The Domain of the Riemann Zeta Function on the Complex Plane

By

Armando M. Evangelista Jr.

armando781973@icloud.com

On

June 20, 2019

ABSTRACT

The Riemann zeta function is one of the most studied complex functions in mathematics. Riemann in his 1859 paper *On the Number of Prime Numbers less than a Given Quantity* [1] claimed that the analytic continuation of the zeta function extends its domain over the entire complex plane except at $z = 1$. But, as I've shown in my two papers: *A Short Disproof of the Riemann Hypothesis* [2] and *Riemann's Functional Equation is Not Valid and Its Implication on the Riemann Hypothesis* [3], the zeta function is only defined on the right half-plane. It is the purpose of this present work to precisely locate the domain of this function on the right half-plane.

The Riemann Zeta Function

The Riemann zeta function is given by the infinite series

(1) $$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

where $z = x + iy$ is the complex variable with real part x and imaginary part y, that is,

$$\Re(z) = x \quad \text{and} \quad \Im(z) = y$$

respectively, and i is the imaginary unit equal to $\sqrt{-1}$. A positive real number associated with any complex quantity is known as its modulus, usually denoted by $|\zeta(z)|$.

Note: Because of my latest paper *The s-Parameter on the Transform Integrals is a Constant* [4], I've replaced $\zeta(s)$ by $\zeta(z)$ and made the accompanying changes.

If $x > 1$ and for every y, the series converges absolutely

$$|\zeta(x+iy)| = \left|\sum_{n=1}^{\infty} n^{-(x+iy)}\right| \le 1 + 2^{-x} + 3^{-x} + 4^{-x} + \cdots = \sum_{n=1}^{\infty} n^{-x}$$

while if $0 < x \le 1$ and $y \ne 0$, the series is said to be conditionally convergent. It may converge if x is large enough. But, how large?

If the series in (1) is expressed as

$$\zeta(z) = \sum_{n=1}^{\infty} e^{-z\log n} = \sum_{n=1}^{\infty} e^{-x\log n - iy\log n} = \sum_{n=1}^{\infty} n^{-x}(\cos y\log n - i\sin y\log n)$$

where $\log n$ is the natural logarithm of n; then, its modulus $|\zeta(x+iy)|$ is

$$|\zeta(x+iy)| = \sqrt{\left(\sum_{n=1}^{\infty} n^{-x}\cos y\log n\right)^2 + \left(\sum_{n=1}^{\infty} n^{-x}\sin y\log n\right)^2}$$

or

$$(2) \qquad |\zeta(x+iy)| = \sqrt{\sum_{n=1}^{\infty} n^{-2x} + 2\sum_{n=2}^{\infty} n^{-x}\cos y\log n + 2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} n^{-x}(n+k)^{-x}\cos y\log\left(\frac{n+k}{n}\right)}$$

One can also express (2) as

$$|\zeta(x+iy)| = \sqrt{S_1 + S_2 + S_3} = \sqrt{S}$$

where,

$$S_1 = \sum_{n=1}^{\infty} n^{-2x}, \quad S_2 = 2\sum_{n=2}^{\infty} n^{-x}\cos y\log n, \quad S_3 = 2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} n^{-x}(n+k)^{-x}\cos y\log\left(\frac{n+k}{n}\right),$$

and $S = S_1 + S_2 + S_3$.

The first series S_1 is independent of y and converges absolutely if $x > ½$; the second series S_2 converges absolutely if $x > 1$; and the double sum S_3 converges quickly due to its highly damped coefficients.

The value of the sum S must be greater than or equal to zero (i.e., $S \geq 0$) in order for (1) to be valid; since if $S < 0$, the modulus $|\zeta(x+iy)|$ will not be a real number.

It is known thatis $|\zeta(x+iy)|$ always greater than zero if $x > 1$ and for every y. The sum $S_1 + S_2$, will converge to a value A, and is unlikely to reduce $|\zeta(x+iy)|$ to zero due to S_1 providing a fixed positive value for a given value of x: that is, $S_1 > |A|$ such that $|\zeta(x+iy)| = \sqrt{S_1 + A} > 0$. Hence, the zeta function has no zeros if $x > 1$.

If $½ < x \leq 1$, S_1 converges absolutely and the sum $S_1 + S_2$ may converge for some $y \neq 0$. Hence,is $|\zeta(x+iy)|$ conditionally convergent if $½ < x \leq 1$ and $y \neq 0$. One can also conclude that the zeta function has no zeros in this region for the same reason given above.

If $0 < x \leq ½$ and for every y, S_1 diverges and it doesn't matter whether the sum $S_1 + S_2$ converge or not, the zeta function is not defined on this region. It is also interesting to note that $\zeta\left(\frac{1}{2} + iy\right)$ is **undefined** on this region which proves that the Riemann Hypothesis is false.

If $x \leq 0$ and for every y, S_1 diverges very rapidly and $\zeta(z)$ is not defined on this region.

Therefore, the **domain** of $\zeta(z)$ is at $x > ½$

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \begin{cases} \frac{1}{2} < x \leq 1 \text{ and } y \neq 0 \\ x > 1 \end{cases}$$

It has no zeros in its entire domain and its modulus is always greater than zero,

$$|\zeta(z)| > 0 \begin{cases} \text{If } \frac{1}{2} < x \leq 1 \text{ and } y \neq 0 \\ \text{If } x > 1 \text{ and for every } y \end{cases}$$

REFERENCES

[1] Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity.*

[2] Evangelista, Armando M. (2018). *A Short Disproof of the Riemann Hypothesis.* https://doi.org/10.5281/zenodo.2649214

[3] Evangelista, Armando M. (2018). *Riemann's Functional Equation is Not Valid and Its Implication on the Riemann Hypothesis.* https://doi.org/10.5281/zenodo.1490765

[4] Evangelista, Armando M. (2019). *The s-Parameter on the Transform Integrals is a Constant.*

https://zenodo.org/record/3244311#.XQpvLy17H9M

LINKS:

- https://en.wikipedia.org/wiki/Riemann_zeta_function
- https://en.wikipedia.org/wiki/Riemann_hypothesis