The Domain of the Riemann Zeta Function on the Complex Plane

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ABSTRACT

The Riemann zeta-function is one of the most studied complex functions in mathematics. Riemann in his 1859 paper *On the Number of Prime Numbers less than a Given Quantity* [1] claimed that the analytic continuation of the zeta-function extends its domain over the entire complex plane except at $s = 1$. But, as I've shown in my two papers: *A Short Disproof of the Riemann Hypothesis* [2] and *Riemann's Functional Equation is Not Valid and Its Implication on the Riemann Hypothesis* [3], the zeta-function is only defined on the right half-plane. It is the purpose of this present work to precisely locate the domain of this function on the right half-plane.

The Riemann Zeta-Function

The Riemann zeta-function is given by the infinite series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\frac{1}{n^s} = \sum_{n=1}^{\infty}\frac{1}{n^s}, \tag{1}$$

where s (= $\sigma + \omega i$) is the complex variable with real part σ and imaginary part ω, that is,

$$\Re(s) = \sigma \text{ and } \Im(s) = \omega,$$

respectively, and i is the imaginary unit equal to $\sqrt{-1}$. A positive real number associated with any complex quantity is known as its modulus, usually denoted by $|\zeta(s)|$. The quantity σ has a damping effect on $\zeta(s)$ while ω acts as a filter that can remove some of its components. Thus, the values of σ and ω can have a great effect on the convergence of the infinite series in (1).

If $\sigma > 1$ and for every ω , the series converges absolutely

$$|\zeta(\sigma+\omega i)| = \left|\sum_{n=1}^{\infty} n^{-\sigma+\omega i}\right| \le 1+2^{-\sigma}+3^{-\sigma}+4^{-\sigma}+5^{-\sigma}+\cdots+n^{-\sigma} = \sum_{n=1}^{\infty} n^{-\sigma},$$

while if $0 < \sigma \le 1$ and $\omega \ne 0$, the series is said to be conditionally convergent. It may converge if σ is large enough. But, how large?

If the series in (1) is expressed as

$$\zeta(s) = \sum_{n=1}^{\infty} e^{-s\log n} = \sum_{n=1}^{\infty} e^{-\sigma\log n - i\omega\log n} = \sum_{n=1}^{\infty} n^{-\sigma}(\cos\omega\log n - i\sin\omega\log n),$$

where log n is the natural logarithm of n; then, its modulus $|\zeta(\sigma+\omega i)|$ is

$$|\zeta(\sigma + \omega i)| = \sqrt{\left(\sum_{n=1}^{\infty} n^{-\sigma}\cos\omega\log n\right)^2 + \left(\sum_{n=1}^{\infty} n^{-\sigma}\sin\omega\log n\right)^2},$$

or,

$$|\zeta(\sigma+\omega i)| = \sqrt{\sum_{n=1}^{\infty} n^{-2\sigma} + 2\sum_{n=2}^{\infty} n^{-\sigma}\cos\omega\log n + 2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} n^{-\sigma}(n+k)^{-\sigma}\cos\omega\log\left(\frac{n+k}{n}\right)}. \tag{2}$$

One can also express (2) as

$$\left|\zeta(\sigma+\omega i)\right| = \sqrt{S_1 + S_2 + S_3} = \sqrt{S},$$

where,

$$S_1 = \sum_{n=1}^{\infty} n^{-2\sigma},\ \ S_2 = 2\sum_{n=2}^{\infty} n^{-\sigma}\cos\omega\log n,\ \ S_3 = 2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty} n^{-\sigma}(n+k)^{-\sigma}\cos\omega\log\left(\frac{n+k}{n}\right),$$

and $S = S_1 + S_2 + S_3$.

The first series S_1 is independent of ω and converges absolutely if $\sigma > ½$; the second series S_2 converges absolutely if $\sigma > 1$; and the double sum S_3 converges quickly due to its highly damped coefficients. The value of the sum S must be greater than or equal to zero (i.e., $S \geq 0$) in order for (1) to be valid; since if $S < 0$, the modulus $\left|\zeta(\sigma+\omega i)\right|$ will not be a real number.

It is known that $\left|\zeta(\sigma+\omega i)\right|$ is always greater than zero if $\sigma > 1$ and for every ω. The sum $S_1 + S_2$, will converge to a value say, A, and is unlikely to reduce $\left|\zeta(\sigma+\omega i)\right|$ to zero due to S_1 providing a fixed positive value for a given value of σ: that is, $S_1 > |A|$ such that $\left|\zeta(\sigma+\omega i)\right| = \sqrt{S_1 + A} > 0$. Hence, the zeta function has no zeros if $\sigma > 1$.

If $½ < \sigma \leq 1$, S_1 converges absolutely and the sum $S_1 + S_2$ may converge for some $\omega \neq 0$. Hence, $\left|\zeta(\sigma+\omega i)\right|$ is conditionally convergent if $½ < \sigma \leq 1$ and $\omega \neq 0$. One can also conclude that the zeta function has no zeros in this region for the same reason given above.

If $0 < \sigma \leq ½$ and for every ω, S_1 diverges and it doesn't matter whether the sum $S_1 + S_2$ converge or not, the zeta function is not defined on this region. It is also interesting to note that $\zeta\left(\frac{1}{2} + \omega i\right)$ is **undefined** on this region which proves that the Riemann Hypothesis is false.

If $\sigma \leq 0$ and for every ω, S_1 diverges very fast and $\zeta(s)$ is not defined on this region.

Therefore, the **domain of** $\zeta(s)$ **is on** $\boldsymbol{\sigma > ½}$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\begin{cases} \frac{1}{2} < \sigma \leq 1 \text{ and } \omega \neq 0, \\ \sigma > 1. \end{cases}$$

It has no zeros in its entire domain and its modulus is always greater than zero,

$$\left|\zeta(s)\right| > 0 \begin{cases} \text{If } \frac{1}{2} < \sigma \leq 1 \text{ and } \omega \neq 0, \\ \text{If } \sigma > 1 \text{ and for every } \omega. \end{cases}$$

REFERENCES

[1] Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity.*

[2] Evangelista, Armando M. (2018). *A Short Disproof of the Riemann Hypothesis.* https://doi.org/10.5281/zenodo.2649214

[3] Evangelista, Armando M. (2018). *Riemann's Functional Equation is Not Valid and Its Implication on the Riemann Hypothesis.* https://doi.org/10.5281/zenodo.1490765

LINKS:

- https://en.wikipedia.org/wiki/Riemann_zeta_function
- https://en.wikipedia.org/wiki/Riemann_hypothesis