic correction has been disregarded, however. For the adjustment, the above-mentioned equation (1) was extended by an additional term with the factor $\sin^2 2 \varphi$, which corresponds to the spherical function of the fourth degree; the adjustment, however, yielded the coefficient of this latter term so inaccurately that there was assumed for it the value $-0.000\ 007$, which E. Wiechert and G. H. Darwin found from geophysical considerations.

The gravity measurements on which the computations are based refer throughout to the Vienna System. The value of gravity for Vienna was derived from the absolute gravity measurements carried out at the observatory in Vienna by v. Oppolzer in 1884 and at the observatory in Munich by v. Orff in 1877; all other stations have been connected with Vienna by relative measurements. But since the absolute measurements mentioned are not to be regarded as completely unobjectionable, in the years 1898-1904 there was carried out a new absolute gravity determination in Potsdam, which has already been discussed in section 73 and section 75. These measurements were carried out with a considerably higher accuracy, and therefore it appeared justified to refer all gravity measurements to this new system. The conversion from the Vienna to the Potsdam System was carried out on the basis of an adjustment of the relative measurements carried out between 20 main stations of the earth, which is represented in *Verhandlungen der 16. allgemeinen Konferenz der Internationalen Erdmessung*, III. Teil, 1911, pp. 1-26. The adjustment yielded the following relation between the two systems

$$\text{Vienna System } -0.016 \text{ gal} = \text{Potsdam System}.$$

The result of Helmert's computations forms the following formula for the normal gravity at sea level:

$$\gamma_0 = 978.030 (1 + 0.005\ 302 \sin^2 \varphi - 0.000\ 007 \sin^2 2 \varphi) \quad (2)$$

or in another form

$$\gamma_0 = 980.616 (1 - 0.002\ 644 \cos 2 \varphi + 0.000\ 007 \cos^2 2 \varphi). \quad (3)$$

Since the computation carried out in 1901 the number of the gravity stations has considerably increased; in the third part of *Verhandlungen der 16. allgemeinen Konferenz der Internationalen Erdmessung*, 1911, a total of 2736 stations is already listed. Consequently, in the Zentralbureau der Internationalen Erdmessung in Potsdam there was undertaken a new working up of the whole material, which we have already mentioned briefly in section 64, p. 319. The results of this work have been published by Helmert in the paper, "Neue Formeln für den Verlauf der Schwerkraft im Meeresniveau beim Festlande," *Sitzungsbericht der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, math.-physik. Klasse, 1915, pp. 676-685. In the case of these computations, the measuring material was organized several times in a different manner,

partly by separating the continental stations from the coastal stations, partly by making a selection of the stations with respect to their position in regard to mountains and steep coasts. On the other hand, an attempt was also made of adding to the above equation (2) a spherical function of the third degree and a term depending on the longitude; the coefficient of the spherical function of the third degree, however, resulted so uncertainly that this term cannot be considered as real. As result, Helmert found the following formula for gravity at sea level on land:

$$g_0 = 978.052 \{ 1 + 0.005\,285 \sin^2 \varphi - 0.000\,007 \sin^2 2 \varphi + 0.000\,018 \cos^2 \varphi \cos 2 (\lambda + 17^\circ) \} . \quad (4)$$

The same formula holds for the coast; the value of 978.068, however, is to be set for the main factor.

Formula (4) with the omission of the term depending on the longitude is to be designated, likewise, as the normal part of gravity at sea level. We thus have for land:

$$\gamma_0 = 978.052 (1 + 0.005\,285 \sin^2 \varphi - 0.000\,007 \sin^2 2 \varphi) , \quad (5)$$

which we can write also in the form

$$\gamma_0 = 980.636 (1 - 0.002\,636 \cos 2 \varphi - 0.000\,007 \sin^2 2 \varphi) . \quad (6)$$

Details of the computations are contained in the treatise, "Die Erdgestalt und die Hauptträgheitsmomente *A* und *B* der Erde im Äquator aus Messungen der Schwerkraft," von Dipl.-Ing. A. Berroth in *Gerlands Beiträge zur Geophysik*, XIV. Band, Heft 3, 1916.

The first derivation of a gravity formula on the basis of isostatically reduced gravity measurements was carried out by Bowie in 1912 with the help of 124 stations in the United States, about which there is a detailed report in *Spec. Publ.* No. 12, Washington, 1912, mentioned already in the previous section 82, p. 407. There resulted hence the formula

$$\gamma_0 = 978.038 (1 + 0.005\,302 \sin^2 \varphi - 0.000\,007 \sin^2 2 \varphi) . \quad (7)$$

A new computation was carried out likewise by Bowie in 1917 (cf. *Spec. Publ.* No. 40, Washington, 1917, p. 134, mentioned in section 82, p. 407) on the basis of 216 stations in the United States, 42 in Canada, 73 in India and 17 in Europe, a total of 348 stations. This new formula reads:

$$\gamma_0 = 978.039 (1 + 0.005\,294 \sin^2 \varphi - 0.000\,007 \sin^2 2 \varphi) . \quad (8)$$

We see that the newly added 224 stations have not brought about noticeable changes compared with formula (7) of 1912.

Then there follow extensive computations by W. Heiskanen in Helsinki, the first one in 1924. All stations lying in one degree square, formed by circles of longitude and latitude, were comprised here as one station, and of stations so comprised there were available: 355 in Europe, Caucasia, Algeria, coastal stations in Africa and on the Red Sea, 234 in America and 87 in Asia, therefore a total of 656 stations. Formulae were derived for the individual territories as well as for their totality; for the latter there was found

$$\gamma_0 = 978.048 (1 + 0.005\,293 \sin^2 \varphi - 0.000\,007 \sin^2 2 \varphi) . \quad (9)$$

Cf. in this connection: *Veröffentlichung des Finnischen Geodätischen Instituts* No. 4, "Untersuchungen über Schwerkraft und Isostasie," von W. Heiskanen. Helsinki 1924, p. 90.

A further computation was published by Heiskanen in 1928. Here also, the individual stations are

comprised in degree squares, whereby 841 such collective stations resulted. The formula derived therefrom reads:

$$\gamma_0 = 978.044 (1 + 0.005\,301 \sin^2 \varphi - 0.000\,007 \sin^2 2 \varphi) . \quad (10)$$

In the conference of the International Geodetic and Geophysical Union in Prague in 1927 the question was discussed as to whether it is appropriate to set up an international gravity formula which agrees with the flattening of the international ellipsoid of the earth (cf. first half-volume, p. 46). Under this point of view, Heiskanen carried out a second computation with the same material, by taking the flattening $\alpha = 1:297$ as a basis, from which there followed:

$$\gamma_0 = 978.049 (1 + 0.005\,289 \sin^2 \varphi - 0.000\,007 \sin^2 2 \varphi) . \quad (11)$$

The question of the international gravity formula forms also the object of a theoretical investigation by G. Cassinis of Pisa in *Bulletin Géodésique* No. 26, 1930, pp. 40-49, on the basis of which the following formula is suggested:

$$\gamma_0 = 978.049 (1 + 0.005\,2884 \sin^2 \varphi - 0.000\,0059 \sin^2 2 \varphi) . \quad (12)$$

This formula was accepted in the conference of the International Union in Stockholm, 1930, as the international gravity formula, and Cassinis has computed tables of the values of the international normal gravity on the basis of this formula in *Bulletin Géodésique* No. 32, 1931, pp. 313-326.

Finally, there exists another computation of the most recent time, which appeared as a *Publication of the Isostatic Institute of the International Union* in 1938, "Investigations of the gravity formula," by Dr. W. Heiskanen, Helsinki, 1938. Here again, the individual stations are collected in degree squares so that 1802 such collective stations, all reduced isostatically, and, besides, 383 isostatically reduced and 211 not isostatically reduced measurements on sea by Vening Meinesz could be used.

The result is

$$\gamma_0 = 978.051 (1 + 0.005\,3027 \sin^2 \varphi - 0.000\,059 \sin^2 2 \varphi) . \quad (13)$$

The flattening

$$\alpha = 1:298.3$$

corresponds to this gravity formula.

In Helmert's formula (4), p. 409, the development in series has been carried further one more step, since an additional term depending on the geographic longitude is indicated there. The computations by Heiskanen also have been carried out, on several occasions, by taking into account this term; thus, besides the above formula (13) there has been set up, in 1938, the more complete formula

$$\gamma_0 = 978.0524 (1 + 0.005\,2970 \sin^2 \varphi - 0.000\,0059 \sin^2 2 \varphi + 0.000\,0276 \cos^2 \varphi \cos 2 (\lambda + 25^\circ)) \quad (14)$$

The carrying of the term depending on the longitude no longer corresponds to the terrestrial spheroid; widely extended anomalies of gravity are rather expressed already by this term. The equator obtains hereby a flattening, so that the spheroid changes to a triaxial ellipsoid, in the case of which the flattening of the meridians is not constant. From formula (14) we find for the flattening of the meridians a maximum value of $\alpha = 1:295.3$, a minimum value of $\alpha = 1:300.2$, and a mean value of $\alpha = 1:297.8$.

On the other hand, the difference between the major and the minor semiaxis of the equatorial ellipse $a_1 - a_2 = 352$ m.

In the foregoing, we only have given the total result of the computations by Heiskanen from the year 1938. In particular, the computation has been carried out separately for stations on land, for stations at sea, at high latitudes, in the neighborhood of the equator and, besides, for various continents. Heiskanen arrives therefrom at the conclusion that the undulations of the geoidal surface are so considerable that it is not possible to set up a gravity formula which adapts itself to the gravity measurements in all parts of the earth's surface. It is therefore advisable to retain, for the present, the international gravity formula (12).

Section 84. Measurements by Means of the Torsion Balance

In 1896, it was proved by Baron Eötvös, professor in Budapest, on the basis of eight years' experiments, that the behavior of gravity on the earth's surface can be given by means of the torsion balance. Since the measurements with this instrument can be carried out conveniently and are capable of great accuracy, a valuable auxiliary means is herewith given for the supplementing of the pendulum measurements.

On this subject, the following papers have been published by Eötvös:

"Untersuchungen über Gravitation und Erdmagnetismus," *Ann. d. Physik u. Chemie*, Band 59, 1896, pp. 354-400.

"Bestimmung der Gradienten der Schwerkraft und ihrer Niveauflächen mit Hilfe der Drehwaage," *Verhandlungen der 15. Konferenz der Internationalen Erdmessung*, 1. Teil, 1908, pp. 337-395.

"Bericht über geodätische Arbeiten in Ungarn, besonders über Beobachtungen mit der Drehwaage," *Verhandlungen der 16. Konferenz der Internationalen Erdmessung*, 1. Teil, 1910, pp. 319-350.

There is to be mentioned further:

J. B. Messerschmitt, *Die Schwerebestimmung an der Erdoberfläche*, Braunschweig 1908, pp. 142-144.

J. B. Messerschmitt, "Eine neue Methode zur Bestimmung der Krümmungsverhältnisse des Geoids," *Zeitschrift für Vermessungswesen*, 1909, pp. 543-548.

F. R. Helmert, "Die Schwerkraft und die Massenverteilung der Erde," *Enzykl. d. math. Wiss.*, Band VI, 1, pp. 166-172.

A. Berroth, "Schweremessungen," *Handbuch der Physik*, Band II, Berlin 1926, pp. 465-473.

H. Schmehl und K. Jung, "Figur, Schwere und Massenverteilung der Erde," *Handbuch der Exper.-Physik*, Band 25, II. Teil, Leipzig 1931, pp. 304-312.

The basic idea of the measuring procedure is the following. As shown in Fig. 1, a rod loaded at its two ends by two equal weights P_1 and P_2 is suspended at its center of gravity O by means of a thin wire AO , so that the axis of the wire coincides with the direction of the vertical passing through O . Due to the curvature of the level surfaces the directions of the vertical at P_1 and P_2 are no longer exactly parallel to AO . Therefore, if we divide the gravity acting on P_1 into three components which are parallel to AO and OP_1 as well as perpendicular to these two directions, then the last-mentioned very small component will tend to cause a rotation of P_1 around AO . A corresponding component acts on P_2 , and if these two components are equal to one another, then the position of equilibrium is not disturbed. But due to the irregularities of the mass distribution in the neighborhood of the torsion balance a small difference of the two components appears by which a small rotation of the balance then occurs. Since the torsional resistance of the suspension wire counteracts the rotation, then the latter ceases as soon as torsion and gravity are in equilibrium. If the torsional resistance is known, then we can conclude from the size of the angle of rotation what the gravity is.

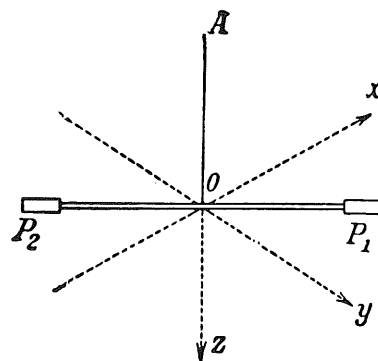


Fig. 1.

After these general discussions we will treat the mathematical theory of the torsion balance. To do so, we introduce a spatial rectangular system of coordinates whose zero point lies at O and whose x -axis is directed downward along the plumb line at O while the x - and y -axes lie on the plane perpendicular to it. Let the rod axis P_1P_2 make with the x -axis an arbitrary angle α .

Let a mass particle dm of the rod or of one of the two weights have the coordinates x , y and z . We can replace the gravity acting on dm by three force components acting on the point O in the direction of the three coordinate axes and by three force couples lying on the coordinate planes. The first do not interest us

here, and of the latter there is considered for the torsion only the force couple lying on the xy -plane, which causes a rotation of dm around the z -axis. If we denote the components of gravity in the direction of the x - and the y -axes by g_x and g_y , then the moment of the force couple mentioned is

$$df = (xg_y - yg_x) dm,$$

and if we extend this consideration to all mass elements of the torsion balance, then the whole turning moment of the balance is

$$F = \int (xg_y - yg_x) dm. \quad (1)$$

Since all mass elements have only a small distance from the zero point O , then, within this small region, a uniform change of gravity can be assumed; therefore, if g_{x0} and g_{y0} are the components of gravity at O , we have for an arbitrary point of the balance with the coordinates x , y and z

$$\left. \begin{aligned} g_x &= g_{x0} + \left(\frac{\partial g_x}{\partial x}\right)_0 x + \left(\frac{\partial g_x}{\partial y}\right)_0 y + \left(\frac{\partial g_x}{\partial z}\right)_0 z + \dots \\ g_y &= g_{y0} + \left(\frac{\partial g_y}{\partial x}\right)_0 x + \left(\frac{\partial g_y}{\partial y}\right)_0 y + \left(\frac{\partial g_y}{\partial z}\right)_0 z + \dots \end{aligned} \right\} \quad (2)$$

We have, however, since the z -axis coincides with the direction of gravity at O ,

$$g_{x0} = 0 \quad \text{and} \quad g_{y0} = 0. \quad (3)$$

We have further according to section 61, p. 307,

$$g_x = \frac{\partial W}{\partial x} \quad \text{and} \quad g_y = \frac{\partial W}{\partial y}, \quad (3a)$$

therefore, we have:

$$\left. \begin{aligned} \frac{\partial g_x}{\partial x} &= \frac{\partial^2 W}{\partial x^2} & \frac{\partial g_x}{\partial y} &= \frac{\partial^2 W}{\partial x \partial y} & \frac{\partial g_x}{\partial z} &= \frac{\partial^2 W}{\partial x \partial z} \\ \frac{\partial g_y}{\partial x} &= \frac{\partial^2 W}{\partial x \partial y} & \frac{\partial g_y}{\partial y} &= \frac{\partial^2 W}{\partial y^2} & \frac{\partial g_y}{\partial z} &= \frac{\partial^2 W}{\partial y \partial z} \end{aligned} \right\} \quad (4)$$

If we substitute (3) and (4) in (2), then we obtain:

$$\left. \begin{aligned} g_x &= \left(\frac{\partial^2 W}{\partial x^2}\right)_0 x + \left(\frac{\partial^2 W}{\partial x \partial y}\right)_0 y + \left(\frac{\partial^2 W}{\partial x \partial z}\right)_0 z + \dots \\ g_y &= \left(\frac{\partial^2 W}{\partial x \partial y}\right)_0 x + \left(\frac{\partial^2 W}{\partial y^2}\right)_0 y + \left(\frac{\partial^2 W}{\partial y \partial z}\right)_0 z + \dots \end{aligned} \right\} \quad (5)$$

and equation (1) then changes to

$$\begin{aligned} F &= \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2}\right)_0 \int xy dm + \left(\frac{\partial^2 W}{\partial x \partial y}\right)_0 \int (x^2 - y^2) dm \\ &\quad + \left(\frac{\partial^2 W}{\partial y \partial z}\right)_0 \int xz dm - \left(\frac{\partial^2 W}{\partial x \partial z}\right)_0 \int yz dm. \end{aligned}$$

In the last two terms we can also set

$$\frac{\partial^2 W}{\partial y \partial z} = \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial^2 W}{\partial x \partial z} = \frac{\partial g}{\partial x}$$

so that we obtain:

$$F = \left\{ \begin{aligned} & \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \int x y \, d m + \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 \int (x^2 - y^2) \, d m \\ & + \left(\frac{\partial g}{\partial y} \right)_0 \int x z \, d m - \left(\frac{\partial g}{\partial x} \right)_0 \int y z \, d m, \end{aligned} \right\} \quad (6)$$

where the integrations are to be extended to the whole mass of the balance.

Now we introduce a new coordinate system ξ, η, ζ , which is rigidly connected with the balance. Let the ζ -axis coincide with the z -axis, while the ξ -axis lies in the axis OP_1 of the rod. Then we have for every mass element according to Fig. 2

$$\begin{aligned} x &= \xi \cos \alpha - \eta \sin \alpha \\ y &= \xi \sin \alpha + \eta \cos \alpha, \end{aligned}$$

and by introducing these values into (6) we obtain:

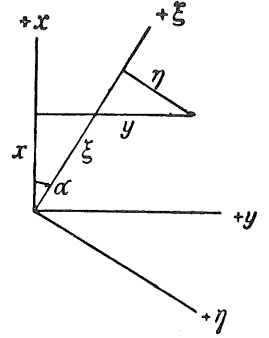


Fig. 2.

$$F = \left\{ \begin{aligned} & \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \frac{1}{2} \sin 2\alpha \int (\xi^2 - \eta^2) \, d m + \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \cos 2\alpha \int \xi \eta \, d m \\ & + \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 \cos 2\alpha \int (\xi^2 - \eta^2) \, d m - \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 2 \sin 2\alpha \int \xi \eta \, d m \\ & + \left(\frac{\partial g}{\partial y} \right)_0 \cos \alpha \int \xi \zeta \, d m - \left(\frac{\partial g}{\partial y} \right)_0 \sin \alpha \int \eta \zeta \, d m \\ & - \left(\frac{\partial g}{\partial x} \right)_0 \sin \alpha \int \xi \zeta \, d m - \left(\frac{\partial g}{\partial x} \right)_0 \cos \alpha \int \eta \zeta \, d m. \end{aligned} \right\} \quad (7)$$

If we assume that the balance is constructed exactly symmetrically, then we must have

$$\int \xi \eta \, d m = 0$$

since equally large positive and negative values of η belong to each value of ξ . We have likewise

$$\int \xi \zeta \, d m = 0 \quad \text{and} \quad \int \eta \zeta \, d m = 0,$$

and then we will have

$$F = \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \frac{1}{2} \sin 2\alpha \int (\xi^2 - \eta^2) \, d m + \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 \cos 2\alpha \int (\xi^2 - \eta^2) \, d m. \quad (8)$$

But according to section 62, p. 310, the moment of inertia of the torsion balance is

$$\int (\xi^2 + \eta^2) \, d m = \kappa \quad (9)$$

If we set

$$\int (\xi^2 - \eta^2) dm = \kappa' \quad \text{and} \quad \kappa' = \kappa (1 - \varepsilon), \quad (10)$$

then, if η remains very small everywhere, and hence the weights are also constructed accordingly, the quantity ε will likewise be small, so that we can set approximately

$$\kappa' = \kappa + \dots \quad (11)$$

Then we have

$$F = \frac{1}{2} \kappa \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \sin 2\alpha + \kappa \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 \cos 2\alpha. \quad (12)$$

If the balance has been turned out of the position of rest by the angle Θ through gravity, then by the torsional force there occurs, likewise, a turning moment, which can be expressed by $\tau\Theta$, where τ is a torsional constant depending on the length and thickness, as well as on the elasticity of the suspension wire. If the two forces keep the equilibrium, then we have

$$F = \tau\Theta;$$

therefore, we obtain

$$\Theta = \frac{\kappa}{2\tau} \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \sin 2\alpha + \frac{\kappa}{\tau} \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 \cos 2\alpha. \quad (13)$$

For the practical execution of the measurement, the supporting device, at which the suspension wire is fastened, must be provided with a fixed horizontal circle, around the center of which the supporting device can be turned. The reading pointer of the circle indicates then the direction of the rod axis with respect to an arbitrarily chosen starting direction, i.e. the angle α . Besides, an auxiliary means by which the angle Θ between the position of the rod axis in each case and its position free from torsion could be measured must be provided at the supporting device. But since the latter [position free from torsion] is not known, then there is read an angle ϑ which deviates from Θ by a constant amount ϑ_0 so that $\Theta = \vartheta - \vartheta_0$. If we read off the angle ϑ in the case of an arbitrary direction α of the instrument, and if the constants κ and τ of the instrument are known, then, according to (13) there follows an equation

$$\vartheta = \vartheta_0 + \frac{\kappa}{2\tau} \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \sin 2\alpha + \frac{\kappa}{\tau} \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 \cos 2\alpha, \quad (14)$$

in which there occur the three unknowns

$$\vartheta_0, \quad \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \quad \text{and} \quad \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0.$$

If we repeat the reading for two other settings α' and α'' , then we obtain altogether three equations for the computation of the unknowns.

Second form of the torsion balance

Another construction of the torsion balance by which further quantities depending on gravity can be determined was used by Eötvös. One of the two weights is suspended here below the rod axis at the interval h

according to the illustration in Fig. 3. For this construction also there holds the general equation (7), which is simplified again by the fact that we will have

$$\int \xi \eta \, dm = 0 \quad \text{and} \quad \int \eta \zeta \, dm = 0 .$$

The third integral $\int \xi \zeta \, dm$, however, is to be treated separately. If we imagine the weight P_1 in Fig. 3 moved upward by the length h , and denote the coordinates of a mass element of the upper weight by ξ, η, ζ' and those of the corresponding element of the lower weight by ξ, η, ζ , then we have

$$\zeta = \zeta' + h ,$$

and we obtain

$$\int \xi \zeta \, dm = \int \xi \zeta' \, dm + h \int \xi \, dm .$$

But we have

$$\int \xi \zeta' \, dm = 0 \quad \text{and} \quad \int \xi \, dm = l M ,$$

where M denotes the mass of the weight P_1 ; therefore, we have

$$\int \xi \zeta \, dm = h l M . \quad (15)$$

With this, we can simplify equation (7) also for the second form of the instrument and obtain finally

$$\vartheta = \vartheta_0 + \left. \begin{aligned} & \frac{\kappa}{2\tau} \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 \sin 2\alpha + \frac{\kappa}{\tau} \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 \cos 2\alpha \\ & - \frac{M h l}{\tau} \left(\frac{\partial g}{\partial x} \right)_0 \sin \alpha + \frac{M h l}{\tau} \left(\frac{\partial g}{\partial y} \right)_0 \cos \alpha . \end{aligned} \right\} \quad (16)$$

If the constants of the instrument κ, τ, M, h and l are given, the following quantities occur as unknowns in this equation

$$\vartheta_0, \quad \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0, \quad \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0, \quad \left(\frac{\partial g}{\partial x} \right)_0 \quad \text{and} \quad \left(\frac{\partial g}{\partial y} \right)_0 \quad (17)$$

and therefore the readings ϑ are to be determined for five different directions α if we aim to determine the values of the unknowns at the point of observation.

In Fig. 4, p. 416, a torsion balance of the second form is illustrated partly as a whole, partly in cross section. On a tripod there rises a column on whose pivots projecting on top, the instrument can be mounted and adjusted vertically by means of adjusting screws. A horizontal circle divided into thirds of degrees, around the axis of which the torsion balance can be turned, forms the lowest part of the instrument. For protection against exterior influences, the casing of the torsion balance consists of double-walled and, partly, of three-walled brass tubes. The platinum wire used for the suspension has a length of 65 cm and a thickness of 0.04 mm and is fastened at the head of the casing and at the rod by means of small brass plates. The rod is a thin-walled brass tube of 40-cm length and 0.5-cm diameter, in which a platinum cylinder with a weight of 30 g is inserted at one end, while a similar cylinder with a weight of 26 g hangs down at the other end on a wire 65 cm long. At the center of the tube there projects upward a rod 10 cm long, which carries a small mirror and on which the torsion wire is attached. Opposite the mirror, the casing is interrupted by a small window to which a small aiming telescope, supported by an arm, is directed. The latter is refracting in order to simplify the observation in a limited space, e.g. a tent. On the telescope there is attached a scale divided into half millimeters, which is visible through reflection in the mirror of the telescope.

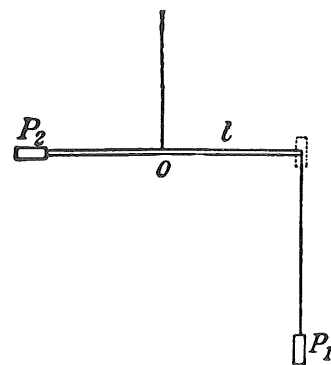


Fig. 3.

In the case of a rotation of the plummet around the suspension wire by a small angle, the mirror image of the scale moves by double the angle. If the scale is read with the help of the thread in the telescope and if the reading for the torsion-free position of the plummet is known, then from this and from the distance of the mirror from the scale there follows the angle of torsion.

Determination of the constants

Before carrying out the measurements, the different constants of the instrument which occur in equations (14) and (16), namely the quantities

$$\frac{\kappa}{\tau} \quad \text{and} \quad \frac{M h l}{\tau},$$

are to be determined further. Of these, M , h and l can be found directly by measurement, while κ and τ must be determined in an indirect way.

At first there follows the quotient $\frac{\kappa}{\tau}$, if we let the torsion balance carry out small horizontal oscillations around its equilibrium position in two positions perpendicular to one another and measure the time of vibration. This method can be used likewise for the torsion balance of the second form; however, strictly speaking, observations of oscillation in four positions of the plummet perpendicular to one another have to be carried out. But, as a rule, two measurements are also sufficient for this, as in the case of the instrument of the first form.

The determination of the torsional constants τ is carried out with the help of an experiment by Cavendish. For this purpose, a lead ball with a weight of about 13 kg, by whose attracting effect the torsion balance is deflected a little, is set up

alternately on one and on the other side of the torsion balance at a distance of about 10 cm from the platinum cylinder. The constant τ can be computed from the size of the deflection if the masses and distances are known.

We give here the results of a determination of the constant which Eötvös communicates in *Verhandlungen der 15. allgemeinen Konferenz der Internationalen Erdmessung 1906, I. Teil, Berlin 1908, pp. 345-346*. There is found here

$$\frac{\kappa}{\tau} = 41,896 \quad \tau = 0.5035,$$

and since there was given for the examined instrument

$$M = 25.43 \text{ g} \quad h = 56.6 \text{ cm} \quad l = 20.0 \text{ cm}$$

there followed

$$\frac{M h l}{\tau} = 57,173.$$

These values of the constant are to be substituted in equation (16). For practical use it is advisable further to introduce instead of the angles ϑ and ϑ_0 the corresponding scale readings. If n_0 is the unknown scale reading in the case of an untwisted wire, n the scale reading for the angle of deflection $\vartheta - \vartheta_0$ and D the distance of the scale from the mirror, then

$$\vartheta - \vartheta_0 = \frac{n_0 - n}{2D}.$$

With $D = 1232$ mm and the above constants, equation (16) changes then to

$$\left. \begin{aligned} n_0 - n = & + 0.05162 \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 10^9 \sin 2\alpha + 0.10323 \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 10^9 \cos 2\alpha \\ & - 0.14087 \left(\frac{\partial g}{\partial x} \right)_0 10^9 \sin \alpha + 0.14087 \left(\frac{\partial g}{\partial y} \right)_0 10^9 \cos \alpha. \end{aligned} \right\} \quad (18)$$

The computation of the observations can be based on this equation.

Example of a measurement with the torsion balance

For the determination of the four unknowns

$$\left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0, \quad \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0, \quad \left(\frac{\partial g}{\partial x} \right)_0, \quad \left(\frac{\partial g}{\partial y} \right)_0,$$

to which the unknown n_0 is added, five observations with different azimuths α are required, from which five equations of the form of equation (18) result. If the observations are carried out with the azimuths $0^\circ, 72^\circ, 144^\circ, 216^\circ$ and 288° , and the scale readings thereby found are denoted by n_1, n_2, \dots, n_5 , then we obtain for the four unknowns the equations

$$\left. \begin{aligned} 10^9 \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 &= + 4.5543 (n_5 - n_2) - 7.3691 (n_4 - n_3) \\ 10^9 \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 &= - 1.1972 (n_4 + n_3 - 2 n_1) + 3.1342 (n_5 + n_2 - 2 n_1) \\ 10^9 \left(\frac{\partial g}{\partial x} \right)_0 &= - 2.7011 (n_5 - n_2) - 1.6694 (n_4 - n_3) \\ 10^9 \left(\frac{\partial g}{\partial y} \right)_0 &= + 2.2976 (n_4 + n_3 - 2 n_1) - 0.8776 (n_5 + n_2 - 2 n_1). \end{aligned} \right\} \quad (19)$$

For setting the instrument to the azimuth $\alpha = 0^\circ$ a compass attached to the casing is used, while the additional azimuths were then set with the horizontal circle.

We take the following example from the report by Eötvös on pp. 348-349. mentioned already on p. 416. On December 2, 1906, the following measurement was carried out in a room of the Physical Institute:

α	0°	72°	144°	216°	288°
n	204.5	200.7	193.2	183.2	190.1

By substituting these values in equations (19) we obtain:

$$10^9 \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right)_0 = + 66.40$$

$$10^9 \left(\frac{\partial^2 W}{\partial x \partial y} \right)_0 = + 10.19$$

$$10^9 \left(\frac{\partial g}{\partial x} \right)_0 = + 21.02$$

$$10^9 \left(\frac{\partial g}{\partial y} \right)_0 = + 66.83.$$

The whole series of observations lasted from the middle of September 1906 to the end of April 1907. The daily observations yielded for the unknown values which deviated only rarely by more than one unit of the order of 10^{-9} from the mean values.

Thus far, the assumption has been made implicitly that the measurements take place only in plane terrain. In hilly terrain, the results of measurement are still to be freed from the influence of the irregularities of the terrain, which we shall however not discuss here further.

More recent torsion balance of Schweydar

For the simplification of the measurements, a double balance was constructed by Eötvös, in the case of which two balances are used in a parallel manner side by side, so that the two hanging weights are on opposite sides. Instead of the five measurements in the case of the simple balance, only three measurements are required here.

The torsion balance of the Geodetic Institute in Potsdam, which was described by Hecker in *Zeitschrift f. Instr.*, 1910, pp. 6-14, contained important innovations. Instead of the direct reading, a device for photographic recording was used here. In addition, the instrument was connected with a clockwork by means of which the balance was automatically turned further by a definite angle at certain time intervals. The attachment of the photographic registration had however the inconvenience that the symmetric arrangement of the masses, as the double balance of Eötvös shows it, could not be preserved.

The torsion balance of Schweydar, which is illustrated in Fig. 5, and which is manufactured by Askania-Werke in Berlin-Friedenau, shows again a perfectly symmetric construction in spite of the photographic registration, since the recording apparatus is placed in the vertical axis of the instrument. The frame for the sensitized plate is attached at the head of the outer tube mantle. Instead of the photographic registration, however, there can also be carried out a direct observation, for which the motion of the turning arms becomes visible on a small ground-glass plate below the plate frame.

The automatic turning of the instrument has likewise been retained in the case of Schweydar's construction. The motion is carried out by means of a clockwork and is interrupted at definite points of a horizontal circle by means of a clock for one hour each. It appeared that the balances come sufficiently to rest within the space of one hour in order to permit the reading or, as the case may be, the registration.

A detailed description of the instrument is given by Schweydar in *Zeitschrift für Instr.*, 1921, pp. 175-183.

In addition to this large torsion balance, a small torsion balance has been constructed by Askania-Werke in Berlin-Friedenau, besides also a three-plummet balance.

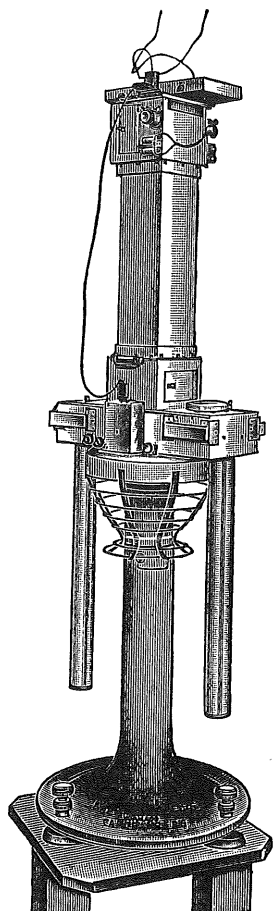


Fig. 5.

In order to recognize the meaning of the quantities measured with the torsion balance, we must in addition premise a few developments for the potential of gravity.

According to (11), section 61, p. 307, we have

$$W = V + \frac{1}{2} (x^2 + y^2) \omega^2, \quad (1)$$

and hence

$$\frac{\partial W}{\partial x} = \frac{\partial V}{\partial x} + x \omega^2, \quad (2)$$

We have further from (8) and (4), section 61, pp. 307 and 306,

$$\frac{\partial V}{\partial x} = f \int \frac{a-x}{u^3} dm.$$

In order to obtain the second differential quotient $\frac{\partial^2 W}{\partial x^2}$ we take from p. 306

$$\frac{\partial \left(\frac{1}{u} \right)}{\partial x} = \frac{a-x}{u^3}$$

and form hence

$$\begin{aligned} \frac{\partial^2 \left(\frac{1}{u} \right)}{\partial x^2} &= -\frac{1}{u^3} + \frac{3(a-x)}{u^5} \frac{\partial \left(\frac{1}{u} \right)}{\partial x} \\ \frac{\partial^2 \left(\frac{1}{u} \right)}{\partial x^2} &= -\frac{1}{u^3} + \frac{3(a-x)^2}{u^5} \end{aligned}$$

Then we have according to (1)

$$\begin{aligned} \frac{\partial^2 W}{\partial x^2} &= \frac{\partial^2 V}{\partial x^2} + \omega^2 = f \int \left(-\frac{1}{u^3} + \frac{3(a-x)^2}{u^5} \right) dm + \omega^2 \\ &= f \int \frac{-u^2 + 3(a-x)^2}{u^5} dm + \omega^2. \end{aligned}$$

If we carry out the same differentiation also for y and z , then we obtain

$$\begin{aligned} \frac{\partial^2 W}{\partial x^2} &= f \int \frac{-u^2 + 3(a-x)^2}{u^5} dm + \omega^2 \\ \frac{\partial^2 W}{\partial y^2} &= f \int \frac{-u^2 + 3(b-y)^2}{u^5} dm + \omega^2 \\ \frac{\partial^2 W}{\partial z^2} &= f \int \frac{-u^2 + 3(c-z)^2}{u^5} dm \end{aligned}$$

and there follows hence

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 2 \omega^2. \quad (3)$$

Through the point P_0 of a level surface we now lay a plane tangential to the latter and introduce a rectangular coordinate system x, y, z whose xy -plane lies on the tangential plane, while the z -axis coincides with the normal at P_0 . If the potential of gravity at P_0 is equal to W_0 , then we have for a point P of the level surface lying very near P_0 , whose coordinates shall be x, y, z , the following development in series according to Taylor's theorem

$$W = W_0 + \frac{\partial W}{\partial x}x + \frac{\partial W}{\partial y}y + \frac{\partial W}{\partial z}z + \frac{1}{2}\frac{\partial^2 W}{\partial x^2}x^2 + \frac{1}{2}\frac{\partial^2 W}{\partial y^2}y^2 + \frac{1}{2}\frac{\partial^2 W}{\partial z^2}z^2 + \frac{\partial^2 W}{\partial x \partial y}xy + \frac{\partial^2 W}{\partial x \partial z}xz + \frac{\partial^2 W}{\partial y \partial z}yz + \dots \quad (4)$$

or in shorter notation:

$$W = W_0 + W_x x + W_y y + W_z z + \frac{1}{2}W_{xx}x^2 + \frac{1}{2}W_{yy}y^2 + \frac{1}{2}W_{zz}z^2 + W_{xy}xy + W_{xz}xz + W_{yz}yz + \dots \quad (5)$$

But since the point P lies on the level surface of the point P_0 , we have $W = W_0$ and

$$\frac{\partial W}{\partial x} = W_x = 0 \quad \frac{\partial W}{\partial y} = W_y = 0 \quad \frac{\partial W}{\partial z} = W_z = -g. \quad (6)$$

Equation (5) then assumes the form

$$W_z z + \frac{1}{2}(W_{xx}x^2 + W_{yy}y^2 + W_{zz}z^2) + W_{xy}xy + W_{xz}xz + W_{yz}yz = 0. \quad (7)$$

If x and y are regarded as quantities of the first order, then according to (7) z is already of the second order, and therefore the terms in z^2 , xz and yz can be omitted in (7). Then

$$W_z z + \frac{1}{2}(W_{xx}x^2 + W_{yy}y^2) + W_{xy}xy = 0 \quad (8)$$

is the equation of the level surface passing through point P_0 .

We now introduce polar coordinates in the xy -plane by setting

$$x = p \cos \alpha \quad y = p \sin \alpha, \quad (9)$$

then (8) changes to

$$W_z z + \frac{1}{2}W_{xx}p^2 \cos^2 \alpha + \frac{1}{2}W_{yy}p^2 \sin^2 \alpha + W_{xy}p^2 \sin \alpha \cos \alpha = 0$$

or according to (6)

$$2z = \frac{p^2}{g}(W_{xx} \cos^2 \alpha + W_{yy} \sin^2 \alpha + 2W_{xy} \sin \alpha \cos \alpha). \quad (10)$$

If we assume for α a definite value, then (10) is the equation of the arc of a vertical section corresponding to this angle α .

Through point P we lay a circle which is tangent to the xy -plane at P_0 . This circle is then at the same time the circle of curvature of the arc of the vertical section. If ρ_α is the radius of the circle, then we have the proportion

$$z : p = p : (2\rho_\alpha - z),$$

therefore

$$p^2 = z(2\rho_\alpha - z).$$

Since z is a quantity of the second order, then for this we can also write simply

$$p^2 = 2\rho_\alpha z$$

and with this, (10) will be

$$\frac{1}{\rho_\alpha} = -\frac{1}{g}(W_{xx}\cos^2\alpha + W_{yy}\sin^2\alpha + 2W_{xy}\sin\alpha\cos\alpha). \quad (11)$$

For the determination of the planes of principal curvature we obtain by differentiation with respect to α

$$-W_{xx}2\cos\alpha\sin\alpha + W_{yy}2\sin\alpha\cos\alpha + W_{xy}2\cos 2\alpha = 0$$

and therefrom, if we denote the direction angle for the planes of principal curvature by α_0 and $\alpha_0 + 90^\circ$,

$$\tan 2\alpha_0 = -\frac{2W_{xy}}{W_{yy} - W_{xx}}. \quad (12)$$

We use this equation in order to eliminate $2W_{xy}$ from (11) and obtain

$$\frac{1}{\rho_0} = \frac{1}{g}\left(W_{xx}\cos^2\alpha_0 + W_{yy}\sin^2\alpha_0 + W_{xx}\frac{\sin^2 2\alpha_0}{2\cos 2\alpha_0} - W_{yy}\frac{\sin^2 2\alpha_0}{2\cos 2\alpha_0}\right).$$

In order to have $2\alpha_0$ everywhere, we use the relations

$$\sin^2\alpha_0 = \frac{1}{2} - \frac{1}{2}\cos 2\alpha_0 \quad \cos^2\alpha_0 = \frac{1}{2} + \frac{1}{2}\cos 2\alpha_0$$

and with this there follows

$$\frac{1}{\rho_0} = \frac{W_{yy} + W_{xx}}{2g} - \frac{W_{yy} - W_{xx}}{2g}\sec 2\alpha_0. \quad (13)$$

From (12) we find two values of $2\alpha_0$, which differ from one another by 180° . Consequently, (13) yields two values for $\frac{1}{\rho_0}$. If we take for α_0 the *acute* angle which follows from (12), then we have from (13) the two reciprocal radii of principal curvature

$$\left. \begin{aligned} \frac{1}{\rho_1} &= \frac{W_{yy} + W_{xx}}{2g} - \frac{W_{yy} - W_{xx}}{2g}\sec 2\alpha_0 \\ \frac{1}{\rho_2} &= \frac{W_{yy} + W_{xx}}{2g} + \frac{W_{yy} - W_{xx}}{2g}\sec 2\alpha_0 \end{aligned} \right\} \quad (14)$$

We can bring these expressions also into another form by eliminating the factor $\sec 2\alpha_0$ with the help of (12). If we set $W_{yy} - W_{xx} = W_\Delta$, we have

$$\tan 2\alpha_0 = -\frac{2W_{xy}}{W_\Delta} \text{ therefore } \sec 2\alpha_0 = \frac{\sqrt{4W_{xy}^2 + W_\Delta^2}}{W_\Delta} \quad (15)$$

and we will have hence

$$\left. \begin{aligned} \frac{1}{\rho_1} &= \frac{W_{yy} + W_{xx}}{2g} - \frac{\sqrt{4W_{xy}^2 + W_\Delta^2}}{2g} \\ \frac{1}{\rho_2} &= \frac{W_{yy} + W_{xx}}{2g} + \frac{\sqrt{4W_{xy}^2 + W_\Delta^2}}{2g} \end{aligned} \right\} \quad (16)$$

According to p. 417 we find from measurements with the torsion balance the four quantities

$$W_{yy} - W_{xx} = W_\Delta, \quad W_{xy}, \quad W_{zx} = -\frac{\partial g}{\partial x}, \quad W_{zy} = -\frac{\partial g}{\partial y}, \quad (17)$$

besides, from the gravity measurements there is known, in addition, $W_{zz} = -g$.

From these values we can calculate at first α_0 according to (12) or (15), as the case may be. If we form further the difference of the two equations (16), then we obtain

$$g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = -\sqrt{4W_{xy}^2 + W_\Delta^2}, \quad (18)$$

from which the difference of the two reciprocal radii of principal curvature can likewise be determined.

Finally, we form the sums of the two equations (14) or (16), with which we obtain

$$g \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) = W_{yy} + W_{xx}$$

But according to (3) we have

$$W_{yy} + W_{xx} = 2\omega^2 - W_{zz},$$

and since $W_{zz} = -\frac{\partial g}{\partial z}$, we have

$$g \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) = \frac{\partial g}{\partial z} + 2\omega^2. \quad (19)$$

Therefore, if it is possible to determine physically the quantity $\frac{\partial g}{\partial z}$, i.e. the change of gravity with the height, then according to (19) we could also determine the sum of the two reciprocal radii of principal curvature, and then from (18) and (19) the two radii of principal curvature themselves would also be known.

Finally, the measured quantities give information about the direction of the maximum change of gravity and about the latter itself. If we denote the maximum change of gravity or the *gradient* of gravity by G , then we have:

$$G = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}, \quad (20)$$

and for the direction φ of the gradient we have

$$\tan \varphi = \frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial x}}. \quad (21)$$

By this, valuable information about the behavior of gravity is already obtained.

Now we will show further how we can determine from measurements with the torsion balance at two points the difference of gravity at the two points.

We imagine that at two points P_1 and P_2 measurements with the torsion balance are carried out, and that the points lie so near each other that between them a uniform change of the measuring results found with the torsion balance can be assumed. Now we lay the coordinate system introduced in Fig. 1, section 84, p. 411, in such a way that the x -axis passes through the two points. From the measurements we obtain then the quantity $\frac{\partial g}{\partial x}$ for the two points. Now

$$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial z} dz,$$

and hence we have

$$g_2 - g_1 = \int_{x_1}^{x_2} \frac{\partial g}{\partial x} dx + \int_{z_1}^{z_2} \frac{\partial g}{\partial z} dz.$$

But since between the two points a uniform behavior of the two quantities $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial z}$ was assumed, then we can also set

$$g_2 - g_1 = \frac{1}{2} \left(\left(\frac{\partial g}{\partial x} \right)_1 + \left(\frac{\partial g}{\partial x} \right)_2 \right) (x_2 - x_1) + \frac{1}{2} \left(\left(\frac{\partial g}{\partial z} \right)_1 + \left(\frac{\partial g}{\partial z} \right)_2 \right) (z_2 - z_1). \quad (22)$$

Since the two values of $\frac{\partial g}{\partial z}$ cannot be determined by means of the torsion balance, then we will introduce for this the normal change of gravity with the height as an approximate value. According to (1), section 81, p. 400, we have

$$\frac{\partial g}{\partial z} = \frac{2g}{r}, \quad (23)$$

if r denotes the earth's radius. With this, (22) becomes

$$g_2 - g_1 = \frac{1}{2} \left(\left(\frac{\partial g}{\partial x} \right)_1 + \left(\frac{\partial g}{\partial x} \right)_2 \right) (x_2 - x_1) + \frac{2g}{r} (z_2 - z_1). \quad (24)$$

The smaller the difference of elevation $z_2 - z_1$ is the more accurately will the correction term $\frac{2g}{r}(z_2 - z_1)$ be determined.

In order to show the accuracy of the relative gravity measurement by means of the torsion balance, in the following we put together a few results found by Eötvös and the results of pendulum measurements:

	Δg Torsion Balance	Δg Pendulum
Kuvin-Hidegkút	+ 40 mgal	+ 39 mgal
Livada-Kuvin	+ 1	0
Pankota-Livada	— 4	— 8
Kuvin-Arad	+ 16	+ 17
Pankota-Arad	+ 14	+ 9
(through Kuvin)		
Pankota-Arad	+ 9	+ 9
(through Kurtics)		

We shall become acquainted with a further application of the torsion balance for the determination of deflections of the vertical in the following chapter in section 99.

Although measurements with the torsion balance are used at the present time primarily as exploratory methods in the service of industry we see from the foregoing that they can also furnish valuable contributions for the solution of geodetic problems.

The theory and the practical use of the torsion balance, as well as the evaluation of the results of measurement, are treated in detail in H. Haalck, *Lehrbuch der angewandten Geophysik*, Berlin, 1934, pp. 38-101.

Section 86. Theory of Geometric Leveling

Gravity and the level surfaces of gravity are of fundamental significance for geometric leveling, whose theory we will therefore discuss more thoroughly in the following.

The method of leveling with equal target ranges backward and forward yields correct differences of elevation as long as we can regard the earth's surface and the level surfaces of the earth as spherical surfaces. This method, however, needs a more thorough examination if the spheroidal shape of the level surfaces is to be taken into account.

In volume II, second half-volume, ninth edition, 1933, section 23, pp. 117-121, we have already made a short note about it, in which the influence of the ellipsoidal shape of the earth was examined collectively. With the help of the results found in the preceding chapters we can now treat the theory of leveling more thoroughly.

Influence of the curvature of the level surfaces

The first difference from spherical computation of leveling follows from the fact that the curvature of the level surface is not exactly the same as we sight backward and sight forward. We understand immediately that the change of curvature is greatest in the direction of the meridian, and therefore we will limit ourselves to this case.

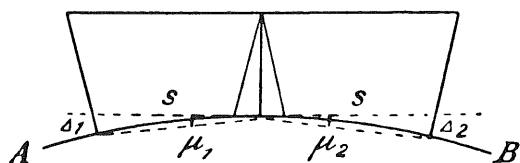


Fig. 1.

In Fig. 1, in which AB represents the meridian section of a level surface, the difference of curvature will manifest itself in the fact that the two magnitudes Δ_1 and Δ_2 , which are equal to one another for the sphere, deviate from one another for the spheroidal level surface. If we denote

the depression angles at the backsight and foresight by μ_1 and μ_2 , then, if the leveling progresses in the

direction from south to north, we have according to equation (16), section 2, p. 12, for $\alpha = 0$:

$$\left. \begin{aligned} \mu_1 &= \frac{s}{2N}(1 + \eta^2) + \frac{s^3}{2N^2}\eta^2 t \\ \mu_2 &= \frac{s}{2N}(1 + \eta^2) - \frac{s^3}{2N^2}\eta^2 t, \end{aligned} \right\} \quad (1)$$

where we have, as always:

$$\eta^2 = e'^2 \cos^2 \varphi \quad t = \tan \varphi.$$

To a rough approximation we can then set

$$\Delta_1 = s \sin \mu_1 \quad \Delta_2 = s \sin \mu_2$$

so that we have:

$$\Delta_1 - \Delta_2 = s(\sin \mu_1 - \sin \mu_2),$$

or else

$$\Delta_1 - \Delta_2 = s \left(\mu_1 - \mu_2 - \frac{1}{6}(\mu_1^3 - \mu_2^3) \right). \quad (2)$$

Since in equations (1) the terms of fifth order are already neglected, then we see, if this omission is also continued on, that in (2) the term $\frac{1}{6}(\mu_1^3 - \mu_2^3)$ can likewise remain disregarded. We thus have

$$\Delta_1 - \Delta_2 = \frac{s^3}{N^2} \eta^2 t. \quad (3)$$

We already see hence that the amount $\Delta_1 - \Delta_2$, i.e. the error which we commit in the usual computation of leveling, is insignificant in the case of the individual setting up of the instrument. However, the question still remains open whether the sum of the amounts (3) can lead to a noticeable error in the case of a long leveling line.

Since $\frac{s}{N} = \frac{\Delta \varphi}{2}$ can be set with sufficient accuracy, if $\Delta \varphi$ denotes the difference of latitude of the two staff points, then we can also write equation (3) in the form:

$$\Delta_1 - \Delta_2 = \frac{s^2}{2N} \eta^2 t \Delta \varphi,$$

and then we have for the whole leveling line

$$\Sigma(\Delta_1 - \Delta_2) = \frac{s^2}{2N} \Sigma \eta^2 t \Delta \varphi,$$

in which we can understand by N , with sufficient accuracy, a constant mean value of the radius of curvature in the prime vertical of the terrestrial ellipsoid. Since the aiming range s must always be very small in comparison to N , then $\Delta \varphi$ can also be understood as a differential $d\varphi$, and we will have:

$$\Sigma(\Delta_1 - \Delta_2) = \frac{s^2}{2N} \int_{\varphi_0}^{\varphi_n} \eta^2 t d\varphi = \frac{s^2}{4N} e'^2 \int_{\varphi_0}^{\varphi_n} \sin 2\varphi d\varphi$$

or

$$\Sigma(\Delta_1 - \Delta_2) = -\frac{s^2}{8N} e'^2 (\cos 2\varphi_n - \cos 2\varphi_0). \quad (4)$$

In order to arrive at a numerical example, we will compute the error term (4) for a leveling line extending from the equator to the pole, so that we will have then $\varphi_0 = 0^\circ$ $\varphi_n = 90^\circ$. There follows hence

$$\Sigma(\Delta_1 - \Delta_2) = +\frac{s^2}{4N} e'^2,$$

and with $s = 50$ m, $N = 6,390,000$ m, $e'^2 = 1:150$, we obtain

$$\Sigma(\Delta_1 - \Delta_2) = +0.00065 \text{ mm}.$$

This estimate shows that the change of curvature of the level surfaces along a leveling line can in any case remain disregarded.

The theoretical closure error of a leveling loop

We have already recognized in section 61, p. 308, that the level surfaces of gravity are not parallel to one another, and that for two neighboring surfaces the expression

$$gdz = \text{const.} \quad (5)$$

holds, in which dz denotes the distance between the surfaces at an arbitrary point and g the gravity at the same point. This equation (5) also forms the basis for the rigorous computation of geometric levelings.

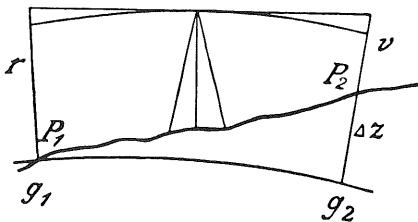


Fig. 2.

In Fig. 2 the leveling instrument is set up between two turning points P_1 and P_2 , where the readings r and v are found at the two staffs. If Δz denotes the distance of point P_2 from the level surface of point P_1 and if g_1 and g_2 are the values of gravity at the two points, then we have according to (5):

$$r g_1 = (v + \Delta z) g_2 \quad \text{or} \quad r g_1 - v g_2 = \Delta z g_2.$$

This can also be written in the form:

$$(r - v) g_2 + r(g_1 - g_2) = \Delta z g_2$$

or

$$\Delta z = r - v + r \frac{g_1 - g_2}{g_2}. \quad (6)$$

With respect to the small difference of gravity at the two turning points P_1 and P_2 we can assume immediately that the last term in (6) is insignificant. But we will also examine if in the case of the summation of the magnitudes Δz in an extended leveling line the last term in (6) can reach any significance. Then we obtain:

$$\Sigma \Delta z = \Sigma(r - v) + \Sigma r \frac{g_1 - g_2}{g_2}.$$

For an approximate calculation it is sufficient to assume a constant mean value for r , e.g. $r = 2$ m; besides, we can replace the last term by an integral and find:

$$\Sigma r \frac{g_1 - g_2}{g_2} = r \int \frac{dg}{g},$$

where the integral is to be extended over the whole leveling line. We will take only the normal part of gravity into consideration here and have then according to section 83, p. 408, and section 81, p. 400,

$$g = g^{\circ}_{45} \left(1 - \beta \cos 2\varphi - \frac{2H}{R} \right), \quad (7)$$

if φ denotes the geographic latitude, H the elevation above sea level and R the mean radius of the earth, while the coefficient $\beta = 0.0026$. According to this, we have

$$dg = g^{\circ}_{45} \left(2\beta \sin 2\varphi d\varphi - \frac{2dH}{R} \right),$$

and if terms of the second order are neglected, we also have

$$\frac{dg}{g} = 2\beta \sin 2\varphi d\varphi - \frac{2dH}{R}. \quad (8)$$

With this, we will have

$$\Sigma r \frac{g_1 - g_2}{g_2} = 2r\beta \int_{\varphi_0}^{\varphi_n} \sin 2\varphi d\varphi - \frac{2r}{R} \int_{H_0}^{H_n} dH,$$

where the indices 0 and n refer to the starting and end points of the line.

The first term integrated is equal to $r\beta (\cos 2\varphi_n - \cos 2\varphi_0)$, and this yields for a line running from the equator to the pole, with $r = 2$ m, an amount of about 10 mm; the second term with $H_2 - H_1 = 1000$ m, reaches only the value of 0.6 mm.

We therefore see that the last term in (6) is insignificant in comparison with the accuracy of measurement, and that in leveling the difference backsight minus foresight always yields the elevation of the turning point in front above the level surface of the turning point in back.

After this preliminary examination we return to the basic equation (5), p. 426, which we will now apply to a closed leveling loop. From the measurement we obtain the distances Δz between the individual level surfaces, and the question now arises whether the sum of the Δz 's for the whole loop becomes equal to zero, as we assume in the usual computation. Each layer bound by two neighboring level surfaces is intersected by the leveling line twice in the opposite direction, and if the level distances hereby measured on the way there and on the way back are denoted by Δz and $\Delta z'$, then we have according to (5), p. 426,

$$\begin{aligned} g_1 \Delta z_1 &= -g_1' \Delta z_1' \\ g_2 \Delta z_2 &= -g_2' \Delta z_2' \\ &\dots \end{aligned}$$

and there follows hence for the whole loop

$$\Sigma g \Delta z = 0. \quad (9)$$

If we introduce, in addition, an arbitrary constant value g_0 , which we imagine to be chosen in such a way that $g - g_0$ remains as small as possible for the whole loop, then we will have

$$g_0 \sum \Delta z + \sum (g - g_0) \Delta z = 0$$

or

$$\sum \Delta z = -\frac{1}{g_0} \sum (g - g_0) \Delta z. \quad (10)$$

This expression represents the theoretical closure error of a leveling loop.

Because of the lack of knowledge of the true values of g the computation of the theoretical closure error according to (10) will be possible only in very rare cases; however, we can determine the closure error always insofar as it depends on the normal part of gravity. According to the above equation (7) we have, if we disregard the small change of gravity with elevation,

$$g = g_{45}^\circ (1 - \beta \cos 2\varphi) \quad \beta = 0.0026,$$

hence we will have

$$g - g_{45}^\circ = -g_{45}^\circ \beta \cos 2\varphi$$

and

$$\sum \Delta z = +0.0026 \sum \cos 2\varphi \Delta z. \quad (11)$$

As an example we mention a computation by Helmert in *Veröffentlichung des Geodätischen Instituts in Potsdam und des Zentralbureaus der Internationalen Erdmessung*, "Die Schwerkraft im Hochgebirge, insbesondere in den Tiroler Alpen in geodätischer und geologischer Beziehung" by F. R. Helmert, Berlin, 1890. In this, there is treated a leveling loop which, starting from Bozen, runs through Brixen, Brenner, Innsbruck, Landeck, Mals, Meran and back to Bozen with a total length of 356 km. For 37 stations of this loop, gravity measurements were carried out by Lieutenant Colonel von Sterneck in the years 1887 and 1888 so that the satisfactory computation of the theoretical closure error became hereby possible. The elevations of the stations vary between 250 m and 1500 m. The results of the computations by Helmert are the following:

$$\begin{aligned} \text{Measured } \sum \Delta z &= -0.180 \text{ m.} \\ \text{Theoretical closure error} &= -0.024 \text{ m.} \\ \text{Normal closure error} &= -0.007 \text{ m.} \end{aligned}$$

In the present case, the measured closure error thus originates for the most part from measuring errors. From this, we must by no means conclude, however, that the influence of gravity must always be so small; in the opinion of Helmert, theoretical closure errors of the order of a decimeter in leveling loops of this kind are rather very conceivable.

Section 87. The Orthometric Correction of Leveling

In Fig. 1 AC denotes sea level and B the summit of a mountain, to which from A a leveling line is carried up; the level distances $\Delta z_1, \Delta z_2, \dots$ are found hereby. Likewise, there is presented in Fig. 1 the result of a second leveling line conducted from C to B .

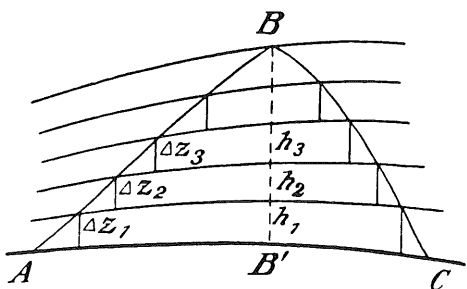


Fig. 1.

From this, we see at first that due to the nonparallel position of the level surfaces, the sum of the measured differences of elevation Δz in the line AB does not agree with the sum of the measured differences of elevation in the line CB . Secondly, however, Fig. 1 also teaches that the correct elevation above sea level $B'B = H$ of point B does not result directly from either the line AB , or from the line CB . According to Goulier, we designate the quantity which is to be added to every leveling line in order that the measured difference of elevation becomes equal to the difference of the elevations above sea level of the two end points as the orthometric correction of the line.

For the computation of the elevation above sea level H of the point B we start likewise from equation (5), section 86, p. 426. For if g_1, g_2, \dots denote the values of gravity at the measured level distances

$\Delta z_1, \Delta z_2, \dots$ and g_1', g_2', \dots the corresponding values at the differences of elevation h_1, h_2, \dots then we have according to (5), section 86, p. 426,

$$h_1 = \frac{g_1}{g_1'} \Delta z_1$$

$$h_2 = \frac{g_2}{g_2'} \Delta z_2$$

$$\dots \dots \dots ,$$

consequently, we will have

$$BB' = H = \Sigma \frac{g}{g'} \Delta z. \quad (1)$$

Strictly speaking, in addition to the values of gravity along the leveling line, the values below the earth's surface at the plumb line of the point in question are therefore necessary also for the satisfactory computation of the elevation above sea level. The rigorous computation is therefore not possible at all; only an approximate computation will rather be practicable always, since the last-mentioned gravity values are determined by approximation.

If special gravity measurements at the leveling points are not available, then there remains only the taking into account of the normal part of gravity. For the determination of the orthometric correction, in Fig. 2 we have represented once again two consecutive turning points P and P' of a leveling line, from which the magnitude Δz is found from the backsight and the foresight.

The quantity v is the orthometric correction of Δz , so that $\Delta z + v$ indicates the actual difference of elevation of the points P and P' . We imagine at first, at an arbitrary elevation, two neighboring level surfaces with the distance of elevation ΔH , for which the deviation from the parallel position is expressed by dv . If g and $g + dg$ are the values of gravity at the lower level surface, then we have according to (5), section 86, p. 426,

$$g \Delta H = (g + dg) (\Delta H + dv).$$

We are to bear in mind here that, it is true, ΔH is to be thought also as an infinitely small quantity, but that dv , in comparison to it, is an infinitely small quantity of the second order. Therefore, the product $dg \Delta H$ must be retained while the product $dg dv$ can be neglected. With this, we obtain:

$$g dv = - dg \Delta H \quad dv = - \Delta H \frac{dg}{g}. \quad (2)$$

From (8), section 86, p. 427, we take

$$\frac{dg}{g} = 2 \beta \sin 2 \varphi d \varphi,$$

where the last term in (8), section 86, p. 427, is omitted, since the change of g with the elevation can be neglected. With this, we will have

$$dv = - 2 \beta \sin 2 \varphi d \varphi \Delta H. \quad (3)$$

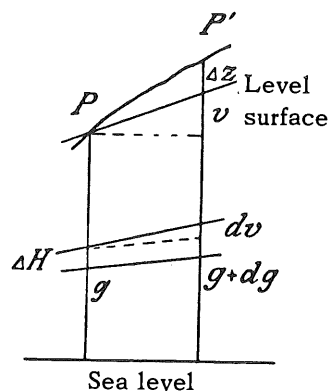


Fig. 2.

If we imagine this quantity dv set up for all layers between sea level and the level surface passing through P , then their sum is equal to the orthometric correction v sought for. Since in this summation the product $\sin 2 \varphi d\varphi$ is constant, we will have

$$v = -2 \beta H \sin 2 \varphi d\varphi. \quad (4)$$

This is the correction for an individual, measured difference of elevation. The correction for the whole leveling line is obtained by summation of the individual corrections or by means of the integral

$$V = -2 \beta \int_{\varphi_0}^{\varphi_n} H \sin 2 \varphi d\varphi, \quad (5)$$

in which φ_0 and φ_n are the latitudes of the two end points of the line.

We designate formula (5), which is based only on the *normal* gravity at sea level, as the spheroidal-orthometric correction.

Practical application of equation (5) is appropriately carried out with the help of a graphical method indicated by Ch. Lallemand, in which the value of V is found by area computation. For if we use a rectangular coordinate system, in which the values

$$\begin{aligned} x &= -\beta \cos 2 \varphi \\ y &= +H \end{aligned}$$

of the individual points of the leveling line are plotted as abscissae and ordinates, then we obtain, according to Fig. 3, a curve which extends from

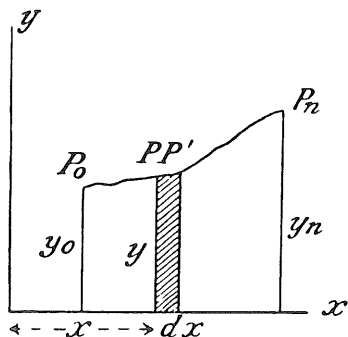


Fig. 3.

to

$$\begin{aligned} x_0 &= -\beta \cos 2 \varphi_0 & y_0 &= H_0 \\ x_n &= -\beta \cos 2 \varphi_n & y_n &= H_n \end{aligned}$$

For the shaded area element in Fig. 3 the area then is:

or

$$\begin{aligned} y dx &= -H d(\beta \cos 2 \varphi) \\ y dx &= +2 \beta H \sin 2 \varphi d\varphi. \end{aligned} \quad (6)$$

The area element, disregarding the sign, is therefore equal to the orthometric correction of the difference of elevation PP' in Fig. 3, and we only have to form the whole area between the ordinates y_0 and y_n in order to find the correction for the line from P_0 to P_n .

The literature which has originated on the theory of geometric leveling since about 1870 is reported by Ch. Lallemand in *Verhandlungen der vom 21. bis zum 29. Oktober 1887 auf der Sternwarte zu Nizza abgehaltenen Konferenz der Permanenten Kommission der Intern. Erdm.*, Berlin 1888, Annexe Nr. V, f pp. 1-2; further by F. Lorber in *Das Nivellieren*, Wien 1894, p. 465. The orthometric correction was published in the form of the above equation (5) for the first time by Ch. Lallemand in *Annales des points et chaussées*, 6. sér., Tome XIV, 1887, 2^e sem., p. 500. A thorough critical treatment of the theory of leveling is found in F. R. Helmert, *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, Band II, Leipzig 1884, pp. 500-550.

Orthometric correction according to Helmert

As we have seen, formula (5), above, is based only on the normal gravity, so that all anomalies of gravity are not taken into account. Since a simple and convenient means for the measurement of gravity is found

in the static gravimeters, the orthometric correction can be improved by the introduction of the measured values of gravity. On this basis, Helmert has developed a new method of reduction in the treatise, "Die Schwerkraft im Hochgebirge," *Veröffentlichung des Kgl. Preussischen Geodätischen Instituts und Zentralbureaus der Internationalen Erdmessung*, Berlin 1890, pp. 27-28.

Below a point P , in whose neighborhood the terrain is assumed to be horizontal, we imagine a certain level surface above which the point P has the elevation h . Let the gravity at P be g . If we go down along the plumb line of P to the point P_0 of the level surface, then we find for P_0 the gravity g_0 according to (11), section 81, p. 403,

$$g_0 = g + \frac{2h}{r}g - \frac{3}{2} \frac{\Theta}{\Theta_m} \frac{h}{r}g.$$

For a point P_m at the mean elevation $\frac{h}{2}$ we then have

$$g_m = g_0 - \frac{h}{r}g,$$

because the slab between the points P_0 and P does not exert an attraction on the point P_m lying at the mean elevation. Therefore, we also have

$$g_m = g + \frac{h}{r}g - \frac{3}{2} \frac{\Theta}{\Theta_m} \frac{h}{r}g = g \left(1 + \frac{h}{r} \left(1 - \frac{3}{2} \frac{\Theta}{\Theta_m} \right) \right)$$

or, if we set

$$1 - \frac{3}{2} \frac{\Theta}{\Theta_m} = \kappa$$

$$g_m = g \left(1 + \kappa \frac{h}{r} \right). \quad (7)$$

We obtain hence the potential difference between the level surfaces at P_0 and P

$$\Delta W = g \left(1 + \kappa \frac{h}{r} \right) h. \quad (8)$$

At a neighboring point, which lies higher than P by the amount dz according to the leveling, and whose elevation above the level surface of P_0 is therefore equal to $h + dh$, we have the gravity

$$(g + dg) \left(1 + \kappa \frac{h + dh}{r} \right)$$

and the potential difference

$$\Delta W' = g \left(1 + \kappa \frac{h}{r} \right) h + g dz. \quad (9)$$

On the other hand, however, the potential difference according to (8) is also

$$\Delta W' = (g + dg) \left(1 + \kappa \frac{h + dh}{r} \right) (h + dh),$$

and consequently, we have the equation

$$(g + dz) \left(1 + \kappa \frac{h + dh}{r} \right) (h + dh) = g \left(1 + \kappa \frac{h}{r} \right) h + g dz. \quad (10)$$

We obtain hence, if quantities of the second order are neglected,

$$dh = dz - \frac{h}{g} dg - 2 \frac{h}{r} \kappa dh - \frac{h^2}{r} g \kappa dg.$$

The last term here, which numerically is very small, can be neglected. Besides, in the third term, dh can be replaced by dz , so that we obtain

$$dh = dz - \frac{h}{g} dg - \frac{2\kappa h}{r} dz. \quad (11)$$

If we change from differentials to finite quantities, then we have, finally, for two points P_1 and P_2

$$H_2 - H_1 = \sum_1^2 \delta z - \frac{2\kappa}{r} \sum_1^2 h \delta z - \frac{2}{1} \frac{h}{g} \delta g. \quad (12)$$

On the question of the orthometric correction of the Swiss land leveling, in which this correction plays an important role, an authoritative report was prepared by F. Baeschlin: *Untersuchungen über die Reduktion der Präzisions-Nivellements*. Im Auftrage der Abt. f. Landestop. des schweiz. Milit.-Dep. bearbeitet von Dipl.-Ing. F. Baeschlin, Professor der Geodäsie an der Eidgen. Techn. Hochschule, Zürich, Bern 1926. In this, the various possibilities of orthometric correction are submitted to a criticism, which leads to the result that for the Swiss precision leveling Helmert's reduction alone is considered.

A further improvement of the reduction method is developed in the paper: *Veröffentlichung der Schweiz. Geodätischen Kommission*, "Nivellement und Schwere als Mittel zur Berechnung wahrer Meereshöhen," von Th. Niethammer, 1932. Schweizerische Landestopographie in Bern. The computation of the leveling line Biasca-S. Bernardino-Reichenau, which rises to within 2000 m above the two starting stations, shows only inconsiderable differences between the new method of reduction and Helmert's method.

DEFLECTIONS OF THE VERTICAL

Section 88. General Information About Deflections of the Vertical

In Chapter VII we have established that in general we can represent the geoid by a spheroid, which very closely approximates a flattened ellipsoid of rotation. The systematic deviations mentioned already in section 58, p. 296, which result in the calculation of degree-measurements, and which cannot be explained by errors of astronomic and geodetic measurements alone, teach that the geoid cannot be replaced completely by an ellipsoid of rotation.

These deflections rather come about by the fact that the normals to the surface of the geoid, or the directions of the vertical, do not coincide exactly with the normals to the surface of the spheroid or the ellipsoid.

In Fig. 1 below, the mutual position of the physical surface of the earth, a mean ellipsoid, and the geoid is represented in a rough schematic manner. The line for the ellipsoid is drawn straight, insofar as only

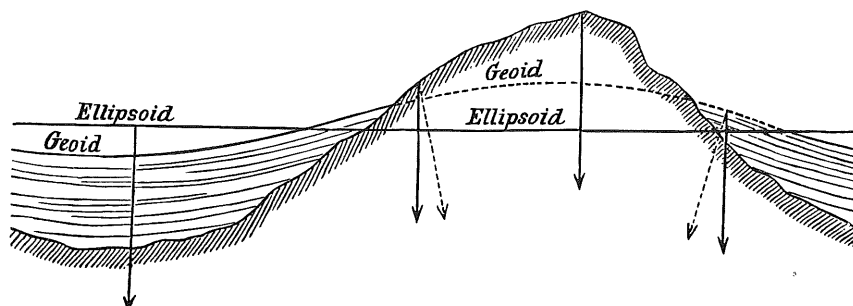


Fig. 1.
Ellipsoid and geoid.

a small part of the earth's surface shall be considered and the drawing serves only to illustrate the relative curvatures between the mean ellipsoid and the geoid.

The solid arrow lines represent the geometric normals to the ellipsoid, and the dotted arrow lines represent the physical plumb lines, which are perpendicular to the surface of the geoid. The small angle between a normal to the ellipsoid and the direction of gravity is the *deflection of the vertical*; if the direction of gravity coincides with the normal to the ellipsoid, as is assumed in Fig. 1 above sea level and at the height of the mountain, then the deflection of the vertical is equal to zero.

It follows from this consideration that deflections of the vertical have always only relative significance, namely for two reasons: First, for the determination of the deflections of the vertical we must assume a definite comparison ellipsoid, which we have introduced in Fig. 1 as a mean ellipsoid, e.g., the Bessel ellipsoid. According to the assumption of the ellipsoid we obtain different deflections of the vertical.

But, in addition, a definite position of the comparison ellipsoid with respect to the geoid must also be assumed; e.g., we can assume at a definite point that the normal to the ellipsoid coincides with the direction of the vertical so that at this point the comparison ellipsoid and the geoid are tangent to one another. If we assume another point as point of tangency, then we obtain other values for all deflections of the vertical.

Therefore, the values of the deflections of the vertical depend on the choice of the comparison ellipsoid or the *reference ellipsoid* and on the orientation of this ellipsoid with respect to the geoid.

The determination of the deflections of the vertical is carried out by the combination of astronomic and geodetic measurements. By means of the astronomic measurements, i.e. by the determination of the geographic

latitude and longitude of arbitrary points of the earth's surface, the position of the direction of the vertical of these points with respect to the equator and with respect to a starting meridian, e.g. the meridian of Greenwich, is established. On the other hand, by means of the geographic latitudes and longitudes computed from geodetic measurements the directions of the normals to the ellipsoid of these points can be determined if we start from an astronomically determined central point. The differences of the astronomically and geodetically determined values yield the deflections of the vertical of the points in latitude and longitude.

We also see from this that the deflections of the vertical are only relative values, since in the computation of geodetic latitudes and longitudes a definite ellipsoid and a definite central point must be taken as a basis.

In Chapter VII of the first half-volume we have treated in detail the astronomic measurements; we now pass over to set forth the determination of the deflections of the vertical.

Section 89. Basic Formulae for the Computation of the Deflections of the Vertical from Astronomic and Geodetic Measurements

Assuming, in the following, a reference ellipsoid which has a definite position with respect to the geoid, according to the ideas put forward in the preceding section the deflection of the vertical at a point is the angle which the physical plumb line of this point makes with the corresponding normal to the ellipsoid.

We will first treat the simple case of the deflection of the vertical in the meridian, i.e. we will assume that the plumb line deviates from the normal to the ellipsoid but is on the plane of the meridian of the ellipsoid.

This case is represented in Fig. 1. At a point J of the ellipsoid we have the normal to the ellipsoid JZ with the ellipsoidal or geodetic latitude φ , and the plumb line JZ' with the astronomical latitude φ' . The plumb line JZ' is perpendicular to the surface of the geoid, which is indicated by the dotted line GJG' . The angle ξ between JZ and JZ' is the deflection of the vertical in the meridian, and we will count ξ in the positive sense, if the plumb line JZ' deviates from JZ toward the North Pole, as is assumed in Fig. 1. We also say in this case that the deviation of the zenith ξ is north or the deflection of the vertical ξ is south, and we have here according to Fig. 1:

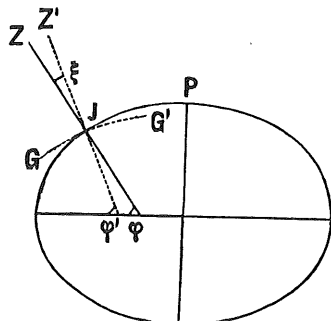


Fig. 1.

$$\xi = \varphi' - \varphi. \quad (1)$$

The deflection of the vertical in general, i.e. not only in the meridian, can be determined in two different ways: First, we indicate the absolute deflection of the vertical Θ and its azimuth ε , or second, we determine the two components ξ and η of the deflection of the vertical to the north and east, namely:

$$\xi = \Theta \cos \varepsilon \quad \eta = \Theta \sin \varepsilon. \quad (2)$$

We will find these two equations, which probably are understood immediately from the foregoing, confirmed again by the following Fig. 2, to which we now pass over.

In Fig. 2 let Z be the geodetic zenith and Z' the astronomic zenith. P is the pole which is related to both zeniths. J is a point of the earth's surface on which geodetic and astronomic observations are made. By astronomic observations there is determined the latitude φ' , the geographic longitude L' and an azimuth α' , and the point in question is to find relations between these quantities φ' , L' , α' and the corresponding geodetic values φ , L , α , which we would have obtained if the zenith were not at Z' , but at Z .

1. *Deflection of the vertical in latitude, ξ*

The meridional component ξ of the deflection of the vertical is easily determined.

The complement of the latitude is always equal to the arc between the pole and the zenith, and hence $ZP = 90^\circ - \varphi$, $Z'P = 90^\circ - \varphi'$, as is already entered in Fig. 2.

Now since the triangle PZZ' has at Z' only the small ordinate η , we can assume, with sufficient accuracy, the projection ξ of the side $ZZ' = \Theta$ as the difference of the two sides PZ and PZ' , and hence:

$$\xi = (90^\circ - \varphi) - (90^\circ - \varphi')$$

or
$$\xi = \varphi' - \varphi.$$

This is again the same equation which we have already found directly in (1).

II. Deflection of the vertical in longitude, $\eta \sec \varphi$

In the comparison of geographic longitudes we have to bear in mind that all astronomic longitude determination rests on local time determination. If L' is the astronomically determined geographic longitude of point J , referred to a starting point J_0 lying west of J (e.g. Greenwich), then this means the same as: A star T , which culminates at J_0 at the time t_0 , culminates at J at the time:

$$t' = t_0 - L'. \quad (4)$$

This culmination takes place upon the transit of the star through the circle of declination PZ' ; on the other hand, the transit through the circle of declination PZ , which belongs to the geodetic zenith, occurs later, and, in fact, by the angular amount ZZ' ; or the geodetic culmination takes place at the time:

$$t = t_0 - L' + ZPZ'. \quad (4a)$$

Now if L is the geographic longitude of the point of observation J , which corresponds to the geodetic zenith Z , then we have according to (4):

$$t = t_0 - L. \quad (5)$$

From (4a) and (5) there follows:

$$ZPZ' = L' - L, \quad (6)$$

as is already entered in Fig. 2.

In order to express $L' - L$ by η , we only need to consider again the narrow elongated triangle $PZ'Z$, or the narrow right triangle which originates by the projection of Z' on the side PZ . We obtain thereby:

$$\sin(L' - L) = \frac{\sin \eta}{\sin(90^\circ - \varphi')} = \frac{\sin \eta}{\cos \varphi'}.$$

Since $L' - L$ and η are both small here, and also φ' varies from φ only by a little, we can form from the foregoing equation:

$$L' - L = \lambda = \eta \sec \varphi. \quad (7)$$

III. Deflection of the vertical in azimuth, $\eta \tan \varphi$

In the case of astronomic azimuth measurement we deal with the horizontal angle between the direction

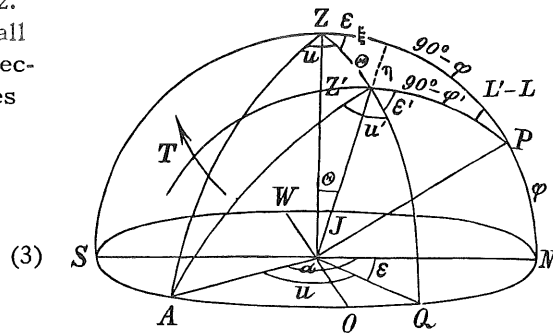


Fig. 2.

- Z = geodetic zenith, corresponding to the ellipsoid;
- φ = geodetic (ellipsoidal) latitude;
- Z' = astronomic zenith, corresponding to the geoid;
- φ' = astronomic latitude (latitude).

to the pole P and the direction to a geodetic target point, which we assume in Fig. 2 as point A , lying at the horizon. The astronomic azimuth measurement will therefore give the angle at the astronomic zenith Z' , which is designated in Fig. 2 as a sum $\varepsilon' + u'$. The vertical theodolite axis is thereby directed to the astronomic zenith Z' or to the physical plumb line JZ' , and the result of the measurement $\varepsilon' + u'$ is influenced by the deflection of the vertical.

On the other hand, if we wish to learn that azimuth which we would have obtained without the deflection of the vertical, i.e. the geodetic azimuth, then we must imagine the vertical theodolite axis directed to the geodetic zenith JZ , and with this, we obtain the azimuth which is represented at Z as a sum $\varepsilon + u$, and on the horizontal plane at J as an angle $\alpha = \varepsilon + u$. For the comparison, we therefore have now:

$$\text{Geodetic azimuth } \alpha = \varepsilon + u \quad (8)$$

$$\text{Astronomic azimuth } \alpha' = \varepsilon' + u' \quad (9)$$

$$\text{Difference } \alpha - \alpha' = (\varepsilon - \varepsilon') + (u - u'). \quad (10)$$

Of these two parts $\varepsilon - \varepsilon'$ and $u - u'$ the second part $u - u'$ is always very small and mostly to be neglected, if the geodetic object point A lies at the horizon itself, or, at least, has only a small angle of elevation or depression.

The difference $u - u'$ is to be compared with the error of a horizontal angle measurement, which occurs by not setting the axis of the theodolite exactly vertical but somewhat oblique.

We have already developed previously (Volume II, 1st half-volume, 1931, p. 356)* the error formula valid for this, in substance as we now make the development in connection with Fig. 3, which differs from Fig. 2, p. 435, only by the fact that the geodetic object point A is no longer assumed at the horizon, but with an angle of elevation h .

By using now an equation of cotangents of the first half-volume, section 33, p. 16, on the spherical triangle $ZZ'A$, Fig. 3, we obtain:

$$\cot(90^\circ - h) \sin \Theta = \cos \Theta \cos u + \sin u \cot(180^\circ - u'). \quad (11)$$

By treating Θ as small, we obtain:

$$\begin{aligned} \Theta \tan h &= \cos u - \sin u \cot u' \\ \Theta \tan h &= \frac{\cos u \sin u' - \sin u \cos u'}{\sin u'} = \frac{\sin(u' - u)}{\sin u'}. \end{aligned}$$

and hence:

$$u' - u = \Theta \sin u \tan h. \quad (12)$$

If the angle of elevation h is small, as is usually the case with geodetic object points, then $\Theta \tan h$ is a small quantity of second order, which we neglect, or reserve for special consideration.

Therefore, there still remains the first part of formula (10), i.e. $\varepsilon - \varepsilon'$ to be examined, and for this, we make a quite similar development as just now (11) to (12), again with regard to the spherical triangle $ZZ'P$.

We therefore take again an equation of cotangents from the first half-volume, section 33, p. 16, and find by its application to the triangle $ZZ'P$:

$$\begin{aligned} \cot(90^\circ - \varphi) \sin \Theta &= \cos \Theta \cos \varepsilon + \sin \varepsilon \cot(180^\circ - \varepsilon') \\ \Theta \tan \varphi &= \cos \varepsilon - \sin \varepsilon \cot \varepsilon' \\ \Theta \tan \varphi &= \frac{\cos \varepsilon \sin \varepsilon' - \sin \varepsilon \cos \varepsilon'}{\sin \varepsilon'} = \frac{\sin(\varepsilon' - \varepsilon)}{\sin \varepsilon}, \end{aligned}$$

and hence:

$$\varepsilon' - \varepsilon = \Theta \sin \varepsilon \tan \varphi. \quad (13)$$

* Not translated.

Instead of the absolute deflection of the vertical Θ we can also introduce here the transverse component $\eta = \Theta \sin \varepsilon$, and by considering again, with the omission discussed in the case of (12), the difference of azimuth $\alpha - \alpha'$ itself, we have:

$$\alpha' - \alpha = \eta \tan \varphi. \quad (14)$$

Summary of basic formulae for the deflection of the vertical

S y m b o l s		Geodet.	Astron.	
Geographic latitude or polar altitude		φ	φ'	
Geographic longitude counted in positive sense from west to east		L	L'	}
Azimuth counted in positive sense from north to east		α	α'	
Absolute deflection of the vertical or deviation of the zenith	$= \Theta$			}
Southward deflection of the vertical or northward deviation of the zenith	$= \xi$			
Westward deflection of the vertical or eastward deviation of the zenith	$= \eta$			

(15)

F o r m u l a e

$$\xi = \varphi' - \varphi \quad (16)$$

$$\eta = (L' - L) \cos \varphi \quad (17)$$

$$\eta = (\alpha' - \alpha) \cot \varphi. \quad (18)$$

The two equations (17) and (18) yield the control equation:

$$\alpha' - \alpha = (L' - L) \sin \varphi. \quad (19)$$

For the determination of the deflection of the vertical ξ in the meridian there is only *one* means, that is, according to (16) the comparison of astronomic and geodetic latitudes. For the transverse deviation η , however, we can use, according to (17) and (18), the comparison of longitudes $L' - L$ or the comparison of azimuths $\alpha' - \alpha$; or if we have both, then a very desirable check results, according to equation (19).

This equation (19) $\alpha' - \alpha = (L' - L) \sin \varphi$ is called the Laplace equation; it is very important, because it gives a relation between the two differences $\alpha' - \alpha$ and $L' - L$, resulting from the deflection of the vertical, without which the amounts of the deflection of the vertical ξ and η themselves need to be known.

The utilization of the Laplace equation for astronomic-geodetic net adjustment will follow in section 92.

Influence of the deflection of the vertical on angle measurements

In connection with Fig. 3, p. 436, which has furnished us the basic formulae for the deflection of the vertical in latitude, longitude and azimuth, we can at once also make a study of the changes which the angle measurement with the theodolite undergoes through the deflection of the vertical.

Horizontal angles. Since the vertical axis of the theodolite is adjusted by the level according to the physical plumb line, it stands oblique to the normal to the ellipsoid in the case of deflection of the vertical. The deviations of the ellipsoidal directions therefore occur to the same extent as if the vertical axis stood oblique because of faulty setting (cf. Vol. II, 1st half-volume, 1931, p. 356).*

We have already determined above in the case of (12), p. 436, the influence of such an oblique setting on the measurement of horizontal angles, namely:

$$u' - u = \Theta \sin u \tan h. \quad (20)$$

If we introduce, instead of Θ , the two components ξ and η , then we have to set:

$$u = \alpha - \varepsilon$$

* Not translated.

and obtain

$$u' - u = \Theta \tan h (\sin \alpha \cos \varepsilon - \cos \alpha \sin \varepsilon)$$

or according to (2), p. 434,

$$u' - u = (\xi \sin \alpha - \eta \cos \alpha) \tan h. \quad (21)$$

We also have recognized already that this reduction will mostly amount to little on the ellipsoid, because in the case of extended triangulations the elevation or depth angles h are also usually small. But in high mountains, where strong inclinations occur and also rather large deflections of the vertical are to be assumed, a considerable amount can result from equation (21).

Since the computation of a land triangulation is always based on a reference ellipsoid, then, strictly speaking, all measured horizontal angles would have to be reduced to this ellipsoid by means of equation (21) before starting the computation of the triangles. However, the deflections of the vertical of the triangle points with respect to the reference ellipsoid concerned must thereby be known, which is in general not the case. For this reason, we have hitherto disregarded this reduction, which in general will remain beyond measuring accuracy.

In *Spezialberichte über den Bau des Simplontunnels*, Erster Teil, "Die Bestimmung der Richtung, der Länge und der Höhenverhältnisse" by M. Rosenmund, Bern, 1901, the deflections of the vertical of the triangulation points were computed from the attraction of the mountain masses; the experiment was thus made to reduce the measured angles to the spheroid. The result was that the average amount of the closure error of triangles decreased from 3.1" to 1.7".

Elevation angles. For the reduction of a measured elevation angle h' to the ellipsoid we start from the triangle AZZ' in Fig. 3, p. 436. This triangle yields:

$$\sin h' = \sin h \cos \Theta + \cos h \sin \Theta \cos u,$$

and since Θ is small:

$$\sin h' = \sin h + \Theta \cos h \cos u.$$

But we have:

$$\sin h' = \sin (h + (h' - h)) = \sin h + (h' - h) \cos h;$$

therefore

$$h' - h = \Theta \cos u = \Theta \cos (\alpha - \varepsilon).$$

There follows hence again with the help of equation (2):

$$h' - h = \xi \cos \alpha + \eta \sin \alpha, \quad (22)$$

or, if the zenith distance z' is measured:

$$z' - z = -\xi \cos \alpha - \eta \sin \alpha. \quad (23)$$

This reduction, for instance, becomes necessary if the measured elevation angles are to be used for the computation of a trigonometric elevation measurement, since the computation can likewise be carried out only by basing it on a reference ellipsoid (cf. Volume II, 2nd half-volume, 1933, p. 140).^{*} But we must not disregard the fact that the computed differences of elevation refer likewise to the reference ellipsoid and obtain other values for any other reference ellipsoid. What relation these ellipsoidal differences of elevation bear to the differences of elevation above sea level shall be discussed more closely in section 100.

The reviser has given an example of such a computation in *Zeitschrift für Vermessungskunde*, 1900, pp. 113 and following.

^{*} Not translated.

Let the two end points P_1 and P_2 of a geodetic line on the ellipsoid have the geographic latitudes φ_1 and φ_2 as well as the longitudes L_1 and L_2 and the difference of longitude $l = L_2 - L_1$; further let α_1 and α_2 be the two azimuths of the geodetic line, and its length be equal to s .

We move the starting point of this geodetic line in the direction of the meridian by $d\varphi_1$, we increase further the azimuth α_1 by $d\alpha_1$, and finally, we extend the line by ds . Instead of φ_1 , α_1 and s we have then $\varphi_1 + d\varphi_1$, $\alpha_1 + d\alpha_1$ and $s + ds$.

By so doing, a displacement of the end point P_2 and a change of the azimuth α_2 will also take place, and instead of φ_2 , l and α_2 we shall now obtain the new values $\varphi_2 + d\varphi_2$, $l + dl$ and $\alpha_2 + d\alpha_2$.

All this is represented in Fig. 1.

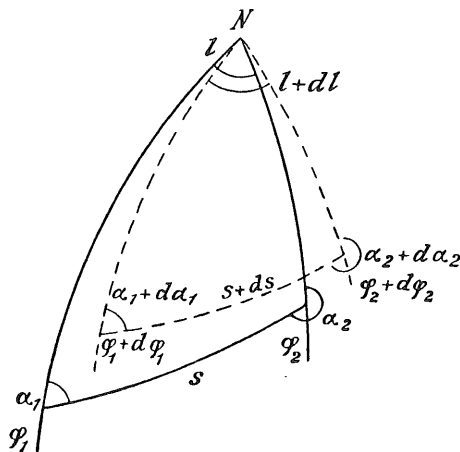


Fig. 1.

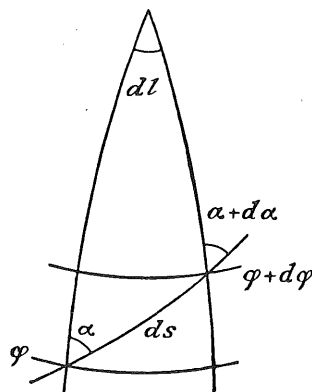


Fig. 2.

Now we are to find the relations between the differentials $d\varphi_1$, $d\alpha_1$ and ds on the one hand, as well as $d\varphi_2$, dl and $d\alpha_2$ on the other.

For this, we start from the differential equations of the geodetic line, with which we have become acquainted in section 7, p. 29, and which we put down together once more along with Fig. 2. With the symbols of Fig. 2 we have:

$$d\varphi = \frac{1}{M} \cos \alpha \, ds \quad (1)$$

$$dl = \frac{1}{N} \frac{\sin \alpha}{\cos \varphi} \, ds \quad (2)$$

$$d\alpha = \frac{1}{N} \sin \alpha \tan \varphi \, ds. \quad (3)$$

These three equations give us the change of latitude, longitude and azimuth, if an arbitrary point of the geodetic line is displaced by the differential length ds in the azimuth α .

In the solution of the above-mentioned problem we will repeatedly make use of equations (1) to (3).

1. Extension of s by ds

We begin with the simplest part of the problem, that is with the determination of the influence of a change of length ds of the geodetic line, where φ_1 and α_1 remain unchanged. The three equations (1) to (3) are then to be used directly; we must only bear in mind that instead of α we have to introduce the azimuth α_2 whereby $\alpha = \alpha_2 - 180^\circ$. However, $d\alpha = d\alpha_2$. We have then:

$$\left. \begin{aligned} d\varphi_2 &= -\frac{1}{M_2} \cos \alpha_2 ds \\ dl &= -\frac{1}{N_2} \frac{\sin \alpha_2}{\cos \varphi_2} ds \\ d\alpha_2 &= -\frac{1}{N_2} \sin \alpha_2 \tan \varphi_2 ds \end{aligned} \right\} \quad (4)$$

II. Rotation of the geodetic line at P_1 by $d\alpha_1$

If at point P_1 the azimuth α_1 is increased by $d\alpha_1$, the length s of the geodetic line being unchanged, then the end point P_2 describes an arc element $m d\alpha_1$, according to section 10, p. 41, where m is the reduced length of the geodetic line. The azimuth of this arc element is equal to $\alpha_2 - 90^\circ$, since it lies perpendicularly to the geodetic line at P_2 .

According to this, we have to replace ds by $m d\alpha_1$ and α by $\alpha_2 - 90^\circ$ in the differential formulae (1) to (3) and obtain then:

$$\left. \begin{aligned} d\varphi_2 &= +\frac{m}{M_2} \sin \alpha_2 d\alpha_1 \\ dl &= -\frac{m \cos \alpha_2}{N_2 \cos \varphi_2} d\alpha_1 \end{aligned} \right\} \quad (5)$$

However, equation (3) is not sufficient in the present case in order to determine the change of azimuth $d\alpha_2$. We have already investigated this more closely in section 11, p. 47, and can take directly from equation (9), section 11, p. 48:

$$d\alpha_2 = \left(\left(\frac{dm}{ds} \right)_1 - \frac{m}{N_2} \cos \alpha_2 \tan \varphi_2 \right) d\alpha_1. \quad (5a)$$

The negative sign had to be introduced here for the second part, because the azimuth α_2 is different by 180° with respect to section 11. The index in the case of the differential quotient $\frac{dm}{ds}$ indicates that the latter is to be computed for point P_1 .

III. Displacement of P_1 by $d\varphi_1$

The treatment of the change of φ_1 in the case of unchanged length s and unchanged azimuth α_1 turns out somewhat more complicated. Therefore, we deal here with a displacement of the geodetic line in the case of constant azimuth α_1 , whereby P_1 moves along its meridian. In Fig. 3 we imagine this displacement carried out in the following manner:

The geodetic line is first rotated around P_2 so that P_1 moves to P'_1 , whereby the line $P_2P'_1$ intersects the meridian of P_1 at Q_1 . Then we extend the line Q_1P_2 beyond P_2 by the length P'_1Q_1 as far as Q_2 . Finally, we rotate the line Q_1Q_2 around Q_1 , until it reaches here the azimuth α_1 again, whereby Q_2 obtains then the final position P'_2 .

Now we have to express mathematically these different motions.

Upon the first rotation around P_2 the azimuth of Q_1P_2 becomes equal to $\alpha_1 + d\alpha_1$, while Q_1 receives the geographic latitude $\varphi_1 + d\varphi_1$. According to the fundamental theorem found in section 11, p. 49, we can indicate the change of azimuth at once. For we have according to (14), section 11, p. 49:

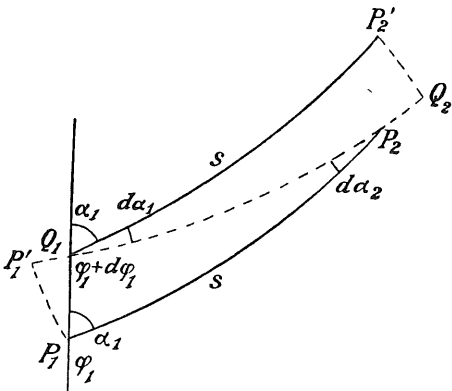


Fig. 3.

$$d\alpha_1 = \frac{1}{m} \sin \alpha_1 \left(\frac{dm}{ds} \right)_2 P_1 Q_1 = \frac{M_1}{m} \sin \alpha_1 \left(\frac{dm}{ds} \right)_2 d\varphi_1. \quad (6)$$

In the case of this rotation, the azimuth α_2 changes at the same time by the angle:

$$d\alpha_2 = \frac{P_1 P_1'}{m} = \frac{M_1}{m} \sin \alpha_1 d\varphi_1. \quad (7)$$

In the case of the displacement from P_2 to Q_2 we will have:

$$P_2 Q_2 = P_1' Q_1 = M_1 \cos \alpha_1 d\varphi_1, \quad (8)$$

and the azimuth of this linear element is equal to $\alpha_2 - 180^\circ$. In the equations (1) to (3) we therefore can replace ds by $M_1 \cos \alpha_1 d\varphi_1$ and α by $\alpha_2 - 180^\circ$.

The last rotation of the geodetic line around the point Q_1 brings about a displacement of point Q_2 by the length:

$$Q_2 P_2' = m d\alpha_1,$$

or according to (6):

$$Q_2 P_2' = M_1 \sin \alpha_1 \left(\frac{dm}{ds} \right)_2 d\varphi_1 \quad (9)$$

with the azimuth $\alpha_2 + 90^\circ$. This can likewise be introduced in equations (1) to (3).

After these general preliminary remarks we can turn to our actual problem.

For the computation of $d\varphi_2$ we have to apply equation (1) consecutively to (8) and (9). The displacement (8) yields for φ_2 the change:

$$-\frac{M_1}{M_2} \cos \alpha_1 \cos \alpha_2 d\varphi_1$$

and the rotation (9):

$$-\frac{M_1}{M_2} \sin \alpha_1 \sin \alpha_2 \left(\frac{dm}{ds} \right)_2 d\varphi_1.$$

The total change of φ_1 is thus:

$$d\varphi_2 = -\frac{M_1}{M_2} \left(\sin \alpha_1 \sin \alpha_2 \left(\frac{dm}{ds} \right)_2 + \cos \alpha_1 \cos \alpha_2 \right) d\varphi_1. \quad (10)$$

For the determination of dl , accordingly, we have to apply equation (2) to (8) and (9). We then obtain at once from the displacement (8) the change of longitude:

$$-\frac{M_1 \cos \alpha_1}{N_2 \cos \varphi_2} \sin \alpha_2 d\varphi_1,$$

and from the rotation (9):

$$+\frac{M_1 \sin \alpha_1}{N_2 \cos \varphi_2} \cos \alpha_2 \left(\frac{dm}{ds} \right)_2 d\varphi_1.$$

The total change of l is thus:

$$d l = \frac{M_1}{N_2 \cos \varphi_2} \left(\sin \alpha_1 \cos \alpha_2 \left(\frac{d m}{d s} \right)_2 - \cos \alpha_1 \sin \alpha_2 \right) d \varphi_1 . \quad (11)$$

The azimuth α_2 is influenced not only by the displacement (8) and the rotation (9) but also by the first rotation around P_2 (Fig. 3, p. 440), whereby, according to (7), the azimuth α_2 is increased by the angle:

$$\frac{M_1}{m} \sin \alpha_1 d \varphi_1$$

The displacement (8) yields further, according to (3), for α_2 the change:

$$- \frac{M_1}{N_2} \sin \alpha_2 \cos \alpha_1 \tan \varphi_2 d \varphi_1 .$$

Finally, if we carry out the counterclockwise rotation around Q_1 by the angle $d \alpha_1$ according to (6), then we obtain the change of α_2 belonging to this from (5a), namely:

$$- \left(\frac{M_1}{m} \left(\frac{d m}{d s} \right)_1 \left(\frac{d m}{d s} \right)_2 \sin \alpha_1 - \frac{M_1}{N_2} \left(\frac{d m}{d s} \right)_2 \sin \alpha_1 \cos \alpha_2 \tan \varphi_2 \right) d \varphi_1 .$$

These three angular amounts are to be added and yield as the final total change of α_2 :

$$d \alpha_2 = \left(\frac{M_1}{m} \sin \alpha_1 - \frac{M_1}{N_2} \sin \alpha_2 \cos \alpha_1 \tan \varphi_2 - \frac{M_1}{m} \left(\frac{d m}{d s} \right)_1 \left(\frac{d m}{d s} \right)_2 \sin \alpha_1 \right. \\ \left. + \frac{M_1}{N_2} \left(\frac{d m}{d s} \right)_2 \sin \alpha_1 \cos \alpha_2 \tan \varphi_2 \right) d \varphi_1 . \quad (12)$$

IV. Displacement of both end points P_1 and P_2

Let us in addition consider the case in which the end points P_1 and P_2 of the geodetic line are displaced in longitude and latitude by the quantities $d L_1$, $d \varphi_1$ and $d L_2$, $d \varphi_2$. The change of the two azimuths α_1 and α_2 caused hereby shall be determined.

We begin with a displacement of point P_2 by $d L_2$ and $d \varphi_2$ and determine the change $d \alpha_1$ of the azimuth α_1 . From equations (4) and (5) we obtain at first:

$$d \varphi_2 = - \frac{1}{M_2} \cos \alpha_2 d s + \frac{m}{M_2} \sin \alpha_2 d \alpha_1 \\ d L_2 = - \frac{1}{N_2 \cos \varphi_2} \sin \alpha_2 d s - \frac{m \cos \alpha_2}{N_2 \cos \varphi_2} d \alpha_1$$

There follows hence, if we eliminate $d s$,

$$d \alpha_1 = \frac{M_2}{m} \sin \alpha_2 d \varphi_2 - \frac{N_2}{m} \cos \varphi_2 \cos \alpha_2 d L_2 . \quad (13)$$

By interchanging the indices we also obtain hence immediately:

$$d\alpha_2 = \frac{M_1}{m} \sin \alpha_1 d\varphi_1 - \frac{N_1}{m} \cos \varphi_1 \cos \alpha_1 dL_1. \quad (14)$$

In order to be able to determine the influence of a displacement of point P_1 on the azimuths, we have from (13), if we set $-dL_1$ instead of $+dL_2$,

$$\frac{\partial \alpha_1}{\partial L_1} = + \frac{N_2}{m} \cos \varphi_2 \cos \alpha_2.$$

On the other hand, we have according to (6):

$$\frac{\partial \alpha_1}{\partial \varphi_1} = + \frac{M_1}{m} \sin \alpha_1 \left(\frac{dm}{ds} \right)_2.$$

With this, we obtain:

$$d\alpha_1 = \frac{M_1}{m} \sin \alpha_1 \left(\frac{dm}{ds} \right)_2 d\varphi_1 + \frac{N_2}{m} \cos \varphi_2 \cos \alpha_2 dL_1 \quad (15)$$

and, accordingly,

$$d\alpha_2 = \frac{M_2}{m} \sin \alpha_2 \left(\frac{dm}{ds} \right)_1 d\varphi_2 + \frac{N_1}{m} \cos \varphi_1 \cos \alpha_1 dL_2. \quad (16)$$

The above equations (13) to (16) play a role if, in the case of intercalation of trigonometric points, the adjustment on the ellipsoid is based directly on geographic coordinates.

Formulae for shorter distances

The above formulae (10), (11) and (12) are not very convenient for numerical application. But since we shall only rarely be in a position where we have to make use of these rigorous formulae, we shall in addition carry out a few reductions which are admissible for shorter distances.

For the reduced length of the geodetic line we have found in equation (14), section 10, p. 44, the expression correct to the fifth order:

$$m = r \sin \frac{s}{r}, \quad (17)$$

in which the mean radius of curvature r holds for the arithmetic mean of the latitudes φ_1 and φ_2 .

According to (17), section 10, p. 45, the differential quotient $\frac{dm}{ds}$ is expressed by the following, if we neglect terms of the fifth order:

$$\frac{dm}{ds} = 1 - \frac{s^2}{2N_1^2} (1 + \eta_1^2) + \frac{s^4}{24N_1^4} + \dots$$

If we change here, as in section 10, p. 43, from φ_1 to the mean latitude φ , then we can take from there, likewise with the omission of the terms of the fifth order:

$$-\frac{s^2}{2N_1^2} (1 + \eta_1^2) = -\frac{s^2}{2N^2} (1 + \eta^2) + \dots = -\frac{s^2}{2r^2} + \dots$$

Therefore, we have with the same accuracy

$$\frac{d m}{d s} = 1 - \frac{s^3}{2 r^2} + \frac{s^5}{24 r^4} + \dots$$

or

$$\frac{d m}{d s} = \cos \frac{s}{r} + \dots \quad (18)$$

We see that in the case of the above omission of the terms of the fifth order the difference between the values of $\frac{d m}{d s}$ vanishes at the two end points of the geodetic line.

With the reductions (17) and (18) which correspond to purely spherical computation with the spherical radius r , as already noted in section 10, p. 44, also the formulae (10), (11) and (12), pp. 441 and 442, can be simplified with a small loss of accuracy, by making use here of the relations of section 23.

If we introduce (18) in (10), then the expression within parentheses changes to:

$$\sin \alpha_1 \sin \alpha_2 \cos \frac{s}{r} + \cos \alpha_1 \cos \alpha_2.$$

If we imagine this referred to the spherical auxiliary triangle of section 23, p. 110, then we obtain according to the spherical-trigonometric formulae of the first half-volume, section 33, p. 16, bearing in mind that we have to set now $\alpha' = \alpha_2 - 180^\circ$:

$$\sin \alpha_1 \sin \alpha_2 \cos \sigma + \cos \alpha_1 \cos \alpha_2 = -\cos \lambda.$$

Now we have according to (20), section 26, p. 121, neglecting the terms of fourth order:

$$\lambda = l \left(1 + \frac{1}{2} e^2 \cos^2 \varphi \right) = l \left(1 + \frac{1}{2} e^2 \cos^2 \psi \right). \quad (19)$$

Likewise, we have according to (19), section 26, p. 121, and (28), first half-volume, section 38, p. 51, with the same omission:

$$\sigma = \frac{s}{r} \left(1 + \frac{1}{2} e^2 \sin^2 \varphi \right) = \frac{s}{r} \left(1 + \frac{1}{2} e^2 \sin^2 \psi \right),$$

where it is indifferent if φ and ψ are replaced by φ_1 or φ_2 or, as the case may be, ψ_1 or ψ_2 . If we neglect the terms of fourth order also further, then we find easily that we can set:

$$\cos \lambda = \cos l \quad \text{and} \quad \cos \sigma = \cos \frac{s}{r},$$

and then we obtain:

$$d \varphi_2 = + \frac{M_1}{M_2} \cos l d \varphi_1. \quad (20)$$

We can refer, accordingly, the expression within parentheses of equation (11) to the spherical auxiliary triangle of section 23, p. 110, and find according to the spherical-trigonometric formulae of the first half-volume, section 33, p. 16:

$$\sin \alpha_1 \cos \alpha_2 \cos \sigma - \cos \alpha_1 \sin \alpha_2 = \sin \lambda \sin \psi_2.$$

According to equation (19) we have:

$$\sin \lambda = \sin l \left(1 + \frac{1}{2} e^2 \cos^2 \varphi \right). \quad (21)$$

If we develop further in equation (5), section 23, p. 110, $\sin \psi = \sqrt{1 - \cos^2 \psi}$, where we neglect again terms of the fourth order, then we have:

$$\sin \psi = \sin \varphi \left(1 - \frac{1}{2} e^2 \cos^2 \varphi \right), \quad (22)$$

whence there follows:

$$\sin \lambda \sin \psi_2 = \sin l \sin \varphi_2,$$

and with this, we will have:

$$d l = \frac{M_1}{N_2} \sin l \tan \varphi_2 d \varphi_1. \quad (23)$$

Finally, if we introduce expression (18) in equation (12), then we obtain:

$$d \alpha_2 = \left(\frac{M_1}{m} \sin \alpha_1 \sin^2 \frac{s}{r} + \frac{M_1}{N_2} \tan \varphi_2 \left(\sin \alpha_1 \cos \alpha_2 \cos \frac{s}{r} - \cos \alpha_1 \sin \alpha_2 \right) \right) d \varphi_1.$$

In this, the second expression within parentheses is to be replaced by $\sin l \sin \varphi_2$ according to the above; if we introduce at the same time $m = r \sin \frac{s}{r}$, then we have:

$$d \alpha_2 = \left(\frac{M_1}{r} \sin \alpha_1 \sin \frac{s}{r} + \frac{M_1}{N_2} \tan \varphi_2 \sin \varphi_2 \sin l \right) d \varphi_1. \quad (24)$$

On the other hand, there follows from the spherical auxiliary triangle:

$$\sin \alpha_1 \sin \sigma = \sin \lambda \cos \psi_2,$$

and since according to the above:

$$\begin{aligned} \sin \sigma &= \sin \frac{s}{r} \left(1 + \frac{1}{2} e^2 \sin^2 \varphi_2 \right) \\ \sin \lambda &= \sin l \left(1 + \frac{1}{2} e^2 \cos^2 \varphi_2 \right) \end{aligned}$$

and according to (14) and (10), section 22:

$$\begin{aligned} \cos \psi_2 &= \cos \varphi_2 \left(1 + \frac{1}{2} e^2 \right) \left(1 - \frac{1}{2} e^2 \cos^2 \varphi_2 \right) \\ \cos \psi_2 &= \cos \varphi_2 \left(1 + \frac{1}{2} e^2 \sin^2 \varphi_2 \right), \end{aligned}$$

then we will have:

$$\sin \alpha_1 \sin \frac{s}{r} = \sin l \cos \varphi_2 \left(1 + \frac{1}{2} e^2 \cos^2 \varphi_2 \right). \quad (25)$$

At the same time, we can also replace the two quotients $\frac{M_1}{r}$ and $\frac{M_1}{N_2}$ by development in series. If we neglect here again the terms of fourth order, then the difference of the latitudes φ_1 and φ_2 is not considered, and the formulae (24) and (25), first half-volume, section 38, p. 50, yields:

$$\frac{M_1}{r} = \frac{1}{V} = 1 - \frac{1}{2} e^2 \cos^2 \varphi_2 + \dots \quad \frac{M_1}{N_2} = \frac{1}{V^2} = 1 - e^2 \cos^2 \varphi_2 + \dots \quad (26)$$

If we now introduce (25) and (26) in (24), then we obtain:

$$\begin{aligned} d\alpha_2 &= (\sin l \cos \varphi_2 + \tan \varphi_2 \sin \varphi_2 \sin l (1 - e^2 \cos^2 \varphi_2)) d\varphi_1 \\ d\alpha_2 &= \frac{\sin l}{\cos \varphi_2} (1 - e^2 \sin^2 \varphi_2 \cos^2 \varphi_2) d\varphi_1, \end{aligned} \quad (27)$$

where only terms of fourth and higher order are neglected.

Summary of formulae

$$\left. \begin{aligned} d\varphi_2 &= \frac{\partial \varphi_2}{\partial \varphi_1} d\varphi_1 + \frac{\partial \varphi_2}{\partial s} ds + \frac{\partial \varphi_2}{\partial \alpha_1} d\alpha_1 \\ dl &= \frac{\partial l}{\partial \varphi_1} d\varphi_1 + \frac{\partial l}{\partial s} ds + \frac{\partial l}{\partial \alpha_1} d\alpha_1 \\ d\alpha_2 &= \frac{\partial \alpha_2}{\partial \varphi_1} d\varphi_1 + \frac{\partial \alpha_2}{\partial s} ds + \frac{\partial \alpha_2}{\partial \alpha_1} d\alpha_1 \end{aligned} \right\} \quad (28)$$

$$\left. \begin{aligned} \frac{\partial \varphi_2}{\partial \varphi_1} &= \frac{M_1}{M_2} \cos l & \frac{\partial \varphi_2}{\partial s} &= -\frac{\rho}{M_2} \cos \alpha_2 & \frac{\partial \varphi_2}{\partial \alpha_1} &= \frac{m}{M_2} \sin \alpha_2 \\ \frac{\partial l}{\partial \varphi_1} &= \frac{M_1}{N_2} \frac{\sin l}{\cot \varphi_2} & \frac{\partial l}{\partial s} &= -\frac{\rho}{N_2} \frac{\sin \alpha_2}{\cos \varphi_2} & \frac{\partial l}{\partial \alpha_1} &= -\frac{m}{N_2} \frac{\cos \alpha_2}{\cos \varphi_2} \\ \frac{\partial \alpha_2}{\partial \varphi_1} &= \frac{\sin l}{\cos \varphi_2} (1 - e^2 \sin^2 \varphi_2 \cos^2 \varphi_2) & \frac{\partial \alpha_2}{\partial s} &= -\frac{\rho}{N_2} \frac{\sin \alpha_2}{\cot \varphi_2} \\ & & \frac{\partial \alpha_2}{\partial \alpha_1} &= \frac{dm}{ds} - \frac{m}{N_2} \cos \alpha_2 \tan \varphi_2. \end{aligned} \right\} \quad (29)$$

Section 91. Influence of a Change of the Ellipsoidal Constants

For the completion of the above differential formulae we will in addition consider the case in which the ellipsoidal constants, in particular, the major semiaxis a and the flattening α , are changed likewise by small quantities da and $d\alpha$. The changes of φ_2 , l and α_2 resulting therefrom shall now be determined. We note further that here, also, α_2 is again to be computed in the sense of Fig. 1, section 90, p. 439, which we have to take into account in the use of the formulae of section 18.

$$1. \text{ Determination of } \frac{d\varphi_2}{da} \text{ and } \frac{d\varphi_2}{d\alpha}.$$

We can use equation (25) developed in section 18, p. 78, limiting ourselves to the first term, in the form

$$\frac{\varphi_1 - \varphi_2}{V_2^2} = \frac{V_2}{c} s \cos \alpha_2 + \dots$$

or

$$\varphi_2 = \varphi_1 - \frac{V_2^3}{a} \sqrt{1-e^2} s \cos \alpha_2 + \dots$$

This abbreviation is sufficient if we wish to neglect terms of the third order in the differentiation with respect to a . Then we have immediately

$$\frac{d\varphi_2}{da} = + \frac{V_2^3}{a^2} \sqrt{1-e^2} s \cos \alpha_2. \quad (1)$$

For the computation of the differential quotients $\frac{d\varphi_2}{da}$ we start best from the formulae for the mean latitude developed in section 21. According to equation (33) indicated there on p. 99 we have

$$\varphi_2 = \varphi_1 + \frac{1}{M} s \cos \alpha + \dots \quad (2)$$

where $\alpha = \frac{\alpha_1 + \alpha_2 - 180^\circ}{2}$ and M refers to the mean latitude $\frac{\varphi_1 + \varphi_2}{2}$. We cite further from the first half-volume, pp. 48-51, the basic equations

$$\frac{1}{M} = \frac{W^3}{a(1-e^2)}, \quad \frac{1}{N} = \frac{W}{a} \quad \text{and} \quad a = 1 - \sqrt{1-e^2}, \quad (3)$$

where we prefer now the function W , instead of the function V mostly used by us, because of its simpler relation to e^2 . With this, (2) changes to

$$\varphi_2 = \varphi_1 + \frac{W^3}{(1-e^2)} \frac{s}{a} \cos \alpha + \dots, \quad (4)$$

from which we find by differentiation:

$$\frac{d\varphi_2}{da} = \frac{1}{a(1-e^2)^3} s \cos \alpha \left(3 W^3 (1-e^2) \frac{dW}{de} \frac{de}{da} + 2 W^3 e \frac{de}{da} \right) - \frac{W^3}{(1-e^2)} \frac{s}{a} \sin \alpha \frac{d\alpha}{da}. \quad (5)$$

But we have

$$\frac{dW}{de} = -\frac{e}{W} \sin^2 \varphi \quad \frac{de}{da} = \frac{\sqrt{1-e^2}}{e}. \quad (5a)$$

We have further according to section 21, p. 98, equations (28) and (26):

$$\alpha = \alpha_1 + \frac{s}{2N} \sin \alpha \tan \varphi + \dots$$

or

$$\alpha = \alpha_1 + \frac{W}{2a} s \sin \alpha \tan \varphi,$$

therefore,

$$\frac{d\alpha}{da} = -\frac{\sqrt{1-e^2}}{W} \frac{s}{2a} \sin \alpha \sin^2 \varphi \tan \varphi.$$

If we introduce all this in equation (5), then we obtain:

$$\frac{d \varphi_2}{d \alpha} = -\frac{3 W}{\sqrt{1-e^2}} \frac{s}{a} \cos \alpha \sin^2 \varphi + \frac{2 W^3}{(1-e^2) \sqrt{1-e^2}} \frac{s}{a} \cos \alpha + \frac{W^2}{2 \sqrt{1-e^2}} \frac{s^2}{a^2} \sin^2 \alpha \sin^2 \varphi \tan \varphi ,$$

and if we introduce again the two radii of curvature M and N :

$$\frac{d \varphi_2}{d \alpha} = -\frac{3}{\sqrt{1-e^2}} \frac{s}{N} \cos \alpha \sin^2 \varphi + \frac{2}{\sqrt{1-e^2}} \frac{s}{M} \cos \alpha + \frac{1}{2 \sqrt{1-e^2}} \frac{s^2}{N^2} \sin^2 \alpha \sin^2 \varphi \tan \varphi . \quad (6)$$

According to the first half-volume, pp. 48-51, we have:

$$\frac{1}{N} = \frac{1}{M} \frac{(1-e^2)}{(1-e^2 \sin^2 \varphi)} = \frac{1}{M} \frac{(1-e^2)}{(1-e^2 + e^2 \cos^2 \varphi)} ,$$

$$\frac{1}{N} = \frac{1}{M} \frac{1}{1 + \frac{e^2 \cos^2 \varphi}{1-e^2}} = \frac{1}{M} \left(1 - \frac{e^2 \cos^2 \varphi}{1-e^2} \right) + \dots$$

or

It follows hence that, if the terms of third order are neglected, we can set

$$\frac{1}{N} = \frac{1}{M} + \dots$$

If we substitute this in the first term on the right-hand side of equation (6) and take into account that according to section 21, p. 98, equation (26) and equation (27)

$$l = \frac{s \sin \alpha}{N \cos \varphi} + \dots \quad \varphi_2 - \varphi_1 = \frac{s}{M} \cos \alpha + \dots \quad (6a)$$

then we obtain from (6):

$$\frac{d \varphi_2}{d \alpha} = -3 (\varphi_2 - \varphi_1) \sin^2 \varphi + 2 (\varphi_2 - \varphi_1) + \frac{1}{2} l^2 \sin^3 \varphi \cos \varphi . \quad (7)$$

The quantities $\sqrt{1-e^2}$ occurring in the denominator are likewise omitted here in (6), since they obviously affect only the terms of third order.

2. Determination of $\frac{d l}{d \alpha}$ and $\frac{d l}{d \alpha}$.

If we set up equation (26) of section 18, p. 78, for the geographic latitude φ_2 , then we obtain

$$l \cos \varphi_2 = -\frac{V_2}{a} \sqrt{1-e^2} s \sin \alpha_2 + \dots ,$$

and hence by differentiation:

$$\frac{d l}{d \alpha} = + \frac{V_2}{a^2} \sqrt{1-e^2} s \frac{\sin \alpha_2}{\cos \varphi_2} . \quad (8)$$

We find the differential quotient $\frac{dl}{da}$ from section 18, p. 78, equation (26), in which we introduce, however, at once the function W instead of the radius of curvature N . Then

$$l = W_1 \frac{s}{a} \frac{\sin \alpha_1}{\cos \varphi_1} + W_1^2 \frac{s^2}{a^2} \frac{\sin \alpha_1}{\cos \varphi_1} \cos \alpha_1 \tan \varphi_1 + \dots \quad (9)$$

We must carry here the second term in order not to neglect in the result terms of the second order. The differentiation yields

$$\frac{dl}{da} = \frac{s}{a} \frac{\sin \alpha_1}{\cos \varphi_1} \frac{dW_1}{de} \frac{de}{da} + 2 W_1 \frac{s^2}{a^2} \frac{\sin \alpha_1}{\cos \varphi_1} \cos \alpha_1 \tan \varphi_1 \frac{dW_1}{de} \frac{de}{da},$$

and with the expressions (5a):

$$\frac{dl}{da} = -\frac{s}{a W_1} \frac{\sin \alpha_1}{\cos \varphi_1} \sin^2 \varphi_1 - 2 \sqrt{1-e^2} \frac{s^2}{a^2} \frac{\sin \alpha_1}{\cos \varphi_1} \cos \alpha_1 \tan \varphi_1 \sin^2 \varphi_1.$$

In both terms, we can either omit or insert the quantity W_1 , since only the terms of third order neglected here are influenced by it. For the same reason, we can also omit the factor $\sqrt{1-e^2}$. Therefore, we will now write the equation as follows:

$$\begin{aligned} \frac{dl}{da} = & -W_1 \frac{s}{a} \frac{\sin \alpha_1}{\cos \varphi_1} \sin^2 \varphi_1 - W_1^2 \frac{s^2}{a^2} \frac{\sin \alpha_1}{\cos \varphi_1} \cos \alpha_1 \tan \varphi_1 \sin^2 \varphi_1 \\ & - W_1 \frac{s^2}{a^2} \frac{\sin \alpha_1}{\cos \varphi_1} \cos \alpha_1 \tan \varphi_1 \sin^2 \varphi_1. \end{aligned}$$

The first two terms, however, yield according to (9) $-l \sin^2 \varphi_1$; if we introduce also l in the third term, then we have

$$\begin{aligned} \frac{dl}{da} &= -l \sin^2 \varphi_1 - l \sin^2 \varphi_1 \frac{s}{a} \cos \alpha_1 \tan \varphi_1 \\ \text{or} \quad \frac{dl}{da} &= -l \sin^2 \varphi_1 \left(1 + \frac{s}{a} \cos \alpha_1 \tan \varphi_1 \right). \end{aligned} \quad (10)$$

For the simplification of the expression within parentheses we start from the identity

$$\cos \varphi_2 = \cos (\varphi_1 + (\varphi_2 - \varphi_1))$$

for which we write with the use of equation (6a):

$$\cos \varphi_2 = \cos \varphi_1 - (\varphi_2 - \varphi_1) \sin \varphi_1 = \cos \varphi_1 - \frac{s}{M} \cos \alpha_1 \sin \varphi_1. \quad (10a)$$

But since $\frac{1}{M}$ agrees with $\frac{1}{a}$ to terms of third order according to the first half-volume, section 38, p. 49, equation (17), then we also have

$$\cos \varphi_2 = \cos \varphi_1 \left(1 - \frac{s}{a} \cos \alpha_1 \tan \varphi_1 \right),$$

or with the same accuracy:

$$\frac{\cos \varphi_1}{\cos \varphi_2} = 1 + \frac{s}{a} \cos \alpha_1 \tan \varphi_1.$$

Substituted in (10) this yields:

$$\frac{d l}{d a} = -l \sin^2 \varphi_1 \frac{\cos \varphi_1}{\cos \varphi_2}. \quad (11)$$

3. Determination of $\frac{d \alpha_2}{d a}$ and $\frac{d \alpha_2}{d a}$.

We can write equation (27) of section 18, p. 78, for $\alpha_1 - \alpha_2$ thus:

$$\begin{aligned} \alpha_1 - \alpha_2 &= -180^\circ + \frac{V_2}{a} \sqrt{1 - e^2} s \sin \alpha_2 \tan \varphi_2 + \dots, \\ \alpha_2 &= \alpha_1 + 180^\circ - \frac{V_2}{a} \sqrt{1 - e^2} s \sin \alpha_2 \tan \varphi_2 + \dots, \end{aligned}$$

or

from which the differentiation yields:

$$\frac{d \alpha_2}{d a} = \frac{V_2}{a^2} \sqrt{1 - e^2} s \sin \alpha_2 \tan \varphi_2 + \dots \quad (12)$$

For the development of the differential quotient $\frac{d \alpha_2}{d a}$ we use likewise equation (27) of section 18, p. 78, where higher order terms are still to be taken into account, however. We have

$$\alpha_2 = \alpha_1 + 180^\circ + W_1 \frac{s}{a} \sin \alpha_1 \tan \varphi_1 + W_1^2 \frac{s^2}{2 a^2} \sin \alpha_1 \cos \alpha_1 (1 + 2 \tan^2 \varphi_1 + e^2 \cos^2 \varphi_1) + \dots$$

In the last term we have set without further ado, e^2 in place of e'^2 , since this change affects only the terms of fifth order. Now we have further

$$\begin{aligned} \frac{d \alpha_2}{d a} &= \frac{s}{a} \sin \alpha_1 \tan \varphi_1 \frac{d W_1}{d e} \frac{d e}{d a} + W_1 \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 (1 + 2 \tan^2 \varphi_1) \frac{d W_1}{d e} \frac{d e}{d a} \\ &\quad + W_1^2 e \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \cos^2 \varphi_1 \frac{d e}{d a}. \end{aligned}$$

We have already indicated in equation (5a), p. 447, the differential quotients $\frac{d W_1}{d e}$ and $\frac{d e}{d a}$, which values we substitute now:

$$\begin{aligned} \frac{d \alpha_2}{d a} &= -\frac{\sqrt{1 - e^2}}{W_1} \frac{s}{a} \sin \alpha_1 \tan \varphi_1 \sin^2 \varphi_1 - \sqrt{1 - e^2} \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \sin^2 \varphi_1 \\ &\quad - 2 \sqrt{1 - e^2} \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \tan^2 \varphi_1 \sin^2 \varphi_1 + W_1^2 \sqrt{1 - e^2} \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \cos^2 \varphi_1. \end{aligned}$$

If we wish to renounce again the terms of third order, then we can neglect the quantity W_1 and the factor $\sqrt{1 - e^2}$. Then we obtain:

$$\begin{aligned}\frac{d\alpha_2}{d\alpha} = & -\frac{s}{a} \sin \alpha_1 \tan \varphi_1 \sin^2 \varphi_1 - \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \sin^2 \varphi_1 \\ & - 2 \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \sin^2 \varphi_1 \tan^2 \varphi_1 + \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \cos^2 \varphi_1\end{aligned}$$

or

$$\begin{aligned}\frac{d\alpha_2}{d\alpha} = & -\sin^3 \varphi_1 \left(\frac{s}{a} \frac{\sin \alpha_1}{\cos \varphi_1} + \frac{s^2}{a^2} \frac{\sin \alpha_1 \cos \alpha_1 \tan \varphi_1}{\cos \varphi_1} \right) - \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \sin^2 \varphi_1 \tan^2 \varphi_1 \\ & - \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \sin^2 \varphi_1 + \frac{s^2}{a^2} \sin \alpha_1 \cos \alpha_1 \cos^2 \varphi_1.\end{aligned}$$

The first term on the right-hand side is equal to $-l \sin^3 \varphi_1$, accurate to terms of second order, according to section 18, p. 78, equation (26). For the other three terms we recall that according to section 18, p. 78, equations (25) and (26).

$$\varphi_2 - \varphi_1 = \frac{s}{a} \cos \alpha_1 + \dots \quad \text{and} \quad l = \frac{s}{a} \frac{\sin \alpha_1}{\cos \varphi_1} + \dots$$

If we substitute all this, then we find

$$\frac{d\alpha_2}{d\alpha} = -l \sin^3 \varphi_1 - l(\varphi_2 - \varphi_1) \sin^3 \varphi_1 \tan \varphi_1 - l(\varphi_2 - \varphi_1) \sin^2 \varphi_1 \cos \varphi_1 + l(\varphi_2 - \varphi_1) \cos^3 \varphi_1,$$

whence there follows:

$$\frac{d\alpha_2}{d\alpha} = -l \sin^3 \varphi_1 (1 + (\varphi_2 - \varphi_1) \tan \varphi_1) + l(\varphi_2 - \varphi_1) \cos^3 \varphi_1 (1 - \tan^2 \varphi_1).$$

According to equation (10a), p. 449, we have

$$\frac{\cos \varphi_2}{\cos \varphi_1} = 1 - (\varphi_2 - \varphi_1) \tan \varphi_1 \quad \text{or} \quad \frac{\cos \varphi_1}{\cos \varphi_2} = 1 + (\varphi_2 - \varphi_1) \tan \varphi_1.$$

With this, we will have

$$\frac{d\alpha_2}{d\alpha} = -l \frac{\sin^3 \varphi_1 \cos \varphi_1}{\cos \varphi_2} + l(\varphi_2 - \varphi_1) \cos^3 \varphi_1 (1 - \tan^2 \varphi_1).$$

The second term is still somewhat simplified if we add to the first term the factor $\frac{\sin \varphi_2}{\sin \varphi_1}$, and hence write:

$$\frac{d\alpha_2}{d\alpha} = -l \frac{\sin^2 \varphi_1 \sin \varphi_2 \cos \varphi_1}{\cos \varphi_2} \frac{\sin \varphi_1}{\sin \varphi_2} + l(\varphi_2 - \varphi_1) \cos^3 \varphi_1 (1 - \tan^2 \varphi_1).$$

Corresponding to equation (10a) we obtain easily:

$$\frac{\sin \varphi_1}{\sin \varphi_2} = 1 - (\varphi_2 - \varphi_1) \cot \varphi_1,$$

and hence we will have

$$\begin{aligned}\frac{d\alpha_2}{d\alpha} = & -l \frac{\sin^2 \varphi_1 \sin \varphi_2 \cos \varphi_1}{\cos \varphi_2} + l(\varphi_2 - \varphi_1) \frac{\sin \varphi_1 \sin \varphi_2 \cos^2 \varphi_1}{\cos \varphi_2} \\ & + l(\varphi_2 - \varphi_1) \cos^3 \varphi_1 - l(\varphi_2 - \varphi_1) \sin^2 \varphi_1 \cos \varphi_1.\end{aligned}$$

But since in the terms of second order the difference between φ_1 and φ_2 can be neglected, then there remains:

$$\frac{d\alpha_2}{d\alpha} = -l \frac{\sin^2 \varphi_1 \sin \varphi_2 \cos \varphi_1}{\cos \varphi_2} + l(\varphi_2 - \varphi_1) \cos^3 \varphi_1. \quad (13)$$

Let us once more summarize the result of this section. At the same time we will express, in addition, the factors $\frac{V^2}{a} \sqrt{1-e^2}$ and $\frac{V^2}{a} \sqrt{1-e^2}$ by the radii of curvature M and N with the help of the relations of the first half-volume, section 38, p. 50, and also insert the necessary ρ 's. Then we obtain:

$$d\varphi_2 = \frac{\rho}{a M_2} s \cos \alpha_2 da + \left(2(\varphi_2 - \varphi_1) - 3(\varphi_2 - \varphi_1) \sin^2 \varphi + \frac{l^2}{2\rho} \sin^3 \varphi \cos \varphi \right) da. \quad (14)$$

$$dl = \frac{\rho}{a N_2} \frac{s \sin \alpha_2}{\cos \varphi_2} da - l \sin^2 \varphi_1 \frac{\cos \varphi_1}{\cos \varphi_2} da. \quad (15)$$

$$d\alpha_2 = \frac{\rho}{a N_2} s \sin \alpha_2 \tan \varphi_2 da - \left(\frac{l \sin^2 \varphi_1 \sin \varphi_2 \cos \varphi_1}{\cos \varphi_2} - \frac{l}{\rho} (\varphi_2 - \varphi_1) \cos^3 \varphi_1 \right) da. \quad (16)$$

In the coefficient of da in equation (14) we understand by φ the arithmetic mean $\frac{\varphi_1 + \varphi_2}{2}$.

The above differential formulae of sections 90 and 91 substantially agree with the formulae which Helmert gave in "Lotabweichungen. Heft I: Formeln und Tafeln sowie einige numerische Ergebnisse für Norddeutschland," Veröff. d. Kgl. Geodätischen Instituts, Berlin, 1886, and whose development is given in Helmert, *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, Band I, Leipzig, 1880, pp. 279-296. In section 91 we have limited ourselves to derive only the simple formulae valid for shorter distances, which, however, will mostly suffice for practical use. The formulae of section 90 have been developed also in the 3rd Edition of this volume, Stuttgart, 1890, pp. 539-545, although only for short geodetic lines. Purely spherical formulae, whose transformation to the ellipsoid is carried out with the help of the relations of section 23, are first set up here.

Section 92. Adjustment of an Astronomic-Geodetic Net

We imagine that for the computation of the deflections of the vertical astronomic latitude, longitude and azimuth determinations have been carried out on individual points of a triangulation net. To these purely astronomically measured results there are then added further the lengths of the shortest connecting lines of the points on the reference ellipsoid, which can be computed with the help of the triangulation net. It is to be borne in mind here that the astronomic measurements are referred to the geoid, the lengths of the geodetic lines to the reference ellipsoid. By means of the deflections of the vertical, however, the latitude, longitude and azimuth measurements can likewise be transformed to the reference ellipsoid, according to section 89, so that all these quantities form then a uniform system.

If we consider at first two points and their connecting line, then there are measured for them:

1. the two geographic latitudes,
2. the astronomic difference of longitude,
3. the two azimuths of the geodetic line connecting the points,
4. the length of the geodetic line.

However, it must be noted here that only the geographic latitudes and the difference of longitude are directly measured quantities. The azimuths of the geodetic lines, on the other hand, are composed of the astronomically measured azimuth and the connecting angles computed from the triangles. The length of the geodetic line is likewise the result of the computation from the angles of the triangles and the base lines.

Strictly speaking, the azimuths as well as the length of the geodetic line are therefore not to be regarded as measured quantities. If in the following we treat, nevertheless, also these quantities as measurements and subject them also to an adjustment, then it is obvious that we deal here only with an approximate adjustment.

It is evident that only three of the six measured quantities are necessary for the determination of the mutual position of the two points on the ellipsoid. But since to this are added the two components of the deflection of the vertical of one point, while those of the other point can be assumed arbitrarily, then only one of the six measurements is excessive, and a condition equation which can be used for an adjustment must therefore be available. Such a condition equation is to be set up for every geodetic line for which the above-mentioned six measured results are available.

Further condition equations result in the case in which several such geodetic lines are joined together into a polygon. As in the polygons of the land survey, in each closed polygon there exist three condition equations, namely one which is formed by the sum of the angles between the consecutive geodetic lines, and two which express the coincidence of the starting point with the end point in longitude and latitude. These are the same three equations which also occur in the case of a triangulation chain arranged in the shape of a wreath; cf. Volume I, 8th edition, 1935, pp. 305-306.

All these equations form collectively the basis of the astronomic-geodetic net adjustment introduced by Helmert. The setting up of the equations and the method of adjustment are represented by Helmert in the *Veröffentlichung des Geodätischen Instituts* in Potsdam: "Lotabweichungen. Heft I, Formeln und Tafeln, sowie einige numerische Ergebnisse für Norddeutschland. Der allgemeinen Konferenz der Internationalen Erdmessung im Oktober 1886 zu Berlin gewidmet." Berlin, 1886.

After this, we pass over to the setting up of the condition equations.

We consider two points P_1 and P_2 and denote the longitudes, latitudes and azimuths measured astronomically at them by $L_1', L_2', \varphi_1', \varphi_2', \alpha_1', \alpha_2'$, while we imagine s' to be the length of the geodetic line found from the triangle chains. These measured quantities shall receive the corrections $\delta L_1', \delta L_2', \dots, \delta s'$ by means of the adjustment. For these corrections, we set up, at first, error equations by introducing as unknowns the ellipsoidal coordinates φ_1 and φ_2 , as well as L_1 and L_2 .

We begin the adjustment similarly to that of a purely geodetic triangulation net by assuming approximate values $\varphi_1^\circ, \varphi_2^\circ$ and L_1°, L_2° of the unknowns. From them, we compute very rigorously the two azimuths α_1° and α_2° as well as the distance s° , by using for it the spheroidal formulae for the mean latitude from section 21, p. 99. These computed values thus form a correlated system on the reference ellipsoid. By means of the adjustment we then have to determine, in addition, only the corrections $d\varphi_1, d\varphi_2$ and dL_1, dL_2 of the preliminary values of the unknowns, to which then the corrections $d\alpha_1, d\alpha_2$ and ds also belong. The final ellipsoidal values are then

$$\left. \begin{aligned} \varphi_1 &= \varphi_1^\circ + d\varphi_1 & \alpha_1 &= \alpha_1^\circ + d\alpha_1 \\ \varphi_2 &= \varphi_2^\circ + d\varphi_2 & \alpha_2 &= \alpha_2^\circ + d\alpha_2 \\ L_1 &= L_1^\circ + dL_1 & s &= s^\circ + ds \\ L_2 &= L_2^\circ + dL_2 \end{aligned} \right\} \quad (1)$$

These final ellipsoidal values differ from the measured values corrected by the adjustment only because of the deflections of the vertical. According to section 89, p. 437, equations (16) to (18) we then have

$$\left. \begin{aligned} \varphi_1^\circ + d\varphi_1 + \xi_1 &= \varphi_1' + \delta\varphi_1' \\ \varphi_2^\circ + d\varphi_2 + \xi_2 &= \varphi_2' + \delta\varphi_2' \\ L_1^\circ + dL_1 + \eta_1 \sec \varphi_1 &= L_1' + \delta L_1' \\ L_2^\circ + dL_2 + \eta_2 \sec \varphi_2 &= L_2' + \delta L_2' \\ \alpha_1^\circ + d\alpha_1 + \eta_1 \tan \varphi_1 &= \alpha_1' + \delta\alpha_1' \\ \alpha_2^\circ + d\alpha_2 + \eta_2 \tan \varphi_2 &= \alpha_2' + \delta\alpha_2' \\ s^\circ + ds &= s' + \delta s' \end{aligned} \right\} \quad (2)$$

In the last equation the deflections of the vertical do not occur, since the distance between the two points on the ellipsoid and on the geoid is the same to within infinitesimal quantities.

We will not directly utilize these error equations for the adjustment, but use them only for the setting up of equations of the deflections of the vertical.

We can express the quantities $d\varphi_2$, dL_2 and $d\alpha_2$ by $d\varphi_1$, dL_1 , $d\alpha_1$ and ds with the help of the differential formulae (28), section 90, p. 446, in which we now have to set $l = L_2 - L_1$. At the same time, we will insert further the relations (14) to (16), section 91, p. 452, in order to express also the changes of φ_2 , L_2 and α_2 , which result from the changes da and $d\alpha$ of the earth's dimensions. With the use of the simple notation of coefficients indicated by Helmert, *ibid.*, we then have the three equations

$$\left. \begin{aligned} -a\varphi_2 &= p_1 d\varphi_1 + p_3 ds + p_4 d\alpha_1 + p_5 \frac{da}{a} + p_6 d\alpha \\ -\cos\varphi_2 dL_2 &= -\cos\varphi_2 dL_1 + q_1 d\varphi_1 + q_3 ds + q_4 d\alpha_1 + q_5 \frac{da}{a} + q_6 d\alpha \\ -\cot\varphi_2 d\alpha_2 &= r_1 d\varphi_1 + r_3 ds + r_4 d\alpha_1 + r_5 \frac{da}{a} + r_6 d\alpha \end{aligned} \right\} \quad (3)$$

From equations (3) we eliminate, with the help of equations (2), the quantities $d\varphi_1$, $d\varphi_2$, dL_1 , and so on, and obtain

$$\left. \begin{aligned} \xi_2 &= \varphi_2' - \varphi_2^\circ + \delta\varphi_2' + p_1(\varphi_1' - \varphi_1^\circ + \delta\varphi_1' - \xi_1) + p_3(s' - s^\circ + \delta s') \\ &\quad + p_4(\alpha_1' - \alpha_1^\circ + \delta\alpha_1' - \eta_1 \tan \varphi_1) + p_5 \frac{da}{a} + p_6 d\alpha \\ \eta_2 &= \cos\varphi_2(L_2' - L_2^\circ - L_1' + L_1^\circ + \delta L_2' - \delta L_1') + \eta_1 \sec\varphi_1 \cos\varphi_2 \\ &\quad + q_1(\varphi_1' - \varphi_1^\circ + \delta\varphi_1' - \xi_1) + q_3(s' - s^\circ + \delta s') + q_4(\alpha_1' - \alpha_1^\circ + \delta\alpha_1' - \eta_1 \tan \varphi_1) \\ &\quad + q_5 \frac{da}{a} + q_6 d\alpha \\ \eta_2 &= \cot\varphi_2(\alpha_2' - \alpha_2^\circ + \delta\alpha_2') + r_1(\varphi_1' - \varphi_1^\circ + \delta\varphi_1' - \xi_1) + r_3(s' - s^\circ + \delta s') \\ &\quad + r_4(\alpha_1' - \alpha_1^\circ + \delta\alpha_1' - \eta_1 \tan \varphi_1) + r_5 \frac{da}{a} + r_6 d\alpha \end{aligned} \right\} \quad (4)$$

Instead of the transverse component η of the deflection of the vertical we will introduce the length component of the deflection of the vertical, which is equal to $\eta \sec \varphi$ according to section 89, p. 435, equation (7). If we denote it by λ , then we have

$$\lambda_1 = \eta_1 \sec \varphi_1 \quad \lambda_2 = \eta_2 \sec \varphi_2. \quad (5)$$

By using, according to Helmert, the new coefficients p_1 , $p_2 \dots$, and so on, we obtain:

$$\left. \begin{aligned} \xi_2 &= w_\varphi + \delta\varphi_2' + p_1 \delta\varphi_1' - p_1 \xi_1 - p_2 \lambda_1 + p_3 \delta s' + p_4 \delta\alpha_1' \\ &\quad + p_5 \frac{da}{a} + p_6 d\alpha \\ \lambda_2 &= w_L + \delta L_2' - \delta L_1' + q_1 \delta\varphi_1' - q_1 \xi_1 - q_2 \lambda_1 + q_3 \delta s' + q_4 \delta\alpha_1' \\ &\quad + q_5 \frac{da}{a} + q_6 d\alpha \\ \lambda_2 &= w_\alpha + \operatorname{cosec} \varphi_2 \delta\alpha_2' + r_1 \delta\varphi_1' - r_1 \xi_1 - r_2 \lambda_1 + r_3 \delta s' + r_4 \delta\alpha_1' \\ &\quad + r_5 \frac{da}{a} + r_6 d\alpha \end{aligned} \right\} \quad (6)$$

where the absolute terms w have the following meaning:

$$\left. \begin{aligned} w_\varphi &= \varphi_2' - \varphi_2^\circ + p_1(\varphi_1' - \varphi_1^\circ) + p_3(s' - s^\circ) + p_4(\alpha_1' - \alpha_1^\circ) \\ w_L &= L_2' - L_2^\circ - L_1' + L_1^\circ + q_1(\varphi_1' - \varphi_1^\circ) + q_3(s' - s^\circ) + q_4(\alpha_1' - \alpha_1^\circ) \\ w_\alpha &= \operatorname{cosec} \varphi_2(\alpha_2' - \alpha_2^\circ) + r_1(\varphi_1' - \varphi_1^\circ) + r_3(s' - s^\circ) + r_4(\alpha_1' - \alpha_1^\circ) \end{aligned} \right\} \quad (7)$$

By comparison of the two systems (4) and (6) we find immediately the relations between the previous and the new coefficients. We have

$$\left. \begin{array}{lll} p_1 = p_1 & p_2 = p_4 \sin \varphi_1 & p_3 = p_3 \\ p_4 = p_5 & p_5 = p_5 & p_6 = p_6 \\ q_1 = q_1 \sec \varphi_2 & q_2 = q_4 \sin \varphi_1 \sec \varphi_2 - 1 & q_3 = q_3 \sec \varphi_2 \\ q_4 = q_4 \sec \varphi_2 & q_5 = q_5 \sec \varphi_2 & q_6 = q_6 \sec \varphi_2 \\ r_1 = r_1 \sec \varphi_2 & r_2 = r_4 \sin \varphi_1 \sec \varphi_2 & r_3 = r_3 \sec \varphi_2 \\ r_4 = r_4 \sec \varphi_2 & r_5 = r_5 \sec \varphi_2 & r_6 = r_6 \sec \varphi_2 \end{array} \right\} \quad (8)$$

From the last two equations (6) we can now eliminate the component of the deflection of the vertical λ corresponding to equations (17) to (19), section 89, p. 437, and by so doing, we obtain the Laplace equation in the form extended by Helmert. By subtraction we find

$$0 = w_L - w_\alpha + \delta L_2' - \delta L_1' - \operatorname{cosec} \varphi_2 \delta \alpha_2' + (q_1 - r_1) \delta \varphi_1' + (q_3 - r_3) \delta s' + (q_4 - r_4) \delta \alpha_1' - (q_1 - r_1) \xi_1 - (q_2 - r_2) \lambda_1 + (q_5 - r_5) \frac{d a}{a} + (q_6 - r_6) d a. \quad (9)$$

This equation contains, apart from the absolute terms, mainly the corrections of the measured quantities. Since deflections of the vertical have always only relative importance, the deflections of the vertical ξ_1 and λ_1 of point P_1 must either be assumed as known or be replaced by the values $\xi_1 = 0$, $\lambda_1 = 0$. We have to set the corrections $\frac{d a}{a}$ and $d a$ of the ellipsoidal constants equal to zero if the Bessel ellipsoid is to be retained; in the other case, if, for instance, the international ellipsoid shall be introduced, we will substitute the corresponding numerical values. With this, equation (9) is then prepared to the extent that it can be used for the adjustment.

Now we still have to indicate the values of the coefficients (8). From equations (8) and (3) we find by comparison with equations (29), section 90, p. 446, and (14) to (16), p. 452:

$$\left. \begin{array}{ll} p_1 = -\frac{M_1}{M_2} \cos l & p_4 = -\frac{m}{M_2} \sin \alpha_2 \\ p_2 = -\frac{m}{M_2} \sin \alpha_2 \sin \varphi_1 & p_5 = -\frac{\rho}{M_2} s \cos \alpha_2 \\ p_3 = +\frac{\rho}{M_2} \cos \alpha_2 & p_6 = -(\varphi_2 - \varphi_1) (2 - 3 \sin^2 \varphi) - \frac{l^2}{2 \rho} \sin^3 \varphi \cos \varphi \end{array} \right\} \quad (10)$$

$$\left. \begin{array}{ll} q_1 = -\frac{M_1}{N_2} \sin l \tan \varphi_2 & q_4 = +\frac{m \cos \alpha_2}{N_2 \cos \varphi_2} \\ q_2 = +\frac{m \sin \varphi_1}{N_2 \cos \varphi_2} \cos \alpha_2 - 1 & q_5 = -\frac{\rho s \sin \alpha_2}{N_2 \cos \varphi_2} \\ q_3 = +\frac{\rho \sin \alpha_2}{N_2 \cos \varphi_2} & q_6 = +l \frac{\sin^2 \varphi_1}{\cos \varphi_2} \cos \varphi_1 \end{array} \right\} \quad (11)$$

$$\left. \begin{array}{ll} r_1 = -\frac{\sin l}{\sin \varphi_2 \cos \varphi_2} (1 - e^2 \sin^2 \varphi_2 \cos^2 \varphi_2) & r_4 = r_2 \frac{1}{\sin \varphi_1} \\ r_2 = -\frac{d m \sin \varphi_1}{d s \sin \varphi_2} + \frac{m \sin \varphi_1}{N_2 \cos \varphi_2} \cos \alpha_2 & r_5 = q_5 \\ r_3 = q_3 & r_6 = l \frac{\sin^2 \varphi_1}{\cos \varphi_2} \cos \varphi_1 - \frac{l(\varphi_2 - \varphi_1) \cos^3 \varphi_1}{\rho \sin \varphi_2} \end{array} \right\} \quad (12)$$

Here, the previous remark should be repeated that, apart from l and s , the quantities without the index are to be computed for the mean geographic latitude $\frac{\varphi_1 + \varphi_2}{2}$.

In order to have together all that is necessary for the setting up of the Laplace equation (9), in the following we reproduce once more also the spheroidal formulae for the mean latitude from section 21, p. 102:

$$\left. \begin{aligned} \log s \sin \alpha &= \log \frac{l \cos \varphi}{[2]} - [3] l^2 \sin^2 \varphi + [4] (\varphi_2 - \varphi_1)^2 \\ \log s \cos \alpha &= \log \frac{(\varphi_2 - \varphi_1)}{[1]} \cos \frac{l}{2} - [5] l^2 \cos^2 \varphi - [6] (\varphi_2 - \varphi_1)^2 \\ \log \Delta \alpha &= \log l \sin \varphi + [7] l^2 \cos^2 \varphi + [8] (\varphi_2 - \varphi_1)^2 \end{aligned} \right\} \quad (13)$$

Here we have $\alpha = \frac{\alpha_1 + \alpha_2 \pm 180^\circ}{2}$ and $\Delta \alpha = \alpha_2 - \alpha_1 \pm 180^\circ$. After the computation of α and $\Delta \alpha$ we thus have the two equations

$$\alpha_1 = \alpha - \frac{\Delta \alpha}{2} \quad \alpha_2 = \alpha + \frac{\Delta \alpha}{2} \pm 180^\circ$$

for the determination of α_1 and α_2 .

Section 93. Example of a Net Adjustment

For a practical application of the computational procedure indicated in the previous section, we take from A. Börsch, "Lotabweichungen Heft III," *Veröffentlichung des Kgl. Preuss. Geodätischen Instituts*, Berlin, 1906, the sector from the astronomic-geodetic net of first order north of the European longitude degree-measurement at 52° latitude represented in Fig. 1. At the three points (9) Rauenberg, (19) Springberg and (22) Schönsee, longitudes and latitudes as well as azimuths have been measured astronomically; we designate such

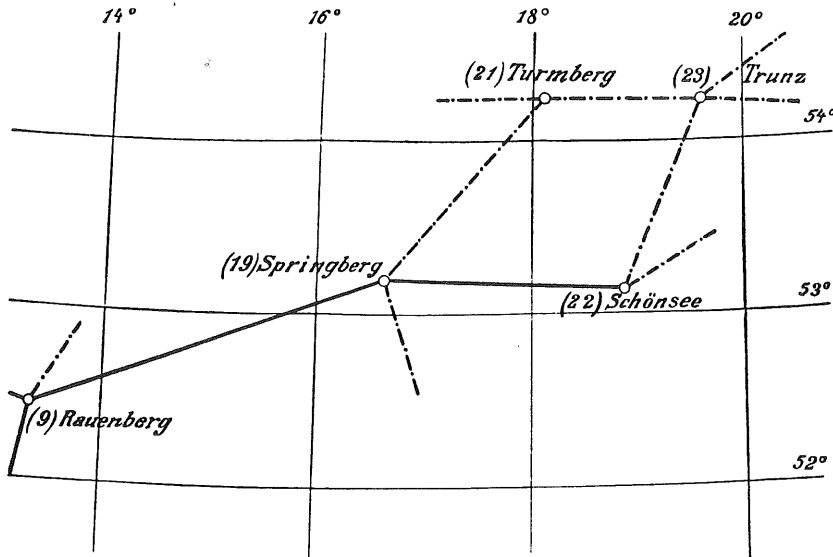


Fig. 1.

points according to Helmert as *Laplace points*. The two remaining points, (21) Turmberg and (23) Trunz have only latitude and azimuth measurements. It will be shown later how these latter points for which no Laplace equations can be set up are nevertheless to be utilized in the net adjustment. The numerical data in "Lotabweichungen Heft III" have still undergone a few changes, which were required by the introduction of the international meter as well as by a new adjustment of the European longitude net. The final working up of the astronomic-geodetic net north of the European longitude degree-measurement is contained in *Veröffentlichung des Geodätischen Instituts*, N.F. Nr. 68, "Lotabweichungen Heft V," von L. Krüger, Berlin, 1916. The following numerical data are taken from this latter publication.

We begin with the line

(9) Rauenberg - (19) Springberg.

From the chains of triangles of the Prussian Land Survey, the following has been computed for the length of the geodetic line connecting the two points:

$$s'_{9.19} = 233,626.6 \text{ m.} \quad (\log s'_{9.19} = 5.368 \ 5231).$$

$$\begin{aligned} \varphi'_9 &= 52^\circ 27' 12.19'' & \varphi'_{19} &= 53^\circ 11' 00.94'' \\ L'_9 &= 13 \ 22 \ 06.03 & L'_{19} &= 16 \ 36 \ 59.22 \end{aligned}$$

have been measured astronomically. Besides, at the two points azimuth measurements have been carried out for every triangle side. From them and the angles of the triangle chains, the following two azimuths were found:

$$\alpha'_{9.19} = 68^\circ 18' 42.11'' \quad \alpha'_{19.9} = 250^\circ 53' 54.48''.$$

As approximate values on the ellipsoid the following were assumed:

$$\begin{aligned} \varphi^\circ_9 &= 52^\circ 27' 12.021'' & \varphi^\circ_{19} &= 53^\circ 11' 07.036'' \\ L^\circ_9 &= 13 \ 22 \ 07.690 & L^\circ_{19} &= 16 \ 37 \ 03.314 \end{aligned}$$

and there was computed hence:

$$\begin{aligned} \alpha^\circ_{9.19} &= 68^\circ 18' 39.064'' & \alpha^\circ_{19.9} &= 250^\circ 53' 58.466'' \\ s^\circ_{9.19} &= 233,624.9 \text{ m} & (\log s^\circ_{9.19} &= 5.368 \ 5192). \end{aligned}$$

With these, the coefficients (10) to (12), section 92, p. 455, are now to be computed. However, we have not carried out this computation, but have permitted ourselves to take the values of the coefficients from the above discussion. We therefore pass over at once to the equations for the deflections of the vertical and have according to (6) and (7), section 92, p. 454:

Deflections of the vertical

$$\left. \begin{aligned} \xi_{19} &= -6.19'' & + \delta \varphi'_{19} - 0.9983 \delta \varphi'_9 - 0.0106 \delta s'_{9.19} + 0.0346 \delta \alpha'_{9.19} \\ & & + 0.9983 \xi_9 - 0.0274 \lambda_9 + 2473 \frac{d a}{a} - 355 d a \\ \lambda_{19} &= -2.62 + \delta L'_{19} - \delta L'_9 - 0.0755 \delta \varphi'_9 - 0.0509 \delta s'_{9.19} - 0.0200 \delta \alpha'_{9.19} \\ & & + 0.0755 \xi_9 + 1.0158 \lambda_9 + 11890 \frac{d a}{a} + 7475 d a \\ \lambda_{19} &= -8.97 + 1.2491 \delta \alpha'_{19.9} - 0.1180 \delta \varphi'_9 - 0.0509 \delta s'_{9.19} - 1.2682 \delta \alpha'_{9.19} \\ & & + 0.1180 \xi_9 + 1.0055 \lambda_9 + 11890 \frac{d a}{a} + 7433 d a. \end{aligned} \right\} \quad (1)$$

From the difference of the last two equations there follows the

Laplace equation:

$$\begin{aligned} -6.35'' &= + \delta L'_{19} - \delta L'_9 - 1.2491 \delta \alpha'_{19.9} + 0.0425 \delta \varphi'_9 + 1.2482 \delta \alpha'_{9.19} \\ & \quad - 0.0425 \xi_9 + 0.0103 \lambda_9 + 42 d a. \end{aligned} \quad (2)$$

We will now write down immediately the results of measurement and computation for the remaining four lines of Fig. 1, p. 456.

Springberg	$\varphi'_{19} = 53^\circ 11' 00.94''$	$\varphi^\circ_{19} = 53^\circ 11' 07.036''$
(19)	$L'_{19} = 16 \ 36 \ 59.22$	$L^\circ_{19} = 16 \ 37 \ 03.314$
	$\alpha'_{19.22} = 90 \ 14 \ 42.11$	$\alpha^\circ_{19.22} = 90 \ 14 \ 44.106$
Schönsee	$\varphi'_{22} = 53 \ 09 \ 26.22$	$\varphi^\circ_{22} = 53 \ 09 \ 27.047$
(22)	$L'_{22} = 18 \ 53 \ 52.89$	$L^\circ_{22} = 18 \ 54 \ 03.858$
	$\alpha'_{22.19} = 272 \ 04 \ 17.25$	$\alpha^\circ_{22.19} = 272 \ 04 \ 24.408$
	$\log s'_{19.22} = 5.183 \ 8542$	$\log s^\circ_{19.22} = 5.183 \ 8535$
	$s'_{19.22} = 152,705.0 \text{ m}$	$s^\circ_{19.22} = 152,705.1 \text{ m}$

Deflections of the vertical

$$\left. \begin{aligned}
 \xi_{22} &= +5.22'' + \delta \varphi'_{22} - 0.9992 \delta \varphi'_{19} + 0.0012 \delta s'_{19.22} + 0.0239 \delta \alpha'_{19.22} \\
 &\quad + 0.9992 \xi_{19} - 0.0192 \lambda_{19} - 179 \frac{d a}{a} - 43 d a \\
 \lambda_{22} &= -6.57 + \delta L'_{22} - \delta L'_{19} - 0.0530 \delta \varphi'_{19} - 0.0538 \delta s'_{19.22} + 0.0014 \delta \alpha'_{19.22} \\
 &\quad + 0.0530 \xi_{19} + 0.9988 \lambda_{19} + 8214 \frac{d a}{a} + 5264 d a \\
 \lambda_{22} &= -5.96 + 1.2496 \delta \alpha'_{22.19} - 0.0829 \delta \varphi'_{19} - 0.0538 \delta s'_{19.22} - 1.2478 \delta \alpha'_{19.22} \\
 &\quad + 0.0829 \xi_{19} + 0.9989 \lambda_{19} + 8214 \frac{d a}{a} + 5265 d a
 \end{aligned} \right\} \quad (3)$$

Laplace equation:

$$\begin{aligned}
 +0.61'' &= +\delta L'_{22} - \delta L'_{19} - 1.2496 \delta \alpha'_{22.19} + 0.0299 \delta \varphi'_{19} + 1.2492 \delta \alpha'_{19.22} \\
 &\quad - 0.0299 \xi_{19} - 0.0001 \lambda_{19} - 1 d a
 \end{aligned} \quad (4)$$

From this equation we still have to eliminate the deflections of the vertical ξ_{19} and λ_{19} of point (19), for which we use the first and the third equation of system (1). Then there results the new Laplace equation:

$$\left. \begin{aligned}
 +0.43'' &= +\delta L'_{22} - \delta L'_{19} + 0.0299 \delta \varphi'_9 - 1.2496 \delta \alpha'_{22.19} + 1.2492 \delta \alpha'_{19.22} \\
 &\quad - 0.0005 \delta \alpha'_{19.9} - 0.0005 \delta \alpha'_{9.19} + 0.0003 \delta s'_{9.19} \\
 &\quad - 0.0299 \xi_9 + 0.0007 \lambda_9 - 75 \frac{d a}{a} + 9 d a
 \end{aligned} \right\} \quad (5)$$

(19) Springberg - (21) Turmberg

Springberg	$\varphi'_{19} = 53^\circ 11' 00.94''$	$\varphi^\circ_{19} = 53^\circ 11' 07.036''$
(19)	$L'_{19} = 16 \ 36 \ 59.22$	$L^\circ_{19} = 16 \ 37 \ 03.314$
	$\alpha'_{19.21} = 40 \ 06 \ 36.00$	$\alpha^\circ_{19.21} = 40 \ 06 \ 37.330$
Turmberg	$\varphi'_{21} = 54 \ 13 \ 26.78$	$\varphi^\circ_{21} = 54 \ 13 \ 31.874$
(21)	$L'_{21} = 18 \ 07 (35.26)$	$L^\circ_{21} = 18 \ 07 \ 35.260$
	$\alpha'_{21.19} = 221 \ 19 \ 38.46$	$\alpha^\circ_{21.19} = 221 \ 19 \ 35.662$
	$\log s'_{19.21} = 5.183 \ 9164$	$\log s^\circ_{19.21} = 5.183 \ 9146$
	$s'_{19.21} = 152,726.8 \text{ m}$	$s^\circ_{19.21} = 152,726.6 \text{ m}$

The astronomic longitude measurement is missing for the point Turmberg; the imagined value $L'_{21} = 18^\circ 07' (35.26'')$ indicated above has been taken over from the value L°_{21} merely for the more convenient setting up of the further computation; any other arbitrary value could also be introduced for it.

$$\left. \begin{aligned} \xi_{21} &= +0.96'' & + \delta \varphi'_{21} - 0.9995 \delta \varphi'_{19} - 0.0243 \delta s'_{19.21} + 0.0158 \delta \alpha'_{19.21} \\ & & + 0.9995 \xi_{19} - 0.0127 \lambda_{19} + 3710 \frac{d a}{a} - 214 d a \\ \lambda_{21} &= (+4.33) + \Delta L'_{21} - \delta L'_{19} - 0.0364 \delta \varphi'_{19} - 0.0365 \delta s'_{19.21} - 0.0307 \delta \alpha'_{19.21} \\ & & + 0.0364 \xi_{19} + 1.0246 \lambda_{19} + 5568 \frac{d a}{a} + 3568 d a \\ \lambda_{21} &= +5.44 + 1.2326 \delta \alpha'_{21.19} - 0.0554 \delta \varphi'_{19} - 0.0365 \delta s'_{19.21} - 1.2629 \delta \alpha'_{19.21} \\ & & + 0.0554 \xi_{19} + 1.0111 \lambda_{19} + 5568 \frac{d a}{a} + 3543 d a \end{aligned} \right\} \quad (6)$$

In the second equation $\Delta L'_{21}$ has been introduced as a correction of the above imagined observation L'_{21} ; as follows already from the above, the whole equation is to be considered only as a computational equation and not to be used for the computation of the deflections of the vertical following later.

(21) Turmberg - (23) Trunz

Turmberg	$\varphi'_{21} = 54^\circ 13' 26.78''$	$\varphi^\circ_{21} = 54^\circ 13' 31.874''$
(21)	$L'_{21} = 18 \ 07 (35.26)$	$L^\circ_{21} = 18 \ 07 \ 35.260$
	$\alpha'_{21.23} = 89 \ 46 \ 09.00$	$\alpha^\circ_{21.23} = 89 \ 46 \ 09.575$
Trunz	$\varphi'_{23} = 54 \ 13 \ 11.76$	$\varphi^\circ_{23} = 54 \ 13 \ 14.121$
(23)	$L'_{23} = 19 \ 32 (14.88)$	$L^\circ_{23} = 19 \ 32 \ 14.880$
	$\alpha'_{23.21} = 270 \ 54 \ 40.82$	$\alpha^\circ_{23.21} = 270 \ 54 \ 50.738$
	$\log s'_{21.23} = 4.963 \ 8939$	$\log s^\circ_{21.23} = 4.963 \ 8881$
	$s'_{21.23} = 92,022.07 \text{ m}$	$s^\circ_{21.23} = 92,021.25 \text{ m}$

Deflections of the vertical

$$\left. \begin{aligned} \xi_{23} &= +2.72'' & + \delta \varphi'_{23} - 0.9997 \delta \varphi'_{21} + 0.0005 \delta s'_{21.23} + 0.0144 \delta \alpha'_{21.23} \\ & & + 0.9997 \xi_{21} - 0.0117 \lambda_{21} - 47 \frac{d a}{a} - 19 d a \\ \lambda_{23} &= (+0.10) + \Delta L'_{23} - \Delta L'_{21} - 0.0341 \delta \varphi'_{21} - 0.0552 \delta s'_{21.23} + 0.0004 \delta \alpha'_{21.23} \\ & & + 0.0341 \xi_{21} + 0.9997 \lambda_{21} + 5079 \frac{d a}{a} + 3343 d a \\ \lambda_{23} &= -11.32 + 1.2326 \delta \alpha'_{23.21} - 0.0518 \delta \varphi'_{21} - 0.0552 \delta s'_{21.23} - 1.2321 \delta \alpha'_{21.23} \\ & & + 0.0518 \xi_{21} + 0.9997 \lambda_{21} + 5079 \frac{d a}{a} + 3343 d a \end{aligned} \right\} \quad (7)$$

(22) Schönsee - (23) Trunz

Schönsee	$\varphi'_{22} = 53^\circ 09' 26.22''$	$\varphi^\circ_{22} = 53^\circ 09' 27.047''$
(22)	$L'_{22} = 18 \ 53 \ 54.21$	$L^\circ_{22} = 18 \ 54 \ 03.858$
	$\alpha'_{22.23} = 19 \ 18 \ 14.28$	$\alpha^\circ_{22.23} = 19 \ 18 \ 19.943$
Trunz	$\varphi'_{23} = 54 \ 13 \ 11.76$	$\varphi^\circ_{23} = 54 \ 13 \ 14.121$
(23)	$L'_{23} = 19 \ 32 (14.88)$	$L^\circ_{23} = 19 \ 32 \ 14.880$
	$\alpha'_{23.22} = 199 \ 48 \ 56.45$	$\alpha^\circ_{23.22} = 199 \ 49 \ 06.169$
	$\log s'_{22.23} = 5.098 \ 8259$	$\log s^\circ_{22.23} = 5.098 \ 8204$
	$s'_{22.23} = 125,552.8 \text{ m}$	$s^\circ_{22.23} = 125,551.1 \text{ m}$

$$\left. \begin{aligned}
 \xi_{23} &= -1.62'' + \delta \varphi'_{23} - 0.9998 \delta \varphi'_{22} - 0.0304 \delta s'_{22,23} + 0.0067 \delta \alpha'_{22,23} \\
 &\quad + 0.9998 \xi_{22} - 0.0053 \lambda_{22} + 3821 \frac{d a}{a} - 203 d a \\
 \lambda_{23} &= (+11.13) + \Delta L'_{23} - \delta L'_{22} - 0.0154 \delta \varphi'_{22} - 0.0187 \delta s'_{22,23} - 0.0316 \delta \alpha'_{22,23} \\
 &\quad + 0.0154 \xi_{22} + 1.0253 \lambda_{22} + 2350 \frac{d a}{a} + 1505 d a \\
 \lambda_{23} &= -4.83 + 1.2326 \delta \alpha'_{23,22} - 0.0234 \delta \varphi'_{22} - 0.0187 \delta s'_{22,23} - 1.2640 \delta \alpha'_{22,23} \\
 &\quad + 0.0234 \xi_{22} + 1.0116 \lambda_{22} + 2350 \frac{d a}{a} + 1494 d a .
 \end{aligned} \right\} \quad (8)$$

Deflections of the vertical with reference to Rauenberg as central point

For the numerical computation of the deflections of the vertical and also for the setting up of the polygon equations it is advantageous to convert the above systems of deflections of the vertical in such a way that they yield the deflections of the vertical of the individual points with reference to Rauenberg as central point, e.g., we have to substitute in (3) for ξ_{19} and λ_{19} the values from (1) so that instead of ξ_{19} and λ_{19} , ξ_9 and λ_9 now occur. Equations (6), (7) and (8) are to be converted likewise, whereby the following systems result:

Springberg (19)

$$\left. \begin{aligned}
 \xi_{19} &= -6.19'' + \delta \varphi'_{19} - 0.9983 (\delta \varphi'_9 - \xi_9) - 0.0274 \lambda_9 - 0.0106 \delta s'_{9,19} \\
 &\quad + 0.0346 \delta \alpha'_{9,19} + 2473 \frac{d a}{a} - 355 d a \\
 \lambda_{19} &= -2.62 + \delta L'_{19} - \delta L'_9 - 0.0755 (\delta \varphi'_9 - \xi_9) + 1.0158 \lambda_9 - 0.0509 \delta s'_{9,19} \\
 &\quad - 0.0200 \delta \alpha'_{9,19} + 11.890 \frac{d a}{a} + 7475 d a \\
 \lambda_{19} &= -8.97 - 0.1180 (\delta \varphi'_9 - \xi_9) + 1.0055 \lambda_9 - 0.0509 \delta s'_{9,19} - 1.2682 \delta \alpha'_{9,19} \\
 &\quad + 1.2491 \delta \alpha'_{19,9} + 11,890 \frac{d a}{a} + 7433 d a .
 \end{aligned} \right\} \quad (9)$$

Turmberg (21)

$$\left. \begin{aligned}
 \xi_{21} &= -5.11'' + \delta \varphi'_{21} - 0.9963 (\delta \varphi'_9 - \xi_9) - 0.0402 \lambda_9 - 0.0100 \delta s'_{9,19} \\
 &\quad - 0.0243 \delta s'_{19,21} + 0.0507 \delta \alpha'_{9,19} - 0.0158 (\delta \alpha'_{19,9} - \delta \alpha'_{19,21}) \\
 &\quad + 6031 \frac{d a}{a} - 663 d a \\
 \lambda_{21} &= (+1.27) + \Delta L'_{21} - \delta L'_9 - 0.1147 (\delta \varphi'_9 - \xi_9) + 1.0395 \lambda_9 - 0.0526 \delta s'_{9,19} \\
 &\quad - 0.0365 \delta s'_{19,21} - 0.0499 \delta \alpha'_{9,19} + 0.0307 (\delta \alpha'_{19,9} - \delta \alpha'_{19,21}) \\
 &\quad + 17,840 \frac{d a}{a} + 11,213 d a \\
 \lambda_{21} &= -3.97 - 0.1746 (\delta \varphi'_9 - \xi_9) + 1.015 \lambda_9 - 0.0521 \delta s'_{9,19} - 0.0365 \delta s'_{19,21} \\
 &\quad - 1.2803 \delta \alpha'_{9,19} + 1.2629 (\delta \alpha'_{19,9} - \delta \alpha'_{19,21}) + 1.2326 \delta \alpha'_{21,19} \\
 &\quad + 17,727 \frac{d a}{a} + 11,039 d a .
 \end{aligned} \right\} \quad (10)$$

Schönsee (22)

$$\left. \begin{aligned}
 \xi_{22} &= -0.79'' + \delta \varphi'_{22} - 0.9952 (\delta \varphi'_9 - \xi_9) - 0.0467 \lambda_9 - 0.0096 \delta s'_{9,19} \\
 &\quad + 0.0012 \delta s'_{19,22} + 0.0589 \delta \alpha'_{9,19} - 0.0239 (\delta \alpha'_{19,9} - \delta \alpha'_{19,22}) \\
 &\quad + 2063 \frac{d a}{a} - 541 d a \\
 \lambda_{22} &= -9.51 + \delta L'_{22} - \delta L'_9 - 0.1283 (\delta \varphi'_9 - \xi_9) + 1.0131 \lambda_9 - 0.0514 \delta s'_{9,19} \\
 &\quad - 0.0538 \delta s'_{19,22} - 0.0167 \delta \alpha'_{9,19} - 0.0014 (\delta \alpha'_{19,9} - \delta \alpha'_{19,22}) \\
 &\quad + 20,221 \frac{d a}{a} + 12,711 d a \\
 \lambda_{22} &= -15.43 - 0.2006 (\delta \varphi'_9 - \xi_9) + 1.0021 \lambda_9 - 0.0517 \delta s'_{9,19} - 0.0538 \delta s'_{19,22} \\
 &\quad - 1.2640 \delta \alpha'_{9,19} + 1.2478 (\delta \alpha'_{19,9} - \delta \alpha'_{19,22}) + 1.2496 \delta \alpha'_{22,19} \\
 &\quad + 20,296 \frac{d a}{a} + 12,660 d a .
 \end{aligned} \right\} \quad (11)$$

$$\begin{aligned}
 \xi_{23} &= -2.34'' + \delta \varphi'_{23} - 0.9940 (\delta \varphi'_9 - \xi_9) - 0.0521 \lambda_9 - 0.0094 \delta s'_{9.19} \\
 &\quad - 0.0239 \delta s'_{19.21} + 0.0005 \delta s'_{21.23} + 0.0657 \delta \alpha'_{9.19} \\
 &\quad - 0.0306 (\delta \alpha'_{19.9} - \delta \alpha'_{19.21}) - 0.0144 (\delta \alpha'_{21.19} - \delta \alpha'_{21.23}) \\
 &\quad + 5775 \frac{d a}{a} - 811 d a \\
 \lambda_{23} &= (+1.20) + \Delta L'_{23} - \delta L'_9 - 0.1486 (\delta \varphi'_9 - \xi_9) + 1.0378 \lambda_9 - 0.0529 \delta s'_{9.19} \\
 &\quad - 0.0373 \delta s'_{19.21} - 0.0552 \delta s'_{21.23} - 0.0478 \delta \alpha'_{9.19} \\
 &\quad + 0.0298 (\delta \alpha'_{19.9} - \delta \alpha'_{19.21}) - 0.0004 (\delta \alpha'_{21.19} - \delta \alpha'_{21.23}) \\
 &\quad + 23,120 \frac{d a}{a} + 14,530 d a \\
 \lambda_{23} &= -15.55 - 0.2261 (\delta \varphi'_9 - \xi_9) + 1.0126 \lambda_9 - 0.0526 \delta s'_{9.19} - 0.0377 \delta s'_{19.21} \\
 &\quad - 0.0552 \delta s'_{21.23} - 1.2772 \delta \alpha'_{9.19} + 1.2616 (\delta \alpha'_{19.9} - \delta \alpha'_{19.21}) \\
 &\quad + 1.2326 \delta \alpha'_{23.21} + 1.2321 (\delta \alpha'_{21.19} - \delta \alpha'_{21.23}) + 23,112 \frac{d a}{a} \\
 &\quad + 14,343 d a .
 \end{aligned} \tag{12}$$

Trunz (23) through Schönsee

$$\begin{aligned}
 \xi_{23} &= -2.33'' + \delta \varphi'_{23} - 0.9939 (\delta \varphi'_9 - \xi_9) - 0.0520 \lambda_9 - 0.0093 \delta s'_{9.19} \\
 &\quad + 0.0015 \delta s'_{19.22} - 0.0304 \delta s'_{22.23} + 0.0656 \delta \alpha'_{9.19} \\
 &\quad - 0.0305 (\delta \alpha'_{19.9} - \delta \alpha'_{19.22}) - 0.0067 (\delta \alpha'_{22.19} - \delta \alpha'_{22.23}) \\
 &\quad + 5776 \frac{d a}{a} - 811 d a \\
 \lambda_{23} &= (+1.22) + \Delta L'_{23} - \delta L'_9 - 0.1487 (\delta \varphi'_9 - \xi_9) + 1.0378 \lambda_9 - 0.0528 \delta s'_{9.19} \\
 &\quad - 0.0552 \delta s'_{19.22} - 0.0187 \delta s'_{22.23} - 0.0478 \delta \alpha'_{9.19} \\
 &\quad + 0.0298 (\delta \alpha'_{19.9} - \delta \alpha'_{19.22}) + 0.0316 (\delta \alpha'_{22.19} - \delta \alpha'_{22.23}) \\
 &\quad + 23,116 \frac{d a}{a} + 14,528 d a \\
 \lambda_{23} &= -20.46 - 0.2263 (\delta \varphi'_9 - \xi_9) + 1.0127 \lambda_9 - 0.0525 \delta s'_{9.19} - 0.0544 \delta s'_{19.22} \\
 &\quad - 0.0187 \delta s'_{22.23} - 1.2772 \delta \alpha'_{9.19} + 1.2617 (\delta \alpha'_{19.9} - \delta \alpha'_{19.22}) \\
 &\quad + 1.2326 \delta \alpha'_{23.22} + 1.2640 (\delta \alpha'_{22.19} - \delta \alpha'_{22.23}) + 22,930 \frac{d a}{a} \\
 &\quad + 14,289 d a .
 \end{aligned} \tag{13}$$

From the last two systems (12) and (13) we also form the corresponding Laplace equations, namely

$$\begin{aligned}
 -16.75'' &= +\Delta L'_{23} - \delta L'_9 + 0.0775 (\delta \varphi'_9 - \xi_9) + 0.0252 \lambda_9 - 0.0003 \delta s'_{9.19} \\
 &\quad + 0.0004 \delta s'_{19.21} + 1.2294 \delta \alpha'_{9.19} - 1.2318 (\delta \alpha'_{19.9} - \delta \alpha'_{19.21}) \\
 &\quad - 1.2326 \delta \alpha'_{23.21} - 1.2325 (\delta \alpha'_{21.19} - \delta \alpha'_{21.23}) + 8 \frac{d a}{a} + 187 d a .
 \end{aligned} \tag{12a}$$

$$\begin{aligned}
 -21.68 &= +\Delta L'_{23} - \delta L'_9 + 0.0776 (\delta \varphi'_9 - \xi_9) + 0.0251 \lambda_9 - 0.0003 \delta s'_{9.19} \\
 &\quad - 0.0008 \delta s'_{19.22} + 1.2294 \delta \alpha'_{9.19} - 1.2319 (\delta \alpha'_{19.9} - \delta \alpha'_{19.22}) \\
 &\quad - 1.2326 \delta \alpha'_{23.22} - 1.2324 (\delta \alpha'_{22.19} - \delta \alpha'_{22.23}) + 186 \frac{d a}{a} + 239 d a .
 \end{aligned} \tag{13a}$$

These two equations have only a formal significance because of the missing longitude determinations at the point Trunz; therefore, they cannot be used as condition equations for the net adjustment, as equations (2) and (5).

The polygon equations

The two systems of equations (12) and (13) for Trunz must yield for this point the same values of ξ_{23} and λ_{23} ; therefore, we can set the corresponding values from (12) and (13) equal to one another. We then obtain three equations which are free from the deflections of the vertical and in which the correction $\delta L'_{23}$ of the nonmeasured longitude of Trunz is also missing. Therefore, there remain three equations which contain

only the corrections of the measured quantities, and which can thus be used, like the Laplace equations, as condition equations. We designate these equations which occur in the case of each closed polygon of geodetic lines as *polygon equations*.

Now we will carry this out for the first two equations of (12) and (13), but instead of the third equations of (12) and (13), use the two equations (12a) and (13a). Then we obtain:

$$\left. \begin{aligned} + 0.01'' &= -0.0239 \delta s'_{19.21} + 0.0005 \delta s'_{21.23} - 0.0015 \delta s'_{19.22} + 0.0304 \delta s'_{22.23} \\ &\quad + 0.0306 (\delta \alpha'_{19.21} - \delta \alpha'_{19.22}) - 0.0144 (\delta \alpha'_{21.19} - \delta \alpha'_{21.23}) \\ &\quad + 0.0067 (\delta \alpha'_{22.19} - \delta \alpha'_{22.23}) - 1 \frac{d a}{a} + 0 d \alpha \\ + 0.02 &= -0.0373 \delta s'_{19.21} - 0.0552 \delta s'_{21.23} + 0.0552 \delta s'_{19.22} + 0.0187 \delta s'_{22.23} \\ &\quad - 0.0298 (\delta \alpha'_{19.21} - \delta \alpha'_{19.22}) - 0.0004 (\delta \alpha'_{21.19} - \delta \alpha'_{21.23}) \\ &\quad - 0.0316 (\delta \alpha'_{22.19} - \delta \alpha'_{22.23}) + 4 \frac{d a}{a} + 2 d \alpha \\ + 4.93 &= + 0.0004 \delta s'_{19.21} + 0.0008 \delta s'_{19.22} + 1.2318 (\delta \alpha'_{19.21} - \delta \alpha'_{19.22}) \\ &\quad - 1.2325 (\delta \alpha'_{21.19} - \delta \alpha'_{21.23}) + 1.2324 (\delta \alpha'_{22.19} - \delta \alpha'_{22.23}) \\ &\quad - 1.2326 (\delta \alpha'_{23.21} - \delta \alpha'_{23.22}) - 178 \frac{d a}{a} - 52 d \alpha. \end{aligned} \right\} \quad (14)$$

These three polygon equations form together with the two Laplace equations (2), p. 457, and (5), p. 458, the condition equations for the net adjustment.

Now we still have to divide the corrections $\delta \alpha'$ of the azimuth of the geodetic lines into the correction of the astronomically measured azimuth and the correction of the connecting angle. We will divide the latter once more into two directions and combine the correction to the direction for the astronomic target point with that to the azimuth. If we thus denote by δ_i the correction of the astronomically measured azimuth and of the connecting direction at the station P_i and by v_{ik} the correction to the direction for the geodetic line, then we have

$$\delta \alpha'_{ik} = \delta_i + v_{ik}. \quad (15)$$

According to this, we thus have, e.g.,

$$\delta \alpha'_{19.9} = \delta_{19} + v_{19.9} \quad \delta \alpha'_{9.19} = \delta_9 + v_{9.19}, \text{ and so on.}$$

At the same time, we round the coefficients at the right-hand side off to three decimal places and combine those which deviate from one another only by one unit of the third decimal place.

In order to adapt the very small coefficients of the first two equations (14) approximately to those of the remaining equations, it is advisable to multiply the first by 33.3. We achieve a further simplification if we divide the last equation (14) by 1.232.

With these variations we obtain the final condition equations combined as follows:

$$\left. \begin{aligned} - 6.35'' - 0.0425 (\delta \varphi'_9 - \xi_9) - 0.0103 \lambda_9 - 42 d \alpha \\ &= \delta L'_{19} - \delta L'_9 - 1.249 (\delta_{19} - \delta_9 + v_{19.9} - v_{9.19}) \\ + 0.43 - 0.0299 (\delta \varphi'_9 - \xi_9) - 0.0007 \lambda_9 + 75 \frac{d a}{a} - 9 d \alpha \\ &= \delta L'_{22} - \delta L'_{19} - 1.250 (\delta_{22} - \delta_{19} + v_{22.19} - v_{19.22}) \\ + 0.33 + 33 \frac{d a}{a} &= + 1.020 (v_{19.21} - v_{19.22}) - 0.480 (v_{21.19} - v_{21.23}) \\ &\quad + 0.223 (v_{22.19} - v_{22.23}) - 0.797 \delta s'_{19.21} + 1.013 \delta s'_{22.23} \\ &\quad + 0.017 \delta s'_{21.23} - 0.050 \delta s'_{19.22} \\ + 0.67 - 133 \frac{d a}{a} - 67 d \alpha &= - 0.993 (v_{19.21} - v_{19.22}) - 1.053 (v_{22.19} - v_{22.23}) \\ &\quad - 1.243 \delta s'_{19.21} - 1.840 \delta s'_{21.23} + 1.840 \delta s'_{19.22} + 0.623 \delta s'_{22.23} \\ + 4.00 + 145 \frac{d a}{a} + 42 d \alpha &= + v_{19.21} - v_{19.22} - v_{21.19} + v_{21.23} + v_{22.19} \\ &\quad - v_{22.23} - v_{23.21} + v_{23.22}. \end{aligned} \right\} \quad (16)$$

Before the adjustment there still arises the difficult problem of determining the weights for the different corrections, for which the accuracy of the individual measurements must be reconsidered. If we begin with the geodetic measurements, then we have first to determine the mean error of the unit of weight for the triangulation chains involved. If the value of this mean error is not known from the net adjustment, then there only remains the computation from the triangle closure errors. For the geodetic lines represented in Fig. 1, p. 456, the following triangle chains of the Prussian Land Survey are used:

Coast survey [Küstenvermessung]	$m = \pm 0.77''$
Chain of 1865	$m = \pm 0.64$
Weichsel chain	$m = \pm 0.57$
Chain of 1867	$m = \pm 0.52,$

for which the mean error of an angle of a triangle was taken from Jordan-Steppes' *Das deutsche Vermessungswesen*, Stuttgart, 1882, p. 134. Since each of the five geodetic lines passes through several chains, then we will simply form from the four mean errors a mean value, which results in

$$m = \pm 0.62''. \quad (1)$$

The mean error for the lengths of the geodetic lines would have to be computed in connection with the adjustment of the triangle chains in a similar way as was shown in Vol. I, 8th edition, 1935, section 72, p. 255, for a diagonal of the triangulation net of Hannover. Such a computation, however, will hardly be feasible in all rigor since the geodetic lines must in general be computed from several chains, already existing in an adjusted state. Therefore, we will only use an approximate method for the determination of the mean error.

If we choose, from the triangulation net between the points Springberg and Schönsee, as is represented in "Lotabweichungen, Heft III," a simple chain which connects the two points in the shortest way, then we obtain a chain of eleven triangles. Instead of this real chain we will imagine a hypothetical chain which consists of eleven equilateral triangles of equal size and runs rectilinearly; from the distance between the two points a side length of 29 km results for these imaginary triangles.

The propagation of error has been determined in the first half-volume, section 29, pp. 187-192,* for such an ideal triangulation chain. According to equation (11) there, section 29, p. 189,* we can set under the assumption that the first triangle side is free from error:

$$M_1 = \pm m \frac{s}{\rho} \cot 60^\circ \sqrt{\frac{4n^3 - 3n^2 + 5n}{3}}, \quad (2)$$

where n is the number of the triangle sides connecting the two points. In the case under consideration we have $n = 6$, $s = 29,000$ m and $m = \pm 0.62''$, from which the mean error for the length of the geodetic line results in ± 0.82 m. In view of the fact that the two end points are not connected only by a simple triangulation chain, we will assume the mean error somewhat smaller and set:

$$M = \pm 0.7 \text{ m}. \quad (3)$$

The remaining geodetic lines were treated similarly, and there resulted altogether the following squares of the mean errors:

Geod. Line	Square of Mean Error	}	(4)
Springberg-Schönsee	0.5		
Springberg-Turmberg	0.5		
Turmberg-Trunz	0.1		
Schönsee-Trunz	0.4		

* Not translated.

The above computation is based on the simplest case of an ideal triangulation chain, in which in each triangle only the three angles are measured. The problem becomes more difficult if direction measurements are assumed, or if diagonals of quadrilaterals are added. A thorough treatment of this problem is found in Paul Simon's *Gewichtsbestimmung für Seitenverhältnisse in schematischen Dreiecksnetzen*, Berlin, 1889.

Now we pass over to determine the mean error for the direction of the geodetic lines. From the triangulation chains there is computed, in addition to the length of the line, also the angle which it makes with the first and the last triangle side. We will treat the determination of the mean error of this angle likewise only as a first approximation.

We assume in Fig. 1 that from the triangulation chain illustrated the geodetic line $P_1 P_n$ is computed. Through the errors of measurement, there will be caused, in addition to a length displacement of point P_n ,

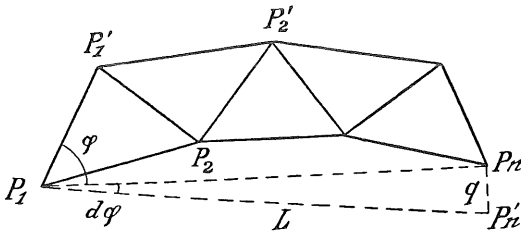


Fig. 1.

also a transverse displacement $P_n P'_n$, by which the angle φ is changed by the amount $d\varphi$. If we consider only the line path $P_1 P_2 \dots P_n$ instead of the triangulation chain and also disregard the curvature of the earth's surface, then we can apply the error theory of the traverse line from Volume II, 1st half-volume, 1931, section 118.* According to equation (8), p. 554, *ibid.*, the mean transverse displacement for the end point of a straight polygon line joined on one side, with equal sides, is

$$q = \pm m \frac{L}{\rho} \sqrt{\frac{n(2n-1)}{6(n-1)}},$$

where L denotes the length of the line, n the number of the polygon points and m the mean error of an angle. From this, there results for the connecting line $P_1 P_n$ a mean twisting which is to be set equal to

$\frac{q}{L} \rho$. Therefore, we have

$$m_\varphi = \pm m \sqrt{\frac{n(2n-1)}{6(n-1)}}. \quad (5)$$

We will apply this to our five geodetic lines while we assume the triangulation chains concerned as running straight.

For the line Rauenberg-Springberg we have $n = 9$; therefore the surd in (5) is equal to 1.79. With the help of (1), p. 463, we have then

$$m_\varphi = \pm 1.11''.$$

If we consider further that the two end points P_1 and P_n are also connected by the line $P_1 P'_1 P'_2 \dots P'_n$, and that the chain consists in part of double triangles, it appears appropriate to decrease the value of m_φ somewhat further, for which reason we will assume approximately

$$m_\varphi = \pm 1.00''.$$

This is the mean error of the angle between the geodetic line and the adjoining triangle side.

Now we will in addition separate from this the mean error of the connecting direction $P_1 P'_1$, whose square is equal to $\frac{1}{2} m^2$, and hence, according to (1) equal to 0.19. Therefore, the square of the mean error for the direction of the geodetic line Rauenberg-Springberg at its two end points is approximately equal to 0.80.

We list at once the squares of the mean errors for the direction of the geodetic lines. We have

* Not translated.

Geod. Line	Square of Mean Error	
Rauenberg-Springberg	0.80	}
Springberg-Schönsee	0.60	
Springberg-Turmberg	0.50	
Turmberg-Trunz	0.20	
Schönsee-Trunz	0.50	

(6)

These accuracy data, too, can be regarded as sufficient only for the theoretical example under consideration. For accurate computations, it will be necessary to take the form of the triangle chains more rigorously into account. Examples for this are found in L. Krüger's "Lotabweichungen, Heft V," Berlin, 1916, pp. 65-72.

The accuracy of the astronomic azimuth determination at the three Laplace points Rauenberg, Springberg and Schönsee can be determined from the observation series at each station. We will limit ourselves to introduce mean values by following the publication: "Die europäische Längengradmessung in 52° Breite von Greenwich bis Warschau," II. Heft, Berlin, 1896, and assuming for Rauenberg the mean square of the error 0.21, for the remaining points 0.33. In addition, there is still to be examined if the azimuth measurement refers directly to a triangle side or to an auxiliary target; however, in view of the fact that in the case of our weight determinations extensive omissions are admissible, we will disregard this distinction here.

As was already noted on p. 462, we will in addition combine these mean azimuth errors with the mean error of the connecting direction whose square is equal to 0.19 according to (1), p. 463. Rounding off, we therefore have the following mean squares of the error of the astronomical orientations

Rauenberg	(9)	0.40	}	(7)
Springberg	(19)	0.50		
Schönsee	(22)	0.50		

The mean errors for the astronomic longitudes are found best by adjustment of all measured differences of longitude. Such an adjustment was published by van de Sande-Bakhuyzen in *Verhandlungen der vom 12. bis 18. September 1893 in Genf abgehaltenen Konferenz der permanenten Kommission der Internationalen Erdmessung*, Berlin, 1894, pp. 101-114. A new adjustment of the Central European longitude net was published by Th. Albrecht in *Astronomische Nachrichten*, Band 167, 1905, pp. 145-162. The adjustment of the longitude net is fashioned similarly as the adjustment of the angles measured at a station between several targets; as we can introduce, after the station adjustment, the adjusted directions into the adjustment of the triangulation net only with the help of Bessel's weight equations (cf. Volume I, 8th edition, 1935, sections 54 to 56), so weight equations would also have to be set up for the adjusted longitudes, in order to be able to adjust them once more in the astronomic-geodetic net. By analogy to the approximate weights of direction (Volume I, 8th edition, 1935, section 91), however, we can also determine an approximate weight for each adjusted longitude and then regard the individual longitudes as independent of one another. This weight determination has been carried out for the points of the European longitude measurement and for the points of the astronomic-geodetic net lying north of it, and we take from: "Lotabweichungen, Heft V," p. 89, the mean squares of the error of the astronomic longitudes for Rauenberg, Springberg and Schönsee 0.31, 0.33 and 0.17. Rounding off, we therefore have the following squares of the mean errors of longitude

Rauenberg	(9)	0.30	}	(7a)
Springberg	(19)	0.30		
Schönsee	(22)	0.20		

Now we summarize once more the whole weight determinations of pp. 463-465 in the table on the following page:

Reciprocal Weights

Corr.	Recip- rocal Weight	Corr.	Recip- rocal Weight	Correction	Recip- rocal Weight	Corr.	Recip- rocal Weight
$\delta L'_9$	0.30	δ_9	0.40	$v_{9.19} , v_{19.9}$	0.80	$\delta s'_{19.21}$	0.50
$\delta L'_{19}$	0.30	δ_{19}	0.50	$v_{19.21} , v_{21.19}$	0.50	$\delta s'_{19.22}$	0.50
$\delta L'_{22}$	0.20	δ_{22}	0.50	$v_{19.22} , v_{22.19}$	0.60	$\delta s'_{21.23}$	0.10
				$v_{21.23} , v_{23.21}$	0.20	$\delta s'_{22.23}$	0.40
				$v_{22.23} , v_{23.22}$	0.50		

After everything has been prepared we can turn to the adjustment of the condition equations (16) of section 93, p. 462. In order to obtain simpler numbers, we will in addition multiply the reciprocal weights mentioned above by 10. We have computed the five normal equations with those and tabulated them clearly as follows:

k_1	k_2	k_3	k_4	k_5		$\delta \varphi'_9 - \xi_9$	λ_9	$\frac{d a}{a}$	$d a$
+ 45.00	- 10.81	.	.	.	- 6.35	-0.0425	-0.0103	.	- 42
- 10.81	+ 39.37	- 9.32	+ 15.35	- 15.00	+ 0.43	-0.0299	-0.0007	+ 75	- 9
	- 9.32	+ 20.90	- 6.74	+ 17.03	+ 0.33	.	.	+ 33	.
	+ 15.35	- 6.74	+ 52.63	- 22.51	+ 0.67	.	.	- 133	- 67
	- 15.00	+ 17.03	- 22.51	+ 36.00	+ 4.00	.	.	+ 145	+ 42

The solution of the normal equations yielded the following values of the correlates:

$$\begin{aligned}
 k_1 &= -0.1442 - 0.0013 (\delta \varphi'_9 - \xi_9) - 0.0003 \lambda_9 + 1 \frac{d a}{a} - 1 d a \\
 k_2 &= -0.0124 - 0.0014 \quad , \quad -0.0001 \quad , \quad + 5 \quad , \quad . \\
 k_3 &= -0.1538 - 0.0003 \quad , \quad . \quad - 2 \quad , \quad - 1 d a \\
 k_4 &= + 0.0998 + 0.0003 \quad , \quad . \quad - 2 \quad , \quad - 1 \quad , \\
 k_5 &= + 0.2410 - 0.0003 \quad , \quad . \quad + 6 \quad , \quad + 1 \quad , .
 \end{aligned}$$

With the help of these correlates and of the coefficients of the condition equations (16) of p. 462 the corrections were computed.

$$\begin{aligned}
 \delta L'_9 &= + 0.43'' + 0.0038 (\delta \varphi'_9 - \xi_9) + 0.0008 \lambda_9 - 3 \frac{d a}{a} + 3 d a \\
 \delta L'_{19} &= - 0.40 + 0.0004 \quad , \quad - 0.0004 \quad , \quad - 11 \quad , \quad - 3 \quad , \\
 \delta L'_{22} &= - 0.02 - 0.0028 \quad , \quad - 0.0002 \quad , \quad + 9 \quad , \quad . \\
 \delta_9 &= - 0.72 - 0.0064 \quad , \quad - 0.0013 \quad , \quad + 6 \quad , \quad - 5 \quad , \\
 \delta_{19} &= + 0.82 - 0.0008 \quad , \quad + 0.0009 \quad , \quad + 22 \quad , \quad + 6 \quad , \\
 \delta_{22} &= + 0.08 + 0.0087 \quad , \quad + 0.0007 \quad , \quad - 29 \quad , \quad . \\
 v_{9.19} &= - 1.44 - 0.0127 \quad , \quad - 0.0025 \quad , \quad + 11 \quad , \quad - 9 \quad , \\
 v_{19.9} &= + 1.44 + 0.0127 \quad , \quad + 0.0025 \quad , \quad - 11 \quad , \quad + 9 \quad , \\
 v_{19.22} &= . - 0.0053 \quad , \quad - 0.0004 \quad , \quad + 1 \quad , \quad - 5 \quad , \\
 v_{22.19} &= + 0.70 + 0.0068 \quad , \quad + 0.0005 \quad , \quad + 7 \quad , \quad + 11 \quad , \\
 v_{19.21} &= - 0.08 - 0.0042 \quad , \quad - 0.0003 \quad , \quad + 29 \quad , \quad + 4 \quad , \\
 v_{21.19} &= - 0.84 + 0.0021 \quad , \quad + 0.0002 \quad , \quad - 24 \quad , \quad - 3 \quad , \\
 v_{21.23} &= + 0.33 - 0.0008 \quad , \quad - 0.0001 \quad , \quad + 10 \quad , \quad + 1 \quad , \\
 v_{23.21} &= - 0.48 + 0.0005 \quad , \quad . \quad - 11 \quad , \quad - 2 \quad , \\
 v_{22.23} &= - 0.51 + 0.0030 \quad , \quad + 0.0002 \quad , \quad - 35 \quad , \quad - 9 \quad , \\
 v_{23.22} &= + 1.20 - 0.0014 \quad , \quad - 0.0001 \quad , \quad + 28 \quad , \quad + 6 \quad , \\
 \delta s'_{19.21} &= - 0.01 - 0.0003 \quad , \quad . \quad + 15 \quad , \quad + 11 \quad , \\
 \delta s'_{19.22} &= + 0.96 + 0.0024 \quad , \quad + 0.0002 \quad , \quad - 15 \quad , \quad - 8 \quad , \\
 \delta s'_{21.23} &= - 0.19 - 0.0005 \quad , \quad . \quad + 3 \quad , \quad + 2 \quad , \\
 \delta s'_{22.23} &= - 0.37 - 0.0007 \quad , \quad - 0.0001 \quad , \quad - 10 \quad , \quad - 7 \quad , .
 \end{aligned}$$

The correctness of these corrections was checked by introducing them into the condition equations (16), p. 462.

After the Laplace equation and the polygon equations have thus been brought into agreement by the adjustment, we have to determine the deflections of the vertical for the five points. Before, we will however indicate also the azimuth corrections of the geodetic lines according to (15), section 93, p. 462. We have

$$\begin{aligned}\delta \alpha'_{9.19} &= -2.16'' - 0.0191 (\delta \varphi'_9 - \xi_9) - 0.0038 \lambda_9 + 17 \frac{d a}{a} - 14 d a \\ \delta \alpha'_{19.9} &= +2.26 + 0.0119 \quad ,, \quad + 0.0034 \quad ,, + 12 \quad ,, + 15 \quad ,, \\ \delta \alpha'_{19.22} &= +0.82 - 0.0061 \quad ,, \quad + 0.0005 \quad ,, + 23 \quad ,, + 1 \quad ,, \\ \delta \alpha'_{22.19} &= +0.78 + 0.0155 \quad ,, \quad + 0.0012 \quad ,, - 22 \quad ,, + 11 \quad ,, \\ \delta \alpha'_{19.21} &= +0.74 - 0.0050 \quad ,, \quad + 0.0006 \quad ,, + 51 \quad ,, + 10 \quad ,, \\ \delta \alpha'_{21.19} &= -0.84 + 0.0021 \quad ,, \quad + 0.0002 \quad ,, - 24 \quad ,, - 3 \quad ,, \\ \delta \alpha'_{21.23} &= +0.33 - 0.0008 \quad ,, \quad - 0.0001 \quad ,, + 10 \quad ,, + 1 \quad ,, \\ \delta \alpha'_{23.21} &= -0.48 + 0.0005 \quad ,, \quad \quad \quad - 11 \quad ,, - 2 \quad ,, \\ \delta \alpha'_{22.23} &= -0.51 + 0.0030 \quad ,, \quad + 0.0002 \quad ,, - 35 \quad ,, - 9 \quad ,, \\ \delta \alpha'_{23.22} &= +1.20 - 0.0014 \quad ,, \quad - 0.0001 \quad ,, + 28 \quad ,, + 6 \quad ,.\end{aligned}$$

If we start with the computation of the deflections of the vertical of point (19) Springberg, then we have to introduce all corrections into equations (9), section 93, p. 460; for $\delta \varphi'_{19}$ and $\delta s'_{9.19}$ we have to substitute zero. With this, there follows:

$$\begin{aligned}\xi_{19} &= -6.26'' - 0.999 (\delta \varphi'_9 - \xi_9) - 0.027 \lambda_9 + 2474 \frac{d a}{a} - 355 d a \\ \lambda_{19} &= -3.41 - 0.078 \quad ,, \quad + 1.015 \quad ,, + 11882 \quad ,, + 7469 \quad ,, \\ \lambda_{19} &= -3.41 - 0.079 \quad ,, \quad + 1.014 \quad ,, + 11883 \quad ,, + 7470 \quad ,.\end{aligned}$$

Both computations of λ_{19} show sufficient agreement.

By calculating the systems (10), (11), (12) and (13) of section 93 likewise, we obtain, with sufficient checks, the following values of the deflections of the vertical:

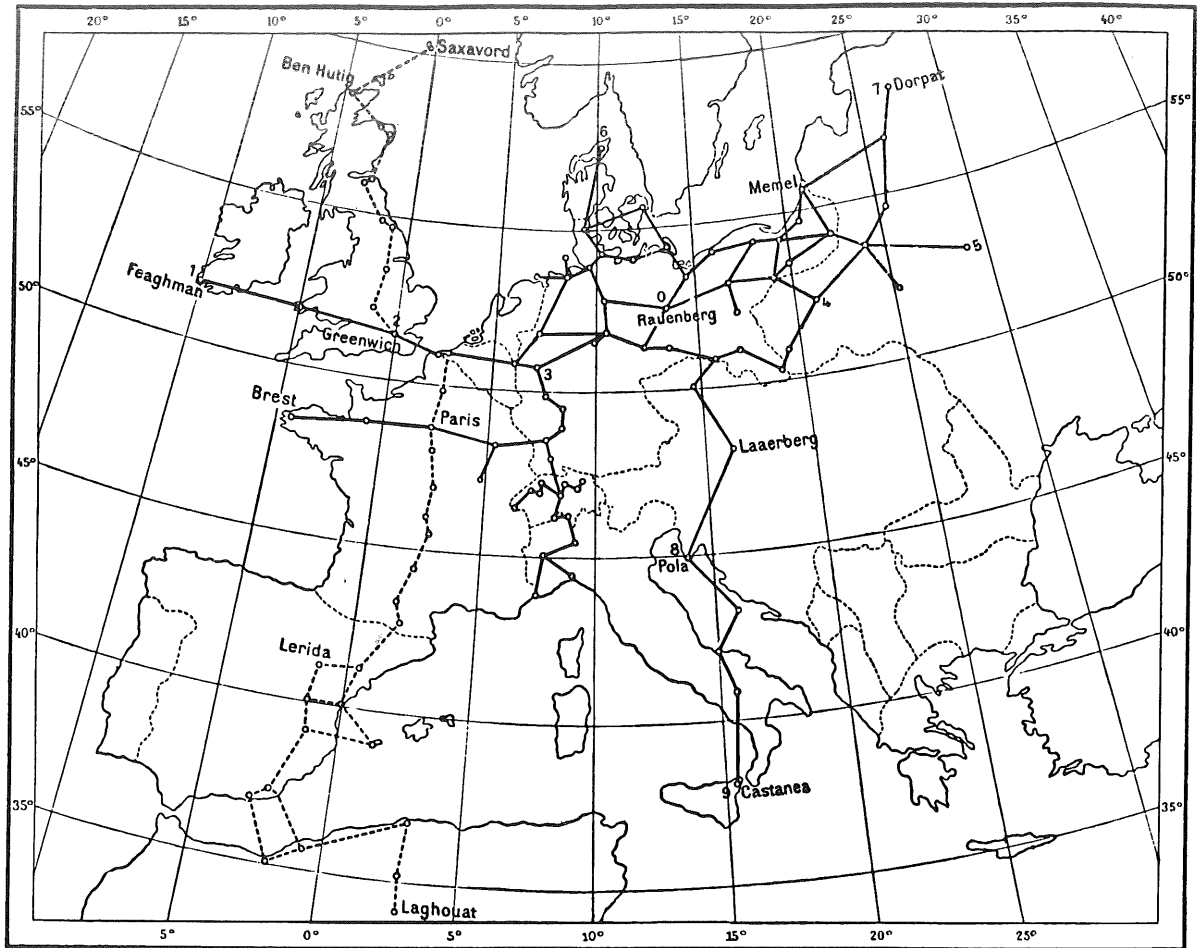
$$\begin{aligned}\xi_{19} &= -6.26'' - 0.999 (\delta \varphi'_9 - \xi_9) - 0.027 \lambda_9 + 2474 \frac{d a}{a} - 355 d a \\ \lambda_{19} &= -3.41 - 0.078 \quad ,, \quad + 1.014 \quad ,, + 11882 \quad ,, + 7470 \quad ,, \\ \xi_{21} &= -5.24 - 0.998 \quad ,, \quad - 0.040 \quad ,, + 6032 \quad ,, - 664 \quad ,, \\ \lambda_{21} &= -0.32 - 0.126 \quad ,, \quad + 1.024 \quad ,, + 17624 \quad ,, + 11064 \quad ,, \\ \xi_{22} &= -0.95 - 0.997 \quad ,, \quad - 0.047 \quad ,, + 2064 \quad ,, - 542 \quad ,, \\ \lambda_{22} &= -9.98 - 0.135 \quad ,, \quad + 1.012 \quad ,, + 20234 \quad ,, + 12709 \quad ,, \\ \xi_{23} &= -2.51 - 0.996 \quad ,, \quad - 0.052 \quad ,, + 5777 \quad ,, - 812 \quad ,, \\ \lambda_{23} &= -12.90 - 0.176 \quad ,, \quad + 1.021 \quad ,, + 22983 \quad ,, + 14359 \quad ,.\end{aligned}$$

From these equations we can compute the deflections of the vertical of the five points with reference to an arbitrary reference ellipsoid, as well as for arbitrary values of ξ_9 and λ_9 .

If we deal with a land survey, then we will use the above equations of the deflections of the vertical with the omission of $\delta \varphi'_9$ to determine the values of ξ_9 and λ_9 for the starting point Rauenberg in such a way that all deflections of the vertical are as small as possible. If the Bessel ellipsoid is to be retained here, then $d a$ and $d a$ have to be set equal to zero. With this, the orientation of the land triangulation net on the ellipsoid is then given.

On the other hand, if the equations of the deflections of the vertical refer to a large territory, say to a whole continent, then we will also determine at the same time, with the help of these equations, the corrections $d a$ and $d a$ of the earth's dimensions in such a way that the final ellipsoid adapts itself as well as possible to the geoid within the continent concerned. We shall discuss these problems more thoroughly later in section 96.

The geodetic lines treated in the previous example belong to the astronomic-geodetic net of first order, which has been worked up by the Geodetic Institute in Potsdam. The following publications have hitherto been published about it:



F. R. Helmert, "Lotabweichungen, Heft I: Formeln und Tafeln, sowie einige numerische Ergebnisse für Norddeutschland," Berlin, 1886. Börsch and Krüger, "Die europäische Längengradmessung in 52° Breite von Greenwich bis Warschau, II. Heft," Berlin, 1896. A. Börsch and L. Krüger, "Lotabweichungen, Heft II: Geodätische Linien südlich der europäischen Längengradmessung in 52° Breite," Berlin, 1902. A. Börsch, "Lotabweichungen, Heft III: Astronomisch-geodätisches Netz I. Ordnung nördlich der europäischen Längengradmessung in 52° Breite," Berlin, 1906. A. Börsch, "Lotabweichungen, Heft IV: Verbindung der russisch-skandinavischen Breitengradmessung mit dem astronomisch-geodätischen Netz in Norddeutschland," Berlin, 1909. L. Krüger, "Lotabweichungen, Heft V: Ausgleichung des astronomisch-geodätischen Netzes I. Ordnung nördlich der europäischen Längengradmessung in 52° Breite," Berlin, 1916.

The sketch reproduced above, which contains all geodetic lines of the net, was designed according to this. In addition, there are entered the astronomic points of the latitude measurement extending from Saxavord in the Shetland Islands to Laghouat in North Africa. The numbering of individual points refers to the computations in section 96 following.

The adjustment of this astronomic-geodetic net has thus far been carried out only for the longitude degree-measurement as well as for the part lying north of it. To the latter there belongs our example of p. 456 whose adjustment results agree in the main with the net adjustment in "Lotabweichungen, Band V." In the latter publication, the results of the longitude degree-measurement have also been adjusted to the net adjustment, so that the two adjustments yield a uniform system of deflections of the vertical.

On p. 452 we already pointed out that the adjustment explained in the previous sections is to be regarded only as an approximate adjustment. As the basis for a rigorous adjustment the author has set up a new theory which is published in *Verhandlungen der in Helsinki 1936 abgehaltenen 9. Tagung der Baltischen Geodätischen Kommission*, Helsinki, 1937, pp. 114-119.

The basic idea of this theory is that a triangulation chain is replaced by a geodetic line connecting its end points and by the two connecting angles between the first or, as the case may be, the last triangle side and the geodetic line. If we assume that the astronomic azimuths are measured for these two triangle sides, then we have the case which exists in the astronomic-geodetic net adjustment. Now the main point is to introduce the geodetic line s and the two connecting angles φ and ψ in Fig. 1 into the net adjustment in such a way that the adjustment becomes the same as in the case of the direct introduction of the measuring elements of the triangulation chain itself.

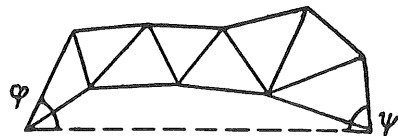


Fig. 1.

If we denote the latter by $l_1, l_2 \dots$ and the corrections which they receive in the adjustment of the chain by $v_1, v_2 \dots$, then we can set up s, φ and ψ as linear functions of the quantities $l + v$

$$\left. \begin{aligned} s &= f_1 (l_1 + v_1) + f_2 (l_2 + v_2) + \dots \\ \varphi &= f'_1 (l_1 + v_1) + f'_2 (l_2 + v_2) + \dots \\ \psi &= f''_1 (l_1 + v_1) + f''_2 (l_2 + v_2) + \dots \end{aligned} \right\} \quad (1)$$

or

$$\left. \begin{aligned} s &= s_0 + f_1 v_1 + f_2 v_2 + \dots \\ \varphi &= \varphi_0 + f'_1 v_1 + f'_2 v_2 + \dots \\ \psi &= \psi_0 + f''_1 v_1 + f''_2 v_2 + \dots \end{aligned} \right\} \quad (2)$$

Through the adjustment the three quantities obtain the values

$$\left. \begin{aligned} s &= s_0 + \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \\ \varphi &= \varphi_0 + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n \\ \psi &= \psi_0 + \gamma_1 v_1 + \gamma_2 v_2 + \dots + \gamma_n v_n \end{aligned} \right\} \quad (3)$$

There result further according to Volume I, 8th edition, section 109, the reciprocal weights

$$\left. \begin{aligned} \frac{1}{P_s} &= [\alpha \alpha] = [f f] - \frac{[a f]^2}{[a a]} - \frac{[b f \cdot 1]^2}{[b b \cdot 1]} - \frac{[c f \cdot 2]^2}{[c c \cdot 2]} - \dots \\ \frac{1}{P_\varphi} &= [\beta \beta] = [f' f'] - \frac{[a f']^2}{[a a]} - \frac{[b f' \cdot 1]^2}{[b b \cdot 1]} - \frac{[c f' \cdot 2]^2}{[c c \cdot 2]} - \dots \\ \frac{1}{P_\psi} &= [\gamma \gamma] = [f'' f''] - \frac{[a f'']^2}{[a a]} - \frac{[b f'' \cdot 1]^2}{[b b \cdot 1]} - \frac{[c f'' \cdot 2]^2}{[c c \cdot 2]} - \dots \end{aligned} \right\} \quad (4)$$

Let us still compute, to these, the additional weight coefficients $[\alpha \beta]$, $[\alpha \gamma]$ and $[\beta \gamma]$, for which we have according to Volume I, 1935, p. 470,*

$$\left. \begin{aligned} [\alpha \beta] &= [f f'] - \frac{[a f][a f']}{[a a]} - \frac{[b f \cdot 1][b f' \cdot 1]}{[b b \cdot 1]} - \frac{[c f \cdot 2][c f' \cdot 2]}{[c c \cdot 2]} - \dots \\ [\alpha \gamma] &= [f f''] - \frac{[a f][a f'']}{[a a]} - \frac{[b f \cdot 1][b f'' \cdot 1]}{[b b \cdot 1]} - \frac{[c f \cdot 2][c f'' \cdot 2]}{[c c \cdot 2]} - \dots \\ [\beta \gamma] &= [f' f''] - \frac{[a f'][a f'']}{[a a]} - \frac{[b f' \cdot 1][b f'' \cdot 1]}{[b b \cdot 1]} - \frac{[c f' \cdot 2][c f'' \cdot 2]}{[c c \cdot 2]} - \dots \end{aligned} \right\} \quad (5)$$

* Not translated.

We can also write equations (4) and (5) in an abbreviated manner if ν denotes the number of the condition equations in the triangulation chain

$$\left. \begin{aligned} [\alpha\alpha] &= [f f \cdot \nu] & [\beta\beta] &= [f' f' \cdot \nu] & [\gamma\gamma] &= [f'' f'' \cdot \nu] \\ [\alpha\beta] &= [f f' \cdot \nu] & [\alpha\gamma] &= [f f'' \cdot \nu] & [\beta\gamma] &= [f' f'' \cdot \nu] \end{aligned} \right\} \quad (6)$$

The six weight coefficients can thus be found at once in connection with the reduction of the normal equations.

Instead of n corrections v_1, v_2, \dots, v_n of the triangulation chain we imagine now three hypothetical corrections $\lambda_1, \lambda_2, \lambda_3$, through which the three quantities s, φ and ψ obtain the values

$$\left. \begin{aligned} s &= s_0 + \alpha_1' \lambda_1 + \alpha_2' \lambda_2 + \alpha_3' \lambda_3 \\ \varphi &= \varphi_0 + \beta_1' \lambda_1 + \beta_2' \lambda_2 + \beta_3' \lambda_3 \\ \psi &= \psi_0 + \gamma_1' \lambda_1 + \gamma_2' \lambda_2 + \gamma_3' \lambda_3 \end{aligned} \right\} \quad (7)$$

These three values (7) will entirely agree with the values (3) if

$$\left. \begin{aligned} [\alpha' \alpha'] &= [\alpha\alpha] & [\beta' \beta'] &= [\beta\beta] & [\gamma' \gamma'] &= [\gamma\gamma] \\ [\alpha' \beta'] &= [\alpha\beta] & [\alpha' \gamma'] &= [\alpha\gamma] & [\beta' \gamma'] &= [\beta\gamma] \end{aligned} \right\} \quad (8)$$

Six condition equations thus exist between the nine coefficients α', β', γ' of (7). Therefore, we can assume arbitrarily three coefficients with which the remaining six are then determined by means of equations (8).

We set $\beta_1' = 0, \gamma_1' = 0$ and $\gamma_2' = 0$, and thus have

$$\left. \begin{aligned} s &= s_0 + \alpha_1' \lambda_1 + \alpha_2' \lambda_2 + \alpha_3' \lambda_3 \\ \varphi &= \varphi_0 + \beta_2' \lambda_2 + \beta_3' \lambda_3 \\ \psi &= \psi_0 + \gamma_3' \lambda_3 \end{aligned} \right\} \quad (9)$$

These six coefficients can easily be computed, for we have for them the equations

$$\gamma_3'^2 = [\gamma\gamma] \quad \beta_3' \gamma_3' = [\beta\gamma] \quad [\alpha_3' \gamma_3'] = [\alpha\gamma],$$

whereby γ_3', β_3' and α_3' are determined. We have further

$$\beta_2'^2 + \beta_3'^2 = [\beta\beta],$$

with which β_2' becomes known, and so on.

The corrections λ_1, λ_2 and λ_3 have the weight 1 here, so that any further weight determination for s, φ and ψ vanishes.

The three values s, φ and ψ can be inserted into the astronomic-geodetic net adjustment in the form (9). We are to set in (2), section 92, p. 453,

$$\delta s' = \alpha_1' \lambda_1 + \alpha_2' \lambda_2 + \alpha_3' \lambda_3.$$

Further, $\delta \alpha_1'$ is composed of the correction of the azimuth measured astronomically at the beginning of the chain and the quantity $\beta_2' \lambda_2 + \beta_3' \lambda_3$, and $\delta \alpha_2'$ from the correction of the azimuth measured astronomically at the end of the chain and the quantity $\gamma_3' \lambda_3$.

If these corrections $\delta s', \delta \alpha_1'$ and $\delta \alpha_2'$ together with the corrections of the astronomic longitudes and latitudes $\delta L_1', \delta L_2', \delta \varphi_1', \delta \varphi_2'$ are introduced into the Laplace equations and the polygon equations, then the net adjustment can be carried out very rigorously.

We have hereby assumed that the triangulation chain is measured completely independent of the

neighboring chains. This, for instance, would not be the case if direction measurements are available for two adjacent chains, and the directions of the common sides hold for both chains. But here, too, the above method can be applied without further consideration if we introduce the common directions into the adjustment with half the weight. (Cf. in this connection "Jahresbericht des Direktors des Geodätischen Instituts für die Zeit von April 1939 bis März 1940," Potsdam, 1940, p. 7.)

Astronomic measurements at all triangulation points. In working up the Finnish main triangulation net, the attempt was made for the first time to measure astronomic longitudes, latitudes and azimuths at *all* triangulation points so that all triangulation points are also, at the same time, Laplace points. In this case, the Laplace equations must be introduced into the net adjustment, which, naturally, leads to a considerable work increase. On the other hand, the setting up of the Laplace equations is simplified due to the small length of the triangle sides compared with the long geodetic lines of the Helmert adjustment. An example of such an adjustment is contained in: *Veröffentlichung d. Finn. Geod. Inst.*, No. 8, "Ausgleichung einer Dreieckschette mit lauter Laplaceschen Punkten," by V. R. Ölander, Helsinki, 1927.

Additional remarks about the introduction of the Laplace equations into the net adjustment are contained in:

W. Jenne, "Einbeziehung Laplacescher Gleichungen in die geodätische Netzausgleichung nach bedingten Beobachtungen unter Anwendung des Entwicklungsverfahrens," *Mitteilungen des Reichsamts für Landesaufnahme*, 1933-34, pp. 286-299.

J. Schive, "Die Mitnahme der Laplace-Gleichung in die Netzausgleichung," *Astronomische Nachrichten*, Band 259, No. 6197, pp. 81-84.

The significance of the introduction of Laplace equations into the adjustment of an extended, in particular elongated, triangulation net lies in the fact that the astronomic azimuth measurements have a favorable effect here. By the accumulation of the inaccuracies in the angle measurement there are caused changes in form deformations in the case of elongated nets, which are counteracted by the astronomically measured azimuths in connection with the astronomic longitude measurements. For the net worked up in "Lotabweichungen, Heft V" (cf. p. 468), G. Schütz has repeated the adjustment by omitting the Laplace equations, whereby a deformation of the net clearly occurs. These computations are contained in *Veröffentlichungen d. Preuss. Geod. Inst.*, Neue Folge No. 101, "Systematische Fehler in geodätischen Netzen," by G. Förster and G. Schütz, Potsdam, 1929.

The Bowie method of the net adjustment

W. Bowie, the former director of the Coast and Geodetic Survey has indicated another form of adjustment of a triangulation net with Laplace points; this method was developed in detail by Adams and applied to the western half of the main triangulation net in the United States.

The net consists of a series of triangulation chains which run, in the main, in the direction of the meridians and parallels and intersect in junctions so that 15 closed loops are formed. In the normal case which we will assume here for the sake of simplicity, we imagine that in each junction there also lies a Laplace point as well as a base line. For the computational preparation, we combine at first the triangles adjoining into a junction as a junction net, which is adjusted independently, so that with the help of the base line contained in it, also all triangle sides of the junction net can be computed.

Then there follows the orientation of the junction net with the help of the astronomically measured azimuth. First, however, this azimuth must be reduced to the ellipsoid, for which the Laplace equation (18), section 89, p. 437, in the form

$$\alpha = \alpha' + (L - L') \sin \varphi \quad (10)$$

is used. In this, α and L denote the geodetic values, α' and L' the astronomic. We call the azimuth α obtained according to the above equation the *Laplace azimuth*.

In order to carry out this reduction, the geodetic longitude for the Laplace points is required. For its determination, the triangle chains between the junction nets are first adjusted individually, so that the lengths of the corresponding geodetic lines can be computed hence. Then, one of the Laplace points which is located at a favorable point is chosen as the central point for which the deflection of the vertical is assumed equal to zero, so that the values of longitude, latitude and azimuth, measured here astronomically, hold at the same

time as ellipsoidal values. With the help of this azimuth, all geodetic lines starting from the central point can then be oriented so that also the ellipsoidal longitudes for the neighboring Laplace points result. With these, everything is then known in order to convert the astronomic azimuths α' at these points into geodetic azimuths or Laplace azimuths α with the help of equation (10). With these, the junction nets of these points as well as the remaining geodetic lines starting from them are also oriented. The method is continued until, finally, the ellipsoidal longitudes and latitudes have been found for all junctions. By the fact that we obtain at each Laplace point a new orientation for the next geodetic line with the help of equation (10), the otherwise unavoidable lateral displacement of the consecutive points is prevented.

For the individual loops of the triangulation net there will then result closing errors in longitudes and latitudes, which are to be eliminated by means of an adjustment according to indirect observations. The differences of longitude and latitude between the junction points are regarded here as measured quantities, while the longitudes and latitudes themselves are the unknowns of the adjustment.

We see that the Bowie method can be regarded even less than that of Helmert as a rigorous adjustment in the sense of the method of least squares; however, it is suited for eliminating, with a relatively small expenditure of work, the discrepancies in a large triangulation net in a manner sufficient for the problems of land survey.

More details about the Bowie method of adjustment are contained in the publication: *U. S. Department of Commerce, Coast and Geodetic Survey*, "The Bowie method of triangulation adjustment as applied to the first-order net in the western part of the United States," by Oscar S. Adams, *Spec. Publ. No. 159*, Washington, 1930.

P. Gast gives also a short report about it in *Zeitschrift für Vermessungswesen*, 1927, pp. 270-273.

Section 96. Determination of the Orientation of a Triangulation Net

We have already indicated in section 94, p. 467, in which form the equations of the deflections of the vertical can be used for the determination of the deflections of the vertical ξ_0 and λ_0 of the central point of a land triangulation. If the equations of the deflections of the vertical exist for a rather great number of triangulation points, then ξ_0 and λ_0 shall be determined in such a way that the sum of squares of all deflections of the vertical Θ becomes as small as possible. For this, we pass over from the deflection of the vertical λ in longitude to the transverse component η , for which we have according to section 89, p. 435,

$$\eta = \lambda \cos \varphi \quad (1)$$

and have then according to (2), section 89, p. 434,

$$\Theta = \sqrt{\xi^2 + \eta^2}. \quad (2)$$

Therefore, the sum of all $\xi^2 + \eta^2$ is to be made as small as possible.

We will show this computation by an example by putting together a number of equations of deflections of the vertical from the net represented on p. 468. We use for this the following points:

- | | |
|----------------|---------------|
| (0) Rauenberg, | (5) Bobruisk, |
| (1) Feaghmain, | (6) Teglhoi, |
| (2) Greenwich, | (7) Dorpat, |
| (3) Bonn, | (8) Pola, |
| (4) Warschau, | (9) Castanea. |

For most of these points the equations of the deflections of the vertical could be taken from the preliminary computation given in "Lotabweichungen, Heft IV," pp. 96-97. For point (1) Feaghmain, the equations resulted from "Längengradmessung, II. Heft," p. 191, with reference to Greenwich; with the help of the reduction equation indicated in "Lotabweichungen, Heft IV," p. 94, these equations were reduced to the zero point Rauenberg. The equations for the points (8) Pola and (9) Castanea were set up according to the data from "Lotabweichungen, Heft II" in connection with the point Schneekoppe.

By passing over at once, according to equation (2), from the λ 's to η and setting

$$10,000 \frac{d a}{a} = u \quad 1000 d a = v$$

we obtain the following equations:

$$\begin{aligned} \xi_0 &= \quad + 1.00 \xi_0 \quad . \quad . \quad . \\ \xi_1 &= -4.15'' + 0.95 \xi_0 + 0.33 \eta_0 - 1.03 u - 0.50 v \\ \xi_2 &= -7.25 + 0.98 \xi_0 + 0.18 \eta_0 - 1.62 u - 0.12 v \\ \xi_3 &= -0.50 + 1.00 \xi_0 + 0.09 \eta_0 - 0.68 u + 0.06 v \\ \xi_4 &= -0.35 + 0.99 \xi_0 - 0.11 \eta_0 - 0.17 u - 0.04 v \\ \xi_5 &= -6.70 + 0.97 \xi_0 - 0.22 \eta_0 - 0.13 u - 0.26 v \\ \xi_6 &= -6.50 + 1.00 \xi_0 + 0.04 \eta_0 + 1.80 u + 0.01 v \\ \xi_7 &= -2.50 + 0.98 \xi_0 - 0.19 \eta_0 + 1.86 u - 0.10 v \\ \xi_8 &= -9.44 + 1.00 \xi_0 - 0.01 \eta_0 - 2.73 u + 0.84 v \\ \xi_9 &= -4.03 + 1.00 \xi_0 - 0.03 \eta_0 - 5.12 u + 2.46 v \\ \eta_0 &= \quad + 1.00 \eta_0 \quad . \quad . \quad . \\ \eta_1 &= +0.02 - 0.32 \xi_0 + 0.95 \eta_0 - 5.11 u - 3.21 v \\ \eta_2 &= +2.42 - 0.18 \xi_0 + 0.98 \eta_0 - 2.92 u - 1.83 v \\ \eta_3 &= -5.01 - 0.08 \xi_0 + 1.00 \eta_0 - 1.38 u - 0.86 v \\ \eta_4 &= -1.86 + 0.11 \xi_0 + 1.00 \eta_0 - 1.68 u + 1.06 v \\ \eta_5 &= -4.99 + 0.22 \xi_0 + 0.97 \eta_0 + 3.42 u + 2.15 v \\ \eta_6 &= -0.31 - 0.04 \xi_0 + 0.97 \eta_0 - 0.68 u - 0.43 v \\ \eta_7 &= -6.76 + 0.19 \xi_0 + 0.98 \eta_0 + 2.91 u + 1.83 v \\ \eta_8 &= -7.98 - 0.06 \xi_0 + 1.00 \eta_0 + 0.10 u + 0.07 v \\ \eta_9 &= -1.01 - 0.03 \xi_0 + 1.08 \eta_0 + 0.47 u + 0.33 v . \end{aligned}$$

We have also carried here, in addition, the corrections u and v of the earth's dimensions, although in view of the small extent of the net they have no importance.

Now we have to treat these 20 equations like error equations and obtain the following normal equations by assuming equal weights:

$$\begin{aligned} + 9.986 \xi_0 - 0.110 \eta_0 - 3.389 u + 4.753 v - 42.919 &= 0 \\ - 0.110 \xi_0 + 10.118 \eta_0 - 5.224 u - 0.939 v - 26.094 &= 0 \\ - 3.389 \xi_0 - 5.224 \eta_0 + 102.541 u + 19.818 v + 5.165 &= 0 \\ + 4.753 \xi_0 - 0.939 \eta_0 + 19.818 u + 30.892 v - 39.002 &= 0 . \end{aligned}$$

For the solution of the normal equations we set $u = 0$ and $v = 0$ and have then from the first two equations

$$\begin{aligned} + 9.986 \xi_0 - 0.110 \eta_0 - 42.919 &= 0 \\ - 0.110 \xi_0 + 10.118 \eta_0 - 26.094 &= 0 . \end{aligned}$$

The solution of these equations yields

$$\begin{aligned} \xi_0 &= + 4.33'' & \eta_0 &= + 2.53'' \\ \text{or} & & \lambda_0 &= + 4.15'' . \end{aligned}$$

With these, the ellipsoidal latitude and longitude can be computed for the central point Rauenberg from the astronomic latitude and longitude measured there; in addition, the measured azimuth can be reduced to the ellipsoid. Besides, through the equations of the deflections of the vertical in which, likewise, $u = 0$ and $v = 0$ are to be set, the deflections of the vertical of the remaining nine points are also known, with which all points have their final position on the ellipsoid.

Helmert found from general investigations the values

$$\xi_0 = + 5'' \quad \lambda_0 = + 4'',$$

which for Central Europe yield the most favorable connection of the reference ellipsoid to the geoid. (*Verhandlungen der vom 21. bis 29. Oktober 1887 auf der Sternwarte zu Nizza abgehaltenen Konferenz der Permanenten Kommission der Internationalen Erdmessung*, Berlin, 1888. Annex No. Ia, F. R. Helmert's "Bericht über Lotabweichungen." Cf. also *Verhandlungen 1888 in Salzburg*, Berlin, 1889, pp. 18 and following.) With this, our above result agrees very well.

In a major computation, Berroth has derived in *Zeitschrift für Vermessungswesen*, 1924, pp. 41-56 and 81-98, the deflections of the vertical for the central points of the land triangulation net in Prussia, Bavaria, Austria and Hungary on the best oriented Bessel ellipsoid. He finds hereby for Rauenberg

$$\xi_0 = + 2.40'' \quad \lambda_0 = + 3.32''.$$

The orientation of the Prussian triangulation net was carried out on the basis of an azimuth measured by Baeyer in 1859 at the point Rauenberg to St. Mary's Church in Berlin, for which the value $\alpha' = 19^\circ 46' 4.87''$ was found. For the latitude of Rauenberg, the value $\varphi' = 52^\circ 27' 12.021''$ had already been determined in 1853 by transformation from the Berlin Observatory. ("Hauptdreiecke Band I," 2. Auflage, Berlin, 1870, Vorrede, p. V.)

A remeasurement of the azimuth 1886/87 by the Geodetic Institute yielded $\alpha' = 19^\circ 46' 8.70''$. If we assume the value $\lambda_0 = + 4''$ for the deflection of the vertical in longitude at Rauenberg and with this reduce the newer astronomic azimuth to the ellipsoid, then we obtain according to (7) and (14), section 89, p. 435 and p. 437,

$$\alpha = \alpha' - \lambda \sin \varphi$$

and the numerical computation yields

$$\begin{array}{r} \alpha' = 19^\circ 46' 8.70'' \\ \lambda \sin \varphi = \quad \quad 3.17 \\ \hline \alpha = 19^\circ 46' 5.53''. \end{array}$$

We see that the error in Baeyer's azimuth of 1859, which is taken as a basis for the orientation of the triangulation net without reduction to the ellipsoid, cancels for the most part so that the orientation on the ellipsoid is correct to within approximately 0.7". Moreover, this error is of no importance, since in the net greatly extended to the east and west, in which no Laplace points are used, rather great deformations must be assumed.

For the point Potsdam, Helmerdturm nominally assumed as the new central point of the German triangulation net, which was newly determined astronomically in 1917 and connected with the point Rauenberg by a triangulation chain, geodetic coordinates have been computed by E. Kohlschütter in *Zeitschrift für Vermessungswesen*, 1924, pp. 321-324.

Section 97. Computation of the Dimensions of the Terrestrial Spheroid

If we aim to determine the shape and size of the terrestrial spheroid from astronomic-geodetic measurements, then the deflections of the vertical [Lotabweichungen] which refer to the geoid must be freed from the effect of the irregularities of the masses at the earth's surface in accordance with the reduction of gravity measurements. The thus reduced deflections of the vertical refer then to the spheroid, which according to section 65, pp. 322 and 323, can be regarded in a very far-reaching approximation as an ellipsoid of rotation.

For the reduction of the deflections of the vertical we shall adopt the same method as in the case of the reduction of gravity measurements; only now we have to determine, in the first place, the horizontal component of the attraction of the masses instead of the vertical component.

First we consider the influence of the visible masses in the neighborhood of a station. For this, we imagine again the terrain around the station divided into sections by means of concentric circles and radial lines so that we obtain a great number of vertical columns between sea level and the earth's surface. The base of such a column is represented by the section $ABCD$ in Fig. 1, p. 475, which is determined by the two radii a_1 and a_2 and by the two azimuths α_1 and α_2 . In order to determine the attraction of the vertical column rising on the base $ABCD$ to the surface of the terrain, we consider a mass particle dm

belonging to this column at the elevation h above sea level, whose projection is represented by hachures in Fig. 1. If we denote by $d v$ the volume of the mass particle and by Θ its density, then we have

$$d v = a d \alpha d a d h, \quad (1)$$

and hence

$$d m = \Theta a d \alpha d a d h. \quad (2)$$

If f is the constant of attraction, then the force exerted by the mass particle $d m$ on a unit mass at point O is according to Newton's law of gravitation

$$d K = f \Theta \frac{a}{a'^2} d \alpha d a d h, \quad (3)$$

where a' denotes the distance of the mass particle from O . But

$$a' = \sqrt{a^2 + h^2}, \quad (4)$$

and hence we have

$$d K = f \Theta \frac{a}{a^2 + h^2} d \alpha d a d h. \quad (5)$$

For the deviation of the plumb at O , only the horizontal component of this force is of interest to us.

If a' forms with the horizontal plane the angle of inclination ε , then $\cos \varepsilon = \frac{a}{a'}$; therefore, we will have

$$d K \cos \varepsilon = f \Theta \frac{a^2}{(a^2 + h^2)^{\frac{3}{2}}} d \alpha d a d h$$

or

$$d K \cos \varepsilon = f \Theta \frac{1}{a \left(1 + \frac{h^2}{a^2}\right)^{\frac{3}{2}}} d \alpha d a d h. \quad (6)$$

Since we aim to compute instead of the true value of the deviation of the plumb [Lotablenkung] its two components in and perpendicular to the meridian, we also divide the force just found into the two corresponding components by setting:

$$\begin{aligned} d X &= d K \cos \varepsilon \cos \alpha \\ d Y &= d K \cos \varepsilon \sin \alpha. \end{aligned}$$

The substitution of the values (6) yields then

$$\left. \begin{aligned} d X &= f \Theta \frac{\cos \alpha}{a \left(1 + \frac{h^2}{a^2}\right)^{\frac{3}{2}}} d \alpha d a d h \\ d Y &= f \Theta \frac{\sin \alpha}{a \left(1 + \frac{h^2}{a^2}\right)^{\frac{3}{2}}} d \alpha d a d h. \end{aligned} \right\} \quad (7)$$

These two equations are to be integrated with respect to the three variables α , a and h whereby the limits of integration for the first two variables are α_1 and α_2 or a_1 and a_2 , respectively, while the third is to be integrated from the limit zero to the terrain elevation H above sea level.

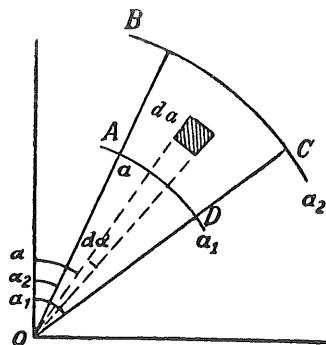


Fig. 1.

If we consider at first a and h as well as their differentials as constant quantities, then the integration with respect to α can easily be carried out, and then there only remain the two double integrals

$$\left. \begin{aligned} X &= \int_0^{\alpha_2} (\sin \alpha_2 - \sin \alpha_1) \int_{a_1}^{a_2} \int_0^H \frac{da dh}{a \left(1 + \frac{h^2}{a^2}\right)^{\frac{3}{2}}} \\ Y &= - \int_0^{\alpha_2} (\cos \alpha_2 - \cos \alpha_1) \int_{a_1}^{a_2} \int_0^H \frac{da dh}{a \left(1 + \frac{h^2}{a^2}\right)^{\frac{3}{2}}} \end{aligned} \right\} \quad (8)$$

If we regard a and da in this again as constant, then we have to determine the integral

$$\int_0^H \frac{dh}{\left(1 + \frac{h^2}{a^2}\right)^{\frac{3}{2}}} = \int_0^H \frac{dh}{\left(1 + \frac{h^2}{a^2}\right) \sqrt{1 + \frac{h^2}{a^2}}} \quad (9)$$

If we set, in order to find at first the indefinite integral,

$$\left. \begin{aligned} 1 + \frac{h^2}{a^2} &= x^2 \quad \text{or} \quad h = a \sqrt{x^2 - 1}, \\ \text{therefore} \quad dh &= \frac{a^2 x}{h} dx = \frac{a x}{\sqrt{x^2 - 1}} dx, \end{aligned} \right\} \quad (10)$$

with which the indefinite integral becomes:

$$\int \frac{dh}{\left(1 + \frac{h^2}{a^2}\right) \sqrt{1 + \frac{h^2}{a^2}}} = a \int \frac{dx}{x^2 \sqrt{x^2 - 1}}. \quad (11)$$

This integral can be determined at once by a further substitution. For if we set

$$x = \frac{1}{\cos y} \quad \text{or} \quad \sqrt{x^2 - 1} = \tan y, \quad \text{and hence} \quad dx = \frac{\sin y}{\cos^2 y} dy, \quad (12)$$

then we will have

$$a \int \frac{dx}{x^2 \sqrt{x^2 - 1}} = a \int \cos y dy = a \sin y. \quad (13)$$

Now since we have according to (12)

$$\sin y = \frac{\sqrt{x^2 - 1}}{x}$$

then we obtain:

$$a \int \frac{dx}{x^2 \sqrt{x^2 - 1}} = a \frac{\sqrt{x^2 - 1}}{x}. \quad (14)$$

If we pass over to the original variable h again, with the help of (10), then there results

$$\int \frac{dh}{\left(1 + \frac{h^2}{a^2}\right) \sqrt{1 + \frac{h^2}{a^2}}} = \frac{h}{\sqrt{1 + \frac{h^2}{a^2}}},$$

and the definite integral is:

$$\int_0^H \frac{d h}{\left(1 + \frac{h^2}{a^2}\right) \sqrt{1 + \frac{h^2}{a^2}}} = \frac{H}{\sqrt{1 + \frac{H^2}{a^2}}}. \quad (15)$$

With this, bearing in mind (9), equations (8) change to

$$\left. \begin{aligned} X &= f \Theta (\sin \alpha_2 - \sin \alpha_1) H \int_{a_1}^{a_2} \frac{d r}{a \sqrt{1 + \frac{H^2}{a^2}}} \\ Y &= -f \Theta (\cos \alpha_2 - \cos \alpha_1) H \int_{a_1}^{a_2} \frac{d a}{a \sqrt{1 + \frac{H^2}{a^2}}} \end{aligned} \right\} \quad (16)$$

Now there only remains a last integration, for which we have first to determine the indefinite integral

$$\int \frac{d a}{a \sqrt{1 + \frac{H^2}{a^2}}} = \int \frac{d a}{\sqrt{H^2 + a^2}}$$

For this, we set

$$a + \sqrt{H^2 + a^2} = x, \quad \text{therefore} \quad d x = \frac{\sqrt{H^2 + a^2} + a}{\sqrt{H^2 + a^2}} d a = \frac{x}{\sqrt{H^2 + a^2}} d a \quad (17)$$

and obtain:

$$\int \frac{d a}{\sqrt{H^2 + a^2}} = \int \frac{d x}{x} = \log \text{nat } x,$$

with which the definite integral becomes:

$$\int_{a_1}^{a_2} \frac{d a}{a \sqrt{1 + \frac{H^2}{a^2}}} = \log \text{nat} \frac{a_2 + \sqrt{H^2 + a_2^2}}{a_1 + \sqrt{H^2 + a_1^2}}. \quad (18)$$

The final expressions for the two force components are therefore

$$\left. \begin{aligned} X &= f \Theta H (\sin \alpha_2 - \sin \alpha_1) \log \text{nat} \frac{a_2 + \sqrt{H^2 + a_2^2}}{a_1 + \sqrt{H^2 + a_1^2}} \\ Y &= -f \Theta H (\cos \alpha_2 - \cos \alpha_1) \log \text{nat} \frac{a_2 + \sqrt{H^2 + a_2^2}}{a_1 + \sqrt{H^2 + a_1^2}} \end{aligned} \right\} \quad (19)$$

Since we can bring the fraction in the last factor of X and Y into the form

$$\frac{a_2 \left(1 + \sqrt{1 + \frac{H^2}{a_2^2}}\right)}{a_1 \left(1 + \sqrt{1 + \frac{H^2}{a_1^2}}\right)}$$

then we see that in most cases $\frac{a_2}{a_1}$ can be set for it. We thus obtain:

$$\left. \begin{aligned} X &= f \Theta H (\sin \alpha_2 - \sin \alpha_1) \log \operatorname{nat} \frac{a_2}{a_1} \\ Y &= -f \Theta H (\cos \alpha_2 - \cos \alpha_1) \log \operatorname{nat} \frac{a_2}{a_1} \end{aligned} \right\} \quad (20)$$

After we have set up these two horizontally acting force components, we must also compute, for the determination of the deviation of the plumb, the attraction of the terrestrial body directed to the center of the earth, which we will denote by Z . The attraction of a sphere on a mass point lying outside acts in such a way as if the whole mass of the sphere were combined at its center. If we denote by r the earth's radius and by Θ_m the mean density of the earth, then we have according to the law of gravitation:

$$Z = \frac{4}{3} f \frac{r^3 \pi \Theta_m}{r^2} = \frac{4}{3} f r \pi \Theta_m. \quad (21)$$

With this, we can indicate the deviation of the plumb at O , which is caused by the mass column rising on the base $ABCD$ (Fig. 1, p. 475). If the components of this deviation of the plumb are expressed by $\Delta \xi$ and $\Delta \eta$, then we have:

$$\left. \begin{aligned} \Delta \xi = \frac{X}{Z} &= \frac{3}{4} \frac{\Theta H \rho}{\Theta_m r \pi} (\sin \alpha_2 - \sin \alpha_1) \log \operatorname{nat} \frac{a_2}{a_1} \\ \Delta \eta = \frac{Y}{Z} &= -\frac{3}{4} \frac{\Theta H \rho}{\Theta_m r \pi} (\cos \alpha_2 - \cos \alpha_1) \log \operatorname{nat} \frac{a_2}{a_1}, \end{aligned} \right\} \quad (22)$$

where, because of the small amount of $\Delta \xi$ and $\Delta \eta$, the angles themselves were set instead of the trigonometric tangent.

Now we will in addition substitute in (22) the numerical values of the constants, for which we have:

$$\begin{aligned} \Theta &= 2.8 & r &= 6,370,000 \text{ m} \\ \Theta_m &= 5.6 & \rho &= 206,265'' \end{aligned}$$

and obtain:

$$\left. \begin{aligned} \Delta \xi &= +0.00386'' H (\sin \alpha_2 - \sin \alpha_1) \log \operatorname{nat} \frac{a_2}{a_1} \\ \Delta \eta &= -0.00386'' H (\cos \alpha_2 - \cos \alpha_1) \log \operatorname{nat} \frac{a_2}{a_1}. \end{aligned} \right\} \quad (23)$$

These two values are to be computed for each terrain section around the point O , and then the two components of the deviation of the plumb for the point O are equal to the algebraic sum of all values of $\Delta \xi$ or of $\Delta \eta$, as the case may be, or

$$\xi = [\Delta \xi] \quad \eta = [\Delta \eta]. \quad (24)$$

According to the above method, computations of deviations of the plumb have repeatedly been carried out, among other things in the work, *Die bayerische Landesvermessung in ihrer wissenschaftlichen Grundlage*, München 1873, which we have followed in the above exposition. While we refer, for further investigations, to F. R. Helmert, *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, Band II, sections 40 and 41, we limit ourselves to mention a new computation which was carried out by the land survey in the United States of North America and published in two papers by John F. Hayford in the years 1909 and 1910.

From these, we will discuss more closely a few further details.

In order to fashion the computations, which were extended to a great number of points, as conveniently as possible, there was chosen, for the radii necessary for the division of the terrain, a system which greatly simplified the equations (23). The division was carried out in such a way that for each individual section we have

$$\sin \alpha_2 - \sin \alpha_1 = 0.25 \quad \text{and} \quad \frac{a_2}{a_1} = 1.426. \quad (25)$$

If we compute H in feet (1 ft = 0.304 8006 m), then we will have according to the first equation (23):

$$\Delta \xi = + 0.000\ 1000'' H, \quad (26)$$

whereby any multiplication thus vanishes.

According to the directions (25) a diagram is constructed on celluloid, which can be laid on the topographic map. The computation extends hereby from circles with a radius of 8 m to approximately 4000 km. Within each single section, the mean elevation of the terrain is estimated on the basis of the elevation curves, from which $\Delta \xi$ results directly according to (26).

The same diagram can at the same time be used for the determination of the values of $\Delta \eta$, after its zero line has been turned in the east-west direction.

Those sections which fell on the sea level were treated according to the same method, whereby H was to be introduced in the negative sense, however. But in order to take into account also the water mass reaching to sea level, whose density is to be assumed equal to 1:2.6 of the mean density at the earth's surface, the quantity $0.615 H \left(\text{i.e. } H - \frac{1}{2.6} H \right)$ was introduced instead of H .

The comparison of the computed deviations of the plumb with the deflections of the vertical found from astronomic-geodetic measurements shows, to be sure, that the values of both series have in general the same signs, but that the absolute values of the first are by far larger than the latter. These differences result from the fact that thus far we have only taken into account the visible mass irregularities which are compensated to a great extent by subterraneous density defects, as we know from section 82, p. 404. We must therefore add an isostatic correction to the values computed according to (23).

The isostatic reduction of the deflections of the vertical

Under the assumption of Pratt's hypothesis of isostasy (cf. section 82, p. 404), we introduce again a surface of compensation at the depth T below sea level. If we denote the changes of $\Delta \xi$ and $\Delta \eta$ due to the density defect $\Delta \Theta$ by $\Delta \xi_0$ and $\Delta \eta_0$, then the sums $\Delta \xi + \Delta \xi_0$ and $\Delta \eta + \Delta \eta_0$ are the corrections which we must subtract from the astronomically determined deflections of the vertical ξ and η in order to obtain the deflections of the vertical reduced to the spheroid.

For the determination of $\Delta \xi_0$ and $\Delta \eta_0$ we can start from the equations (22), p. 478, and have then

$$\left. \begin{aligned} \Delta \xi_0 &= - \frac{3}{4} \frac{\Delta \Theta T \rho}{\Theta_m r \pi} (\sin \alpha_2 - \sin \alpha_1) \log \text{nat} \frac{a_2 + \sqrt{T^2 + a_2^2}}{a_1 + \sqrt{T^2 + a_1^2}} \\ \Delta \eta_0 &= + \frac{3}{4} \frac{\Delta \Theta T \rho}{\Theta_m r \pi} (\cos \alpha_2 - \cos \alpha_1) \log \text{nat} \frac{a_2 + \sqrt{T^2 + a_2^2}}{a_1 + \sqrt{T^2 + a_1^2}}, \end{aligned} \right\} \quad (27)$$

where in the last factor the more rigorous form of (19) was retained because of the large value of T . According to equation (2), section 82, p. 404, we have

$$\Delta \Theta T = \Theta H,$$

and hence, we can write the equations (27) also in the form:

$$\left. \begin{aligned} \Delta \xi_0 &= -\frac{3}{4} \frac{\Theta H \rho}{\Theta_m r \pi} (\sin \alpha_2 - \sin \alpha_1) \log \text{nat} \frac{a_2 + \sqrt{T^2 + a_2^2}}{a_1 + \sqrt{T^2 + a_1^2}} \\ \Delta \eta_0 &= +\frac{3}{4} \frac{\Theta H \rho}{\Theta_m r \pi} (\cos \alpha_2 - \cos \alpha_1) \log \text{nat} \frac{a_2 + \sqrt{T^2 + a_2^2}}{a_1 + \sqrt{T^2 + a_1^2}} \end{aligned} \right\} \quad (28)$$

These two equations agree with the previous equations (22), p. 478, except for the signs and the last factor. We can therefore indicate the sums $\Delta \xi + \Delta \xi_0$ and $\Delta \eta + \Delta \eta_0$, which we have to form for each section, at once from $\Delta \xi$ or, as the case may be, $\Delta \eta$ alone. For if we set

$$F = \frac{\Delta \xi + \Delta \xi_0}{\Delta \xi} = 1 - \frac{\log \frac{a_2 + \sqrt{T^2 + a_2^2}}{a_1 + \sqrt{T^2 + a_1^2}}}{\log \frac{a_2}{a_1}}, \quad (29)$$

then for a definite value of T we can set up a table, from which for every value a_1 (because $a_2 = 1.426 a_1$) the factor F can be taken. Then we have

$$\left. \begin{aligned} \Delta \xi + \Delta \xi_0 &= F \Delta \xi \\ \Delta \eta + \Delta \eta_0 &= F \Delta \eta \end{aligned} \right\} \quad (30)$$

The deviations of the plumb [Lotablenkungen] determined by means of (30) are the angles between the geoidal directions of the vertical and the normals to the spheroid, so far as the assumptions of the above theory can be regarded as correct. By means of the deflections of the vertical [Lotabweichungen] resulting from the astronomic-geodetic measurements, on the other hand, we have found the angles between the geoidal directions of the vertical and the normals to the reference ellipsoid taken as a basis for the geodetic computations. Therefore, if we subtract for each point the deviation of the plumb [angle between the direction of gravity and the normal to the spheroid] from the deflection of the vertical [angle between the direction of gravity and the normal to the ellipsoid], then we obtain as reduced deflections of the vertical the angles between the normals to the spheroid and the reference ellipsoid. Since the spheroid can likewise be considered as an ellipsoid of rotation according to section 65, p. 323, the reduced deflections of the vertical can now be used to determine the dimensions of the spheroid according to the method already described in section 96, p. 472.

The computations by Hayford were based on different values for the depth of compensation T , whereby it turned out that the value $T = 122.2$ km yielded a spheroid which adapts itself best to the reduced deflections of the vertical. For this spheroid there were obtained

$$a = 6,378,388 \text{ m} \quad a = 1:297.0.$$

The extensive computations are published in the two papers:

Department of Commerce and Labour, Coast and Geodetic Survey, Geodesy, The Figure of the Earth and Isostasy from Measurements in the United States, by John F. Hayford, Washington, 1909.

— — — *Supplementary Investigation in 1909 of the Figure of the Earth and Isostasy*, by John F. Hayford, Washington, 1910. (Cf. *Zeitschr. f. Verm.*, 1911, pp. 534-541.)

The question concerning the accuracy of the determination of T , which could not be answered by Hayford's method of computation, gave rise to Helmert's investigation "Über die Genauigkeit der Dimensionen des Hayfordschen Ellipsoids" in *Sitzungsberichte d. Kgl. Preuss. Ak. d. Wiss.*, math.-phys. Klasse, 1911, pp. 10-19. Helmert finds for the depth of the surface of compensation the value $T = 123.5 \pm 9.4$ km.

New computations of the earth's dimensions

Since we have reported in the introduction to the first half-volume on pp. 9-11 about the newer

operations of degree-measurements, we will in addition communicate the results of some computations which have been carried out more recently. Concerning computational works at the Geodetic Institute in Potsdam, there exist two publications by Helmert:

F. R. Helmert, "Die Grösse der Erde," *Sitzungsberichte d. Kgl. Preuss. Ak. d. Wiss., math.-phys. Klasse*, 1906, pp. 525-537. F. R. Helmert, "Geoid und Erdellipsoid," *Zeitschrift der Gesellschaft für Erdkunde zu Berlin*, 1913.

Retaining the value $\alpha = 1:299.15$ found by Bessel for the flattening, the following values result for the major semiaxis:

1. Russo-Scandinavian latitude degree-measurement

$$a = 6,378,455 \text{ m} \pm 127 \text{ m},$$

2. West European-African latitude degree-measurement

$$a = 6,377,935 \text{ m} \pm 155 \text{ m},$$

3. European longitude degree-measurement at 52° latitude

$$a = 6,378,057 \text{ m} \pm 105 \text{ m},$$

4. Russian longitude degree-measurement at $47\frac{1}{2}^\circ$ latitude

$$a = 6,377,350 \text{ m} \pm 650 \text{ m}.$$

If we form the mean from these four values, then there results with Bessel's flattening

$$a = 6,378,150 \text{ m}.$$

If the flattening $\alpha = 1:296.7$ found from gravity measurements is used (cf. section 64, 319), then the mean value of a is only a little smaller.

Now a few non-European degree-measurements are added to this:

5. the South African latitude degree-measurement

$$a = 6,378,307 \text{ m} \pm 179 \text{ m} \quad \text{for} \quad \alpha = 1:298.3,$$

6. the Indian longitude degree-measurements

$$a = 6,378,358 \text{ m} \pm 182 \text{ m} \quad \text{for} \quad \alpha = 1:298.3,$$

7. degree-measurements in the United States

$$a = 6,378,388 \text{ m} \pm 53 \text{ m} \quad \text{for} \quad \alpha = 1:297.0.$$

The question of the best values of the earth's dimensions at the present time is answered by Helmert by the formation of the mean of the six values of a resulting from the measurements in Europe, Asia and Africa, taking into account their mean errors. This yields

$$a = 6,378,192 \text{ m} \pm 94 \text{ m}.$$

If we include the North American measurements, whose weight is approximately equal to four times the weight of the above mean value, then there results

$$a = 6,378,350 \text{ m},$$

a value which is only 38 m smaller than that found in North America.

Further computations are carried out by W. Heiskanen in:

Veröff. d. Finn. Geod. Inst., No. 6, "Die Erddimensionen nach den europäischen Gradmessungen," by W. Heiskanen, Helsinki, 1926.

These computations are based on

1. the West European meridian arc,
2. the Russo-Scandinavian meridian arc,
3. the deflections of the vertical in Central Europe.

The new working up of these European degree-measurements is especially valuable because in contrast to the previous computations it introduces the topographic-isostatic reduction of the deflections of the vertical. By using the flattening $a = 1:297$ Heiskanen finds

$$a = 6,378,397 \pm 72 \text{ m,}$$

and hence a value of the equatorial radius, which almost accurately agrees again with the North American value. We have already indicated in the first half-volume, p. 46, that the North American ellipsoid has been accepted in 1924 as the *International Ellipsoid*. This assumption finds renewed justification by the computations of Heiskanen.

In the United States of North America, the computation of triangulations is based, as of old, on the terrestrial dimensions which Clarke computed in 1866 (cf. first half-volume, p. 8); the conversion to the new dimensions would have been connected with very great inconveniences in view of the large extent of the operations carried out to date.

The above-mentioned new terrestrial dimensions, however, have found acceptance in Finland in the new land triangulation, which was begun in 1918. The International Ellipsoid is also taken as a basis in the computations of the Baltic Geodetic Commission. In the first half-volume, pp. 86 and 87, we have reported about the auxiliary tables available for this.

The triaxial ellipsoid of the earth

We have already seen in section 83, p. 410, that the formula for normal gravity adapts itself better for the gravity measurements if we carry a term with the longitude in addition to the principal term dependent on the latitude, whereby an elliptic form of the equator is caused.

Several times there has also been made the attempt of determining the difference of the semiaxes of the equator from degree-measurements. We have a first computation by General T. F. von Schubert in *Mém. de l'Acad. imp. des sciences de St. Pétersbourg*, Series VIII, Tome I, 1859, No. 6, p. 1. The difference between the two equatorial semiaxes is here equal to 717 m, whereby the major semiaxis has the longitude $41^{\circ} 4'$ east of Greenwich. In addition, A. R. Clarke carried out two computations. The first is published in *Mem. of the R. A. Soc.*, vol. XXIX, 1861, where the difference of the semiaxes amounts to 1945 m, while the major axis lies at $15^{\circ} 34'$ east of Greenwich. In a second computation, in *Philosophical Magazine*, August 1878, Clarke found 464 m and $8^{\circ} 15'$ west of Greenwich.

In a treatise by W. Heiskanen: "Ist die Erde ein dreiachsiges Ellipsoid?" in *Gerl. Beitr. zur Geophysik*, Band XIX, 1928, pp. 356-377, the question is investigated on the basis of gravity measurements as well as on the basis of degree-measurements in Europe and in America. Both investigations lead to almost the same result that the difference of the two equatorial semiaxes amounts to approximately 200 m, and that the major semiaxis lies nearly at the meridian of Greenwich.

Finally, W. Heiskanen has once again treated the problem in *Veröff. d. Finn. Geod. Inst.*, No. 12, "Über die Elliptizität des Erdäquators," Helsinki, 1929. Here there is introduced into the equations of the deflections of the vertical of the European degree-measurements a term dependent on the longitude, from which the value $165 \pm 57 \text{ m}$ results for the difference of the semiaxes of the equator, whereby the major semiaxis lies $38^{\circ} \pm 10^{\circ}$ east of Greenwich.

The developments of formulae for geodetic computations on the triaxial ellipsoid are contained in *Veröff. d. Preuss. Geod. Inst.*, Neue Folge No. 98, "Untersuchungen über ein allgemeines Ellipsoid," von H. Schmehl, Potsdam, 1927.

The large systems of the deflection of the vertical hitherto treated have for their aim, in the first place, the determination of a reference ellipsoid which adapts itself as well as possible to the surface of the geoid or, as the case may be, the terrestrial spheroid in the survey region. At the same time, the remaining deflections of the vertical give a general picture of the course of the geoidal surface with respect to the reference ellipsoid found.

For a thorough study of the geoid, deflections of the vertical must be determined at such small intervals that the deflections of the vertical for the intermediate points can be interpolated with sufficient accuracy. In general, it is assumed that determinations of the deflections of the vertical at intervals of 5 km will be sufficient for this. If one of the measured points is at the same time a point of the astronomic-geodetic net of first order, then by means of measurements the shape and position of the geoidal surface with respect to the reference ellipsoid can be determined.

From this, there already follows that because of the great number of astronomic measurements such a special study can be considered only for a region of a moderate extent.

Astronomic levelling

In Fig. 1 let the points P and Q be two very closely neighboring points of the physical surface of the earth. Let point P have the elevation above sea level H , which is counted from point P' of the surface of the geoid. We assume further that $P''Q''$ denotes the surface of an arbitrarily assumed reference ellipsoid lying below P' at the depth N . The two straight lines PP'' and QQ'' are imagined as normals to the ellipsoid; these two straight lines, however, will deviate only a little from the normals to the level surfaces and the geoid because of the small inclination of the different surfaces to each other.

If at P the deflection of the vertical Λ with respect to the reference ellipsoid is determined, then Λ denotes the angle between the normal to the ellipsoid and the deflection of the vertical at P , and hence also the angle between the level surface passing through P and a parallel surface to the reference ellipsoid. The deflection of the vertical Λ' at P' , which refers to the geoid deviates somewhat from this. However, if we make the assumption that the elevation above sea level H of the point P is not very considerable, then we can regard the level surface at P as parallel to the surface of the geoid and neglect the difference between Λ and Λ' .

With the help of the distance ds there results then the elevation of the geoid between the two points P' and Q'

$$dN = Q'Q'' - P'P'' = \Lambda ds. \quad (1)$$

This equation (1) can be applied successively to a whole series of points, which form, in their totality, a length profile of the physical surface of the earth. The summation of all values dN yields then the total elevation of the geoid between the two end points. If we denote these by P_1 and P_2 , then we have

$$N_2 - N_1 = \int_{P_1}^{P_2} \Lambda ds. \quad (2)$$

This is the basic idea of *astronomic levelling* named after Helmert.

The actual execution of this method turns out simplest in meridional profiles, since the meridional components of the deflection of the vertical ξ can be determined very quickly and with sufficient accuracy by means of latitude determinations. We choose, for this, a series of points which lie nearly at the same

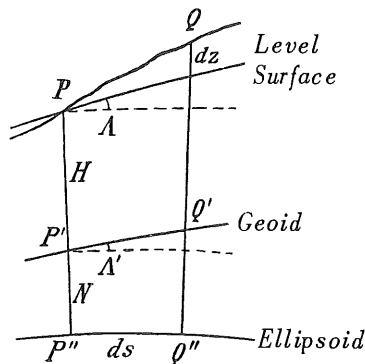


Fig. 1.

meridian and are also connected with each other by triangulation measurements. If the geographic latitude is determined at these points by astronomic measurements, then, by assuming an arbitrary value for the deflection of the vertical of one of these points we can also determine all remaining deflections of the vertical. We base the further working up best on a graphical representation by plotting the geographic latitudes of the points as abscissae and their deflections of the vertical as ordinates. If we connect the end points of the ordinates by a smooth curve, then we can read from it the deflections of the vertical at small intervals of latitude, say every minute. For these intervals, the values ξds are then computed, and by summation or mechanical quadrature there results the value of the integral (2).

Such astronomic levellings have already been carried out for a rather large number, of which we will indicate a few examples:

“Bestimmung der Polhöhe und der Intensität der Schwerkraft auf 22 Stationen von der Ostsee bei Kolberg bis zur Schneekoppe,” *Veröff. d. Kgl. Preuss. Geod. Inst. Berlin*, 1896.

Veröffentlichungen des Grossherz. Hessischen Kommissars für die Intern. Erdm., herausgegeben von Paul Fenner, Heft II, Darmstadt, 1909.

“Astronomisches Nivellement durch Württemberg etwa entlang dem Meridian $9^{\circ} 4'$ östlich von Greenwich,” bearbeitet von E. Hammer, *Veröff. d. Kgl. Württ. Komm. f. d. Intern. Erdm.*, Stuttgart, 1901.

“Zweites astronomisches Nivellement durch Württemberg im Meridian $8^{\circ} 33'$ östlich von Greenwich,” bearbeitet von E. Hammer, *Württ. Veröff. f. d. Intern. Erdm.*, Stuttgart, 1909.

Geoid profiles at a parallel are carried out less simply, since the determination of the component of the deflection of the vertical η is possible only by astronomic longitude or azimuth measurements. Such profiles, however, obtain a special significance for the connection of different meridian profiles, whereby the determination of a rather large section of area of the geoid becomes then possible.

As we have already indicated above, the forementioned theory is not entirely error-free, since, it is true, the deflection of the vertical Λ measured at the earth's surface at point P in Fig. 1, p. 483, indicates the angle between the level surface belonging to it and the reference ellipsoid, but not the angle Λ' between the surface of the geoid and the reference ellipsoid. At a rather large elevation above sea level H , the difference between Λ and Λ' , to which a curvature of the direction of the vertical passing through P thus corresponds, will already be noticeable. Instead of equation (1) we have then the correct equation

$$dN = \Lambda' ds = \Lambda ds - (\Lambda - \Lambda') ds \quad (3)$$

or

$$dN = \Lambda ds - dE, \quad (4)$$

if we set

$$(\Lambda - \Lambda') ds = dE \quad \int_{P_1}^{P_2} (\Lambda - \Lambda') ds = E \quad (5)$$

Then we have

$$N_2 - N_1 = \int_{P_1}^{P_2} \Lambda ds - E. \quad (6)$$

The computation of the correction term E , by which the curvature of the plumb line is taken into account has been treated by Helmert in the two publications: “Zur Bestimmung kleiner Flächenstücke des Geoids aus Lotabweichungen mit Rücksicht auf Lotkrümmung,” Erste Mitteilung, *Sitzungsberichte der Kgl. Preuss. Ak. d. Wiss. zu Berlin*, math.-physik. Klasse, 1900, pp. 964-982; Zweite Mitteilung. *Ibid.* 1901, pp. 958-975. According to this, we have the following theory.

The same correction term E , which occurs in (6) plays also a role if the point in question is to determine the difference of the elevations above sea level of two points P_1 and P_2 by geometric levelling. If we level, in Fig. 1, p. 483, from P to Q , then we obtain the elevation dz of Q above the level surface of P . The difference of the elevations above sea level is however, as we see from Fig. 1, p. 483,

$$dH = dz + \Lambda ds - \Lambda' ds = dz + dE,$$

and for the two points P_1 and P_2 we will then have

$$H_2 - H_1 = \int_{P_1}^{P_2} dz + E. \quad (7)$$

This equation can be utilized for the computation of E if, in addition, the acceleration of gravity g as well as its potential W is introduced. According to equation (18), section 61, p. 308, we have

$$\begin{aligned} dW &= -g dz, \\ \text{therefore} \quad W_2 - W_1 &= -\int_{P_1}^{P_2} g dz. \end{aligned} \quad (8)$$

Now we will understand by g_m an arbitrary constant mean value of gravity, which deviates as little as possible from the values of gravity measured between P_1 and P_2 , and write (8) in the form

$$W_2 - W_1 = -g_m \int_{P_1}^{P_2} dz - \int_{P_1}^{P_2} (g - g_m) dz,$$

and we find hence

$$-\int_{P_1}^{P_2} dz = \frac{W_2 - W_1}{g_m} + \int_{P_1}^{P_2} \frac{g - g_m}{g_m} dz. \quad (9)$$

This substituted in (7) yields

$$E = H_2 - H_1 + \frac{W_2 - W_1}{g_m} + \int_{P_1}^{P_2} \frac{g - g_m}{g_m} dz,$$

and if, in addition, a quantity W_0 is introduced:

$$E = \int_{P_1}^{P_2} \frac{g - g_m}{g_m} dz + \left(H_2 - \frac{W_0 - W_2}{g_m} \right) - \left(H_1 - \frac{W_0 - W_1}{g_m} \right). \quad (10)$$

W_0 shall be here the potential of gravity at sea level. If we denote further by \bar{g}_1 and \bar{g}_2 gravity below the two points P_1 and P_2 at half the elevation above sea level $\frac{H_1}{2}$ and $\frac{H_2}{2}$, then we can also set:

$$W_0 - W_2 = H_2 \bar{g}_2 \quad W_0 - W_1 = H_1 \bar{g}_1.$$

Consequently, we will have

$$E = \int_{P_1}^{P_2} \frac{g - g_m}{g_m} dz + H_2 \frac{g_m - \bar{g}_2}{g_m} - H_1 \frac{g_m - \bar{g}_1}{g_m}. \quad (11)$$

In the practical application of this expression for the correction term E the first term can be computed with sufficient certainty if the value of g along the profile from P_1 to P_2 is sufficiently known. However, the determination of the two further terms becomes more difficult, since the values of \bar{g}_1 and \bar{g}_2 are beyond accurate computation, and can only be found approximately.

Assuming that the terrain in the greater vicinity of a point can be regarded as horizontal, normal gravity for the point at the elevation above sea level H is according to (11), section 81, p. 403,

$$\gamma = \gamma_0 - \frac{2H}{r} \gamma_0 + \frac{3}{2} \frac{\Theta}{\Theta_m} \frac{H}{r} \gamma_0, \quad (12)$$

where r denotes the mean radius of the earth, Θ the mean density of the slab of terrain lying between the point and sea level and Θ_m the mean density of the whole terrestrial body.

For a point lying on the same plumb line but at sea level we obtain the normal gravity γ' , if we take into account the attracting effect of the slab lying between the two points; we have therefore

$$\gamma' = \gamma_0 - \frac{3}{2} \frac{\Theta}{\Theta_m} \frac{H}{r} \gamma_0. \quad (13)$$

The normal gravity at the mean elevation $\frac{H}{2}$ is thus

$$\bar{\gamma} = \frac{\gamma + \gamma'}{2} = \gamma_0 - \frac{H}{r} \gamma_0. \quad (14)$$

The true value \bar{g} of gravity will deviate more or less from the normal value $\bar{\gamma}$; therefore a correction $\delta \bar{g}$ is still to be added to (14). The latter, however, is only partly accessible to the computation.

For practical application, the above formulae (6) and (11) have been brought by Helmert into still another form, in which preliminary values E^* and N^* are first computed, which are free from the uncertainty of the determination of the mean values \bar{g} . In order to obtain the final values N we still have to compute small corrections which contain the above-mentioned quantities $\delta \bar{g}$. For these details we refer to the publications by Helmert already mentioned, as well as to the treatise by Galle, "Das Geoid im Harz," from which we will communicate a few further remarks.

The geoid in the Hartz Mountains

Already in the years 1873 and 1874, astronomic measurements for the determination of the deflections of the vertical in latitude were carried out at 10 stations in the region of the Hartz by order of the then president of the Royal Prussian Geodetic Institute, Lieutenant General z. D. Dr. Baeyer. Further measurements soon followed these first ones, and in 1881, at 43 stations there were already known the deflections of the vertical in latitude, which were used by C. G. Andrae for the determination of the form of the geoid in the Hartz and in the western part of the Thuringian Forest. This work is described in *Problèmes de haute géodésie*, 3^e cahier, Copenhagen, 1883. Since deflections of the vertical in longitude were not available, only meridian profiles of the geoid could be determined; however, Andrae introduced for the elevation of the profiles in the opposite direction an assumption which later proved approximately correct.

After Helmert had assumed in 1886 the directorship of the Geodetic Institute, there was also planned the determination of components of the deflection of the vertical in the eastern part in order to establish a connection of the meridional profiles. For this purpose, azimuth measurements were carried out at two parallel circles north and south of the Brocken at 13 stations, which were connected with the Brocken by means of a triangulation net by the trigonometric section of the Prussian Land Survey at the request of the Geodetic Institute.

The whole material was submitted to a thorough working by Galle whose results are communicated in two *Veröffentlichungen des Geodätischen Instituts*:

A. Galle, "Lotabweichungen im Harz und seiner weiteren Umgebung," Berlin, 1908, and A. Galle, "Das Geoid im Harz," Berlin, 1914.

Helmert gave a short report about these treatises in the publication: "Die Bestimmung des Geoids im Gebiete des Harzes," *Sitzungsber. d. Kgl. Preuss. Ak. d. Wiss.* 1913, pp. 550-560.

The basis for the determination of the deflections of the vertical is formed by a reference ellipsoid whose size and position with respect to the geoid is assumed by Helmert according to the investigations mentioned already on p. 474 in such a way that the point Rauenberg near Berlin receives the component of the deflection of the vertical $\xi = + 5''$ $\lambda = + 4''$ while the major semiaxis of the Bessel ellipsoid is enlarged by 0.0001 of its value. The position of the points on the reference ellipsoid is based on the system of the measurement of a degree of longitude at 52° latitude.

For the majority of the points which belongs to the triangulation net of the Prussian Land Survey, it was more convenient to take the geodetic longitudes and latitudes first from the lists of the land survey and to carry out the attachment to the longitude degree-measurement and Helmert's reference ellipsoid in another way. This attachment, however, could be reached directly for a few other points which belong to the Saxon triangulation net. For this, there was used the line Leipzig-Grossenhain of the longitude degree-measurement whose length and direction could also be computed from the Saxon net. From this, there resulted at once the correction to the scale and direction for the Saxon net. Since, furthermore, the components of the deflection of the vertical of Grossenhain with reference to Helmert's ellipsoid are also known from the equations of the longitude degree-measurement, then the components of the deflection of the vertical for the above-mentioned points of the Saxon net could also be computed.

The points of the Prussian triangulation net lying around the point Brocken were combined at first into a net which could be submitted to a uniform adjustment. With the help of the azimuth Brocken-Fallstein taken from the longitude degree-measurement, all azimuths of this Brocken net were found and by means of the attachment to the base line of Göttingen, whose correction to the scale of the longitude degree-measurement is also known already, there also resulted all side lengths.

The components of the deflection of the vertical of the point Brocken are according to the equations of the longitude degree-measurement for Helmert's reference ellipsoid

$$\xi = + 13.338'' \quad \lambda = + 4.134''.$$

With these starting values, the deflections of the vertical for all points of the Brocken net could be computed.

For the points of the Prussian Land Survey not belonging to the Brocken net, the corrections to the scale and direction could be determined in a similar way, so that also for these points the deflections of the vertical in the system of the longitude degree-measurement were found.

With these results, the curves of equal deflections of the vertical ξ in latitude were projected on a map. At the edges of the territory, individual uncertainties which made the supplementary insertion of 14 more stations necessary turned out here; the number of the points thus grew to 114.

For the interpolation of the values of gravity which are needed for the computation of E in equation (11) on p. 485, it would have suggested itself to project the lines of equal gravity on a map with the help of the measured values of gravity; because of the great dependence of gravity on elevation, it proved however more appropriate to subtract from the measured values of gravity their normal values, and to represent on the map only the disturbances of gravity.

With this, everything was prepared for the working up of the meridian profiles. East and west of the Brocken there were chosen 5 meridians each at $10'$ interval of longitude, at which the elevation of the geoid was computed for every $1'$ of geographic latitude. For these meridians, the profiles of these deflections of the vertical were drawn by means of the map of the deflections of the vertical ξ and the values of ξ were read from it for every $0.5'$ of latitude. After the computation of the products ξds there resulted, by mechanical quadrature, the sum of the ξds 's with reference to the northernmost point of each meridian as the starting point.

Then there followed the computation of the preliminary correction term E^* , with which the preliminary elevations of the geoid above the same starting point were found. In general, a rising of the geoid from north to south by about 2 m turned out here in the profiles.

After the 11 meridian profiles of the geoid were thus determined, their connection with the help of the

eastern components of the deflection of the vertical η was still missing. Of the 27 measured values of η , 20 lie in the neighborhood of two parallel circles which, it is true, do not intersect all meridian profiles, but are sufficient for their connection and also yield, in part, condition equations for an adjustment. These two parallel profiles were worked up similarly to the meridian profiles, and after the adjustment was carried out, all geoidal elevations could now be referred to a common starting point.

The absolute elevation of the geoid above the reference ellipsoid was based on previous computations by L. Krüger, which treat the profile of the geoid in the meridian of the Hartz and in which the elevation of the geoid above the reference ellipsoid was assumed equal to zero at the northern boundary of Schleswig-Holstein. For the northern parallel profile ($52^{\circ} 5'$) in the meridian of the Brocken there follows hence the elevation of the geoid $N^* = 2.250$ m, with which all remaining values N^* could then be computed.

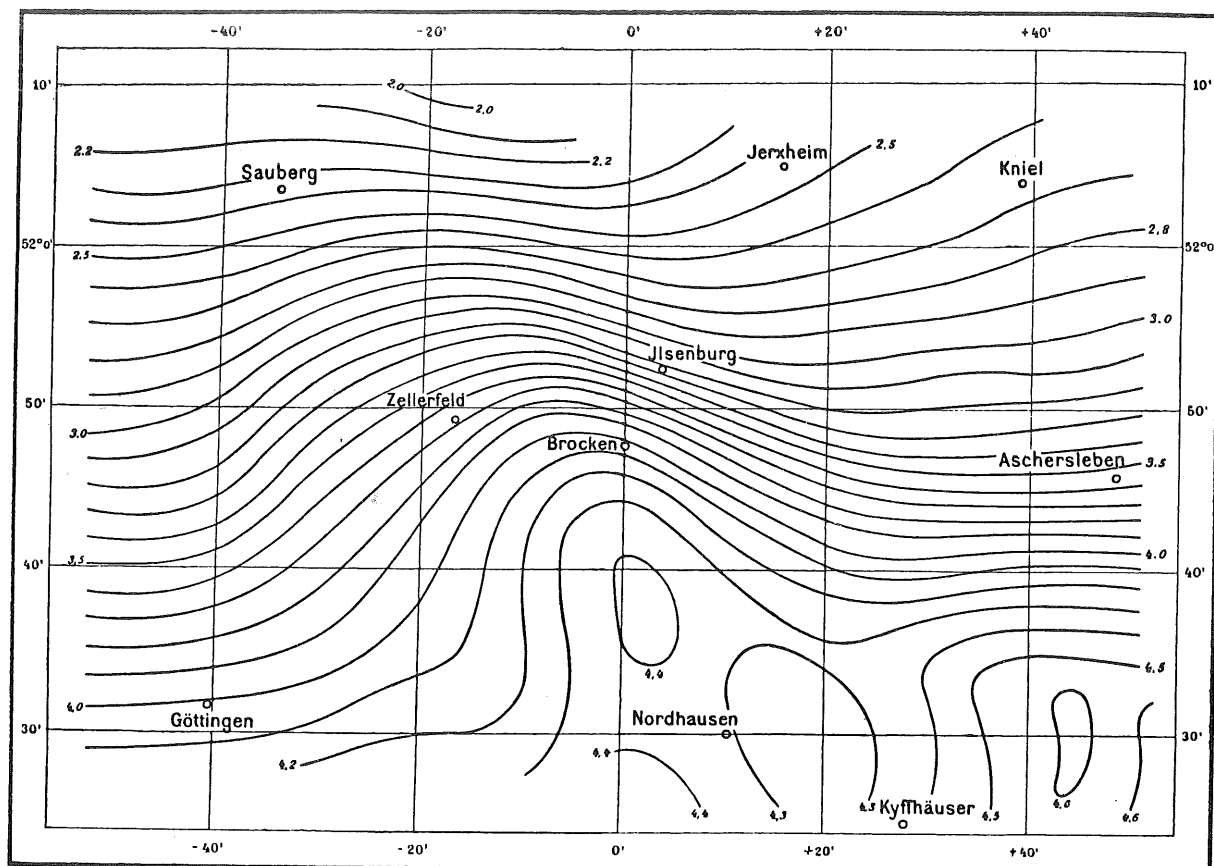


Fig. 2. The geoid in the Hartz.

As was already shown on p. 486 above, the preliminary elevations of the geoid N^* still need a small correction which depends on the gravity below the earth's surface and therefore cannot be determined perfectly accurately. However, the uncertainties thereby arising cannot exert a considerable influence on the elevations N since the corrections are very small, and reach the amount of 10 mm only in one case.

To illustrate the results there was designed a map at the scale 1:250,000, on which the elevation lines of the geoid with respect to the reference ellipsoid are represented to every decimeter. This map, which is reproduced at a reduced scale in Fig. 2, shows that the geoid in the territory represented rises from north to south by about 2 m and reaches its greatest height south of the Brocken.

Therefore, only small elevation changes of the geoid with respect to the reference ellipsoid occur here, and closer investigation teaches that the surface of the geoid remains convex everywhere at this point.

An additional astronomic levelling was carried out in the years 1916-1918 by the Swiss Geodetic Commission for the determination of the profile of the geoid in the meridian of St. Gothard. The two publications have appeared about it:

Also in this work, the curvatures of the plumb line have been taken into account. However, there was used here a new theory, which is developed by Niethammer and which, compared with Helmert's theory, shows the advantage that it is based on the measured gravity values instead of normal gravity.

The geoid remains in the course of the whole profile below the surface of the reference ellipsoid; the deepest point of the geoid lies 2.64 m below the ellipsoid. The geoid reaches the highest point in the massif of St. Gotthard; here, the influence of the curvature of the plumb line is also greatest; it amounts to 0.43 m.

Section 99. Measurement of Deflections of the Vertical by Means of the Torsion Balance

The torsion balance of Eötvös, about whose theory we have already learned in section 84, can also be used for the determination of deflections of the vertical within an astronomic-geodetic net.

We treat at first the transformation of the quantities found with the torsion balance to another direction of the x - and the y -axis. If the axes of the new system make with those of the old system the angle α and if we denote the coordinates in the new system by x' and y' , then we have

$$\left. \begin{aligned} x &= x' \cos \alpha - y' \sin \alpha \\ y &= x' \sin \alpha + y' \cos \alpha, \end{aligned} \right\} \quad (1)$$

while we keep $z = z'$.

Since the potential W is a function of x , y and z , but x and y are functions of x' and y' according to (1), then we obtain

$$\frac{\partial W}{\partial x'} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial x'} \quad \text{and} \quad \frac{\partial W}{\partial y'} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial y'} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial y'},$$

or according to (1)

$$\left. \begin{aligned} \frac{\partial W}{\partial x'} &= \frac{\partial W}{\partial x} \cos \alpha + \frac{\partial W}{\partial y} \sin \alpha \\ \frac{\partial W}{\partial y'} &= -\frac{\partial W}{\partial x} \sin \alpha + \frac{\partial W}{\partial y} \cos \alpha. \end{aligned} \right\} \quad (2)$$

By passing now over to the second differential quotients, we have

$$\frac{\partial^2 W}{\partial x'^2} = \frac{\partial}{\partial x} \frac{\partial W}{\partial x'} \frac{\partial x}{\partial x'} + \frac{\partial}{\partial y} \frac{\partial W}{\partial x'} \frac{\partial y}{\partial x'}$$

and with the equations (2) and (1)

$$\frac{\partial^2 W}{\partial x'^2} = \frac{\partial^2 W}{\partial x^2} \cos^2 \alpha + \frac{\partial^2 W}{\partial y^2} \sin^2 \alpha + \frac{\partial^2 W}{\partial x \partial y} \sin 2\alpha. \quad (3)$$

We obtain likewise from the equation

$$\frac{\partial^2 W}{\partial y'^2} = \frac{\partial}{\partial x} \frac{\partial W}{\partial y'} \frac{\partial x}{\partial y'} + \frac{\partial}{\partial y} \frac{\partial W}{\partial y'} \frac{\partial y}{\partial y'}$$

the further relation

$$\frac{\partial^2 W}{\partial y'^2} = \frac{\partial^2 W}{\partial x^2} \sin^2 \alpha + \frac{\partial^2 W}{\partial y^2} \cos^2 \alpha - \frac{\partial^2 W}{\partial x \partial y} \sin 2\alpha. \quad (4)$$

Thus we will have according to (3) and (4)

$$\frac{\partial^2 W}{\partial y'^2} - \frac{\partial^2 W}{\partial x'^2} = \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right) \cos 2\alpha - \frac{\partial^2 W}{\partial x \partial y} 2 \sin 2\alpha. \quad (5)$$

With this, the first of the quantities found with the torsion balance can be transformed to a new direction of the x - and the y -axis.

Further we still determine

$$\frac{\partial^2 W}{\partial x' \partial y'} = \frac{\partial}{\partial x} \frac{\partial W}{\partial x'} \frac{\partial x}{\partial y'} + \frac{\partial}{\partial y} \frac{\partial W}{\partial x'} \frac{\partial y}{\partial y'},$$

for which we obtain again with the use of equations (2) and (1)

$$\frac{\partial^2 W}{\partial x' \partial y'} = \frac{\partial^2 W}{\partial x \partial y} \cos 2\alpha + \left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \right) \frac{1}{2} \sin 2\alpha. \quad (6)$$

Since the z -axis is the same in both systems, then we still can indicate directly from (2):

$$\left. \begin{aligned} \frac{\partial^2 W}{\partial x' \partial z'} &= \frac{\partial^2 W}{\partial x \partial z} \cos \alpha + \frac{\partial^2 W}{\partial y \partial z} \sin \alpha \\ \frac{\partial^2 W}{\partial y' \partial z'} &= -\frac{\partial^2 W}{\partial x \partial z} \sin \alpha + \frac{\partial^2 W}{\partial y \partial z} \cos \alpha \end{aligned} \right\} \quad (7)$$

By introducing now a reference ellipsoid, we will assume that this ellipsoid deviates only little from the level surface hitherto considered, and that therefore the quantities found with the torsion balance also hold for the ellipsoid. We will refer the coordinate system from now on to the ellipsoid, whose normal shall coincide with the z -axis, while the x -axis shall lie in the north-south direction.

In Fig. 1 there is represented a vertical section through the z -axis in the north-south direction, so that ξ denotes the meridional component of the deflection of the vertical. If g and g_x indicate gravity and its meridional component, then we have according to Fig. 1

$$\sin \xi = -\frac{g_x}{g}.$$

We have likewise for the east-west component η of the deflection of the vertical the value

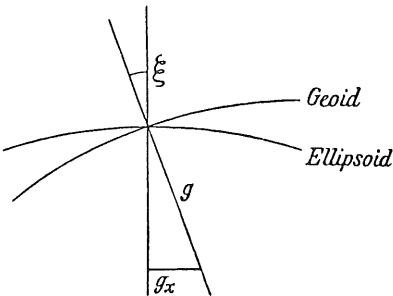


Fig. 1.

$$\sin \eta = -\frac{g_y}{g}.$$

For g_x and g_y we have according to (3a), section 84, p. 412, the values

$$g_x = \frac{\partial W}{\partial x} \quad g_y = \frac{\partial W}{\partial y},$$

and since we can also set ξ and η instead of $\sin \xi$ and $\sin \eta$, we have

$$\xi = -\frac{1}{g} \frac{\partial W}{\partial x} \quad \eta = -\frac{1}{g} \frac{\partial W}{\partial y}. \quad (8)$$

We will now assume that the measurements with the torsion balance have taken place at small intervals at a rather large number of points. At first we consider two of these points P_1 and P_2 and introduce, besides the original coordinate system x_y , a second one $x'y'$ whose x' -axis falls in the direction P_1P_2 . With the help of equation (6) we can then compute from the measurements at P_1 and P_2 the quantities

$$\left(\frac{\partial^2 W}{\partial x' \partial y'} \right)_1 \quad \text{and} \quad \left(\frac{\partial^2 W}{\partial x' \partial y'} \right)_2 \quad (9)$$

We have now

$$\int_{x'_1}^{x'_2} \frac{\partial^2 W}{\partial x' \partial y'} dx' = \int_{x'_1}^{x'_2} \frac{\partial}{\partial x'} \frac{\partial W}{\partial y'} dx' = \left(\frac{\partial W}{\partial y'} \right)_2 - \left(\frac{\partial W}{\partial y'} \right)_1.$$

But if the distance between the two points is so small that the change of the quantity $\frac{\partial^2 W}{\partial x' \partial y'}$ takes place uniformly from one point to the other, then we have on the other hand

$$\int_{x'_1}^{x'_2} \frac{\partial^2 W}{\partial x' \partial y'} dx' = \frac{1}{2} \left\{ \left(\frac{\partial^2 W}{\partial x' \partial y'} \right)_1 + \left(\frac{\partial^2 W}{\partial x' \partial y'} \right)_2 \right\} (x'_2 - x'_1).$$

If we denote the distance between the points $x'_2 - x'_1$ by $s_{1.2}$ and the value of the above integral by $T_{1.2}$, then

$$T_{1.2} = \frac{1}{2} \left\{ \left(\frac{\partial^2 W}{\partial x' \partial y'} \right)_1 + \left(\frac{\partial^2 W}{\partial x' \partial y'} \right)_2 \right\} s_{1.2} \quad (10)$$

can be computed from the two values (9). Therefore, also

$$\left(\frac{\partial W}{\partial y'} \right)_2 - \left(\frac{\partial W}{\partial y'} \right)_1 = T_{1.2} \quad (11)$$

is to be regarded as known.

The two differential quotients $\left(\frac{\partial W}{\partial y'}\right)_2$ and $\left(\frac{\partial W}{\partial y'}\right)_1$, however, can easily be expressed by $\left(\frac{\partial W}{\partial y}\right)_2$ and $\left(\frac{\partial W}{\partial y}\right)_1$. For if $\alpha_{1.2}$ is the direction angle of the length P_1P_2 , then we have according to (2), p. 489:

$$\frac{\partial W}{\partial y'} = -\frac{\partial W}{\partial x} \sin \alpha_{1.2} + \frac{\partial W}{\partial y} \cos \alpha_{1.2},$$

therefore,
$$T_{1.2} = -\left\{\left(\frac{\partial W}{\partial x}\right)_2 - \left(\frac{\partial W}{\partial x}\right)_1\right\} \sin \alpha_{1.2} + \left\{\left(\frac{\partial W}{\partial y}\right)_2 - \left(\frac{\partial W}{\partial y}\right)_1\right\} \cos \alpha_{1.2}.$$

According to (8), however, we have

$$\left.\begin{aligned}\left(\frac{\partial W}{\partial x}\right)_2 - \left(\frac{\partial W}{\partial x}\right)_1 &= -g(\xi_2 - \xi_1) = \varphi_{1.2} \\ \left(\frac{\partial W}{\partial y}\right)_2 - \left(\frac{\partial W}{\partial y}\right)_1 &= -g(\eta_2 - \eta_1) = \psi_{1.2},\end{aligned}\right\} \quad (12)$$

where the quantities $\varphi_{1.2}$ and $\psi_{1.2}$ are introduced for abbreviation. Consequently, we have the equation

$$T_{1.2} = -\varphi_{1.2} \sin \alpha_{1.2} + \psi_{1.2} \cos \alpha_{1.2}, \quad (13)$$

in which $T_{1.2}$ and $\alpha_{1.2}$ are to be regarded as known.

If there still exists a third observation point P_3 , then we can also compute $T_{2.3}$ and $T_{3.1}$, and have then two additional equations:

$$\left.\begin{aligned}T_{2.3} &= -\varphi_{2.3} \sin \alpha_{2.3} + \psi_{2.3} \cos \alpha_{2.3} \\ T_{1.3} &= -\varphi_{3.1} \sin \alpha_{3.1} + \psi_{3.1} \cos \alpha_{3.1}.\end{aligned}\right\} \quad (14)$$

Besides, however, we have according to (12)

$$\left.\begin{aligned}\varphi_{3.1} &= -\varphi_{1.2} - \varphi_{2.3} \\ \psi_{3.1} &= -\psi_{1.2} - \psi_{2.3}.\end{aligned}\right\} \quad (15)$$

For the determination of the six unknowns $\varphi_{1.2}$, $\psi_{1.2}$, $\varphi_{2.3}$, $\psi_{2.3}$ and $\varphi_{3.1}$, $\psi_{3.1}$ there thus exist the five equations (13), (14) and (15); five unknowns can therefore be expressed by the sixth. If we denote $\varphi_{1.2}$ by t , then we obtain:

$$\left.\begin{aligned}\varphi_{1.2} &= t \\ \psi_{1.2} &= \frac{T_{1.2}}{\cos \alpha_{1.2}} + t \tan \alpha_{1.2}\end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned}\varphi_{2.3} &= \frac{T_{2.3} \cos \alpha_{3.1} + (T_{3.1} - \varphi_{1.2} \sin \alpha_{3.1} + \psi_{1.2} \cos \alpha_{3.1}) \cos \alpha_{2.3}}{\sin(\alpha_{3.1} - \alpha_{2.3})} \\ \psi_{2.3} &= \frac{T_{2.3} \sin \alpha_{3.1} + (T_{3.1} - \varphi_{1.2} \sin \alpha_{3.1} + \psi_{1.2} \cos \alpha_{3.1}) \sin \alpha_{2.3}}{\sin(\alpha_{3.1} - \alpha_{2.3})}.\end{aligned}\right\} \quad (17)$$

If an observation point P_4 is connected with the points P_2 and P_3 , so that $T_{2.4}$ and $T_{3.4}$ can be computed, then, with the above formulae, after interchanging the indices, the values $\varphi_{3.4}$ and $\psi_{3.4}$ can also be indicated. If the measured points form a chain of triangles as in Fig. 2, then we obtain finally the relative deflections of the vertical for the end points of each triangle side in the form

$$\left. \begin{aligned} \varphi_{ik} &= a_{ik} + b_{ik} t \\ \psi_{ik} &= c_{ik} + d_{ik} t. \end{aligned} \right\}$$

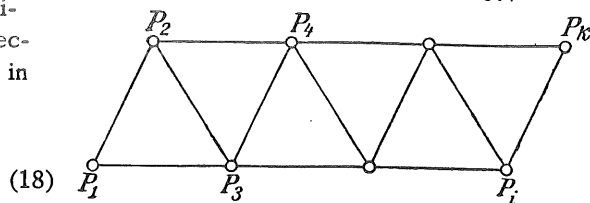


Fig. 2.

If we denote the relative components of the deflection of the vertical between two points P_i and P_k by ξ_{ik} and η_{ik} , then we have

$$\xi_{ik} = \xi_k - \xi_i \quad \text{and} \quad \eta_{ik} = \eta_k - \eta_i,$$

therefore, according to (12), by inserting the factor ρ

$$\xi_{ik} = -\frac{\varphi_{ik}}{g} \rho \quad \text{and} \quad \eta_{ik} = -\frac{\psi_{ik}}{g} \rho, \quad (19)$$

with which the components of the deflection of the vertical are found in seconds of arc.

Now the question is still the unknown t , in order to obtain the true values of the deflections of the vertical. For this, either ξ_{ik} or η_{ik} must be determined for two arbitrary points P_i and P_k from astronomic-geodetic measurements. We obtain therefrom, with the help of (19), the corresponding value of φ_{ik} or ψ_{ik} , and if, for this, a_{ik} and b_{ik} or c_{ik} and d_{ik} is known from the measurements with the torsion balance, then we find the value of t from (18).

For the practical test of the method reproduced above, Eötvös carried out, in the part of the Hungarian plain lying east of Arad, a large number of measurements with the torsion balance, in the case of which the observed points lay about 2 km apart from each other. The points were grouped into a closed triangulation chain so that between every two points the quantities φ_{ik} and ψ_{ik} could be computed. The results of this computation have been communicated in the first part of the report on the 15th general conference of Internationale Erdmessung, already mentioned several times, to which a supplement in the first part of the report on the 16th conference is to be mentioned.

In order to be able to explain the computational procedure by a numerical example, we take from these publications the measurements carried out with the torsion balance at the three points Nos. 2141 (P_1), 2140 (P_2) and 2103 (P_3), so far as they are required for the computation of the deflection of the vertical. According to this, we have:

$$\left. \begin{aligned} P_1: \quad \frac{\partial^2 W}{\partial x \partial y} &= +1.4 \cdot 10^{-9} & \frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} &= +13.8 \cdot 10^{-9} \\ P_2: \quad \frac{\partial^2 W}{\partial x \partial y} &= +6.0 \cdot 10^{-9} & \frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} &= +20.6 \cdot 10^{-9} \\ P_3: \quad \frac{\partial^2 W}{\partial x \partial y} &= +4.1 \cdot 10^{-9} & \frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} &= +14.8 \cdot 10^{-9}. \end{aligned} \right\} \quad (20)$$

The azimuths and distances are not communicated in the publications; for the present purpose it is sufficient to take these values from the plan of site. We find

$$\alpha_{2.3}^* = 357^\circ \quad \alpha_{3.1} = 95^\circ, \quad \text{therefore} \quad \alpha_{3.1} - \alpha_{2.3} = 98^\circ \quad (21)$$

$$s_{2.3} = 2.15 \text{ km} \quad s_{3.1} = 2.15 \text{ km}. \quad (22)$$

Now we have to transform at first the measured values into the directions P_2P_3 and P_3P_1 with the help of equation (6). For the point P_2 and the direction to P_3 we have $2\alpha = 354^\circ$, therefore

$\cos 2\alpha = 0.994$ and $\frac{1}{2} \sin 2\alpha = -0.052$, and with this, equation (6) yields

$$\frac{\partial^2 W}{\partial x' \partial y'} = +4.89 \cdot 10^{-9}.$$

The remaining values were computed likewise so that there result the values assembled in the following:

Point	Direction to	$\frac{\partial^2 W}{\partial x' \partial y'}$
P_2	P_3	$+4.89 \cdot 10^{-9}$
P_3	P_2	$+3.30 \cdot 10^{-9}$
P_3	P_1	$-5.33 \cdot 10^{-9}$
P_1	P_3	$-2.58 \cdot 10^{-9}$

Therefore, we have:

$$\left. \begin{array}{l} \text{Mean value for the side } P_2 P_3: +4.10 \cdot 10^{-9} \\ \text{Mean value for the side } P_3 P_1: -3.96 \cdot 10^{-9} \end{array} \right\} \quad (23)$$

With these two values we obtain according to equation (10), where the distances s are to be introduced in centimeters corresponding to all measured data of Eötvös' experiments,

$$T_{2.3} = +0.88 \cdot 10^{-3} \quad \text{and} \quad T_{3.1} = -0.85 \cdot 10^{-3}. \quad (24)$$

We take now from the publications by Eötvös the values of $\varphi_{1.2}$ and $\psi_{1.2}$, i.e.

$$\varphi_{1.2} = -3.71 - 2.60 t \quad \psi_{1.2} = -7.19 - 2.71 t \quad (25)$$

and we will compute the quantities $\varphi_{2.3}$ and $\psi_{2.3}$ according to equations (17). The substitution of the above numerical values in equation (17) yields

$$\varphi_{2.3} = +3.38 + 2.83 t \quad \psi_{2.3} = +0.71 - 0.15 t, \quad (26)$$

which agrees sufficiently with the values computed by Eötvös.

According to this scheme, the relative values φ_{ik} and ψ_{ik} have been computed by Eötvös for the whole system of points. If we set $\varphi_k = \varphi_i + \varphi_{ik}$ and $\psi_k = \psi_i + \psi_{ik}$ and assume for a suitably chosen point $\varphi = 0$, as well as $\psi = 0$ for another point, then the values of φ and ψ can be found for all points by summation of the relative values. For the three points treated above we take from the reports

$$\varphi_1 = -3.81 + 0.58 t \quad \psi_1 = +6.39 + 2.87 t.$$

With (25) and (26) we obtain then further

$$\begin{array}{ll} \varphi_2 = -7.52 - 2.02 t & \psi_2 = -0.80 + 0.16 t \\ \varphi_3 = -4.14 + 0.81 t & \psi_3 = -0.09 + 0.01 t. \end{array}$$

Now there still remains to be shown how the quantity t can be determined. For this purpose, Eötvös also carried out astronomic latitude determinations at several points of his net, which we reproduce in the following. Retaining the astronomic latitude of the first point Pankota, the latitudes of the remaining points were computed by geodetic means so that the meridional components of the deflections of the vertical were found with reference to the first point as zero point.

Station	Astr. Latitude	Geod. Latitude	Deflection of the Vertical ξ	$\varphi \cdot 10^3$
Pankota	46° 21' 07.10''	46° 21' 07.10''	0.0''	+ 35.92 + 28.80 t
Világos	46 15 58.71	46 15 58.38	+ 0.3	+ 8.78 + 15.24 t
Kuvín	46 09 58.75	46 10 01.64	- 2.9	- 1.22 - 0.22 t
Paulis	46 06 16.09	46 06 24.80	- 8.7	+ 12.09 - 9.31 t
Zábrány	46 04 32.85	46 04 40.79	- 7.9	+ 1.90 - 13.70 t
Mikalaka	46 19 04.91	46 19 06.46	- 1.5	- 2.72 + 0.83 t
Nagyhalom ..	46 10 25.40	46 10 27.87	- 2.5	+ 9.22 - 0.89 t

For the computation of t we can use the astronomic-geodetic measurements of two arbitrary stations; we choose, for this, the stations Pankota and Zábrány and have then according to equation (19)

$$\xi_z - \xi_p = -\frac{1}{g}(\varphi_z - \varphi_p)g$$

or with the above numerical values and with $g = 981$ cm

$$-7.9 = -\frac{206.265}{981}(-34.02 - 42.50 t),$$

from which there results

$$t = -1.6840.$$

With this, the components of the deflection of the vertical ξ for all seven points are computed from the values of φ according to equation (19) and put together in the following table. The first column holds for the zero point of the φ 's, the second for the zero point Pankota; in the last one, finally, the values found by astronomic-geodetic means are compared once more.

Station	Torsion Balance		Astr.-Geod.
	ξ	$\xi - \xi_0$	$\xi - \xi_0$
Pankota	+ 2.6''	0.0''	0.0''
Világos	+ 3.5	+ 0.9	+ 0.3
Kuvín	+ 0.1	- 2.5	- 2.9
Paulis	- 5.9	- 8.5	- 8.7
Zábrány	- 5.2	- 7.8	- 7.9
Mikalaka	+ 0.8	- 1.8	- 1.5
Nagyhalom	+ 0.3	- 2.3	- 2.5

The agreement cannot be expected better, especially because the astronomic-geodetic deflections of the vertical have a mean error of approximately $\pm 0.4''$.

Also the component of the deflection of the vertical in the prime vertical could be examined at least in one case, namely between the two points Kuvín and Mikalaka, which were connected with each other by azimuth measurements. From these astronomic as well as the geodetic measurements there followed between the points mentioned

$$\eta = -5.3''.$$

On the other hand, there was determined from the measurements with the torsion balance, taking the value found above $t = -1.6840$ as a basis,

$$\eta = -5.5''$$

and hence, likewise a perfectly satisfactory agreement.

We have communicated a few additional remarks with respect to measurements with the torsion balance in *Zeitschrift für Vermessungswesen* 1913, pp. 474-483 and pp. 505-517.

Section 100. Trigonometric Elevation Measurement

In Volume II, 2nd half-volume, 1933, section 30,* we have developed the basic formulae of trigonometric elevation measurement, taking the earth's surface imagined spherically as a basis. Now we have to examine how far those formulae are valid also for the ellipsoidal surface of the earth and in what respect we can compare the elevation differences found trigonometrically with the results of geometric levelling. At the same time, we are to take into account that the measurement of the elevation angles or zenith distances refers to the direction of the vertical, not to the normal to the ellipsoid.

These relations are represented in Fig. 1. We imagine that at points A and A' the deflections of the vertical are known with respect to a given reference ellipsoid. Since we have treated the theory of refraction exhaustively in Volume II, 2nd half-volume, 1933, sections 27, 30, 36-39,* then we will assume now that the measurement at the two points A and A' has yielded directly the zenith distances ζ and ζ' of the connecting straight line AA' . We compute therefrom the ellipsoidal zenith distances z and z' with the help of the components of the deflection of the vertical ξ, η and ξ', η' according to (23), section 89, p. 438. If we denote the components of the deflection of the vertical in the directions AA' and $A'A$ by Λ and Λ' , then we have according to Fig. 1

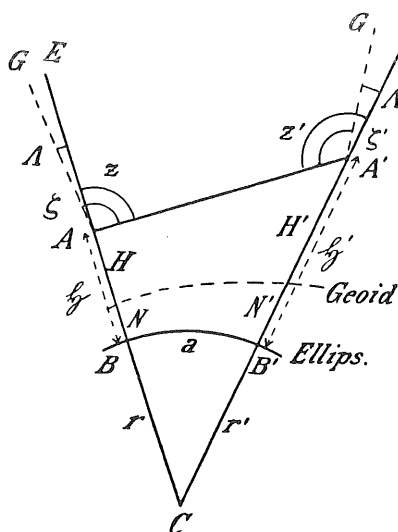


Fig. 1.

$$z = \zeta - \Lambda \quad z' = \zeta' + \Lambda'. \quad (1)$$

From the geographic coordinates of points A and A' there is known further the ellipsoidal distance $BB' = a$.

With regard to Fig. 1 there is now still to be considered that the normals to the ellipsoid at A and A' as a rule do not lie in a plane. The two normal planes at A and A' , which pass through the straight line AA' will instead make a small angle ν which can become in the most favorable case, according to equation (10), section 3, p. 17,

$$\nu = \frac{\eta^2}{2} \frac{s}{N} \rho$$

with $\eta^2 = e'^2 \cos^2 \varphi$. For $s = 100,000$ m this angle reaches the amount of $5''$; therefore, if we imagine in Fig. 1 the two directions of the vertical at A and A' projected on a plane parallel to them, then no noticeable distortion will thereby arise.

The difference of the two lengths $BC = r$ and $B'C = r'$, which does not occur in the case of the spherical development of the trigonometric elevation formula, needs a further examination. For this, we have represented once again in Fig. 2 the points B, B' and C with the chord k and the two angles of depression μ and μ' and find

$$r = k \frac{\cos \mu'}{\sin(\mu + \mu')} \quad r' = k \frac{\cos \mu}{\sin(\mu + \mu')},$$

therefore,

$$r - r' = k \frac{\cos \mu' - \cos \mu}{\sin(\mu + \mu')},$$

and after a simple transformation:

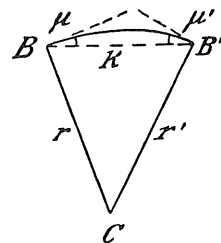


Fig. 2.

* Not translated.

$$r - r' = k \frac{\sin \frac{\mu - \mu'}{2}}{\cos \frac{\mu + \mu'}{2}}.$$

Since the angles of depression μ and μ' are very small, and we are only interested in obtaining an estimated value for $r - r'$, then we can also set for this:

$$r - r' = \frac{\mu - \mu'}{2} k. \quad (2)$$

The difference $r - r'$ will no doubt be greatest if the two points B and B' lie on a meridian; therefore, we will only pursue this case further.

According to equation (16), section 2, p. 12, we have for $\alpha = 0^\circ$ and for $\alpha = 180^\circ$

$$\mu = \frac{s}{2N}(1 + \eta^2) - \frac{s^2}{2N^2}\eta^2 t, \quad (3)$$

$$\mu' = \frac{s}{2N'}(1 + \eta'^2) - \frac{s^2}{2N'^2}\eta'^2 t', \quad (4)$$

where the quantities N , η^2 and t are imagined for the latitudes φ and φ' of B and B' . In order to reduce everything to the latitude φ , we have from equations (o) and (m) in the first half-volume, section 40, p. 63,

$$\frac{1}{N'} = \frac{1}{N} \left(1 - \frac{s}{N}\eta^2 t\right) \quad \eta'^2 = \eta^2 - 2\frac{s}{N}\eta^2 t,$$

where we have replaced $\varphi' - \varphi$ by $\frac{s}{N}$. With this, we obtain from (4):

$$\mu' = \frac{s}{2N}(1 + \eta^2) - 2\frac{s^2}{2N^2}\eta^2 t, \quad (5)$$

and from (3) and (5):

$$\mu - \mu' = \frac{s^2}{2N^2}\eta^2 t. \quad (6)$$

Now we can introduce this into (2) and at the same time also replace the chord k by the arc length s . Then we will have

$$r - r' = \frac{s^3}{4N^2}\eta^2 t. \quad (7)$$

For a distance of 50 km this expression reaches in the maximum not quite the amount of 3 mm; the assumption $r = r'$ is thus perfectly justified.

If we now pass over to the results of trigonometric elevation measurement, then the solution of the triangle $AA'C$ in Fig. 1, p. 496, yields, with the assumption $r = r'$, the ellipsoidal difference of elevation

$$h = \xi' - \xi, \quad (8)$$

where the distances of the two points A and A' from the reference ellipsoid are denoted by ξ and ξ' . With this, the efficiency of trigonometric elevation measurement is exhausted.

According to the position of the reference ellipsoid with respect to the geoid, the quantity h can deviate very considerably from the difference of the elevations above sea level $H' - H$. If N and N' are the elevations of the geoid above the reference ellipsoid below the two points, then we have

$$\begin{aligned}\xi &= H + N \\ \xi' &= H' + N',\end{aligned}$$

and hence

$$H' - H = \xi' - \xi - (N' - N). \quad (9)$$

According to equation (6), section 98, p. 484, however, we have

$$N' - N = \int_A^{A'} \mathcal{A} ds - E, \quad (10)$$

where E is the correction term required by the curvature of the plumb line. With this, (4) becomes

$$H' - H = \xi' - \xi - \int_A^{A'} \mathcal{A} ds + E. \quad (11)$$

We see from this that trigonometric elevation measurement is suited only in connection with astronomic levelling for the determination of elevations above sea level.

The above equation (11) also leads to an important relation between trigonometric elevation measurement, astronomic and geometric levelling. In sections 86 and 87 we have so far included gravity for the computation of elevations above sea level from geometric levelling. This computation, however, can also be carried out with the help of the curvature of the plumb line, as we have already seen in equation (7), section 98, p. 485. According to equation (7), section 98, p. 485, we have

$$H' - H = \int_A^{A'} dz + E, \quad (12)$$

where the integral represents the result of geometric levelling between the points A and A' while E has the same meaning as in (11). If we eliminate E from (11) and (12), then we obtain

$$\xi' - \xi = \int_A^{A'} \mathcal{A} ds + \int_A^{A'} dz, \quad (13)$$

an equation which is independent of the curvature of the plumb line.

Equation (13) has been indicated for the first time by Helmert in the treatise "Zur Bestimmung kleiner Flächenstücke des Geoids aus Lotabweichungen mit Rücksicht auf Lotkrümmung," Erste Mitteilung, *Sitzungsber. d. Kgl. Preuss. Ak. d. Wiss. zu Berlin*, math.-phys. Klasse, 1900, p. 975.

In addition, we will discuss the question of what result is obtained if we do not take into account the deflections of the vertical in the computation of the trigonometric elevation measurement, and thus introduce the measured zenith distances directly into the computation, as it mostly happens in practice. We shall examine here only the case of the reciprocal zenith distances. According to Volume II, 2nd half-volume, 1933, equation (4), section 30, p. 142,* with $z = \zeta - \Lambda$, $z' = \zeta' + \Lambda'$ we have, for this, the basic equation

$$h = a \left(1 + \frac{H + H'}{2r} \right) \tan \frac{(\zeta' - \zeta) + (\Lambda' + \Lambda)}{2} \quad (14)$$

* Not translated.

where we shall disregard again refraction. If the deflections of the vertical are thus entirely disregarded, then this is equivalent to assuming

$$\Lambda' + \Lambda = 0. \quad (15)$$

This presupposes again a definite position of the reference ellipsoid with respect to the geoid. If we add the further assumption that the deflection of the vertical changes in proportion to the distance, thus that we can set

$$\Lambda = p + q s$$

then we have for point A

$$\Lambda = p,$$

and for point A'

$$\Lambda' = p + q a,$$

therefore,

$$\Lambda + \Lambda' = 2p + q a. \quad (16)$$

On the other hand, the integral will be

$$\int_A^{A'} \Lambda ds = p a + \frac{1}{2} q a^2 = \frac{1}{2} a (2p + q a), \quad (17)$$

and if $\Lambda + \Lambda' = 0$ then according to (16), the value of the integral (17) becomes also equal to zero. As equation (11) shows, the trigonometric elevation measurement yields directly in this case, disregarding the always very small correction term E , differences of elevations above sea level.

The condition that the deflections of the vertical change in proportion to the distance will in reality mostly be satisfied approximately – at least in the case of not too large distances – and on this there rests the practical applicability of trigonometric elevation measurement without taking into account the deflections of the vertical.

The method of Villarceau and Bruns for the determination of the geoid

The procedure of trigonometric elevation measurement forms an essential part of the method indicated by Yvon Villarceau and later treated in detail by H. Bruns for the determination of the geoid independently of all hypothetical assumptions. This method was published by Villarceau in the treatise "Nouveau théorème sur les attractions locales," *Compt. rend. hebd. des séances de l'acad. des sciences*, Vol. 67, Paris, 1868, pp. 1275-1281. A detailed discussion of the auxiliary means available and the aims to be reached was carried out by Bruns in the fundamental treatise, *Die Figur der Erde. Ein Beitrag zur europäischen Gradmessung*, Berlin, 1878.

The following kinds of measurement are available for the rigorous solution of the problem:

1. Astronomic place determinations (latitudes, longitudes, azimuths).
2. Triangulations (horizontal angles, base lines).
3. Trigonometric elevation measurements (zenith distances).
4. Geometric levellings.
5. Gravity measurements.

The foundation of the method is formed by a triangulation net with angle and base-line measurement whose side lengths are computed by being based on a preliminary reference ellipsoid. It is further assumed that all triangulation points are at the same time astronomic stations by measuring at all points the latitude and an azimuth, while for particular points astronomic longitude determinations are added.

The connection of these geodetic and astronomic measurements makes possible the determination of the constants of a reference ellipsoid which adapts itself as well as possible to the surface of the geoid, as we have learned in section 97, p. 480. At the same time, the deflections of the vertical of all points with respect to this ellipsoid become also known thereby.

At the triangulation points there must also be measured for all triangle sides the zenith distances. The latter refer at first to the directions of the vertical, but they can also be transformed to the normals to the ellipsoid according to equation (23), section 89, p. 438. If we assume then arbitrarily for one of the points the elevation ξ_0 above the ellipsoid, then the zenith distances yield, in connection with the lengths of the triangle sides, the ellipsoidal elevations ξ of all points according to the basic formulae of trigonometric elevation measurement.

Now we still lack the determination of the elevation of the geoid with respect to the reference ellipsoid, for which geometric levelling and gravity measurements can be used. If we imagine all triangulation points connected with each other and with the sea coast by levelling lines and if the course of gravity is sufficiently known for the computation of the orthometric corrections, then we obtain the elevation above sea level H for all points, and the difference $\xi - H$ yields for each point the elevation N of the geoid above the reference ellipsoid. Since *one* ellipsoidal elevation ξ_0 was at first assumed arbitrarily, then we can change this afterwards in such a way that the sum of all N 's becomes equal to zero, whereby the reference ellipsoid coincides in height as well as possible with the surface of the geoid.

We see from this that the above mentioned kinds of measurement are perfectly sufficient in order to determine the shape of the surface of the geoid free from all assumptions; however, the execution of this method meets with many difficulties. At first, the astronomic place determinations at all triangulation points would cause an extraordinary increase of work. This holds, to a still greater extent, for the measurements of zenith distances, if these are to be found as free as possible from abnormal influences of refraction. But even though this should be achieved, the theory of refraction is at the present time not yet adequate for sufficiently taking refraction into account. The method of the reciprocal and simultaneous measurement of zenith distances at more than two points which are connected with each other by a triangle or a central system appears most promising for this. In the case of this arrangement of the measurements, the computation of the coefficients of refraction is possible without the knowledge of the meteorological elements, as Helmert has shown in *Die math. u. phys. Theorie d. höh. Geod.*, Vol. II, 1884, pp. 599-607. We have represented these theories also in our Volume II, second half-volume, 1933, pp. 633-636,* where we have also referred to applications used thus far.

* Not translated.

Chapter X

PERIODIC VARIATIONS OF THE VERTICAL AND THE MOVEMENT OF THE POLE

Section 101. Variability of the Direction of the Vertical Under the Influence of the Sun and the Moon

In section 61 we have developed the shape of the surface of the geoid under the influence of the attractive force of the terrestrial body and centrifugal force and disregarded thereby completely the effect of exterior forces. It is obvious, however, that the attractive effect of the sun and the moon and – to a far lesser extent, however – also that of the planets must likewise influence the shape of the geoid. In the following, we will therefore examine the changes of the directions of the vertical on the earth's surface and the changes of the geoidal surface connected with them, which result from exterior influences.

The following developments refer, at first, directly to the moon, but can also be applied at once to the sun. To take into account further celestial bodies has no practical significance, however.

In Fig. 1 let E be the earth, which we imagine as a sphere with the radius a while M represents the mass of the moon. The mass of the moon M exerts, on a unit mass at point O on the earth's surface, an attractive effect whose horizontal component is equal to

$$f \frac{M}{r'^2} \sin z'$$

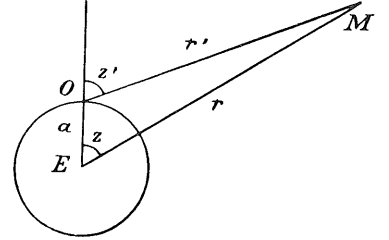


Fig. 1.

if f denotes the constant of attraction. The absolute attraction of the mass point O does not interest us, however, but the difference of the attractive forces which the moon exerts on point O and the center of gravity of the earth E , and which acts toward a deformation of the terrestrial body. We denote this difference as the *tidal force* of the moon at point O , whose horizontal component is therefore equal to

$$f \frac{M}{r'^2} \sin z' - f \frac{M}{r^2} \sin z \quad (1)$$

According to Fig. 1 we have

$$\left. \begin{aligned} r \sin z &= r' \sin z' \\ r \cos z &= a + r' \cos z' \end{aligned} \right\} \quad (2)$$

and hence

$$\begin{aligned} r \sin^2 z &= r' \sin z \sin z' \\ r \cos^2 z &= a \cos z + r' \cos z \cos z' . \end{aligned}$$

But since we can set with sufficient accuracy

$$\sin z \sin z' = \sin^2 z \quad \text{and} \quad \cos z \cos z' = \cos^2 z$$

then we also will have

$$r = r' + a \cos z. \quad (3)$$

It follows from (2) that

$$\frac{\sin z'}{r'^2} - \frac{\sin z}{r^2} = r \sin z \left(\frac{1}{r'^3} - \frac{1}{r^3} \right).$$

But according to (3) we have

$$\frac{1}{r'^3} = \frac{1}{(r - a \cos z)^3} = \frac{1}{r^3 \left(1 - \frac{a}{r} \cos z \right)^3} = \frac{1}{r^3} \left(1 + 3 \frac{a}{r} \cos z \right), \quad (3a)$$

therefore,

$$\frac{\sin z'}{r'^2} - \frac{\sin z}{r^2} = 3 \frac{a}{r^3} \sin z \cos z$$

and the horizontal component of the tidal force is thus

$$3 f M \frac{a}{r^3} \sin z \cos z. \quad (4)$$

We also obtain this expression from the potential

$$V = f \frac{M}{2} \frac{a^2}{r^3} (3 \cos^2 z - 1), \quad (5)$$

for since the line element at O in the horizontal direction and on the vertical plane of the moon is equal to $-a dz$, then we have

$$\frac{1}{a} \frac{\partial V}{\partial z} = 3 f M \frac{a}{r^3} \sin z \cos z. \quad (6)$$

At the point O there acts now in the horizontal direction the force (6), in the vertical direction the force of gravity g , so that at O the vertical will assume the direction of the resultant of both forces. There thus originates at O , through the tidal force, a deflection of the vertical of magnitude

$$\frac{1}{a g} \frac{\partial V}{\partial z} = 3 \frac{f M}{g} \frac{a}{r^3} \sin z \cos z. \quad (7)$$

Instead of the constant of attraction f we can also introduce the force of gravity g in (5) and (6), for we have according to (6), section 64, p. 315, in the first approximation

$$g = f \frac{E}{a^2},$$

if E denotes the mass of the earth. We thus have

$$f = g \frac{a^2}{E}, \quad (8)$$

and hence, according to (5)

$$V = \frac{g}{2} \frac{M}{E} \frac{a^4}{r^3} (3 \cos^2 z - 1) \quad (9)$$

and the deflection of the vertical will be according to (7)

$$\frac{1}{a} \frac{\partial V}{\partial z} = 3 \frac{M}{E} \frac{a^3}{r^3} \sin z \cos z. \quad (10)$$

Now we have in round numbers

$$\frac{M}{E} = \frac{1}{82}, \quad \frac{a}{r} = \frac{1}{60}, \quad \text{therefore} \quad \frac{M}{E} \frac{a^3}{r^3} = \frac{1}{18 \cdot 10^6},$$

and hence the deflection of the vertical will be

$$\frac{1}{g} \frac{\partial V}{\partial z} = \frac{\varrho}{12 \cdot 10^6} \sin z \cos z = 0.017'' \sin 2z. \quad (11)$$

The deflection of the vertical caused by the moon thus amounts to a maximum of nearly 0.02".

For the effect of the sun we can indicate the result of the corresponding development at once, if we divide the factor 0.017" by 2.18 corresponding to the mass of the sun and its distance from the earth. Then, the amount of the deflection of the vertical caused by the sun is equal to

$$0.008'' \sin 2z. \quad (12)$$

The potential V in (5) or, as the case may be, (9) represents the disturbance of the potential W of the force of gravity effected by the tidal force. Now since according to (18), section 61, p. 308,

$$\frac{dW}{dh} = g \quad \text{or} \quad dh = \frac{dW}{g}$$

then we have in the expression

$$\frac{V}{g} = \frac{f}{2} \frac{M}{g} \frac{a^2}{r^3} (3 \cos^2 z - 1)$$

or

$$\frac{V}{g} = \frac{1}{2} \frac{M}{E} \frac{a^4}{r^3} (3 \cos^2 z - 1)$$

the elevation and depression of the geoidal surface effected by the tidal force. The expression reaches its minimum for $z = 90^\circ$ or 270° , its maximum for $z = 0$ or 180° , and the difference between the maximum and the minimum is

$$\frac{3}{2} \frac{M}{E} \frac{a^4}{r^3} = \frac{3}{2} \frac{a}{18 \cdot 10^6} = 0.54 \text{ m.}$$

For the sun there follows the corresponding value of 0.25 m.

We will now divide the deflection of the vertical (10) caused by the tidal force into a northward and a westward directed component. For this, we have in these two directions the line elements

$$a d\varphi \quad \text{and} \quad a \cos \varphi d\lambda, \quad (13)$$

where φ and λ denote the geographic latitude and the western longitude of the point O . If A is the azimuth of the moon counted from the south, then we have for the two components of the deflection of the vertical from (13) and (10)

$$\left. \begin{aligned} \frac{1}{a g} \frac{\partial V}{\partial \varphi} &= -3 \frac{M}{E} \frac{a^3}{r^3} \sin z \cos z \cos A \\ \frac{1}{a g \cos \varphi} \frac{\partial V}{\partial \lambda} &= +3 \frac{M}{E} \frac{a^3}{r^3} \sin z \cos z \sin A. \end{aligned} \right\} \quad (14)$$

For an arbitrary azimuth A' , which shall likewise be counted from the south according to astronomic usage, we then obtain the deflection of the vertical

$$\eta = \frac{1}{a g} \left(-\frac{\partial V}{\partial \varphi} \cos A' + \frac{1}{\cos \varphi} \frac{\partial V}{\partial \lambda} \sin A' \right) \quad (15)$$

or

$$\eta = 3 \frac{M}{E} \frac{a^3}{r^3} \sin z \cos z \cos (A' - A). \quad (16)$$

From equations (5) or, as the case may be, (9), pp. 502 and 503, it already follows that the potential of the tidal force is subjected to periodic variations, which depend on the change of the zenith distance z . This shows still more clearly if we introduce the declination and the hour angle of the moon instead of the zenith distance z . For this, we use the nautical triangle of Fig. 2 known from spherical astronomy, in which P denotes the celestial north pole, Z the zenith of the point of observation and M the position of the moon. The great circle PZ is the meridian of the place of observation, and hence, A is the azimuth of the moon counted from the south. In Fig. 2, φ denotes further the geographic latitude of the place of observation and δ the declination of the moon. Then we have according to Fig. 2

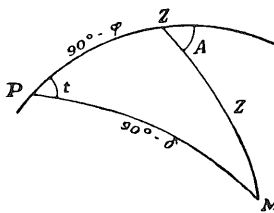


Fig. 2.

$$\left. \begin{aligned} \cos z &= \sin \delta \sin \varphi + \cos \delta \cos \varphi \cos t \\ \sin z \sin A &= \cos \delta \sin t \\ \sin z \cos A &= -\sin \delta \cos \varphi + \cos \delta \sin \varphi \cos t. \end{aligned} \right\} \quad (17)$$

From the first equation of (17) we obtain

$$\cos^2 z = \sin^2 \varphi \sin^2 \delta + \cos^2 \varphi \cos^2 \delta \cos^2 t + \frac{1}{2} \sin 2 \varphi \sin 2 \delta \cos t.$$

If we set here further

$$\cos^2 t = \frac{1}{2} + \frac{1}{2} \cos 2 t$$

then we have after a simple transformation

$$\begin{aligned} 3 \cos^2 z - 1 &= \frac{9}{2} \left(\sin^2 \varphi - \frac{1}{3} \right) \left(\sin^2 \delta - \frac{1}{3} \right) + \frac{3}{2} \cos^2 \varphi \cos^2 \delta \cos 2 t \\ &\quad + \frac{3}{2} \sin 2 \varphi \sin 2 \delta \cos t. \end{aligned}$$

If we introduce this in (5), then we obtain for the potential of the tidal force the expression

$$V = \frac{3}{4} f M \frac{a^2}{r^3} \left\{ 3 \left(\sin^2 \varphi - \frac{1}{3} \right) \left(\sin^2 \delta - \frac{1}{3} \right) + \cos^2 \varphi \cos^2 \delta \cos 2 t + \sin 2 \varphi \sin 2 \delta \cos t \right\}. \quad (18)$$

We have hereby divided V into three individual terms of which the first, due to the term $\sin^2 \delta$, has a period of fourteen days, the second a period of half a lunar day, the third a period of one lunar day.

In addition, we can also transform the two components (14) of the deflection of the vertical. If we use again equations (17) as well as the goniometric relation $\cos^2 t = \frac{1}{2} + \frac{1}{2} \cos 2t$, then

$$\left. \begin{aligned} \text{the northern component of the} \\ \text{deflection of the vertical} &= -\frac{3}{4} \frac{M}{E} \frac{a^3}{r^3} \left\{ \sin 2\varphi \cos^2 \delta \cos 2t - 2 \cos 2\varphi \sin 2\delta \cos t \right. \\ &\quad \left. + \sin 2\varphi (1 - 3 \sin^2 \delta) \right\}, \\ \text{and the western component of} \\ \text{the deflection of the vertical} &= +\frac{3}{2} \frac{M}{E} \frac{a^3}{r^3} \left\{ \cos \varphi \cos^2 \delta \sin 2t + \sin \varphi \sin 2\delta \sin t \right\}. \end{aligned} \right\} \quad (19)$$

In (18) we have set up the potential only for the effect of the moon. To this must be added a second expression of the same form which contains the mass, the declination and the hour angle of the sun, as well as its distance from the earth. The sum of both expressions yields then the total potential of the tidal force.

Change of the force of gravity

We have thus far taken into account only the horizontal component of the tidal force. Now we consider, in addition, briefly, its vertical component, which represents a variation in the force of gravity under the influence of the sun and the moon. According to equation (1), p. 501, for the vertical component, which we will denote by Δg , we have the expression

$$\Delta g = f M \left(\frac{\cos z'}{r'^2} - \frac{\cos z}{r^2} \right). \quad (20)$$

According to the second equation (2), p. 501, we have

$$\frac{\cos z'}{r'^2} = \frac{r \cos z}{r'^3} - \frac{a}{r'^3},$$

and according to (3a), p. 502,

$$\frac{1}{r'^3} = \frac{1}{r^3} \left(1 + 3 \frac{a}{r} \cos z \right).$$

If we set this into (20) and neglect terms in $\frac{1}{r^4}$, then we obtain

$$\Delta g = f M \frac{a}{r^3} (3 \cos^2 z - 1). \quad (21)$$

In this, we can again replace the constant of attraction f by the force of gravity g with the help of (8), p. 502, and have then

$$\Delta g = g \frac{M}{E} \frac{a^3}{r^3} (3 \cos^2 z - 1). \quad (22)$$

M denotes here the mass of the moon, E the mass of the earth, a the radius of the earth and r the distance of the moon from the center of the earth.

For the rounded off numerical values mentioned on p. 503 and for $g = 978$ gal we obtain the influence of the moon:

$$\Delta g_m = 0.000\,0543 (3 \cos^2 z - 1). \quad (23)$$

The variation of the value of the force of gravity under the influence of the moon thus amounts to a maximum of 0.11 mgal.

The influence of the sun results again as on p. 503 if we divide the numerical coefficient in (23) by 2.18. Therefore, we will have

$$\Delta g_s = 0.000\ 0249 (3 \cos^2 z - 1) \quad (24)$$

and for this, there follows the maximum value equal to 0.05 mgal.

Section 102. The Horizontal Pendulum

The amount of the deviation of the plumb caused by the influence of the moon and the sun, as we have indicated in the preceding section 101 in (11) and (12), p. 503, was first computed by C. A. F. Peters in *Astronomische Nachrichten*, No. 507, pp. 33-42, in 1844. However, the attempt was made already before this time to determine and to measure the deviations of the plumb by an experimental method. The first experiments of this kind originate from Gruithuizen in Munich from the beginning of the nineteenth century. A lead plumb 10 ft in length which carried a scale at the lower end was observed by him by means of a telescope, and he believed he noticed hereby motions which originated from the influence of the celestial bodies. However, if we consider the angular quantities indicated in (11) and (12) in section 101, p. 503, then we see that the variations of the plumb observed by Gruithuizen were to be attributed to other causes.

The measurements carried out by Hengler, a scholar of Gruithuizen, for which he constructed a new instrument, the *astronomic pendulum balance*, mean a considerable progress. A description of this instrument as well as that of the experiments has been published by Hengler in *Dinglers polytechnisches Journal*, Vol. 43,

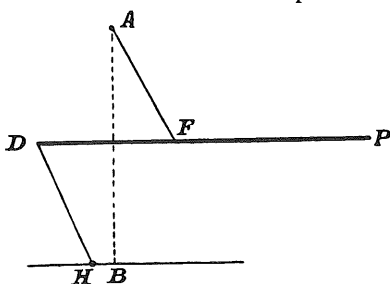


Fig. 1.

1832, pp. 81-92. A rod DP in Fig. 1 is fastened by two threads AF and DH at two fixed points A and H , whereby the connecting line of the two points A and H makes a small angle of inclination with the direction of the plumb AB . At the position of rest, the rod will regulate itself in such a way that its center of gravity lies in the vertical plane passing through A and H . If, by external causes, a small inclination of the axis of rotation AH occurs perpendicular to this plane, then the rod will follow this inclination and adjust itself in the new vertical plane of the axis of rotation. With this instrument, Hengler carried out a series of observations in which the attraction of the sun and the moon was doubtlessly expressed;

these experiments, however, were little noticed.

The horizontal pendulum of Zöllner

The basic idea of Hengler's pendulum balance was carried further by various scholars during the next decades and then was made the basis by Zöllner of the construction of a *horizontal pendulum*, which he has described in the *Berichte der K. Sächs. Ges. d. Wiss. zu Leipzig*, math.-phys. Klasse, 1869 and 1871. (These papers are also contained in *Zöllners Wissenschaftliche Abhandlungen* Band 4.) The instrument illustrated in Fig. 2 consists of a column which is carried by a tripod with adjusting screws, and at which, at c and c' , the suspension points for the pendulum are fastened. The latter is formed by a cylindrical lead weight A , weighing 3 kg, which is suspended at b and b' with two fine watch springs. At B a counterweight corresponding to the weight of the pendulum is fastened to the column. The general mounting of the instrument is carried out by means of the three foot screws; in addition, however, the inclination of the axis of rotation cc' of the pendulum can be separately adjusted with the help of the foot screw d . For the measurement of the motion of the pendulum there is used the mirror C and a horizontal scale set up before the latter at a distance of a few meters. If the scale is observed in the mirror by means of a telescope, then, from the apparent motions of the scale and from its distance from the mirror, the rotations of the pendulum can be determined. From the latter, the changes of inclination of the axis of rotation perpendicular to its original vertical plane can be computed, as we shall see later.

In more recent times, Zöllner's horizontal pendulum has also been used for measurements of the Geodetic Institute in Potsdam, about which there is a report in *Veröffentlichung des Zentralbureaus der Internationalen Erdmessung*, Neue Folge Nr. 38, "Lotschwankung und Deformation der Erde durch Flutkräfte" by W. Schweydar, Berlin, 1921. In the case of this instrument, special value is laid on a stable mounting, and therefore a hollow iron column with a height of 50 cm and a diameter of 20 cm on a strong iron baseplate is chosen as the support. Instead of the watch springs used by Zöllner, platinum-iridium wires, artificially aged, with a thickness of 0.04 mm, serve as suspension wires. Besides, there are fastened to the column two horizontal pendulums which make an angle of nearly 90° so that both components of the deflection of the vertical can be measured at the same time.

The horizontal pendulum of v. Rebeur-Paschwitz

The fear that the torsion as well as the tension of the wires in the case of Zöllner's horizontal pendulum could exert an unfavorable influence led E. v. Rebeur-Paschwitz to construct a new horizontal pendulum, in the case of which the kind of the suspension approaches more that of the vertical pendulum.

Fig. 3 shows a schematic illustration of this horizontal pendulum. On a baseplate, which can be adjusted by means of three screws as in the case of Zöllner's instrument, there rise two columns *A* and *B*, which carry the points *S* and *S'*. The pendulum *C*, a light body in the form of an isosceles triangle made of thin brass tubes or of aluminum, carries two agate pans with which it is put on the points *S* and *S'*. For the regulation of the position of the center of gravity, the screw *D* can be moved a little to-and-fro. At *N* there is fastened to the pendulum body, and inclined toward it by 45° , a small mirror which nearly coincides with the axis of rotation, and which is used for the measurement of the motions of the pendulum. The many adjusting devices which are necessary for the precise adjusting of the points and of the agate pans are omitted in Fig. 3. However, at *h* and *h'* there are visible, in addition, two points whose connecting line passes through the points *S* and *S'*. By means of these points *h* and *h'*, the pendulum can be hung on the bearings of a special support in order that its duration of oscillation, when oscillating around a horizontal axis, can be determined. This duration

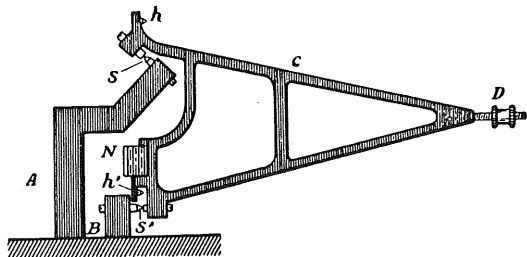


Fig. 3.

of oscillation, whose determination is also provided for in the case of Zöllner's pendulum, is needed for the reduction of the measurements, to which we shall refer later on. With regard to the further details of the pendulum we refer to a treatise by Hecker, "Das Horizontalpendel" in *Zeitschr. f. Instr.*, 1896, pp. 2-16.

On the same baseplate the instrument contains two pendulums which make an angle of nearly 90° .

V. Rebeur-Paschwitz reached a considerable increase of the measuring accuracy by the use of a photographic recording instead of the visual observations hitherto used. Fig. 4 shows the arrangement of the devices necessary for this, whose designations correspond to those of Fig. 3. C_1 and C_2 are the two pendulum bodies whose mirrors N_1 and N_2 are parallel to one another in the normal position of the pendulum. Opposite them there are two prisms P_1 and P_2 , before which a third mirror N' is placed, lying however somewhat deeper. As the light source for the photographic recording there is used a small lamp *L*, which stands about 3 m in front of the apparatus. From here the light rays pass through a fine slit, penetrate a

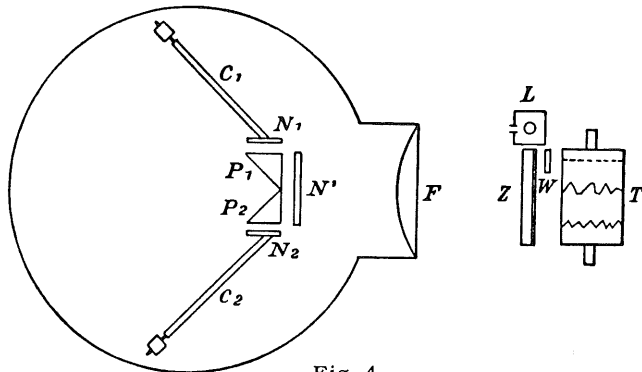


Fig. 4.

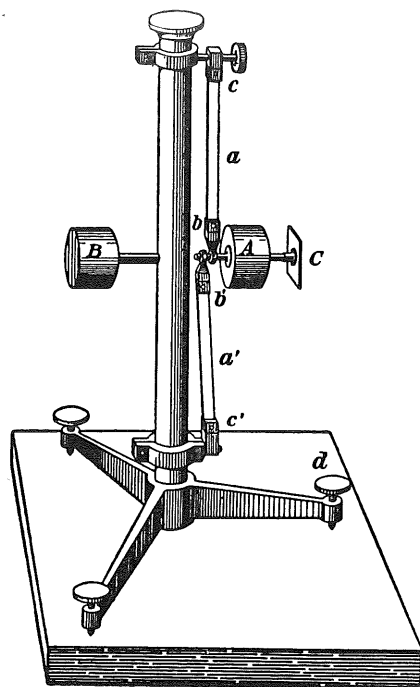


Fig. 2.

lens F of a focal length of 3 m and then are reflected at the three mirrors. The reflected rays are focused again by the lens F , in its focal length, as three images of the slit. Of these, a cylindrical lens Z projects three light points, which fall on a drum T covered with sensitized paper. The latter describes a full revolution within 24 hours by means of a clockwork, so that on the sensitized paper two curves are projected, which reproduce the oscillations of the pendulum. To the fixed mirror N' there corresponds a line, which shows an interruption every hour, since a screen W is then inserted by the clockwork each time for 2 minutes.

The horizontal pendulum of the Geodetic Institute

The horizontal pendulum of the Geodetic Institute illustrated in Fig. 5, which is provided likewise with two pendulum bodies intersecting at an angle of 90° , shows considerable improvements in comparison to v. Rebeur's construction. The baseplate is formed here by a heavy iron plate in the form of an isosceles triangle with a side length of about 50 cm. On it there rise the two pendulum stools, at which a very fine lateral correction of the pendulum axes is possible by means of sensitive adjusting devices. The pendulums each consist of two thin brass tubes connected with each other at right angles and are suspended similarly as in Fig. 3. The prisms, which stand on special columns in front of the two pendulum mirrors, can be corrected from the recording apparatus by means of long keys in such a way that the light points fall on the recording drum. This correction is necessary, since the light point already leaves the drum at an inclination of the pendulum axis of a few seconds perpendicular to the pendulum plane. The fixed mirror can be corrected likewise. The latter consists, in addition, of two parts twisted somewhat toward each other, so that on the drum there originate two lines

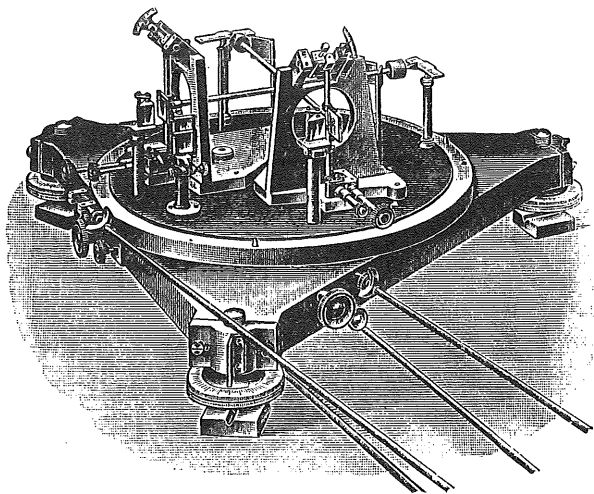


Fig. 5.

of abscissae - one at each end of the drum - which make it possible to take into account the stretch of the paper due to air humidity. The drum is moved, after a full revolution, automatically by 8 mm in the direction of the axis, so that the renewal of the paper is necessary only every second day.

On the baseplate there is put a copper cylinder, which is closed on top by a glass plate, and which carries also the above-mentioned lenses.

Theory of the horizontal pendulum

Let the axis of rotation be inclined by a small angle i toward the direction of the plumb. In the case of the oscillation of the pendulum, the center of gravity describes then an arc of a circle whose plane lies perpendicular to the axis of rotation, and hence, is inclined likewise by the angle i toward the horizontal plane. This plane takes the place of the vertical plane in the case of the ordinary pendulum with a horizontal axis of rotation.

The theory of the horizontal pendulum can easily be reduced to that of the usual physical pendulum according to these basic concepts. All considerations of the previous section 69 hold also for the horizontal pendulum, if we introduce, instead of the force of gravity g , the force which is effective in the plane of vibration, and, in fact, in the direction of its largest gradient. This force is equal to $g \cos (90^\circ - i)$ or equal to $g \sin i$.

Therefore, the differential equation for the motion of the horizontal pendulum is according to (10), section 69, p. 335,

$$\frac{d^2 \alpha}{dt^2} (a^2 + k^2) = -g a \sin i \sin \alpha,$$

if the same notation as previously is retained.

The integration yields according to (8a), section 68, p. 330,

$$\left(\frac{d\alpha}{dt}\right)^2 = \frac{2ga \sin i (\cos \alpha - \cos \varphi)}{a^2 + k^2}.$$

If we require now the length l of a mathematical pendulum which, in the case of a horizontal axis of rotation, oscillates as rapidly as the horizontal pendulum in the case of an inclined axis, then we have for the first, according to (8a), section 68, p. 330, the equation

$$\left(\frac{d\alpha}{dt}\right)^2 = \frac{2g}{l} (\cos \alpha - \cos \varphi).$$

The length of the mathematical pendulum is therefore

$$l = \frac{a^2 + k^2}{a \sin i}.$$

If, however, the same horizontal pendulum would oscillate with a horizontal axis, i.e. with $i = 90^\circ$, then a mathematical pendulum with the length

$$l_0 = \frac{a^2 + k^2}{a}$$

would correspond to it.

If we denote the duration of oscillation of the horizontal pendulum in the case of an inclined axis by T , and in the case of a horizontal axis by T_0 , then we have

$$\frac{T_0^2}{T^2} = \frac{l_0}{l} \quad \text{or} \quad \frac{T_0^2}{T^2} = \sin i. \quad (1)$$

Now we will assume further that the axis of rotation is somewhat inclined toward the vertical plane in which the angle i lies, and we will examine which rotation the pendulum will carry out in order to reach the new position of equilibrium.

In Fig. 6 let OP be the original axis of rotation of the pendulum inclined toward the direction of the plumb by the angle i , and OS the normal from the center of gravity S to the axis of rotation. Now we assume the axis to be turned toward the plane ZPS by the small angle $POP' = \psi$. The center of gravity of the pendulum, in order to reach again the position of equilibrium, will move then from S to S' and, hence, show a deflection α , which we will compute from ψ .

At first we obtain from the triangle $PP'S$, if we denote the angle at Z by β ,

$$PP' = \psi \cos i = \psi$$

and from the triangle $PP'Z$

$$PP' = \beta \sin i,$$

and hence, we will have

$$\beta = \frac{\psi}{\sin i}.$$

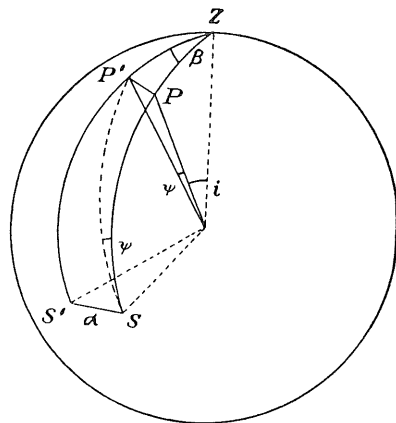


Fig. 6.

We have further from the triangle $SS'Z$

$$\alpha = \beta \sin (90^\circ + i) = \beta \cos i = \beta.$$

Therefore, we also have

$$\alpha = \frac{\psi}{\sin i} \quad \text{or} \quad \psi = \alpha \sin i. \quad (2)$$

The way the horizontal pendulum is used is therefore the following: The pendulum is first suspended in a special device with a horizontal axis of rotation, with which we can determine the duration of oscillation T_0 as in the case of the pendulum measurements. After the instrument has then been set up ready for use with a small inclination i of the axis of rotation, the duration of oscillation T for this axis is likewise determined. From T_0 and T there follows the angle of inclination i according to (1). With this, we have found the constants of reduction.

Let the distance of the pendulum mirror from the recording drum be equal to s in millimeters. An angular deflection of the pendulum of magnitude $\frac{\rho}{2s}$ in seconds corresponds then to a change of ordinate of 1 mm on the drum. The change of inclination of the axis of rotation perpendicular to the original vertical plane is thus

$$\frac{\rho}{2s} \sin i. \quad (3)$$

The ordinate changes measured in millimeters are to be multiplied by this factor in order to yield the corresponding changes of inclination.

The bifilar gravimeter

The measurement of the change of gravity (cf. section 101, p. 505) under the influence of the sun and the moon is carried out with the help of the bifilar gravimeter of Tomaschek, which is described in *Annalen der Physik*, Band 15, 1932, p. 787. It consists of a loaded circular disc which is suspended on two threads diametrically fastened at its edge. Besides, the disc hangs on a spiral spring which is fastened rigidly at its center. If we turn the spring by an arbitrary angle around its axis, then the disc participates in this rotation, whereby, at the same time, a twisting of the two suspension threads occurs however. The latter counteracts the rotation so that a tension of the spring occurs at the same time. The disc will then adjust itself in such a way that there exists equilibrium, i.e., that the rotation moment of the bifilar suspension is equal to the rotation moment of the extended spring. If the force of gravity changes, then the rotation moment of the bifilar suspension changes with it, and there occurs a small turn of the disc, which can be measured by means of a mirror arrangement. We obtain hereby a measurement for the change of gravity.

of the Tidal Force

As follows from equation (18), section 101, p. 504, the motion of the direction of the vertical under the influence of the sun and the moon is extremely complicated so that it becomes difficult to compare the measured results found with the horizontal pendulum with the theory. It is therefore proper to divide the expression for the potential, by the development in series, into individual terms, which can be computed theoretically, but which, on the other hand, can also be determined from pendulum measurements, and thus make possible a comparison of the theory with the observations.

According to equation (5), section 101, p. 502, the potential of the tidal force is

$$V = f \frac{M}{2} \frac{a^2}{r^3} (3 \cos^2 z - 1). \quad (1)$$

According to Fig. 1 we introduce a rectangular system of coordinates ABC whose zero point lies at the earth's center and whose axes are fixed in the terrestrial body. We assume that the axis C passes through the North Pole while A and B lie at the equator.

Let a second system of axes XYZ have the same zero point; we assume however that the XY -plane coincides with the plane of the moon's orbit, and the X -axis lies on the intersection of the equatorial plane and the moon's orbit.

Further let M be the moon and l the longitude of the moon measured in its orbit from X . Let J be the inclination of the moon's orbit with respect to the equator and χ the distance between the points X and A .

If we denote by M_1, M_2, M_3 the direction cosines of the moon with respect to the three axes ABC , so that

$$M_1 = \cos MA \quad M_2 = \cos MB \quad M_3 = \cos MC$$

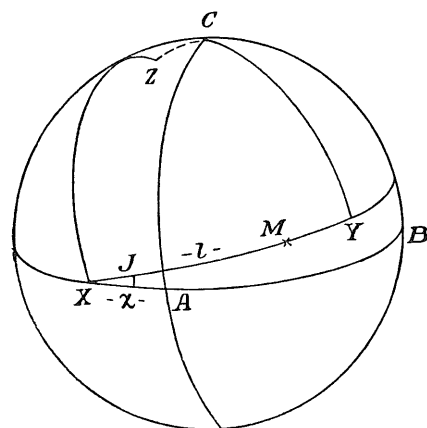


Fig. 1.

then we have, according to Fig. 1, from the triangles AMX , BMX and CMX

$$\left. \begin{aligned} M_1 &= \cos l \cos \chi + \sin l \sin \chi \cos J \\ M_2 &= -\cos l \sin \chi + \sin l \cos \chi \cos J \\ M_3 &= \sin l \sin J \end{aligned} \right\} \quad (2)$$

We set now for abbreviation

$$\cos \frac{J}{2} = p \quad \sin \frac{J}{2} = q; \quad \text{therefore} \quad p^2 + q^2 = 1, \quad (3)$$

then we have

$$\left. \begin{aligned} M_1 &= \cos l \cos \chi (p^2 + q^2) + \sin l \sin \chi (p^2 - q^2) = p^2 \cos (\chi - l) + q^2 \cos (\chi + l) \\ M_2 &= -\cos l \sin \chi (p^2 + q^2) + \sin l \cos \chi (p^2 - q^2) = -p^2 \sin (\chi - l) - q^2 \sin (\chi + l) \\ M_3 &= 2 p q \sin l. \end{aligned} \right\} \quad (4)$$

If we denote further by $\xi \eta \zeta$ the cosines for the direction to the zenith of the place of observation with respect to the axes ABC (not entered in Fig. 1), then we have

$$\cos z = \xi M_1 + \eta M_2 + \zeta M_3 \quad (5)$$

and there follows hence

$$\cos^2 z - \frac{1}{3} = 2 \xi \eta M_1 M_2 + 2 \frac{\xi^2 - \eta^2}{2} \frac{M_1^2 - M_2^2}{2} + 2 \eta \zeta M_2 M_3 + 2 \xi \zeta M_1 M_3 \left. \vphantom{\cos^2 z} \right\} + \frac{3}{2} \frac{\xi^2 + \eta^2 - 2 \zeta^2}{3} \frac{M_1^2 + M_2^2 - M_3^2}{3} \quad (6)$$

Now we will introduce definite assumptions for the position of the axes A and B at the equator. Let A lie on the meridian of the place of observation, and let B lie 90° east of A . If φ is the latitude of the place of observation, then we have

$$\begin{aligned} \xi &= \cos \varphi & \eta &= 0 & \zeta &= \sin \varphi, \\ \text{therefore} \quad \xi^2 - \eta^2 &= \cos^2 \varphi & \xi \eta &= \eta \zeta = 0 & 2 \xi \zeta &= \sin 2 \varphi \\ & \cdot \frac{1}{3} (\xi^2 + \eta^2 - 2 \zeta^2) &= \frac{1}{3} - \sin^2 \varphi. \end{aligned}$$

Then

$$\cos^2 z - \frac{1}{3} = \frac{M_1^2 - M_2^2}{2} \cos^2 \varphi + M_1 M_3 \sin^2 \varphi + \frac{M_1^2 + M_2^2 - 2 M_3^2}{2} \left(\frac{1}{3} - \sin^2 \varphi \right) \quad (7)$$

For this, we find from (4)

$$\left. \begin{aligned} M_1^2 - M_2^2 &= p^4 \cos 2(\chi - l) + 2 p^2 q^2 \cos 2 \chi + q^4 \cos 2(\chi + l) \\ M_1 M_3 &= -p^3 q \sin(\chi - 2l) + p q (p^2 - q^2) \sin \chi + p q^3 \sin(\chi + 2l) \\ \frac{M_1^2 + M_2^2 - 2 M_3^2}{3} &= \frac{1}{3} - M_3^2 = \frac{1}{3} (p^4 - 4 p^2 q^2 + q^4) + 2 p^2 q^2 \cos 2l. \end{aligned} \right\} \quad (8)$$

Now let c be the mean distance of the moon from the earth and e the eccentricity of the moon's orbit. We introduce further the quantities X , Y , Z by setting

$$X = \left(\frac{c(1-e^2)}{r} \right)^{\frac{3}{2}} M_1 \quad Y = \left(\frac{c(1-e^2)}{r} \right)^{\frac{3}{2}} M_2 \quad Z = \left(\frac{c(1-e^2)}{r} \right)^{\frac{3}{2}} M_3 \quad (9)$$

or

$$M_1 = X \left(\frac{r}{c(1-e^2)} \right)^{\frac{3}{2}} \quad M_2 = Y \left(\frac{r}{c(1-e^2)} \right)^{\frac{3}{2}} \quad M_3 = Z \left(\frac{r}{c(1-e^2)} \right)^{\frac{3}{2}}. \quad (10)$$

Then, according to (7), we will have

$$\cos^2 z - \frac{1}{3} = \left(\frac{r}{c(1-e^2)} \right)^3 \left\{ \frac{X^2 - Y^2}{2} \cos^2 \varphi + X Z \sin 2 \varphi + \frac{X^2 + Y^2 - 2 Z^2}{2} \left(\frac{1}{3} - \sin^2 \varphi \right) \right\}$$

and hence, according to (1),

$$V = \frac{3}{2} f \frac{M}{(1-e^2)^3 c^3} \left\{ \frac{X^2 - Y^2}{2} \cos^2 \varphi + X Y \sin 2 \varphi + \frac{X^2 + Y^2 - 2 Z^2}{2} \left(\frac{1}{3} - \sin^2 \varphi \right) \right\} \quad (11)$$

whereby, according to (8),

$$\left. \begin{aligned} X^2 - Y^2 &= \left(\frac{c(1-e^2)}{r} \right)^3 \left\{ p^4 \cos 2(\chi - l) + 2 p^2 q^2 \cos 2 \chi + q^4 \cos 2(\chi + l) \right\} \\ X Y &= \left(\frac{c(1-e^2)}{r} \right)^3 \left\{ -p^3 q \sin(\chi - 2l) + p q (p^2 - q^2) \sin \chi + p q^3 \sin(\chi + 2l) \right\} \\ \frac{1}{3} (X^2 + Y^2 - 2 Z^2) &= \left(\frac{c(1-e^2)}{r} \right)^3 \left\{ \frac{1}{3} (p^4 - 4 p^2 q^2 + q^4) + 2 p^2 q^2 \cos 2l \right\} \end{aligned} \right\} \quad (12)$$

For the further transformation of equation (11) we introduce three new auxiliary functions, namely

$$\varphi(\alpha) = \left(\frac{c(1-e^2)}{r} \right)^3 \cos(2l + \alpha) \quad \psi(\alpha) = \left(\frac{c(1-e^2)}{r} \right)^3 \cos \alpha \quad R = \left(\frac{c(1-e^2)}{r} \right)^3. \quad (13)$$

Then we have according to (12)

$$\left. \begin{aligned} X^2 - Y^2 &= p^4 \varphi(-2\chi) + 2p^2 q^2 \psi(2\chi) + q^4 \varphi(2\chi) \\ XZ &= -p^3 q \varphi(90^\circ - \chi) + p q (p^2 - q^2) \psi(\chi - 90^\circ) + p q^3 \varphi(\chi - 90^\circ) \\ \frac{1}{3}(X^2 + Y^2 - 2Z^2) &= \frac{1}{3}(p^4 - 4p^2 q^2 + q^4) R + 2p^2 q^2 \varphi(0). \end{aligned} \right\} \quad (14)$$

We will now develop the functions $\varphi(\alpha)$, $\psi(\alpha)$ and R in series according to (13), for which we must make use of a few concepts from theoretical astronomy. We start here first from the assumption that the moon moves around the earth on an invariable elliptic path.

In Fig. 2, the two arcs of great circles denote the moon's orbit and the ecliptic on the celestial sphere. The points γ and X are the points of intersection of these two great circles with the celestial equator, while Ω is the point of intersection of the moon's orbit with the ecliptic or the *ascending node of the moon's orbit*. In the latter, M denotes the moon and

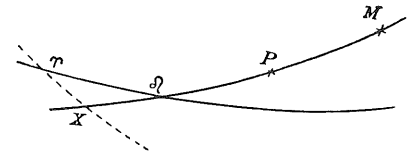


Fig. 2.

P the perigee, i.e., the point of the elliptic path of the moon which lies nearest to the earth. Then we have

$$\begin{aligned} l &= XM = \text{longitude of the moon in its orbit, counted from } X. \\ \tilde{\omega} &= XP = \text{longitude of the perigee.} \end{aligned}$$

In addition to the actual position of the moon at a definite time, we consider an imaginary mean position which the moon would assume at the same time if it moved in its orbit from P with a uniform mean velocity. The apparent distance of this point from the perigee P is the *mean anomaly* of the moon, which we will call μ , and then we have

$$\sigma = \tilde{\omega} + \mu = \text{mean longitude of the moon in its orbit, counted from } X.$$

The equation of the path ellipse of the moon is

$$\frac{c(1-e^2)}{r} = 1 + e \cos(l - \tilde{\omega})$$

and hence, we have

$$\left. \begin{aligned} R &= \left(\frac{c(1-e^2)}{r} \right)^3 = 1 + \frac{3}{2} e^2 + 3e \cos(l - \tilde{\omega}) + \frac{3}{2} e^2 \cos 2(l - \tilde{\omega}) + \dots \\ \varphi(\alpha) &= R \cos(2l + \alpha) = \left(1 + \frac{3}{2} e^2 \right) \cos(2l + \alpha) + \frac{3}{2} e \left\{ \cos(3l + \alpha - \tilde{\omega}) \right. \\ &\quad \left. + \cos(l + \alpha - \tilde{\omega}) \right\} + \frac{3}{4} e^2 \left\{ \cos(4l + \alpha - 2\tilde{\omega}) + \cos(\alpha + 2\tilde{\omega}) \right\} + \dots \\ \psi(\alpha) &= R \cos \alpha. \end{aligned} \right\} \quad (15)$$

Now it is convenient to replace l by σ , since σ increases in proportion to the time. For this, we have from the theory of the motion of a celestial body on an elliptic path

$$l = \sigma + 2e \sin(\sigma - \tilde{\omega}) + \frac{5}{4} e^2 \sin 2(\sigma - \tilde{\omega}) + \dots$$

We substitute this in the individual terms of (15) and obtain

$$\begin{aligned}
 \cos(2l + \alpha) &= (1 - 4e^2) \cos(2\sigma + \alpha) - 2e \cos(\sigma + \alpha + \tilde{\omega}) + 2e \cos(3\sigma + \alpha - \tilde{\omega}) \\
 &\quad + \frac{3}{4} e^2 \cos(\alpha + 2\tilde{\omega}) + \frac{13}{4} e^2 \cos(4\sigma + \alpha - 2\tilde{\omega}) + \dots \\
 \cos(3l + \alpha - \tilde{\omega}) &= \cos(3\sigma + \alpha - \tilde{\omega}) - 3e \cos(2\sigma + \alpha) + 3e \cos(4\sigma + \alpha - 2\tilde{\omega}) + \dots \\
 \cos(l + \alpha + \tilde{\omega}) &= \cos(\sigma + \alpha + \tilde{\omega}) + e \cos(2\sigma + \alpha) - e \cos(\alpha + 2\tilde{\omega}) + \dots \\
 \cos(4l + \alpha - 2\tilde{\omega}) &= \cos(4\sigma + \alpha - 2\tilde{\omega}) + \dots
 \end{aligned}$$

If we introduce all this into (15), then we find

$$\left. \begin{aligned}
 \varphi(\alpha) &= \left(1 - \frac{11}{2} e^2\right) \cos(2\sigma + \alpha) - \frac{1}{2} e \cos(\sigma + \alpha + \tilde{\omega}) + \frac{7}{2} e \cos(3\sigma + \alpha - \tilde{\omega}) \\
 &\quad + \frac{17}{2} e^2 \cos(4\sigma + \alpha - 2\tilde{\omega}) + \dots \\
 \psi(\alpha) &= \left(1 + \frac{3}{2} e^2\right) \cos \alpha + \frac{3}{2} e (\cos(\sigma + \alpha - \tilde{\omega}) + \cos(\sigma - \alpha - \tilde{\omega})) \\
 &\quad + \frac{9}{4} e^2 (\cos(2\sigma + \alpha - 2\tilde{\omega}) + \cos(2\sigma - \alpha - 2\tilde{\omega})) + \dots \\
 R &= \left(1 + \frac{3}{2} e^2\right) + 3e \cos(\sigma - \tilde{\omega}) + \frac{9}{2} e^2 \cos 2(\sigma - \tilde{\omega}) + \dots
 \end{aligned} \right\} \quad (16)$$

We have to introduce these developments in series (16) into (14), whereby we must substitute for α the quantities -2χ , 2χ , $90^\circ - \chi$, etc., indicated in (14). We limit ourselves however to retaining only those terms which have practical significance for our present purpose. Under this assumption we obtain

$$X^2 - Y^2 = \left(1 - \frac{11}{2} e^2\right) p^4 \cos 2(\chi - \sigma) + \left(1 - \frac{3}{2} e^2\right) 2 p^2 q^2 \cos 2\chi + \frac{7}{2} e p^4 \cos(2\chi - 3\sigma + \tilde{\omega}) \quad (17)$$

$$XY = -\left(1 - \frac{11}{2} e^2\right) p^3 q \sin(\chi - 2\sigma) + \left(1 - \frac{3}{2} e^2\right) p q (p^2 - q^2) \sin \chi \quad (18)$$

$$\frac{1}{3} (X^2 + Y^2 - 2Z^2) = \left(1 - \frac{11}{2} e^2\right) 2 p^2 q^2 \cos 2\sigma. \quad (19)$$

For simplification we will introduce instead of X , Y , Z new quantities X_0 , Y_0 , Z_0 by setting

$$X^2 = X_0^2 (1 - e^2)^3 \quad Y^2 = Y_0^2 (1 - e^2)^3 \quad Z^2 = Z_0^2 (1 - e^2)^3. \quad (20)$$

We will further replace the constant of attraction f by the force of gravity g as in the previous section 101. We have according to (8), section 101, p. 502,

$$f = g \frac{a^2}{E};$$

therefore, (11), p. 512, changes into

$$V = \tau g a \left\{ \frac{1}{2} (X_0^2 - Y_0^2) \cos^2 \varphi + X_0 Y_0 \sin 2\varphi + \frac{1}{2} (X_0^2 + Y_0^2 - 2Z_0^2) \left(\frac{1}{3} - \sin^2 \varphi \right) \right\}, \quad (21)$$

where

$$\tau = \frac{3}{2} \frac{M}{E} \left(\frac{a}{c} \right)^3$$

Now we have in addition to change the three functions (20), whose values result from (17) to (19), in such a way that all their terms contain only the cosine function with positive sign. This can easily be reached by inserting π or, as the case may be, $\frac{\pi}{2}$. In addition, if we carry out at the same time the division by $(1 - e^2)^3$, which becomes necessary according to (20), then we obtain

$$X_0^2 - Y_0^2 = \left\{ \left(1 - \frac{5}{2}e^2\right) p^4 \cos 2(\chi - \sigma) + \left(1 + \frac{3}{2}e^2\right) 2 p^2 q^2 \cos 2\chi + \frac{7}{2}e p^4 \cos(2\chi - 3\sigma + \tilde{\omega}) \right\} \quad (22)$$

$$X_0 Y_0 = \left\{ \left(1 - \frac{5}{2}e^2\right) p^3 q \cos\left(\chi - 2\sigma + \frac{1}{2}\pi\right) + \left(1 + \frac{3}{2}e^2\right) p q (p^2 - q^2) \cos\left(\chi - \frac{1}{2}\pi\right) \right\} \quad (23)$$

$$\frac{1}{3}(X_0^2 + Y_0^2 - 2Z_0^2) = \left(1 - \frac{5}{2}e^2\right) 2 p^2 q^2 \cos 2\sigma. \quad (24)$$

We have thus far counted the mean longitude of the moon in its orbit and the longitude of the perigee according to Fig. 2, p. 513, from the point of intersection X of the moon's orbit with the equator. We will now count the same quantities once again from another starting point which has in the orbit the same distance from the ascending node Ω as the vernal equinox Υ in the ecliptic. Let then

$p = \Upsilon \Omega + \Omega P$ = longitude of the perigee in the moon's orbit counted from the vernal equinox Υ .
 $s = p + \mu$ = mean longitude of the moon in its orbit,

where μ denotes again the mean anomaly of the moon.

In addition, the quantities χ , σ and $\tilde{\omega}$, which increase in proportion to the time, are now to be expressed by the sidereal time or mean time, as well as by the mean longitude of the moon and the longitude of the lunar perigee. In Fig. 3, illustrated in the margin, let M again denote the moon's center and P the perigee, while A and X have the same meaning as in Fig. 1, p. 511.

Let further

N = longitude of the node Ω
 ν = right ascension of the axis X
 i = inclination of the moon's orbit
 ω = obliquity of the ecliptic
 g = sidereal time
 ξ = longitude of point X in the moon's orbit counted from Υ .

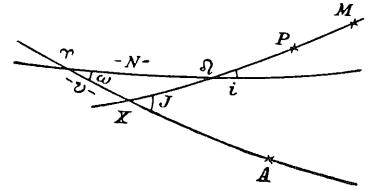


Fig. 3.

Then we have

$$g = A \Upsilon$$

and according to the above definition of the longitude in the orbit we have

$$\xi = \Upsilon \Omega - \Omega X. \quad (25)$$

Now since $s - p$ and according to p. 513 also $\sigma - \tilde{\omega}$ is equal to the mean anomaly of the moon, then we have

$$s - p = \sigma - \tilde{\omega}. \quad (26)$$

According to p. 513 and according to the above definition we have:

$$\begin{aligned}\tilde{\omega} &= X P \\ p &= \Upsilon \Omega + \Omega P.\end{aligned}$$

There follows hence with the help of (25)

$$\left. \begin{aligned}\tilde{\omega} &= p - \xi \\ \sigma &= s - \xi.\end{aligned} \right\} \quad (27)$$

We have further

$$\chi = X A = A \Upsilon - X \Upsilon = g - \nu. \quad (28)$$

We assume here that the sidereal time g is computed consecutively from the beginning of the observation. Since the observations take place in mean time, then it is better to express g by mean time t and by the mean longitude h of the sun. Then we have

$$\chi = t + h - \nu. \quad (29)$$

Now we have to introduce (27) and (29) into the functions (22) to (24) and, after this, to set up the potential V of the tidal force according to (21), which results as the sum of a series of individual terms.

The solar tides

The expression for the potential of the tidal force originating from the sun can forthwith be taken from the foregoing. The eccentricity of the apparent path of the sun is equal to 0.0168, and hence so small that most terms which depend on the eccentricity can be neglected. For the sun, the above coefficient

$$\tau = \frac{3}{2} \frac{M}{E} \left(\frac{a}{c} \right)^3 \text{ is to be replaced by}$$

$$\tau_1 = \frac{3}{2} \frac{S}{E} \left(\frac{a}{c_1} \right)^3,$$

where S denotes the mass of the sun and c_1 the mean distance of the sun from the earth. Here we have

$$\frac{\tau_1}{\tau} = 0.46035,$$

by which the lunar terms are to be multiplied.

If we denote by $V_{\text{☾}}$ and $V_{\text{☉}}$ the terms resulting from the moon, or from the sun, respectively, then

$$V = V_{\text{☾}} + V_{\text{☉}}$$

is the potential of the tidal force resulting from the attraction of the moon and the sun.

The theory of the tides was first treated in all thoroughness by G. H. Darwin in the treatise, "The Harmonic Analysis of Tidal Observations," report of a committee consisting of Professors G. H. Darwin and J. C. Adams for the harmonic analysis of tidal observations, *British Association Report* for 1883, pp. 49-118. It is also reprinted in *Scientific Papers* by Sir George Howard Darwin, Vol. I, "Oceanic Tides and Lunar Disturbance of Gravity," Cambridge, 1907, pp. 1-69. A further exposition is found in Börgen's "Die harmonische Analyse der Gezeitenbeobachtungen," *Ann. d. Hydrographie u. marit. Met.*, 1884. For the foregoing treatment of the subject, Darwin's publications were used primarily.

By the developments of the previous section 103 the tidal force is divided into a sum of simultaneously acting individual forces, each of which assigns to the plumb an elliptic motion of a special form and period. From the combined effect of the individual motions there results, consequently, an exceedingly complicated total motion of the plumb.

For the individual terms of the potential, (21), section 103, p. 514, as far as they possess practical significance, special symbols have been adopted; in the following, we will indicate them from (21) to (24), section 103, pp. 514 and 515.

We omit here the last term in (21), section 103, p. 514, which is hardly considered to be taken into account numerically.

1. Semidiurnal lunar terms

$$\left. \begin{aligned} M_2 &= \tau g a \frac{1}{2} \cos^2 \varphi \left(1 - \frac{5}{2} e^2 \right) p^4 \cos 2 (\chi - \sigma) \\ K_2 &= \tau g a \frac{1}{2} \cos^2 \varphi \left(1 + \frac{3}{2} e^2 \right) 2 p^2 q^2 \cos 2 \chi \\ N &= \tau g a \frac{1}{2} \cos^2 \varphi \frac{7}{2} e p^4 \cos (2 \chi - 3 \sigma + \tilde{\omega}). \end{aligned} \right\} \quad (1)$$

2. Diurnal lunar terms

$$\left. \begin{aligned} O &= \tau g a \sin 2 \varphi \left(1 - \frac{5}{2} e^2 \right) p^3 q \cos \left(\chi - 2 \sigma + \frac{1}{2} \pi \right) \\ K_1 &= \tau g a \sin 2 \varphi \left(1 + \frac{3}{2} e^2 \right) p q (p^2 - q^2) \cos \left(\chi - \frac{1}{2} \pi \right). \end{aligned} \right\} \quad (2)$$

For the sun, τ is to be replaced by τ_1 ; in addition, we will furnish the individual elements with the index 1. Then we have

3. Semidiurnal solar terms

$$\left. \begin{aligned} S_2 &= \tau_1 g a \frac{1}{2} \cos^2 \varphi \left(1 - \frac{5}{2} e_1^2 \right) p_1^4 \cos 2 (\chi_1 - \sigma_1) \\ K_2 &= \tau_1 g a \frac{1}{2} \cos^2 \varphi \left(1 + \frac{3}{2} e_1^2 \right) 2 p_1^2 q_1^2 \cos 2 \chi. \end{aligned} \right\} \quad (3)$$

4. Diurnal solar terms

$$\left. \begin{aligned} P &= \tau_1 g a \sin 2 \varphi \left(1 - \frac{5}{2} e_1^2 \right) p_1^3 q_1 \cos \left(\chi_1 - 2 \sigma_1 + \frac{1}{2} \pi \right) \\ K_1 &= \tau_1 g a \sin 2 \varphi \left(1 + \frac{3}{2} e_1^2 \right) p_1 q_1 (p_1^2 - q_1^2) \cos \left(\chi_1 - \frac{1}{2} \pi \right). \end{aligned} \right\} \quad (4)$$

If we substitute the values for χ , $\tilde{\omega}$ and σ from (27) and (29), section 103, p. 516, then we obtain

1. Semidiurnal lunar terms

$$\left. \begin{aligned} M_2 &= \frac{1}{2} \tau g a \cos^2 \varphi \left(1 - \frac{5}{2} e^2 \right) \cos^4 \frac{J}{2} \cos 2 (t + (h - s) + (\xi - \nu)) \\ K_2 &= \frac{1}{4} \tau g a \cos^2 \varphi \left(1 + \frac{3}{2} e^2 \right) \sin^2 J \cos 2 (t + (h - \nu)) \\ N &= \frac{7}{4} \tau g a \cos^2 \varphi e \cos^4 \frac{J}{2} \cos (2 t + 2 (h - s) + 2 (\xi - \nu) + (s - p)). \end{aligned} \right\} \quad (5)$$

2. Diurnal lunar terms

$$\left. \begin{aligned} O &= \frac{1}{2} \tau g a \sin 2 \varphi \left(1 - \frac{5}{2} e^2 \right) \cos^2 \frac{J}{2} \sin J \cos \left(t + (h - \nu) - 2(s - \xi) + \frac{1}{2} \pi \right) \\ K_1 &= \frac{1}{2} \tau g a \sin 2 \varphi \left(1 + \frac{3}{2} e^2 \right) \sin J \cos J \cos \left(t + (h - \nu) - \frac{1}{2} \pi \right). \end{aligned} \right\} \quad (6)$$

In the solar terms, $q_1 = \sin \omega$, $p_1 = \cos \omega$, and besides, $\nu = 0$, $\xi = 0$; therefore, $\chi_1 = t + h$, $\sigma_1 = h$. With these, we will have

3. Semidiurnal solar terms

$$\left. \begin{aligned} S_2 &= \frac{1}{2} \tau_1 g a \cos^2 \varphi \left(1 - \frac{5}{2} e_1^2 \right) \cos^4 \frac{\omega}{2} \cos 2 t \\ K_2 &= \frac{1}{4} \tau_1 g a \cos^2 \varphi \left(1 + \frac{3}{2} e_1^2 \right) \sin^2 \omega \cos 2(t + h). \end{aligned} \right\} \quad (7)$$

4. Diurnal solar terms

$$\left. \begin{aligned} P &= \frac{1}{2} \tau_1 g a \sin 2 \varphi \left(1 - \frac{5}{2} e_1^2 \right) \sin \omega \cos^2 \frac{\omega}{2} \cos \left(t - h + \frac{1}{2} \pi \right) \\ K_1 &= \frac{1}{2} \tau_1 g a \sin 2 \varphi \left(1 + \frac{3}{2} e_1^2 \right) \sin \omega \cos \omega \cos \left(t + h - \frac{1}{2} \pi \right). \end{aligned} \right\} \quad (8)$$

Then

$$V = V_{\zeta} + V_{\odot}, \quad (9)$$

if we set

$$\left. \begin{aligned} V_{\zeta} &= M_2 + K_2 \zeta + N + K_1 \zeta + O + \dots \\ V_{\odot} &= S_2 + K_2 \odot + K_1 \odot + P + \dots \end{aligned} \right\} \quad (10)$$

On the basis of the above formulae we are in a position to compute the individual terms and thus also the total potential for a given interval of time.

Now we will bring also the formulae for the computation of the disturbances of the plumb, about which we have learned in (14) to (16), section 101, p. 504, into another form.

Quite generally, we can express an arbitrary term of the potential V in the following manner:

Let V_0 be the part of the argument of the cosine function dependent on the longitudes h , s and ξ or, as the case may be, on ν at an arbitrary time T_0 and i the hourly change of the whole argument; then an arbitrary term in V has the form

$$V = \tau g a f(\varphi) \frac{1}{2} C \cos(i t + \varrho T_0 + V_0). \quad (11)$$

Here ρ is equal to 0, equal to 1 or equal to 2, further $f(\varphi)$ is equal to $\cos^2 \varphi$, $\sin 2 \varphi$ or equal to 1. In the course of a year, the quantity C can be regarded as constant. Finally, t denotes the number of hours since the time T_0 .

The disturbance of the plumb in the azimuth α is then for this term according to (15), section 101, p. 504,

$$\eta = \tau \left\{ - \frac{d f(\varphi)}{d \varphi} \frac{1}{2} C \cos(i t + \varrho T_0 + V_0) \cos \alpha + \frac{1}{\cos \varphi} f(\varphi) \frac{\varrho}{2} C \sin(i t + \varrho T_0 + V_0) \sin \alpha \right\}. \quad (12)$$

If we introduce two auxiliary quantities p and A by means of the equations

$$\left. \begin{aligned} p \sin A &= \frac{\varrho}{2} \frac{1}{\cos \varphi} f(\varphi) \sin \alpha \\ p \cos A &= -\frac{1}{2} \frac{df(\varphi)}{d\varphi} \cos \alpha \end{aligned} \right\} \quad (13)$$

then we will have

$$\eta = \tau C p \cos (i t + \varrho T_0 + V_0 - A)$$

or, if we combine ϱT_0 and V_0

$$\eta = \tau C p \cos (i t + V_0 - A). \quad (14)$$

If we denote the individual terms of the disturbance of the plumb η by the same letters as those of the potential V , then we have

$$\eta = M_2 + S_2 + N + K_2 + K_1 + O + P \dots, \quad (15)$$

whereby in K_2 and K_1 the parts of the moon and the sun are combined.

Section 105. The Evaluation of the Horizontal Pendulum Measurements

The deflections of two horizontal pendulums set up perpendicular to each other yield the motions of the plumb in the two measured directions. The point in question is now to divide, likewise, these measured motions of the plumb into their individual elements, and hence, to determine the size and position of the individual ellipses. This problem is solved with the help of harmonic analysis, which was developed first by Lord Kelvin in the years 1868-1878 in its application to the tides of the ocean. A detailed theory of harmonic analysis was then published by G. H. Darwin in 1883, and referring to this, C. Börgen has given the fundamental exposition already mentioned at the end of section 103, p. 516. C. Börgen, in the treatise: "Über eine neue Methode, die harmonischen Konstanten der Gezeiten abzuleiten," *Ann. d. Hydrogr. u. marit. Met.*, 1894, published another method which leads more quickly to the goal in numerical computation.

These methods, designed only for the observation of the tides of the ocean, were also used when the evaluation of horizontal pendulum measurements were involved. At first, one limited oneself to determine the semidiurnal lunar term M_2 , numerically most significant. But when it was recognized that for the study of the effect of the moon and the sun, the examination of other terms is also necessary, the above indicated methods were applied to the horizontal pendulum measurements to a greater extent. An exposition especially adapted to this purpose is contained in *Veröffentlichung des Geodätischen Instituts in Potsdam*, "Harmonische Analyse der Lotstörungen durch Sonne und Mond" von Dr. Wilhelm Schweydar, Potsdam, 1914.

The basic idea of harmonic analysis is the following: In order to find at first the terms depending on the sun, the observations by solar time are combined and the mean of all measurements which correspond to the same hour angle of the sun is formed. If the series of observations extends over a rather long period, then various hour angles of the moon will correspond to every hour angle of the sun. Consequently, in taking the mean value the lunar terms will cancel, and the solar terms alone will remain.

Conversely, in the same way we will carry out a grouping of the measurements by hour angles of the moon in order to determine the lunar terms. The method can even be organized still further so that we can also separate the semidiurnal terms from the diurnal. For carrying out the method we refer to the above indicated publication by W. Schweydar.

Harmonic analysis yields an arbitrary term of η in the form

$$\eta = R \cos (i t - \zeta). \quad (1)$$

In order to make the measured values of different years comparable with one another, the results are brought into another form. Instead of the original amplitude R , which depends on the variable quantities J and ν , there is introduced the reduced amplitude H , which refers to mean values of J and ν .

We compute further the quantity V_0 for the time of the beginning of the observations and add it to all values of ζ . According to theory we shall have

$$\zeta = -V_0 + A;$$

in reality, however, there will occur deviations from it, which we will denote by κ . Consequently, we are to set

$$\zeta + V_0 = A + \kappa$$

and then we have the reduced values

$$\eta' = H \cos (i t - \kappa - A), \quad (2)$$

which can be compared with other series of observations.

Of the newer measurements with the horizontal pendulum, those which v. Rebeur-Paschwitz carried out in Wilhelmshaven and Potsdam in 1889, in Puerto Orotava on Teneriffa during the years 1890-1891 and in Strassbourg from 1892-1893 with the horizontal pendulum constructed by him and described on p. 507 deserve to be mentioned first. V. Rebeur-Paschwitz succeeded for the first time in determining unobjectionably the motion of the plumb caused by the changing attraction of the moon; the accuracy of the measurements, however, was not yet sufficient in order to draw further conclusions from the results. On the other hand, in these measurements there already appeared the inconvenience that in the course of a day the ground is subjected, by the changes of the temperature and by solar radiation, to periodic warping, which is expressed in the horizontal pendulum and which considerably exceeds the motions of the plumb caused by the tidal force of the moon.

In order to avoid this difficulty, measurements were carried out by Hecker in the Geodetic Institute in Potsdam with the horizontal pendulum described on p. 508 and illustrated in Fig. 5 on p. 508 in a chamber located 25 m below the ground next to the well installation of the Institute. Becker reported about the results of these measurements in *Veröffentlichungen des Geodätischen Instituts*, "Beobachtungen an Horizontalpendeln über die Deformation des Erdkörpers unter dem Einfluss von Sonne und Mond," I. Heft, Berlin, 1907, II. Heft, Berlin, 1911. There appeared, in fact, a very considerable decrease of the influence of the solar radiation, which remained however still greater than the influence of the tidal force of the moon. If the solar radiation were effective every day in the same manner, then we could easily separate its effect from that of the tidal force of the moon due to the diversity of the periods. Although this does not prove true, the influence of the solar radiation becomes weakened in the case of continuous recording extending over a whole year so that usable results could nevertheless be derived from the measurements mentioned.

At the sixteenth conference of the Internationale Erdmessung in London in 1909, at which these questions were likewise discussed, further horizontal pendulum measurements at a greater depth below the earth's surface were suggested. The first experiments which were undertaken in Příbram in Bohemia in the mines at a depth of 1100 m did not lead to the goal due to the stronger motion of the mountains; it could be established, however, that the influence of the solar radiation is imperceptible at this depth.

Measurements at the ore mines in Freiburg, Saxony, were carried out to a rather great extent from December 1910 to December 1915. On this subject, a detailed report has been prepared by W. Schweydar in *Veröffentlichung der Internationalen Erdmessung*, "Lotschwankung und Deformation der Erde durch Flutkräfte, gemessen mit zwei Horizontalpendeln im Bergwerk in 189 m Tiefe bei Freiburg i. Sa.," Berlin, 1921.

For the measurement there were used two Zöllner horizontal pendulums with a length of about 25 cm, which were set up at the azimuth 140° and 230° (counted from the south). We have already given a short description of these instruments in section 102, p. 507.

In the working up of the measurements, the material was divided into five parts of 371 days within which harmonic analysis was applied. From this, there resulted at first, for the individual terms, the amplitude R and the phase ζ according to equation (1), p. 519, from which the reduced amplitudes H and the

reduced phases $A + \kappa$ were then computed, e.g., the values assembled in the following were found for the term M_2 which was the largest occurring numerically.

Pendulum I. Azimuth $50^\circ 29'$

Section	R	ζ	H	$\kappa + A$	A
1	0.00 554''	121.1°	0.00 570''	65.9°	57.4°
2	560	163.6	580	62.3	57.4
3	603	208.2	625	62.3	57.4
4	578	250.7	598	61.0	57.4
5	539	283.8	554	60.6	57.4
Mean			0.00 585''	62.4°	57.4°

Pendulum II. Azimuth $139^\circ 35'$

Section	R	ζ	H	$\kappa + A$	A
1	0.00 480''	191.8°	0.00 494''	136.6°	132.3°
2	415	72.7	430	136.0	132.3
3	464	348.4	481	132.0	132.3
4	396	32.6	410	131.4	132.3
5	384	359.8	395	136.6	132.3
Mean			0.00 442''	134.5°	132.3°

The mean values can only be formed from H and $\kappa + A$ but not from R and ζ , to which we have already referred on p. 520.

In the same manner, the values of the other terms S_2 , K_2 , N , K_1 , O and P were determined according to equation (15), section 104, p. 519. At the same time, the diurnal wave, which is caused by the above-mentioned solar radiation, has been taken into account also by a further term S_1 . In the following table we have indicated the mean values of the individual terms, as they are found from the whole 5-year series of observations.

	Azimuth $50^\circ 29'$			Azimuth $139^\circ 35'$		
	H	$\kappa + A$	A	H	$\kappa + A$	A
M_2	0.00 585''	62.4°	57.4°	0.00 442''	134.5°	132.3°
S_2	258	23.8	57.4	428	124.6	132.3
K_2	077	47.8	57.4	093	136.5	132.3
N	133	57.2	57.4	083	136.8	132.3
K_1	407	98.4	77.7	420	83.8	107.2
S_1	148	251.6		159	101.7	
O	341	110.1	77.7	273	60.8	107.2
P	121	114.5	77.7	218	89.5	107.2

For the term S_1 , the value of A is not listed, since this term is not to be attributed to the tidal force.

The above results were used further for the determination of the motions of the plumb in the direction of the meridian and of the prime vertical. If we denote the disturbances of the plumb in the above two azimuths by η_1 and η_2 and the azimuths by α_1 and α_2 , and further the northern and the western component by η_n and η_w , then we have according to (15) and (14), section 101, p. 504,

$$\begin{aligned}\eta_1 &= -\eta_n \cos a_1 + \eta_w \sin a_1 \\ \eta_2 &= -\eta_n \cos a_2 + \eta_w \sin a_2.\end{aligned}$$

There follows hence

$$\eta_n = \frac{\eta_1 \sin a_2 - \eta_2 \sin a_1}{\sin(a_1 - a_2)} \quad \eta_w = \frac{\eta_1 \cos a_2 - \eta_2 \cos a_1}{\sin(a_1 - a_2)}$$

or, if we limit ourselves to η_n , in simplified notation

$$\eta_n = p \eta_1 + q \eta_2.$$

But now we have

$$\begin{aligned}\eta_1 &= H_1 \cos(i t - \kappa_1') & \kappa_1' &= \kappa_1 + A_1 \\ \eta_2 &= H_2 \cos(i t - \kappa_2') & \kappa_2' &= \kappa_2 + A_2,\end{aligned}$$

and hence we will have

$$\eta_n = p H_1 \cos(i t - \kappa_1') + q H_2 \cos(i t - \kappa_2').$$

If we set

$$\kappa_2' = \kappa_1' + \Delta \kappa',$$

then

$$\eta_n = p H_1 \cos(i t - \kappa_1') + q H_2 \cos(i t - \kappa_1' - \Delta \kappa')$$

or

$$\eta_n = (p H_1 + q H_2 \cos \Delta \kappa') \cos(i t - \kappa_1') + q H_2 \sin \Delta \kappa' \sin(i t - \kappa_1').$$

We can contract this expression by introducing the new quantities H_n and κ_n through the equations

$$\begin{aligned}p H_1 + q H_2 \cos \Delta \kappa' &= H_n \cos(180^\circ + \kappa_n - \kappa_1') \\ q H_2 \sin \Delta \kappa' &= H_n \sin(180^\circ + \kappa_n - \kappa_1')\end{aligned}$$

Then we will have

$$\eta_n = H_n \cos(i t - \kappa_n - 180^\circ),$$

with which we have also expressed the disturbance of the plumb in the meridian in the form of equation (2), p. 520. We can also find a corresponding expression in the same manner for η_w .

The conversion yielded for the northern and for the western component the following values as the mean values of the whole series of measurements:

	Azimuth 180°			Azimuth 90°		
	<i>H</i>	<i>A</i> + κ	<i>A</i>	<i>H</i>	<i>A</i> + κ	<i>A</i>
<i>M</i> ₂	0.00 418''	192.5°	180°	0.00 605''	89.2°	90°
<i>S</i> ₂	391	148.9	180	309	85.4	90
<i>K</i> ₂	085	171.2	180	085	92.7	90
<i>N</i>	096	196.9	180	125	82.2	90
<i>K</i> ₁	095	41.1	180	580	91.6	90
<i>S</i> ₁	207	88.6		057	187.0	
<i>O</i>	177	353.1	180	401	90.6	90
<i>P</i>	101	71.0	180	229	99.5	90

On the basis of the evaluation of the observations, p. 522, the terms of expression (15), p. 519, can now be assembled with the insertion of the term S_1 for the influence of the solar radiation as functions of the time, whereby their coefficients are mean values of the whole series of observations. We refer for this to the publication by W. Schweydar and, in addition, reproduce from the latter only the two illustrations, Fig. 1 and Fig. 2, from which the motion of the plumb during two days results. We see hence, on one hand, in what a complicated form the influence of the tidal force of the moon and of the sun on the geoid manifests itself; on the other hand, there is also revealed with it the amazing efficiency of the horizontal pendulum which makes it possible to determine variations of the plumb of a few thousandths of a second of arc.

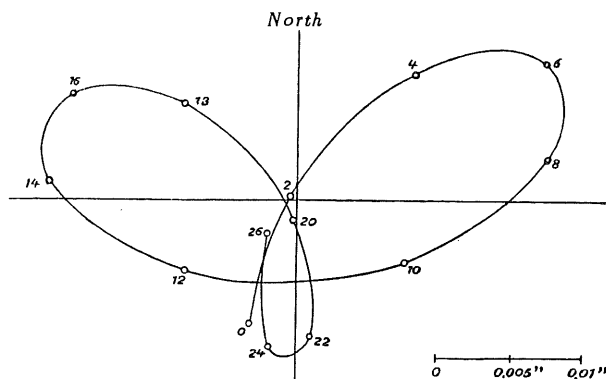


Fig. 1.
Motion of the plumb, 1912, Jan. 2.0 h - Jan. 3.2 h.

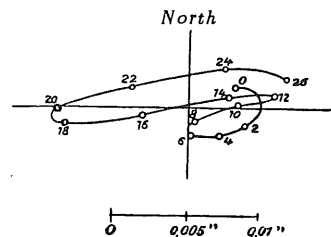


Fig. 2.
Motion of the plumb, 1912,
Jan. 9.0 h - Jan. 10.2 h.

Now it is of interest to compute the motions of the plumb also purely theoretically according to equations (5) to (8), section 104, pp. 517 and 518, in order to make the comparison of the observational results with the theory possible. For this, we bring the theoretically computed terms of η likewise into the form $H \cos (it - \kappa')$, where κ' is now equal to A while it was equal to $A + \kappa$ for the measured values. In the following tables, the results of observation and theory are arranged side by side in a readily visible manner.

	Azimuth 50.5°				Azimuth 139.6°			
	Observed H	κ'	Theoretical H	κ'	Observed H	κ'	Theoretical H	κ'
M_2	0.00 585''	62.4°	0.00 914''	57.4°	0.00 442''	134.5°	0.00 874''	132.3°
S_2	258	23.8	424	57.4	428	124.6	406	132.3
K_2	077	47.8	116	57.4	093	136.5	111	132.3
N	133	57.2	177	57.4	083	136.8	169	132.3
K_1	407	98.4	566	77.4	420	83.8	486	107.2
O	341	110.1	402	77.4	273	60.8	345	107.2
P	121	114.5	187	77.4	218	89.5	161	107.2
S_1	148	251.6	—	—	159	101.7	—	—

	West				North			
	Observed H	κ'	Theoretical H	κ'	Observed H	κ'	Theoretical H	κ'
M_2	0.00 605''	89.2°	0.00 997''	90°	0.00 418''	192.5°	0.00 774''	180°
S_2	309	85.4	463	90	391	148.9	360	180
K_2	085	92.7	126	90	085	171.2	098	180
N	125	82.2	193	90	096	196.9	150	180
K_1	580	91.6	716	90	095	41.1	189	180
O	401	90.6	509	90	177	353.1	134	180
P	229	99.5	237	90	101	71.0	062	180
S_1	057	187.0	—	—	207	88.6	—	—

Upon comparison of the observed and the theoretical values it is obvious at first glance that the theoretically computed amplitudes H are larger throughout than the amplitudes found from the observations. This phenomenon, which has also become evident in the case of all previous horizontal pendulum measurements, indicates that the assumption of the absolutely rigid terrestrial body on which the theory of section 104 is based does not prove correct in reality, and that the earth is rather to be regarded as an elastic body. The above differences offer now the possibility to draw valuable conclusions as to the degree of elasticity of the earth, which we will discuss in greater detail in the following.

Section 106. Determination of the Elasticity of the Earth

If the earth were a completely rigid body, then the horizontal pendulum fastened on the earth's surface would have to indicate in full size the variations of the direction of the vertical caused by the tidal force of the moon and the sun. On the other hand, if the terrestrial body were completely flexible, upon every change of the direction of the vertical it would at once adjust itself to the new position of equilibrium, and consequently, the horizontal pendulum would not show any deflection. The comparison of the motions of the plumb determined with the horizontal pendulum and the motions of the plumb computed under the assumption of an absolutely rigid terrestrial body shows that the earth yields in part to the influence of the tidal force, and hence represents an elastic body. The ratio of the two disturbances of the plumb yields a measure for the elasticity of the earth. We will denote this ratio by γ and have therefore

$$\gamma = \frac{\text{measured disturbance of the plumb}}{\text{theoretical disturbance of the plumb}}.$$

We can find the value of γ from the amplitudes H of the observed and of the theoretical variations of the plumb, and the ratio γ would have to have the same value for all terms of η . In the earlier measurements in Strassbourg and Potsdam, which we have already mentioned on p. 520, as well as in the measurements in Freiburg in the Breisgau, and in Dorpat, it became evident now that the value of γ in the north-south direction is a different one than in the east-west direction. A different elasticity of the earth was concluded therefrom for these two directions without being able to indicate for this a satisfactory explanation, however. Only the principal term M_2 of the tidal force, however, was derived from these earlier measurements, so that they did not furnish very extensive material for the further discussion of the question. Referring to the measurements in Freiburg, Saxony, Schweydar has made this phenomenon the object of exhaustive studies, which he has laid down in *Veröffentlichung des Geodätischen Instituts in Potsdam*, "Theorie der Deformation der Erde durch Flutkräfte," Potsdam, 1916. Schweydar proves that, because of the attractive effect of the elevated and depressed water masses, the tides of the ocean exert a perceptible influence on the motion of the plumb, and in fact, not only at the coasts, but also within the continents, and that the difference found in the values of γ can be explained by this influence.

The investigations of Schweydar indicated further that the values of γ determined from the diurnal terms must be nearly free from the influence of the tides of the ocean, while this influence in the case of the semidiurnal terms, in particular in the case of the lunar term M_2 , is especially great. On the other hand, only the semidiurnal terms could hitherto be derived to a rather great accuracy from the horizontal pendulum measurements, and only the discussion of the measurements of Freiburg yielded the possibility of determining reliably also the diurnal terms.

In the treatise, *Lotschwankungen und Deformation der Erde durch Flutkräfte*, Berlin, 1921, Schweydar tries to determine more accurately the influence of the tides of the ocean on the motion of the plumb from the Freiburg measurements and finds that the diurnal terms K_1 , O and P are influenced by the tides of the ocean almost entirely in the direction of the meridian. Consequently, the east-west components of these three terms originate almost entirely from the tidal force and are therefore especially suited for the determination of γ . We find from the table on p. 523

for K_1	$\gamma = 0.81,$
for O	$\gamma = 0.79,$
for P	$\gamma = 0.97.$
	$\gamma = 0.86.$

As the mean value we thus have

Under plausible assumptions, Schweydar also reaches the point of determining the ellipses of oscillation corresponding to the tides of the ocean for the two more accurately determined terms K_1 and O , so that there also results, in addition, the small influence in the east-west direction. By this more accurate computation there is found for γ the final value

$$\gamma = 0.841 \quad (1)$$

which we would have obtained directly from all terms if the earth were without oceans.

Concept of elasticity of form or rigidity of a body. Let E be the modulus of elasticity of length, i.e., the force which would have to act at the end of a rod of cross section 1 in order to double the length of the rod.

Let further be σ the modulus of transverse contraction which is defined by the equation

$$\sigma = \frac{\Delta d}{d} : \frac{\Delta l}{l}$$

$\frac{\Delta l}{l}$ denotes here the relative extension of the rod and $\frac{\Delta d}{d}$ the relative reduction of its cross section.

According to Lamé

$$n = \frac{1}{2} \frac{E}{1 + \sigma} \quad (2)$$

is then the constant of rigidity of the body.

For example, for steel $E = 21,800 \text{ kg/qmm}$ and $\sigma = 0.31$, therefore we have for steel

$$n = 8.2 \times 10^{11} \text{ c/g/s} \quad (3)$$

in units of the cm-g-sec system.

The constant of rigidity of the earth

The theoretical principles for the determination of the constant of rigidity from horizontal pendulum measurements were first worked up by W. Schweydar in the treatise, "Ein Beitrag zur Bestimmung des Starrheitskoeffizienten der Erde" in *Gerlands Beiträge zur Geophysik*, Vol. IX, 1907, pp. 41-77. This theory rests on Wiechert's law for the density of the earth, according to which the earth consists of a homogeneous nucleus of density 8 and a likewise homogeneous rock mantle, about 1500 km thick, of density 3. A special coefficient of rigidity was quoted for the nucleus and the mantle each.

Schweydar gave a new treatment of the problem in *Veröffentlichung des Geodätischen Instituts*, "Theorie der Deformation der Erde durch Flutkräfte," Potsdam, 1916. Here it is assumed at first that the density ρ as well as the rigidity n are an arbitrary function of the distance from the earth's center. On this basis there are developed the changes of form of the terrestrial body if small exterior deforming forces act on it. For the density there is then introduced the law which Roche has derived in the paper, "Mémoire sur la loi de densité à l'intérieur de la Terre" (*Académie des Sciences de Montpellier*, 1848). According to this, we have

$$\rho = \rho_0 \left(1 - \beta \left(\frac{r}{a} \right)^2 \right) \quad \rho_0 = 10.1 \quad \beta = 0.764. \quad (4)$$

Besides, the assumption is made that the constant of rigidity n is subject to a law of the same form, so that

$$n = n_0 \left(1 - \eta \left(\frac{r}{a} \right)^2 \right) \quad (5)$$

can be set, where the constants n_0 and η are yet to be determined.

Now let W_0 be the potential of gravity at a point of the earth's surface, which we can imagine, for the purpose in question, as a sphere. We denote the potential of the tidal force by V . By the effect of the tidal force there occurs a deformation of the sphere, which changes the potential of gravity by ΔW . We assume that, due to deformation, a point of the earth's surface is moved in the radial direction by u , which has as a consequence a change of the potential of gravity by the amount $u \frac{\partial W_0}{\partial r}$.

According to this, the potential of all forces which act on a point of the earth's surface is

$$R = W_0 + u \frac{\partial W_0}{\partial r} + \Delta W + V. \quad (6)$$

The theory of elasticity shows that we can express the elastic displacement u and the disturbance of the potential of gravity in the form

$$u = k \frac{V}{g} \quad \Delta W = h V \quad (7)$$

where h and k are functions of the distribution of density and elasticity in the earth. In addition, we have according to (18), section 61, p. 308,

$$\frac{\partial W}{\partial r} = -g. \quad (8)$$

Consequently,

$$R = W_0 + V(1 + h - k). \quad (9)$$

$V(1 + h - k)$ thus represents the total disturbance of the potential of gravity at an arbitrary point of the earth's surface due to the effect of the tidal force.

For an infinitesimal displacement of the point in an arbitrary direction on the earth's surface, the linear element is equal to $a d\psi$, if $d\psi$ denotes the central angle. Therefore, the horizontal component of the force which corresponds to the potential $V(1 + h - k)$ is equal to $\frac{1}{a} \frac{\partial V}{\partial \psi} (1 + h - k)$, and there results a deflection of the plumb

$$\frac{1}{a g} \frac{\partial V}{\partial \psi} (1 + h - k). \quad (10)$$

If the terrestrial body were absolutely rigid, then the deflection of the plumb would be equal to

$$\frac{1}{a g} \frac{\partial V}{\partial \psi} \quad (11)$$

we thus have according to p. 525

$$\gamma = 1 + h - k. \quad (12)$$

The development of h and k with the help of the theory of elasticity leads to the result

$$k = \frac{Z}{N} \quad h = \frac{Z_1}{N}, \quad (13)$$

where Z , Z_1 and N are the following functions of η .

$$\left. \begin{aligned} Z &= \frac{1}{e} (a + b\eta + c\eta^3 + \dots) + (a' + b'\eta + \dots) + e(a'' + b''\eta + \dots) + \dots \\ Z_1 &= \frac{1}{e} (a_1 + b_1\eta + c_1\eta^3 + \dots) + (a_1' + b_1'\eta + \dots) + e(a_1'' + b_1''\eta + \dots) + \dots \\ N &= \frac{1}{e^2} (p + q\eta + r\eta^2 + \dots) + \frac{1}{e} (p' + q'\eta + \dots) + (p'' + q''\eta + \dots) + e(p''' + q''' \eta + \dots) + \dots \end{aligned} \right\} \quad (14)$$

Here we have

$$e = \frac{36}{5} \frac{a g \rho_0}{n_0} \frac{\beta^2}{1 - 0.6 \beta} \quad \text{or} \quad e = \frac{489.7}{n_0} 10^7 \text{ c/g/s}, \quad (15)$$

while $a b c$, etc., are determinate coefficients which can be indicated numerically.

If the above expressions (13) to (15) are introduced into (12), then we obtain an equation with the two unknowns n_0 and η . We can now replace n_0 by the coefficient of rigidity \bar{n} at the earth's surface, for which, according to (5), p. 525, the equation

$$\bar{n} = n_0 (1 - \eta)$$

exists. The value of \bar{n} has been determined several times from the velocity of the earth waves at the earth's surface. Schweydar uses the value determined by Haussmann and Zeissig

$$\bar{n} = 3.08 \times 10^7 \text{ c/g/s}.$$

Then we will have

$$e = \frac{489.7}{\bar{n}} \times 10^7 (1 - \eta) \text{ c/g/s} \quad \text{or} \quad e = m (1 - \eta), \quad (16)$$

where $m = 159$.

If we introduce (13) to (16) as well the value (1) into (12), then we obtain an equation of the form

$$u + v\eta + w\eta^2 + \dots = m(u' + v'\eta + \dots) + m^2(u'' + v''\eta + \dots) + m^3(u''' + v''' \eta + \dots), \quad (17)$$

in which the coefficients $u v w$, etc., are again known. The equation contains only η as the single unknown, and we find hence by successive approximation

$$\eta = 0.90, \quad (18)$$

$$\text{and with this, we will have} \quad n_0 = 30.8 \times 10^{11} \quad (19)$$

$$\text{and} \quad n = 30.8 \times 10^{11} \left(1 - 0.90 \frac{r^2}{a^2}\right) \text{ c/g/s}. \quad (20)$$

Therefore, the rigidity of the earth at the center is $n_0 = 30.8 \times 10^{11} \text{ c/g/s}$ and at the earth's surface $\bar{n} = 3.1 \times 10^{11} \text{ c/g/s}$. We see from (3) that the terrestrial body at the center is almost four times as strong as steel.

In the investigations about the mathematical shape of the earth hitherto carried out, we have started implicitly from the assumption that the rotation of the earth takes place constantly around the main axis corresponding to the moment of inertia C , or the geometrical axis of the terrestrial body. We will now investigate more closely to what extent this assumption proves correct by communicating some data about the theory of the rotation of the earth.

The motion of the terrestrial body in space is composed of a motion of its center of gravity and a rotation around the center of gravity. Both motions originate from a starting motion and from the reciprocal attraction of the earth, the sun, the moon and the planets.

For the rotary motion of the earth around its center of gravity, which alone occupies us here, we imagine the earth as a rigid body. This motion then takes place at each moment as if the earth rotates around an axis which changes its position in space with time; we call this axis the instantaneous axis of rotation. Let there be denoted by O the center of gravity of the terrestrial body, by OJ the direction of the instantaneous axis of rotation for a time t and by ω the angular velocity of the rotation for the same time. In addition, we also introduce a coordinate system Ox, Oy, Oz which corresponds to that used in section 63, p. 311; the directions of the axes of coordinates thus coincide with the three main axes.

If we imagine the rotary motion around the axis OJ , lying in an arbitrary position, divided into three components which refer to these three axes of coordinates as axes of rotation, and if p, q, r are the angular velocities of these three components, then we have

$$\left. \begin{aligned} p &= \omega \cos(J, x) \\ q &= \omega \cos(J, y) \\ r &= \omega \cos(J, z) \\ p^2 + q^2 + r^2 &= \omega^2. \end{aligned} \right\} \quad (1)$$

and

By (J, x) , (J, y) and (J, z) there are understood here the three angles between the instantaneous axis of rotation OJ and the axes of coordinates.

By the introduction of the principal moments of inertia A, B, C of section 62, p. 310, the equations of motion of the terrestrial body named after Euler read then

$$\left. \begin{aligned} A \frac{dp}{dt} + (C - B)qr &= L \\ B \frac{dq}{dt} + (A - C)rp &= M \\ C \frac{dr}{dt} + (B - A)pq &= N, \end{aligned} \right\} \quad (2)$$

in which L, M, N are the moments of rotation of the external forces acting on the terrestrial body.

In order to obtain a first approximation we can neglect taking into account the external forces. Since we already know, besides, from section 64, p. 315, that the two moments of inertia A and B can be set equal to one another, then we obtain the simple equations:

$$\left. \begin{aligned} A \frac{dp}{dt} + (C - A)qr &= 0 \\ A \frac{dq}{dt} - (C - A)rp &= 0 \\ C \frac{dr}{dt} &= 0. \end{aligned} \right\} \quad (3)$$

From the third equation there follows directly that r is equal to a constant; therefore, we set $r = r_0$. If we set further for abbreviation

$$\frac{A - C}{A} = \mu,$$

then

$$\begin{aligned}\frac{d p}{d t} - \mu r_0 q &= 0 \\ \frac{d q}{d t} + \mu r_0 p &= 0.\end{aligned}$$

We find hence

$$\frac{d^2 p}{d t^2} + \mu^2 r_0^2 p = 0,$$

and this integrated yields

$$p = m \sin(\mu r_0 t + \varepsilon),$$

where m and ε are the two constants of integration.

Thus we will have

$$\frac{d p}{d t} = m \mu r_0 \cos(\mu r_0 t + \varepsilon) = \mu r_0 q,$$

and hence

$$q = m \cos(\mu r_0 t + \varepsilon).$$

Therefore, all together, the three components of the angular velocity with respect to the principal axes of inertia are

$$\left. \begin{aligned} p &= m \sin(\mu r_0 t + \varepsilon) \\ q &= m \cos(\mu r_0 t + \varepsilon) \\ r &= r_0. \end{aligned} \right\} \quad (4)$$

From this, we conclude at first

$$\omega^2 = p^2 + q^2 + r^2 = m^2 + r_0^2;$$

the angular velocity of the rotary motion around the instantaneous axis of rotation is thus invariant.

Since r and ω are constant, it follows further from the third equation of (1) that the angle (J, z) is likewise invariant, and hence, that the instantaneous axis of rotation describes a circle around the geometrical axis of the terrestrial body. We designate this motion as the *free nutation* of the earth since it is independent of the effect of external forces.

Fig. 1, in which the three axes of coordinates are at the same time the principal axes of inertia, is designed according to this. In addition, if we introduce the angle α for the determination of the arc of the great circle $z J$ with respect to the xz -plane, as is indicated in Fig. 1, then we obtain from the spherical triangles $z x J$ and $z J y$

$$\left. \begin{aligned} \cos(J, x) &= \sin(J, z) \cos \alpha \\ \cos(J, y) &= \sin(J, z) \sin \alpha. \end{aligned} \right\} \quad (5)$$

From this, equations (1) yield

$$\begin{aligned} p &= \omega \sin(J, z) \cos \alpha \\ q &= \omega \sin(J, z) \sin \alpha, \end{aligned}$$

and since

$$p^2 + q^2 = \omega^2 \sin^2(J, z) = m^2$$

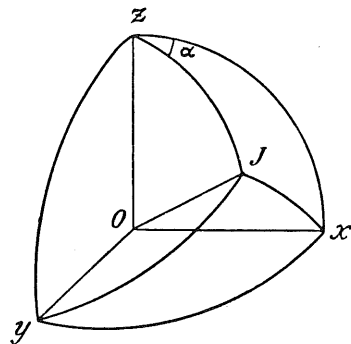


Fig. 1.

then we will have

$$\begin{aligned} p &= m \cos \alpha \\ q &= m \sin \alpha, \end{aligned}$$

and the comparison with (4) yields

$$\alpha = \mu r_0 t + \varepsilon. \quad (6)$$

We see therefrom that the instantaneous axis of rotation rotates with uniform velocity around the z -axis. The velocity of this rotation is

$$\frac{d\alpha}{dt} = \mu r_0.$$

If we denote the time, in which the instantaneous axis of rotation rotates around the z -axis once completely, by T , then we have accordingly

$$T = \frac{2\pi}{\mu r_0} = \frac{2\pi}{\frac{A-C}{A} r_0}. \quad (7)$$

The quantity $\frac{A-C}{A}$ is known from the determinations of the constants of precession and nutation, and from Bauschinger, *Die Bahnbestimmung der Himmelskörper*, Leipzig, 1906, p. 83, we take the value

$$\frac{C-A}{A} = 0.003\,295.$$

For the component r_0 of the angular velocity we can introduce the angular velocity ω itself, since, in accordance with experience, the instantaneous axis of rotation deviates only very little from the z -axis, and if we take as a basis a stellar day as the unit of time, then

$$\frac{2\pi}{\omega} = 1 \text{ stellar day} = 0.9973 \text{ mean solar days.}$$

Therefore, we obtain from (7)

$$T = \frac{0.9973}{0.003\,295} = 303 \text{ mean solar days.}$$

The instantaneous axis of rotation thus describes, around the principal axis of rotation, a conical surface with a period of about 10 months, which we designate as Euler's period.

Observed variations of latitude

The displacements of the two poles on the earth's surface must manifest themselves by variations of the polar altitudes or geographical latitudes. But since it is a question only of exceedingly small changes, which are to be determined only by means of very accurate measurements, then it is natural that only during the last decades the more general attention of the astronomers was directed to this phenomenon.

It is true, we do not lack indications from older times in which changes of latitude are represented as probable; however, in most cases it is a question here, in the first place, of local irregularities in the refraction of rays and of irregular errors of an instrumental kind. The following values were thus found by Airy for the latitude of Greenwich from the measurements of 1836-1860 according to Tisserand:

1836—1841	$\varphi = 51^{\circ} 28' 38.43''$
1842—1848	38.17
1851—1860	37.92 .

The cause of the decrease occurring here, however, is thought by Airy himself to lie in changes of the methods of observation.

Quite a number of examples for the change of latitude were compiled by Fergola in 1872. According to Tissérand, Fergola indicates the following values:

Washington 1845—1846		$\varphi = 38^{\circ} 53' 39.25''$
	1861—1864	38.78
Paris	before 1825	48 50 13.0
	1851—1854	11.2
Milan	1811	45 27 60.7
	1871	59.19
Rome	1807—1812	41 53 54.26
	1866	54.09
Naples	1820	40 51 46.63
	1871	45.41 .

These observational results, too, however, do not merit too much confidence for the most part in regard to their accuracy.

In the case of all these data it is a question of determining gradually progressing, and hence, secular, changes of latitude. The first more reliable data about periodical variations of latitude are referred to the observatory of Pulkovo, whose latitude was derived by Peters from 279 measurements during the years 1842 and 1843. There follows hence an amplitude of 0.08" and a period which corresponds well to that of Euler; however, Peters leaves the question open as to whether these changes, for instance, are to be attributed to the change of the seasons.

The investigations by Peters were continued by Nyrén who, in addition to older measurements by W. Struve and those by Peters, discusses also measurements by Gylden from the years 1863-1870 and his own measurements from the years 1871-1873. There resulted hence the following values for the amplitude of the variations of latitude in Pulkovo:

W. Struve	0.040''
Peters	0.101
Gylden	0.125
Nyrén	0.058 .

A detailed critical discussion of these investigations is also found in Helmert, *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, Band II, Leipzig, 1884, pp. 394-399. Since, however, these isolated measurements have at the present time only just a historical significance, we will not discuss them in greater detail, and turn to the new investigations with respect to the question of the variations of latitude.

At the seventh general conference of the European degree-measurement in October 1883 in Rome, a motion was brought forward by the Italian delegate, Fergola, to investigate by means of latitude determinations at several places, suitably chosen, if the poles of the axis of rotation of the earth are to be regarded as fixed on the earth's surface, or if they are subject to minor variations. Although this motion had the approval of the conference and various observatories declared themselves willing to carry out suitable latitude measurements, yet several years passed until the matter was started. Measurements by Küstner in Berlin, from which it followed undoubtedly that the latitude of the Berlin observatory was $0.20''$ smaller in the spring of 1885 than in the spring of 1884, gave a new impulse to this. Therefore, at the instigation of the Internationale Erdmessung, there were carried out, from the beginning of 1889 to April 1890, simultaneous latitude measurements in Berlin, Potsdam and Prague, from which a perfectly parallel course resulted for the latitudes of these three stations. The variability of the latitudes could thus no longer be questioned; at the same time, in view of the great distance between the stations, the assumption of local causes for these changes also no longer held good. In order to remove any doubt with respect to the latter question, Helmert suggested the execution of measurements at a station 180 degrees of longitude distant, which were to be compared with simultaneous measurements in Europe. True, there still existed doubts if the results hitherto obtained justified such a costly enterprise, nevertheless in the spring of 1891, by order of the Internationale Erdmessung, Marcuse was sent to Honolulu where a total of 1800 latitude determinations were carried out during the time from 1 June 1891 to 18 May 1892. The result of these measurements is compared with the simultaneous measurements at the Berlin observatory in Fig. 1 below from which it is seen that the changes of latitude in Berlin and Honolulu show indeed the same amount but an opposite course, and thus can only be attributed to displacements of the poles.

After this important question was decided upon, the problem of establishing a permanent latitude service had to be approached in order to be able to follow constantly the motions of the poles of the earth. The collaboration of the observatories did not appear sufficient for this because of the diversity of the methods of measurement; it would also have led to difficulties to find a series of observatories in a location suited for the solution of the problem. The Zentralbureau der Internationalen Erdmessung was therefore charged with the preliminary work for the establishment of special latitude stations.

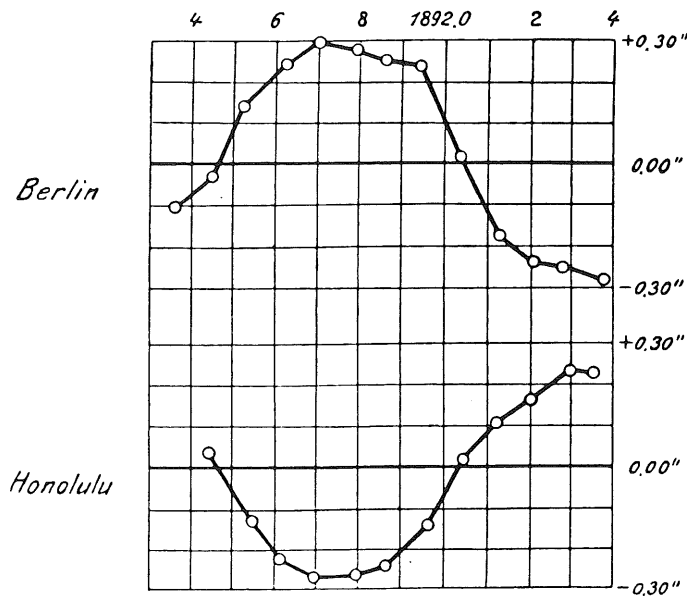


Fig. 1.

For the choice of the stations, there is decisive, in the first place, the requirement that they must lie exactly on the same circle of latitude in order to make possible the use of the same pairs of stars at all stations in the application of the Horrebow-Talcott method, and to make hereby impossible inaccuracies in the

declinations of the stars, which is likely to happen. A distribution of the stations as symmetrical as possible within the chosen parallel circle had further to be aimed at. By taking into account these conditions as well as all other circumstances involved for the observations themselves, and above all, the meteorological and seismological conditions, the following four stations lying on the parallel circle $+39^{\circ} 8'$ were finally selected.

Mizusawa, Japan	$\lambda = -141^{\circ} 8'$
Carloforte, Italy	$- 8 19$
Gaithersburg, Eastern North America	$+ 77 12$
Ukiah, Western North America	$+ 123 13 .$

Besides, by the voluntary cooperation of the topographic battalion of the Russian general staff and the observatory in Cincinnati, there were added the following two stations likewise lying exactly on the above-mentioned parallel circle:

Tschardjui, Central Asia	$\lambda = - 63^{\circ} 29'$
Cincinnati, Central North America	$+ 84 25 .$

With respect to the measuring procedure there had been planned from the outset the use of the Horrebow-Talcott method, which we have treated in detail in Volume II, section 109.* For the four main stations of the International Latitude Service, four large zenith telescopes of exactly the same kind were furnished by the mechanic Wanschaff in Berlin; the observation houses necessary for this were also erected uniformly according to a plan designed in the Zentralbureau in Potsdam. The equipment of the two volunteer stations also followed according to the same points of view, although the zenith telescopes designed for them were not exactly equivalent to the remaining ones.

The preparations were carried out with all possible speed so that the measurements could be started at all six stations during the last third of the year 1899 and have continued since then without interruption — on most stations even during the World War.

The assembling of the results of the measurements and the deriving of the motions of the pole took place in the Zentralbureau der Internationalen Erdmessung in Potsdam, from which the publication of the measured and computed material also started. There have been published: *Resultate des Internationalen Breitendienstes*, Band I, Berlin, 1903, Band II, 1906, Band III, 1909, Band IV, 1911, Band V, 1916. Volume I has been published by Th. Albrecht, Volumes II to IV by Th. Albrecht and B. Wanach, and Volume V by B. Wanach alone.

After the World War, an additional volume was worked up in the Prussian Geodetic Institute: *Ergebnisse des Internationalen Breitendienstes von 1912.0 to 1922.7*, by B. Wanach and H. Mahnkopf, Potsdam, 1932. After this, the care of the latitude service passed to Japan, where the results until 1931.0 were published in the volume: *Results of the International Latitude Service from 1922.7 to 1931.0*, Vol. VII, by Hisashi Kimura, Mizusawa, 1935.

Already before the establishment of the International Latitude Service, the continual latitude measurements of various observatories had been worked up in the Zentralbureau der Internationalen Erdmessung for the derivation of the motion of the pole, and the procedure used hereby was now retained for the measurements of the six latitude stations. At first, there were determined hereby suitable mean values of the latitude for the six stations and their deviations from the individual values of the latitude were formed. These deviations were plotted graphically and adjusted by line paths. The adjusted deviations $\Delta \varphi$ were taken therefrom for all tenths of a year and taken as a basis for the further computation.

For the derivation of the motion of the pole, in Fig. 2 there is introduced a rectangular system of coordinates whose zero point P_0 corresponds to a mean position of the pole while the axis of abscissae lies on the meridian of Greenwich; let the coordinates of the instantaneous pole P be x and y . The deviation $\Delta \varphi$ of the instantaneous latitude of a station S from its mean latitude is then expressed by the projection of PP_0 on the meridian of the station, and according to this, there results the following error equation, if the correction v is inserted at the same time

$$\Delta \varphi + v = x \cos \lambda + y \sin \lambda, \quad (1)$$

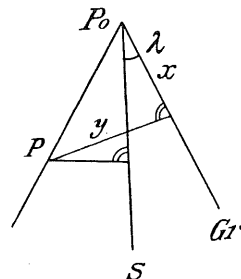


Fig. 2.

* Not translated.

if λ denotes the geographic longitude of the station with respect to Greenwich. The values of $\Delta \varphi$ are determined for every tenth of the year for the six stations, and from the six error equations, the two unknowns for each of these instants can then be computed.

For the first computations of the motion of the pole from the measurements of the various observatories not arranged systematically, the above form of the error equations was sufficient. But when the change was made to work up the results of the International Latitude Service carried out more accurately and uniformly, it turned out that the above expression did not satisfactorily reproduce the variations of latitude of the individual stations. A better agreement resulted when, at the suggestion of Prof. Kimura in Tokyo, an additional term independent of the longitude of the station was introduced, so that the error equations were now set down in the form

$$\Delta \varphi + v = x \cos \lambda + y \sin \lambda + z. \quad (2)$$

In this form, the measurements as well as the derivation of the motion of the pole were carried out until the end of the year 1905. However, just by the introduction of the quantity z a new difficulty had arisen since from the measurements hitherto carried out the validity of this term was established only for the circle of latitude of the observation stations. In particular, the applicability of the found values of z to the southern hemisphere had to appear doubtful throughout. In order to obtain information about this question, two further latitude stations were set up on the southern parallel $-31^{\circ} 55'$, namely

Bayswater, West Australia $\lambda = -115^{\circ} 55'$
 Oncativo, Argentina $+ 63^{\circ} 42'$,

at which measurements were carried out in the same arrangement as on the northern parallel since the beginning of the year 1906. The two stations were chosen with a difference of longitude of 180° , since the arithmetic mean of the corresponding $\Delta \varphi$'s of the two stations yields then immediately the quantity z according to the above equation (2). On the other hand, the difference of the two $\Delta \varphi$'s yields the expression $x \cos \lambda + y \sin \lambda$, which can be compared with the value of the northern parallel appropriate at that time.

The result of the measurements of the two years 1906 and 1907 showed that the values of z on the northern and on the southern parallel agree within the limits $\pm 0.02''$, and also that for the values of the expression $x \cos \lambda + y \sin \lambda$ a fundamental difference on the two parallel circles does not seem to exist.

About the cause of the term z there have already been set up many assumptions, in which, for instance, this quantity is attributed, among other things, to small terms which remained disregarded in the computation of the apparent star places or to disturbances of refraction. However, the measurements on only two parallel circles are not sufficient in order to examine these assumptions more closely; for this, additional measurements on further parallel circles would have to be included.

In conclusion, in Figs. 3 to 6 we give representations of the motion of the pole from the beginning of 1909 to the beginning of 1930. Fig. 3 is taken from a publication by Th. Albrecht in *Astronomische Nachrichten*, Band 201, Nr. 4802, 1915, while Figs. 4 to 6 originate from the publications of the International Latitude Service. Figs. 3 and 4 show a relatively regular course of the motion of the pole, where the amplitudes fluctuate approximately between $0.1''$ and $0.3''$ or in linear measure between 3 m and 9 m. However, the motion of the pole in Figs. 5 and 6 is very irregular in part, and especially in the first half of 1927 there appear great disturbances in the motion curve.

The points 1922.7 do not coincide exactly in Fig. 4 and in Fig. 5, which is to be attributed, in the first place, to differences in the astronomical fundamental data.

The Chandler period

The consideration of Figs. 3 to 6 shows that the motion of the pole does not correspond to the theory developed in the preceding section 107, since neither a circular path curve nor the Euler period of 10 months is observed. If we approach the question of the period, then we will only take Fig. 3 and Fig. 4 as a basis for this, and hence, the time from 1909.0-1922.7, whence it follows that the pole has carried out 11.5 revolutions during these 13.7 years, and with this, we obtain a period of 435 days.

This new period deviating considerably from the Euler period of 303 days was indicated for the first

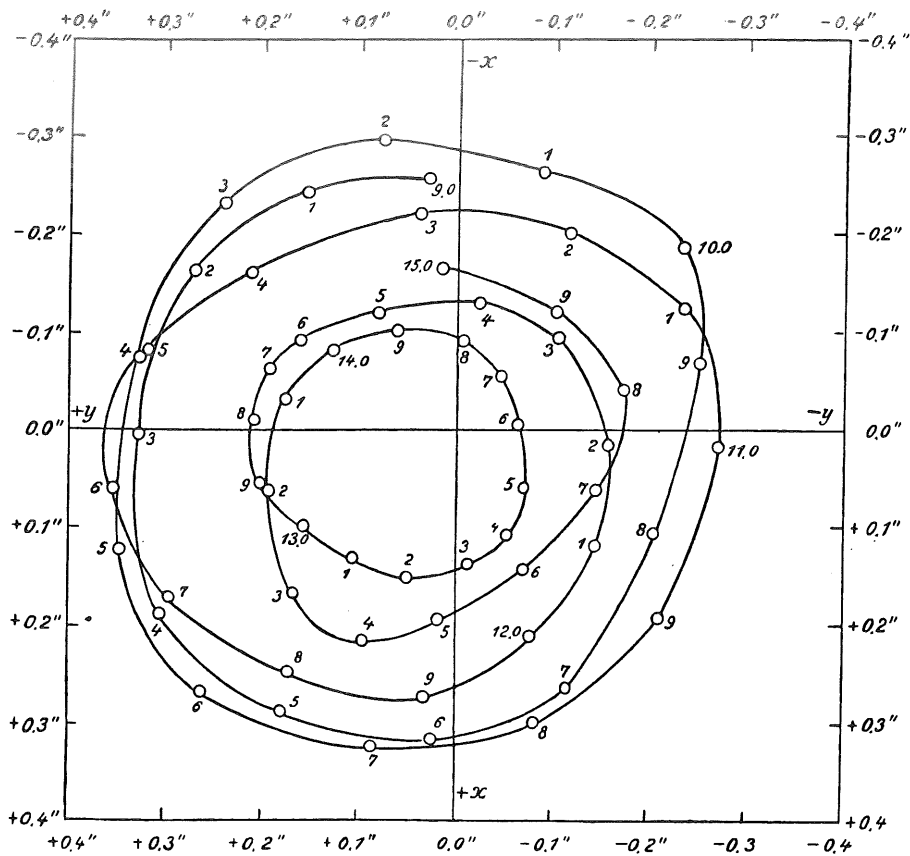


Fig. 3.

Motion of the pole from 1909.0-1915.0.

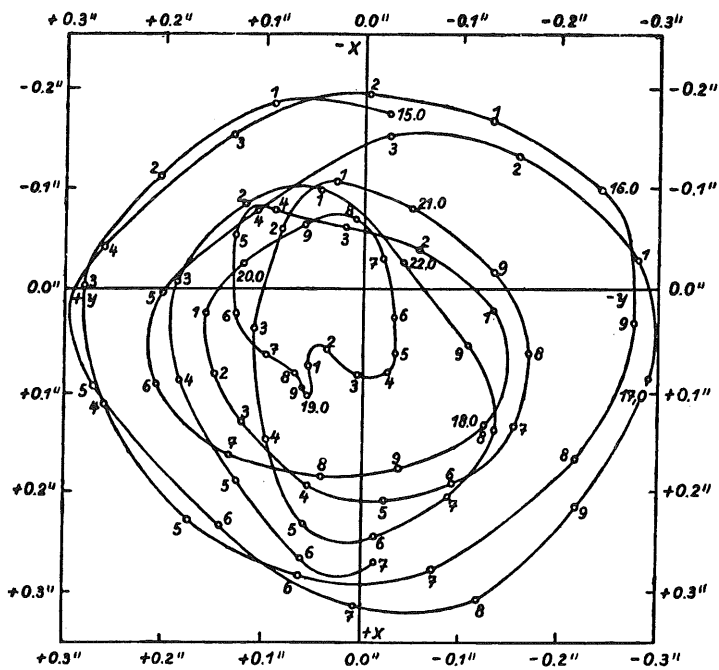


Fig. 4.

Motion of the pole from 1915.0-1922.7.

time by the American astronomer Chandler in *Astronomical Journal*, Vol. XI, who critically examined a great number of latitude determinations from the years 1840-1891 and thereby reached a period of 427 days; we designate this period of approximately 14 months as the Chandler period.

The discrepancy between the Euler and the Chandler periods was explained by S. Newcomb, who showed in 1892 that the Euler period, which presupposes a complete rigidity of the earth, must become longer if we can regard the earth as an elastically yielding body.

The motion of the pole then takes place in the following manner: Since the axis of rotation does not coincide with the axis of inertia of the earth, then it describes around the latter a revolution of the duration of Euler's period. At the same time the terrestrial body has a tendency to deform itself in such a way that the axis of inertia coincides with the axis of rotation. Since the flexibility of the earth, however, is a very limited one, then the axis of inertia will displace itself only a little in the direction toward the axis of rotation. The latter now describes again a revolution around the new axis of inertia in correspondence with the Euler period, and this motion again has as a consequence a displacement of the axis of inertia, and so on. The consequence of this elastic flexibility of the earth is the extension of the Euler period from 303 to 435 days.

We call this motion the *free* oscillation of the axis of rotation.

The extension of the Euler period, however, does not depend entirely alone on the elasticity of the earth. The position of the axis of rotation is also changed, in addition, by displacements of the masses on the earth's surface, e.g. such as from a meteorological or also a seismic origin, and, hereby, there is added to the free oscillation also a *forced* oscillation of the axis of rotation.

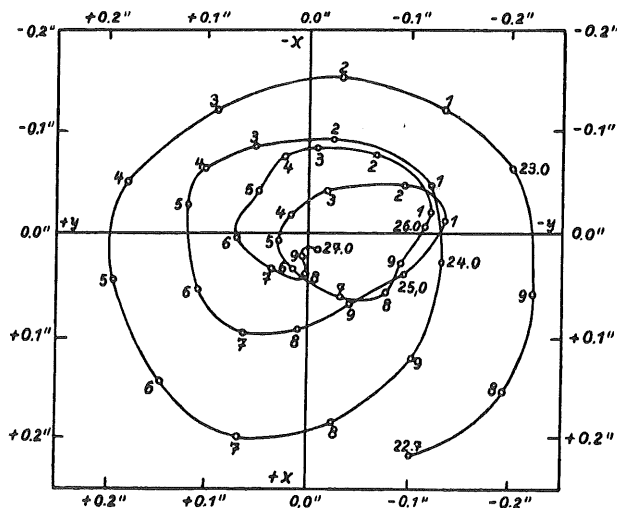


Fig. 5.

Motion of the pole from 1922.7-1927.0.

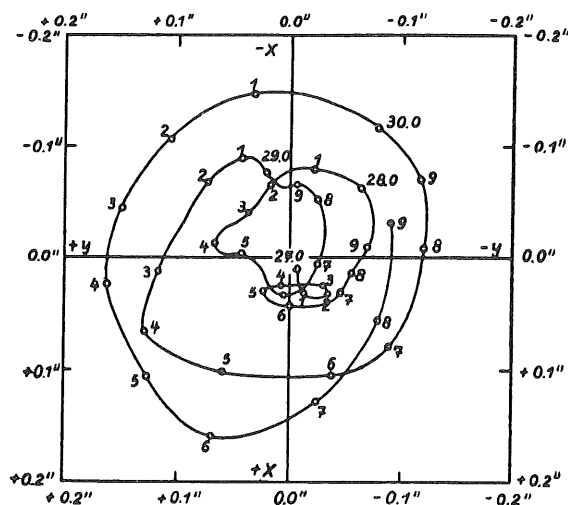


Fig. 6.

Motion of the pole from 1927.0-1930.9.

The accurate determination of the Chandler period becomes hereby exceedingly difficult. B. Wanach believed in *Result. d. Intern. Breitendienstes*, Band V, he could indicate with great certainty the value of 432.8 days for the period originating only from the elasticity of the earth and constant in its nature; however, it has turned out later that the separation of the free from the forced oscillation cannot be carried out unobjectionably, and that, therefore, the duration of the Chandler period can only be determined with an uncertainty of several days.

In addition to the publications of the Zentralbureau der Internationalen Erdmessung, every year there has been published a preliminary report about the measurements of the International Latitude Service and about the motion of the pole in *Astr. Nachr.*

The duration of the period of the motion of the pole is submitted to a critical investigation in the *Veröffentlichung des Zentralbureaus der Internationalen Erdmessung*, "Die Chandlersche und die Newcombsche Periode der Polbewegung" by B. Wansch. Berlin, 1919.

A comprehensive exposition is found in the paper: *Die Polhöenschwankungen* by Dr. E. Przybyllok, Braunschweig, 1914.

Since the extension of the Euler period from 303 days to 435 days is to be attributed almost exclusively to the elasticity of the earth, then a means for the determination of the elasticity of the earth offers itself likewise in this. However, the *polar currents* are further to be taken into account here. In the case of the shift of the axis of rotation, the oceans will yield at once to the tendency of the terrestrial body to adapt itself to the new position of the axis, and the polar currents thus arising cause likewise an extension of the period of the motion of the pole.

The English physicist Love has set up a simple relation between the quantity h introduced already in section 106, p. 526, and the motion of the pole. Let T_0 be the Euler and T the Chandler period of the motion of the pole. Let further α be the flattening of the earth and m the ratio of the centrifugal force at the equator to the force of gravity at the equator, and hence, according to section 64, p. 319,

$$m = \frac{\alpha \omega^2}{g_1}. \quad (1)$$

The relation found by Love reads then

$$1 - \frac{T_0}{T} = \frac{m}{2\alpha - m} h. \quad (2)$$

According to p. 319 we have

$$\alpha = \frac{1}{297} \quad m = 0.0034679 = \frac{1}{288}.$$

For T_0 we introduce the value found on p. 530

$$T_0 = 303 \text{ mean solar days}$$

while we assume according to p. 534

$$T_0 = 435 \text{ mean solar days}$$

With this, there follows from equation (2):

$$h = 0.282.$$

On the other hand, Schweydar has shown that by taking into account the polar currents, between the quantities h and η (cf. p. 525) there exists the relation

$$h - 0.6 \frac{\mu}{\rho_m} (1 + h) = \frac{Z_2}{N_1} \quad (3)$$

Here $\mu = 1.03$ the density of the ocean and $\rho_m = 5.51$ the mean density of the earth according to the law of Roche. The quantities Z_2 and N_1 are represented by the following equations

$$\left. \begin{aligned} Z_2 &= \frac{1}{e} (a_2 + b_2 \eta + c_2 \eta^2 + \dots) + (a_2' + b_2' \eta + \dots) + e (a_2'' + b_2'' \eta + \dots) \\ N_1 &= \frac{1}{e^2} (p_1 + q_1 \eta + r_1 \eta^2 + \dots) + \frac{1}{e} (p_1' + q_1' \eta + \dots) + (p_1'' + q_1'' \eta + \dots) \\ &\quad + e (p_1''' + q_1''' \eta + \dots), \end{aligned} \right\} \quad (4)$$

whose coefficients are given numerical values.

With the above value of h and the values of μ and ρ_m (3) passes then over into

$$0.138 N_1 = Z_2. \quad (5)$$

This equation contains again the two unknowns η and n_0 , of which we can eliminate n_0 as on p. 527 with the help of the rigidity \bar{n} at the earth's surface determined from the velocity of the earth waves. With this, there would then result an equation for the determination of η . However we have preferred to find, by way of gradual approximation, that value of η which satisfies equation (5). For this, there followed $\eta = 0.887$, which we will round off to

$$\eta = 0.89, \quad (6)$$

and with this we will have according to (16), section 106, p. 527,

$$n_0 = \frac{\bar{n}}{1 - \eta} = 28.0.$$

We thus obtain as our result the expression:

$$n = 28.0 \times 10^{11} \left(1 - 0.89 \frac{r^2}{a^2} \right) \text{ c/g/s.} \quad (7)$$

In order to obtain a judgment about the reliability of the result, it was established that an increase of T by one day causes a decrease of n_0 by 0.4 units. Therefore, even if we are to consider an inaccuracy of several days in T , n_0 will hereby be influenced only by a few units.

With respect to the above-mentioned polar currents we mention further the *Veröffentlichung des Geodätischen Instituts*, "Über die sogenannte Polflut in der Ost- und Nordsee" by E. Przybyllok, Berlin, 1919.

Combined evaluation of horizontal pendulum measurements and the period of the motion of the pole

The two methods for the determination of the elasticity of the earth represented above and in section 106 use seismometric observations for the elimination of the quantity n_0 . A combined determination of the unknowns n_0 and η independent of this follows immediately if we combine the value of γ determined from the horizontal pendulum measurements with the observed period T of the motion of the pole. From (12) and (13), section 106, p. 526, in

$$(1 - \gamma) N = Z - Z_1 \quad (8)$$

we have an equation with the two unknowns n_0 and η , which are contained in N , Z and Z_1 .

On the other hand, we obtain a value of h from the period of the motion of the pole with the help of (2), p. 537, and equation (3), p. 537,

$$h - 0.6 \frac{\mu}{\rho_m} (1 + h) = \frac{Z_2}{N_1} \quad (9)$$

yields then a second relation between the unknowns n_0 and η . The two equations (8) and (9) thus make simultaneous determination of n_0 and η possible without the aid of further auxiliary measurements.

Such a computation is likewise carried out by Schweydar in "Theorie der Deformation der Erde durch Flutkräfte," Potsdam, 1916, pp. 34-38. The values

$$\gamma = 0.827 \quad T = 434.1 \quad T_0 = 304.8,$$

however, have been used here, while above we have taken the values

$$\gamma = 0.841 \quad T = 435 \quad T_0 = 303$$

as a basis.

Schweydar finds as the result of his computation

$$n = 29.0 \times 10^{11} \left(1 - 0.91 \frac{r^2}{a^2} \right) \text{ c/g/s,}$$

which agrees satisfactorily with the results (20), section 106, p. 527, and (7), p. 538, hitherto communicated.

APPENDIX

AUXILIARY TABLES

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[1]

Coefficients of geodetic formulae (Bessel Ellipsoid)

 φ = geographic latitude $t^2 = \tan^2 \varphi$, $e'^2 \cos^2 \varphi = \eta^2$, $e'^2 \sin^2 \varphi = \eta^2 t^2$.

φ	t^2	Diff.	t^4	Diff.	η^2	Diff.	$\eta^2 t^2$	Diff.	η^4	Diff.
0°	0,0000	+	0,0000	+	0,00672	—	0,00000	+	0,0000451	+
1	0003	3	0000		672	0	000	0	451	0
2	0012	9	0000		671	1	001	1	450	1
3	0027	15	0000		670	1	002	1	449	1
4	0049	22	0000		669	1	003	1	447	2
		28				2		2		2
5°	0,0077		0,0001		0,00667		0,00005		0,0000445	
6	0110	33	0001	0	665	2	007	2	442	3
7	0151	41	0002	1	662	3	010	3	438	4
8	0198	47	0004	2	659	3	013	3	434	4
9	0251	53	0006	2	655	4	016	3	430	4
		60		4		3		4		5
10°	0,0311		0,0010		0,00652		0,00020		0,0000425	
11	0378	67	0014	4	647	5	025	5	419	6
12	0452	74	0020	6	643	4	029	4	413	6
13	0533	81	0028	8	638	5	034	5	407	6
14	0622	89	0039	11	633	5	039	5	400	7
		96		13		6		6		7
15°	0,0718		0,0052		0,00627		0,00045		0,0000393	
16	0822	104	0068	16	621	6	051	6	385	8
17	0935	113	0088	20	614	7	057	6	378	7
18	1056	121	0111	23	608	6	064	7	369	9
19	1186	130	0141	30	601	7	071	7	361	8
		139		34		8		8		9
20°	0,1325		0,0175		0,00593		0,00079		0,0000352	
21	1474	149	0217	42	586	7	086	7	343	9
22	1632	158	0266	49	578	8	094	8	334	9
23	1802	170	0325	59	569	9	103	9	324	10
24	1982	180	0393	68	561	8	111	8	314	10
		192		80		9		9		9
25°	0,2174		0,0473		0,00552		0,00120		0,0000305	
26	2379	205	0566	93	543	9	129	9	295	10
27	2596	217	0674	108	533	10	138	9	285	10
28	2827	231	0799	125	524	9	148	10	274	11
29	3073	246	0944	145	514	10	158	10	264	10
		260		167		10		10		10
30°	0,3333		0,1111		0,00504		0,00168		0,0000254	
31	3610	277	1303	192	494	10	178	10	244	10
32	3905	295	1525	222	483	11	189	11	234	10
33	4217	312	1779	254	473	10	199	10	223	11
34	4550	333	2070	291	462	11	210	11	213	10
		353		334		11		11		10
35°	0,4903		0,2404		0,00451		0,00221		0,0000203	
36	5279	376	2786	382	440	11	232	11	193	10
37	5678	399	3225	439	429	11	243	11	184	9
38	6104	426	3726	501	417	12	255	12	174	10
39	6558	454	4300	574	406	11	266	11	165	9
		483		657		12		12		10
40°	0,7041		0,4957		0,00394		0,00278		0,0000155	
41	7557	516	5710	753	383	11	289	11	146	9
42	8107	550	6573	863	371	12	301	12	138	8
43	8696	589	7562	989	359	12	313	12	129	9
44	9326	630	8697	1135	348	11	324	11	121	8
45	1,0000	674	1,0000	1303	0,00336	12	0,00336	12	0,0000113	8

Coefficients of geodetic formulae (Bessel Ellipsoid) [2]

φ = geographic latitude $t^2 = \tan^2 \varphi$, $e'^2 \cos^2 \varphi = \eta^2$, $e'^2 \sin^2 \varphi = \eta^2 t^2$.

φ	t^2	Diff.	t^4	Diff.	η^2	Diff.	$\eta^2 t^2$	Diff.	η^4	Diff.
45°	1,000	+	1,000	+	0,00336	—	0,00336	+	0,0000113	—
46	1,072	0,072	1,160	0,150	324	12	348	12	105	8
47	1,150	0,078	1,322	0,172	313	11	359	11	098	7
48	1,233	0,083	1,521	0,199	301	12	371	12	091	7
49	1,323	0,090	1,751	0,240	289	12	383	12	084	7
		0,097		0,266		11		11		7
50°	1,420		2,017		0,00278		0,00394		0,0000077	
51	1,525	0,105	2,326	0,309	266	12	406	12	071	6
52	1,638	0,113	2,684	0,358	255	11	417	11	065	6
53	1,761	0,123	3,101	0,417	243	12	429	12	059	6
54	1,894	0,133	3,588	0,487	232	11	440	11	054	5
		0,146		0,572		12		11		5
55°	2,040		4,160		0,00221		0,00451		0,0000049	
56	2,198	0,158	4,831	0,671	210	11	462	11	044	5
57	2,371	0,173	5,623	0,792	199	11	473	11	040	4
58	2,561	0,190	6,559	0,936	189	10	483	10	036	4
59	2,770	0,209	7,672	1,113	178	11	494	11	032	4
		0,23		1,33		10		10		4
60°	3,00		9,00		0,00168		0,00504		0,0000028	
61	3,25	0,25	10,59	1,59	158	10	514	10	025	3
62	3,54	0,29	12,51	1,92	148	10	524	10	022	3
63	3,85	0,31	14,84	2,33	138	10	533	9	019	3
64	4,20	0,35	17,67	2,83	129	9	543	10	017	2
		0,40		3,48		9		9		3
65°	4,60		21,15		0,00120		0,00552		0,0000014	
66	5,04	0,44	25,45	4,30	111	9	561	9	012	2
67	5,55	0,51	30,80	5,35	103	8	569	8	011	1
68	6,13	0,58	37,53	6,73	94	9	578	9	009	2
69	6,79	0,66	46,06	8,53	86	8	586	8	007	2
		0,76		10,9		7		7		1
70°	7,55		57,0		0,00079		0,00593		0,0000006	
71	8,43	0,88	71,1	14,1	71	8	601	8	005	1
72	9,47	1,04	89,7	18,6	64	7	608	7	004	1
73	10,70	1,23	114,5	24,8	57	7	614	6	003	1
74	12,16	1,46	147,9	33,4	51	6	621	7	003	0
		1,77		46		6		6		1
75°	13,93		194		0,00045		0,00627		0,0000002	
76	16,09	2,16	259	65	039	6	633	6	002	0
77	18,76	2,67	352	103	034	5	638	5	001	1
78	22,13	3,37	490	138	029	5	643	5	001	0
79	26,47	4,34	700	210	025	4	647	4	001	0
		5,69		334		5		5		1
80°	32,16		1034		0,00020		0,00652		0,0000000	
81	39,88	7,70	1589	555	016	4	655	3		
82	50,63	10,77	2563	974	013	3	659	4		
83	66,33	15,70	4400	1837	010	3	662	3		
84	90,52	24,19	8194	3794	007	3	665	3		
						2		2		
85°	131		17068		0,00005		0,00667			
86	205		41824		003	2	669	2		
87	364		132561		002	1	670	1		
88	820		672458		001	1	671	1		
89	3282		∞		000	1	672	1		
90°	∞				0,00000	0	0,00672	0		

φ	t^2	Diff.	t^4	Diff.	η^2	Diff.	η^4	Diff.	$\eta^2 t^2$	Diff.
44° 0'	0,933	11	0,870	20	0,003477	20	0,00001209		0,003242	
10	0,944	11	0,890	21	457	19	1195	14	261	19
20	0,955	11	0,911	22	438	20	1182	13	281	20
30	0,966	11	0,933	22	418	19	1168	14	301	20
40	0,977	11	0,955	22	399	20	1155	13	320	19
50	0,988	12	0,977	23	379	19	1142	13	340	20
								13		20
45° 0'	1,000	12	1,000	23	0,003360	20	0,00001129		0,003360	
10	1,012	12	1,023	24	340	20	1116	13	379	19
20	1,024	12	1,047	25	320	19	1103	13	399	20
30	1,036	12	1,072	25	301	20	1090	13	418	19
40	1,048	12	1,097	26	281	19	1077	13	438	20
50	1,060	12	1,123	27	262	20	1064	13	457	19
								13		20
46° 0'	1,072	13	1,150	27	0,003242	19	0,00001051		0,003477	
10	1,085	13	1,177	28	223	20	1039	12	496	19
20	1,098	13	1,205	28	203	19	1026	13	516	20
30	1,110	13	1,233	29	184	20	1014	12	535	19
40	1,123	14	1,262	30	164	19	1001	13	555	20
50	1,137	13	1,292	30	145	20	0989	12	574	19
								12		20
47° 0'	1,150	13	1,322	31	0,003125	19	0,00000977		0,003594	
10	1,163	14	1,353	32	106	20	0965	12	613	19
20	1,177	14	1,385	33	086	19	0953	12	633	20
30	1,191	14	1,418	34	067	20	0941	12	652	19
40	1,205	14	1,452	34	047	19	0929	12	672	20
50	1,219	14	1,486	35	028	20	0917	12	691	19
								12		20
48° 0'	1,233	15	1,521	36	0,003008	19	0,00000905		0,003711	
10	1,248	15	1,557	37	0,002989	19	0893	12	730	19
20	1,263	15	1,594	38	970	20	0882	11	750	20
30	1,278	15	1,632	39	950	20	0870	12	769	19
40	1,293	15	1,671	40	931	19	0859	11	788	20
50	1,308	15	1,711	40	911	19	0847	12	808	19
								11		20
49° 0'	1,323	16	1,751	42	0,002892	19	0,00000836		0,003827	
10	1,339	16	1,793	43	873	20	0825	11	846	19
20	1,355	16	1,836	43	853	19	0814	11	866	20
30	1,371	16	1,879	44	834	19	0803	11	885	19
40	1,387	16	1,924	45	815	19	0792	11	904	20
50	1,404	16	1,970	46	796	20	0782	10	924	19
								11		20
50° 0'	1,420	17	2,017	48	0,002776	19	0,00000771		0,003943	
10	1,437	17	2,065	50	757	19	0760	11	962	19
20	1,454	18	2,115	51	738	19	0750	10	981	20
30	1,472	18	2,166	52	719	19	0739	11	0,004001	19
40	1,489	17	2,218	53	699	20	0728	11	020	20
50	1,507	18	2,271	55	680	19	0718	10	039	19
								10		20
51° 0'	1,525	18	2,326	56	0,002661	19	0,00000708		0,004058	
10	1,543	19	2,382	57	642	19	0698	10	077	19
20	1,562	18	2,439	59	623	19	0688	10	096	20
30	1,580	19	2,498	60	604	19	0678	10	115	19
40	1,599	20	2,558	62	585	19	0668	10	134	20
50	1,619	19	2,620	64	566	19	0658	9	153	19
										20
52° 0'	1,638		2,684		0,002547		0,00000649		0,004172	

φ	t^2	Diff.	t^4	Diff.	η^2	Diff.	η^4	Diff.	$\eta^2 t^2$	Diff.
52° 0'	1,638	20	2,684	65	0,002547	19	0,00000649	10	0,004172	19
10	1,658	20	2,749	67	528	19	0639	9	191	19
20	1,678	20	2,816	67	509	19	0630	9	210	19
30	1,698	20	2,884	68	490	19	0620	10	229	19
40	1,719	21	2,955	71	471	19	0611	9	248	19
50	1,740	21	3,027	72	452	19	0601	10	267	19
		21		74		18		9		19
53° 0'	1,761	21	3,101	76	0,002434	19	0,00000592	9	0,004286	18
10	1,782	22	3,177	78	415	19	0583	9	304	19
20	1,804	22	3,255	78	396	19	0574	9	323	19
30	1,826	22	3,336	81	377	19	0565	9	342	19
40	1,849	22	3,418	82	359	18	0556	9	360	18
50	1,871	22	3,502	84	340	19	0548	8	379	19
		23		87		19		9		19
54° 0'	1,894	24	3,589	89	0,002321	18	0,00000539	9	0,004398	18
10	1,918	23	3,678	91	303	19	0530	8	416	19
20	1,941	23	3,769	91	284	19	0522	8	435	19
30	1,965	24	3,863	94	266	18	0513	9	453	18
40	1,990	25	3,959	96	247	19	0505	8	472	19
50	2,015	25	4,058	99	229	18	0497	8	490	18
		25		102		18		8		19
55° 0'	2,040	25	4,160	104	0,002211	19	0,00000489	8	0,004509	18
10	2,065	26	4,264	108	192	18	0481	8	527	18
20	2,091	26	4,372	108	174	18	0473	8	545	18
30	2,117	26	4,483	111	156	18	0465	8	564	19
40	2,144	27	4,596	113	137	19	0457	8	582	18
50	2,171	27	4,712	116	119	18	0449	8	600	18
		27		119		18		8		18
56° 0'	2,198	28	4,831	123	0,002101	18	0,00000441	7	0,004618	18
10	2,226	28	4,954	126	083	18	0434	8	636	18
20	2,254	28	5,080	126	065	18	0426	8	654	18
30	2,283	29	5,210	130	047	18	0419	7	672	18
40	2,312	29	5,344	134	029	18	0412	7	690	18
50	2,341	29	5,481	137	0,002011	18	0404	8	708	18
		30		142		18		7		18
57° 0'	2,371	31	5,623	145	0,001993	18	0,00000397	7	0,004726	18
10	2,402	31	5,768	145	975	18	0390	7	744	18
20	2,433	31	5,917	149	957	18	0383	7	762	18
30	2,464	31	6,071	154	940	17	0376	7	777	17
40	2,496	32	6,229	158	922	18	0369	7	799	18
50	2,528	32	6,392	163	904	18	0362	7	815	18
		33		167		17		6		17
58° 0'	2,561	33	6,559	172	0,001887	18	0,00000356	7	0,004832	18
10	2,594	34	6,731	178	869	17	0349	7	850	18
20	2,628	34	6,909	182	852	17	0343	6	867	17
30	2,663	35	7,091	188	834	18	0336	7	885	18
40	2,698	35	7,279	188	817	17	0330	6	902	17
50	2,734	36	7,473	194	800	17	0324	6	920	18
		36		199		18		6		17
59° 0'	2,770	37	7,672	205	0,001782	17	0,00000318	6	0,004937	17
10	2,807	37	7,877	212	765	17	0312	6	954	17
20	2,844	37	8,089	217	748	17	0306	6	971	17
30	2,882	38	8,306	225	731	17	0300	6	988	17
40	2,921	39	8,531	230	714	17	0294	6	0,005005	17
50	2,960	39	8,761	239	697	17	0288	6	022	17
		40		239		17		6		17
60° 0'	3,000		9,000		0,001680		0,00000282		0,005039	

[5]

Coefficients of geodetic formulae (International Ellipsoid).

φ	$\log V_i^2$	Diff.	η_i^2	Diff.	η_i^4	Diff.	$\eta_i^2 t_i^2$	Diff.
44° 0'	0,001518	8	0,003502	19	0,00001226	13	0,003266	19
10	510	9	483	20	1213	14	285	20
20	501	8	463	20	1199	14	305	20
30	493	9	443	19	1185	13	325	19
40	484	8	424	20	1172	13	344	20
50	476	9	404	20	1159	14	364	20
45° 0'	0,001467	8	0,003384	20	0,00001145	13	0,003384	20
10	459	9	364	19	1132	13	404	19
20	450	8	345	20	1119	13	423	20
30	442	9	325	20	1106	13	443	20
40	433	8	305	20	1093	13	463	20
50	425	9	285	20	1080	13	483	20
46° 0'	0,001416	8	0,003265	19	0,00001067	13	0,003503	19
10	408	9	246	19	1054	13	522	19
20	399	8	227	20	1041	13	541	20
30	391	9	207	20	1028	12	561	20
40	382	8	187	19	1016	12	581	19
50	374	9	168	20	1004	13	600	20
47° 0'	0,001365	8	0,003148	19	0,00000991	12	0,003620	19
10	357	9	129	20	0979	12	639	20
20	348	8	109	20	0967	13	659	20
30	340	9	089	20	0954	12	679	20
40	331	8	069	19	0942	12	699	19
50	323	9	050	20	0930	12	718	20
48° 0'	0,001314	8	0,003030	19	0,00000918	11	0,003738	19
10	306	9	011	20	0907	12	757	20
20	297	8	0,002991	19	0895	12	777	19
30	289	9	972	20	0883	12	796	20
40	280	8	952	19	0871	11	816	19
50	272	9	933	20	0860	11	835	20
49° 0'	0,001263	8	0,002913	20	0,00000849	12	0,003855	20
10	255	9	893	19	0837	11	875	19
20	246	8	874	19	0826	11	894	19
30	238	8	855	20	0815	11	913	20
40	230	9	835	19	0804	11	933	19
50	221	8	816	20	0793	11	952	20
50° 0'	0,001213	9	0,002796	19	0,00000782	11	0,003972	19
10	204	8	777	19	0771	10	991	19
20	196	8	758	20	0761	11	0,004010	20
30	188	9	738	19	0750	11	030	19
40	179	9	719	19	0739	11	049	19
50	170	8	700	20	0728	11	068	20
51° 0'	0,001162	8	0,002680	19	0,00000717	10	0,004088	19
10	154	8	661	19	0707	9	107	19
20	146	8	642	19	0698	10	126	19
30	138	8	623	19	0688	10	145	19
40	129	8	604	20	0678	10	164	20
50	121	8	584	19	0668	10	184	19
52° 0'	0,001113		0,002565		0,00000658		0,004203	

φ	$\log V_i^2$	Diff.	η_i^2	Diff.	η_i^4	Diff.	$\eta_i^2 t_i^2$	Diff.
52° 0'	0,001113		0,002565		0,00000658		0,004203	
10	104	9	546	19	0648	10	222	19
20	096	8	527	19	0639	9	241	19
30	088	8	508	19	0629	10	260	19
40	080	8	489	19	0620	9	279	19
50	072	9	470	19	0610	10	298	19
						9		
53° 0'	0,001063		0,002451		0,00000601		0,004317	
10	055	8	432	19	0592	9	336	19
20	047	8	414	18	0583	9	354	18
30	039	8	395	19	0574	9	373	19
40	031	8	376	19	0565	9	392	19
50	022	9	357	19	0556	9	411	19
		8		18				18
54° 0'	0,001014		0,002339		0,00000547		0,004429	
10	006	8	320	19	0538	9	448	19
20	0,000998	8	301	19	0529	9	467	19
30	990	8	283	18	0521	8	485	18
40	982	8	264	19	0513	8	504	19
50	974	8	245	19	0504	9	523	19
		8		18		8		18
55° 0'	0,000966		0,002227		0,00000496		0,004541	
10	958	8	208	19	0488	8	560	19
20	950	8	189	19	0479	9	579	19
30	942	8	171	18	0471	8	597	18
40	934	8	153	18	0463	8	615	18
50	926	8	134	19	0455	8	634	19
		8		18		7		18
56° 0'	0,000918		0,002116		0,00000448		0,004652	
10	910	8	098	18	0440	8	670	18
20	902	8	080	18	0433	7	688	18
30	894	8	062	18	0425	8	706	18
40	887	7	044	18	0417	8	724	18
50	879	8	026	18	0410	7	742	18
		8		18		7		18
57° 0'	0,000871		0,002008		0,00000403		0,004760	
10	863	8	0,001990	18	0396	7	778	18
20	856	7	972	18	0389	7	796	18
30	848	8	954	18	0382	7	814	18
40	840	8	936	18	0375	7	832	18
50	832	8	918	18	0368	7	850	18
		7		18		7		18
58° 0'	0,000825		0,001900		0,00000361		0,004868	
10	817	8	883	17	0355	6	885	17
20	809	8	865	18	0348	7	903	18
30	802	7	848	17	0342	6	920	17
40	794	8	830	18	0335	7	938	18
50	786	8	813	17	0329	6	955	17
		7		18		7		18
59° 0'	0,000779		0,001795		0,00000322		0,004973	
10	772	7	778	17	0316	6	990	17
20	764	8	761	17	0310	6	0,005007	17
30	756	8	743	18	0304	6	025	18
40	749	7	726	17	0298	6	042	17
50	742	7	709	17	0292	6	059	17
		8		17		5		17
60° 0'	0,000734		0,001692		0,00000287		0,005076	

Auxiliary tables for interpolation.

0' 0''	0,0000'	0' 20''	0,0333'	0' 40''	0,0667'
1	0,0017	21	0,0350	41	0,0683
2	0,0033	22	0,0367	42	0,0700
3	0,0050	23	0,0383	43	0,0717
4	0,0067	24	0,0400	44	0,0733
5	0,0083	25	0,0417	45	0,0750
6	0,0100	26	0,0433	46	0,0767
7	0,0117	27	0,0450	47	0,0783
8	0,0133	28	0,0467	48	0,0800
9	0,0150	29	0,0483	49	0,0817
0' 10''	0,0167'	0' 30''	0,0500'	0' 50''	0,0833'
11	0,0183	31	0,0517	51	0,0850
12	0,0200	32	0,0533	52	0,0867
13	0,0217	33	0,0550	53	0,0883
14	0,0233	34	0,0567	54	0,0900
15	0,0250	35	0,0583	55	0,0917
16	0,0267	36	0,0600	56	0,0933
17	0,0283	37	0,0617	57	0,0950
18	0,0300	38	0,0633	58	0,0967
19	0,0317	39	0,0650	59	0,0983
0' 20''	0,0333'	0' 40''	0,0667'	1' 00''	0,1000'

0''	0,000'	20''	0,333'	40''	0,667'
1	0,017	21	0,350	41	0,683
2	0,033	22	0,367	42	0,700
3	0,050	23	0,383	43	0,717
4	0,067	24	0,400	44	0,733
5	0,083	25	0,417	45	0,750
6	0,100	26	0,433	46	0,767
7	0,117	27	0,450	47	0,783
8	0,133	28	0,467	48	0,800
9	0,150	29	0,483	49	0,817
10''	0,167'	30''	0,500'	50''	0,833'
11	0,183	31	0,517	51	0,850
12	0,200	32	0,533	52	0,867
13	0,217	33	0,550	53	0,883
14	0,233	34	0,567	54	0,900
15	0,250	35	0,583	55	0,917
16	0,267	36	0,600	56	0,933
17	0,283	37	0,617	57	0,950
18	0,300	38	0,633	58	0,967
19	0,317	39	0,650	59	0,983
20''	0,333'	40''	0,667'	1'	1,000'

φ	$\log [4]$	Diff.	$\log [5]$	Diff.	$\log [6]$	Diff.	$\log [7]$	Diff.	$\log [8]$	Diff.
0°	4.62581	—	4.62284	+	2.930 _n	—	4.93266	—	5.10777	—
1	62581	0	62284	0	930 _n	0	93266	0	10777	0
2	62579	2	62285	1	929 _n	1	93266	0	10777	0
3	62575	4	62286	1	928 _n	1	93265	1	10777	0
4	62570	5	62287	1	926 _n	2	93265	0	10776	1
5	62564	6	62289	2	924 _n	2	93264	1	10776	0
		8		2		3		1		1
6°	4.62556	—	4.62291	+	2.921 _n	—	4.93263	—	5.10775	—
7	62547	9	62293	2	917 _n	4	93262	1	10774	1
8	62537	10	62296	3	913 _n	4	93261	1	10774	0
9	62525	12	62299	3	908 _n	5	93259	2	10772	2
10	62511	14	62302	3	903 _n	5	93257	2	10771	1
11	62497	14	62306	4	898 _n	5	93255	2	10770	1
		16		5		6		1		1
12°	4.62481	—	4.62311	+	2.892 _n	—	4.93254	—	5.10769	—
13	62464	17	62315	4	885 _n	7	93251	3	10768	1
14	62445	19	62319	4	877 _n	8	93249	2	10766	2
15	62426	19	62324	5	869 _n	8	93247	2	10764	2
16	62405	21	62329	5	860 _n	9	93244	3	10763	1
17	62383	22	62335	6	850 _n	10	93241	3	10761	2
		24		6		10		3		2
18°	4.62359	—	4.62341	+	2.840 _n	—	4.93238	—	5.10759	—
19	62335	24	62347	6	829 _n	11	93235	3	10757	2
20	62309	26	62353	6	817 _n	12	93232	3	10754	3
21	62282	27	62360	7	804 _n	13	93229	3	10752	2
22	62254	28	62367	7	790 _n	14	93225	4	10750	2
23	62225	29	62375	8	775 _n	15	93222	3	10748	2
		30		7		16		4		2
24°	4.62195	—	4.62382	+	2.759 _n	—	4.93218	—	5.10746	—
25	62164	31	62390	8	742 _n	17	93214	4	10743	3
26	62133	31	62398	8	724 _n	18	93210	4	10740	3
27	62099	34	62406	8	704 _n	20	93206	4	10738	2
28	62066	33	62415	9	683 _n	21	93202	4	10735	3
29	62031	35	62423	8	660 _n	23	93198	4	10732	3
		35		9		25		4		3
30°	4.61996	—	4.62432	+	2.635 _n	—	4.93194	—	5.10729	—
31	61960	36	62441	9	609 _n	26	93189	5	10726	3
32	61923	37	62450	9	580 _n	29	93185	4	10723	3
33	61885	38	62459	9	548 _n	32	93180	5	10720	3
34	61847	38	62469	10	613 _n	35	93175	5	10717	3
35	61808	39	62478	9	475 _n	38	93171	4	10714	3
		39		10		43		5		3
36°	4.61769	—	4.62488	+	2.432 _n	—	4.93166	—	5.10711	—
37	61729	40	62498	10	384 _n	48	93161	5	10708	3
38	61869	40	62508	10	329 _n	55	93156	5	10704	4
39	61648	41	62518	10	266 _n	63	93151	5	10701	3
40	61607	41	62528	10	191 _n	75	93146	5	10698	3
41	61565	42	62538	10	100 _n	91	93141	5	10695	3
		41		10				5		4
42°	4.61524	—	4.62548	+	1.984 _n	—	4.93136	—	5.10691	—
43	61482	42	62559	11	1.824 _n		93131	5	10688	3
44	61440	42	62569	10	1.568 _n		93126	5	10685	3
45	61398	42	62580	11	0.855 _n		93121	5	10681	4
46	61356	42	62590	10	1.356		93116	5	10678	3
47	61313	43	62600	10	1.720		93111	5	10675	3
48°	4.61271	42	4.62610	10	1.915		4.93106	5	5.10671	4

 $\log [3] = 4.62872$ constant.

cf. section 21.

φ	$\log [4]$	Diff.	$\log [5]$	Diff.	$\log [6]$	Diff.	$\log [7]$	Diff.	$\log [8]$	Diff.
48°	4.61271	—	4.62610	+	1.915	+	4.93106	—	5.10671	—
49	61229	42	62620	10	2.049	134	93101	5	10668	3
50	61187	42	62630	10	151	102	93096	5	10664	4
51	61145	42	62641	11	233	82	93091	5	10661	3
52	61104	41	62651	10	301	68	93086	5	10658	3
53	61063	41	62660	9	360	59	93081	5	10655	3
54°	4.61022	41	4.62670	10	2.411	51	4.93076	5	5.10651	4
55	60981	41	62680	10	458	47	93071	5	10648	3
56	60941	40	62689	9	497	39	93066	5	10645	3
57	60902	39	62699	10	534	37	93062	4	10642	3
58	60863	39	62708	9	567	33	93057	5	10639	3
59	60825	38	62717	9	598	31	93053	4	10636	3
60°	4.60787	38	4.62726	9	2.626	28	4.93048	5	5.10633	3
61	60750	37	62735	9	652	26	93043	5	10630	3
62	60714	36	62744	9	676	24	93040	3	10627	3
63	60678	36	62752	8	698	22	93035	5	10624	3
64	60644	34	62760	8	718	20	93031	4	10622	2
65	60611	33	62768	8	737	19	93027	4	10619	3
66°	4.60578	33	4.62775	7	2.755	18	4.93024	3	5.10617	2
67	60546	32	62783	8	772	17	93020	4	10614	3
68	60515	31	62790	7	787	15	93016	4	10612	2
69	60486	29	62797	7	802	15	93013	3	10610	2
70	60456	30	62804	7	815	13	93009	4	10607	3
Special table from 43° to 55°.										
43° 0'	4.61482	—	4.62559	+	1.824 _n	—	4.93131	—	5.10688	—
10	61475	7	62561	2	1.791 _n	33	93130	1	10687	1
20	61468	7	62562	1	1.754 _n	37	93129	1	10687	0
30	61461	7	62564	2	1.715 _n	39	93129	0	10686	1
40	61454	7	62566	2	1.671 _n	44	93128	1	10686	0
50	61447	7	62568	2	1.622 _n	49	93127	1	10685	1
44° 0'	4.61440	7	4.62569	1	1.568 _n	54	4.93126	1	5.10685	0
10	61433	7	62571	2	1.505 _n	63	93125	1	10684	1
20	61426	7	62573	2	1.432 _n	73	93125	1	10683	1
30	61419	7	62574	1	1.344 _n	88	93124	1	10683	0
40	61412	7	62576	1	1.236 _n	108	93123	0	10683	1
50	61405	7	62578	2	1.084 _n	152	93123	1	10682	1
45° 0'	4.61398	7	4.62580	2	0.855 _n	229	93122	1	10682	1
10	61391	7	62581	1	0.339 _n	516	4.93121	1	5.10681	0
				2			93120	1	10681	1
20	61384	7	62583	—	0.442	—	93119	—	10680	—
30	61377	7	62585	+	0.889	+	93118	+	10680	+
40	61370	7	62586	447	1.104	215	93118	1	10679	1
50	61363	7	62588	2	1.248	144	93117	0	10679	1
46° 0'	4.61356	7	4.62590	2	1.356	108	93117	1	10678	0
10	61348	8	62591	1	1.441	85	4.93116	1	5.10678	1
20	61341	7	62593	2	1.513	72	93115	1	10677	0
30	61334	7	62595	2	1.575	62	93114	1	10677	1
40	61327	7	62597	2	1.627	52	93113	1	10676	0
50	61320	7	62598	1	1.676	49	93112	0	10676	1
47° 0'	4.61313	7	4.62600	2	1.720	44	93112	1	10675	0
							4.93111	—	5.10675	—

cf. section 21.

φ	$\log [4]$		$\log [5]$		$\log [6]$		$\log [7]$		$\log [8]$	
47° 0'	4.61313	—	4.62600	+	1.720	—	4.93111	—	5.10675	—
10	61306	7	62602	2	1.759	39	93110	1	10674	1
20	61299	7	62603	1	1.795	36	93109	1	10673	1
30	61292	7	62605	2	1.829	34	93108	1	10673	0
40	61285	7	62607	2	1.859	30	93107	1	10672	1
50	61278	7	62608	1	1.888	29	93107	0	10672	0
		7		2		27		1		1
48° 0'	4.61271	7	4.62610	2	1.915	26	4.93106	1	5.10671	1
10	61264	7	62612	2	1.941	26	93105	1	10670	0
20	61257	7	62614	1	1.965	24	93104	1	10670	1
30	61250	7	62615	2	1.987	22	93103	1	10669	0
40	61243	7	62617	2	2.009	22	93102	1	10669	0
50	61236	7	62619	2	2.029	20	93102	0	10668	1
		7		1		20		1		0
49° 0'	4.61229	7	4.62620	2	2.049	19	4.93101	1	5.10668	1
10	61222	7	62622	2	2.068	19	93100	1	10667	0
20	61215	7	62624	2	2.086	18	93099	1	10667	1
30	61208	7	62625	1	2.103	17	93098	1	10666	0
40	61201	7	62627	2	2.120	17	93097	1	10666	0
50	61194	7	62629	2	2.136	16	93097	0	10665	1
		7		1		15		1		1
50° 0'	4.61187	7	4.62630	2	2.151	15	4.93096	1	5.10664	0
10	61180	7	62632	2	2.166	15	93095	1	10664	0
20	61173	7	62634	2	2.180	14	93094	1	10663	1
30	61166	7	62636	2	2.194	14	93093	1	10663	0
40	61159	7	62637	1	2.207	13	93092	1	10662	1
50	61152	7	62639	2	2.220	13	93092	0	10662	0
		7		2		13		1		1
51° 0'	4.61145	7	4.62641	2	2.233	12	4.93091	1	5.10661	0
10	61138	6	62642	1	2.245	12	93090	1	10661	0
20	61132	7	62644	2	257	11	93089	1	10660	1
30	61125	7	62646	2	268	11	93088	1	10660	0
40	61118	7	62647	1	280	12	93087	0	10659	1
50	61111	7	62649	2	291	11	93087	1	10658	1
		7		2		10		1		0
52° 0'	4.61104	7	4.62651	1	2.301	11	4.93086	1	5.10658	1
10	61097	7	62652	2	312	10	93085	1	10657	0
20	61090	7	62654	2	322	10	93084	1	10657	1
30	61083	7	62656	2	332	10	93083	1	10656	0
40	61076	6	62657	1	341	9	93082	1	10656	1
50	61070	7	62659	2	351	10	93082	0	10655	1
		7		1		9		1		0
53° 0'	4.61063	7	4.62660	2	2.360	9	4.93081	1	5.10655	1
10	61056	7	62662	2	369	9	93080	1	10654	0
20	61049	7	62664	2	378	9	93079	1	10654	1
30	61042	7	62665	1	387	8	93078	1	10653	1
40	61035	7	62667	2	395	8	93078	0	10652	1
50	61029	6	62669	2	403	8	93077	1	10652	0
		7		1		8		1		1
54° 0'	4.61022	7	4.62670	2	2.411	8	4.93076	1	5.10651	0
10	61015	6	62672	1	419	8	93075	1	10651	1
20	61009	7	62673	2	427	8	93074	0	10650	0
30	61002	7	62675	2	435	8	93074	1	10650	1
40	60995	7	62677	2	442	7	93073	1	10649	0
50	60989	6	62678	1	450	8	93072	1	10649	1
		7		2		8		1		
55° 0'	4.60981		4.62680		2.458		4.93071		5.10648	

cf. section 21.

$$\tan \psi = \sqrt{1 - e^2} \tan \varphi, \quad \varphi - \psi = 345,32538 \sin(\varphi + \psi) \\ [2.538 \ 2285]$$

φ	$\varphi - \psi$	Diff.	φ	$\varphi - \psi$	Diff.	φ	$\varphi - \psi$	Diff.
0°	0' 0,0''	+	45° 0'	5' 45,33''	—	53° 0'	5' 32,10''	—
1	0 12,0	12,0''	10	45,32	0,01''	10	31,50	0,56''
2	0 24,0	12,0	20	45,31	0,01	20	30,98	0,56
3	0 36,0	12,0	30	45,28	0,03	30	30,40	0,58
4	0 48,0	12,0	40	45,24	0,04	40	29,81	0,59
5	1 0,0	11,7	50	45,20	0,04	50	29,21	0,60
					0,07			0,62
6°	1' 11,7''	11,7''	46° 0'	5' 45,13''	0,07''	54° 0'	5' 28,59''	0,62''
7	1 23,4	11,6	10	45,06	0,08	10	27,97	0,63
8	35,0	11,5	20	44,98	0,10	20	27,34	0,65
9	46,5	11,4	30	44,88	0,11	30	26,69	0,66
10	57,9	11,3	40	44,77	0,12	40	26,03	0,67
11	2' 9,2	11,0	50	44,65	0,13	50	25,36	0,67
12°	2' 20,2''	11,0''	47° 0'	5' 44,52''	0,14''	55° 0'	5' 24,69''	4,3''
13	31,2	10,7	10	44,38	0,15	56	20,4	4,7
14	41,9	10,5	20	44,23	0,17	57	15,7	5,1
15	52,4	10,3	30	44,06	0,18	58	10,6	5,5
16	3' 2,7	10,1	40	43,88	0,19	59	5,1	5,8
17	12,8	9,9	50	43,69	0,20	60	4' 59,3	6,2
18°	3' 22,7''	9,6''	48° 0'	5' 43,49''	0,21''	61°	4' 53,1''	6,5''
19	32,3	9,4	10	43,28	0,22	62	46,6	7,0
20	41,7	9,1	20	43,06	0,24	63	39,6	7,2
21	50,8	8,8	30	42,82	0,25	64	32,4	7,6
22	59,6	8,5	40	42,57	0,26	65	24,8	7,9
23	4' 8,1	8,2	50	42,31	0,27	66	16,9	8,2
24°	4' 16,3''	7,9''	49° 0'	5' 42,04''	0,28''	67°	4' 8,7''	8,5''
25	24,2	7,6	10	41,76	0,29	68	0,2	8,8
26	31,8	7,3	20	41,47	0,31	69	3' 51,4	9,1
27	39,1	6,9	30	41,16	0,31	70	42,3	9,4
28	46,0	6,6	40	40,85	0,33	71	32,9	9,6
29	52,6	6,2	50	40,52	0,34	72	23,3	9,9
30°	4' 58,8''	5,9''	50° 0'	5' 40,18''	0,35''	73°	3' 13,4''	10,1''
31	5' 4,7	5,4	10	39,83	0,37	74	3,3	10,4
32	10,1	5,2	20	39,46	0,37	75	2' 52,9	10,5
33	15,3	4,7	30	39,09	0,39	76	42,4	10,8
34	20,0	4,3	40	38,70	0,39	77	31,6	10,9
35	24,3	4,0	50	38,31	0,41	78	20,7	11,1
36°	5' 28,3''	3,5''	51° 0'	5' 37,90''	0,42''	79°	2' 9,6''	11,3''
37	31,8	3,1	10	37,48	0,44	80	1' 58,3	11,4
38	34,9	2,8	20	37,04	0,44	81	46,9	11,6
39	37,7	2,3	30	36,60	0,45	82	35,3	11,6
40	40,0	1,9	40	36,15	0,47	83	23,7	11,8
41	41,9	1,5	50	35,68	0,48	84	11,9	11,8
42°	5' 43,4''	1,0''	52° 0'	5' 35,20''	0,49''	85°	1' 0,1''	12,0''
43	44,4	0,7	10	34,71	0,50	86	0' 48,1	11,9
44	45,1	+ 0,2	20	34,21	0,51	87	36,2	12,1
45	45,3	— 0,2	30	33,70	0,52	88	24,1	12,0
46	45,1	0,6	40	33,18	0,53	89	12,1	12,1
47	5' 44,5		50	32,65		90	0' 0,0	

cf. section 22 p. 108.

φ	$\log(\sigma_1)$		$\log(\sigma_2)$		$\log(\sigma_3)$		$\log(\sigma_4)$		$\log(\sigma_5)$	
30°	5.52260 _n	—	5.99435 _n	—	3.587	—	3.712	+	3.976 _n	+19
31	5.49609 _n	2651	5.98512 _n	893	3.560	27	3.803	91	3.995 _n	14
32	5.46727 _n	2882	5.97613 _n	929	3.530	30	3.885	82	4.009 _n	13
33	5.43583 _n	3144	5.96647 _n	966	3.498	32	3.959	74	4.022 _n	13
34	5.40134 _n	3449	5.95644 _n	1003	3.462	36	4.029	70	4.035 _n	11
		3805		1042		40		65		
35°	5.36329 _n	—	5.94602 _n	—	3.422	—	4.094	+	4.046 _n	+ 8
36	5.32097 _n	4232	5.93520 _n	1082	3.378	44	4.155	61	4.054 _n	7
37	5.27346 _n	4751	5.92398 _n	1122	3.329	49	4.214	59	4.061 _n	4
38	5.21948 _n	5398	5.91235 _n	1163	3.272	57	4.270	56	4.065 _n	+ 3
39	5.15717 _n	6231	5.90029 _n	1206	3.206	66	4.323	53	4.068 _n	— 1
		7357		1249		78		52		
40°	5.08360 _n	—	5.88780 _n	—	3.128	—	4.375	+	4.067 _n	— 3
41	4.99460 _n	8900	5.87485 _n	1295	3.032	96	4.426	51	4.064 _n	4
42	4.88154 _n	11306	5.86143 _n	1342	2.908	124	4.474	48	4.058 _n	10
43	4.72741 _n	15413	5.84754 _n	1389	2.732	176	4.523	49	4.048 _n	15
44	4.48516 _n	24225	5.83316 _n	1438	2.431	301	4.569	46	4.033 _n	20
				1490				46		
45°	3.88570 _n	—	5.81826 _n	—	— ∞	—	4.615	+	4.013 _n	—26
46	4.18121	+	5.80283 _n	1543	2.431 _n	+	4.660	45	3.987 _n	35
47	4.58102	39981	5.78685 _n	1598	2.732 _n	301	4.705	45	3.952 _n	45
48	4.78513	20411	5.77031 _n	1654	2.908 _n	176	4.749	44	3.907 _n	62
49	4.92311	13798	5.75317 _n	1714	3.032 _n	124	4.792	43	3.845 _n	—86
		10420		1775		86		43		
50°	5.02731	—	5.73542 _n	—	3.128 _n	—	4.835	+	3.759 _n	127
51	5.11089	8358	5.71703 _n	1839	3.206 _n	78	4.877	42	3.632 _n	219
52	5.18054	6965	5.69797 _n	1906	3.272 _n	66	4.920	43	3.413 _n	607
53	5.24012	5958	5.67821 _n	1976	3.329 _n	57	4.962	42	2.806 _n	
54	5.29296	5194	5.65772 _n	2049	3.378 _n	49	5.004	42	3.202	
55	5.33799	4593	5.63647 _n	2125	3.422 _n	44	5.046	42	3.617	

Special table for $\log(\sigma_1)$.

φ	0'	10'	20'	30'	40'	50'	Diff.
45°	3.88570 _n	3.58776 _n	1.73612 _n	3.61198	3.87962	4.05674	+
46	4.18221	4.27944	4.35882	4.42590	4.48399	4.53521	12547
47	4.58102	4.62244	4.66024	4.69500	4.72718	4.75712	4581
48	4.78513	4.81142	4.83619	4.85963	4.88185	4.90298	2801
49	4.92311	4.94234	4.96074	4.97838	4.99532	5.01162	2013
							1569
50°	5.02731	5.04244	5.05705	5.07118	5.08483	5.09806	1283
51	5.11089	5.12334	5.13542	5.14717	5.15859	5.16971	1083
52	5.18054	5.19109	5.20138	5.21141	5.22121	5.23077	935
53	5.24012	5.24925	5.25819	5.26692	5.27548	5.28386	820
54	5.29206	5.30009	5.30797	5.31569	5.32327	5.33070	729
55	5.33799						

In the neighborhood of $\varphi = 45^\circ$, where (σ_1) passes through zero, we need $\log(\sigma_1)$ only to a very few places; therefore, in spite of the inequality of the differences we can interpolate with sufficient accuracy proportionally or graphically.

[13] Coefficients for Jordan's solution of the geodetic mean problem.

φ	$\log(\lambda_1)$		$\log(\lambda_2)$		$\log(\lambda_3)$		$\log(\lambda_4)$		$\log(\lambda_5)$	
30°	5.99325 _n	+	5.99435 _n	—	4.065 _n	+	4.012	—	4.013 _n	+
31	6.00207 _n	882	5.98542 _n	893	4.084 _n	19	4.003	9	4.035 _n	22
32	6.01087 _n	880	5.97613 _n	929	4.104 _n	20	3.987	16	4.054 _n	19
33	6.01964 _n	877	5.96647 _n	966	4.123 _n	19	3.966	21	4.073 _n	19
34	6.02837 _n	873	5.95644 _n	1003	4.142 _n	19	3.938	28	4.090 _n	17
		867		1042		18		38		16
35°	6.03704 _n		5.94602 _n		4.160 _n		3.900		4.106 _n	
36	6.04566 _n	862	5.93520 _n	1082	4.177 _n	17	3.850	50	4.121 _n	15
37	6.05420 _n	854	5.92398 _n	1122	4.194 _n	17	3.782	68	4.134 _n	13
38	6.06265 _n	845	5.91235 _n	1163	4.211 _n	17	3.685	97	4.146 _n	12
39	6.07098 _n	833	5.90029 _n	1206	4.227 _n	16	3.545	140	4.157 _n	11
		829		1249		16		—		9
40°	6.07927 _n		5.88780 _n		4.243 _n		3.262		4.166 _n	
41	6.08745 _n	818	5.87485 _n	1295	4.258 _n	15	1.409	+	4.174 _n	8
42	6.09550 _n	805	5.86143 _n	1342	4.273 _n	15	3.331		4.181 _n	7
43	6.10343 _n	793	5.84754 _n	1389	4.287 _n	14	3.657	326	4.185 _n	4
44	6.11124 _n	781	5.83316 _n	1438	4.301 _n	14	3.861	204	4.188 _n	3
		768		1490		13		152		±
45°	6.11892 _n		5.81826 _n		4.314 _n		4.013		4.189 _n	
46	6.12646 _n	754	5.80283 _n	1543	4.327 _n	13	4.138	125	4.188 _n	1
47	6.13387 _n	741	5.78685 _n	1598	4.340 _n	13	4.245	107	4.185 _n	3
48	6.14113 _n	726	5.77031 _n	1654	4.352 _n	12	4.340	95	4.179 _n	6
49	6.14825 _n	712	5.75317 _n	1714	4.364 _n	12	4.427	87	4.169 _n	10
		697		1775		12		79		13
50°	6.15522 _n		5.73542 _n		4.376 _n		4.506		4.156 _n	
51	6.16203 _n	681	5.71703 _n	1839	4.387 _n	11	4.580	74	4.139 _n	17
52	6.16869 _n	666	5.69797 _n	1906	4.398 _n	11	4.650	70	4.117 _n	22
53	6.17519 _n	650	5.67821 _n	1976	4.408 _n	10	4.716	67	4.087 _n	30
54	6.18153 _n	634	5.65772 _n	2049	4.418 _n	10	4.781	64	4.049 _n	38
55°	6.18771 _n	618	5.63647 _n	2125	4.428 _n	10	4.842	61	3.999 _n	50

Special table for $\log(\sigma_1) = \log(\lambda_1)$.

φ	0'	10'	20'	30'	40'	50'	Diff.
45°	5.81826 _n	5.81572 _n	5.81317 _n	5.81061 _n	5.80803 _n	5.80544 _n	—
46	5.80283 _n	5.80021 _n	5.79757 _n	5.79491 _n	5.79224 _n	5.78956 _n	261
47	5.78685 _n	5.78414 _n	5.78140 _n	5.77865 _n	5.77589 _n	5.77311 _n	271
48	5.77031 _n	5.76749 _n	5.76466 _n	5.76182 _n	5.75895 _n	5.75607 _n	280
49	5.75317 _n	5.75026 _n	5.74733 _n	5.74438 _n	5.74141 _n	5.73842 _n	290
							300
50°	5.73542 _n	5.73240 _n	5.72936 _n	5.72631 _n	5.72323 _n	5.72014 _n	311
51	5.71703 _n	5.71390 _n	5.71075 _n	5.70759 _n	5.70440 _n	5.70120 _n	323
52	5.69797 _n	5.69473 _n	5.69146 _n	5.68818 _n	5.68488 _n	5.68156 _n	335
53	5.67821 _n	5.67485 _n	5.67147 _n	5.66806 _n	5.66464 _n	5.66119 _n	347
54	5.65772 _n	5.65424 _n	5.65073 _n	5.64720 _n	5.64364 _n	5.64007 _n	360
55	5.63647 _n						

cf. section 26 p. 124.

I.

$$\text{Abcissa} = A \Delta \varphi + B \Delta \varphi^2 + C \varphi^2 - D \Delta \varphi \varphi^2 - E \Delta \varphi^3 - F \Delta \varphi^2 \varphi^2 + G \varphi^4.$$

φ_0	$\log A$	$\log B$	$\log C$	$\log D$	$\log E$	$\log F$	$\log G$	Meridian arc
	$(+ A \Delta \varphi)$	$(+ B \Delta \varphi^2)$	$(+ C \varphi^2)$	$(- D \Delta \varphi \varphi^2)$	$(- E \Delta \varphi^3)$	$(- F \Delta \varphi^2 \varphi^2)$	$(+ G \varphi^4)$	m
47° 0'	1.489 6397.637	3.875 180	5.573 4506	9.394 196	7.174 063	5.24563	4.11845	5 206 717,134
47 30	1.489 6777.260	3.874 637	5.572 8667	9.493 061	7.282 566	5.24505	4.10502	5 262 298,751
48 0	1.489 7156.325	3.873 980	5.572 1495	9.573 438	7.369 234	5.24433	4.09108	5 317 885,233
48 30	1.489 7534.718	3.873 180	5.571 2984	9.641 127	7.441 366	5.24348	4.07662	5 373 476,563
49 0	1.489 7912.325	3.872 245	5.570 3131	9.699 564	7.503 116	5.24249	4.06162	5 429 072,732
49 30	1.489 8289.026	3.871 175	5.569 1928	9.750 950	7.557 074	5.24137	4.04605	5 484 673,729
50 0	1.489 8664.710	3.869 969	5.567 9369	9.796 778	7.604 960	5.24012	4.02989	5 540 279,543
50 30	1.489 9039.260	3.868 626	5.566 5445	9.838 111	7.647 991	5.23872	4.01311	5 595 890,160
51 0	1.489 9412.565	3.867 146	5.565 0147	9.875 733	7.687 032	5.23719	3.99569	5 651 505,565
51 30	1.489 9784.508	3.865 528	5.563 3466	9.910 234	7.722 743	5.23553	3.97758	5 707 125,743
52 0	1.490 0154.975	3.863 770	5.561 5392	9.942 074	7.755 629	5.23372	3.95876	5 762 750,675
52 30	1.490 0523.855	3.861 871	5.559 5911	9.971 617	7.786 088	5.23177	3.93918	5 818 380,311
53 0	1.490 0891.034	3.859 830	5.557 5012	9.999 155	7.814 436	5.22968	3.91880	5 874 014,723
53 30	1.490 1256.404	3.857 646	5.555 2681	0.024 929	7.840 934	5.22745	3.89757	5 929 653,797
54 0	1.490 1619.845	3.855 316	5.552 8902	0.049 135	7.865 791	5.22507	3.87543	5 985 297,540
54 30	1.490 1981.255	3.852 840	5.550 3660	0.071 939	7.889 185	5.22255	3.85234	6 040 945,925
55 0	1.490 2340.517	3.850 216	5.547 6937	0.093 481	7.911 266	5.21987	3.82819	6 096 598,930

cf. section 28 p. 142 and section 34 p. 177.

II.
 Ordinate $y = Hl - J \Delta \varphi l - K \Delta \varphi^2 l - L \varphi^3 - M \Delta \varphi^3 + N \Delta \varphi^3 l$
 $= Hl - J \Delta \varphi l - K \Delta \varphi^2 l - L' \varphi^3 - M' \Delta \varphi^3 + N \Delta \varphi^3 l$ for conformal coordinates

φ^0	$\log H$	$\log J$	$\log K$	$\log L$	$\log M$	$\log N$	$\log L'$	$\log M'$
	$(+ H l)$	$(- J \Delta \varphi l)$	$(- K \Delta \varphi^2 l)$	$(- L \varphi^3)$	$(- M \Delta \varphi^3)$	$(+ N \Delta \varphi^3 l)$	$(- L' \varphi^3)$	$(- M' \Delta \varphi^3)$
47° 0'	1.324 7782-584	6.039 3422	0.398 186	9.646 032	4.22936	4.62963	8.752 216	4.88537
47 30	1.320 6909-282	6.042 8835	0.394 199	9.648 952	4.20322	4.63822	8.846 955	4.87606
48 0	1.316 5311-128	6.046 3640	0.390 140	9.651 676	4.17492	4.63675	8.923 131	4.86633
48 30	1.312 2974-025	6.049 7845	0.386 006	9.654 209	4.14441	4.64022	8.986 550	4.85614
49 0	1.307 9883-345	6.053 1460	0.381 798	9.656 546	4.11040	4.64364	9.040 640	4.84548
49 30	1.303 6023-904	6.056 4493	0.377 512	9.658 693	4.07327	4.64699	9.087 602	4.83434
50 0	1.299 1379-939	6.059 6954	0.373 147	9.660 644	4.03205	4.65029	9.128 928	4.82268
50 30	1.294 5935-074	6.062 8849	0.368 702	9.662 405	3.98586	4.65353	9.165 680	4.81048
51 0	1.289 9672-296	6.066 0188	0.364 174	9.663 971	3.93348	4.65671	9.198 637	4.79772
51 30	1.285 2573-908	6.069 0977	0.359 563	9.665 345	3.87315	4.65984	9.228 391	4.78436
52 0	1.280 4621-514	6.072 1225	0.354 866	9.666 525	3.80221	4.66291	9.253 399	4.77037
52 30	1.275 5795-965	6.075 0940	0.350 080	9.667 511	3.71642	4.66593	9.280 023	4.75571
53 0	1.270 6077-329	6.078 0126	0.345 206	9.668 305	3.60825	4.66890	9.302 553	4.74034
53 30	1.265 5444-843	6.080 8792	0.340 240	9.668 901	3.46235	4.67182	9.323 227	4.72421
54 0	1.260 3876-869	6.083 6945	0.335 180	9.669 301	3.23868	4.67468	9.342 240	4.70727
54 30	1.255 1350-852	6.086 4590	0.330 023	9.669 506	2.74472	4.67749	9.359 755	4.68946
55 0	1.249 7845-254	6.089 1735	0.324 767	9.669 512	2.79712	4.68025	9.375 910	4.67070

conformal

cf. section 28 p. 142 and section 34 p. 177.

III.
Difference of latitude $\Delta \varphi = ax - bx^2 - cy^2 - dy^2x + ex^3 - fxy^2 + gy^4$
 $= ax - bx^2 - cy^2 - dy^2x + ex^3 - fxy^2 + gy^4$ for conformal coordinates

φ_0	$\log a$	$\log b$	$\log c$	$\log d$	$\log e$	$\log f$	$\log g$	$\log g'$
47° 0'	(+ a x) 8.510 3602-363	(- b x ²) 9.406 261	(- c y ²) 1.434 2543	(- d y ² x) 4.929 374	(+ e x ³) 1.26939	(- f x ² y ²) 8.15585	(+ g y ⁴) 7.39259	(+ g' y ⁴) 7.67109
47 30	8.510 3222-740	9.405 614	1.441 8071	4.937 493	1.36604	8.17158	7.41195	7.68489
48 0	8.510 2843-675	9.404 834	1.449 3716	4.945 759	1.44492	8.18746	7.43143	7.69884
48 30	8.510 2465-282	9.403 919	1.456 9502	4.954 175	1.51151	8.20351	7.45104	7.71295
49 0	8.510 2087-675	9.402 871	1.464 5452	4.962 743	1.56911	8.21972	7.47078	7.72722
49 30	8.510 1710-974	9.401 688	1.472 1592	4.971 467	1.61983	8.23610	7.49066	7.74166
50 0	8.510 1335-290	9.400 370	1.479 7944	4.980 350	1.66510	8.25267	7.51068	7.75627
50 30	8.510 0960-740	9.398 914	1.487 4535	4.989 394	1.70597	8.26942	7.53084	7.77107
51 0	8.510 0587-435	9.397 323	1.495 1390	4.998 605	1.74318	8.28638	7.55115	7.78605
51 30	8.510 0215-492	9.395 593	1.502 8535	5.007 984	1.77733	8.30349	7.57163	7.80123
52 0	8.509 9845-025	9.393 724	1.510 5994	5.017 536	1.80886	8.32082	7.59226	7.81661
52 30	8.509 9476-145	9.391 714	1.518 3796	5.027 264	1.83812	8.33837	7.61307	7.83220
53 0	8.509 9108-966	9.389 563	1.526 1967	5.037 172	1.86540	8.35613	7.63405	7.84800
53 30	8.509 8743-596	9.387 269	1.534 0534	5.047 266	1.89093	8.37411	7.65522	7.86403
54 0	8.509 8380-155	9.384 830	1.541 9529	5.057 550	1.91492	8.39233	7.67657	7.88029
54 30	8.509 8018-745	9.382 246	1.549 8977	5.068 024	1.93752	8.41078	7.69814	7.89678
55 0	8.509 7659-483	9.379 514	1.557 8911	5.078 697	1.95887	8.42947	7.71988	7.91353

conformal

cf. section 28 p. 142 and section 34 p. 177.

IV.
Difference of longitude $l = hy + ixy + kyx^2 - l y^3 - m y^3 x + n y x^3$
 $= hy + ixy + kyx^2 - l' y^3 - m' y^3 x + n' y x^3$ for conformal coordinates

φ_0	$\log h$	$\log i$	$\log k$	$\log l$	$\log m$	$\log n$	$\log l'$	$\log m' = \log n$
	$(+ h y)$	$(+ i x y)$	$(+ k y x^2)$	$(- l y^3)$	$(- m y^3 x)$	$(+ n y x^3)$	$(- l' y^3)$	$(- m' y^3 x)$
47° 0'	8.675 2217-416	1.900 1458	5.282 280	4.647 949	8.54857	8.58670	4.80515	8.58670
47 30	8.679 3090-718	1.911 8239	5.296 969	4.667 218	8.56992	8.60723	4.81985	8.60723
48 0	8.683 4688-872	1.923 5863	5.311 869	4.686 582	8.59148	8.62799	4.83475	8.62799
48 30	8.687 7025-975	1.935 4362	5.326 672	4.706 048	8.61327	8.64899	4.84985	8.64899
49 0	8.692 0116-655	1.947 3781	5.342 285	4.725 623	8.63529	8.67022	4.86516	8.67022
49 30	8.696 3976-096	1.959 4156	5.357 808	4.745 312	8.65756	8.69170	4.88069	8.69170
50 0	8.700 8620-061	1.971 5529	5.373 548	4.765 122	8.68007	8.71344	4.89643	8.71344
50 30	8.705 4064-926	1.983 7939	5.389 509	4.785 060	8.70284	8.73544	4.91239	8.73544
51 0	8.710 0327-704	1.996 1430	5.405 696	4.805 132	8.72587	8.75770	4.92857	8.75770
51 30	8.714 7426-092	2.008 6045	5.422 113	4.825 345	8.74917	8.78024	4.94499	8.78024
52 0	8.719 5378-486	2.021 1827	5.438 764	4.845 706	8.77275	8.80307	4.96164	8.80307
52 30	8.724 4204-035	2.033 8823	5.455 657	4.866 223	8.79661	8.82618	4.97854	8.82618
53 0	8.729 3922-671	2.046 7081	5.472 796	4.886 903	8.82077	8.84960	4.99567	8.84960
53 30	8.734 4555-157	2.059 6646	5.490 187	4.907 752	8.84522	8.87333	5.01307	8.87333
54 0	8.739 6123-131	2.072 7572	5.507 836	4.928 781	8.86999	8.89738	5.03071	8.89738
54 30	8.744 8649-148	2.085 9908	5.525 750	4.949 995	8.89509	8.92175	5.04863	8.92175
55 0	8.750 2156-746	2.099 3708	5.543 935	4.971 404	8.92051	8.94646	5.06681	8.94646

cf. section 28 p. 142 and section 34 p. 177.

conformal

V.
Meridian convergence $l = Pl + Ql\Delta\varphi - Rl\Delta\varphi^2 + S'l^3 - T'l^3\Delta\varphi - Ul\Delta\varphi^3$
 $= Pl + Ql\Delta\varphi - Rl\Delta\varphi^2 + S'l^3 - T'l^3\Delta\varphi - Ul\Delta\varphi^3$ for conformal coordinates

φ_0	$\log P$	$\log Q$	$\log R$	$\log S$	$\log T$	$\log U$	$\log S'$
	(+ P l)	(+ Q l Δ φ)	(- R l Δ φ²)	(+ S l³)	(- T l³ Δ φ)	(- U l Δ φ³)	(+ S' l³)
47° 0'	9.864 1274-638	4.519 3582	8.934 247	8.427 078	3.19488	3.11236	8.429 774
47 30	9.867 6308-843	4.515 2582	8.937 751	8.422 356	3.20913	3.10826	8.425 008
48 0	9.871 0734-581	4.511 0858	8.941 193	8.417 428	3.22254	3.10408	8.420 015
48 30	9.874 4561-424	4.506 8394	8.944 576	8.412 293	3.23516	3.09984	8.414 844
49 0	9.877 7798-629	4.502 5178	8.947 900	8.406 948	3.24705	3.09552	8.409 449
49 30	9.881 0455-153	4.498 1193	8.951 165	8.401 392	3.25823	3.09112	8.403 843
50 0	9.884 2539-665	4.493 6424	8.954 374	8.395 622	3.26875	3.08664	8.398 023
50 30	9.887 4060-555	4.489 0854	8.957 526	8.389 635	3.27865	3.08208	8.391 986
51 0	9.890 5025-944	4.484 4467	8.960 622	8.383 429	3.28796	3.07745	8.385 731
51 30	9.893 5443-701	4.479 7245	8.963 664	8.377 001	3.29670	3.07272	8.379 254
52 0	9.896 5321-441	4.474 9168	8.966 652	8.370 349	3.30490	3.06792	8.372 552
52 30	9.899 4666-546	4.470 0220	8.969 586	8.363 470	3.31258	3.06302	8.365 615
53 0	9.902 3486-165	4.465 0379	8.972 468	8.356 359	3.31976	3.05804	8.358 465
53 30	9.905 1787-227	4.459 9625	8.975 298	8.349 014	3.32645	3.05296	8.351 071
54 0	9.907 9576-446	4.454 7936	8.978 077	8.341 430	3.33269	3.04779	8.343 440
54 30	9.910 6860-332	4.449 5289	8.980 806	8.333 606	3.33847	3.04253	8.335 567
55 0	9.913 3645-194	4.444 1661	8.983 484	8.325 535	3.34382	3.03716	8.327 448

cf. section 28 p. 142 and section 34 p. 177.

conformal

VI.
Meridian convergence $\gamma = py + qyx + ryyx^2 - sy^3 - ty^3x + uyx^3$
 $= py + qyx + ryyx^2 - sy^3 - ty^3x + uyx^3$ for conformal coordinates

φ_0	$\log p$	$\log q$	$\log r$	$\log s$	$\log t$	$\log u$	$\log s'$	$\log t'$
	$(+py)$	$(+qyx)$	$(+ryx^2)$	$(-sy^3)$	$(-ty^3x)$	$(+ux^2)$	$(-s'y^3)$	$(-t'y^3x)$
47° 0'	8.539 3492-054	2.036 6492	5.260 311	4.669 276	8.54465	8.59641	4.78382	8.59641
47 30	8.546 9999-561	2.044 8005	5.276 100	4.687 478	8.56611	8.61639	4.79959	8.61639
48 0	8.554 5423-453	2.053 0974	5.292 004	4.705 821	8.58779	8.63663	4.81551	8.63663
48 30	8.562 1587-399	2.061 5423	5.308 151	4.761 308	8.60970	8.65713	4.83159	8.65713
49 0	8.569 7915-284	2.070 1387	5.324 424	4.742 943	8.63185	8.67790	4.84784	8.67790
49 30	8.577 4431-248	2.078 8890	5.340 869	4.761 732	8.65423	8.69893	4.86427	8.69893
50 0	8.585 1159-726	2.087 7967	5.357 491	4.780 681	8.67687	8.72024	4.88087	8.72024
50 30	8.592 8125-481	2.096 8653	5.374 297	4.799 794	8.69976	8.74183	4.89765	8.74183
51 0	8.600 5353-648	2.106 0978	5.391 293	4.819 077	8.72291	8.76372	4.91463	8.76372
51 30	8.608 2869-793	2.115 4977	5.408 483	4.838 536	8.74632	8.78589	4.93180	8.78589
52 0	8.616 0699-927	2.125 0697	5.425 876	4.858 176	8.77002	8.80838	4.94917	8.80838
52 30	8.623 8870-581	2.134 8164	5.443 477	4.878 002	8.79399	8.83117	4.96676	8.83117
53 0	8.631 7408-836	2.144 7420	5.461 292	4.898 021	8.81826	8.85428	4.98455	8.85428
53 30	8.639 6342-384	2.154 8511	5.479 330	4.918 244	8.84283	8.87771	5.00257	8.87771
54 0	8.647 5699-577	2.165 1478	5.497 596	4.938 673	8.86771	8.90148	5.02082	8.90148
54 30	8.655 5509-480	2.175 6364	5.516 107	4.959 325	8.89288	8.92559	5.03951	8.92559
55 0	8.663 5801-940	2.186 3220	5.534 845	4.980 178	8.91843	8.95006	5.05804	8.95006

cf. section 28 p. 142 and section 34 p. 177

conformal

Change from a west system (x, y) to an east system (x', y') .

x	φ_0	x_0	y_0	$\log(1 - \cos 2\gamma_0)$	$\log \sin 2\gamma_0$	$\log s_2$	u_2	$\log s_3$	u_3
m		m	m						
5 180 000	46° 30'	5 152 233,647	115 129,255	6.858 1339.9	8.579 5038.3	1.4518-10	+ 93° 15,9'	4.20-20	- 89°
5 236 000	47 0	5 207 809,275	114 069,260	6.865 2639.2	8.583 0665.4	1.4417	+ 93 17,5	4.20	- 88
5 291 000	47 30	5 263 389,432	113 000,516	6.872 2670.2	8.586 5673.9	1.4435	+ 93 19,1	4.20	- 88
5 347 000	48 0	5 318 974,111	111 923,105	6.879 1417.7	8.590 0073.2	1.4392	+ 93 20,7	4.21	- 88
5 402 000	48 30	5 374 563,306	110 837,105	6.885 9037.5	8.593 3873.2	1.4349	+ 93 22,3	4.21	- 88
5 458 000	49 0	5 430 157,009	109 742,600	6.892 5509.3	8.596 7084.6	1.4305	+ 93 23,9	4.21	- 88
5 514 000	49 30	5 485 755,209	108 639,672	6.899 0814.6	8.599 9715.6	1.4261	+ 93 25,4	4.22	- 88
5 569 000	50 0	5 541 357,897	107 528,402	6.905 4990.6	8.603 1775.1	1.4216	+ 93 26,9	4.22	- 88
5 625 000	50 30	5 596 965,059	106 408,875	6.911 8019.1	8.606 3269.6	1.4170	+ 93 28,4	4.22	- 88
5 681 000	51 0	5 652 576,681	105 281,175	6.917 9833.7	8.609 4208.7	1.4123	+ 93 29,9	4.23	- 88
5 736 000	51 30	5 708 192,750	104 145,386	6.924 0672.6	8.612 4601.3	1.4075	+ 93 31,4	4.23	- 88
5 792 000	52 0	5 763 813,247	103 001,595	6.930 0418.2	8.615 4454.3	1.4027	+ 93 32,8	4.23	- 87
5 847 000	52 30	5 819 438,155	101 849,888	6.935 9101.4	8.618 3774.1	1.3978	+ 93 34,3	4.24	- 87
5 903 000	53 0	5 875 067,456	100 690,352	6.941 6704.1	8.621 2569.0	1.3928	+ 93 35,7	4.24	- 87
5 959 000	53 30	5 930 701,127	99 523,073	6.947 3258.7	8.624 0845.0	1.3877	+ 93 37,2	4.24	- 87
6 014 000	54 0	5 986 339,148	98 348,140	6.952 8795.8	8.626 8609.0	1.3825	+ 93 38,6	4.24	- 87
6 070 000	54 30	6 041 981,494	97 165,642	6.958 3392.7	8.629 5868.5	1.3772	+ 93 40,0	4.25	- 87
6 125 000	55 0	6 097 628,144	95 975,669	6.963 7028.5	8.632 2628.8	1.3718	+ 93 41,3	4.25	- 87
6 181 000	55 30	6 153 279,068	94 778,311	6.968 9496.8	8.634 8896.1	1.3663	+ 93 42,7	4.25	- 87
	56 0	6 208 934,241	93 573,657	6.974 1108.3	8.637 4676.2	1.3607	+ 93 44,1	4.25	- 87

cf. section 38 p. 209.

Sphere u	Ellipsoid φ	Differences		$\log m$	Diff.	k	Diff.
46° 0'	46° 1' 19,1599 "	10' 1,4235 "	—	14.46	—	8,79 "	—
46 10	46 11 20,5834	10 1,4060	0,0175	13.40	1.06	8,37	0,42 "
46 20	46 21 21,9894	10 1,3885	0,0175	12.40	1.00	7,95	0,42
46 30	46 30 24,3779	10 1,3711	0,0174	11.46	0.94	7,54	0,41
46 40	46 41 24,74900	10 1,35362	0,0175	10.559	0.90	7,141	0,40
46 50	46 51 26,10262	10 1,33616	0,01746	9.709	0.850	6,754	0,387
			0,01746		0.805		0,376
47° 0'	47° 1' 27,43878 "	10' 1,31870 "	—	8.904	—	6,378 "	—
47 10	47 11 28,75748	10 1,30124	0,01746	8.146	0.758	6,012	0,366 "
47 20	47 21 30,05872	10 1,28378	0,01746	7.431	0.715	5,657	0,355
47 30	47 31 31,34250	10 1,26633	0,01745	6.759	0.672	5,313	0,344
47 40	47 41 32,60883	10 1,24889	0,01744	6.129	0.630	4,979	0,334
47 50	47 51 33,85772	10 1,23144	0,01745	5.539	0.590	4,655	0,324
			0,01744		0.551		0,312
48° 0'	48° 1' 35,08916 "	10' 1,21400 "	—	4.988	—	4,343 "	—
48 10	48 11 36,30316	10 1,19657	0,01743	4.476	0.513	4,041	0,302 "
48 20	48 21 37,49973	10 1,17915	0,01742	3.998	0.477	3,749	0,292
48 30	48 31 38,67888	10 1,16173	0,01742	3.556	0.442	3,469	0,280
48 40	48 41 39,84061	10 1,14433	0,01740	3.148	0.408	3,199	0,270
48 50	48 51 40,98494	10 1,12693	0,01740	2.772	0.376	2,940	0,259
			0,01739		0.345		0,248
49° 0'	49° 1' 42,11187 "	10' 1,10954 "	—	2.427	—	2,692 "	—
49 10	49 11 43,22141	10 1,09217	0,01737	2.112	0.315	2,454	0,238 "
49 20	49 21 44,31358	10 1,07480	0,01737	1.825	0.287	2,227	0,227
49 30	49 31 45,38838	10 1,05746	0,01734	1.566	0.259	2,012	0,215
49 40	49 41 46,44584	10 1,04012	0,01734	1.332	0.234	1,807	0,205
49 50	49 51 47,48596	10 1,02280	0,01732	1.123	0.209	1,613	0,194
			0,01731		0.187		0,184
50° 0'	50° 1' 48,50876 "	10' 1,00549 "	—	0.936	—	1,429 "	—
50 10	50 11 49,51425	10 0,98820	0,01729	0.772	0.164	1,257	0,172 "
50 20	50 21 50,50245	10 0,97093	0,01727	0.628	0.144	1,096	0,161
50 30	50 31 51,47338	10 0,95367	0,01726	0.503	0.125	0,946	0,150
50 40	50 41 52,42705	10 0,93643	0,01724	0.396	0.107	0,806	0,140
50 50	50 51 53,36348	10 0,91922	0,01721	0.305	0.091	0,678	0,128
			0,01721		0.076		0,117
51° 0'	51° 1' 54,28270 "	10' 0,90201 "	—	0.229	—	0,561 "	—
51 10	51 11 55,18471	10 0,88484	0,01717	0.167	0.062	0,454	0,107 "
51 20	51 21 56,06955	10 0,86767	0,01717	0.118	0.049	0,359	0,095
51 30	51 31 56,93722	10 0,85055	0,01712	0.079	0.039	0,275	0,084
51 40	51 41 57,78777	10 0,83343	0,01712	0.050	0.029	0,202	0,073
51 50	51 51 58,62120	10 0,81634	0,01709	0.029	0.021	0,141	0,061
			0,01707		0.014		0,051
52° 0'	52° 1' 59,43754 "	10' 0,79927 "	—	0.015	—	0,090 "	—
52 10	52 12 0,23681	10 0,78224	0,01703	0.006	0.009	0,051	—0,039 "
52 20	52 22 1,01905	10 0,76523	0,01701	+0.002	0.004	0,023	—0,028
52 30	52 32 1,78428	10 0,74823	0,01700	0.000	0.002	0,006	—0,017
52 40	52 42 2,53251	10 0,73128	0,01695	0.000		0,000	—0,006
52 50	52 52 3,26379	10 0,71435	0,01693	—0.002		0,006	+0,006
53° 0'	53° 2' 3,97814 "			—0.002		0,023	+0,017

cf. section 48 p. 252 and section 49 p. 255.

Sphere u	Ellipsoid φ	Differences		$\log m$	Diff.	k	Diff.
52° 0'	52° 1' 59,43754"	+	—	+	—	0,090"	—
52 10	52 12 0,23681	10' 0,79927"	0,01703	0-015	0-009	0,051	—0,039"
52 20	52 22 1,01905	10 0,78224	0,01701	0-006	0-004	0,023	—0,028
52 30	52 32 1,78428	10 0,76523	0,01700	0-002	0-002	0,006	—0,017
52 40	52 42 2,53251	10 0,74823	0,01695	0-000		0,000	+0,006
52 50	52 52 3,25379	10 0,73128	0,01693	±		0,000	+0,006
		10 0,71435	0,01690	0-000	0-002	0,006	+0,017
53° 0'	53° 2' 3,97814"	+	—	+	—	0,023"	—
53 10	53 12 4,67559	10' 0,69745"	0,01688	0-002	0-004	0,051	+0,028"
53 20	53 22 5,35616	10 0,68057	0,01684	0-006	0-009	0,091	0,040
53 30	53 32 6,01989	10 0,66373	0,01681	0-015	0-014	0,091	0,051
53 40	53 42 6,66681	10 0,64692	0,01677	0-029	0-021	0,142	0,062
53 50	53 52 7,29696	10 0,63015	0,01675	0-050	0-029	0,204	0,074
		10 0,61340	0,01672	0-079	0-029	0,278	0,085
54° 0'	54° 2' 7,91036"	+	—	+	—	0,363"	—
54 10	54 12 8,50704	10' 0,59668"	0,01667	0-119	0-050	0,460	+0,097"
54 20	54 22 9,08705	10 0,58001	0,01664	0-169	0-063	0,569	0,109
54 30	54 32 9,65042	10 0,56337	0,01661	0-232	0-077	0,689	0,120
54 40	54 42 10,19718	10 0,54676	0,01658	0-309	0-092	0,820	0,131
54 50	54 52 10,72736	10 0,53018	0,01652	0-401	0-109	0,964	0,144
		10 0,51366	0,01650	0-510	0-128	0,964	0,154
55° 0'	55° 2' 11,24102"	+	—	+	—	1,118"	—
55 10	55 12 11,73818	10' 0,49716"	0,01645	0-368	0-147	1,285	+0,167"
55 20	55 22 12,21889	10 0,48071	0,01642	0-785	0-168	1,463	0,178
55 30	55 32 12,68318	10 0,46429	0,01638	0-953	0-191	1,653	0,190
55 40	55 42 13,13109	10 0,44791	0,01633	1-144	0-214	1,855	0,202
55 50	55 52 13,56267	10 0,43158	0,01630	1-358	0-240	2,068	0,213
		10 0,41528	0,01624	1-598	0-267	2,068	0,225
56° 0'	56° 2' 13,97795"	+	—	+	—	2,293"	—
56 10	56 12 14,37699	10' 0,39904"	0,01621	1-865	0-296	2,531	+0,238"
56 20	56 22 14,75982	10 0,38283	0,01617	2-161	0-325	2,780	0,249
56 30	56 32 15,12648	10 0,36666	0,01611	2-486	0-356	3,041	0,261
56 40	56 42 15,47703	10 0,35055	0,01608	2-842	0-389	3,314	0,273
56 50	56 52 15,81150	10 0,33447	0,01602	3-231	0-423	3,599	0,285
		10 0,31845	0,01598	3-654	0-459	3,599	0,297
57° 0'	57° 2' 16,12995"	+	—	+	—	3,896"	—
57 10	57 12 16,43242	10' 0,30247"	0,01593	4-113	0-496	4,205	+0,309"
57 20	57 22 16,71896	10 0,28654	0,01588	4-609	0-535	4,526	0,321
57 30	57 32 16,98962	10 0,27066	0,01584	5-144	0-575	4,859	0,333
57 40	57 42 17,24444	10 0,25482	0,01578	5-719	0-616	5,205	0,346
57 50	57 52 17,48348	10 0,23904	0,01574	6-335	0-659	5,563	0,358
		10 0,22330	0,01567	6-994	0-704	5,563	0,370
58° 0'	58° 2' 17,70678"	+	—	+	—	5,933"	—
58 10	58 12 17,91441	10' 0,20763"	0,01563	7-698	0-750	6,315	+0,382"
58 20	58 22 18,10641	10 0,19200	0,01558	8-448	0-798	6,710	0,395
58 30	58 32 18,28283	10 0,17642	0,01552	9-246	0-846	7,117	0,407
58 40	58 42 18,44373	10 0,16090	0,01547	10-092	0-898	7,536	0,419
58 50	58 52 18,58916	10 0,14543	0,01541	10-990	0-951	7,967	0,431
59° 0'	59° 2' 18,71918	10 0,13002		11-941	1-002	8,410	0,443
				12-943			

cf. section 48 p. 252 and section 49 p. 255

[23] Reduced longitudes of Gauss' conformal spherical projection

Spheroidal longitude = l (Ellipsoid)

Spherical longitude = λ (Sphere).

$$\lambda = \alpha l \quad l = \frac{1}{\alpha} \lambda$$

$$\alpha = 1,000\,452\,918$$

$$\frac{1}{\alpha} = 0,999\,547\,287 = 1 - 0,000\,452\,713$$

$$\lambda - l = + 0,000\,452\,918\,l$$

$$l - \lambda = - 0,000\,452\,713\,\lambda$$

$$\log \alpha = 0.000\,1966\,553$$

$$\log \frac{1}{\alpha} = 9.999\,8033\,447$$

Ellipsoid l	$\lambda - l$	Sphere λ	$l - \lambda$
1''	+ 0,000 453''	1''	- 0,000 453''
2''	0,000 906''	2''	0,000 905''
3''	0,001 359''	3''	0,001 358''
4''	0,001 812''	4''	0,001 811''
5''	0,002 265''	5''	0,002 264''
6''	+ 0,002 718''	6''	- 0,002 716''
7''	0,003 170''	7''	0,003 169''
8''	0,003 623''	8''	0,003 622''
9''	0,004 076''	9''	0,004 074''
10''	0,004 529''	10''	0,004 527''
20''	+ 0,009 058''	20''	- 0,009 054''
30''	0,013 588''	30''	0,013 581''
40''	0,018 117''	40''	0,018 109''
50''	0,022 646''	50''	0,022 636''
60''	0,027 175''	60''	0,027 163''
1'	+ 0,027 175''	1'	- 0,027 163''
2'	0,054 350''	2'	0,054 326''
3'	0,081 525''	3'	0,081 488''
4'	0,108 700''	4'	0,108 651''
5'	0,135 875''	5'	0,135 814''
6'	+ 0,163 050''	6'	- 0,162 977''
7'	0,190 226''	7'	0,190 189''
8'	0,217 401''	8'	0,217 302''
9'	0,244 576''	9'	0,244 465''
10'	0,271 751''	10'	0,271 628''
20'	+ 0,543 502''	20'	- 0,543 256''
30'	0,815 252''	30'	0,814 883''
40'	1,087 003''	40'	1,086 511''
50'	1,358 754''	50'	1,358 139''
60'	1,630 505''	60'	1,629 767''
1°	+ 1,630 505''	1°	- 1,629 767''
2°	3,261 010''	2°	3,259 534''
3°	4,891 514''	3°	4,889 300''
4°	6,522 019''	4°	6,519 067''
5°	8,152 524''	5°	8,148 834''
6°	+ 9,783 029''	6°	- 9,778 601''
7°	11,413 534''	7°	11,408 368''
8°	13,044 038''	8°	13,038 134''
9°	14,674 543''	9°	14,667 901''
10°	16,305 048''	10°	16,297 668''

cf. section 48 p. 249 and section 52 p. 264.