



Elegance with substance

Thomas Colignatus

Samuel van Houten Genootschap

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ELEGANCE WITH SUBSTANCE

Elegance with Substance

Mathematics and its education designed for Ladies and Gentlemen

What is wrong with mathematics education and how it can be righted

On the political economy of mathematics and its education

Thomas Colignatus

Samuel van Houten Genootschap

Colignatus is the preferred name of Thomas Cool in science. He is an econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008).

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Samuel van Houten Genootschap

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Prologue to the 1st edition in 2009

Our pupils and students are best treated as ladies and gentlemen with elegance and substance. Providing them with equal mathematics is our much valued objective.

Ideally mathematics would be perfect and unchanging and just be there to be discovered. Mathematics however is as much art as discovery. It is made. It is a creation, in the way that cavemen carved their scores in bones and that we create virtual reality with supercomputers. In the interaction between what we do and what we understand almost all of the weight is on what we do, which then imprints on our mind. It appears tedious and hard work to go a bit in the reverse direction, to even get where we are now, let alone develop a notion of perfection.

Given this fragile and historic nature of mathematics it should not come as a surprise that what we currently call mathematics actually appears, on close inspection, to be often cumbersome or even outright irrational. Clarity and understanding are frequently blocked by contradictions and nonsense that are internal to current mathematics itself. Who has a problem mastering mathematics should not be surprised.

Over the years, while teaching mathematics and writing my notes that now result in these pages, there were many moments that I felt frustrated and at times even quite annoyed about the straightjacket of current mathematical conventions. One is supposed to teach mathematics but it is precisely the textbook that blocks this prospect. For many pupils and students the goal is impossible from the outset not because of their lack of capability but because of awkward conventions that only came about in a historical process.

The flip side is that this is a Garden of Eden for didactic development. What is awkward can be hammered into something elegant. What is irrational can be turned rational and consistent. What is dark and nonsensical can be thrown out and replaced by clarity. There is beauty and satisfaction in redesign.

This didactic reconsideration also changes what we call 'mathematics'. The interaction between what we do and what we understand shifts to a new equilibrium, a higher optimum at a more agreeable level for both students and teacher. It will still be mathematics since it can be recognized as mathematics. It will be stronger and more efficient mathematics too but it will no longer be the old one.

The criterion for change lies in elegance with substance. Elegance without substance creates a dandy. Elegance ought to signal substance. Mathematics concentrates on the elegance and specific fields of study like economics concentrate on the substance. But mathematics needs to have some substance of itself too. The criterion is tricky since some people see it in the present mathematical conventions too, where awkwardness A plus awkwardness B gives inconsistency C . However, we will compare the old ways with the suggestions of the new ways and let the criterion speak for itself. This should open some eyes. Otherwise we just stay in the Garden of Eden.

Which leaves me to thank my own teachers and colleagues who trained and helped me in the old ways. A redesign starts from something and when the old is replaced then this implies that it was valuable to start with. I thank in particular my pupils and students for what they taught me.

Prologue to the 2nd edition in 2015

This 2nd edition is almost the same as the 1st edition. *Elegance with Substance* (EWS) has a strong structure, a wealth of information, and not too many pages. It argues its case succinctly. This clarity will be lost when new issues are included.

The results from 2009 generally haven't changed. When developments in 2009-2015 supplement some of the results then a footnote explains where more can be found.

Major new developments in 2009-2015 have been:

- *Conquest of the Plane* (COTP) (2011c) presents a *proof of concept* for EWS and creates a *primer* for a re-engineered course in mathematics for highschool or first year of higher education. Gill (2012) reviewed both EWS and COTP favourably and advised to read with an open mind. At the review site of the European Mathematical Society, Gamboa (2011) states about COTP: "Once the reader becomes familiar with the notations and the author style, he/she will enjoy the book. I am convinced that this work will help the students to recognize what they should know but they ignore, (...)"
- *Foundations of Mathematics. A Neoclassical Approach to Infinity* (FMNAI) (2015f) argues the case for having set theory and number theory in the highschool programme. A major deduction is that the transfinite by Georg Cantor are based upon ill-defined constructs, comparable to Russell's paradox, so that these can be eliminated. As a result the theory for the classroom is not as complex as commonly perceived.

The memo *What a mathematician might wish to know about my work* (2013) has been included on page 125 in the Appendices. The *Introduction* already explains about my background in econometrics and teaching mathematics, but this memo emphasizes again that my interest is not abstraction for the purpose of abstraction itself, but to create scope for betterment of society. Obviously, society will not be improved when an applied theory is unsound, and thus there is a role for mathematics, but here the sounding board is not only abstraction itself but also empirical science.

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I. Introduction

1. Natural limitations to a noble art

A distinction that comes natural to us is between empirical reality and abstract thought. The first is the subject of the empirical sciences, the latter the realm for mathematics and ideal philosophy. This distinction comes with the observation that mathematicians are little trained in empirical issues.

Our subject is the education in mathematics.

Didactics, and in particular the didactics of mathematics, deals with real pupils and students. Didactics requires a mindset that is sensitive to empirical observation – which is not what mathematicians are trained for.

2. As far as the mind can reach

Mathematics is a great liberating force. No dictator forces you to accept the truth of the Pythagorean Theorem. You are free to check it for yourself. You may even object to its assumptions and invent non-Euclidean geometry. Mathematical reasoning is all about ideas and deductions and about how far your free mind will get you – which is amazingly far.

But you have to be aware of reality if you say something about reality.

The education in mathematics is not without some empirical study. It is proper to recall the Van Streun (2006) *In Memoriam* of A.D. de Groot. It is a painful point however that such exceptions prove the rule.

For the record

The stock market crash in Autumn 2008 caused criticism on mathematicians and the 'rocket scientists' by Mandelbrot & Taleb (2009), Taleb (2009) and Salmon (2009). The mathematicians involved overlooked the difference between their models and reality. Accents differ a bit: Mandelbrot more on other solutions on the assumptions on the law of large numbers, Taleb more on risk, Salmon more on correlation. It remains amazing that the mathematicians at the banks and hedge funds did not issue a warning somewhere in the process and it would be obvious that those cannot evade part of the responsibility. Of course, there is a lot of blame to go around. Like the rest of the world, Taleb (2009) and Salmon (2009) are also critical on economists and lawyers in bank management and financial regulation. Fortunately, I am one those economists who issued a warning.

With respect to ecological collapse, Tinbergen & Hueting (1991) presented an approach to monitor how the economy affects the environment and to keep account of ecological survival. Their economic approach pays attention to statistics and real risks as indicated by ecologists. Alternatives came notably from modellers with a mathematical mindset who put emphasis on elegant form and easy notions of risk. Those models suggest that there are no relevant risks on the ecology, which is an agreeable suggestion for most policy makers. Final responsibility falls on those policy makers and society who allow this to happen but it remains strange that those modellers think that they contribute more than only their own assumptions and deductions. See THAEES, Colignatus (2009, 2015).

With respect to logic and democracy, Colignatus (2007ab, 2008b), updated from 1981 / 1990, considers statements by mathematicians that have been accepted throughout academia and subsequently society on the basis of mathematical authority. It appears however that those statements mix up true mathematical results with interpretations about reality. When these interpretations are modelled mathematically, the statements reduce to falsehoods. Society has been awfully off-track on basic notions of logic, civic discourse and democracy. Even in 2007, mathematicians working on voting theory wrote a *Letter to the governments of the EU member states* advising the use of the Penrose Square Root Weights (PSRW) for the EU Council of Ministers. See Colignatus (2007c) on their statistical inadequacy and their misrepresentation of both morality and reality.

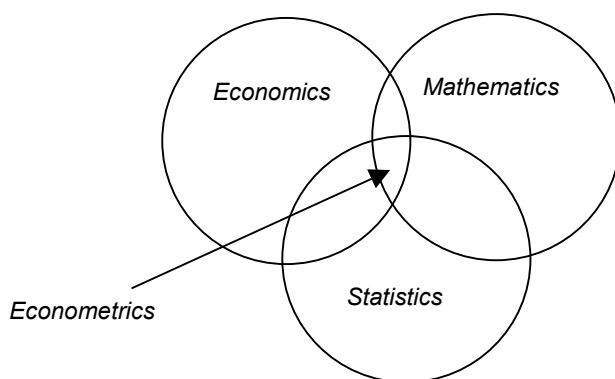
Over the millenia a tradition and culture of mathematics has grown that conditions mathematicians to, well, what mathematicians do. Which is not empirical analysis. Psychology will play a role too in the filtering out of those students who will later become mathematicians. After graduation, mathematicians either have a tenure track in (pure) mathematics or they are absorbed into other fields such as physics, economics or psychology. They tend to take along their basic training in abstraction and then try to become empirical scientists – but within the framework of their basic training.

The result is comparable to what happens when mathematicians become educators in mathematics. They succeed easily in replicating the conditioning and in the filtering out of new recruits who adapt to the treatment. For other pupils it is hard pounding.

Definition of econometrics

My own training in mathematics, as a student of econometrics starting Autumn 1973, was with the students of mathematics, physics and astronomy. Thus I do not feel any shortcomings here. My mathematical results e.g. in Colignatus (2007ab) are quite nice even though not developed axiomatically. I have limited affection for pure mathematics but am aware of the hesitations on their part. At least I have the training not to claim more than can be proven, to distinguish fact and hypothesis. For me, however, this holds both in mathematics and about reality. For readers not familiar with the notion of econometrics, I can usefully replicate the diagram by Rijken van Olst, see **Figure 1**.

Figure 1. The Rijken van Olst diagram for econometrics



Some see econometrics as a specialisation but actually it is a generalisation that allows one to work on all angles. Specialists in one of the angles can get deeper results and generalisation comes with modesty, but this generalisation is the only way if we want to tackle reality in scientific fashion.

Beams of light (through a glass darkly)

One of the beauties of a sound education in mathematics is that you learn to see that a good argumentation exposes the dependency on assumptions. Officially, mathematicians are aware of this. They are the first to admit “well, if you change one of the assumptions, of course you may get another result”. They will say the same, in reconstruction, for assumptions on reality, whether it is the stock market crash, ecological collapse, destruction of democracy, corruption of the subject of mathematical logic, or even mathematics education itself. If only that they would be aware of it sooner and that society would not be swayed so easily by their seeming competence.

On occasion there is a mathematician who is not only officially aware of mathematical shortcomings but who also goes a long way in developing answers. Writing this book got me to reading Krantz (2008) *Through a Glass Darkly* at arXiv again, and it was gratifying indeed. It is advised reading for proper digestion of this present book.

From his conclusions:

“So it may be time to re-assess our goals, and our milieu, and indeed our very *lingua franca*, and think about how to fit in more naturally with the flow of life. Every medical student takes a course on medical ethics. Perhaps every mathematics graduate student should take a course on communication. This would include not only good language skills, but how to use electronic media, how to talk to people with varying (non-mathematical) backgrounds, how to seek the right level for a presentation, how to select a topic, and many of the other details that make for effective verbal and visual skills. Doing so would strengthen us as individuals, and it would strengthen our profession. We would be able to get along more effectively as members of the university, and also as members of society at large. Surely the benefits would outweigh the inconvenience and aggravation, and we would likely learn something from the process. But we must train ourselves (in some instances re-train ourselves) to be welcoming to new points of view, to new perspectives, to new value systems. These different value systems need not be perceived as inimical to our own. Rather they are complementary, and we can grow by internalizing them.”

In such a future, didactics in mathematical education may come about more naturally. In the mean time however we are confronted with the current situation and the current stock of mathematicians. This is what this book is about.

Understanding the main advice

Please do not understand me wrong. This is a book about the education in mathematics, not an evaluation of mathematics by itself and what they have done since Sumer 5000 years ago. We will not look into what mathematicians have done positively in all kinds of areas and neither will we look into what horrors the empirical sciences have wrought by applying inadequate math. These other issues are not relevant here. We will merely consider the current state of the education in mathematics. This book is about solutions to the problems that we find there.

Please do neither misjudge me. My nature is quite cheerful and I tend to rise each morning in good humour and expectation of the interesting events that the day will bring. I have had my share of things but while these add to experience they don't affect my nature and *savoir vivre*. When I employ expressions like “the dismal state of math education” and “let parliament take action” then you might imagine a dishevelled character waving a protest banner. While in truth I am enjoying music and a cup of coffee, carefully compose this text with shades and dashes, and find satisfaction in completing a rational argument to its proper conclusion. Do not read more in the text.

This said, let us get down to business and consider education in mathematics. The subject is fascinating and enlightening. There is a Garden of Eden for all kinds of improvements and advancement indeed. It is liberating to see what causes the viscosity and to see what can be done about it.

3. An art and an industry

Mathematics is as much an ideal art as an industry. The art targets the intellectual jolt when an insight strikes the mind. The Pythagorean Theorem causes a sense of wonder. Alongside there is the industry of creation, application and teaching. Struik's (1977) *Concise history of mathematics* clarifies that mathematics developed within society as all other arts and sciences. When this math gets taught there are similar influences. Ernest (2000) *Why teach mathematics?* recognizes at least five interest groups in the teaching of mathematics and uses these labels: Industrial Trainers, Technological Pragmatists, Old Humanists, Progressive Educators and Public Educators. His opening statement reads:

"First of all I want to argue that school mathematics is neither uniquely defined nor value-free and culture-free. School mathematics is not the same as academic or research mathematics, but a recontextualised selection from the parent discipline, which itself is a multiplicity (Davis and Hersh 1980). Some of the content of school mathematics has no place in the discipline proper but is drawn from the history and popular practices of mathematics, such as the study of percentages (Ernest 1986). Which parts are selected and what values and purposes underpin that selection and the way it is structured must materially determine the nature of school mathematics. Further changes are brought about by choices about how school mathematics should be sequenced, taught and assessed. Thus the nature of school mathematics is to a greater or lesser extent open, and consequently the justification problem must accommodate this diversity. So the justification problem should address not only the rationale for the teaching and learning of mathematics, but also for the selection of what mathematics should be taught and how, as these questions are inseparable from the problem."

The approach in this book

This book will consider the two faces of the ideal art versus the industry. Our subject is the *education* in mathematics but the ideal art will be guiding and it may be that we first have to change *mathematics itself* before we can adapt its education. Apparently this does not fit in easily with the Ernest categories.

To understand what we will do, consider the case of the decimal separator that can be either the comma (France) or the point (England). The long standing choice by the International Organization for Standardization (ISO) has been the comma but since 2006 it compromised by allowing the point as well. A Technological Pragmatic approach is that anything works as long as it is standard, even when the standard is double. Here however we will ask which of the two is better as seen from mathematical elegance. Practical considerations have to weigh in too but a change of an ISO standard should be no restriction, and neither the change of textbooks in other subjects that use mathematics. For us the ideal art will be guiding. In this example the decimal marker is not much of mathematics but the idea in this book is that we are willing to change anything if it gives better math. How does the industry deal with the decimal marker? Of the industry, we primarily meet with teachers of mathematics and the authors of textbooks. They follow their country. Highschool math and didactics in principle are a different world from universities per se (see below on developing brains). Professors of mathematics may already tend to use decimal points even though they live in decimal comma countries. Highschool mathematics in comma countries implicitly assume that

there is no internet or that their students are more versatile than professors and can deal with both comma's in textbooks and points in internet resources.

The industry

The organizational structure in the education in mathematics is somewhat Byzantine. There is a forest of governments, committees, mathematical associations, exam boards, textbook authors, institutes of education of teachers, journals, a self-created almost world government style *International Commission on Mathematical Instruction* (ICMI), and what have you. Attitudes range from 'teaching to the test' to the Ernest five groups. Each tree holds on to its roots in order to survive. Suggestions for redesign have to convince that forest. Most suggestions in this book may seem rather bold so that adoption will not be very likely. There is no alternative but to convince that forest. The following arguments and structure of argumentation will be used:

- (1) To show that mathematics fails we do not require statistics but can look at the math itself. Officially we require a statistic that competing textbooks use the same math but for the sake of simplicity we trust that ICMI has had some success, and my small sample has not disproven this.
- (2) A corner stone is that mathematics is man made. It is a building made over time such that all kinds of conventions have crept in. If we were to redesign the building anew, many of those conventions would be deleted. People living in that building – the mathematicians – will mostly not discover by themselves how strange those conventions can be. Others looking on from the outside – for example physicists and economists with mathematical training – can recognize them sooner.
- (3) This book shows that redesign of math will result into better mathematics. Most cases presented here can be understood in principle by anyone with a highschool level of mathematics, and some may require a first year of higher education in a direction that uses mathematics but isn't a mathematics major.
- (4) At the meta-level and by implication, this shows that there is something amiss with the current industry. Improvements are not easy to bring about and the price of current conventions and procedures is very high. Mathematics can be beautiful and contribute to confidence, competence and joy in life. If the mathematical industry does not serve its customers well, it fails its own stated objectives and may meet with public criticism.

Mathematicians will conventionally regard argument 3 as the only convincing one. They might be the first to recognize the improvements in mathematics and didactics presented here. Mathematical tradition clearly is an improvement from alchemy and astrology. Most people will also tend to let the professors and teachers decide on whether these items are improvements indeed. It is tempting to conclude that the system then works: an improvement is proposed, it is recognized, and eventually will be implemented. This approach however takes a risk with respect to potential future changes. With the present failure and analysis on the cause we should rather be wary of that risk. We'd better regulate the industry of mathematics education in robust manner.

If this were a competitive market, where nobody can change the going price, then it would only *seem* chaotic and there would be the invisible hand working for the good. Instead, markets for ideas and education are regarded in economic theory as *monopolistic competition* and in some cases *natural monopolies* and such markets require more regulation. Many see regulation as a restriction of freedom but it actually liberates and enhances quality. Thus, we have cause to consider regulations and changing them.

Implementing change

It would be unwise to leave the restructuring of the industry to the mathematicians themselves. They are not in the position to look at themselves from the outside. They cannot 'think out of the box'. Teachers and professors of mathematics can do their work with love and acuteness but they have not succeeded, internationally and jointly, to cleanse mathematics and the teaching of mathematics from cumbersomeness and irrationality. Instead, the math teachers, having been trained in their conventions, implement those conventions again and condition their students in the same. When students encountered problems and complained about them, they were not listened to and instead subjected to further conditioning. Mental anguish and even tears by damsels in distress carried no weight, mathematical convention was sacred and all blame was put onto the students and their supposed lack of mathematical capability.

Realism suggests that we have a system that actually works. Annually millions of students get their highschool diploma including some math, so apparently the system works to a high degree. Our advanced society could not exist otherwise. But, sobering, do graduates leave school with mathematics or is it only seeming 'mathematics'? That it 'works' and that teachers of mathematics tend to be decent people is by itself no argument to neglect criticism. The evidence in this book carries some weight. Awkwardness and irrationality in 'mathematics' also have consequences for other subjects that use mathematics. We spill a lot of time and energy in education because of the state of mathematics. Many kids suffer. Those who pass their math exams actually are much stunted in their mathematical development. The economy suffers with such low development of mathematical knowledge, skill and attitude. It is rather impossible to quantify the loss and counterproductiveness.

Supposedly, as it is a problem that affects each nation, it would be a task for each national parliament to start the wheels of change. Parents are advised to write their representative. Parliament is not asked to determine the next digit of π but to rearrange the institutional setting so that our kids get math without pain.

The suggestion causes people to raise eyebrows. People elect parliament but seem to dislike it and not regard it as a useful place to resolve bottlenecks. The present situation is a chance for parliament to enhance its standing. Decisive action on the failure of mathematics education will set an example.

4. Limitations to this study

While mathematics has its limitations, this book suffers some too.

Setting, experience, anecdote, bias

Mathematics itself is international. I have taught mathematics for four years at the international college level with students from all over the world. Nevertheless the location was in Holland. From my own foreign exchange student year in California I know that American highschool is very different from the Dutch system. My observations will still be biased by necessity. Though the present discussion tries to be as general as possible my main experience is bound to create some idiosyncracies. Specific references to Holland will be reduced to a minimum. Holland might be used sometimes as an example however when this can be enlightening.

For example, there is now a discussion in Holland about the choice in elementary school between the algorithmic long-division and the "realistic mathematics education" (RME) (pejorative "guesstimate"), where pupils are supposed to find the answer by trial and error relying on their understanding of the problem and self-creation of method. Clearly, teachers in secondary education suffer the consequences of what is done at the

elementary level. But there is no joint responsibility and management of the whole column. Teachers at elementary schools appear to have problems with mathematics themselves. The Minister of Education allows a “Math C” profile level for their certification as teacher, i.e. a level with not very much of math. Hence, those elementary school teachers may tell their pupils that mathematics is very difficult and not worth your effort. While the situation in Holland is not our focus, the example clarifies that it is advisable to consider the whole industry and to keep an open mind for the subtle influences between the ideal art and that industry.

Limitations exist internationally

Braams (2001), on the evaluation of research into K-12 mathematics education:

“A practicing scientist might think that reform efforts could, should, and probably would be guided by a respected body of research into what works and what does not, although within such a body of research there might still be significant differences in research focus, methodology and results. With that in mind I started looking for appropriate research, and this letter is a little report on my search. I'll say right away that the outcome has been entirely negative. (...) To be sure, there are plenty of efforts in mathematics education research. Many of them provide results that are of anecdotal and perhaps of inspirational value. Many appear to be tightly linked to a particular implementation of some reform, limiting their scientific standing. It really looks as if all the recent United States efforts in education research have not produced a single respected comprehensive study of the kind outlined above, let alone a body of authoritative research that provides firm empirical guidance for mathematics pedagogy.

Fortunately we still have our common sense to guide mathematics education. Unfortunately (but it would take us too far afield to discuss it further here) present trends towards discovery-based learning and constructivist pedagogy seem as little rooted in mathematicians' common sense as they are rooted in education research.”

With this in mind, I can usefully express that the method chosen here is logic. I draw information from my own experience and reading but, since this would be anecdotal indeed, all conclusions and advices are only based upon logic. And, OK, upon common sense.

I am aware of the Watkins (1995) paper on the US *Follow Through evaluation*, the Hattie (1999) meta-analyses in particular on the influences on student learning, the Anderson, Reder and Simon (2000) evaluation of applications and misapplications of cognitive psychology to mathematics education. Writing these lines I realize their dates. The point however is context awareness. While this book concentrates on what and how we are teaching when we are teaching mathematics, education is a rich context that will always have to be taken account of.

5. The order of discussion

While teaching I kept notes. When I grew aware of some regularities in those notes the idea arose to collect them more systematically. From a list of potentially more cases some could be selected that were particularly useful for the purpose at hand: proving the need for change. I still feared that I had only issues and no unity. It appeared possible to categorize the notes into more unifying chapters. The regularities materialized but it still was a surprise to suddenly see how the perspectives themselves were linked. At some point the unity simply shone out and it became obvious that the whole should be presented to an international audience. This book retains that effect, you will have the

same surprise. (Though you miss out on the surprise of having to rewrite this *Introduction*.)

This book has a didactic set-up. We already presented the main message. Now we get down to the evidence. We work from the small upwards to the more complex. The small issues should be fun and eye-opening. They prime the mind to become sensitive to the more complex. By allowing readers to digest the examples and arguments the overall reasoning has more chance to be understood.

The chapter on redesigning mathematics itself only gives summaries and then refers to the relevant sources elsewhere. However, the paper on derivatives has been rewritten and is now included as a new chapter of itself.

The first eight chapters number their paragraphs for easier reference to specific points.

This *introduction* summarizes the book. A much shorter summary and condensed abstract are in the appendix, for backup.

Now, however, forget about this Introduction. Let us consider the education in mathematics afresh. Suppose you are a teacher or student facing the blackboard with some texts, formulas, tables and graphs. What to make of them ? Are they clear, how do we communicate effectively ?

II. Issues of notation

6. The decimal point

The decimal notation was invented by Simon Stevin (1548-1620) who aspired at clarity. He would be upset about what is done with his invention. For decimal marker, the British use the point and the French use the comma. The ISO standard followed the French but since 2006 points are accepted notably for texts in English, see Baum (2006). To allow either a comma or a point is a standard, of course, but actually somewhat loose.

This book uses English. Conventionally we would use the dot and be done with it. At issue now is however to consider the matter from the angle of elegance with substance. Let us avoid getting lost by French – British disputes and diplomacies and let us try to determine what we want.

In Europe we see that textbooks use comma's, computers have to be set to comma's, but internet resources in English will use the point. Pupils and students apparently learn how to deal with it (or fade from view). But it is an awkward situation and weak students suffer a needless burden. Perhaps legal documents require a single format and we have to teach students to use that format. But it is not clear why a course in mathematics should suffer the inability of the legal world to adopt a single notation. The best solution remains that the world adopts one notation and be done with it.

There is already a mathematical standard application for the comma. For a two-dimensional point we use the notation $\{x, y\}$. This is clear for the point $\{2, 0\}$ but it gets less clear for $\{2,5, 4,32\}$ so that some start writing $\{2,5 ; 4,32\}$. English readers will not be familiar with these Byzantine consequences and it may open their eyes to the larger problem.

Hence it is best to use the decimal point.

It might be a compromise to use the dot raised to the middle of the text line, like in $2\cdot5$, as I saw this in the medical literature, but this is not advisable since there is no need to change a good practice in the English speaking world of science in general.

7. Brackets

Brackets belong to the most important symbols. Consider $(a + b)(c + d)$ if we did not have brackets. There is also the notation for a function $f(x)$. Thus $a(x)$ could be both a function and $a \times \dots$ which is inconsistent (unless the function would be very specific and a would remain non-numeric).

Most students can learn to deal with context-dependency and most would guess that $a(x)$ is a function. What about $a(c + d)$? Is this a multiplication or a functional expression ?

Some people might object “normally we don’t write sums in a function’s argument, so your example of $a(c + d)$ is a crafted and irrelevant exception”. But what about $f(x + h) - f(x)$ leading to differential calculus ? Mathematicians should admit that they themselves are confused because of the ambiguity of $a(c + d)$ and they are irrational when they don’t admit it.

Issues like these arose in the design of computer algebra languages. Computers are strict and require unambiguous input. The language *Mathematica* (my standard reference) chose to use straight brackets for functions, thus $f[x]$.

A standard reaction by mathematicians is (a) straight brackets are ugly, (b) it is only for computers. Hence, indeed, *Mathematica* later developed the “traditional form” such that the (hidden) input uses straight brackets for clarity while the display has round brackets.

The proper answer is: (ad a) what is ugly is to a large extent a matter of convention, (ad b) people are much like computers. There is no difference in getting stuck. For consistent thought, people require unambiguity too which is something else than saying that they are computers.

When confronted with $f(x)$ people can do more than computers and work around corners. There is the hidden rule that letters like f, g, h, \dots are conventionally used for functions so that the expression would be unambiguous. Or at one point a is defined to be a function so that $a(x)$ can be recognized. It are rules like these that are not explicitly mentioned in textbooks but that students have to figure out if they want to pass. “Read carefully” the standard mathematician might say. The key point remains that this is exactly that: *working around corners*. It puts a burden on the weaker students to acquire that additional competence. They are told that mathematics would be clear and consistent, they are confronted with a clearly inconsistent notation, and when it gives them a hard time then they are told that they are themselves to blame.

The supposed esthetics of the round brackets in the notation of a function is merely conditioning – and that conditioning is so strong that a software firm went a long way to satisfy it. A solution might be to design esthetic brackets that still look different.

Some mathematicians might admit to all of this but continue to torture their weak students, using the argument that they ought to be able to read conventional math papers. Now, clearly, weak students will not tend to read such papers anyway while the smart students who will read the historical papers of Euler etcetera would well be able to adapt on the spot.

PM. The meaning of $f(x)$ thus depends upon context frames. Perhaps it is a key mathematical skill to be able to switch frames quickly – for example since notation may frequently be ambiguous anyway. That skill is no explicit target in math education. We will return to this.

8. Brackets (2)

Conventionally, the notation (x, y) can be used for the two-dimensional point and for the open interval from x to y . In Holland this ambiguity is solved by using $\langle x, y \rangle$ for the open interval. As $[x, y]$ is the closed interval, in France $]x, y[$ gives the open interval.

If (x, y) is a point then something can be said in favour of using $f(x, y)$ for the function on that point. Unfortunately this breaks down for the single dimension $f(x)$ and the other use of round brackets.

The straight brackets in $f[x, y]$ in *Mathematica* might cause a confusion with the closed interval. Hence *Mathematica* has the notation / object `Interval[x, y]` which is a bit inelegant. With function call $f[x, y]$ we might expect $[x, y]$ to be used for the two-dimensional point but instead *Mathematica* uses $\{x, y\}$. For this notation, *Mathematica* has an option to distinguish ordered (default) and unordered sets. Potentially there is a difference between $f[x, y]$ and $f[\{x, y\}]$.

We do not need to resolve these issues here. We merely indicate the problem of the consistent use of brackets and let us hope that an international committee finds a solution.

In this book we adopt *Mathematica*'s notation of the two-dimensional point $\{x, y\}$.

2015: We actually follow *Mathematica* in its versatility: depending upon context $\{a, \dots, b\}$ can also be a *program* with ordered steps, and $\{a, \dots, b\}$ may also be an unordered set.

9. Fractions

There is the expression *two-and-a-half* or $2\frac{1}{2}$ alongside the expression $2\sqrt{2}$. The first is the addition $2 + \frac{1}{2}$ and the latter is the multiplication $2 * \sqrt{2}$. A blank space is multiplication and thus there arises the following issue.

Try to spot when $2 \text{ times } \frac{1}{2} = 1$ turns into $2 \text{ plus } \frac{1}{2} = 2\frac{1}{2}$. How large can the space be ?

$$\begin{array}{r} 2 \quad \frac{1}{2} \\ 2 \quad \frac{1}{2} \\ 2 \quad \frac{1}{2} \\ 2\frac{1}{2} \end{array}$$

$$\begin{array}{r} 2 \quad \frac{1}{2} \\ 2 \quad \frac{1}{2} \\ 2 \quad \frac{1}{2} \\ 2\frac{1}{2} \end{array}$$

$$\begin{array}{r} x \quad y \\ x \quad y \\ x \quad y \\ xy \end{array}$$

Somewhat teasingly too this book will tend to write $2 \frac{1}{2}$ because I actually prefer a bit more space inbetween but clearly this would be confusing since it could read as $2 * \frac{1}{2}$.

The improvement would be to consistently write $2 + \frac{1}{2}$ and to stop using $2\frac{1}{2}$. In the same way we already write $2 + \sqrt{2}$. In intermediate steps we would often use $5 / 2$ rather than $2 + \frac{1}{2}$ but the latter is the best presentation of the end result.

This would fit not only with notation in general but also with the actual calculation, e.g. of $2\frac{1}{2} * 1\frac{1}{4} = (2 + \frac{1}{2}) (1 + \frac{1}{4})$.

In computer languages xy stands for a single variable in the same way as 34 stands for a single number. Mathematical textbooks however can write $a(b + c) = ab + ac$ where the latter are multiplications. It is better that they drop this habit and insert a blank space. For example, with multiplication ab and $a = 3$ and $b = 4$ children play by calculating $34 = 3 * 4 = 12 = 1 * 2 = 2$.

Conversely, one can argue that the use of smaller fonts unambiguously indicates fractions, and that writing a number directly close up to the fractional line (which would anyway be normal for larger numbers) can unambiguously mean that this is addition. On this ground there is no need for change. The latter might be valid – as apparently people with mathematical ability learn to switch frames when we compare $2\frac{1}{2}$ and $2\sqrt{2}$.

However, a lot of math education time is wasted by the current notation of fractions. (a) The switching of frames requires mental space and energy without a contribution of substance. (b) While textbooks have neat typesetting with larger and smaller fonts, and can parse neatly with and without intermediate (half-) blank spaces, the handwriting by students is less accurate and frequently causes confusion. (c) The handwriting by teachers may not be as neat as well but then a hidden algorithm is used: “this calculation should give 2.5 and thus we write $2 \frac{1}{2}$ and then we stop thinking since we have reached the end of the calculation” – while proper reading should give outcome 1.

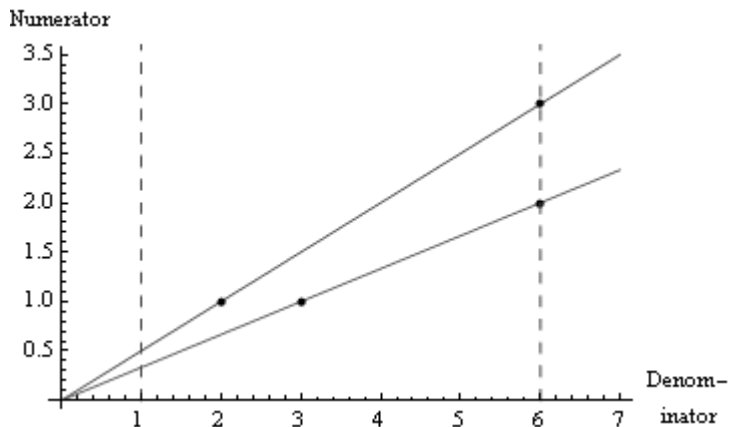
Of course, at the grocery students may see the notation $2\frac{1}{2}$ and thus will have to know what it might mean. But it suffices to explain in class that this is an old notation. Draw a red square around it, explain that it means $2 + \frac{1}{2}$. But don't let them use it themselves.

Fractions are important. Only the current notation is no good.

PM 1. Pupils at elementary school tend to learn about fractions from cutting up pies and cakes. Adding up fractions can become an intricate matter in that manner. A simpler and clearer way is the method in **Figure 2**. Fraction $\frac{1}{2}$ can be denoted by the slope of two steps to the right and 1 up. Fraction $\frac{1}{3}$ is three steps to the right and one up. Adding

them can be done by taking a common multiple of steps, say 6. Extending the lines gives us $3/6 + 2/6$ or $5/6$ as the total. If we look at the vertical at 1 instead then we would get decimal fractions. Van Hiele has explained over the last fifty years that pupils at elementary school can already deal with vectors and co-ordinates. Who sees this display for dealing with fractions will tend to agree that we should not withhold it. Of course, the addition of slopes is different from the addition of vectors, and thus the pupils better grow aware that it matters what labels are on the axes.

Figure 2. Adding $1/2$ and $1/3$



PM 2. We might use a separate symbol for a fraction that has been simplified, as opposed to one that isn't. Thus $2 + 4/6$ might still be unfinished and $2 + 2/3$ FIN or perhaps $2 \dashv 2/3$ would be the result that 'definitely' cannot be simplified further. Normally we would not use the latter as it is rather pedantic but it can be a tool in the communication between teacher and pupil when the topic of discussion is simplification. We might also write "Do $2 + 4/6$ " and "Done $2 + 2/3$ ".

2015:

PM 3. COTP §15.3 has a discussion of proportion space.

PM 4. A better suggestion is to abolish fractions altogether, using $1/x = x^H$, which results into $2 + 2^H$ as a fine notation. x^H is pronounced as "per x" or sometimes "one per x". It uses $H = -1$, the Harremoës operator, to be pronounced as "eta".

See the weblog texts – included in BHRM (2015g).¹

¹ <https://boycottholland.wordpress.com/2014/08/25/confusing-math-in-elementary-school/>
<https://boycottholland.wordpress.com/2012/04/01/english-as-a-dialect-of-mathematics/>
<https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/>
<https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/>

10. The cult of the radical sign

When it has been clarified that $2^2 = 4$ then it is straightforward to explain that $2 = 4^{1/2}$, and subsequently develop exponents in general. This direct development of exponents is a clear and straightforward route. Students can use good practice on this. It takes time and energy to learn to write the exponent at the right height, fractions already were a bit difficult and the notion of this type of inverse has to sink in.

Instead, school mathematics has developed the cult of the radical sign.

The latin word for root was *radix*, it was abbreviated to *r*, this was written as $\sqrt{}$, and subsequently it got generalized into the p -radical sign $\sqrt[p]{x}$. Teachers of mathematics apparently seem unable to imagine a life without this sign. Students are submerged in its use and tricks.

Apparently the apotheosis of this cult is that students are told that $\sqrt[p]{x^q} = x^{q/p}$. But we could have gotten there also via the direct route in the first paragraph, with the aside that $x^{q/p}$ is sometimes written by some people as $\sqrt[p]{x^q}$.

Admittedly, the square root sign is useful in two-dimensional geometry, notably with fast and clear labelling of the lengths of sides using the Pythagorean Theorem. And the notion of a 'root' is fine too. But apart from that it is clutter.

The radical sign has created a life of itself, outside its realm of usefulness, and with counterproductive results. For example, it is considered ugly or unconventional to write $\sqrt{(a+b)}$ with brackets and subsequently a lot of time is spent in having the pupils extend the upper root line to the end of the expression under the root, with hopefully a small drop to indicate that the end has been reached. For example, the equivalence of p -roots and exponents does not sink in fast and students lose time in translating the one to the other, and trying to figure out whether this also means that the properties are transferred. Eventually, good students understand that the radical sign is merely a different way of writing fractions for exponents – but really, what is the mathematical insight? What sense of wonder is this supposed to generate and how is this supposed to contribute to the motivation to learn more?

It is a valid argument that the notion of 'root' best sinks in with the use of a symbol that explicitly is called "root". Indeed, use $\sqrt[2]{}$. Without bound though this is like believing that the notion of an accident best sinks in with the use of a symbol that says 'accident' and is printed on all pictures of an accident. No, this confuses convention and efficiency. A photographer may use stamps "accident" and "art" to categorize his collection but this is not how the pictures and understanding came about.

In judging the cult of the radical sign, we compare the gain in knowledge, skill and attitude with the investment of time and energy. Since the exponential notation has to be mastered as the principal notation anyway, the use of the radical sign adds little. It has its cause mainly in convention. Thus, the radical sign $\sqrt{}$ can be kept for (a) historical reasons, (b) the name "root" and (c) fast and clear notation in geometry. But there it should stop.

Let us eradicate this cult of the radical sign.

11. Archi or Pi

The mathematical symbol π (Greek “pi”) is defined on a circle as the ratio of the circumference to the diameter. This derives from ancient experience that the measurement of the diameter is very practical. In science and education the diameter has lost its relevance however, and it is the radius of the circle that matters.

Angles are commonly measured in 360 degrees or 2π radians. Because of the distinction between diameter and radius, education suffers the perpetual factor 2.

It is useful to define Θ (Greek capital theta, pronounced as *Archi* from Archimede) as the ratio of the circumference to the radius. Thus $\Theta = 2\pi$.

For angles, we can take the plane itself as the unit of account. One *turn* around the circle is a better measure than 360 degrees.

The advantage of using Θ is twofold:

- (1) It is easier to think in terms of whole circles and turns than half circles. As π radians are an angle of 180 degrees, or a half plane, it carries with it a notion of non-completeness. Using $\Theta / 2$ or $2^H \Theta$ carries the notion of only a half turn. Indeed, the symbol π is taken from “perimeter” and it has succeeded only half.
- (2) There is more outward clarity on the linkage with calculus.
The integral of x is $\frac{1}{2} x^2$ or $2^H x^2$.
Thus with radius r the circle circumference is $r \Theta$ and its surface is $\frac{1}{2} r^2 \Theta$ or $2^H r^2 \Theta$.
Admittedly, when you look for it then the relationship from calculus can also be seen when using π but the advantage of Θ is that you don’t really have to look for it since it tends to stand out more by itself.

Independently, Palais (2001ab) came to the same view (see also his animated website ²). Palais introduced the three-legged $\pi\pi$ but this is bound to cause writing and reading errors, let alone confusion, and I remain with Θ .

Here it suffices to point out the mere benefits of using Θ . We will return to trigonometry later on when discussing the measurement of angles, see page 64.

PM 1. Some students confronted with 2π have the tendency to complete this by applying the calculator and returning $2 * 3.14... = 6.28...$ With Θ it would be easier for them to stop, and wonder whether the exact Θ is required or the numerical approximation. However, they will meet much of the same problem when they are confronted with $\Theta / 2$ or $2^H \Theta$. Hence this issue must be dealt with separately.

PM 2. Rather write $x \Theta$ instead of Θx . Current convention is to write $2\pi r$ but there is advantage in recognizing Θ as an indication of the full turn as a unit of measurement.

2015:

PM 3. There is the tau-community who propose to use tau or $\tau = \Theta = 2\pi$. Earlier I considered this option too but rejected it because of the similarity with r , the symbol for the radius. This similarity will cause much confusion, especially in handwriting in exams.

PM 4. There is an animation. ³

² <http://www.math.utah.edu/~palais/cossin.html>

³ <https://boycottholland.wordpress.com/2014/07/14/an-archi-gif-compliments-to-lucas-v-barbosa/>

12. Text, function, table and graph

When exploiting the linkages between text, function, table and graph, the current convention requires an unnecessary switch in orientation.

For the graph, we use x for the horizontal axis and y for the vertical axis.

For the table, convention puts x on top and y below (with no further explanation).

This layout of the standard table causes a switch in orientation with respect to the graph. Students have to glance from the numbers in the table to the graph, check values and, now, in addition, have to translate *up* to *down* and in reverse.

Why ? Merely because of the convention that text lines in a book run from the top of the page to the bottom of the page, and that for functions the x values cause the y values and hence come first. There isn't more to it. But this thoughtless convention comes at a price. Young brains that have few memory places and that need to learn to compact their concepts and actions would be served with the same orientation.

Also, the distinction between cause and effect does not fully correlate with the order of the lines on a page. It is more instructive to create a table that states cause and effect explicitly, see **Table 1**. Suggestion: try this on **Figure 2**.

Table 1. Improved layout of the table used to draw a function

Effect $y = f[x]$	$f[0]$	2^H	...
Cause x	0	2	...

In addition, when a slope is determined with $\Delta y / \Delta x$ then the current convention with x on top and y below causes another reorientation. The format in **Table 1** retains the orientation of numerator and denominator.

PM 1. The latter might be objected to with the argument that it allows thoughtless execution of a (simpler) algorithm. Of course we want students to know what they do. Eventually. But they have to learn to do too. The algorithm is best learned if it isn't cumbersome and actually supports learning. There is great value in learning to perform the algorithm and then look back and wonder: "OK, they told me what it was. What was it again that they said ?"

PM 2. See page 52 for the important issue of *text*.

PM 3. A good standard format is to put text, formula, table and graph on one single page, in four blocks, in that order. Textbooks tend to put the items at random, going for the flashy.

13. Verb versus noun

To ride is a verb and *ride* is a noun. Riding is an activity and a ride is something completed, an abstraction of the activity. Clearly, a ride also implies an activity but there remains that subtle difference.

Computer programmers have noted this distinction early on. In the language Algol a statement " $X := 5$ " means that variable X is set to the value of 5 while a statement " $X = 5$ " would be a logical expression that evaluates to True iff X is 5. Earlier in history there was the distinction between the potential and actual infinite. Above, however, we saw that $2 + \frac{1}{2}$ was seen (also by teachers) not as the result itself but as an instruction to further simplify to $2\frac{1}{2}$. Conversely we saw that students had to learn to recognize 2π as a result on itself instead of an instruction for continued calculation.

The distinction between verb and noun can be stated mathematically as the distinction between a procedure / algorithm and its outcome / result.

Mathematics is full of switches between verb and noun. It is a pity and also rather a shame that this is not pointed out didactically as frequently. As a teacher I noted that pupils and students have a hard time to deal with these switches. There are two conclusions. (1) The first is the general insight that educators in math must pay more attention in general to this distinction and how it affects learning by students. (2) The second is that it will help to introduce innovations at particular spots to support this.

PM 1. Pierre van Hiele (1959) and Dina van Hiele-Geldof developed a theory of learning mathematics, by concrete, ordering and abstract levels:

"the process of learning proceeds through three levels: (1) a pupil reaches the first level of thinking as soon as he can manipulate the known characteristics of a pattern that is familiar to him/her; (2) as soon as he/she learns to manipulate the interrelatedness of the characteristics he/she will have reached the second level; (3) he/she will reach the third level of thinking when he/she starts manipulating the intrinsic characteristics of relations." (FU wiki (2008))

Textbooks should better recognize the points where level jumps tend to occur or are required to occur. The verb / noun distinction is such a point. Sometimes the noun will be the abstract of the verb, sometimes in reverse. The individual learning process may differ from the reconstruction of a general process in more standardized terms.

PM 2. Independently, Gray & Tall developed this distinction into the idea of a 'procept'. Tall (2002) seems to embed the 'procept' into the 2nd Van Hiele level:

"The Symbolic-Proceptual World of symbols in arithmetic, algebra and calculus that act both as PROCesses to do (eg $4+3$ as a process of addition) and conCEPTs to think about (eg $4+3$ as the concept of sum.)"

I have a small problem with this use of vocabulary, in that a concept is not necessarily static and may well be a process too. It is not necessary to limit the distinction between verb and noun to symbols only. It is not entirely clear whether it is really useful to use a new word "procept" to indicate that verbs and nouns are connected, and that processes hopefully give a result and that results tend to be created by processes. That said, the Gray & Tall idea remains important. It points to the phenomenon that mathematics can use *deliberate vagueness* in order to make efficient use of the same symbols.

2015: See Colignatus (2014a) for Van Hiele and Tall.

14. Verb versus noun – square root

A key example is the square root, for example $\sqrt{2}$.

The equation $x^2 = 4$ has two solutions, $x = 2$ and $x = -2$. At this stage in the curriculum, students are not aware of the distinction between a function (for each x there is a single y) and a correspondence (for each x there may be more y). It would be better if they were introduced to this distinction. The solution of $x^2 = 4$ would be easier with the correspondence $\text{Do}\sqrt{}$ or 'take the root' so that $x^2 = 4$ solves into $(x = \text{Do}\sqrt{4}) \rightarrow (x \text{ in } \{2, -2\})$, the solution set. This inverse can be shown by mirroring x^2 along the line $y = x$.

In the current situation punching in $\sqrt{4}$ on the calculator looks like a procedure and the students get confused (a) between the noun / number and verb / procedure, and (b) between solving and simplifying. Students are inclined to take the square root of 4 and to write 'solutions' $\sqrt{4} = 2$ and $\sqrt{4} = -2$, which they check by squaring both sides. In mathematical convention this is false since $\sqrt{4}$ has to be a nonnegative number. $\sqrt{4}$ can be simplified to 2, and simplification is not solving. Thus $\sqrt{4} = -2$ is a false statement.

We can also write $\text{Sim}\sqrt{4} = 2$ for the procedure of simplifying the square root of 4, to distinguish it from $\text{Do}\sqrt{4} \rightarrow \{2, -2\}$. *Mathematica* also has $N[\sqrt{x}]$ to find a numerical value.

For the instruction in the current situation it would help to write the solution of $x^2 = 4$ as $x = \sqrt{4} = 2$ or $x = -\sqrt{4} = -2$. Curiously, this is not really done much. In some books we can see that functions $f[x] = x^2$ and $g[x] = \sqrt{x}$ are discussed but with little discussion of their relation, and in other books there is more discussion but it tends to be confusing.

Currently the radical sign denotes the passive number and equation solving gives the active process. In itself it is a strong distinction. But expressions like "take the root" must then be avoided (which is somewhat difficult since roots are used).

PM 1. Students find it hard to distinguish between the number notion and the procedure that is available on their calculator. Mathematics teachers think that students are confused between exact and approximate results but here it would rather be the distinction between verb and noun. If you recognize $\sqrt{2}$ as information and stop seeing it as a command then *there you are*. See page 38 for the issue of approximation itself.

For exponents in general we would have $\text{DoExp}[y, 1/n]$ so that there is a distinction between the noun / number $4^{1/2} = 2$ and the verb / process $\text{DoExp}[4, 1/2] \rightarrow \{2, -2\}$.

PM 2. It might actually be a suggestion to define $\sqrt[p]{x^q} \equiv \text{DoExp}[x^q, 1 / p]$. This means that the radical sign becomes the solution operator instead of the completed number. That implies that $\sqrt[2]{4} = \text{Do}\sqrt{4} \rightarrow \{2, -2\}$ so that this sign differs from $\sqrt{4} = 2$. It remains to be seen whether the profession is willing to make the change. Likely $\text{Do}\sqrt[p]{x^q} \equiv \text{DoExp}[x^q, 1 / p]$ is a good choice then.

PM 3. Simplest seems the distinction between $4^{1/2} = 2$ and $\text{Do}[4, \frac{1}{2}] = \{2, -2\}$.

2015:

PM 4. The Harremoës operator gives another level: 4^{2^H} . This is not a problem except for word processors.

15. Verb versus noun – division

Western mathematics had to wait till 1200 AD before the zero came from India via Arabia together with the Indian digits – where both “zero” and “cipher” are jointly derived from the Arabic “sifr” = “empty”. Indian numerals are easier to work with than Roman numerals, e.g. try to divide MCM by VII, yet this advance came with the cost that the zero caused a lot of paradoxes. Mathematics solved most problems by forbidding division by zero. However, there is also the issue of algebra.

Dijksterhuis (1990) suggests that the ancient Greeks did not develop algebra – and subsequently analytical geometry – since they used their alphabet to denote numbers. Thus $\alpha + \alpha = \beta$ already had the meaning $1 + 1 = 2$, whence it would be less easy to hit upon the idea to use α as a variable. We too would consider it strange to use e.g. 15 as a variable ranging over $-\infty$ to $+\infty$. This explanation is not entirely convincing since the Greeks did use names like “Plato” or “Aristotle” and thus might have used a name to denote a variable – like “Variabottle” – though the letters in the name then should not again decode to a number. Notation clearly was one of the obstacles to overcome.

Let us now assume that we are familiar with algebra and that someone announces the new invention of the zero.

Let us distinguish the passive division result from the active division process. In the active mode of dividing y by x we may first simplify algebraically under the assumption that $x \neq 0$ while subsequently the result can also be declared valid for $x = 0$. This means *extending the domain*, i.e. not setting $x = 0$. The active notion would be a new concept. Denote it as $y // x$ or $(y x^D)$ where the brackets are relevant to keep y and x together.

There is already an active notion (verb) in taking a ratio $y : x$. But a ratio is not defined for $x = 0$. Mathematicians will tend to regard division y / x as already defined for the passive result without simplification – i.e. defined except for $x = 0$. Others who aren't professional mathematicians will tend to take y / x as an active process (and they might denote $y // x$ for the passive result). All in all, it would not matter much, since we might continue to write y / x and allow both interpretations depending upon context. In that way the paradoxes of division by zero are actually explained, i.e. by confusion of approach or perspective.

Definition of dynamic division

To make this strict, let y / x be as it is used currently in textbooks, and let $y // x = (y x^D)$ be the following process or program, called *dynamic division*:

$y // x \equiv \{ y / x, \text{ unless } x \text{ is a variable and then: assume } x \neq 0, \text{ simplify the expression } y / x, \text{ declare the result valid also for the domain extension } x = 0 \}.$

Thus simplification only holds for variables but not for numbers. Thus $x // x = 1$ but $4 // 0$ generates $4 / 0$ which is undefined. Also x / x is standardly undefined for $x = 0$.

2015: This definition takes y as the numerator and x as the denominator, with domains that might be manipulated. When the denominator is an expression like $(p + 2)$ then one must perform the appropriate algebra. It has been an option in the $\{...\}$ definition above to write “(a) variable” instead of “a variable”, which allows a shift from the syntactic test towards the semantic test of variability, and which also allows *substitution*, like “ $(p + 2)$ is (a) variable”. After ample consideration, already in 2007 and later explicitly in Colignatus (2014b), I think that we are better served with the syntactic test on the denominator, since this directly leads to the question: what is the domain of the denominator ?

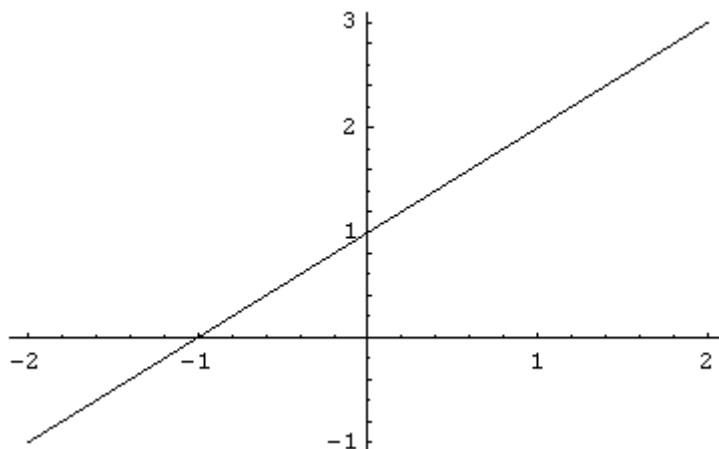
General application

There is no need to be very strict about always writing “/”. Once the idea is clear, we might simply keep on writing “/”. An expression like $(1 - x^2) / (1 - x)$ would be undefined at $x = 1$ but the natural tendency is to simplify to $1 + x$ and not to include a note that $x \neq 1$, since there is nothing in the context that suggests that we would need to be so pedantic, see **Table 2**. The current practice in teaching and math exams is to use the division y / x as a hidden code that must be cracked to find where $x = 0$ but it should rather be the reverse, i.e. that such undefined points must be explicitly provided if those values are germane to the discussion. Standard graphical routines also skip the undefined point, see **Figure 3**, requiring us to give the special point if we really want a hole.

Table 2. Symplication and continuity

<i>Traditional definition overload</i>	<i>With the dynamic quotient</i>
$f(x) = (1 - x^2) / (1 - x) = 1 + x \quad (x \neq 1)$ $f(1) = 2$	$(1 - x^2) // (1 - x) = 1 + x$

Figure 3. Graph of $(1 - x^2) // (1 - x)$



Subtleties

The classic example of the inappropriateness of division by zero is the equation

$$(x - x)(x + x) = x^2 - x^2 = (x - x)x,$$

where division by $(x - x)$ causes $x + x = x$ or $2 = 1$.

This is also a good example for the clarification that the rule that we should never divide by zero actually means that we must distinguish between:

- creation of a fraction by the choice of the *infix* between $(x - x)(x + x)$ and $(x - x)$
- handling of a fraction such as $(x - x)(x + x)$ *infix* $(x - x)$ once it has been created.

The first can be the great sin that creates such nonsense as $2 = 1$, the second is only the application of the rules of algebra. In this case, $x - x$ is a constant (0) and not a variable, so that simplification generates a value Indeterminate for both $/$ and $//$.

Also $a(x + x) / a$ would generate $2x$ for $a \neq 0$ and be undefined for $a = 0$. However, the expression $a(x + x) // a$ gives $2x$, and this result would also hold for $a = 0$, even while it then is possible to write $a = x - x = 0$, since then it is an instant (and not presented as a variable).

Another conclusion is that calculus might use algebra and the dynamic quotient for the differential quotient instead of referring to infinitesimals or limits, see page 89.

Requirements

Clearly, mathematics education already takes account of these kind of aspects in some fashion. In early excercises pupils are allowed to divide $2a / a = 2$ without the definition overload. At a certain stage though the conditions are enforced more strictly. The topic of discussion is not only that this stage can be a bit later but also that the transition can be smoother, also for the rest of the education, by the distinction between $/$ and $//$. For the mathematically inclined pupils or students graduating at highschool one would require that they are aware that x / x is undefined for $x = 0$ and that they can find such points.

III. Opaque or confusing terms

16. Logarithm versus Recovered Exponent

Around 1600, Simon Stevin created many terms in the Dutch language that better clarified the Greek and Latin phrases of then-traditional mathematics. For ‘mathematics’ he coined the Dutch word “wiskunde” – meaning the art of certainty, as ‘mathesis’ means ‘what we have learned’, which should not be something fickle. Nowadays, with probability theory and statistics, the Dutch would also need the word “giskunde” – the art of uncertainty. What anyhow remains is that it helps in education to use *self-explanatory* terms.

John Napier’s term “logarithm” (Log) is singularly opaque. My suggestion is to use the term “recovered exponent” (Rex).⁴

With 10^3 the exponent 3 disappears into the result 1000, while it is recovered by the operation $^{10}\log[1000] = \log[10, 1000] = 3$.

Instead, we would write $^{10}\text{rex}[1000] = \text{rex}[10, 1000] = 3$.

PM 1. Current teaching practice is to use $\log[x]$ with base 10 and $\ln[x]$ with base e . This reflects the phenomenon that it is cumbersome to continuously write the base. Indeed, some graphical calculators curiously don’t have $\log[b, x]$ but require using $\log[x] / \log[b]$. Didactically, though, it would be wise to start with $\text{rex}[b, x]$ and continue writing the base. This helps students in realizing that the function is defined with respect to that base. Eventually they see *themselves* that it is cumbersome to continuously write the base and use the shorter $\text{rex}[x]$ with default base e .

PM 2. Dutch textbooks are prim on e . It is only presented in grade 10 or 11. Compare this to the equally special number π that is introduced much earlier. I would suggest that there remains a difference between being able to ride a bicycle and explaining how in terms of Newtonian physics. (See page 54+.)

2015:

PM 3. *Exp* and *Rex* might be written with upper case, notably for computer input, but human readers will prefer lower case.

⁴ Dutch: “teruggevonden exponent” $\text{tge}[x]$

17. National idiosyncracies

There are idiosyncracies that differ by nation and that cannot be discussed in general but only by example. Each nation would benefit however by cleaning up their clutter.

For example, in Holland, the expression $2 < 3$ is translated inaccurately as “two is smaller than three” while the English language is accurate with “two is less than three”. The Dutch language at school confuses size with order. Dutch students get into problem when considering $-100 < 3$ where -100 clearly is less than 3 but not smaller in absolute size.

The Dutch curiously have a good alternative. Do not say “*twee is kleiner dan drie*” but “*twee is minder dan drie*”. Historically it can be understood, since Dutch grandchildren are “kleinkinderen” which again expresses size rather than order. But it is equally clear that we better avoid this history.

Once I attended a class given by an English math teacher who explained how to distinguish the various polygons – triangle, square, etcetera – by counting their sides rather than vertices. Apparently this was not a didactic gimmick but he had survived his education himself by not knowing that the Greek *gonos* means *corner*. Quite likely the Greeks had already discovered that it may be didactically easier to count vertices anyway since the pointy bits stick out so clearly.

Perhaps it is an idiosyncrasy of the English language that so many of the opaque Greek and Latin terms have survived. It causes great pride in the breasts of the Greek but it may not really help the English pupils. One may suppose that there have been English variants of Simon Stevin who haven't had the impact and it likely would be very beneficial to overcome some needless conservatism here. Admittedly, English since William the Conqueror and William Shakespeare contains both the courtly French and the popular Anglo-Saxon which adds to the richness of the language. Mathematics in the English speaking world would benefit from using Shakespeare's example and use more popular terms alongside the lofty phrases.

2015: I have changed my view a bit: It is better to use the Latin and Greek phrases most of the time since these allow international exchange. The Shakespeare-suggestion of the mix still stands but Simon Stevin's approach to fully replace Latin and Greek is a misconception. Dutch pupils and students have an artificial barrier to English math texts on the internet. The message is to use the Latin and Greek phrases early on so that they become familiar. Obviously with translations provided, but these merely indicate the origin of a word.

18. The vertex of a parabola

Sullivan (1999:496) defines the vertex of the parabola from the intersection of the parabola and its axis of symmetry. Angel (2000:594) has: “The *vertex* is the lowest point on a parabola that opens upward or the highest point on a parabola that opens downward”. The latter definition avoids the opaque terms *concave* (hollow seen from below, h-shaped) and *convex* (bulging seen from below, b-shaped).

Both definitions still take a risk on *vertex*. Mathematicians often grab a word from the language soup and stick it onto their own well-defined notions. It is dubious whether that is the right procedure. The vertex of a parabola is mathematically well-defined but the general notion might be a bit confusing. The English language itself is a bit ambiguous about what a ‘vertex’ means.

Hornby (1985) has:

“highest point; top; point of a triangle, cone, etc. opposite to the base”.

Hornby uses the adjective “highest” which suggests an orientation. This is not really the mathematical intention. The mathematical instruction and the normal English instruction are inconsistent. Sullivan’s students who consult Hornby might be lost. Angel’s students might look for the highest point also in a parabola that opens upward.

For the plural *vertices* Hornby refers to *vertex*, correctly implying e.g. that a triangle might have multiple highest points (notably when one angle is the base), but digressing from the mathematical usage that a vertex in general may be just a corner (or even be defined at liberty depending upon the subject).

Partridge (1979) has etymologically:

“L. uertex, ML v-, a whirl, e.g. a whirlpool, hence, app from a supposed whirling centre, the pole of the heavens, hence a summit (e.g. the crown of the head), the top or crest (...)”

All ambiguity can be solved by using the term “turning point”.

Dutch textbooks use the label “top” to indicate the vertex of a parabola. We can imagine indeed that a hat or cone has a top, whatever the orientation of the object. However, someone may hold the hat upside down, ask you to indicate the top, and thus trick you with the ambiguity. Dutch textbooks do not use the English distinction of opening upwards or downwards but put more emphasis on the orientation by using the distinction between ‘mountain’ and ‘valley’ parabolas. In Dutch highschool mathematical lingo there are *valley parabolas* that have a *top* – while everyone knows that only mountains have tops and valleys have bottoms. The Dutch thus do much worse than the English. Dutch math teachers and exam requirements succeed in mixing up two analogies without noticing that it creates lunacy and increased problems for their pupils. They must be applauded for their wish to avoid the terms *concave* and *convex* but they have not been sufficiently critical on how they did this.

It is advisable to follow Shakespeare and mix lofty language with the popular, so that we indeed can pick pieces from the language soup. But we have to remain critical in picking the right pieces to avoid confusing associations and conventions.

2015: *Concave* and *convex* are okay when used early and normally.

19. Perfection, exactness and approximation

One would hope that *exactness* would be an exact notion. Textbooks still can create some vagueness.

The popular story, true or false, is that Pythagoras thought that everything in the cosmos could be expressed in numbers and all numbers again in ratios (fractions), but was shown, by use of his very own theorem, that this does not hold for $\sqrt{2}$. It is a wonderful story since it shows the power of proof. That someone apparently got murdered for leaking the secret also shows human nature. The story clarifies that $\sqrt{2}$ is the exact number and that it can only be approximated, by fractions or decimals.

The standard story is also that fractions would be more exact than decimals. For example, $1/3$ or 3^H is more exact than 0.333...

Issues however get mixed up. The proper distinction is between $=$ and \approx . Math textbooks persistently use the equality sign where they better use approximation. Let us observe:

- (1) The number 0.25 is just as exact as $25 \cdot 100^H$ and only written differently. The number 0.25 is not simplified to 4^H but is simplified to a decimal form so that it is clearer in relation to other decimal forms. For example 0.25 and 0.20 compare a bit better than 4^H and 5^H .
- (2) If $a = 3^H + 10^{-5}$ then 3^H is only an approximation of a . Thus we cannot hold that 3^H is always the exact number.
- (3) Obviously, 3^H is an exact number seen just by itself. The approximation of a is an exact number. And only $a \approx 3^H$.
- (4) $a \approx 0.333$ differs from $a \approx 0.333...$ with the necessity of approximation if $a = 3^H$.

Students have a tendency to regard 0.333... as more exact or accurate than 3^H , and 3.14... as more exact than π , likely since the digits better relay where the number is located on the real axis. This tendency is pervasive. This is not a simple issue but reflects the difference between engineers and pure mathematicians. An engineer will use $\sqrt{2}$ in intermediate steps, and rejoice when it can be eliminated to simplify a result, but when $\sqrt{2}$ turns up in the final answer then the engineer wants to know where it is at. Students with insecure mathematical skills will resort to piecemeal-engineering and use the calculator on $\sqrt{2}$ in intermediate steps as well.

This is an issue of sensitivity to the meaning of language. The 'exact sciences' are not just mathematics. The percentage of engineers is much larger than the percentage of pure mathematicians. Civilization produced economic growth when the engineers liberated themselves from the reign of the pure mathematicians. My inclination is to let the engineers have the word "exact" and also use the term "perfect number" for 3^H , π and $\sqrt{2}$ that in decimals can only be approximated. Perfection better expresses what is intended. Ask your students. Apologies to the small branch in Number Theory that already employs the "perfect number" label and that will have to switch to "ancient Greek perfect number".

PM. Dutch textbooks use the phrase "solve algebraically" as equivalent to "solve exactly". The phrase derives from the choice between the use of pen and paper versus the use of the graphical calculator. This interpretation of 'algebra' differs from widely held notions about algebra. The calculator may also use algebraic routines and even computer algebra. The phrase "solve exactly" is inaccurate, arbitrary, pedantic and superfluous, and can be ditched. When "perfection" has been defined as above, the statements "solve with perfect numbers" or "solve with perfection" are clear.

20. Slope of a line: more words for the same

The inclination of a line can be measured by its angle or functions thereof, as we already know from trigonometry. The tangent or rise / run is a useful measure, especially when the line has a formula within a system of co-ordinates. Take two points on the line and then deduce that the coefficient of x is $\Delta y / \Delta x$ which is that tangent. It can be mentioned that this particular form of the tangent is also called the difference quotient. Tangent and difference quotient are the same, this is called *slope* to distinguish it from *angle*, and to distinguish this oriented tangent from the general notion. The slope is also the average increase over an interval, or the rise / run, which average must be the same everywhere along the line.

The last paragraph uses different terms and aspects. However, while the above seems like a clear and straightforward approach, textbooks create a wilderness with different compositions and accentuations. Notably:

- (a) slope, defined by Angel (2000:426): "The *slope of a line* is a ratio of the vertical change to the horizontal change between any two selected points on the line"
- (b) difference quotient, defined on itself as $\Delta y / \Delta x$
- (c) average increase over an interval, as a general notion
- (d) tangent, defined with a small drawn right triangle, as the ratio of the length of the opposite side (rise) to the length of the adjacent side (run)
- (e) for Dutch textbooks: coefficient of direction, defined as the coefficient of x in the formula $y = b x + c$. Which is also the definition of 'slope' by Sullivan (1999).

We see the same terms arise as in the first paragraph above but with clear distinctions: (1) The idea that degrees could be used (as in polar co-ordinates) is not mentioned, (2) There is no link either to the known concept of the tangent in a non-directed right triangle, (3) The difference quotient is created out of the blue as a supposedly independent concept, (4) The latter may happen with the average increase as well, (5) There can be idiosyncracies like 'coefficient of direction', (6) These terms and properties can be used in all kinds of combinations, (7) Suddenly there can be an application with the picture of a traffic sign that warns about a "10% slope" in which the measure seems to be a percentage.

It was a discovery when the Morning Star appeared to be the Evening Star – i.e. both the planet Venus. This was a question on nature. It must be doubted whether multiplicity must be increased in the realm of the mind to provoke similar sensations of discovery. A richness in concepts can help understanding but there is also a danger. Overabundance has some curious effects:

- (A) A student may think that something is new, but not see that it is old. The student does not understand it, is not saved from misunderstanding, and has more material to create new blockages to understanding.
- (B) A student may think that something has to be new, but only see the old. (B1) The student concludes not to understand it (and indeed loses understanding). (B2) While understanding, he feels cheated for his time and energy, and loses motivation.

A teacher may entertain her students a whole year with concepts that are essentially the same and most of the students won't notice anything. Is this education ? Would the notion of 'education' not require that you explain that they are being entertained with concepts that are essentially the same ? And for those few who notice the lack of advance: will they not feel cheated ?

My colleague educators will hesitate. Sameness will be 'obvious' for us but this may only be because of our training. The sameness of (a) to (e) may be explained in perhaps less than five minutes to a novice to these terms but then it will be passive knowledge only and for a limited period only. It requires the immersion into the various aspects to acquire active knowledge, skill and attitude. Multiplicity serves a purpose.

The current approaches have some logic as well. With (a) the definition then (b) is the implementation, and a useful stepping stone to the differential quotient. Then (c) is an interpretation that helps to understand what the slope means. The use of the tangent (d) might be seen as confusing. Better not discuss the tangent since some students will start to calculate the angle and say that this is the slope. Finally (e) uses a formula and thus uses an entirely different formalization than (a).

Maybe. Let me refer to the first paragraph on the former page. Check the logic and how it hangs together.

There are two empirical hypotheses that can be tested in practice:

- (i) The current axiom: Students have to be exposed continuously with the various perspectives, even when those are essentially the same, even while students are not told that they are essentially the same, in order to challenge their brains to grow and to bring about the required integration of concepts *themselves*.
- (ii) My conjecture: Those brains are growing and adapting anyway and under smart exposure to the material it is only a matter of time before they will bring about the required integration of concepts.

Most likely, there are different groups (i) and (ii) so that it is rather a matter of determination what student falls into what group. Most likely there are different degrees of 'smart exposure' as well. Admittedly, the latter is a vague concept but in the context of this book it would be clear what I intend. For example, use the first paragraph. For example, when lines and slopes are used formally in other subjects than mathematics then the math class can save time on practice. However, the force of the argument is that current practice is too far into the (i) direction while it could move towards (ii). Instead of beating students about the bush we better streamline the information and offer them the opportunity to work on the steps that are not sufficiently clear yet.

Current practice has grown over time. It may be thought that (ii) has been tried in the past but has failed, no 'smart exposure' has been found, so that experience has shown that students have to take the long circuitous route. I doubt that this is true. There may of course be particular effects. When the chapter on trigonometry contains all kinds of complexities that many students turn averse about, then it might be a psychological gimmick to start lines and slopes with newly defined difference quotients that seem entirely different. The alternative course would then be to rather save the complexities of trigonometry to a later chapter. Likely there are all kinds of options here that have not been tried yet.

Hence: (1) The current approach on the *slope of a line* is a mess, (2) There is a lack of evidence and documentation that the current approach would be the best, (3) There is a clear alternative, stated in the first paragraph on the former page, that purely on logical ground should rather be the null hypothesis and the base to start collecting the evidence. (4) This example on *the slope of a line* is an instance of a more general phenomenon. (5) Personally, though, I would prefer to let "inclination" and "slope" be equivalent, and allow these to be measured by angle or gradient (tangent).

IV. Breaking the chain of understanding

21. Inconsistent names for parameters

Textbooks often use $y = a x + b$ for the line and $y = a x^2 + b x + c$ for the parabola. Do you spot a possible source of confusion ?

Might it not be an idea to use the line $y = b x + c$ instead ?

It is only a small difference, and mathematically irrelevant, but it would didactically help students who associate a with the slope. Are we to make life difficult for them and test their real understanding just now and use that as a criterion for advance, or are we going to help them and allow understanding and skill grow over time ?

It would be an advantage to be able to teach that a parabola with $a = 0$ reduces to a line, without losing time on showing by various substitutions that it does indeed. The Quadratic Formula cannot be used when $a = 0$ and it can be pointed out to students that there exist tricky test questions where they have to check this condition.

Why do those textbooks use the notation $y = a x + b$? Most likely because of the order of the letters of the alphabet and the fact that the line is presented before the parabola.

(There is no direct relation in terms of derivative or integral, as for example holds for velocity v and acceleration a .)

Another textbook uses line $y = m x + c$ instead. This still does not link up to the parabola in a straightforward manner.

Perhaps my colleague math teachers will pose that students have to learn that parameters can be indicated with different letters. In that case, my response is that we should not confuse two learning objectives. The relation between a line and a parabola is one thing. Dealing with arbitrary letters is another thing. Indeed, for the latter it would be useful to see more Greek letters.

PM. It is an option to use standard order $a + b x + c x^2 + d x^3 + \dots$. Of course the c stands nicely for 'constant'. Decisions, decisions. Well, the following reasons are in favour for the standard $y = a x^2 + b x + c$: (1) c for constant, (2) the ubiquity of this choice for the quadratic formula, (3) higher order polynomials are used seldomly and by that time students would have learned to handle arbitrary names for coefficients.

22. The line subservient to the function

When the line is presented as $y = b x + c$ then we need $x = m$ as a 'special line', namely the vertical line with an undefined or infinite tangent.

The proper general formula rather is $k y = b x + c$ so that $k = 0$ and $x = -c / b = m$ just fall under this general framework.

Why is it that textbooks opt for the broken approach instead of the general formula ? The cause must be that students are not presented with the notion of a *correspondence*. Students are only told about functions. With $k y = b x + c$ we find that y cannot be written as a function when $k = 0$ and $x = m$.

It must be doubted that pupils and student would be incapable of understanding the difference between a function and a correspondence. Instead, it need not be doubted that we do wrong in withholding that insight. Since we withhold it, students suffer the difficulty of entertaining a 'distinction' between $y = b x + c$ and $x = m$.

The broken approach to lines actually breaks down in the chapter on linear programming. Here we need the general formula of the line anyway.

The treatment of the line is strange and cumbersome.

Gladwell (2008:239) has a discussion about how a student learns that a vertical line has an infinite slope. The setting does not display any particular deep mathematical insight but is entirely caused by the framing of the question. Presenting lines in this manner combines both their mathematical formulation with difficult notions in the infinite. It would be more enlightening for the student to know that the angle is 4^H turns or $4^H \ominus$ radians ("one-per-four turns" or "archi-per-four radians"). Gladwell's basic story is that students learn more when they are persistent, which is OK. Let us encourage persistence but also allow for a lower slope in clutter and a higher gradient in learning.

PM. One might ask whether also $k y = a x^2 + b x + c$ is more general. In that case $k = 0$ reduces to the 0, 1, 2 solution points of the parabola, and the vertical lines through them. A discussion of this might be part in explaining the difference between a function and a correspondence.

23. Chaos with co-ordinates, complex numbers and vectors

At already an early stage in his mathematical education, the student is introduced to the system of co-ordinates, the x -axis and the y -axis, on which he draws his lines and parabolas.

Likely even earlier, the Pythagorean Theorem has been discussed, i.e. with the $a^2 + b^2 = c^2$ of the sides of a triangle with a right angle.

A logical development would be to consider the addition of co-ordinates, as in $\{1, 2\} + \{3, 4\} = \{4, 6\}$. Arrows can be drawn from $\{0, 0\}$. Subsequently, the lengths of the arrows can be calculated with Pythagoras. Finally, students can be told that they can sound wise and competent by using the phrase “adding vector $A = \{1, 2\}$ to vector $B = \{2, 4\}$ gives vector $C = \{4, 6\}$ ”. Let John come up front, say this, and let the class give him a great applause. Let Mary come up front, say this, and let the class give her a great applause. Perhaps a volunteer? In advance of the class, inform the adjacent teachers that you will be teaching vectors today.

The difficulty doesn't lie in the mathematics but in understanding why this type of calculation and modelling would be so useful for practical applications. The marble that rolls over the deck of a ship however remains a helpful example.

Nothing would thus be simpler than to show that ‘calculation with vectors’ is exactly the same as the ‘calculation with co-ordinates’. The mathematical difficulty starts with multiplication – that leads to matrix algebra.

My sample may be small but I have not seen a textbook that proceeds in this manner.

Rather, textbooks introduce the ‘entirely new concepts’ of complex numbers and vectors. This is another example of ‘More words for the same’ – see page 39. It is destructive. Now with the added zing that the natural growth of the understanding of space and the development from co-ordinates to matrix algebra is broken.

For example, Sullivan (1999) develops matrix algebra from systems of equations. But a linear equation actually is an improduct so it is better to start with vector multiplication in the system of co-ordinates.

PM 1. It would also be simple to show complex numbers as a historically interesting reformulation for the two-dimensional plane, with

$$z = \{x, y\} = x + iy = |z| (\cos \varphi + i \sin \varphi) = |z| \operatorname{Exp}[i \varphi] = \operatorname{Polar}[|z|, \varphi]$$

The implementation of the imaginary number as $i = \sqrt{-1}$ remains problematic with $-1 = i^2 = \sqrt{-1} \sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1$. The implementation $i = \{0, 1\}$ does not suffer this problem. The Harremoës operator of course is $H = \{-1, 0\}$.

PM 2. Students are taught that the Quadratic Formula has no solution for a negative discriminant. Later they are told that there is a complex solution. It should be feasible to mention the complex solution directly. To know that it exists is different than doing exercises with it. Perhaps we need a course Geography of Mathematics with all the countries we never go to but still know about. You learn to wash your hands and only later may have a chance to look under the microscope to see the germs. (2015: A suggestion is to call this course element the *Encyclopedia of Mathematics*.)

24. Needlessly slow on derivatives

The discussion about Superficial Calculus (rules only), Serious Calculus (Cauchy limits) or Deep Calculus (Weierstraß) has a long history. Let us consider the current state and a suggestion for improvement.

Students currently find the turning point of a parabola with a formula that either is merely supplied or derived by moving the parabola so that the turning point is on the horizontal axis. (Thus, a single point of intersection, choosing c in the Quadratic Formula such that Discriminant is zero without actually calculating any c .) The same formula can also be found by differentiation but this is taught only later in the course. The reasoning on this lesson plan must be that students first require some mathematical skill, to be developed on the parabola, before they can grasp the notion of the derivative, which will help them to reflect on their earlier learning on the parabola. There is indeed a small effect of amazement when students discover that the derivative gives the already known result.

I beg to differ. In an alternative lesson plan the rules of differentiation are presented at a much earlier phase. When they are applied to the parabola to find the turning point then also the ancient way to find it can be presented alongside, both for corroboration and historical perspective, and clearly both approaches will sink in much better at the same time.

Slopes are important. That is why they are in the programme. The rules of differentiation are an important discovery not only because they are fairly simple but also because they generate important results and generate them fast. It pays to command those rules as soon as possible. For example, in economics, to differentiate the parabola of profits, set the derivative to zero, and find maximal profits. Why it works? Well, there are levels of understanding. Clearly the slope is zero at a maximum, minimum or inflection point. Why these rules give the slope? Well, we will get to this later on in the course.

Recall that we allow people to drive a car without knowing how it works. People are vaguely aware of the different kinds of electrical current but only sufficiently to prevent appliances to blow up. We play soccer without much knowledge of Newtonian physics and aerodynamics. It is not evident that all of this would be different for mathematics. It is a nice ethic that you want to prove everything but (a) clearly this is *not done now* in highschool, and (b) the selection of *what is proven now* is arbitrary, superficial, traditionalistic, unconvincing. It is valuable that pupils feel that some argument is given, and an explanation helps memory. But an explanation 'derivatives help to find the slope' may be as adequate as the explanation in biology 'people breathe because they need oxygen'. Eventually a lot can be explained and proven but soon it becomes a specialty and it runs against economic laws that everyone can be a specialist in everything. Thus, in the same way, we can teach how to find the derivative without detailing why it works. It is already quite a mathematical competence to know the rules and how to apply them.

The true story about the current situation is that students first memorize the rule for the turning point before they discover that they had better memorized the rules of differentiation for finding such points in general. Thus there is more memorizing than needed and less time spent in competence.

Admittedly, for calculus in the English speaking world, I have available only Hughes-Hallett et al. (2000) for universities and colleges. This course in calculus would be separate from a course in algebra (e.g. Sullivan (1999) and Angel (2000)). For highschools I have to rely on my experience in Holland. Dutch highschools have four tracks of math: D for the advanced level (somewhat linking up to above English sources),

B for normal math (always taken by D too, including Serious Calculus), A for economists (Superficial Calculus) and C for students of art (no derivatives).

In Holland the main distinction between A/C and B/D thus has already been made. At issue here is only the order of presentation. My suggestion is to always start with the rules, in track A and B/D alike, already when discussing the parabola, and only later provide a more formal justification for the B/D group. We want these students to get serious mathematics – which however means that we also want to enhance elegance with substance, and avoid a cumulation of cumbersome convolutions.

I can understand the mathematical urge to introduce some more formal math at the highschool level, albeit not Weierstraß then at least Cauchy. We see the same with Hughes-Hallett et al. (2000), where first the formal definition is presented (though Cauchy only) while only the subsequent chapter provides the short-cut rules. The driving force in this reasoning is the urge by (pure) mathematicians that the derivative needs a good definition *before* it can be applied. It seems to be part of the mathematical ethic and decency not to use something that hasn't been clearly defined first. They are not fundamentalist on this, they are willing to compromise, they don't insist on Weierstraß and accept Cauchy, and they let the A/C tracks go their way. Nevertheless, the urge is there. In my view this urge is didactically unwise, not only for the B/D track but also for the A/C track, since all tracks get the rules on differentiation too late. Much time in the early phases of the current programme is lost on fractions and radical signs. It is much better to spend that time on learning the important rules of differentiation.

Perhaps course designers also feel that when students know the rules they would not be interested any more in the formal definition. Indeed, once the formal definition is presented it is hardly used anymore and all attention goes to the rules and their application. Nevertheless, students in the B/D track would most certainly have the attitude to be interested – as it also is an interesting subject. But a bit later.

With Van Hiele: first concreteness, then ordering, then analysis.

Below, we will look a bit deeper into the formal definition of the derivative.

PM. Dutch students in the B/D track have only derivatives and integrals of a single variable and miss out on the distinction between partial and total derivatives. The latter should rather be in the program.

V. Like the stepmother in the fairy tale

25. Probability and statistics

This may be a Dutch idiosyncrasy. In the Dutch highschool programme probability and statistics are put in Track A and are not included in Track B (see the tracks on page 44). Perhaps the physics professors want to be able to develop quantum mechanics by themselves. Physicists may have limited understanding of probability and statistics:

“There appears to exist a strange miscommunication between physics and mathematics. Gill quotes Suppes: “For those familiar with the applications of probability and mathematical statistics in mathematical psychology or mathematical economics, it is surprising indeed to read the treatments of probability even in the most respected texts of quantum mechanics. ... What is surprising is that the level of treatment in both terms of mathematical clarity and mathematical depth is surprisingly low. Probability concepts have a strange and awkward appearance in quantum mechanics, as if they had been brought within the framework of the theory only as an afterthought and with apology for their inclusion.” (P. Suppes, 1963). Gill suggests that this is still the case in 1998.” (Colignatus (2005:81) footnote 64.)

Students from both the A and the B tracks are not introduced to the ‘abstract’ notation of elementary probability. A textbook need not mention the formal definition and notation for the conditional probability $P[X | Y] = P[X, Y] / P[Y]$ while this would be important for proper understanding. Even worse, students are submitted to complex language constructions that supposedly code for conditional probabilities. Thus they have to learn both the concept of conditional probability and dubious linguistic codes.

The following example translates well. A textbook has a crosstable on injuries at a sports club. Let X = “an arbitrary club member is younger than 20 years” and Y = “an arbitrary club member has more than one injury”. A is the probability that an arbitrary member of the club “is younger than 20 years and has more than one injury”. B is the probability that an arbitrary member of the club “that has more than one injury, is younger than 20 years”. Thus $A = P[X, Y]$ and $B = P[X | Y]$. One awkward point is that the language construct uses a comma for the conditional while the mathematical convention uses the comma for the joint probability. Students are encouraged to write “ P [that has more than one injury, is younger than 20 years]” which will require some unlearning again later on. Another awkward point is that the clear statement “the probability that a member is younger than 20 years *given* that he or she has more than one injury” is not used. The textbook uses a construct that admittedly might be used. We should hope that people who use that construct indeed intend the conditional probability. However, the construct will be rather unfamiliar for students in a first course on the subject. To avoid the ambiguity and parallel learning of both mathematics and language, it is much better to concentrate on the mathematics and use language for clear communication. The expression “given that” provides that clarity and indeed links up to the formal expression of conditionality.

26. Ambiguous dice

There is a distinction between a perfect die with probabilities $1/6 = 6^{-1}$ and empirical dice of which the probabilities per die have to be determined empirically and that could be approximated by observed frequencies under similar conditions.

Many discussions and test questions don't mention the label "perfect" and expect students to be able to determine from the context whether it applies or not.

We note a subtle shift in learning goal. The math course is no longer targetted at math itself but apparently on 'reading well' – with always the gamble on what the author really intended.

Supposedly when the exam question is about a die factory doing quality tests then we might presume students to be so smart as to understand that factories cannot produce perfect dice. A question like "John throws two dice. What is the chance that he throws less than 4?" is already tricky on language students who will hold that two dice are less than four dice so that the probability must be 1. Assuming that the digit codes for the outcome, the question might presume perfect dice so that John is only an imaginary figure created for literary purposes. Otherwise we would not know what that chance could be since we have not been able to test those dice. Perhaps there is a hidden convention that factories are real and person names imaginary.

It is advisable to distinguish the learning of math from the learning of context. For the math section there could be a statement like "all dice are perfect unless it is explicitly stated that they are real" or "all dice are real unless it is explicitly stated that they are perfect". For the context section there could be a statement like "determine from the context whether dice are perfect or real".

27. Mathematics and economics

Textbooks on mathematics must develop a position with respect to textbooks on economics. Economics is often seen as a useful application of mathematics – though historically many impluses went the other way – and thus textbooks on mathematics contain such topics. An example from economics might occasionally be used to highlight a mathematical point, though hopefully mathematics is supposed to support economics and not the other way around.

The Cambridge economist Alfred Marshall (1842-1924) created the diagram of demand and supply, put the cause *price* on the vertical axis and the effects *demand* and *supply* on the horizontal axis. Textbooks in economics faithfully copy him to this day.

The international scientific and mathematical convention is to put the cause on the horizontal axis and the effect on the vertical axis.

It would be obvious that textbooks in economics better adapt to the international scientific standard. It would reduce the confusion for their students between the classrooms in economics and mathematics.

It would be rather simple for economics to adapt. They could start in textbooks for highschool, and trust that these students will not read the historical books and the international journals. When the train gets going then it will be as simple to adapt the textbooks for university and college. At that level, students will be sufficiently versed in the subject to understand the older literature.

Teachers of mathematics apparently are confused themselves too and don't seem to realize the inverted use in economics. They are a bit like a hair-dresser who offers his services but appears to know only one cut. Of course there is the Cournot model where companies set quantities rather than prices. However, the common discussion is about the competitive model where the price is given. Diagrams in economics then have a horizontal line. Discussion of this case in math textbooks creates the curious situation that they want to draw a vertical line and still reproduce the economics diagram. They manage to somehow talk around it, but obviously at great confusion for the student.

Mathematicians should urge the economists to adapt. In the mean time textbooks in mathematics better (a) keep following the international scientific standard, (b) refrain from messing up economic models, (c) explain to students about cause and effect and (d) explain the differences in conventions in mathematics and (old) textbooks in economics.

28. A shopping list on content

Textbooks in mathematics clutter with the dust of ages and the efforts by mathematicians to understand something about mathematics and to formulate it clearly. While the sand flows in the hour-glass, and hour-glasses themselves slide through our fingers, time in class is spent on tradition, and hardly any time is left to discuss new things that would actually be useful to discuss. There is a balance between tradition and adaptability. Let us see what could be included in the mathematics programme, preferably in highschool.

(a) Logic and set theory

Sullivan (1999) fortunately contains some set theory but curiously logic and set theory have disappeared from Dutch highschools – only to resurface a bit in the new programme for track C. The formal representations of logic and sets are crucial results for the history of mankind but curiously they are not mentioned. To me it feels like a criminal act – a ‘white-board crime’ – to withhold these results from students. Apparently, set theory already belonged to the exam programme for a while in the Dutch past but then was reduced to needlessly complex issues of notation. It is advisable to try again. I must refer to Colignatus (2007a) *A Logic of Exceptions* (ALOE) since this redesigns logic. Thus it is advisable not to start with traditional logic. ALOE has been written for first year students in higher education – but it is no mere textbook but re-engineers logic. Also, fuzzy logic deserves attention too.

2015: FMNAI (2015f) – see page 67 below – clears confusions in set theory, number theory and infinity, so that it would be feasible to present these in "neoclassical" manner in highschool too. FMNAI has first year students as part of its intended readership too.

(b) The axiomatic method

The axiomatic method is a topic of content rather than a way of teaching. As content, an axiomatic system is a rational reconstruction of a body of experience with a lot of irrationality. Teaching this content will increase competence in reasoning. As a didactic method, it is of dubious quality. The next section will say more on the method. Traditional mathematics has a tendency to fuse the two. The Van Hiele theory however, reduced to rough simplicity, has the levels of concreteness, ordering and analysis. (What is it, what can you do with it, how does it work and why does it work ?) Analysis only comes at a later stage. This amends the traditional way of the education in mathematics. Possibly pupils with mathematical ability have a fast route to the analytical phase so that the earlier phases might be neglected more but that is a different kind of discussion.

In my own highschool days (I am from 1954) there was much more reliance on the axiomatic method or at least the Form with definition, theorem, proof. Checking those books again this method does not strike me as so didactically useful indeed. It is hard to tell, of course, since my analytical capabilities must have been influenced by that background, for better or worse. I think anyway that we have strayed too far from abandoning the topic itself. Hughes-Hallett et al. (1999) for example present the rule of L'Hopital and then proceed with "To justify this result (...)". It is a nice literary trick. One might turn it more formal.

Using the Form in normal discourse is pedantic and should be avoided. But in mathematics the objective is to develop and support reasoning. There the Form is on target. If a proof is given then it would support this notion by providing the Form. When to apply it ? The criterion would be that students have already had the first two stages of Van Hiele and are ready for the analytical phase.

If students would get worried and ask whether they are required to provide such proofs themselves then the answer can be (i) only sometimes, (ii) if so, do not worry, for we can follow the old Greek advice: assume that what needs to be proven is already proven, write down all the properties that you know (also given that assumption), reorder a bit, and the proof will click in place, (iii) remember, the idea behind the mathematical method is the liberating force that no authority can impose a rule but that only you yourself can check its validity – and with this liberty also comes the duty to prove to others what you would like them to believe, which means that you better work on some competence to provide proofs.

2015: An example of proofs that may be tried in elementary school concerns the Harremoës operator.⁵

For elementary school, Kyllian (2013) has a wonderfully insightful proof of the Pythagorean Theorem. Pupils aren't used to think about right angled triangles and thus she uses the diagonal of a rectangle. Rectangles are used later in the proof anyway. With width w and length ℓ pupils already know about surface $s = w \ell$ and circumference $c = 2 w + 2 \ell$, and now they learn the formula for the diagonal d . She assumes that pupils are not used yet to denote squares with an exponent, and fully writes out $d^2 = w^2 + \ell^2$. These are just some of the innovations here. I've seen it in action and kids of 10 years of age rediscover the theorem themselves and are thrilled by this. Kyllian gave me permission to re-use some of these ideas, see *A child wants nice and not mean numbers* (CWNN).⁶

Intermezzo: Is anyone to blame ?

Some mathematicians are inclined to explain the disappearance of the Form to the pressure of social and economic developments. We can refer to Ernest (2000) again. On Dutch history, Goffree et al. (2000) is obviously relevant too. Instead, it is just as well the weak backs of the mathematicians themselves who have not defended their field.

At his retirement after 40 years of teaching Groen (2003) states, in my translation:

"In the last decennia the call has become louder and louder that education must offer knowledge that is directly applicable and useful. Economic use as the measure of all things is incontestable. Talk about products in cases where this never happened before (university graduates, train connections, medical treatment, overnight stay in a hotel) is only considered strange by unwordly and unadjusted poor souls. This had its influence on programmes for mathematics. In the past we were satisfied with the proposition that education in mathematics greatly contributed to Bildung without being able to show concretely what the effects of that forming value were. Nowadays we want to see that forming value expressed into recognizable, profitable applications. This has also led to the almost complete disappearance of the emphasis on theorems, definitions and proofs that existed in the past almost directly from the 7th grade in Lyceum. It has been replaced by quasi socially relevant calculation about heating bills and angles of sight. The return of planar Euclidean geometry as a context for exercises on proofs is an effort to do something about that again, but that only begins in senior highschool. To me this seems dangerously late for the development of the required reasoning capabilities."

Mathematics teachers may also conclude that they as guardians of that Bildung have failed collectively in defending it. Society is liable to be gullible if we would hand them that responsibility afresh. What guarantee is there that they now will steer the right course ? My suggestion is to put the responsibility with a council of not only mathematicians but also the other sciences and humanities, teachers, parents and students.

⁵ <https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/>

⁶ <http://thomascool.eu/Papers/NiceNumbers/Index.html>

(c) Algorithms and computer programming

Algorithms are key in mathematics. A proof for example is an algorithm to check the theorem. An algorithm is a longer chain of logic, possibly extended with text, formulas, tables and graphs, to identify problems and solve them. Landa (1998) is an important source here. Landa's core idea is (a) observe experts, (b) dissect their actions in small steps, (c) analyse these steps, optimize them, formulate everything as an algorithm, (d) allow students to execute this algorithm so that they can perform immediately like experts. Subsequently, students greatly extend their scope when they learn to create algorithms themselves and recognize this as a tool for enhancing their understanding. For explorative cases (deterministic) algorithms are replaced by (probabilistic) heuristics. Note that the terms algorithm and heuristic are not part of the student vocabulary. Using these words may at first put them off. Learning them is part of the understanding.

Textbooks in mathematics generally provide students with algorithms to solve the test questions in the book, but they are modest in discussing algorithmic design. Students will learn a lot from computer programming itself, as part of the course, since the choices for problems and solutions are more varied. Working with the computer is interactive with direct feedback. A programmer has to think about the overall target and the small steps at the same time. You tinker with it till it works. If programming is to be educational it must be done in a serious language and not with drop-down menu clicking or the use of strange codes. A modern course will take a computer algebra language such as *Mathematica* or Maple which allows flexibility to explore the different kinds of programming (functional, object-oriented, rule based).

Current education tends to use the graphical calculator. This is penny wise and pound foolish. It seems a good bargain but it has limited capacity, does not allow good programming, reduces effort to a lot of senseless punching, kills motivation.

(d) Use of language

Textbooks can contribute to a sharper use of language. With *text*, *formulas*, *tables* and *graphs*, the first element does not get sufficient attention. We already have seen the examples of the vertex of a parabola and the perfect die. Textbooks better use sharper language themselves but it would also be an improvement when they provide educational material to increase student awareness. Give students a crummy text and let them work on guessing and editing to what it might mean.

Mathematicians hold the idea that language is vague and formulas will be exact. This idea however runs counter to good didactics since it would imply that we may give up trying. Instead it is better to sharpen language as well.

Reality is not neat. Data have to be collected and pruned to become evidence. Formulas and graphs don't fall from the sky but have to be hunted and crafted. Texts can be very messy. There is no reason to single out language as the element to neglect.

Accuracy also applies to math test questions. It is no rare occurrence that a question is opaque except under a particular interpretation that suddenly gives all that is required. The student then is tested on finding that particular interpretation and not really on mathematical insight. "Reading well" is a soft criterion. Math test questions should provide all information, and actually also some redundancy to allow a double check. Admittedly, it may be difficult to provide all information without giving away the answer but it will generally be clear what kind of questions actually should not be asked. Opaque questions might perhaps be asked to query mathematical creativity – which is anyway a difficult property to test.

(e) Theory of democracy and voting

Democracy is an important concept. The mathematics of voting is somewhat complex. It would be beneficial for society when its citizens understand more about the mathematics behind election results. Students in the USA have a Government class where such aspects can be indicated. Political Science as a subject has not reached highschool in general. Much can be said in favour of including the subject in economics, since the aggregation of preferences into a social welfare function is a topic of Political Economy. See Colignatus (2007b, 2014) *Voting Theory for Democracy* (VTFD) for details and other references – see page 63 below. Most economists will be unfamiliar with the topic and its mathematics though and thus it may well be practical to include it in the mathematics programme.

2015: The UK 2015 general elections were needlessly paradoxical and the difference between proportional representation (PR) and district representation (DR) leaves a lot to question.⁷ The Conservatives with 36.9% of the vote get 50.9% of the seats – and given the turnout of 66.1% they only have 24.4% of the electorate. This paper⁸ compares PR as in Holland with DR as in the UK. There is this note⁹ on the UK Referendum on PR in *Mathematics Teaching* 222, January 5 2011, the journal of the *Association of Teachers of Mathematics* (ATM). There are some comments¹⁰ on some confusing influence from Holland on the understanding in the UK about voting. Mathematicians in Holland still confound the issues.¹¹

(f) New subjects of the last two decades

There seem to be no other new subjects of the last two decades that students should do in depth. This shows the elementary nature of the current curriculum.

However, there are subjects of the category “useful to have seen the major relevance and results”. Such subjects tend to date back longer but apparently take a while to diffuse into textbooks. Those are fractals, chaos as opposite to randomness, cryptography, perhaps cellular automata. Topology with the fixed point, useful for the definition of e . Graphical models with conditional independence are a useful addition, and a combination of graphs and probability theory.

Economics may want to spend more time on finance theory and stock market crashes, possibly desiring mathematical support for the Black-Scholes model for option pricing and the critique by Mandelbrot & Taleb on the too simplistic interpretation of the law of large numbers – while major issues are fat tails and the use of a proper measure of risk.¹²

⁷ https://en.wikipedia.org/wiki/United_Kingdom_general_election,_2015

⁸ <http://mpira.ub.uni-muenchen.de/22782/>

⁹ <http://www.atm.org.uk/Mathematics-Teaching-Journal-Archive/3921>

¹⁰ <https://boycottholland.wordpress.com/2015/05/12/d66-con-on-the-2015-uk-general-elections/>

¹¹ <http://thomascool.eu/Thomas/Nederlands/Wetenschap/Artikelen/2013-02-14-PasOpMetWiskundeOverVerkiezingen.html>

¹² <http://thomascool.eu/Risk/index.html>

29. A shopping list on method

Next to content there is the way how mathematics is taught. Some aspects hold for education in general but some will be specific to mathematics.

(a) Dealing with developing brains

Jolles et al. (2006) “Brain lessons”, its website <http://www.brainandlearning.eu/> and other initiatives around the world provide a fresh angle, alongside other developments on evidence based education.

I was also struck by Gladwell (2000), while reporting the known fact that small kids enjoy watching the same tv program over and over again (e.g. Sesame Street), also mentioning that they see different things each time, which is an angle I had not considered before. The phenomenon can actually also be observed at highschool, where much of the same material is presented in the different grades, over and over again (and continues to give problems). Apparently brains value a decent amount of repetition and in particular when they develop.

We already discussed the aspect of “More words for the same” – see page 39 – and suggested ‘smart exposure’ as an alternative. Thus, brains must be stimulated to grow but they must not be forced on topics for which they are not ripe and that will come about rather by themselves over time. This is a nice general statement and possibly everyone agrees as long we are vague on specifics. Randomized controlled trials would be a way to work out the details, provided that parents will offer their kids to such experiments. A key point of this book is that, when designing such trials, we better don’t do it with mathematics that is inherently cumbersome and irrational, but with the elegance with substance that we expect from good mathematics.

One aspect is cognitive dissonance, see Aronson (1992). It is a pervasive human property and must affect education too. The brain is an information processing machine with conditions of energy efficiency, and one of the cheapest ways to deal with new information is to neglect it. One example might be textbooks used in 9th grade and 10th grade. Dutch textbooks are not by subject but collect the material used in a grade for the different subjects. In 10th grade it might be instructive to run through the textbooks of the 9th grade again, and refresh what already should be known. The kids might consider this childish though and below their standards. Some might argue that a whole new book provides the chance to create a new environment afresh, a new start, a new dawn, and when much of the same material is treated again then this gives pupils a chance who missed out the last time. Perhaps. An alternative is to arrange textbooks by subject, such that a discussion at the level of the 9th grade is followed by a discussion of more advanced aspects at the level of the 10th grade. This avoids the cognitive dissonance that it would be childish to look into the book of last year, repetition comes about naturally, and we can save a lot of time on actual repetition because of these two effects. Of course kids would have more books. Are we *penny wise, pound foolish* ?

2015: Colignatus (2011e) on brain / mind has been included in SMOJ and FMNAI.

(c) Overall didactic awareness

Overall didactic awareness: it seems obvious but may amount to a paradigm shift in the teaching of mathematics. Textbooks of mathematics still suffer from the tradition that Euclid’s axiomatic (re-) construction of geometry defines the Nature of Mathematics and is The Way, not only for Presenting Results but also for Teaching and Learning.

We already mentioned the Van Hiele approach to allow room for levels of understanding. We also mentioned Landa's algorithmic and heuristic approach as subjects to learn, but they also are methods of teaching. Including with other writers on didactics, research on the brain and cognitive psychology, there is a strong alternative to The Way.

A textbook example

Old ways die hard. An example may be taken from a Dutch textbook where the derivative of a^x is introduced. It is not stated first that $(a^x)' = a^x \text{ rex}[a]$ – see page 35 – but it is derived formally. The differential quotient gives an expression where the natural logarithm cannot be used yet since e has not yet been defined. The purpose of the exercise precisely is the definition of e . The book solves the problem by defining $f(x) = a^x$ and then presents the solution that $f'(x) = f'(0) a^x$. For a reminder, note:

$$f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \cdot a^x = f'(0) \cdot a^x$$

The original problem of finding the derivative of a^x is temporarily unsolved and dropped from consideration. The section proceeds with determining e and only the next section completes with determining that $f'(0) = \text{rex}[a]$. Hence it is proven in general that $(a^x)' = a^x \text{ rex}[a]$.

The reasoning is sound and will appeal to the mathematically trained who wants to check. Possibly the mathematical ethic also requires that we should not discuss things that have not been defined properly. Possibly it saves time and energy doodling about with concepts only to find out later that they are ill defined / not defined / not definable at all.

An alternative approach

However, it does not save time when the abstract soup prevents understanding. Here I would rather follow Van Hiele and allow the students to first play around with what it all means both concretely and in terms of interrelationships, before concluding with the proof why things actually are so. Thus:

- (1) There is a fixed point in differentiation with $f'[x] = f[x]$.
- (2) In particular there is a number e such that $(e^x)' = e^x$ – on the computer $\text{Exp}[x]$.
- (3) There is only one such fixed point in differentiation. The number $e = 2.718\dots$ is as special to mathematics as Θ .
- (4) We define $\text{rex}[x] = \text{rex}[e, x]$, such that $\text{rex}[e, e] = 1$.
- (5) For all exponential functions we find $(a^x)' = a^x \text{ rex}[a]$.
- (6) Check that $(e^x)' = e^x \text{ rex}[e] = e^x$ indeed, since $\text{rex}[e] = \text{rex}[e, e] = 1$.
- (7) All positive numbers can be expressed as powers of e .
- (8) Graphics and exercises, to explore what it means.
- (9) Provide the proof using above proper differential, to show why. Calculate e .
- (10) Graphics and exercises to let it sink in, so that we do it in full understanding.

To me it would be obvious to proceed in this manner. But I referred to a serious textbook and they mess it up. They also clutter the argument by first discussing translations of logarithmic functions, suggesting that it seems like the major point of the chapter while this is a minor topic that may come in an appendix. You don't have to be able to translate a logarithmic function to master e .

2015: COTP is a proof of concept. See also page 93.

(c) Class and quality

At university thirty years ago I attended my math lectures in an oratorium with possibly 150 students and after the lecture we had our practica in smaller groups.

Class size depends upon national regulations and possibilities of scheduling. In Holland highschool teachers of mathematics accept a class size with maximal 30 pupils. Apparently it works somewhat, witness the state of the Dutch economy (with natural gas resources). I would still hold that math is not the same as French or geography. Learning to think and to reason and catching the subtleties in the personal route towards understanding are served by a class of maximal 15 pupils or students.

Allowing only 15 pupils or students requires more math teachers. There can be savings in (a) a quality program requires less contact hours, (b) shift non-core-business such as repetition of exercises back to the subject fields such as economics and physics where those actually belong, (c) recruit (good) older grade students to help younger grade students, (d) rely on more independent work with the computer, (e) relieve the task of checking exams, by more computerized tests. If a class of 30 students has an hour of geography and subsequently an hour of mathematics, then it can be split and we need two teachers of mathematics where we now schedule one. In practice a class might have for example 25 students, 15 would go the contact hour, some might consult their student-assistant if she is scheduled to be available, others work on the computer.

Overall, though, some increase in the number of math teachers seems advisable. Mathematics is important, and good mathematics saves on the demands on other subjects. It is said that there is a shortage of teachers of mathematics but this is a use of language that is low on analysis. The better statement is that salaries are too low and that more must be done to let it become education in mathematics indeed.

(d1) Support and testing by computer – the problem

The world abounds with computer programs and materials for mathematics. This should not be surprising since computers were developed by mathematicians and computer science engineers. Nevertheless, the relation of mathematics and the education in mathematics to the computer is actually rather a problem.

We have e.g. Excel, Java, typesetting LaTeX, html or xml with MathML, *Mathematica*, Maple and Maple T.A., Matlab, Maxima, Wiris, Derive, Scientific Workplace, open source Sage,¹³ and the graphical calculators as well. All these have their various applications that users often put on the internet. MathBook or OpenMath/MathDox, see RIACA, TU Eindhoven, accept various computer algebra systems and build a layer on top, which seems useful but requires additional attention for the uninitiated and seems unnecessary for who already has a system. Geometry programs are Cabri and free Geogebra.¹⁴ Class management systems are Blackboard/WebCT and open source Moodle. In Holland examination on the computer is already partially allowed for graduation and there are steps to further develop that. Systems that combine class management with instruction and testing are WIMS,¹⁵ Maple T.A. and (likely) Wiris in combination with Moodle.¹⁶ Textbook publishers are starting to provide their own systems. Schools tend to have their own system to administer students and their grades.

The key notion is insularity. Each participant is defined by own objectives, own resources and own restrictions and it appears very difficult to arrive at a common goal, pool resources and overcome the restrictions. To name a few points:

¹³ <http://www.sagemath.org/>

¹⁴ <http://www.geogebra.org/>

¹⁵ <http://wims.math.leidenuniv.nl/wims/>

¹⁶ <http://www.wiris.com/> (only a "basis" is free)

- (i) Countries have their own languages and national regulations.
- (ii) Nations have their own school districts. Education is a sensitive issue for parents.
- (iii) Publishers have their own authors and websites.
- (iv) Teachers have their own students and particular issues.
- (v) Programmers have their own computer languages.
- (vi) Associations of mathematics must be diplomatic about sensitivities.
- (vii) As this book shows, issues need not be simple, with different grades, levels of understanding and competence, aspects of didactics.
- (viii) It is not correct to only consider mathematicians since it are governments and national parliaments who determine how important they judge this issue and how many resources they make available.
- (ix) Since mathematics rather is an international venture it actually is the international community that is responsible.

Educators are peddlers and drugdealers. First you are encouraged to “graduate” from elementary school if your life is to be any good, but once you have done so then you are told that you have to graduate from highschool. That done, you are told that you better graduate from college or university if you want to have some perspective. With that document secured, you are told that the minimum is a Ph D.. Eventually you may discover that you may have learned a lot but still know very little. Plenty dealers around to peddle a course that you really should take. Eventually you may discover when you are teaching the subject that only then you really get to understand it. The moral is that we may as well be relaxed about all this, even concerning mathematics.

Perhaps the situation compares with soccer clubs that do not co-operate to form one super club. Soccer clubs are focussed on competition and thus mathematicians would be a more agreeable lot, perhaps only a bit more critical than soccer clubs on the aspects where they disagree. But let us see what can be done for mathematics.

(d2) Support and testing by computer – a direction for solution

2015: Developments in ICT are fast in some aspects and slow in other aspects. Though ICT might require an update of the text of 2009 – for example with the rise of the tablet and the advance of computerised testing with Maple T.A.¹⁷ – this update on specifics would lead too far for the present purposes of the book. Relevant are the general considerations.

Given the importance, there is a separate chapter on this, see page 75. However, at this point it is more relevant to develop the underlying notions:

- (1) In education, feedback is important and differs in kind and intensity depending upon the individual. Nowadays the teacher gives feedback, students look in the booklet with answers and they ask around. The idea is that the computer will be a great tool to take away the tedium and to provide new levels of interactivity. Mathematics will continue to require much testing with pen and paper and teachers will want to see what their students are doing in that manner to better judge their knowledge, skill and attitude. But at various points even multiple choice questions can be used if only for preparation and to set entrance levels. (i) Teachers will have to take the psychological barrier and recognize that the feedback from written tests is relevant but limited. Computer tests can be relatively smart by responding to the level of the student and by monitoring how often the same kind of multiple choice test is done in an effort to pass purely by randomness. (ii) The second barrier to take is organization. Schools and universities have to create computer test rooms, with special supervisors who check on identity card, login procedure, mobiles and usb-sticks if it is a formal test. This applies for all subjects but also mathematics. Since the creation / modification of test questions is fairly simple for mathematics (e.g. plug

¹⁷ In Holland: <http://www.methakamminga.nl/>

in numbers selected in random manner) students may be allowed to do tests at liberty. It seems rather strange, but a major bottleneck towards advance in quality in teaching of mathematics are the costs of such test supervisors and other concierges for school opening hours. This relates to my economic analysis in DRGTPE. In economics, everything hangs together. (iii) Rather general experience with WIMS, and also my own, shows that students don't use its availability on the internet and its possibility of feedback if they cannot earn points. Hence, procedures are designed such that students learn that it is wise to do such testing especially if they lack in competence. One option is to require an entrance test in advance of a written test, which entrance test is done on the computer under supervision (otherwise a friend might use the internet). Another option is that a failed exam is counted as a worse failure if there is no record of sufficient advance self-testing. Another option is to give the new system a chance, let students get used to it, create an attitude and culture that they use computer feedback, and subsequently talk with the students who don't and their parents.

- (2) Schools can best stop using graphical calculators since what those can do can hardly be called mathematics. Proper is the switch to mini laptops with open source linux, open-office, open source Sage / Python and free Geogebra. This will support instruction and feedback from interactivity. Feedback from actual tests will not be automated yet. It is a start and we can work from there. See page 75. See below though for decisions on quality versus price (open source or commercial software – *penny wise pound foolishness*).
- (3) The use of those mini laptops during official examination will be problematic since students would be free to put anything on the hard disk or perhaps even create a wireless connection. Reformatting and reinstalling is tedious and actually somewhat unfriendly towards the hard working student who includes all kinds of material. Alternatives are (a) the use of the common test room, (b) have a sample of mini laptops in minimal configuration purely for such tests.
- (4) There are three additional advantages of using mini laptops: (a) programming – see page 52, (b) integration with other subjects such as economics and physics, since computer algebra is much more versatile than the graphical calculator, (c) overall mathematical accuracy. Above we saw the distinction between $f(x)$ as multiplication $f \times x$ (dropping the brackets) and $f(x)$ as the function call $f[x]$. Who works with a computer algebra system will see many more cases where accuracy can be improved.
- (5) Computer programmers are insufficiently aware of the golden rule in programming:

Do not program to others what you would not want to be programmed to yourself.

The rule should be basic to the education of programmers. Perhaps the basic education for programmers is to engage them in social activities (since programming tends to come to them naturally anyway – see Krantz (2008) again).

- (6) The integration of computer algebra in mathematics education is no small issue. A small example is notation. A capable mathematician and teacher of mathematics can switch relatively easy between the various notations, e.g. between the various books, the textbook, graphical calculator, the computer algebra system, and, indeed the writings of the students. Students however are learning mathematics and rely on consistent notation. Students are very sensitive to differences between the textbook and computer programs. The choice of a program is a crucial one and not easily changed. See page 75. Textbooks will have to adapt to the computer as well.
- (7) Textbooks are rather expensive but that is also because they nowadays provide their own websites and software. If publishers had done that much earlier then software

producers would not have stepped in – and now they are competing for market share, actually driving up costs and reducing quality. There is an increasing tendency to refer to free sources on the internet. The internet seems to provide an abundance of applications indeed. This is rather an illusion. Many applications are in Java and thus very specific, not easy to adapt, and not suited as building bricks for a more complete system. The only sound step is to switch to using a computer algebra program, see page 75. This conclusion does not disqualify or diminish the efforts by teachers and other producers of those other programs and their discussions of manuals and didactic qualities. Indeed, when we consider the various resources created e.g. in Holland by e.g. the Freudenthal Institute,¹⁸ Mathadore¹⁹ or Kennisnet²⁰ even apart from the main three commercial publishers and other sources, the fragmentation seems to prove the need for a single working environment. In fact, this is already obvious for the last 15 years if not earlier.

- (8) For computer algebra we can distinguish between the mathematical language – that would be uniform over the world – and the computer program that interpretes this language and evaluates this. Current programs tend to proprietize mathematics by using slightly different codings. That menus differ and that different programs have different capacities and layouts would be acceptable and subject to competition in the market place. However, a criterion should be that there is a uniform, text based, simple language for mathematics, that can be used as input and output. See Colignatus (1999, 2000). Personally, I am in favour of using *Mathematica* as the base of that mathematical language, and hope that there can be put a shell on top of Sage / Python, or whatever. I imagine that others think otherwise. The Sage language does not strike me as sufficiently elegant for doing mathematics on the computer. But it is an improvement upon graphical calculators and we may work from there. See page 75.
- (9) For senior highschool and up, mathematics would likely be done in English for most countries in the world. With this complexity of mathematics it might not pay to translate all of it. This would affect the other subjects like economics and physics that use mathematics. Likely those subjects face the same kind of problems with respect to computer assisted support and testing. Countries face tough decisions about the costs of maintaining their national languages in education. My advice is to be relaxed about it since national identity is very strong and will not be rocked by this influx of English.
- (10) Best is to design a mechanism to transport applications to the public domain. Applications written in the uniform mathematical language would be put on the internet as an open source contribution. This however creates an unbalance between the investments and costs for the producer and the use by the free-riding world. For quality we require higher investments but those costs will not be covered – as already is the case. Computer assisted education has been in the doldrums for decades because of the inability of society to create the proper market structure. The solution is that countries contribute funds to either a national authority or an international authority that (i) awards contributions and (ii) tenders projects with the objective to put results into the public domain. The use of applications can be monitored and good use can be properly awarded again. Countries can do so on a national basis but then have to accept that other nations ride free on them.
- (11) The latter is actually derivative of a more general proposal. The economy will benefit much if individual creativity is released in more areas than just programming for mathematics. We may for example consider the situation of scientific publishing,

¹⁸ <http://www.fi.uu.nl/nl/> (July 2015: link relocates)

¹⁹ <http://www.mathadore.nl/> (July 2015: link likely changed to <http://www.math4all.nl/>)

²⁰ http://digischool.kennisnet.nl/community_wi (July 2015: perhaps try <http://www.digischool.nl/>)

where governments subsidize universities but the output disappears behind the gates of publishers in the private sector. Similarly, the publication of textbooks for mathematics can be managed differently. Texts would be in the public domain, awarded for that, publishers could compile courses, and be awarded for that again.

- (12) Current computer keyboards have a layout that is little better than QWERTY with a special pad for data punchers. Nowadays they could add some rows with the most relevant mathematical symbols for easy access. And a key to toggle between the Latin and Greek alphabets. Apparently the standing of mathematics is low even amongst the engineers who make the computers – it is time to enhance it.
- (13) One important feature of current computer algebra systems is that they tend to concentrate on 'getting the solution' and not on showing the intermediate steps at a level understandable for a human reader. Indeed, for $7 + 5$ that becomes 12 it is a topic of discussion what the level of understanding must be. A low level would be to show a counting list 7, 8, ..., 12. An intermediate level might use step $7 + 3 + 2 = 10 + 2 = 12$. A higher level would provide no further explanation. A highest level might actually provide some group theory. Thus, it might be argued that computer algebra for education is different than current systems. A counter argument would be that also current systems require step by step explanations, since it ought to be possible to check complex results, and since we ought to avoid the 'I don't know why but the computer gave it' mentality. It might be that current computer algebra systems and programming methods must be fundamentally redesigned to facilitate level-sensitive stepwise checking. In the mean time, though, it is possible to proceed pragmatically.
- (14) Just this June 2009, the Dutch minister of education launched its MIA (innovation agenda) and Wikiwijs, a wiki for teachers to write their own books.²¹ For reasons explained in this book, all this will not work so well for mathematics.

2015: We see this expectation somewhat confirmed. A positive surprise is that there are 2830 submissions for mathematics. However, there are "only" 16 "star collection" math course materials, for a large part developed by order from another foundation, <http://www.vo-content.nl>, financed by schools contributing 7 euros per student per year. I took a small sample. (a) A movie taken from NTR School TV explains that Pi is the ratio between circumference and ... sometimes the radius (minute 1) and sometimes the diameter (later on). I sent an email to the math co-ordinator but it came back as undeliverable.²² The "D-test" there doesn't explain what approximation of Pi must be used. I used 3.14 and failed two questions by 0.01 difference. One other question gives a plain wrong answer. (b) While the latter uses the digital environment of film and interactive testing, I looked at a course proposal by four students at TU Eindhoven for beta-didactics "Where is the ball ?" (translated), that proposes that highschool students use photographs and (graphical) perspective theory to locate a ball on a soccer field. This doesn't seem quite relevant for the current programme. Why didn't they look at EWS 2009 or COTP 2011 ?

A pleasant surprise however is *Rekentuin*²³ or English *Math(s)Garden*,²⁴ a commercial spin-off of a University of Amsterdam project on adaptive learning. This still is basic technology but at least it has attention for psychology and EBE. Of course it has $2\frac{1}{2}$ instead of $2 + \frac{1}{2}$ or $2 + 2^H$ which will require reprogramming.

²¹ <http://www.minocw.nl/innovatieinhetonderwijs/index.html> (July 2015: link no longer there, and wikiwijs.nl relocates to <http://www.wikiwijsleermiddelenplein.nl>.)

²² http://maken.wikiwijs.nl/57126/Thema__Omtrek_en_opervlakte__VMBO_KGT_2

²³ <http://www.rekentuin.nl/>

²⁴ <http://www.mathsgarden.com/> (UK English has "maths" and USA English has "math")

VI. Redesigning mathematics itself

30. Introduction

The chapters above rearrange standard material but leave known mathematics intact. The current chapter creatively innovates mathematics, in a way that is relevant for education.

I am actually not interested in doing research in mathematics. My focus for research is on economics, in scientific manner with econometrics. There have been some impulses that set me on a course that eventually caused these new results anyway.

The first case when this happened was when I was still a student of econometrics and followed lectures on philosophy, logic and the methodology of science. The logical paradoxes caused me to write a book on logic. The typescript was shelved in 1981 but turned up again in 2006 when moving house. I found time to type it over and program the logical routines in *Mathematica*. It is now *A Logic of Exceptions* (ALOE), Colignatus (1981 unpublished, 2007a, 2011). See the discussion by Gill (2008). The news is a development of three-valued logic that remains free from Liar paradoxes itself.

The second case was in 1990, at the Central Planning Bureau, when I had cause to consider Kenneth Arrow's Impossibility Theorem with respect to the voting paradoxes. The subject started as the economic question about the social welfare function to use in economic models but ended up in a rejection of Arrow's analysis. Arrow's Theorem is mathematically valid but Arrow's verbal interpretation does not cover it, and when that interpretation is formalized then it fails. See *Voting Theory for Democracy* (VTFD), Colignatus (2007b, 2014). Part of the news is also a suggestion for a compromise voting procedure that many are likely to be able to live with – the Borda Fixed Point method.

The third case arose in 2008 seeing students struggle with trigonometry. I hadn't used the subject for a long while and apparently could approach it afresh. The news is the measure Unit Meter (measure) Around (UMA) alongside degrees and radians. The functions $\text{xur}[\alpha] = \cos[\alpha \ominus]$ and $\text{yur}[\alpha] = \sin[\alpha \ominus]$ eliminate a lot of clutter and tedious calculation.

The fourth case arose in 2007 as well. While teaching mathematics, various questions had come up naturally. Most of those issues belong to the earlier chapters. While retyping ALOE and thinking about paradoxes again, the idea came up to reconsider also the paradoxes of division by zero, in particular in relation to the differential quotient and the problems encountered by students. In economics there is the distinction between statics and dynamics. In 1981 in ALOE I had already applied that distinction to (static) propositions and (dynamic) inference. This also fitted the distinction in programming between identity and equations (=) versus assignment (:=), see page 29. Thus the idea arose to algebraically distinguish the act of dividing (/) from the result after division (/), see page 31. The news is that calculus can be formulated algebraically without use of limits or infinitesimals. A proof of concept is *Conquest of the Plane* (COTP), Colignatus (2011c).

2015: The fifth case is *Foundations of Mathematics. A Neoclassical Approach to Infinity* (FMNAI), Colignatus (2015f), on set theory, number theory and infinity, see page 67. The sixth case is *A child wants nice and not mean numbers* (CWNN) (2015h), see page 68.

31. A Logic of Exceptions (ALOE)

A Logic of Exceptions, Colignatus (1981 unpublished, 2007a, 2011), is intended for use in the first year of college or university. The last two chapters require a more advanced level that is worked up to. For highschool, ALOE is advisable for teachers and textbook authors but for implementation for K12 students the notions in the book need to be translated.

ALOE provides the concepts and tools for sound inference. Discussed are: (1) the basic elements: propositional operators, predicates and sets; (2) the basic notions: inference, syllogism, axiomatics, proof theory; (3) the basic extra's: history, relation to the scientific method, the paradoxes. The new elements in the book are: (4) a logic of exceptions, solutions for those paradoxes, analysis of common errors in the literature, routines in *Mathematica*.

Logic is used not only in science and mathematics but also in business and sometimes in politics and government. Logic and inference however can suffer from paradoxes such as the Liar paradox "*This sentence is false*" or the proof-theoretic variant by Gödel "*This statement is not provable*" or the Russell set paradox of "*The catalogue of all catalogues that don't mention themselves*". ALOE explains and solves those paradoxes, and thereby gives a clarity that was lacking up to now. The author proposes the new approach that a concept, such as the definition of truth or the notion of proof or the definition of a set, also reckons with the exceptions that may pertain to its very definition. The approach to keep exceptions in the back of one's mind is a general sign of intelligence.

A quote from ALOE:

"Since the Egyptians, mankind has been trying to solve the problem of bureaucracy. One frequent approach is the rule of law, say, that a supreme law-giver defines a rule that a bureaucracy must enforce. It is difficult for a law however to account for all kinds of exceptions that might be considered in its implementation. Ruthless enforcement might well destroy the very intentions of that law. Some bureaucrats might still opt for such enforcement merely to play it safe that nobody can say that they don't do their job. Decades may pass before such detrimental application is noticed and revised. There is the story of Catherine the Great regularly visiting a small park for a rest in the open air, so that they put a guard there; and some hundred years after her death somebody noticed that guarding that small park had become kind of silly. When both law-givers and bureaucrats grow more aware of some logic of exceptions then they might better deal with the contingencies of public management. It is a long shot to think so, of course, but in general it would help when people are not only aware of the rigour of a logical argument or rule but also of the possibility of some exception."

The computer environment has these advantages:

- (a) Three-valued logic, that normally is rather opaque, can be handled now with clarity.
- (b) The student can create more complex algorithms using the routines.
- (c) ALOE has no Questions & Answers. But interactive variation is possible.

32. Voting Theory for Democracy (VTFD)

Voting Theory for Democracy, Colignatus (2001, 2007b, 2011g, 2014), can be used in college or university. The last chapters require a more advanced level that is worked up to. For highschool, VTFD is advisable for teachers and textbook authors but for implementation for K12 students the notions in the book need to be translated.

VTFD provides the concepts and tools for democratic decision making. Voting is used not only in politics and government, but also in business - and not only in the shareholders' meetings but also in teams. Voting however can suffer from paradoxes. In some systems, it is possible that candidate *A* wins from *B*, *B* from *C*, and *C* from *A* again. VTFD explains and solves those paradoxes, and thereby it gives a clarity that was lacking up to now. The author proposes the new scheme of 'Pareto Majority' which combines the good properties of the older schemes proposed by Pareto, Borda and Condorcet, while it adds the notion of a 'Borda fixed point'. Many people will likely prefer this new scheme over Plurality voting which is currently the common practice.

The literature on voting theory has suffered from some serious miscommunications in the last 50 years. Nobel Prize winning economists Kenneth Arrow and Amartya Sen created correct mathematical theorems, but gave incorrect verbal explanations. The author emphasises that there is a distinction between 'voting' and deciding. A voting field only becomes a decision by explicitly dealing with the paradoxes. Arrow and Sen did not solve the paradoxes and used them instead to conclude that it was 'impossible' to find a 'good' system. This however is a wrong approach. Once we understand the paradoxes, we can find the system that we want to use.

This book develops the theory of games (with Rasch - Elo rating) to show that decisions can change, even dramatically, when candidates or items are added to the list or deleted from it. The use of the fixed point criterion however limits the impact of such changes, and if these occur, they are quite reasonable. Groups are advised, therefore, to spend time on establishing what budget they will vote on.

See also Colignatus (2008b) *Review of Howard DeLong (1991), "A refutation of Arrow's theorem"*.

The computer environment has these advantages:

- (a) Voting routines are computationally cumbersome but can be handled now with clarity.
- (b) The student can create more complex algorithms using the routines.
- (c) VTFD has no Questions & Answers. But interactive variation is possible.

Chapter X, "*Without time, no morality*", page 81 below, has already been included in DRGTPE and VTFD but is included here again. It shows how important the mathematics of voting theory is and that it warrants a better place in the curriculum than current overindulgence in traditional misconceptions.

33. Trig rerigged (in COTP)

I am not much of a fan of trigonometry. Apparently I am neither too rational, for the smart way would be to neglect it and proceed with the fun stuff. On the other hand, it was a bad itch that felt like scratching. We already discussed the choice of $\Theta = 2\pi$, see page 27. But we can do more.

For students it is a bit confusing that angles are measured counterclockwise. It would be too complex to change this, e.g. also with derivatives. Perhaps there is a moment later on to try it but now we let this rest.

In thinking about angles, people naturally think in turns, half turns, quarter turns. Mathematicians have considered the case, and don't listen. An angle is *defined* as a plane section between two intersecting lines. But it is *measured* (in a dubious distinction with *definition*) with either (a) sine, cosine and tangent, or (b) the arc of the unit circle. A unit circle has radius 1. The circumference can be subdivided in 360 degrees, deriving from the Sumerian measurement of the year and maintained over the ages since 360 is easy to calculate with. Subsequently, it is seen as an "innovation" – the advancement of grade 11 over grade 10 – that said perimeter can also be subdivided in Θ radians.

Most mathematicians would hold that radians and π are dimensionless numbers. For example π would be defined as the ratio of a circumference $2\pi r$ to the diameter $2r$ of any circle. Since numerator and denominator are measured in the same unit of measurement, say the meter, it drops out. I would oppose this, since a 'meter around' is something else than a 'meter in one direction'. Here we have our turns, half turns, quarter turns. The *turn* is its own dimension. But does the meter really disappear? When we consider a unit circle, then that unit has to be something. Everyone can imagine a circle and also imagine a measuring rod, and each image will be quite arbitrary. But it is curious to argue that this would be without a unit of measurement – precisely since such a measuring rod is imagined too. Thus the unit would be "*unit measure around*" (UMA) and not degrees or radians. For communication it helps to use the already existing unit of measurement, the meter. It is still a choice – and I am inclined to prefer it – to let the UMA be the "*unit meter around*" dimension. When drawing a sine function the student can plot out one meter instead of measuring out $\Theta = 6.28\dots$ meters. We can also use a circle with a circumference of 1 meter and a radius $r = \Theta^H \approx 16.16$ cm.

We cannot wholly eliminate the unit circle because of sine and cosine and their neat derivatives. Sine and cosine are OK for triangles in arbitrary orientation too. With co-ordinates, they indicate y and x on the unit circle. Thus let us *call them* so too.

Figure 4 and **Figure 5** give the situation:

- Put *angle* α on the Unit Circumference Circle (UCC) a.k.a. the *angular circle*.
- Take co-ordinates on the Unit (Radius) Circle (UR). On UR there is *arc* $\varphi = \alpha \Theta$.
- Find $x_{ur} = x_{ur}[\alpha] = \cos[\alpha \Theta]$ and $y_{ur} = y_{ur}[\alpha] = \sin[\alpha \Theta]$. These functions thus translate the α turn into the $\{x, y\}$ co-ordinates on the unit circle.

It remains to document this further and to show that exercises become more tractable. I have considered including the paper Colignatus (2008a) in this book but the proof of concept is given in COTP (2011).

NB 1. π not only clutters traditional expressions but those expressions also implicitly use π to indicate the measurement in radians, letting students guess. NB 2. Textbooks manage to write $\sin(x)$ and $\cos(x)$ where x then both signifies the angle and the co-ordinate. NB 3. A definition $\sin[\varphi] = y/r$ doesn't give a function but an equation to solve.

A typical question is: Solve $\cos(\varphi)^2 - \cos(\varphi) = 0$. Solved by: $\cos(\varphi) (\cos(\varphi) - 1) = 0$. Thus $\cos(\varphi) = 0$ or $\cos(\varphi) = 1$. Thus $\varphi = \pi/2 + k\pi$ or $\varphi = 2\pi k$ rad.

This now becomes: Solve $\text{xur}[\alpha]^2 - \text{xur}[\alpha] = 0$. Solved by: $\text{xur}[\alpha] (\text{xur}[\alpha] - 1) = 0$. Thus $\text{xur}[\alpha] = 0$ or $\text{xur}[\alpha] = 1$. Thus $\alpha = \frac{1}{4} + \frac{1}{2}k$ or $\alpha = k$ UMA. Less cryptic: $\alpha = 0, \frac{1}{4}$ or $\frac{3}{4}$, and each subsequent full turn from there.

(Preferably though $\alpha = 4^H + k 2^H$ or $\alpha = k$ UMA. Less cryptic: $\alpha = 0, 4^H$ or $3 \cdot 4^H$, and each subsequent full turn from there.)

Figure 4. Angular circle ($r = \Theta^H$), unit circle ($r = 1$), $x = \text{Xur}$ and $y = \text{Yur}$

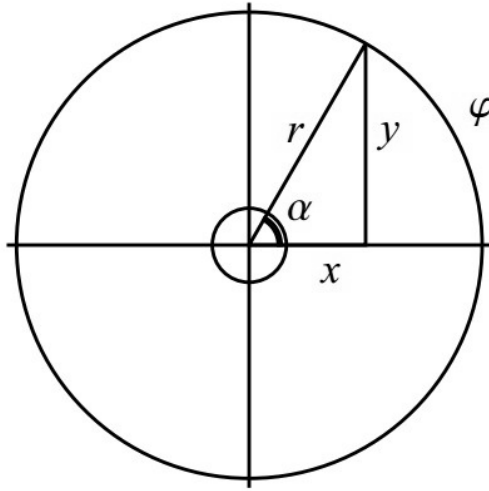
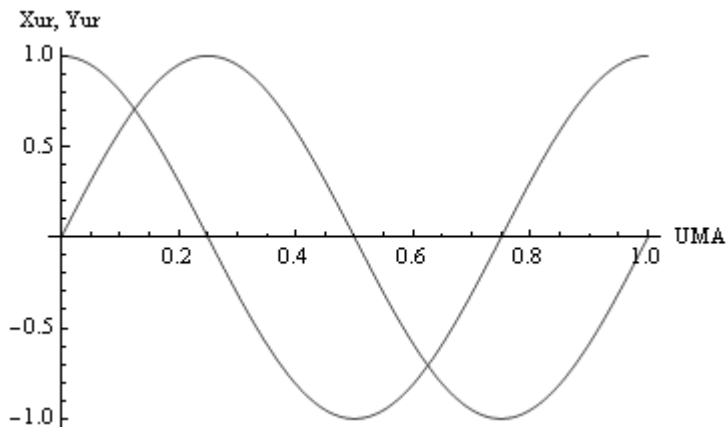


Figure 5. The functional graphs of Xur and Yur



34. The derivative (in COTP)

Calculus can be developed with algebra and without the use of limits and infinitesimals.

Define y / x as the “outcome” of division and $y // x$ as the “procedure” of division (see page 31). Using $y // x$ with x possibly becoming zero will not be paradoxical when the paradoxical part has first been eliminated by algebraic simplification. The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand for the derivative $\lim(\Delta x \rightarrow 0) \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using $\Delta f // \Delta x$ on the formula and then extending the domain with $\Delta x = 0$, and subsequently setting $\Delta x = 0$ is not paradoxical at all. Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $|x|$ traditionally is undefined at $x = 0$ but would algebraically be $\text{sign}[x]$.

This longer discussion can best be put in a separate chapter, see page 89. That discussion improves upon a version of July 2007 on my website.

2015: That latter discussion is *top-down*. The book *Conquest of the Plane* (COTP) (2011c) gives a *bottom up* proof of concept.

35. Set theory, number theory and infinity (FMNAI)

The target readership of *Foundations of Mathematics. A Neoclassical Approach to Infinity* (FMNAI) (2015f) are (1) students with an interest in methodology of science and the foundations of mathematics – for example students in physics, engineering, economics, psychology, thus a broad group that uses mathematics and not only those majoring in mathematics – and (2) fellow teachers of mathematics who are sympathetic to the idea of bringing set theory and number theory into general mathematics education – while avoiding the *New Math* disaster in the 1960s in highschool.²⁵

Readers would be interested in:

- (A) Constructivism with Abstraction, as a scientific methodology
- (B) Particulars about infinity and number theory, within foundations and set theory
- (C) Correction of errors within mathematics on (B) caused by neglect of (A).

Other readers are (3) research mathematicians, but while they would benefit from the last correction in (C), they must mend for that they are not in the prime target groups. They would start with FMNAI:61-72 and then restart at the beginning of FMNAI again for methodology of science.

Set theory and number theory would be crucial for a better educational programme:

- (i) They greatly enhance competence and confidence
- (ii) They open up the mind to logical structure and calculation also in other subjects
- (iii) They are fundamental for learning and teaching themselves.

The world can be amazed that (A) and (B) are not taught systematically in current school and first year higher education. There are two explanations. One is the mentioned *New Math* disaster in the 1960s. Another more hidden cause are the *transfinites* created by Georg Cantor (1845-1918). When a mathematics teacher starts on the topics of set theory and numbers, then infinity comes up too, and then he or she feels obliged to discuss these transfinities. However, her or she also feels doubt whether these should be taught. For highschool and first year students they might be too complex and paradoxical. People in real life have no application for these transfinities and it makes little sense to have transfinities in the highschool diploma. They are relevant purely for mathematicians – and for a particular branch of mathematics as well. Thus, mathematics teaching is stuck. A mathematical *curl* causes so much complexity and irrelevance *that the wonderful basics are not taught*. FMNAI proposes to cut the knot.

FMNAI adds the bitter irony that Cantor's analysis appears to be misguided. Neglect of (A) made generations of mathematicians blind to some crucial errors. This book *Elegance with Substance* already criticised the mathematical blindness to the outer world but now we observe this blindness to the foundations of mathematics itself.

²⁵ https://en.wikipedia.org/wiki/New_Math

36. A child wants nice and not mean numbers (CWNN)

The book *A child wants nice and not mean numbers* (CWNN) (2015h) is another spin-off of EWS 2009.²⁶

It was all caused by Gladwell (2008:228):

“(...) we store digits in a memory loop that runs for about two seconds.”

English numbers are cumbersome to store. Gladwell quotes Stanislas Dehaene:

“(...) the prize for efficacy goes to the Cantonese dialect of Chinese, whose brevity grants residents of Hong Kong a rocketing memory span of about 10 digits.”

Gladwell on addition:

“Ask an English-speaking seven-year-old to add *thirty·seven* plus *twenty·two* in her head, and she has to convert the words to numbers (37 + 22). Only then can she do the math: 2 plus 7 is 9 and 30 plus 20 is 50, which makes 59. Ask an Asian child to add *three·tens·seven* and *two·tens·two*, and then the necessary equation is right there, embedded in the sentence. No number translation is necessary: It's *five·tens·nine*.” (Hyphens replaced by middle dots.)

It is remarkable how comments like these can spark one's imagination and cause one to look deeper into a matter.

CWNN comes with a warning: my background is teaching for highschool and first year of higher education, and not elementary school. Research for the latter is a field of itself. Thus I am not knowledgeable here, and am very hesitant to say anything about this at all. However, from the perspective of highschool one may ask questions about the entry level, and whether some things might not be done already in elementary school. Researchers for elementary school might take an interest in some ideas. With this warning, CWNN proceeds with wondering what might be done in elementary school.

Some propositions are:

- (1) When we take *two·ten·two* for 22 as the proper pronunciation in mathematics (with position *ten* rather than multiple *tens*), then English becomes a dialect. For teaching it is quite a different perspective to identify English as such.
- (2) The positional system is under-utilised in education. A more developed treatment support counting and arithmetic. A sign language with hands and fingers should also take account of the positional system. I looked on the internet, did not (quickly) find such a "positional sign system", and CWNN contains a proposal for research.
- (3) If pupils in First Grade can understand the positional system with *two·ten·two* for 22 then they would also understand the formula $2 \times 10 + 2$, whence they would understand multiplication. Research should clarify what they really understand about multiplication.
- (4) Some more prospects are w.r.t. division, negative numbers, subtractions, geometry, co-ordinates, vectors and the Pythagorean Theorem.
- (5) Elementary school is served by a better understanding of Van Hiele didactics.

²⁶ There first was a memo of three papers in English (2012) and a Dutch book on the local situation (2012a). The new English book CWNN (2015h) skips the Dutch locality and includes some papers that have been written since 2012.

VII. Questions for evidence based education

37. What to test ?

A new trend is evidence based education (EBE), by analogy of evidence based medicine, while the stock market crash and economic-financial crisis has caused the call for evidence based finance. In academic hospitals care, study and training are combined, and by analogy we best get *academic schools* where education of pupils is combined with study on their education and training of their teachers. A good friend of mine has warned though that pupils and students tend to be much too diverse, not only across time and culture, but also in personal histories, to allow for much accuracy even with huge sample sizes. Thus let us be cautious. And let us be aware of the issues of equity involved – which kids will get the increased attention ? Thus, the problem of the sample size is a basic empirical bottleneck, and the best answer is the institutional set-up, with said model of the *academic school*, with the medical school as example for education in general.

The institutes of education themselves can be subject to closer study too. A study on institutional set-up may be easier and more productive than studying specifics (e.g. textbook A page x versus variant B page y).

38. Test questions

Some issues crossed the mind as suggestions for such *research on institutional set-up*:

- (1) School organization depends crucially upon the concierge and other facilitators. Generally their wage costs are out of line, causing a reduction of services such as opening hours. The economic analysis in DRGTPE helps to free resources.
- (2) Schools follow a model developed in medieval times for the elite, with full time learning. Why not allow an integration of work and study at already younger ages ?
- (3) Dronkers ²⁷ observed: In a greying society, the stock of teachers is confronted with fewer students, which might cause schools to allow more students into the higher tracks of education, causing a drop in general quality.
- (4) The greying of society and the rapid development of ICT affects the gap between teachers and students, between what is done and could be done.
- (5) Teenagers apparently have a different biological clock.
- (6) European textbooks still do not deal properly with backgrounds of migrants.
- (7) There can be more democracy at schools, see VTFD.
- (8) Empowerment of teachers will affect quality. Will teachers have influence on what questions are researched in EBE ?
- (9) I already referred to Gladwell (2000) (2008). I found this also illuminating on: (a) organization size of 150 people, (b) enrollment per half-year instead of per year, (c) too long summer vacations (at least in the USA), (d) Asian counting, (e) rice paddies and the impact of persistence on math competence.

²⁷ <http://www.eui.eu/Personal/Dronkers/> - lost the actual reference. Dutch / Flemish: <http://www.o-zon.be/teksten/proefdrukmanifest/profjaapdronkers/index.html>

39. Number sense

The section in EWS 2009 on *number sense* has gotten a life of its own and has become the book *A child wants nice and not mean numbers* (CWNN) (2012, 2015h), see page 68.

A bit more can be said from the angle from EWS. The study of number sense is an active field of research, with valuable outcomes. The critical comment is that researchers on number sense are at risk of taking traditional mathematics as their frame of reference. There is every reason to question this tradition however, and give pride of place to empirical observation. Mathematics may well be in need of re-engineering.

We already observed that English is a dialect of mathematics.²⁸ It makes a huge difference both in class and research whether English is presented as the ideal standard or as a (crummy) dialect. The issue of pronunciation has a quick fix: Use the Cantonese system and sounds for numbers. It would be good EBE to determine whether this would be feasible for an English speaking environment (perhaps begin in Hong Kong).

Other news is: (i) For negative numbers and subtraction, see here.²⁹ (ii) Later on, I realised that fractions abuse the rank order names: e.g. rank order *fifth* is abused for a *fifth*. There is now the proposal to use $1/x = x^H$, and pronounce this as "per-x".³⁰ (iii) See here for an overview of pronunciation, addition, subtraction, multiplication, division.³¹

A deeper issue is that the West writes and reads text from the left to the right while Hindu-Arabic or rather Indian numbers are from the right to the left. Thus *fourteen* is 14.

English already adapted a bit, with *twenty one* and 21. Dutch still has "*een en twintig*" up to "*negen en negentig*". From hundreds onwards Dutch follows the Indian too, for example "*vijf honderd een en twintig*" (521). French of course still has the special "*quatre-vingt*" for 80 and "*quatre-vingt-treize*" (80 + 13) for 93.

There are two ways to consider a number. 37 can be seen as a series of digits only and pronounced as *three·seven* – like specifying a telephone number – or it can be weighed as *thirty·seven* or *three·ten·seven*. Looking at the digits only gives some freedom to consider alternatives. Could we write numbers in the opposite way? Let us use the word **Novel** for when we write <123> for the Indian number 321 (and try not to get confused). To distinguish Novel from the Indian we may also write the digits in mirror image (perhaps as they are intended to be read if in different order). Thus 19 becomes 91. It does not take much time to get used to and **Table 3** contains the first practice.

Table 3. Novel versus Indian, in notation and addition

1231	1234
123	567
88	89
0881	1890

Overflow in Novel is processed neatly in the reading direction. This is straightforward.

Two conclusions from CWNN are: (1) The need to *size up* the number (for speaking) conflicts with any writing order. The whole must be judged anyway. (2) We will not quickly drop the Indian numbers and writing order. But EBE on these aspects will help a lot.

²⁸ <https://boycottholland.wordpress.com/2012/04/01/english-as-a-dialect-of-mathematics/>

²⁹ <https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/>

³⁰ <https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/>

³¹ <https://boycottholland.wordpress.com/2014/08/25/confusing-math-in-elementary-school/>

40. Memory

Yates (1974) relates that society used to be built upon the training of memory. Orators like Cicero are inspiring examples but law makers, lawyers and bureaucracy alike in the ancient and medieval world would require it in mundane fashion. The art of memory tended to rely on the trick to foster memorabilia and associate new matters with those. One could for example visit a temple or church, memorize the statues and their locations, and associate the steps of a mathematical proof with the separate points along the physical walk. The art of memory was embedded in a wider culture of learning, philosophy and ethics, in which, indeed oratory played an important part. However, when the printing press was invented and the abundance of bibles facilitated the rise of Luther and Calvin, with reliance on the bible instead of authority, the protestant iconoclasts did not only destroy the statues in the churches but also their images in memory, since also the classical education was reformed and pruned from the old ways. Society became dependent upon the printing press, a world faded and the art of memory with it.

For evidence based education it would be interesting to determine whether a rekindling of perhaps some modified form of the Art of Memory would not be beneficial.

PM. Symbols and notation in mathematics are also anchors for memory, which explains part of their importance. Writing perhaps started from accounting and subsequently was hijacked by the literary people who now regard anything that isn't text as an abomination. See Barrow (1993).

VIII. Re-engineering the industry

Introduction

Countries differ in histories, regulations, organizations, conventions. I am only vaguely aware how they differ. It is relatively easy to download material on mathematical content from the internet but it is rather more complex to understand the situation elsewhere. My base is Holland and I only tentatively write for an international audience, precisely to get more abstraction. Readers from other countries will go for the abstraction but may nevertheless find some aspects interesting that pertain to Holland.

Goal

Economists distinguish between competitive markets where participants have no influence on price and quality, and non-competitive markets such as oligopoly and monopoly where participants have influence. A hybrid combination is monopolistic competition where products are so special that each seller is a monopolist in the niche while buyers are budget constrained and still have to choose amongst sellers.

Our subject is the education in mathematics in a country. Education is quite specialised and thus non-competitive with many features of monopolistic competition. A market like this cannot be left to itself and requires a market manager and clearing house. Markets for food and medicine are already quite regulated and the same would hold for education. Economics emphasizes the advantages of free enterprise and competition. People should be free to set up a school, appoint teachers, collect materials and enroll students, and hope that employers accept the graduation certificates. But there are standards and the market only works well if properly regulated. Aspects are didactics, quality, norms, levels, standards versus implementations, evidence based education. Projects must be contracted out, managed, evaluated. There are economies of scale and scope while freedom can be enhanced by smart social engineering. For example, products can be acquired centrally and put in the public domain.

It is useful to have a market manager and clearing house for the education in mathematics. There is a letter soup of existing organizations for niches, and their role needs monitoring and evaluation.

Governance

A new design can be this: The Ministry of Education would supervise education in general only. For the branch of the education in mathematics there would be a national institute named *Mathematics Education Name of the Country* (MENC) – like the national statistical offices have managed to call themselves *Statistics Name of the Country*. The MENC runs ME. The MENC will also have the authority to set the standards, specifications and details of the computer algebra language in the open domain that is also used in education, obliterating any claim by commercial parties, also potential claims based upon the past.

The **MENC Council** is open to society. It has seats for (1) representatives of (a) parents, (b) pupils and students, (c) business and labour, (d) the arts and the media, (2) presidents of recognized associations of (e) mathematicians in general, (f) teachers of mathematics, (g) institutes of education (employers), (h) professions who use mathematics, (i) producers of educational materials such as textbook authors and programmers, and their publishers.

The **MENC User Parliament** consists of (a) teachers of mathematics and (b) producers of educational materials such as textbook authors and programmers. Each year a quarter is replaced by elections in the constituencies.

The **MENC Executive** has at least a quarter of its employees in parttime teaching.

Finance

Finance comes from the national treasury. Reasons are: (i) Bildung, (ii) key role for other subjects, (iii) economies of scale and scope, (iv) contribution to the national economy, (v) necessity. The necessity follows from the economic observation above. Improvements don't come about when there are no funds. Teachers are no entrepreneurs. They write and teach and can program software but this remains fragmented in niches when there is no organization and when there are no funds.

A Dutch experience

Bear with me. In Autumn 2008 I proposed to create a *Simon Stevin Institute* (SSI) for this basic infrastructure, see Colignatus (2008c) – when the idea to call it *Mathematics Education Netherlands* (MEN) had not occurred yet. Independently and at almost the same time, Poelman et al. eds. (2008) came with a *Masterplan Wiskunde* (MPW) with main support by (President of the Royal Academy of Sciences) Dijkgraaf, (Social Economic Council chairman) Rinnooy Kan, and (internationally known mathematician) (J.K.) Lenstra. My budget is EUR 10 million per annum and the masterplan requires EUR 18.5 million but does more on female participation (WoMEN ?). One development following that masterplan is the creation of a *Platform Wiskunde Nederland* (PWN) where two mathematical associations KWG and NVvW start working closer together to reduce fragmentation. The main difference is that MEN / SSI opens up the world of mathematics to society at large while MPW considers itself fantastic and wants to do more *public relations* to the multitudes out there who do not understand yet that mathematics is so important. Interestingly, mathematicians have a captive audience of the whole population during six to twelve formative years, but they still manage to foul it up and then conclude that the cause must be not *us* but *them*.

The Dutch Minister of Education, Culture and Science, Plasterk was so kind to react to this suggestion of a MEN / SSI and even kinder to qualify it as “interesting and thoroughly developed” (letter 2009-11-26, BOA/EBV/82918). His reaction is that it would create a new layer of superfluous bureaucracy with respect to the various existing institutes. Clearly I didn't explain sufficiently clear that the MEN / SSI has been targetted to actually reduce bureaucracy. Perhaps this book gives a second chance. Hopefully we have our integrated textbook / computer algebra environment by 2015.

(2015: Not yet.)

I agree with one idea of *public relations*. Other subjects like physics, economics and psychology depend upon mathematics. Their professors will be respected by mathematicians. I move that some of the *masterplan* funds are used to distribute copies of this book to them. In the kind and warm light of reason flowers will grow.

IX. Beating the software jungle

The need for a common computer algebra language

As said:

“The MENC will also have the authority to set the standards, specifications and details of the computer algebra language in the open domain that is also used in education, obliterating any claim by commercial parties, also potential claims based upon the past” (page 73).

This authority is useful (a) to set a common standard, (b) to prevent any confusion or commercial hold-up. Above on page 57+ we already saw the importance of the computer algebra language. Its notation must fit the textbook. It must be uniform across schools for economies of scale (more students) and scope (more applications) but primarily for didactic reasons – in that pupils and students do not switch easily between formats. See how hard it is to switch between traditional 2π and proposed Θ & π . For example, society regulates that cars have (at least) four wheels, mirrors, brakes, drive on the one side of the road, and such. In the same way there is a national committee on spelling the language – not a popular committee though – since it matters both for education but rather also for legal documents. We need similar rules for doing mathematics on the computer.

The problem

There is the distinction between the single common language and various commercial engines that can interpretate the language and evaluate it to produce results. The engines are the place for commercial competition. The problem that occurs is that commercial companies start mixing the two.

The major topic of this chapter is the commercial appropriation of the language of mathematics. The computer algebra languages are mainly created in the USA where there is a strong litigation culture. Such companies have a tendency to evade conflicts of copyright by creating new issues of copyright. By consequence it becomes rather impossible to do mathematics on the computer without paying for copyrights.

There is a good language available

In 1993 I selected the commercial computer program *Mathematica* because it seemed better, closer to the language of mathematics. I have been using this program consistently since then. Looking at alternatives again in 1999 and 2009 still gives the same conclusion. The language used in the *Mathematica system for doing mathematics on the computer* is a straightforward implementation of the age-old mathematical conventions. There are some particulars but that is because people differ from computers, or that computers differ from other environments.

When mathematics adapts to the environment – speech, wax or clay tablets, papyrus, blackboard, printing press, typewriter, computer – then this does not imply copyrights for any particular firm. Mathematics is free for common use and without copyrights.

Above, the open source Sage / Python language has been noted. The suggestion was to start using this and work from there. The key words are “work from there”. In other words, Sage / Python is not perfect. In particular, as a language Sage / Python appears rather

ugly. The main question then is: why not use *Mathematica* ? Because it would be copyright protected ? Would you really be able to copyright mathematics ?

Other people have (developed) a preference for other (cheaper) computer algebra languages such as Maple or Maxima or Wiris. For the present discussion this is immaterial. In the following I shall write "*Mathematica*" and "WRI" (Wolfram Research Inc., the makers of *Mathematica*), and the reader can substitute the personal preference. What is important is that society arrives at a standard computer algebra language for education.

The major policy questions

The major policy questions remain:

- (i) Will society accept appropriation of the language of mathematics by WRI ? Will it accept a possible commercial claim by WRI on the mathematical language used in the *Mathematica system for doing mathematics on the computer* ?
- (ii) If society accepts such a claim, will it accept the associated costs of using *Mathematica*, or incur the costs of alternatives (including the costs of an alternative language for mathematics) ?
- (iii) If society does not accept such a claim, will it stimulate other producers to create engines that use the language of mathematics on the computer ?

These policy questions are answered either explicitly or implicitly. Current decisions are left to the unregulated oligopolistic market. By implication choice (ii) surfaces, with the associated high costs.

In 1999 and 2000 I wrote two papers on these policy questions. I will restate the summaries and provide the proper links to where the papers can be found. They are dated with respect to particulars but still relevant on the analysis and choice criteria.

A suggestion done in 1999

The summary of my paper *Beating the software jungle. Selecting the economics software of the future* Colignatus (1999) reads:³²

"Currently there is a jungle of software for economics, for both professional and educational software, and including the supportive mathematics and statistics. A comparison of 1993 showed and now in 1999 shows again - at least to this author - that *Mathematica* is the most useful and promising software, both for its elegant language and its breadth of application. A problem with *Mathematica* is its current price of about \$1500 for a professional licence. Part of the solution would be to separate the language and interface and the engine. Once the *Mathematica* language is adopted as the lingua franca of science software, for which there are no legal barriers, there can be competition in front ends, interpreters and compilers. Another part of the solution in the short term would be coherent and determined discussion of the economics community (software users and purchasing departments) with Wolfram Research Inc. (WRI), the makers of *Mathematica*. Also, as there might still be a (natural or lock-in) monopoly, there could be regulatory action that creates a public service utility. WRI could name its price for becoming a public utility company, and we might see whether *Mathematica* users are willing to pay that."

The current price of *Mathematica for students* is EUR 160. As said Sage / Python is the available open source program and we may use it to get going. As a language it is pretty ugly but beggars can't be choosers.

³² <http://econpapers.repec.org/paper/wpawuwpgt/9904001.htm>

(2015: The current price of a desktop *Mathematica for students* is EUR 130.)

An analysis done in 2000

The other relevant article is *The Disappointment and Embarrassment of MathML* Colignatus (2000), with summary:³³

“W3C is about to release MathML 2.0. This should have been a joyous occasion, but it appears to be a horror. They created a horrible way to do mathematics on the internet. It is Byzantinely complex, unintuitive, unesthetic, highly undocumented, it requires complex software support, etcetera. A quite perfect alternative already exists in *Mathematica*: simple, elegant, intuitive, highly documented etcetera - and users of Maple may think similarly about Maple. W3C is reinventing the wheel, making it square, and putting the horse behind the cart. Their talk about providing a ‘service to the scientific and educational community’ is pure nonsense, as they precisely do the opposite. The real reason why W3C developed MathML is (a) that they didn’t do their homework, (b) that they didn’t *really* deal with the makers of *Mathematica* (or Maple). We can only solve this situation by have a serious discussion of the copyright status of mathematics. A short run pragmatic solution is to use a `<mathematics use=Mathematica>` and `</mathematics>` bracket in HTML (with possible other values, like Maple). This may be ‘expensive’ in the short run, but much cheaper and beneficial in the longer term. Update: This discussion now includes answers to reactions of others. Readers should keep in focus that this paper concludes to the proposal to the scientific community that we have a discussion on the question: Are we going to accept this gift from W3C, or is it something like the Trojan horse, that will actually destroy the intellectual freedom of mathematics ?”

The paper is dated on some aspects but still valid on the situation and the criteria. For example:³⁴

“The expression $(a+b)^2$ in MathML is to read as (see op.cit. for the explanation):

```
<msup>
  <mfenced>
    <mrow>
      <mi>a</mi>
      <mo>+</mo>
      <mi>b</mi>
    </mrow>
  </mfenced>
  <mn>2</mn>
</msup>
```

Conversely, the *Mathematica* Inputform is: $(a+b)^2$.”

The MathML argument is that the latter is ambiguous between an exponent, an index or a footnote. They neglect (1) that that ‘ambiguity’ does not arise in *Mathematica* (there it is already defined to be InputForm), (2) that *Mathematica* already works on the computer, and (3) that MathML then doesn’t deal with input by people.

The true story of MathML is that the math community is afraid of copyright claims. The reason to recall this is that it may happen again now with Sage / Python. ‘Open source’ sounds like a bargain but society may fall in the trap of *penny wise, pound foolish*.

³³ <http://thomascool.eu/Papers/MathML/OnMathML.html>

³⁴ See also <http://www.w3.org/Math/>

The larger picture is a lack of regulation, at the cost of the freedom of mathematics and the education in mathematics. From this 2000 paper:

“The general idea of this paper is that dealing with the language of mathematics is an issue of market structure. The current W3C solution is to “program around” market structure. The W3C solution would be ‘open’, and all other languages might be turned into property rights. This is an approach that is in direct violation with the tradition of mathematics itself, and that might indeed cause a market structure that we would not want.”

Advance in 2009 from 1999-2000

In 2009 there is no change on the fundamental data since 1999-2000:

- (1) In terms of language, mathematics is free for common use and cannot be put under copyrights. The commercial product *Mathematica* uses a language that is a straightforward implementation of the age-old mathematical conventions.
- (2) There remains the distinction between the single common language and various commercial engines. The engines are the place for commercial competition.
- (3) The news in 2009 is (a) that Maple T.A. has advanced in the field for testing of pupils and students, where WRI, the maker of the *Mathematica* engine, apparently is absent, (b) that Sage / Python now are available as open source environment and engine.
- (4) As language, Sage / Python is no improvement. Well, if something has already been done, it is hard to beat it, especially when you are afraid of copyright issues.
- (5) Sage apparently could be produced quite quickly by use of the various bits and pieces of software that various mathematical programmers had already put on the internet. It still remains quite an enterprise to further develop and support it for a great variety of potential users. It may be doubted whether the open source community can provide the support on the applications that are required for education. The current community of users of Sage seems to be more of the variety of computer-wise math university students and graduates who differ, it may be noted, from junior high pupils.

Apart from the sad conclusion that the news indicates progressed fragmentation, it also reflects the tough choices facing the math community and educators in mathematics.

Comparing some costs

As an example of costs: It might be cheaper when each pupil or student buys a copy of *Mathematica* at EUR 160 than:

- (a) all the work to create Sage / Python (well, OK, it has already been created, but then the subsequent versions)
- (b) suffer the difficulties and limitations of the Sage / Python language and engine (see for example the recent discussion that x / y should stand rather for normal division instead of giving the floor integer)
- (c) suffer the (temporary) differences for pupils and students between Sage / Python and the Maple T.A. testing environment. (If this point has much weight, the overall choice might be Maple instead of Sage / Python. I have not checked what its current price is.)
- (d) create all kinds of applications (such as an own testing environment but e.g. also for economics and physics) but eventually change those again to the language as used

in *Mathematica* anyway because of its more agreeable character. (In this scenario, a language interpreter is put on top of Sage, thus still with a non-integrated engine.)

Relevant are also the costs when we don't do anything. The above assumes the optimistic scenario that Sage / Python is selected so that at least something will happen. It is more likely though that stagnation and fragmentation continue if parliament doesn't re-engineer the industry.

Wolfram Alpha and Google

The May 2009 release of <http://www.wolframalpha.com/> is one additional step in the ongoing discussion. See the examples there. From the website:

"Today's Wolfram|Alpha is the first step in an ambitious, long-term project to make all systematic knowledge immediately computable by anyone. You enter your question or calculation, and Wolfram|Alpha uses its built-in algorithms and growing collection of data to compute the answer."

For the generations of students that are getting used to Google, the inclusion of *Mathematica* into the search engine or the world of searching can have a powerful effect. One consequence is that computer users have a single line *Mathematica* interpreter available now. For example $\sqrt{45}$ is evaluated both perfectly and in numerical approximation. Users may get used to the *Mathematica* input language. Programming via including multiple lines is not possible (yet), however.

Managing the industry

I did not perform a survey in the mathematical industry how they think about these issues. This is beyond my means and a bit beyond the immediate relevance. It is rather useless to ask views when people are not aware of the issues. For Holland, a good point of reference is the *Masterplan Wiskunde* (MPW) by the Dutch academic mathematical community (see on page 74). The plan does not mention computer algebra. It mentions an initiative without additional budget for more co-operation in the exact sciences on computational science, which is something else. As said the plan also mentions more attention from the academia for highschools but one of the major instruments is *public relations*.

Let us state some common sense hypotheses on views in the different layers:

- Kids in elementary school would actually already be able to use computer algebra, as they learn arithmetic and, according to Van Hiele, can master vectors. But teachers at elementary schools will hardly be aware of computer algebra.
- Teachers at highschools are aware of its existence but will still have little use for it. In Holland, highschools got stuck by selecting the graphical calculator. It is hard to get out of this because of the software jungle and the divergence in lock-in interests.
- Professors at university will focus on 'real math' and will see computers as interesting topics for computer science only. For highschool math they rather want to see the same. They are not bothered much by students outside of mathematics, except that if non-mathematics students get math then they must still be taught by real mathematicians.

By consequence the important contribution of computer algebra for highschool pupils and non-mathematics students at university or college is lost. The didactic importance of algorithms, interactivity and feedback, the quality difference in math instruction between graphical calculators and computer algebra, the advantages of computerized testing (at liberty), the integration of subjects: the industry will not be interested.

Which has indeed been the case for the last 15 years.

A caveat

Mathematica's quality got me to use it. And keep using it. Another relevant quote from (1999):

"While the discussion is open minded, it turns out that it still centers around *Mathematica*. The reader should be aware that a lot of my work thus is with *Mathematica*, and I even sell application software for it, see (...), so that I may have a personal lock-in bias. Please check whether I am still level-headed. Please be aware too, that I do not want cross relations with Wolfram Research Inc. (WRI), the only providers of *Mathematica*, the product that my work relies on. So when I suggest to differentiate and to lower the price of the product, to separate the *Mathematica* language from front end and engine, and perhaps cutting up the company, I may still be biased in trying to be friends."

2015: I don't have the idea that something essentially changed since 2009.

The big surprise is the UK, with the decision to have programming in the curriculum.³⁵

³⁵ <http://www.bbc.com/news/technology-29010511>

X. Without time, no morality

Summary ³⁶

Theory shows that voting is subject to paradoxes, while it also appears that a voting result is caused as much by the procedure as by the voters' preferences. From a moral point of view, the choice of the procedure then is the major issue. A key insight is that morality presumes time. In a static world everything is given and there is no place for individuals who have to ponder their moral choices. The real world is dynamic however and the most challenging voting paradoxes concern budget changes. The paper develops a new "Borda Fixed Point" mechanism that provides a better protection to surprises by such budget changes. Under dynamics, Saari's argument on symmetry is less convincing.

Introduction

The currently accepted view is sometimes expressed as that 'there is no ideal voting scheme'. The [former chapter in VTFD] destroyed that view. There is no mathematical reason to think that such an ideal cannot exist. Since Arrow's axioms must be rejected, they do not form an ideal. An ideal still can exist, but apparently it is different than originally thought. Perhaps people have different ideals, but then the non-existence of a common ideal derives from empirically different opinions and not from mathematical reasons. Since people can benefit from co-operation, they can still aspire at a scheme that all can agree upon.

Above analysis does not answer the positive question yet what would be a generally good system. The main point here is that everyone should determine this for oneself. Theory can only help to remain consistent. The following is a suggestion for a scheme that is consistent and that could appeal to many.

Control of natural forces in the social process

One important idea is that time plays a role. The basis for this idea is that, abstractly, morality presupposes time. Without time there would be no morality. In a static world everything is given, and there is no place for an individual who has to ponder his or her moral choices. As economists, we can draw static utility functions and isoquants, but those are abstractions, and they might distract from the real moral problem. The moral problem is that *now* a decision has to be made while the consequences appear *later*. Afterwards, everything can be explained deterministically (which is the meaning of 'explanation'), and by hypothesis, determinism will also hold for the future. Yet, in the mean time forecasts are imperfect, there is fundamental uncertainty, and that creates the possibility of morality (or the illusion of morality).

Economic science is intended to help explain reality. In this reality, we see an evolution of human beings in a social process of natural forces. The basic concept is power, in a continuous process, so that the basic approach uses ratio scales and cardinal utility and not ordinal scales. Other assumptions than cardinality enter the discussion only when the group wants to control power, and for example introduce democracy. A common notion is that economists reject cardinality and interpersonal comparison of utility. However, the concept of 'one person, one vote' actually imposes some interpersonal comparison of utilities. Also comparing orderings of preferences implies some comparison of utilities.

³⁶ *Voting Theory for Democracy*, p 251-260

The proper perspective is rather that cardinality is deficient since people can cheat about their preferences (at least in the current state of technology). The major argument for ordinality is that it limits the room for cheating. If people could not cheat, interpersonal comparison likely would be much more popular amongst economists. The point that ordinality reduces interpersonal comparison thus seems less relevant than the point that cardinal comparisons are unreliable since people can cheat.

For example, when a family goes on holiday and has the choice between Spain or Greece, then little Robby might exaggerate his preference for Greece and say that he might as well die when Spain is selected. When the aggregation of preferences would be cardinal, such a huge negative weight for one option would certainly block it. Imposing ordinality limits the impact of cheating however. In common textbooks on voting theory, cheating comes in relatively late, but it is more adequate to start right away with that notion. The crucial insight is: Arrow's Theorem and the voting paradoxes are the price that we have to pay in order to limit that impact of 'strategic' voting behaviour.

Arrow's original question whether there could exist a generally good voting mechanism remains a valid question, though. As history has shown, mathematicians are proficient in identifying paradoxes and in deriving new impossibilities, and one will not quickly find a suggestion for a generally good system. But it appears that when we consider the issue of time, then a solution tends to suggest itself. To understand this solution, it is useful to first consider three main contenders, i.e. the 'traditional' solutions provided by Plurality, Borda and Condorcet. There are other methods, but their properties are such that they need no consideration here.

Three traditional methods

In **Plurality**, all voters have one vote, and the candidate with the highest number is selected. Note the problems with this method. The criterion of 'highest number' does not imply that the winner must also have more than 50% of the vote. If this is additionally imposed, then this may require more rounds of voting, and then there is the difficult issue whether candidates have to drop out, and if so, how.

Borda's method is to let each voter rank the candidates by importance, then assign weights given by the rank position, to add the weights per candidate for all voters, and then select the candidate with the highest value. Note that the method appears sensitive to preference reversal, see below.

Condorcet's method is to vote on all pairs of candidates, and to select the one who wins from all alternatives. Note that such a "Condorcet winner" does not need to exist. In that case the margins of winning can be used to solve the deadlock - but this increases the sensitivity to who participates.

The following example is taken from Saari (2001ab). Consider a budget of three candidates *A*, *B* and *C*, and let there be 114 voters. When we neglect indifference and use strict preference only, then with 3 candidates there are $3! = 6$ possible ways of ranking them. **Table 4** contains an arbitrary allocation of those voters over such preferences. The highest ranking candidate gets rankorder weight 3, the second gets weight 2, and the least preferred candidate gets weight 1. In the table we can read for example that there are 33 candidates with preference $A > B > C$.

Table 4. Voting example

<i>Number of voters</i>	<i>Candidates and their rank order weight</i>		
<i>Sum</i> 114	<i>A</i>	<i>B</i>	<i>C</i>
33	3	2	1
0	3	1	2
25	2	1	3
17	1	2	3
14	1	3	2
25	2	3	1
<i>Results of the procedures</i>			
<i>Most preferred</i>	33+0 = 33	14+25 = 39	25+17 = 42
<i>Borda</i>	230	242	212
<i>Pairs: A vs B</i>	58	56	-
<i>A vs C</i>	58	-	56
<i>B vs C</i>	-	72	42

The different voting schemes result into different decisions:

- (1) **Plurality:** Voters give one single vote to the candidate of their highest preference. For candidate *A* we consider its column, select the rows with the score 3, and add the associated numbers of voters $33 + 0 = 33$. And so on. Candidate *C* gets most votes, namely 42.
- (2) **Borda:** The votes are weighted with the rank order weight. De column for *A* is multiplied row by row with the number of voters $3 * 33 + 3 * 0 + 2 * 25 + \dots = 230$. Candidate *B* gets most votes, namely 242.
- (3) **Condorcet:** Voting pairwise over *A* versus *B*, there are $33 + 0 + 25 = 58$ voters who give *A* a higher rankorder than *B*. Etcetera. Candidate *A* appears to win from both *B* and *C*, and then is the “Condorcet winner”.

This example shows that *A*, *B* and *C* can all be winners, depending upon the method selected. The properties of the methods then are the true issue.

Above still neglects strategic voting. This could be represented by a change in apparent position. How do we evaluate this ? It appears that the Condorcet approach is least sensitive to cheating since in a pairwise vote there is an incentive to express one's true preferences. Pairwise voting however can be unattractive since there need not be a Condorcet winner, or, when one exists, it may conflict with the preference rankings. One way to solve the complexity of choosing between these methods is to compromise by having a run-off election. The two top outcomes of Plurality or Borda are taken and then subjected to a pairwise vote as in Condorcet. There is one final consideration. Simply taking the two ‘top outcomes’ seems unduly simple, we should consider what these actually are. In France, the election between Chirac, Jospin, Le Pen and others caused Jospin's votes to scatter over all kinds of smaller parties so that he dropped from the race while he was the Condorcet winner of both Chirac and Le Pen. When we are compromising, we should focus on determining the two main contenders.

Borda Fixed point

Let us reconsider the dynamic process that occurs within an economy. We see that under the influence of time, the budget changes continuously. A voting scheme naturally requires that there is a list of candidates, but one cause for paradoxes is that that list is not fixed. For example, in the Borda vote above, *B* is selected, but if *C* decides to withdraw (or gets a heart attack), then we would expect *B* to remain the winner, but

suddenly it is A (see the Condorcet vote A versus B). Remember also the Bush, Gore and Nader case. We could consider a procedure to be better when the choice is less dependent upon changes in the budget.

A way to achieve this is to use the notion of a 'fixed point'. For a function $f: D \rightarrow R$, for some domain D and range R , the point p is a fixed point iff $f[p] = p$. Let us consider this concept for voting.

Let P be the voting procedure, and let $X = \{x_1, \dots, x_n\}$ be the budget with all the candidates. Let the unrefined winner be $w = P[X]$. Let Y be the budget when w does not participate, $Y = X \setminus \{w\}$. Let the 'alternative winner' be $v = P[Y] = v[w]$, i.e. the candidate who wins when the first winner w does not participate. This is not simply the run-off between the winner and the common runner-up, since the selection of the alternative winner requires the recalculation of the preference weights. This alternative winner can be seen as a 'summary' of the opposition to w . The scheme is a compromise since the Condorcet pairwise condition holds for the winner and the alternative winner. While these notions are defined with respect to the unrefined winner, we can generalise this to any winner, and in particular to our optimal winner.

An alternative condition for winning in general is the ability to win from one's strongest opponent. This gives the fixed point condition. Define $f[x] = P[x, P[X \setminus \{x\}]]$, which is the general function 'the vote result of x and its alternative winner'. Then w^* is the solution to the fixed point condition $x = f[x]$:

$$w^* = P[w^*, v[w^*]] = P[w^*, P[X \setminus \{w^*\}]] = f[w^*]$$

When the unrefined winner w is not a fixed point, i.e. when the unrefined winner $w = P[X]$ appears to lose from v , so that $w \neq P[w, v]$, then the search process can start again from v .

It appears that this fixed point voting procedure reduces the dependence upon budget changes. There can still be a dependence, but it is not as large as without the condition.

In **Table 4**, the **Borda Fixed Point** (BordaFP) winner is A . With B the Borda winner, A is the alternative winner when B does not participate, and B loses from A in a pairwise match; starting the search from A , its alternative winner is B , and A wins from B .

More on this can be found in VTFD, Colignatus (2007b, 2014). That book has also been intended as a textbook and it developed *Mathematica* programs for the various voting schemes and data manipulations. Given the complexity of the matter, this working environment has appeared a great advantage.

Relation to Saari's work

Donald Saari (2001ab) showed that Borda's method is the only method that satisfies certain symmetries. His suggestion is that the Borda rule 'therefor is best'. This argument does not convince by itself since 'symmetry' is not by itself a moral category. Dynamics is linked to morality, by the notion that morality presumes time, and thus seems a better angle.

Consider direct symmetry first. Suppose that your preference is $A > B > C$ and that my preference is $C > B > A$. The direct symmetry consideration is that we might both abstain from a vote and stay home, since our preferences strictly oppose each other. Saari noted too that voting cycles can be catalogued under the mathematical concept of rotational symmetry. His subsequent suggestion is that cancellation should hold for all symmetries for all subsets of voters.

What happens when cancellation of 'rotational symmetry' is applied to subsets ? The following is an example by Saari that cancellation isn't trivial then. In **Table 5** there are 48 voters, and B is selected by both Borda and Condorcet. In **Table 6**, 27 voters have been

added who have the mentioned rotational symmetry, with 9 for each subgroup. Now Borda still selects *B*, but Condorcet, and the Borda Fixed Point, select *A*. In Saari's view, Borda satisfies symmetry, and 'hence' is the better method.

My reasoning is a bit different. First of all, note that I myself have used an argument similar to that of Saari. In my view, the typical Condorcet situation of three preferences $A > B > C$, $B > C > A$ and $C > A > B$ results into indifference rather than an inconsistency, and I use this against Arrow's analysis. So I agree with Saari's view that such votes cancel. I applaud Saari's insight that if you apply cancellation for all cycles in all subsets, then the logic is to get rid of Condorcet's method and to use Borda's method.

Table 5. Start with 48 voters: Borda *B*, Condorcet *B*

	<i>Candidates and their rank order weight</i>		
<i>Number of voters</i>	<i>A</i>	<i>B</i>	<i>C</i>
20	3	2	1
28	2	3	1
<i>Borda weighted total</i>	116	124	48
<i>A versus B</i>	20	28	
<i>A versus C</i>	48		0
<i>B versus C</i>		48	0

Table 6. Add 27 'neutral' others: Borda *B*, Condorcet *A*

	<i>Candidates and their rank order weight</i>		
<i>Number of voters</i>	<i>A</i>	<i>B</i>	<i>C</i>
20	3	2	1
28	2	3	1
9	3	2	1
9	1	3	2
9	2	1	3
<i>Borda weighted total</i>	170	178	102
<i>A versus B</i>	38	37	
<i>A versus C</i>	57		18
<i>B versus C</i>		66	9

Secondly, however, my problem remains that there is the phenomenon of budget changes. Note that Saari's example uses a changing electorate rather than a changing budget. My suggestion is that a change in the electorate would require a new vote, while we would want to avoid that in case of a change in the budget. The Borda method would be best, only when the budget would be really given. When it might change, the application of cancellation to all subsets becomes doubtful, since subsets change. There is a fundamental uncertainty with respect to the future. Consider the following example. At a specific point in time, the population of a nation is given, and thus the vote for a President has a specified budget: the population. But, uncertainty sets in again, when people may withdraw from the race. Only a few actually run. Hence, we might well want a rule to deal with possible changes in the budget. Hence, it is not logically required that we cancel votes for all possible subcycles (also for candidates who are not in the race). Saari is very strong on the argument that when we accept cancellation in one case, then we should do so in all cases. I am more sensitive to the exception: when 'if one, then all' does not hold.

Concerning **Table 5** and **Table 6**, my reasoning is – contrary to Saari – that the added votes cannot be neglected. The argument of rotational symmetry breaks down when we compare a winner with the alternative winner – which is a pair – while rotational symmetry requires a third candidate or more. For the pair, the addition has an effect. When we consider unrefined winner *B* and its alternative winner *A*, then the added votes are in favour of *A* and no longer ‘neutral’. While *C* is important since it shows a cycle for a subgroup of voters, another view is that *C* could be neglected since it is not a fixed point. Candidate *C* is a typical example of an irrelevant candidate that can cause a preference reversal in Borda voting. Namely, let us consider **Table 6** under Borda voting, and let *C* decide to drop from the race: then *A* becomes the winner. The Borda Fixed Point method has been developed precisely to deal with that kind of preference reversal.

Thus, when you select your voting method then you must choose between the properties exemplified by this case. (1) Borda is subject to preference reversal. In the example of **Table 6**, when *C* drops out, then there would be switch from *B* to *A*. (2) The Borda Fixed Point method still depends upon the voting field. In this example, when 27 voters drop out, then there is a switch from *A* to *B*.

The choice basically is whether we attach more importance either to the voters or to the candidates. Saari suggests that the candidates are more important, since he cancels the votes of 27 voters and keeps *C* in the race. I would say that the voters are important and that candidate *C* is less relevant. The proper question would be whether the winner is a convincing winner. Of course, *C* can become an important candidate when we add other voters. But then the argument is that those voters count, rather than *C*.

Consider the impact of semantics. While it has been a long standing notion that cycles may also be taken as indifference, so that the votes cancel, Saari now rephrases this as rotational symmetry, and he suggests that acceptance of rotational symmetry implies acceptance of it for all cases and subsets. The label might be a common mathematical label, but I have a problem with that label in the realm of morality (and the implied universality). Human beings seem to have biological preference for symmetry, and by labelling something as ‘symmetry’, it becomes more attractive. When discussing the different voting schemes, we should be aware of such effects, and try to focus on what the properties really mean, and we should make a proper distinction between a property that is universal and a property that is dependent upon the situation. Perhaps it might be analysed as the ‘mathematical frame of mind’ that acceptance of a property for one set also implies acceptance for all other (sub-) sets, but my conclusion is that when we look closer, that there is room for more subtlety. Indeed, it might well be that considerations of symmetry apply to the static situation, but that we need other considerations for dynamics.

Another example for this need for subtlety is that the ‘rotational symmetry’ argument breaks down on the *status quo* (see below).

Saari has also developed an ingenious way to depict voting schemes geometrically. For 3 candidates, this becomes a triangle, and the different procedures can be calculated from that. It appears that these triangles are a good educational tool. However, my experience is that the computer programs (VTFD uses *Mathematica*) are easier to use, since they take away the need for calculations, while they are available for more dimensions and also allow for indifference and not just strict preference. A complex scheme like the Borda Fixed Point also requires more work with the triangle, while in *Mathematica* it is a simple procedure call. It may be noted that above discussion of the Borda Fixed Point method has been simplified by assuming single winners. In practice, there can be ties, complicating the search, and requiring tie-breaking rules.

Pareto

Another consequence of the switch of attention from statics to dynamics is the recognition of a *status quo*.

There appears to exist another wide-spread confusion about ‘majority voting’. This idea is that a majority result would still be democratically valid, even if the winning decision implies a real loss for the opposition. The counter-example is when the majority decides that the minority pays \$1 to the majority: this is not necessarily a morally acceptable situation, even though there is a majority. From a moral point of view, each voting scheme should have two rounds: a first round to select the Pareto improving points compared to the *status quo*, and then a second round to select the winner from those Paretian improvements. The majority rule thus can be regarded as only a tie-breaking rule, namely for the deadlock when there are more Pareto improving points. In elections of persons, the *status quo* can be a vacancy, and in that respect all candidates could be taken as Paretian. But the Paretian pre-condition cannot be skipped in general.

The Paretian condition may require some subtlety. Consider the family choice for a holiday to Greece or Spain, discussed above. If little Robby considers the holiday to Spain to be a deterioration from the *status quo* of not having a holiday at all, then there is moral argument to say that Spain is not a valid option to take a vote on. However, if it can be established in a first round that going on a holiday is unanimously a good idea, then Robby has to accept a possible majority decision in favour of Spain and against Greece.

One argument against the selection of Pareto improving points is that people might also cheat about these points. This argument is not convincing, since Pareto improvement is in one’s own interest. Indeed, little Robby might try to veto Spain by saying that he does not want a holiday, and thus he might be trying to bargain to get everybody to accept Greece. However, this ploy can be prevented by having that first round on having a holiday, since if he really wants a holiday anyhow, then he has to show this then. Careful construction of the voting process thus remains an issue.

A note on cheating

One of the key problems in voting theory is strategic voting behaviour, better known as cheating. In a scheme like Borda, cardinal utility has already been reduced to ordinal utility, so perhaps we should be lenient and allow voters to maximize their utility from the final outcome by manipulating their vote. But our opinion on this does not matter, since the ballot generally is secret and we cannot stop people from voting strategically anyway. In fact, the *Mathematica* programs in VTFD contain routines for cheating. These are simple routines that assume both full information and that others don’t cheat, since the mathematics of cheating while assuming that others cheat too is rather complex, especially when nobody has full information about the true preferences. Given all this, one surmises that election results do not reflect the true state.

Thinking about these issues gave me an idea that might be helpful to elicit the true state. Suppose that each voter is informed in advance that there is a probability p that the ranking order that is submitted will be used by the election computer for strategic voting. If the voter submits his or her true ranking, then this is rewarded with probability p to improve the election result for that voter, and much better than the voter can, since the computer knows all submitted rankings. If the voter submits a strategically adapted ranking, then this is punished with probability p namely to improve the election result for that false ranking. Likely there is a specific value of p that would generate the most truthful election result. Unfortunately, I haven’t had time to develop this idea.

Conclusion

An election result is 'as much' the result of the procedure as of the preferences. Arrow's Impossibility Theorem is complex and full with paradoxes, but the dependence of morality upon time provides a way towards solution.

There are two key conclusions:

- (1) The Pareto condition for the candidates under ballot should not be neglected - i.e. that only those candidates are voted on that are an improvement compared to the *status quo*.
- (2) The Borda Fixed Point can be seen as a compromise between the Borda and Condorcet procedures (on Paretian points), and provides a degree of protection against budget changes.

There is also another conclusion. Voting is complex, and becomes increasingly complex when the numbers of candidates and voters rise (especially when we also include indifference and not just strict preference). Direct election of a President becomes quickly infeasible for the more advanced voting procedures. From this observation we can conclude that it is better to have a proportional parliamentary system, so that the elected professionals can use the advanced voting procedures to select the President. This approach of representation also prevents that there is a different electoral mandate for President versus Parliament. Note that the discussion above, on Arrow's Theorem and the Borda Fixed Point method, considers single seat elections, and not multi-seat elections. But the complexity of direct single seat elections tends to support this conclusion on the overall system of proportional representation and indirect election of the chief executives.

XI. The derivative is algebra

Improving the logical base of calculus on the issue of 'division by zero'

Abstract

Calculus can be developed with algebra and without the use of limits and infinitesimals.

Define y / x as the 'outcome' of division and $y // x$ as the 'procedure' of division (see page 31). Using $y // x$ with x possibly becoming zero will not be paradoxical when the paradoxical part has first been eliminated by algebraic simplification. The Weierstraß $\epsilon > 0$ and $\delta > 0$ and its Cauchy shorthand for the derivative $\lim(\Delta x \rightarrow 0) \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using $\Delta f // \Delta x$ and then setting $\Delta x = 0$ is not paradoxical at all. Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $|x|$ traditionally is undefined at $x = 0$ but would algebraically be $\text{sign}[x]$, and so on.

PM. The present discussion improves upon a version of July 2007 on my website.

Introduction

Since its invention, the zero has been giving trouble. Mathematicians solved the paradoxes by forbidding the division by zero. But the problem persisted in calculus, where the differential quotient relies on infinitesimals that magically are both non-zero before division but zero after it. Karl Weierstraß (1815-1897) is credited with formulating the strict concept of the limit to deal with the differential quotient. However, he did not resolve the paradoxical aspects.

Regard these expressions, three well-known and the fourth a new design.

- (1) The difference quotient $\Delta f / \Delta x = (f[x + \Delta x] - f[x]) / \Delta x$ for $\Delta x \neq 0$. Note that one would see this as a result and not as a procedure.
- (2) The differential quotient or derivative $f'[x] = df / dx = \lim(\Delta x \rightarrow 0) \Delta f / \Delta x$.
- (3) The current "theoretical true meaning of the derivative" with outcome value L :
 $\forall \epsilon > 0 \exists \delta > 0$ so that for $0 < |\Delta x| < \delta$ we have $|\Delta f / \Delta x - L| < \epsilon$.
- (4) The new suggestion: $f'[x] = df / dx = \{\Delta f // \Delta x, \text{ then set } \Delta x = 0\}$. This means first algebraically simplifying the difference quotient, expanding the domain with 0, and then setting Δx to zero. PM. $y // x$ is defined on page 31.

Let us consider the various properties.

The old approaches

The theory of limits is problematic. The limit of e.g. x / x for $x \rightarrow 0$ is said to be defined for the value $x = 0$ on the horizontal axis yet not defined for actually setting $x = 0$ but only for x getting close to it, which is paradoxical since $x = 0$ would be the value we are interested in. Mathematicians get around this by defining a special function $f[x] = x / x$ with split domain but this requires a separate $f[0] = 1$ statement, while it is faster to write $x // x$.

Also, the interpretation given by Weierstraß can be rejected since that definition of the limit still excludes the value (at) $\Delta x = 0$ which actually is precisely the value of interest at the point where the limit is taken.

While the Weierstraß approach uses predicate logic to identify the limit values, the new alternative approach uses algebra, the logic of formula manipulation.

Fermat, Leibniz, Newton, Cauchy and Weierstraß were trained to regard y / x as sacrosanct such that it indeed doesn't have a value for $x = 0$. They worked around that, so that algebraically y / x could be simplified before x got its value. While doing so, they created a new math that appeared useful for other realms. These new results gave them confidence that they were on the right track. Yet, they also created something overly complex and essentially inconsistent. Infinitesimals are curious constructs with no coherent meaning. Bishop Berkeley criticized the use of infinitesimals, that were both quantities and zero: who could accept all that, need, according to him, "not be squeamish about any point in divinity". The standard story is that Weierstraß set the record straight. However, Weierstraß's limit is undefined at precisely the relevant point of interest. 'Arbitrary close' is a curious notion for results that seem perfectly exact. When we look at the issue from this new algebraic angle, the problem in calculus has not been caused by the "infinitesimals" but by the confusion between "/" and "//".

The present discussion can be seen as reviving the Cauchy approach but providing another algebraic interpretation that avoids the use of 'infinitesimals'. The impetus comes from the notion of the dynamic quotient in algebra. We cannot change properties of functions but we can change some interpretations. Undoubtedly, the notion of the limit and Weierstraß's implementation remain useful for specific purposes. That said, the discussion can be simplified and pruned from paradoxes.

Struik (1977) incidently states that Lagrange already saw the derivative as algebraic. See there for details and why contemporaries thought his method unconvincing.

The algebraic approach

In a way, the new algebraic definition is nothing new since it merely codifies what people have been doing since Leibniz and Newton. In another respect, the approach is a bit different since the discussion of 'infinitesimals', i.e. the 'quantities vanishing to zero', is avoided.

The derivative deals with formulas too, and not just numbers (as conventionally). It uses both that $\Delta f // \Delta x$ extends the domain to $\Delta x = 0$ and that the instruction "set $\Delta x = 0$ " subsequently restricts the result to that point.

Since we have been taught not to divide without writing down that the denominator ought to be nonzero, the following explanation will help for the proper interpretation of the derivative: first the expression is simplified for $\Delta x \neq 0$, then the result is declared valid also for the domain $\Delta x = 0$, and then Δx is set to the value 0. The reason for this declaration of validity resides in the algebraic nature of the elimination of a symbol, as in $x // x = 1$, and the algebraic considerations on 'form'.

The true problem is to show why this new definition of df / dx makes sense.

Stepwise explanation of the algebraic approach

Let us create calculus without depending upon infinitesimals or limits or division by zero.

- (1) We distinguish cases $\Delta x \neq 0$ and $\Delta x = 0$, and the (*) implicit or (**) explicit definition of relative error $r[\Delta x]$.

- (2) Let $F[x]$ be the surface under $y = f[x]$ to the horizontal axis from 0 till x , for known F and unknown f that is to be determined (note this order). For example $F[x] = x^2$ gives a surface under some f and we want to know that f .
- (3) Then the change in surface is $\Delta F = F[x + \Delta x] - F[x]$. When $\Delta x = 0$ then $\Delta F = 0$.
- (4) The surface change can be approximated in various ways. Of these $\Delta F \approx y \Delta x = f[x] \Delta x$ is the simplest expression with explicit y . (Alternatives are e.g. $\Delta F \approx f[x + \Delta x] \Delta x$, or inbetween $\Delta F \approx (y + \Delta y / 2) \Delta x$, with $\Delta y = f[x + \Delta x] - f[x]$.)
- (5) The error will be a function of Δx again. We can write ΔF in terms of $y = f[x]$ (to be found) and a general error term $\varepsilon[\Delta x]$, where the latter can also be written as $\varepsilon[\Delta x] = \Delta x r[\Delta x]$ where $r[\Delta x]$ is the relative error. When $\Delta x = 0$ and thus $\varepsilon[\Delta x] = 0$ then the relative error can be seen as undefined and it can be set to zero by definition.
- (6) We have these relations where we multiply by zero and nowhere divide by zero or infinitesimals.

	(*) <i>Implicit definition of r</i>	(**) <i>Explicit definition of r</i>
$\Delta x \neq 0$	$\Delta F = y \Delta x + \varepsilon[\Delta x]$	$r[\Delta x] \equiv \Delta F / \Delta x - y$
$\Delta x = 0$	$\Delta F = 0 = u \Delta x + \varepsilon[\Delta x]$ for any u ; select $u = y$	$r[\Delta x] \equiv 0 = u - y$ for $u = y$

- (7) Simplify $\Delta F / \Delta x$ algebraically for $\Delta x \neq 0$ and determine whether setting $\Delta x = 0$ gives a defined outcome. When the latter is the case, take u as that outcome.
- (8) Thus $u = \{\Delta F // \Delta x, \text{ then set } \Delta x = 0\}$. (Setting a to value b is denoted as $a := b$.)
- (9) We then find $u = y = f[x]$ which can be denoted as $F'[x]$ as well.

For example, the derivative for $F[x] = x^2$ gives $dF / dx = \{(x + \Delta x)^2 - x^2\} // \Delta x$, then $\Delta x := 0\}$ = $\{2x + \Delta x, \text{ then } \Delta x := 0\} = 2x$. This contains a seeming 'division by zero' while actually there is no such division.

The selection of $u = y$ is based upon 'formal identity'. This is a sense of consistency or 'continuity', not in the sense of limits but in the sense of 'same formula', in that (*) and (**) have the same form (each seen per column) irrespective of the value of Δx .

The deeper reason (or 'trick') why this construction works is that (*) evades the question what the outcome of $\varepsilon[\Delta x] // \Delta x$ would be but (**) provides a definition when the error is seen as a formula. Thus, (*) and (**) give exactly what we need for both a good expression of the error and subsequently the 'derivative' at $\Delta x = 0$. The deepest reason (or 'magic') why this works is that we have defined $F[x]$ as the surface (or integral), with both (a) an approximation and (b) an error for *any* approximation that still is accurate for $\Delta x = 0$. When the error is zero then we know that $F[x]$ gives the surface under the $u = y = f[x] = F'[x]$ which is the function that we found.

In summary: The program is $F'[x] = dF / dx = \{\Delta F // \Delta x, \text{ then set } \Delta x = 0\}$. The definitions (*) and (**) give the rationale for extending the domain with $\Delta x = 0$, namely form.

[2011: Select e.g. $\Delta F \approx y^2 \Delta x$ as the approximation. Then (*) suggests form $u = y^2$. But (**) has form $\Delta F / \Delta x - y^2$ and in $\Delta F / \Delta x$ there is no suggestion of a square so that the choice of $u = y^2$ is problematic. The relative error features as a criterion because it allows an identification of its outcome as the y that we are looking for.]

Implications

The proper introduction to calculus is to start with a function that describes a surface and then find the derivative. Since we only use equivalences, this also establishes that the reverse operation on the derivative gives a function for the surface.

The relation to the slope only arises in point (4) above. Traditionally the derivative is created from the question to find the slope at some point of a function. This tradition also suggests a separate development for the integral, e.g. with Riemann sums. Instead, here we find that the slope comes as a fast corollary – seeing that $\Delta F // \Delta x$ would be the tangent if it is defined.

Let us look closer into the difference between starting from slopes or from surfaces.

The derivative of $|x|$ is traditionally undefined at $x = 0$ but would algebraically become $\text{sign}[x]$. For $x \neq 0$, we can consider the various combinations and find the normal result, $\text{sign}[x]$. For $x = 0$ the dynamic quotient gives $(|0 + \Delta x| - |0|) // \Delta x = |\Delta x| // \Delta x = \text{sign}[\Delta x]$. Setting $\Delta x = 0$ gives 0. Hence in general $|x|' = \text{sign}[x]$.

The traditional approach to $|x|$ is a bit complicated. Cauchy naturally gives 0 at 0 too. Traditionally the derivative is used for finding slopes and then the amendment on Cauchy was to hold that the right derivative differs from the left derivative, hence traditionally there is no general derivative. However, there is a multitude of 'tangent' lines at 0, that is, when tangency is not defined as having the same slope as the function (which slope is undefined at 0) but as having a point in common that is no intersection.

In our approach, when we are interested in slopes, then it remains proper to consider these left and right derivatives. We do not need to speak about limits but merely can point to the different values of the derivative $\text{sign}[x]$ in the intervals $(-\infty, 0)$, $[0]$, $(0, +\infty)$. Depending upon the definition of 'tangent': (a) "Tangent" lines that have the point $\{0, 0\}$ in common without intersection then can have slopes from -1 to 1 . (b) "Tangent" lines that have the same slope as the function however have only the three slopes $-1, 0, 1$.

The dynamic quotient is the leading impetus here and the issue starts with algebra so that slopes come in only second. $|x|$ is the surface under some function f . Any approximation of changes in the surface, when the surface value is $|0| = 0$, finds a perfect answer with zero relative error by requiring $f[0] = 0$. The general function appears to be $\text{sign}[x]$. The choice to extend the domain of Δx with value 0 at $x = 0$ derives from a notion of consistency of the *form* of the relative error in the approximation. This is sufficient though not necessary. One could argue that the relative error is not defined when $\Delta x = 0$ but this runs counter to our choice to define it as 0. This choice again relates to the form of the relations in step (6).

Students

Generations of students have been suffering. Teachers of math seem to have overcome their own difficulties (mainly by stopping to think) and thereafter don't seem to notice the inherent vagueness.

Students not only suffer from the vagueness but also from the notation. Many forget to write " $\lim(\Delta x \rightarrow 0)$ " as the first part of each differential quotient, each separate line again and again for each step of the deduction, assuming that stating it once should be sufficient to express that they are taking the limit. Some 'take the limit' so that for them Δx has become 0, and then, just to be sure, they still mention "... + Δx " arguing that it should not matter when you add 0. Those 'official mathematical errors' will be past.

Conversely, if the new notation of dynamic division is adopted also for general purposes, see page 31, then the algebraic origin of the derivative will be sooner recognized,

strengthening the insights in logic and algebra. Time can be won for more relevant issues.

Teachers may be less tempted to distinguish between ‘those who know the truth’ (Deep Calculus, the ε and δ) (who thus actually are wrongfooted) and ‘those who only learn the tricks’ (Superficial Calculus).

Didactics remain an issue. Above nine steps are somewhat elaborate while the short program $\{\Delta f // \Delta x, \text{ then set } \Delta x = 0\}$ sums it up and suffices. Possibly some randomized controlled trials in education would bring more light in the question what explanation works where.

The chain rule

The chain rule is an important result and can found directly as follows.

$$\begin{aligned} df / dx &= \{\Delta f // \Delta x, \text{ then set } \Delta x = 0\} \\ &= \{\Delta f // \Delta g * \Delta g // \Delta x \text{ for } (\Delta x = 0 \Leftrightarrow \Delta g = 0), \text{ then set } \Delta x = 0\} \\ &= \{\Delta f // \Delta g, \text{ then set } \Delta g = 0\} * \{\Delta g // \Delta x, \text{ then set } \Delta x = 0\} \\ &= df / dg * dg / dx \end{aligned}$$

The derivative of an exponential function

[2011: The key deduction on the exponential function is improved upon in COTP §12.1.8.3, notably by moving from the dynamic quotient to the surface identity. The text of this paragraph can remain here for the didactic aspects.]

The derivative of an exponential function follows from the chain rule and the presumption that $\text{Exp}[x] = e^x$ is the fixed point in differentiation:

$$\frac{\partial a^x}{\partial x} = \frac{\partial e^{x \text{rex}[a]}}{\partial x} = e^{x \text{rex}[a]} \cdot \text{rex}[a] = a^x \text{rex}[a]$$

The reasoning thus is (compare with the “alternative approach” on page 55):

- (i) All positive functions can be expressed as an exponential function for any positive base number b , as $f[x] = \text{Exp}[b, \text{rex}[b, f[x]]]$.
- (ii) We presume that in this class of all possible bases there is a fixed point in differentiation. Call this base the number e . Thus by definition $(e^x)' = e^x$.
- (iii) We can calculate e from the property $e^x \equiv d e^x / dx = \{e^x (e^h - 1) // h, \text{ set } h = 0\}$. This gives $1 = \{(e^h - 1) // h, \text{ set } h = 0\}$. Note that 1 thus is the solution to the programme on the RHS. By setting $(e^h - 1) = h$ and solving $e = (1 + h)^{(1/h)}$ we find approximate values of e by taking h close to zero.
- (iv) That there is an actual number e with ‘infinite accuracy’ follows from (iii) and from notions of continuity (‘there are no holes between 2 and 3’).
- (v) From the chain rule we find in general $\text{rex}[a] = \{(a^h - 1) // h, \text{ set } h = 0\}$.

Thus, the dynamic quotient $(a^h - 1) // h = (e^{h \text{rex}[a]} - 1) // h$ does not simplify easily (see page 55). However, when we use the chain rule then we can avoid using this explicit expression and actually find its value by implication.

Some meta-comments are:

- (a) The number e remains an algebraic concept like the number π .
- (b) The procedure is to first presume e and its property, and only then calculate / approximate it, and thus prove its existence by calculation. This summarizes an intricate historical development, but does not invalidate the existence proof.
- (c) In this case approximate values for e are found as we would normally take a limit. But the limit is not applied for the derivative.
- (d) The notion of a limit by itself still has its advantages, e.g. for the limit to infinity, and thus for $1 // 0$ again. It would not be right not to mention limits in education.
- (e) There remains a distinction however between algebraic simplification and extension of the domain on the one hand and the traditional concept of a limit on the other hand. This distinction causes the insight that the derivative is an algebraic notion rather than dependent upon infinitesimals.
- (f) Given that limits can be defined in acceptable manner suggests that calculus can be developed by using limits. Indeed, complex ways can be used for what is simple.

Conclusion

History is a big subject and we should be careful about drawing big historical lines. But the following seems an acceptable summary of the situation where we currently find us after the historical introduction of the zero.

The introduction of the zero in Europe around AD 1200 gave so many problems that once those were getting solved, those solutions, such as that one cannot divide by zero, were codified in stone, and pupils in the schools of Europe would meet with bad grades, severe punishment and infamy if they would sin against those sacrosanct rules. Tragically, a bit later on the historical timeline, division by zero seemed to be important for the differential quotient. Rather than reconsidering what 'division' actually meant, and slightly modifying our concept of division, Fermat, Leibniz, Newton, Cauchy and Weierstraß decided to work around this, creating the concepts of infinitesimals or the limit. In this way they actually complicated the issue and created paradoxes of their own.

The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and the derivative's shorthand $\lim(\Delta x \rightarrow 0) \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken.

Logical clarity and soundness can be restored by distinguishing between the (formal) act of division and the (numerical) result of division. Using $\Delta f // \Delta x$ and then enlarging the domain and setting $\Delta x = 0$ is not paradoxical at all.

The distinction between static and dynamic division suggests that the Weierstraß purity may be overly pedantic for the main body of calculus. The exact definition of the limit is of great value but not necessarily for all of calculus. Indeed, 'most' derivatives can be found without the Weierstraß technical purity and 'many' courses already teach calculus without developing that purity. Thus there is ample cause to bring theory and practice more in line.

[2011: There is a paradox that I may refer to but have not developed further. In the Weierstraß definition of continuity around some x_0 it may be that there is some begging of the question, as the $\varepsilon > 0$ and $\delta > 0$ that are used may require their own infinitesimals.]

Appendix about 2009-2015

The news on the algebraic approach to the derivative is split in two parts.

The text above is a bit edited from COTP, which itself is a small revision of EWS 2009. The differences from 2009 are indicated by "[2011: ...]". Subsection (1) below explains the situation. Subsection (2) discusses the period 2009-2015 in general.

(1) COTP (2011) §15.5.1 The derivative is algebra

Improving the logical base of calculus on the issue of 'division by zero'.

Quote from COTP:

The history of this text is as follows. *A Logic of Exceptions* considers the logical paradoxes. Retyping it in 2007 caused me to consider the paradoxes of division by zero too, out of a sense of completeness. There still was some lingering doubt with respect to the lectures in Analysis that I attended as a student back in [1973/4] and the Weierstraß construction for derivatives and continuity. In logic there is the difference between *implication* and *inference*, and inspired by the difference in economics between *statics* and *dynamics* as ways of analysis I had already in 1980 classified logic into *static implication* and *dynamic inference*. Hence in 2007 the *dynamic quotient* was born. The paradoxes of the derivative and the approach discussed here already got a section in *A Logic of Exceptions* in 2007. A longer paper with the present title *The derivative is algebra* of July 2007 is on my website. (Later this was linked to the difference between *verb* and *noun* in general - and in 2007/8 in a course on the didactics of mathematics I discovered that Gray & Tall had developed the term "procept".) It has been polished up and appeared as chapter XI in *Elegance with Substance* 2009. The chapter can be reproduced here with little additional comment except for the points that the main body of this *Conquest of the Plane* (1) improves on the derivative of the exponential function, (2) extends with the derivatives of Cos and Sin, (3) contains § 2.3.3 and 2.3.4 with extensive definitions for the process of division, (4) does not discuss the relative error that is crucial and is discussed below. You miss some references to pages in *Elegance with Substance* but the relevant concepts like the distinction between *verb* and *noun* are also in this present book.

2015: In 2011 I considered it crucial to point out that the relative error is a criterion for identification of the best approximation. Now in 2015 I think more relaxedly that this ought to be obvious since the original text from 2007 already included the relative error to start with. It causes the dynamic quotient, after all.

The text here is intended for mathematicians, since the creation of the dynamic quotient and its application to calculus are a novel contribution to mathematics. The text is also intended for teachers as it clarifies the difficulties in teaching calculus. The text is not intended as an introduction to calculus for students since that is presented in the body of this book. While the text below develops the mathematical theory it has been a challenge indeed to compose an introduction for students from the bottom up. It is satisfactory to see that it indeed can be done and that calculus in this manner finds a natural place with analytic geometry. The cost is that the introduction above does not discuss the notion of the relative error yet, which is explained in the text below. It plays a role in judging on algebraic form. Reviewing the whole I am again impressed by the contributions of our great mathematicians who allow us to take this journey.

End Quote from COTP.

(2) The period 2009-2015 in general

(i) On Lagrange's interest in the algebraic approach, I was happy to discover Grabiner (2010). His approach is quite different than proposed here, but still.

(ii) The comment "Infinitesimals are curious constructs with no coherent meaning" of course drew comments from the *non-standard-analysis* community. I gave a beginning of a reply in CCPO-WIP (2007j) but given that the paradoxes of the derivative had a solution by the dynamic quotient, my attention focussed on Cantor and the paradoxes of the infinite.

(iii) David Tall adheres to non-standard analysis and wants to explain the derivative by using your fingers to touch the slope and using software to focus in with a magnifying glass. I am a bit perplexed, since the dynamic quotient is related to his "procept" notion. When Tall was in Holland we discussed it. His view was presented in the Dutch journal NAW of the Royal Dutch Mathematical Society (KWG), that allowed me a comment, Colignatus (2012d). Unfortunately, I later had to write *Pierre van Hiele and David Tall: Getting the facts right*, Colignatus (2014a).

(iv) In 2013 I gave a presentation at the research day of the Dutch Association of Teachers of Mathematics (NVVW), see a discussion with presentation slides.³⁷

(v) While EWS and COTP got an open minded review by Gill (2012) in NAW, there was a derogatory and slanderous "review" in *Euclides*, the journal of the NVVW. The "reviewer" did not explain the proposed algebraic approach to the derivative and its didactic properties, but explained the Weierstraß method as if I would not know about this. The editors of *Euclides* did not allow me to protest in the journal, and they have decided to no longer review any books by me (with early victims KWAG and EWVJ / SMOJ). Up to now in 2015 there has been no report in the journal *Euclides* about what the proposal for the algebraic approach is – originally proposed in ALOE 2007. Fellow teachers may have heard rumours about some kind of disagreement but they will have no idea about what the proposal entails. However, Boudri actually read COTP and was critical about the "review".³⁸ The issue is discussed here³⁹ and in planned BHRM (2015g).

(vi) Colignatus (2014b) *Education, division & derivative: Putting a Sky above a Field or a Meadow* tries to look at a more theoretical level whether more can be said about the algebraic approach. In the dynamic quotient $y // x$ the numerator and denominator are taken as variables themselves when judging about $x = 0$, but also, for the stage of simplification, as expressions that may have other variables. The algebra is straightforward but theorists may wonder about the details.

(vii) "Historian" Amir Alexander wrote about infinitesimals. Apparently he first selects his storyline and later looks for data to support this. This might make for good stories but not for good history.⁴⁰ Making the circle round: there is Grabiner's review on Alexander.⁴¹

³⁷ <https://boycottholland.wordpress.com/2013/11/13/calculus/>

³⁸ <http://thomascool.eu/Papers/COTP/2013-03-15-Boudri-over-COTP.pdf>

³⁹ <https://boycottholland.wordpress.com/2014/06/12/slander-squared/>

⁴⁰ <https://boycottholland.wordpress.com/2014/06/14/amir-alexander-and-history-as-storytelling/>

⁴¹ <http://www.maa.org/press/maa-reviews/infinitesimal-how-a-dangerous-mathematical-theory-shaped-the-modern-world>

XII. Neoclassical mathematics for the schools

Abstract, September 6 & December 20 2011 – edited 2015 ⁴²

National Parliaments around the world are advised to each have their own national parliamentary enquiry into the education in mathematics and into what is called 'mathematics'. Current mathematics education namely fails and causes extreme social costs. The failure can be traced to a deep rooted tradition and culture in mathematics itself. Mathematicians are trained for abstract theory but when they teach then they meet with real life pupils and students. Didactics requires a mindset that is sensitive to empirical observation which is not what mathematicians are basically trained for. The recent call by professor Wu to research mathematicians to start participating actively in the education enterprise (see the *AMS Notices* March 2011) calls for the wrong cavalry. We need engineers with an empirical set of mind rather than abstract academics. The mathematics required for schools likely can best be called *Neoclassical Mathematics* and is based upon the books *A Logic of Exceptions*, *Elegance with Substance* and *Conquest of the Plane* and now in 2015 also *Foundations of Mathematics. A Neoclassical Approach to Infinity*.

Introduction

If we want to improve the education in mathematics then we must consider the content, the education of teachers and the tools. Below gives an outline redefinition of the content into Neoclassical mathematics (NM). For the (re-) education of teachers we need the involvement not quite of research mathematicians but rather of the empirical sciences, since education is an empirical issue. The tools follow from these.

This point of view differs from the distinction by Hung-Hsi Wu (2011ab) into *Research mathematics* (RM), *School mathematics* (SM) and *Textbook SM* (TSM). Wu estimates roughly that TSM may contain an error every two pages. Teachers get TSM as basic education and RM at higher education, and never really arrive at some ideal SM. Wu (2011a) is a call for action directed at research mathematicians to co-operate with the teaching community to actually create that SM and its (re-) education of teachers. Wu's call does not mean that only research mathematicians can help out, since also the education community has a stake. However, it is not a call to empirical science.

The distinction between these two views concerns empirics. The education community is insufficiently empirical and research mathematicians may help but might also do damage. Mathematicians are trained for abstract thought but pupils happen to occur in real life. Teachers try to resolve their cognitive dissonance by relying on tradition, but traditional mathematical content is a nightmare. The ideal SM that Wu paints still suffers from the very same blindness to reality. Creating more consistency into a nightmare does not remove the very nightmare itself. This will be illustrated below with a discussion on fractions. A longer exposition and many more cases can be found in *Elegance with Substance* (EWS) and *Conquest of the Plane* (COTP). I admire professor Wu for his insights in and contribution to the education of mathematics. That even professor Wu falls into the trap of underestimating empirical science shows how difficult the subject is.

We can only hope that the issue gets the best of our possible attention and therefor I advise each nation to have an enquiry by its national parliament. Mathematicians should be the first in line to ask Parliaments to help them to carry the burden assigned to them of

⁴² <http://thomascool.eu/Papers/Math/2011-09-06-NeoclassicalMathematics.pdf>, while this edited version of 2015 is also included in FMNAI.

caring for the education in mathematics. Parliaments can be motivated by the properties of mathematics education: the costly investments in manpower and computer programs and equipment, as well as the level of education itself and the economic consequences.

The name "neoclassical mathematics" derives from the foundations of mathematics. We are familiar with the distinction between *logicism*, *formalism* and *constructivism* as those arose around 1900. Classical mathematics (CM) came into two major problems:

(1) The 'division by zero' of the derivative created historically the approaches of (a) exhaustion by Antiphon and Eudoxos, (b) infinitesimals by Archimede, Newton and Leibniz, (c) algebra by Euler and Lagrange, (d) limits by Cauchy and Weierstraß.

(2) With the Liar paradox of the ancient world there came the paradox by Russell and the theorems by Gödel.

These issues (1) and (2) however are resolved by *A Logic of Exceptions* (ALOE), and see the review in the Dutch journal of mathematics NAW, written by professor Gill (2008) of the Dutch Royal Academy of Sciences. Hence it is possible to teach mathematics again in quite classical perspective, using 2000 years of didactic advance as well of course. Research mathematics might continue with the neglect of ALOE but keeping this neglect in research only would not put a burden on education (though possibly on financial markets and such).

An offspring of ALOE is *Conquest of the Plane* (COTP) with a favourable review by Gamboa (2011) on the website of the European Mathematical Society. The book *Elegance with Substance* (EWS) lies in time and purposes between ALOE and COTP, and got a mixed review by Limpens (2010), who is critical of some aspects but in sum appreciates the critical look at mathematics itself. A favourable review of EWS and COTP is again by Gill (2012).⁴³

My suggestion is that the reform could be for K-12 but also the first year of college or university, but when the discussion takes place in the context of professor Wu's paper then it suffices to use the term "schools".

We first consider the fractions and then give an outline of the neoclassical approach.

Fractions, but also division in general

(a) First consider $2\frac{1}{2}$ for "*two and a half*", where the position next to each other means addition. Secondly consider $2a$ for "*two times a*" or $2\sqrt{2}$ for "*two times the square root of two*", where the position next to each other means multiplication. Comparing these, the adjoining positions thus are interpreted differently, and pupils must be trained to see the difference. This also causes that we must make sure that there is a space inbetween $2\frac{1}{2}$ when we want it to reduce to 1. This tradition of different interpretations of positions is curious, but it might be acceptable when we use typesetting with fixed places. The tradition however is asking for problems in handwriting when a pupil may write $2\frac{1}{2}$ as $2\frac{1}{2}$ or conversely, and thus slip into error. The solution is abolish the notation $2\frac{1}{2}$ and to keep $2 + \frac{1}{2}$ so that the "+" nicely reflects the "and" in "*two and a half*" and so that the "+" may also be an end-station. This is similar to the case that $\sqrt{2}$ can be an end-station and need not be expanded into decimals 1.414... It takes a huge amount of time to train pupils now to write $2 + \frac{1}{2}$ as $2\frac{1}{2}$ (and not reduce this to 1), and later again to unlearn this positional approach for $2a$. The only reason for this waste of time is tradition for tradition's sake.

2015: See the suggestion how fractions as we know them can be abolished.⁴⁴ Observe that the pronunciation of fractions also abuses rank order names: traditionally $1 / 5$ is

⁴³ For completeness, let me mention a slandering text that is no real review, see page 96 and <http://thomascool.eu/Papers/COTP/LOWI/Index.html>

⁴⁴ <https://boycottholland.wordpress.com/2014/09/04/with-your-undivided-attention/>

pronounced "a fifth", which means that fractions borrow from a different concept (ranking: 1st, 2nd, 3rd, ...) and that it is better to pronounce $1 / 5 = 5^{-1}$ as "per 5".)⁴⁵

(b) EWS and COTP both present a *proportion space* and defend the point of view of Pierre van Hiele that kids at elementary school would be able to work with vectors and thus a vector space. The proportion and vector spaces need an integrated discussion otherwise there arises confusion.

(c) Another point is that division essentially links up with the algebraic approach to the derivative. Since Cauchy and Weierstraß we have been trained to focus on numerical aspects but Weierstraß already uses predicate logic and it appears that algebra and the logic of the manipulation of the domain create the derivative just as well. Even better, since this eliminates the paradox of 'division by zero' and it avoids the educational combi-load of both limits and the derivative. Limits and infinitesimals are useful, e.g. for the understanding of real numbers and approximations, but not necessarily for the derivative of functions used in K-12 (and likely wider). See EWS, COTP and FMNAI. Hence, a good understanding of division is not only required to survive 3rd grade but also the derivative.

These insights (a), (b) and (c) are missing in Wu (2011c). It merely illustrates the importance of the empirical approach to education, and may cause the reader to look at the other cases mentioned in EWS and developed in COTP.

An outline

Neoclassical mathematics has no precise definition yet but uses ALOE, with an application to education in EWS. The latter is implemented again in COTP. Now there is also FMNAI. Neoclassical mathematics gives a point of view that Aristotle and Euclid supposedly could live with, and that people might find rather natural to understand. Some points are:

(1) The Liar and Gödeliar statements are nonsensical, in a three-valued logic.

(2) Russell's set paradox and Cantor's Conjecture for infinite sets are nonsense too. We may use a *set of all sets*. There are no '*transfinites*'.

For example, Russell's set is $R = \{x \mid x \notin x\}$. This definition can be diagnosed as self-contradictory, whence it is decided that the concept is nonsensical. Using a three-valued logic, the definition is still allowed, i.e. not excluded by a *Theory of Types* (that makes it non-sensical too). Statements using it receive a truthvalue *nonsense* or *Indeterminate*. An example of a set similar to Russell's set but without contradiction is $S = \{x \mid x \notin x \wedge x \in S\}$ which definition uses a small consistency condition, taken from Paul of Venice, see ALOE:127-129. See FMNAI, also for a discussion on infinite regress.

(3) Euclidean space is defined as our notion of space. Non-Euclidean space can only be imagined in Euclidean space.

(4) The natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ are denumerable, with a potential infinite while their 'total' is an actual infinite. The continuum \mathbb{R} or the interval $[0, 1]$ is also an actual infinite. There is a *bijection by abstraction* between \mathbb{N} and \mathbb{R} such that these are 'equally large'. See FMNAI.

(5) Probability and statistics in relation to the *sorites paradox*. (Not developed.)

⁴⁵ <https://boycottholland.wordpress.com/2014/08/25/confusing-math-in-elementary-school/>

Table 7. Comparison of traditional and neoclassical mathematics

	<i>Traditional mathematics (TM)</i>	<i>Neoclassical mathematics (NM)</i>
1	Two-valued logic. What is nonsensical is excluded by restrictions on form	Three-valued logic. Closer to linguistic freedom. What is nonsensical is explicitly called nonsensical (ALOE)
2	Gödel's theorems on undecidability	Under some stronger properties of the proof predicate the Gödelian sentence causes a contradiction so that it can be judged to be as nonsensical as the Liar statement. There remains a similar kind of philosophy: that mathematical activity by mankind has the fundamental uncertainty that some inconsistency may pop up (ALOE)
3	Zermelo Fraenkel axioms of set theory, also to deal with Russell's paradox	Self-reference is allowed, and nonsensical cases like Russell's paradox are recognised for what they are. ZFC is inconsistent (FMNAI)
4	Cantor's Conjecture on the power set	The conjecture holds for finite sets but not for infinite sets. The diagonal argument appears to be nonsense (FMNAI)
5	Difference between denumerable and non-denumerable infinity. There are transfinities	Potential infinity associates with counting, actual infinity associates with the continuum. There is a bijection by abstraction between natural and real numbers. There are no transfinities (FMNAI)
6	Weierstraß for the derivative of regular functions (i.e. used in highschool)	Algebraic definition of derivative and integral for such functions. Limits are useful but not for the derivative. (Possibly Weierstraß for other functions.) (ALOE, EWS, COTP)
7	What is 'space' depends upon axioms	Euclidean space is defined as our notion of space. Non-Euclidean space can only be imagined in Euclidean space (COTP)
8	Arrow's Theorem shows that ideal democracy is impossible	A key property of the ideal of democracy is that it should work. Hence one of Arrow's axioms has to be rejected. This appears to be the axiom of pairwise decision making (VTFD)
9	Mathematics education for highschool and first year of college requires training on traditional concepts	Mathematics education requires a fundamental re-engineering. Much of mathematical content will remain the same but there are key gains in consistency and didactics (EWS, COTP, FMNAI).

(6) Mathematics (abstraction) and engineering (dealing with reality, that mathematicians call "approximation") are discussed in conjunction, to foster sensitivity to the translations. For example, measurement errors due to the constancy of the speed of light do not mean a distortion of space but remain measurement errors.

(7) An encyclopedia of mathematics, e.g. what might result if some assumptions are changed. For example fuzzy logic, the Brouwer-Heyting axioms, incompleteness, computability, transfinites, fractals, chaos theory ...

(8) Democracy is a key concept but generally misrepresented by mathematicians, see my book *Voting Theory for Democracy* (VTFD). Mathematician Kenneth Arrow claimed that reasonable and morally desirable properties caused an inconsistency, and hence that what we ideally expect from democracy would be impossible. This however appears to be unwarranted. See on the VTFD website how some mathematicians are still locked in denial of reasonable analysis on democracy.

A table

Table 7 gives an overview of the differences between traditional and the proposed neoclassical mathematics. The table lists my books in which the points are discussed.

Conclusion

If neoclassical mathematics as indicated above is adopted as school mathematics then professor Wu probably still might be happy that there at least is a SM, and undoubtedly many kids would be happy too. The choices involved will be clear. When research mathematicians (RM) drop the nonsense and look more into engineering with sound standards, and when the empirical sciences look into the education in mathematics, then there will be more cause for hope for improvement. Since so much is at stake and since professionals entertain standards that currently cannot be met without sizeable investments, and since individuals should not try to do the impossible, it is advisable that the national Parliaments investigate the issue.

XIII. A key insight in the didactics of mathematics

Adapted from §15.2 of *Conquest of the Plane* (2011)
Also included in *A child wants nice and not mean numbers* (2015h)

Introduction

This chapter has been adapted from *Conquest of the Plane* (COTP) §15.2. COTP is a *primer* for highschool and first year of higher education. The issue is relevant for mathematics education in general however. The contribution by Pierre van Hiele is generally not recognised for its importance, see Colignatus (2014a). The following reviews the situation and suggests the discovery of a *missing link*.

The didactic approach

Learning goals are generally knowledge, skills and attitude. The didactics are guided by the Van Hiele levels *concrete*, *sorting*, *analysis*, or, with the latter split w.r.t. formality:

- Level 0: visualization and intuition
- Level 1: description, sorting, classification
- Level 2: informal deduction
- Level 3: formal deduction

Importantly, at each level the *same* words may be used but with *different* intentions, complicating discussion and understanding.

Van Hiele (1973:177) gives the following example, and (1973:179) explains: “At each level we are explicitly busy with internally arranging the former level.” (my translations):

- (1) An isosceles triangle is recognized like an oak or mouse are recognized.
- (2) It is recognized that this triangle has the property of at least two equal sides or angles.
- (3) Relations between properties are recognized: at least two equal sides *if and only if* at least two equal sides.
- (4) The logical reasons for these relations are considered: why *if*, and what does it mean to reverse an implication ?

Van Hiele (1973:179) on geometry:

“At the base level we consider space like it appears to us; we can call this *spatial sense* (like *common sense*). At the first level we have the geometric spatial sense. (E.g. measuring degrees of an angle / TC.) At the second level we have mathematical geometric sense; there we study what geometric sense involves. At the next level we study the mathematical logical sense; it then concerns the question why geometric manners of thought belong to mathematics.”

The levels do not provide information about the boundaries of topics, and they are not strong when it comes to finalizing a topic and switching to a next one (that builds upon the earlier). In this book we mostly look at Level 1 and 2, and there are some patches that peek into possibilities for Level 3. The reader should be able to identify the spots.

In moving from one level to the next, Van Hiele (1973:149+) identifies phases:

- (a) intake of information (examples)

- (b) bounded orientation (direct instructions)
- (c) explicitation (making explicit, verbalization in own words of what is known)
- (d) free orientation (extending the relationships in the network)
- (e) integration (summarizing and compacting what has been learned, often old fashioned learning).

Van Dormolen recognizes similar stages: Orientation, Sorting, Abstraction, Explicitation, Processing & Internalisation (OSAEP/I).

We reject Freudenthal's "realistic mathematics education" (RME) in its more extreme interpretation. This is best discussed in separate paragraphs.

It hinges on what counts as experience

Van Hiele and Freudenthal overlap in the starting point in experience. The question remains what kind of experience we choose:

- Working in the plane itself is seen by Freudenthal as too abstract
- while Van Hiele in principle allows the notion that it might be experience too. Mental thought is an abstract process by nature and we can have experience in that.

Modern research on the brain clarifies many aspects of mental processes. Operational definitions of thinking and consciousness however cannot replace the definition of thinking as experienced by the conscious self. When we look for a definition of what thought is, in this experience of being conscious, then we quickly arrive at a Platonic version of ideas. In the mind's eye a triangle has a purity about it that is not caught in any drawing. Also mudd becomes perfect mudd. There is no difference between an image of a triangle and an image of mudd, or even an image of a sunset, in the sense that they are constructed out of the same mental elements that can only be pure. It are these mental ideas that education deals with, and experience in reality is only a tool to reach them. This does not mean that we have to be full Platonists in assigning an indestructible and immortal quality to these ideas. Thought and thinking, consciousness and awareness, are primitive notions for the thinking intellect itself, and up to this day and age of human history they do not generate any additional information for more conclusions than their very experience.

The paradox – seeming contradiction – is that Freudenthal was an abstract thinking mathematician who developed an abstract notion of "realism", while Van Hiele was a practicing school teacher who was open to the relevance of abstraction itself.

There is a difference between:

- designing a mathematical model, as in applied mathematics, by someone who already has a command of mathematical concepts, with the aim to match properties,
- learning to understand and developing a command of those mathematical concepts.

What we can assume and build upon

Students and pupils have sufficient experience with the plane since making drawings in kindergarten. When they think about a triangle it is as abstract as it can get because such thought is abstract by nature. We can draw many triangles on paper but the notion of a triangle in the mind is an entirely different matter, and when the student or pupil thinks about a triangle then it is that notion that is in the mind and not the drawing on the piece of paper. What counts are the lingering notions in their abstract imagination that have to be activated. When we put labels to angles on paper and draw supporting lines then we use paper images to enter new concepts into the mind. It remains an essentially abstract activity, with pen and paper only tools for communication. It distracts and confuses when mental clarification is mixed with the application to reality. Application to reality is relevant but should be dosed wisely.

Finding the proper dose and perspective

My book *A Logic of Exceptions* maintains that the force of logic derives from reality. If a truck approaches and if you do not jump aside then it will hit you. Mimicking this, *A Logic of Exceptions* starts with electrical switches to clarify the constants of propositional logic. In this case we do not need to explain these constants since we presume that students already know them. We only help making them explicit. The empirical examples are only intended to highlight the properties and to pave the road towards formalization. Here the electrical switches do not distract since the case is not presented as an exercise in building electrical circuits. The examples help to focus on the logical properties. Electrical switches are as good an example as language, and in a way a better example since the focus in logic is already so much on language that it helps to provide another angle.

For analytic geometry it may be argued that a bucket and a faucet that adds a liter per minute would be a similar good starting point. This is dubious however. If the objective is to distinguish linear processes from other processes then indeed examples in reality are the stepping stones, but that is another issue than linking up with geometry. The example distracts from the very abstract notion that we want to establish. “Realistic math” might require a student to spend a sizeable part of the lesson time on realistic examples trying to figure out what is the point. When supporters of “realistic math” argue that students of geometry do not understand a linear process without such examples as the bucket, then the reply is that those teachers have not spent sufficient effort in providing the abstract tools to perform the mental process.

It are different mental processes: imagining a bucket and faucet and imaging a graph of a linear function.

- The bucket and faucet have been learned in kindergarten.
- The graph and its geometric interpretation first have to be learned before they can be imagined and linked up to the bucket and faucet.

Once we have the graph then it is OK to say, and indeed we ought to say, that the bucket and faucet are an interpretation and application, and only then there can be that flash of understanding that shows that the link has been achieved. Once an aspect of the plane has been conquered then abstract understanding can be easier related to those other cases from reality, which means that those other examples are relevant for the Van Dormolen Processing & Internalisation stages. *But first we must develop the geometry of that graph, using the mental images of geometry itself.*

The challenge

There is a challenge though. Eudlid's *Elements* and his axiomatics have been the standard for more than two millenia. They are at Van Hiele's highest level. Perhaps 12-year olds can deal with those abstractions, as they actually are rather simple. But it becomes a bit different when we try to incorporate the advances in analytic geometry and calculus. Here are concepts that better be developed at a lower level and Van Hiele then wins from Euclid. Here Freudenthal steps in and resorts to the richness of reality, and at first that seems like a golden solution. Indeed, axiomatic geometry is at Level 3 and not at Level 0 ! However, as explained Freudenthal's approach is not convincing since it neglects that thought is abstract by nature. Rather than going sideways into reality we should focus more on the processes of thought and thinking itself.

A missing link

We should provide for an abundance of words and concepts in the abstract plane, so that the student has enough to hold on to for visualization and intuition. A key observation is:

A missing link in geometry appears to be that those anchors are rather absent.

When you visit a new city then you tend to like it when the streets already have names. Suppose that you would be forced to invent your own labels, like “that crooked street with the blue shop” and then hope that other people understand you. Current textbooks on geometry send out students to conquer the plane but present it as a verbal desert, without conceptual guidance other than the x and y co-ordinates. The Van Hiele Level 0 requires them to visualize and to activate their intuition, yet that also requires a richness of words and concepts – that currently are lacking. Euclidean geometry has a poverty of points and lines that can intersect, be parallel or overlap: and though it is a great exercise in logic it must be admitted that Freudenthal has a point that Euclid’s approach is not so appealing to the average student over the last two millenia. Conventional analytic geometry is an improvement since drawings are supported with formulas, and vice versa, yet again, its richness is only developed over time, and at the Level 0 and 1 there still isn’t much to visualize and intuit and verbalize.

In particular, it will be useful to extend the plane with a nomenclature of “*named lines*”. Chapter 4 of *Conquest of the Plane* opens with them and then builds up – see there to check what this means. A quick reply will be that we already have names, such as $x = 1$, $x = 2$, for vertical lines for example. Those names derive from a formal development however. Instead we rather first create standard names that fit the experience with the plane. This will provide the fertile ground, where the coin can drop when experience is morphed into abstract understanding.

It may be argued that it is fairly simple to draw a line and determine the starting value on the vertical axis and its slope. Exercises and realistic examples then provide for learning. However, experience shows that students later have difficulty with the horizontal and vertical lines. Why a line works as it does tends to remain elusive for them. A conclusion is that it is better to start with named horizontals and verticals and then awaken the motivation that a general formula will be useful.

Thus the didactic suggestion here is that the notion of “*named lines*” can be the missing link that resolves the issues in the choice between dropping Euclid and moving towards analytic geometry and calculus (and not just Descartes but along the lines of Van Hiele). The notion of these named lines caused the very layout of Chapter 4 on lines and subsequently from there the layout of the whole *Conquest of the Plane*.

Co-ordinates and vectors

Pierre Marie van Hiele argued most of his life (May 4 1909 - November 1 2010) in favour of the use of vectors already in elementary school. Though he has been greatly valued for his ideas on the didactics of mathematics, he never succeeded in overcoming the opposing views. Vectors even appear late in highschools. The missing link suggested here of *named lines* is hopefully helpful. Logically, if this is indeed the missing link that has been provided only now, then teachers seem to have been right in resisting Van Hiele’s suggestion, since the picture is complete only now. Alternatively, the suggestion of *named lines* is not really a missing link and only one of the possible bridges, and we are underestimating the capabilities of pupils and students all over the board.

Clearly, the proof of the pudding is in the eating, and only empirical testing will show whether students indeed learn faster following the didactic approach presented here. If this book would be mistaken, and “realistic mathematics education” would still be needed to propel the more practically minded students, then, the lame argument becomes, it would suffice to include it in this book as well, and the advantage of this book would remain to be its logical order and novel concepts.

The importance of motivation

A final point of note is that I do not have clear ideas about what would motivate a pupil in elementary school to be interested in arithmetic and geometry, or a 12 or 14 year old kid

to be interested in analytic geometry and calculus. Van Hiele (1973) rightly remarks that students and pupils hardly can be motivated for what they learn since they do not know yet what they will learn. A common ground is that man is a curious ape and cherishes the flashes of insight. Pupils recognise the moments when they grow in competence. Mathematics is a language and it can be fun to learn a new language and a new world. Paul Goodman (1962, 1973) *Compulsory miseducation* remains sobering though. While my books on mathematics education concentrate on knowledge the didactic setting naturally is a complex whole, in which motivation plays a key role, and it is mandatory to keep that in focus too.

XIV. Residual comments

This chapter collects comments that do not find a natural place in the other parts of the book but that still seem useful to include.

Scope

Kind of mathematics

This book does not cover all angles in mathematics. Its math tends to be a bit literal, with logic, notation and procedure. We should also consider geometry, shape, patterns, symmetry, regularity, order vs chaos, topics in probability and statistics. Pierre van Hiele has been arguing that kids at elementary school can already work with vectors, if only they are allowed to. The scope for improvement is large indeed. Perhaps even the abstractions of category theory. This book stays rather close to the traditional curriculum, it is actually quite conservative and it might well be that a more fundamental change is better.

Notation

This book puts some weight on issues of notation. Notation in itself seems a trivial issue. Mathematics is done in the mind (or subconsciously, with the conscious part mostly in the spectator role). The mind codes addition and other operators in a different manner than we on paper. Notation however is important for communication. It becomes especially important in learning, especially for the weak student. Confusion quickly sets in, and wrong habits are hard to undo. We also have seen the link from notation to the more complex issues. Thus, the notational examples might seem trivial but their triviality also reminds us that those issues should have been solved long ago.

Notation and psychology

Part of the issue can be seen in the Dutch distinction in math tracks, math A and B. These tracks cater to different psychological capacities (that are stimulated by tracking them). Track A relies for understanding on context, tends to a (vague) helicopter view and is less analytical. Track B is less influenced by or sensitive to context, or too sensitive so that it is better reduced, cannot do without an overview but digs analytically deeper. The good mathematician and in particular the econometrician does both, takes the context, makes the model, derives results, has an eye for detail, maintains that helicopter view, and also sees the purely mathematical properties behind all of it. Not everyone can play two instruments. Concessions must be made for practical education, and then issues of notation start playing an important role.

General phenomena and properties

The examples are not just issues by themselves but can be caused by deeper processes, sometimes making them instances of those processes.

In some notational issues that we have considered the underlying property was that you have to develop a local schizophrenia. The key example is writing $2 + \frac{1}{2}$ as $2 \frac{1}{2}$. Some pupils and students can do so but then are conditioned so that they do no longer see what they do. It is a prerequisite of becoming a mathematician. An alternative example is the switching between a verb and a noun. In this case the switch is potentially productive

instead of burdensome. The general property may well be cognitive dissonance, or reflect fundamentally how a brain works.

Mathematicians can be observant of confusions but once they have defined confusion away then they can be less observant in seeing the value in what people continue to tend to do in opposition to those definitions. Pupils and students over many generations have been right about the cumbersomeness and irrationality in mathematics. Supposedly they could not put a finger on precisely what the problem was but that was not their responsibility. What is crucial is that they got not listened to.

The best mathematicians would be happy with what they had learned themselves and would focus on new problems. Teachers face another trap. It is kind of natural to think that when you are the teacher then you must tell others what they must learn. But putting up a wall is something else.

Conjectures and refutations

Pretentions

A reader of an earlier Dutch version thought it pretentious to say “what is called mathematics but actually isn’t so”. But my examples were not refuted. The examples are not just examples, they are cases. They build up to *sufficient evidence* that an enquiry by parliament is desirable. They are called *examples* since other cases can and likely will show up. Each example likely could be handled without parliament sitting in but the total adds up. Math textbooks really should look different from what they are now.

More research

It has been suggested that the issue requires more research, and that I would do this myself. However, my role is limited, and definitely my resources. I already referred to some sources on the history of mathematical education, the policy changes over the decennia, developments in didactics. This is sufficient. My role is precisely to clarify that need for enquiry. The first step is an enquiry by parliament.

Prodding with questions

Another suggestion was that I would pose questions rather than solutions and opinions. This book indeed has a more open style than the original in Dutch that was argued more directly. Indeed, questions are more friendly than direct critique. They are necessary for teaching, also outside of the classroom. Prodding with questions may have a larger effect since readers discover themselves that something may have to change. My reaction is that I am a bit beyond the posing of questions. Questions have been posed for decennia. Kids have been brought to tears for failing math and because this affected their self-image and life. Mathematics allows clarity. Let us use that clarity. Mathematics fails. The education in mathematics does not give what we would expect from it. This is not a criticism per se but an expression of standards. Pointing to successes in the number of students that currently pass their highschool graduation is a bit awkward since they have been taught ‘mathematics’ that isn’t really mathematics. *Lies, damn lies, and statistics.*

Respect for the efforts made

Readers might feel that this book is disrespectful of the efforts by mathematicians and teachers of mathematics. If that is the impression indeed then let me correct it. There is great respect and gratitude. I stated that mathematicians are not quite blind to the issues raised here. They notice the hardship with their students. My problem is merely that they don’t dig deeper. They keep the illogical material and then work on better didactics so that students can learn the illogical more easily. When mathematicians operate like this they are not tuned to the logic of empirical observation but to blind obedience to

traditional authority. My idea is to set them free. It is the mathematical thing to do. Mathematicians will only be convinced if they see that the mathematics can be improved.

A note, hopefully subtle enough

Admittedly, some mathematicians in Sumer, 5000 years ago, and other mathematicians working in the middle ages, 700 years ago, were doing real math, even though their tools were, in the terms of this book, cumbersome and illogical. It would not be correct to say that these were only astrologers and alchemists since there was real effort (at times) at abstract thought and deduction. Without their cumulative effort we would be nowhere. But this is not the topic of discussion. It would become the topic of discussion if you would suggest that highschool math is replaced by Sumerian math, since, as you would hold, the only goal of education would be to teach pupils to think, and Sumerian math would be mathematics too (we agreed on that, in some respects). This kind of discussion tends to become awkward. (E.g. People must have the right to vote. Children are people. Hence children must have the right to vote.) In response, I would rather point to the concrete arguments and amendments given in this book. Current mathematics would be mathematics in the sense that you can think about it abstractly and do deductions on it, but at the same time it would not be real mathematics because of the errors exposed and the kind of attitude that shows from those errors.

Barbarians at the gate

Some circles appear to regard parliament as barbarians. Totally unfit to judge about mathematics education in any way or other. Well, in that case: the barbarians are at the gate! As mathematicians haven't succeeded in bringing their house in order, these barbarians have every reason to think that they belong there too.

Industrial aspects

Bringing about change

We have seen various points that require further development and testing. I argued for change in 2009 but didn't give a blueprint for a new textbook; but 2011 gave COTP. Evidence based education sets new standards. When the mathematics industry starts processing our comments, then critique will turn into a self-critical-attitude. It is too simple though to assume that this will happen just by itself. Society will have to do some prodding.

Education versus didactics

The current infrastructure around the education in mathematics creates its own problems. The capacity for self-organisation is limited, the product ill-defined and the moment of transaction rather vague. In the past there were more competing books, a teacher wrote her own book, found a publisher, and that was it. The size of the market has grown and specialisation is determined by the size of the market. Nowadays there seems to arise a distinction between education and didactics, with real education done in class ("what" to do – e.g. use $2\frac{1}{2}$) and didactics done as a university science executed by Ph D's far away from class ("how" to do – e.g. test the various methods to torture students on learning fractions). The journals on didactics may hardly be read by teachers, and it is a science again to translate results from meta-analyses to possible application in class. Society feels helpless. It has provided the funds to improve on the education in mathematics but it may have created a new bureaucracy. Allowing each his or her ways keeps the peace. The bureaucracy has its uses, for there are 'experts' and people like that idea. In a way it functions. The professional didactics earn their wages, articles are published, websites maintained, teachers are invited for their annual refreshing day. Little seems changed since the tale *Gulliver's Travels* by Jonathan Swift and its parody of the *Royal Society*.

Even the size of the world may not be sufficiently large to create a decent free market for the education in mathematics. Education for highschool may seem rather standard and one textbook on highschool algebra might well suffice for some hundred million highschool students. But the markets are fragmented, likely by their very nature.

Economics of education

There is a subfield in economics, the economics of education. Given my economic comments on the education in mathematics it might be expected of me that I would delve deeper into that. However, I haven't felt the urge. For me it is a matter of logic. (1) The suggestions here would be an improvement. (2) Making the change would come at some cost. (3) The people who would bear the costs (teachers) would not be the ones who would benefit (students, future society). (4) Current society might compensate teachers, in an intertemporal cost-benefit analysis. (5) It is no use doing such calculations if there is no awareness of the issue and no perspective that it will be understood. (6) Such calculations would be difficult anyway since how would we score the effects of writing $2 + \frac{1}{2}$ instead of $2\frac{1}{2}$? (7) Nevertheless, once the awareness of the problem has grown and once objections to a possible changes are based not on the miscomprehension of the content but on true costs, then it would be sensible to see if we can agree on the economic benefits.

Consequences

We mentioned the consequences of the mathematical attitude for ALOE 1981, VTFD 1990, the stock market crash in 2008, and the environment.

Pure mathematicians will hold that they have no involvement with real world data and that it are the other sciences that deal with those. It is a valuable notion. There are also deeper philosophical aspects on how mathematics relates to the world. This 'refutation' however seems to misunderstand this book. This is not what has been argued. This book argues that mathematics suffers from itself. Even these pure mathematicians got lost on logic, $2 + \frac{1}{2}$ and the derivative. Subsequently, what to do with monks who claim to distance themselves from the world but who still want to eat and drink? The Dutch *Masterplan Wiskunde* (MPW) referred to above, page 74, puts a lot of emphasis on the relevance for society – and they don't mind mentioning stochastic diffusion for option pricing just at the moment while the stock market crashed. So we are talking grey areas here and mathematics cannot evade part in the key responsibilities here.

Public relations and the power of math

There is a TV series ("*Numb3rs*") where the hero mathematician helps to catch thugs by the use of mathematical techniques. It is good public relations. The logic is somewhat convoluted though. Undoubtedly mathematical theories can enlighten situations but that does not make mathematics an empirical science. Fortunately, the hero shines out as a person with outstanding ability. The common professor in mathematics would not be able to translate the empirical thug situation to the right mathematical format. Hopefully Hollywood script writers find inspiration to create an interesting series on math education in class. Gladwell (2008:239) contains an example.

A note on Barrow 1993

Barrow (1993:1) on the power of math:

"A mystery lurks beneath the magic carpet of science, something that scientists have not been telling, something too shocking to mention except in rather esoterically refined circle: that at the root of the success of twentieth-century science there lies a deeply 'religious' belief – a belief in an unseen and perfect

transcendental world that controls us in an unexplained way, yet upon which we seem to exert no influence whatsoever.”

I think that Barrow is a bit mistaken on empirical science. Only reality proves what abstraction is relevant for reality. Implications of abstractions are only relevant if the assumptions fit the bill. Barrow also has a Venn diagram of “The material world” and “The mathematical world” with apparently a nonzero intersection. I would consider the latter a category mistake. There can be points that map easily but these are still maps and not identities.

But abstract thought is important. And the philosophical issues are worthy of discussion.

2015: See FMNAI for more on abstraction.

The stock market crash and the classroom

The crash ⁴⁶ clearly has not been caused by mathematics taught in highschool. Teaching math in highschool is interesting because the math is of a fundamental nature and because of the didactics and the interaction with the pupils. Universities carry the burden of complexity and professional integrity. Some mathematicians might hold that highschools should return to hard-core axiomatics to ingrain the proper attitude. Alternatively, this book argues and shows that highschool math already suffers the non-communicative tendencies that are not corrected by universities. Returning more to the axiomatic method would by itself not be the cure.

Taleb (2009) targets ‘the economics establishment’ rather than mathematicians:

“3. People who were driving a school bus blindfolded (and crashed it) should never be given a new bus. The economics establishment (universities, regulators, central bankers, government officials, various organisations staffed with economists) lost its legitimacy with the failure of the system. It is irresponsible and foolish to put our trust in the ability of such experts to get us out of this mess. Instead, find the smart people whose hands are clean.”

For the present analysis, I rather side with Mandelbrot and Taleb (2009) and Salmon (2009) who point to the underlying math and the shared responsibility of mathematicians. Mathematicians appointed to become professors in economics do not, by that act, become economists.

2015: In the July *Newsletter* of the Royal Economic Society, I had a short note (2011a) on the “Ricardian Vice”, i.e. an economic theorist's error to mistake model for reality. A major reference was to Steinsaltz (2011) in the AMS Notices.

Statistics

Perhaps empirically and statistically it does not matter so much. Suppose we adopt all the suggested improvements. This only means that there is more scope for better teaching and learning but now the burden falls on the implementation in the classroom. Then the statistics become of prime importance. Perhaps all improvement disappears in the error fudge of individual diversity. This book takes a logical position while using the available information, but, indeed, it might not be enough. We can likely only tell after changes have been tried.

⁴⁶ https://en.wikipedia.org/wiki/Stock_market_crash#Crash_of_2008.E2.80.932009

The farmer and his animals

At a young age children assume that the farmer exists to look after the animals. Old presumptions die hard. At some age reality gets through and it is seen that the animals are there for the farmer.

A waste processing plant also created aerial dioxine that dropped on meadows around it. Quality controllers on milk wanted to forbid further grazing by cows there. The farmers protested that it was not their fault and that the milk should be allowed. Thus, also consumers are there for the farmer.

It is a good trick of the educational community to have some central exam requirement. For now it are the parents who want their kids to qualify instead of the teachers looking for a job. The professionals set the standards and the demand side has to qualify.

There can be all kinds of arguments of a self-serving nature that can be used to defend the bastion. The simplest is to argue against change since that would be difficult for the pupils and students while in reality it would only be a hassle for the teacher.

The golden rule in education is not to kill the natural interest in learning new things. Somehow this rule is broken regularly. It indicates an imbalance in the distribution of power between demand and supply. A teacher depending upon results would have every incentive to keep kids interested in learning.

Of course, as the literature on incentives shows, they may have unintended effects. This holds as well for current incentives. The master / apprentice relationship seems the most sturdy model. This book is not on that aspect but it can be usefully mentioned.

Freudenthal

The first draft of this book didn't mention Hans Freudenthal (1905-1990). This seemed a bit strange with respect to the references to Holland where the Freudenthal Institute was created for research in the education in mathematics. Some comments may correct the imbalance. Freudenthal (1978) holds that there is no science yet for the education in mathematics, and his book is a 'preface' to something that doesn't exist yet. His observations on 'evidence based education' at that time seem on the mark – though of course I haven't checked since that would be a gigantic task. Freudenthal (1991) is more tedious. Once we consider education in mathematics as an industry, we naturally get details and complexity, with plans, technological term, etcetera. These books are still useful to consider. However, they do not change our current conclusions.

2015: Regrettably, I discovered that Freudenthal took key ideas from his Ph.D. student Pierre van Hiele (1909-2010) and did not refer properly where it all came from. He also misunderstood those ideas and changed them into an amalgam of his own called "*realistic mathematics education*" (RME). One can try to make a case that if Freudenthal created his own amalgam then there would have been no need to refer to Van Hiele, but this is not how science works. The reader is invited to compare the ideas of Van Hiele and Freudenthal and the conclusion of a breach in scientific integrity will be clear. While Van Hiele should have become professor in the education in mathematics, Freudenthal took that position himself, and caused havoc. Freudenthal appears to have been trained to be an abstract thinking mathematician, without understanding what empirical science entails. I plea for a rehabilitation of Van Hiele and for an abolition of the "*Freudenthal Head in the Clouds Realistic Mathematics Institute*" in Utrecht. See this weblog entry ⁴⁷ and the paper Colignatus (2014a).

⁴⁷ <https://boycottholland.wordpress.com/2014/07/06/hans-freudenthal-s-fraud/>

XV. Conclusions

Intermediate conclusions

Mathematics is man made. Education is man made. Pupils and students are people too. We can only say something about education in mathematics when we have the proper empirical attitude and attention for reality.

Education has progressed a lot but in this day and age we don't get good results and there is still much to improve.

We found:

- Spatial sense and understanding are hindered and obstructed by subservience of the line to the function, inconsistent names of parameters, switches in orientation of tables and graphs, opaque or inconsistent terms, cumbersome treatment of derivatives, maltreatment of co-ordinates, vectors, complex numbers and trigonometry.
- Algebraic sense and competence are hindered and obstructed by inconsistent brackets, switches in plus / times with fractions, language idiosyncracies, the cult of the radical sign, intractable terms, untenable conventions of exactness and approximation.
- Logical sense and the competence in reasoning are hindered and obstructed by above confusions and cumbersomeness, the withholding of explicit discussion of logic and set theory, the withholding of the basic calculus of probability, by not supporting the development of mathematical ability in general by means of such formalizations.

What is seen as mathematics appears to be illogical and/or undidactic. Hence it has to be redesigned. It is no use to improve on the didactics of bad material, it better is replaced. We also considered only a number of topics, a selection of ideas that this author found interesting to develop a bit. More can be found. We should allow for the possibility that teachers have more comments and suggestions themselves (though our critique is that either they don't have them or don't follow up on them). The situation is wanting.

This book looks at the result rather than at how this situation could have come about. Still, if the result is inadequate, the conclusion is warranted that some cause is wrong.

One of the most important human characteristics is the preference for what is known and familiar – and mathematicians are only human. They adapt to new developments and are are critical and self-critical, not only with respect to what is discussed but also on how things will change. Nevertheless, key issues got stuck, and the industry as a whole is incapable of freeing itself from grown patterns. New entrants in the industry are conditioned to the blind spots, and pupils and students suffer from them.

The situation is not such that there are no mathematicians to improve on content and that we lack researchers in didactics to improve on that angle. This book will hopefully be read by some in both groups and contribute to improvements. But it would be wrong for governments to think that it would suffice to leave the matter to the industry, and possibly give more subsidies for more of the same. More funds may well mean more outgrowth of awkwardness, cumbersomeness, irrationality. A call for more teaching hours may well

mean more hours to mentally torture the students even more. Given this whole industry and the inadequate result the conclusion is rather that the whole industry is to be tackled.

Indeed, it sounds so well. Mathematicians will hold that only they are capable of deciding what is 'mathematics'. Researchers in the education of 'mathematics' will hold that they do the research and nobody else. Will they regard this book as 'research in the education in mathematics' ? *Quis custodet custodes* ? It will be a mis-judgement to provide the industry with more funds without serious reorganization.

In sum, we have considered the work of men and found them to be men. It is a joy to see all these issues that can be improved upon. Let us hope that mathematicians proceed in this direction indeed. Let economists and the other professions support them.

2015: In Holland the State Secretary on Education Sander Dekker has observed that arithmetic skills are below requirements. He avoids a diagnosis on the Freudenthal "Realistic Mathematics Education" (RME) and thus he doesn't require a reschooling of the 150,000 elementary school teachers. Instead he shifts the burden to the 4,000 teachers of mathematics in secondary education, by requiring an additional arithmetic test for highschool graduation. Apparently he is not aware that creation of arithmetic competence in elementary school is required for later algebraic competence in secondary education. I am sorry to report that there is a breach in the integrity of science in the mathematics education research, so that Mr. Dekker does not get scientifically warranted information. At KNAW there are some abstract thinking mathematicians who think that they know more about mathematics education than empirical scientists, and they don't care about the evidence to the contrary.⁴⁸

Check out the weblog or the book website for developments.⁴⁹

Final conclusion

My final conclusion definitely applies to Holland. I tend not to judge about other countries. But the same cumbersome and illogical issues can also be seen internationally. There is a structure to it. It is part of the economics of regulation. Didactics require a mindset sensitive to empirical observation which is not what mathematicians are trained for. Tradition and culture condition mathematicians to see what they are conditioned to see. The industry cannot handle its responsibility. This must hold internationally, country by country. A parliamentary enquiry is advisable, country by country.

Parents are advised to write their representative – and not only those parents who pay for extra private lessons. The professional associations of mathematics and economics are advised to write their parliament in support of that enquiry.

⁴⁸ <https://boycottholland.wordpress.com/2014/07/16/integrity-of-science-in-dutch-research-in-didactics-of-mathematics/>

⁴⁹ <http://thomascool.eu/Papers/Math/Index.html>

Epilogue

Epilogue 2009

It is useful to be aware of the following. With respect to the Rijken van Olst **Figure 1** it can be observed that an econometrician has more scope to be misunderstood on more angles.

My books ALOE and VTFD referred to above have not received much attention. This holds in general, also for my fellow economists but also for mathematicians, who would be potential readers with respect to the theorems discussed there. Some readers might think that this explains my criticism on mathematics and mathematicians. So let me recall what I wrote, in the *Introduction* about my nature. It is not affected. It is not logical to interpret lack of attention and/or appreciation into something that is targetted at my person and that would have to affect the way how I feel. OK, I miss out on some satisfaction of the meeting of minds but the potential readers who neglect ALOE and VTFD miss out on some good books and fundamental theory, and their attitude and misunderstanding reflects on them rather than on me.

With this established, it is useful to be specific on these points.

- For VTFD I refer to a text on my website.⁵⁰ The mathematicians who clearly did not understand voting theory later participated in the already mentioned *Letter to the governments of the EU member states* advising the use of the Penrose Square Root Weights (PSRW) for the EU Council of Ministers. This letter was misleading in argument and professionally deficient on VTFD, see Colignatus (2007c). In a parallel track, there was a sorry episode with wikipedia – quite sad for its users. The main perpetrator was a math student from MIT.⁵¹
- For ALOE, I can refer to ALOE itself as it explains what happened in 1981. In short, see page 61 above. In ALOE I already applied the distinction in economics on static and dynamic analysis to propositions (static) and inference (dynamic) in logic. We see this return in the distinction between verb and noun. The professor who did not appreciate ALOE in 1981, in the discussion back then did however appreciate that distinction. He later got the Spinoza Award for a project “*Logic in Action*” from 1997 to 2001. It is not clear to me whether there is proper reference.

What happens with all of this is not so material by itself. Though it is relevant to observe that science apparently lacks adequate avenues to channel problems with professional conduct.⁵²

I must confess to one important personal point of interest however, relating to writing this current book in 2009, that might affect it by way of conflicting interests that might contribute to bias. I avow that this is not the case, but, maybe I am not in a position to judge. This concerns my economics book on unemployment, DRGTPE. This book is not

⁵⁰ <http://thomascool.eu/Thomas/English/Science/Letters/SCT-working-group.html>

⁵¹ [http://\(...\) /Letters/2006-03-20-Comments-RfC.pdf](http://(...) /Letters/2006-03-20-Comments-RfC.pdf)

⁵² With theft and peddling drugs you can call the police but when a professor repeats falsehoods ? You can write a paper clarifying this politely, put it on the internet and send the professor the link. Thereafter the professor tells not only falsehoods but also lies. Freedom of speech differs from graft. He says he hasn't read the paper and doesn't have to. But it belongs to the scientific mores that the other party looks into it. You are a competent econometrician and have taken the time to explain the issue. He may say that you are not a mathematician, and then doesn't know his Venn-diagrams. He may say that it is not peer-reviewed, but you approach him to do that. What next ?

getting sufficient attention by my fellow economists. I have a vested interest in getting this afloat.

One element in the current situation is that mathematicians pay insufficient attention to my work in ALOE and VTFD. The books show that standard texts are incorrect but they don't read the books. It would help that they did and subsequently could tell the economists that it is sound indeed. To the effect of 'say, this is good work, why are you ignoring work by one of your own who is capable of something?' It would help not being ignored on all sides. It would help to have some support on the minor confusions in current mathematics and then be able to face the major misunderstandings in current economics on the main problem in society.

In this case mathematicians started ignoring in 1981, perhaps they can be the first to restore this, following Gill (2008). A subtle point is of course my location so that the current situation may have come about by the idiosyncracies in Holland. It is no use to attribute to mathematics in general what happened in this small country (that is, on ALOE and VTFD, not education in general).

In the *Introduction*, I listed some major real world problems in which mathematicians have been busy, the stock market crash, ecological collapse, destruction of democracy, perversion of logic. This book adds education in mathematics. This epilogue adds the indirect contribution to mass unemployment (without stock market crashes).

Mathematicians thus are depicted here like lifeguards, who you'd expect to jump into the water to save a drowning person (mathematical theorem, which is their job), but who don't do so – while it also happens that this person holds on to some papers and yells "save these papers!" (economic theory). Perhaps some information overload? Or merely more interested in their pet theories, the ladies on the beach (other theorems)?

Thus, this book is about both the education in mathematics and what is considered to be mathematics. A basic observation concerns the non-empirical training and attitude by mathematicians. Another key aspect is that ALOE and VTFD change conceptions about what mathematics is. ALOE implies (amongst others) that you must keep account of exceptions even in formal systems. VTFD implies (amongst others) that you should not confuse a theorem with your interpretation of it. A change on these aspects will, as a corollary, have effects on other issues as well. Such as on perceptions of my fellow economists on my analysis on unemployment. I think that it is important to be aware of that corollary. When you go to London for a holiday then it is a corollary that you are in Europe. When waking up in the hotel you might decide that a trip to Paris is actually a nice surprise. It would be not correct to infer that your trip was targetted for Paris – as it would be inaccurate to say that this book and its 'creative destruction of mathematics' was written with the idea to get my fellow economists to study DRGTPE. But there could be a wonderful bonus for the unemployed.

Epilogue 2015

The six years after 2009 allow some stock-taking:

- ALOE: Three-valued logic appeared important for FMNAI (2015f). The logic professor of 1980-81 retired, and I decided to write a longer protest.⁵³
- FMNAI: This book of 2015 creates scope for treatment of set theory and number theory in highschool and first year of higher education for non-math majors. However, developing this analysis caused me to observe a general problem with integrity of science in the field of logic and set theory in Holland.⁵⁴
- VTFD: The book was "reviewed" in the journal *Voting Matters*. The "reviewer" spotted a non-crucial and actually rather dumb error on my part, made some errors himself, and neglected to look into the main contributions. There is now a 4th edition that corrects my error and explains about that "review".
- COTP is a proof of concept for EWS: See page 96 above for developments.
- KWAG (Dutch) and some weblog entries: These develop some analyses for elementary school. Note that arithmetic at elementary school is relevant for competence in algebra in secondary school.
- CWNN: This planned *A child wants nice and not mean numbers* (2015h) discusses mathematics in primary education, now in English.⁵⁵
- DRGTPE: The 2007+ economic crisis continued without attention for DRGTPE. I collected my economic crisis papers till 2012 in *"Common Sense: Boycott Holland"* (CSBH) (2012e) and began the weblog boycottholland.wordpress.com.⁵⁶
- THAEES: This draft got an update in 2015.⁵⁷
- EWVJ / SMOJ: It appears that the Dutch mind is also closed on this.⁵⁸ *The simple mathematics of Jesus* (SMOJ) (2012) observes that mathematics deals with abstraction, assumes that thinking itself consists of abstractions, and considers the possibility that ideas about the divine are as abstract as ideas about lines and circles. The book is a proposal for a multidisciplinary project such that mathematical insights can be balanced by views from other disciplines such as history. Before studying a particular religion it may be advisable to study the notion of religion itself.
- BHRM: This planned book *Boycott Holland: Role of Mathematics and Mathematics Education Research* (2015g) collects my weblog entries on the role of mathematics and its education, exclusive of the special topic of SMOJ. This will also be a good place to discuss the breach of integrity in Dutch mathematics education research.⁵⁹ The major question for editing this 2nd edition of EWS in 2015 was whether to include these other topics and pages. This would have created a book of some 250 pages and would have caused a loss of focus. It is wiser to have separate books. It is one thing to argue that mathematics education needs re-engineering (EWS) and it is

⁵³ <http://thomascool.eu/Papers/ALOE/JFAKvB/Index.html>

⁵⁴ <http://thomascool.eu/Papers/ALOE/2015-05-21-A-breach-of-integrity-on-paradoxes.pdf>

⁵⁵ <http://thomascool.eu/Papers/NiceNumbers/Index.html>

⁵⁶ <https://boycottholland.wordpress.com/about/>

⁵⁷ <http://mpira.ub.uni-muenchen.de/63904/>

⁵⁸ <https://boycottholland.wordpress.com/2015/02/02/the-closed-dutch-mind-on-jesus-too/>

⁵⁹ <https://boycottholland.wordpress.com/2014/07/16/integrity-of-science-in-dutch-research-in-didactics-of-mathematics/>

another thing to clarify some linkages of the world of mathematics to my advice of a boycott of Holland till the censorship of science there is lifted. There are connections indeed, so I hope that you will consider both issues, but readers should not be overburdened. They might actually like the idea that there will be more to read later on.

Perhaps Holland is a tolerant country and perhaps there would be more stagnation elsewhere. Arrogance based upon ignorance is no pretty sight anywhere. There is little that I can do about the situation in Holland anyway. This is another reason to try to prevent misunderstandings and call your attention to *What a mathematician might wish to know about my work*, page 125 below. My overall observation is that there is a Garden of Eden for improvement.

These expositions seem to generate much repetition. Hopefully some repetition also provides for more angles of perspective.

Appendices

What is new in this analysis (in 2009) ?

'New' is taken in comparison to others, and thus includes points also made in my earlier publications on this analysis. New are:

- 1) A list of examples / cases in mathematics that are cumbersome or illogical. Clarification and resolution.
- 2) Associated suggestions for better notation, such as decimal dot in comma-using countries, better brackets, $2 + \frac{1}{2}$ instead of $2\frac{1}{2}$, eradication of the cult of the radical sign, better tables for drawing graphs, $\text{rex}[x]$ instead of $\log[x]$, $\text{DoExp}[y, 1/n]$ for solution by taking roots, $y // x$ for dynamic division, say *turning point* instead of *vertex* of a parabola.
- 3) *New math*: ALOE and VTFD. Highly relevant and easy for education.
- 4) *New math*: A redesign of trigonometry with Θ , unit measure (meter) around (UMA), xur and yur . Much greater ease.
- 5) *New math*: Clarification that the derivative is algebra, as opposed to using limits and infinitesimals. Much greater ease for education.
- 6) Explanation of the fundamental causes. Didactics require a mindset sensitive to empirical observation which is not what mathematicians are trained for. Tradition and culture cause mathematicians to see what they are conditioned to see.
- 7) Suggestions for structural redesign of highschool mathematics.
- 8) Identification and direction of solution of main problems in ICT. Suggestion for a world standard in computer algebra. Creation of computer test rooms. Resolution of the problem of supervision and the costs of concierges for supervision and school opening hours.
- 9) Suggestions of research questions for evidence based education (in mathematics).
- 10) Suggestions for re-engineering the industry of mathematics education.

What is new in this analysis (in 2015) ?

- 11) *New math*: COTP as *proof of concept*, including many new details.
- 12) *New math*: FMNAI on set theory, number theory and the infinite.
- 13) *New math*: CWNN for primary education. E.g. better use of the positional system, the Harremoës operator $x^H = 1 / x$ and its applications.
- 14) Empirical observation of a breach in integrity in mathematics education research.

List of new notions and symbols

Some new notions and symbols are not discussed here but elsewhere, and can be included since they form part of the larger landscape.

x^H	1 / x, "per x", with Harremoës operator $H = -1$, pronounced as "eta", p 25
$2 + 2^H$	$2 + \frac{1}{2}$, instead of $2\frac{1}{2}$, p 24
$(y x^D)$	$y // x$, dynamic division, algebra rather than numbers
$y // x$	dynamic division, p 31
Algebraic derivative	$dF / dx = \{\Delta F // \Delta x, \text{ then set } \Delta x = 0\}$, p 91 and COTP
Θ	2π , pronounced as "Archi" from Archimede, p 27 and COTP ⁶⁰
UMA	Unit Measure (Meter) Around, p 61 and 64
UCC	Unit Circumference Circle, angular circle, p 64
angle	angle α , redefined to be on the angular circle, p 64 (special kind of arc)
arc	arc $\varphi = \alpha \Theta$, standard on the Unit (Radius) Circle, p 64
xur[α]	$\cos[\alpha \Theta]$, horizontal co-ordinate of the arc on the Unit (Radius) Circle, p 64
yur[α]	$\sin[\alpha \Theta]$, vertical co-ordinate of the arc on the Unit (Radius) Circle, p 64
tur[α]	$yur[\alpha] / xur[\alpha]$
rex[x]	$\log[x]$, logarithm, "recovered exponent", p 35
$b x + c$	notation for a line, to fit the quadratic formula, p 41
$f[x]$	follow <i>Mathematica</i> for function calls, p 22
$\{x, y\}$	follow <i>Mathematica</i> for more-dimensional points or sets, p 23
$x = \text{Do}\sqrt{y}$	solve $y = x^2$ for x , p 30
$\text{DoExp}[y, 1/n]$	solve $y = x^n$ for x , p 30
perfection	0.25 is a perfect number if it stands for $25 \cdot 100^H$, p 38 ("exact" is vague)
Proportion space	p 25 and COTP
Named lines	see COTP
Subtraction method [-]	better subtraction by better using the positional system ⁶¹
BordaFP	Borda Fixed Point, election method, p 84 and VTFD
tig	ten, for counting in Dutch, dealing with confusing 13-19, see KWAG
hand	training on the positional system: fingers of the right hand count single digits and fingers of the left hand count the number of right hands, p 68
Novel	notation of numbers in mirror form, maintaining pronunciation, p 70
MENC	Mathematics Education Name of the Country, governance, p 73
$\dagger p$	p is nonsensical, or, <i>not-at-all</i> p , see ALOE
\mathcal{B}	bijection by abstraction between natural and real numbers, see FMNAI
Neoclassical	overall approach, see FMNAI

⁶⁰ <https://boycottholland.wordpress.com/2014/07/14/an-archi-gif-compliments-to-lucas-v-barbosa/>

⁶¹ <https://boycottholland.wordpress.com/2014/08/30/taking-a-loss/>

Abstract

Education in mathematics fails. What is called 'mathematics' often is illogical. Pupils and students are tortured and withheld from proper mathematical insight and competence. Professors and teachers of mathematics apparently cannot diagnose this themselves. The economic consequences are huge. Let each national parliament take action starting with an enquiry.

Summary

Subject: The education in mathematics, its failure and costs, and how to redesign it. Mathematics seen as an art and an industry that requires better regulation. On the political economy of mathematics and its education.

Method: We do not require statistics to show that mathematics education fails but can look at the math itself. Criticism on mathematics itself can only succeed if it results into better mathematics. Similarly for the didactics of mathematics. Proof is provided that the mathematics that is taught often is cumbersome and illogical. It is rather impossible to provide good didactics on what is inherently illogical.

Basic observations: We would presume that school mathematics would be clear and didactically effective. A closer look shows that it is cumbersome and illogical. (1) This is illustrated here with some twenty examples from a larger stock of potential topics. (2) It appears possible to formulate additional shopping lists for improvement on both content and didactic method. (3) Improvements appear possible with respect to mathematics itself, on logic, voting theory, trigonometry and calculus. The latter two improvements directly originate from a didactic approach and it is amazing that they have not been noted earlier by conventional mathematics. (4) What is called mathematics thus is not really mathematics. Pupils and students are psychologically tortured and withheld from proper mathematical insight and competence. Spatial sense and understanding, algebraic sense and competence, logical sense and the competence in reasoning, they all are hindered and obstructed. Mathematics forms a core element in education and destroys much of school life of pupils and students in their formative years.

Basic analysis: This situation arises not because it is only school math, where mathematics must be simpler of necessity, but it arises because of the failure of mathematicians to deliver. The failure can be traced to a deep rooted tradition and culture in mathematics. Didactics requires a mindset that is sensitive to empirical observation which is not what mathematicians are trained for. Psychology will play a role in the filtering out of those students who will later become mathematicians. Their tradition and culture conditions mathematicians to see what they are conditioned to see.

Higher order observations: When mathematicians deal with empirical issues then problems arise in general. The failure in education is only one example in a whole range. The stock market crash in 2008 was caused by many factors, including mismanagement by bank managers and failing regulation, but also by mathematicians and 'rocket scientists' mistaking abstract models for reality (Mandelbrot & Taleb 2009). Another failure arises in the modelling of the economics of the environment where an influx of mathematical approaches causes too much emphasis on elegant form and easy notions of risk and insufficient attention to reality, statistics and real risk (Tinbergen & Hueting 1991). Improvements in mathematics itself appear possible in logic and voting theory,

with consequences for civic discourse and democracy, where the inspiration for the improvement comes from realism (Colignatus 2007). Economics as a science suffers from bad math and the maltreatment of its students – and most likely this is also true for the other sciences. Professors and teachers of mathematics – or at least 99.9% of them – apparently cannot diagnose their collective failure themselves and apparently ‘blame the victims’ for not understanding mathematics. The other scientific professions are advised to verify these points.

Higher order analysis: Application of economic theory helps to understand that the markets for education and ideas tend to be characterized by monopolistic competition and natural monopolies. Regulations are important. Apparently the industry of mathematics education currently is not adequately regulated. The regulation of financial markets is a hot topic nowadays but the persistent failure of mathematics education would rather be high on the list as well. It will be important to let the industry become more open to society. Without adjustment of regulations at the macro-level it is rather useless to try to improve mathematics education and didactics at the micro level. Mathematical tradition and culture creates a mindset, and mathematicians are like lemmings that are set to go into one direction. Trying to micro-manage change with some particular lemmings will not help in any way. An example layout is provided how the industry could be regulated.

Conundrum: Mathematicians might be the first to recognize the improvements in mathematics and didactics presented here. Mathematical tradition clearly is an improvement from alchemy and astrology. Most people will also tend to let the professors and teachers decide on whether these items are improvements indeed. It is tempting to conclude that the system then works: an improvement is proposed, it is recognized, and eventually will be implemented. This approach however takes a risk with respect to potential future changes. With the present failure and analysis on the cause we should rather be wary of that risk. We better regulate the industry of mathematics education in robust manner. The mathematical examples presented here can be understood in principle by anyone with a highschool level of mathematics. They are targetted to explain didactically to a large audience how big the failure in the education in mathematics actually is.

Advice: The economic consequences are huge. National parliaments are advised to do something about this, starting with an enquiry. Parents are advised to write their representative. The professional associations of mathematics and economics are advised to write their parliament in support of that enquiry.

Cover text in 2009

National parliaments around the world are advised to each have their own national parliamentary enquiry into the education in mathematics and into what is called ‘mathematics’. Current mathematics namely fails and causes extreme social costs.

The failure in mathematics and math education can be traced to a deep rooted tradition and culture in mathematics itself. Mathematicians are trained for abstract theory but when they teach then they meet with real life pupils and students. Didactics requires a mindset that is sensitive to empirical observation which is not what mathematicians are basically trained for.

When mathematicians deal with empirical issues then problems arise in general. The stock market crash in 2008 was caused by many factors, including mismanagement by bank managers and failing regulation, but also by mathematicians and ‘rocket scientists’

mistaking abstract models for reality (Mandelbrot & Taleb 2009). Another failure arises in the modelling of the economics of the environment where an influx of mathematical approaches causes too much emphasis on elegant form and easy notions of risk but insufficient attention to reality, statistics and real risk (Tinbergen & Huetting 1991). Errors by mathematicians on realistic assumptions have important consequences for civic discourse and democracy as well (DeLong 1991, Colignatus 2007). The failure in math education is only one example in a whole range.

The discussion of mathematics in this book can be understood by anyone with a decent command of highschool mathematics. While school math should be clear and didactically effective, a closer look shows that it is cumbersome and illogical. (1) This is illustrated with some twenty examples from a larger stock of potential topics. (2) Additional shopping lists for improvement on both content and didactic method can be formulated as well. (3) Improvements appear possible with respect to mathematics itself, on logic, voting theory, trigonometry and calculus. (4) What is called mathematics thus is not really mathematics. Pupils and students are psychologically tortured and withheld from proper mathematical insight and competence. Other subjects, like the education in economics, biology or physics, suffer as well.

Application of economic theory helps us to understand that markets for education and ideas tend to be characterized by monopolistic competition and natural monopolies. Regulations then are important. Apparently the industry of mathematics education currently is not adequately regulated. The regulation of financial markets is a hot topic nowadays but the persistent failure of mathematics education should rather be high on the list as well. It will be important to let the industry become more open to society.

When you want to understand the underlying historical processes that cause the current state of the world then this is the book for you. Mathematics education must be tackled, both as a noble goal of itself and for the larger causes.

Thomas Colignatus (1954) is an econometrician and teacher of mathematics.

What a mathematician might wish to know about my work

March 26 2013⁶² – updated 2015

Abstract

Mathematicians have contributed to confusions in the areas of logic, voting theory and the education of mathematics itself. While mankind may mistake abstract ideas for reality, mathematicians are not immune to this either. Part of my work has been to correct such mistakes. Let mathematicians study those corrections with an open mind, so that we can get better logic, more democracy and proper education in mathematics.

Introduction

Mathematics per se is not my target. Over the years I have written some texts that nevertheless may be of interest to mathematicians, like reformulations of logic, voting theory, calculus and set theory. These texts are not presented in ways that mathematicians may be used to. My work might be called *applied mathematics*, as it is not developed in an axiomatic context but is in the intended interpretation of some axiomatics that may still need to be developed. Hence some explanation is useful for mathematicians about what to expect about my work, since I would like them to study

⁶² <http://thomascool.eu/Papers/Math/2013-03-26-WAMMWTKAMW.pdf>

these books too. Without the explanation below we may expect neglect and misunderstandings from mathematicians, and this would be unfortunate.

I refer to my books ALOE, VTFD, EWS, COTP, CWNN / KWAG, SMOJ / EWVJ, and FMNAI,⁶³ see the references.

Who is interested in logical paradoxes, will benefit from ALOE. Who is interested in democracy and voting theory, will benefit from VTFD. Who is interested in the impact of ancient mathematics and astronomy on religion, will benefit from SMOJ / EWVJ. Who is interested in didactics of geometry and calculus, will benefit from COTP. Who is interested in set theory, number theory and infinity will benefit from FMNAI. Who is interested in education of mathematics, will benefit from EWS and CWNN / KWAG. The latter books also rely on economics, when they discuss the mathematics industry and advise to a parliamentary enquiry.

I am an econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008). I chose to study econometrics because I wanted to find decent solutions for world problems, and I considered the mathematical base as a *conditio sine qua non*. At the interfaculty of econometrics in the 1970s, we as students had our courses in mathematics jointly with students of mathematics, physics and astronomy. Clearly I consider mathematics important but I rather apply it. The didactics of mathematics is an empirical issue as well.

A context of quality

An indication of quality are two favourable reviews by Richard Gill (2008) (2012) in *Nieuw Archief voor Wiskunde* with respect to ALOE and EWS & COTP. The *European Mathematical Society* website has two favourable reviews of ALOE and COTP too. Recently, Christiaan Boudri (2013) at the website of the *Dutch Association of Teachers of Mathematics* NVVW calls for having an open mind here too. Koolstra and Groeneveld mention it in the Dutch math email newsletter in 2013.

A standard textbook is TSOM with ir. Karel Drenth (TUD, sadly deceased) (2000), as perhaps additional confirmation that I would be able to walk the standard path as well.

A point is that Gill is not specialised in logic and didactics as research fields. This also holds for me, though I did study the subjects that I write about. My subjects are elementary, so a specialisation to higher levels is not required. Still, the specialists in logic and didactics will tend to defend their specialisms. Everyone can check elementary errors, but will the specialists acknowledge those ?

Standard ways versus re-engineering

The point is now to explain the path into uncharted territory. There are plenty of cranks in the world, and standard mathematicians might tend to catalogue me as one of those, but if they would walk along with me along these new paths, then they might start to wonder whether the standard ways aren't a bit cranky themselves. An option is that they develop the required axiomatics, if that is required to make the new approach acceptable in the mathematics community.

Some mathematicians advise me: 'Present your proposals in various neat articles, phrased in the language that mathematicians like to read, and be done with it.' I am afraid that it doesn't work like that.

Above books *re-engineer* their subjects. In my analysis, standard views in those subjects are misguided in subtle ways. The best approach towards clarity is to reorganise the subject matter, start from scratch, and build on from there with an open mind.

⁶³ FMNAI is the exception since it actually looks at axiomatics and proposes some new.

Mathematicians contribute to confusion

A key point is that much confusion in these areas has been caused by mathematicians, who neglect the world and who focus on some *if-then* relationships, where perhaps the logic might be right in some respects, but where the assumptions are (subtly) confused about what the discussion is about.

My books intend to set the record straight, and to invite the mathematicians to reconsider their work, so that this fits in the whole, and thus to stop their contribution to confusion.

This observation is only a rephrasing of the earlier statement: my work is in the *intended interpretation*, the world of application. Thus mathematicians who consider such applications ought to benefit from this effort at re-engineering. Mathematicians are not my target, but the areas of application. Mathematicians interested in those fields better avoid confusion.

Hopefully, you see the problem. Once we enter the world of application, one must study reality and not just mathematics. Selecting only bits and pieces to find *if-then* relationships can contribute to confusion. A mathematician better be precise what the contribution is, and help to clarify what the application really is. Read my books and see how mathematicians have gone astray. They provide lessons on how to communicate on real issues, partly using the language of mathematics, but still focussed on the real issues.

Modesty

My books are modest. They are only what they are about, and not something else. It might sound curious to speak about 're-engineering logic, calculus and voting theory' but the purpose of this description is not to sound curious but to indicate what the books do.

An editor at the journal *Euclides* of NVVW worried that this description of re-organising subjects uses 'big words' but agreed with various points and did not specify what would be wrong, so it is not clear why that description would not be correct. I do not claim that I know everything. I do not claim that am infallible. I just explain where mathematicians go astray at key points in key applications.

My books explain where Hans Freudenthal was in error, and suggest a better alternative, without returning to the old ways of teaching Euclid. Many other authors have promising suggestions too. Let parliament abolish the *Freudenthal Institute* in Utrecht and create the *Simon Stevin Institute*, where researchers from various fields can test what works for pupils and students. I am amazed that ICMi has a "*Freudenthal Medal*" while he has been disastrous for the education in mathematics, with his abstract mind in conflict with empirical reality. He rightly said that education is engineering, thus involves reality, but he was no engineer himself. It actually appears that integrity is at issue, see Colignatus (2014a).

My position is *not* that my books explain how things must be, and that Parliament must impose this on the mathematics community. That would be a gross misunderstanding. What works in education is up to the pupils and students themselves. Parliament must step in to make the funds available for research.

Little contact with mathematicians

As a writer, I have hardly any contact with mathematicians. It is my great regret that these books have been written mostly in a context where mathematicians were not willing to discuss the issues. The book on logic was written in my student days, at first in some interaction with some teachers in logic, but fairly soon in bitter antagonism from their side and with their unwillingness to listen to criticism. The book on voting theory first had some interaction with some mathematicians, but fairly soon in bitter antagonism again. For the

book on calculus I have assumed neglect, perhaps I was wrong, and in reality there would have been keen interest. However, one so-called "review" confirms bitter antagonism again.

There have been some contacts. KWAG is in Dutch and contains some responses from the world of mathematics education research, and also from the minister of education, w.r.t. many of above proposals in their early Dutch form. Apparently that early Dutch form was not convincing enough, but since you can see the mathematics content yourself, the conclusion would be straightforward that there may be something amiss with the organisational structure.

The books on the education of mathematics EWS, COTP and CWNN / KWAG have been written with little interaction on these with my direct colleagues teachers of mathematics. My colleagues have all been fine and capable teachers and pleasant to work with. The focus of our work was on teaching the established programme. My ideas developed over time in notes, but I had little reason to discuss them, since this would distract from the overall focus on the established programme. For the same reason the books were written outside of school, and have not been presented at school. Clearly I want to avoid that colleagues might get the wrong impression that I would teach something else than the established programme. The discussion is best done in a context where improvement of the programme is on the agenda. Clearly, I haven't found that environment yet, given the neglect by such commissions (in Holland: cTWO and NOCW).

The books essentially are invitations to be read and discussed, and if someone has a good counterargument, I will be the first to correct.

In the journal *Euclides* two reviewers of EWS and COTP have selected to start slandering. I have filed a protest with the editors. See my website for the slander and my reply. The reply was not published in the journal. They have decided not to review any book of me anymore. This now holds for CWNN / KWAG and SMOJ / EWVJ, and the 2nd edition of EWS. Slander and now censorship. And these people teach mathematics.

The book editors of *Nieuw Archief voor Wiskunde* rejected SMOJ / EWVJ for review, with the argument that it would not fit the readership but that indicates that they have not studied it.

I have submitted some short papers to journals. Editors respond by saying "I do not understand it" and then reject the paper. It would help, and be more decent, to specify what one doesn't understand.

Some original contributions

My books also contain some bits of original contributions to mathematics, but these might perhaps only be understood in the reconstructed framework of those books. They are already available in those books, so there is little advantage in trying to extract them as separate articles. The chance is slim that they might be understood without that reconstructed environment.

Apparently when I want to relay a message A then I also want more internal consistency in the mathematics $M(A)$ that underlies that message, and then I find myself solving problems in $M(A)$. While a research mathematician would focus on $M(A)$ and think about a possible message A only in a second stage or not at all. Perhaps research mathematicians might never discover $M(A)$ since they don't see A in the first place. This partly explains why research mathematicians may have a hard time to accept $M(A)$ anyway. But I do not claim to understand how research mathematicians think. I can only say: the context and my manner of presentation in that context tend to differ from what (research) mathematicians may be used to. If they are inflexible then they miss out on some key findings.

FMNAI in 2015 discusses the foundations of mathematics, and this creates the paradox that this is done by someone who professes that mathematics is not his cup of tea. The paradox is resolved by that I am only tasting the water, and that the presented results are a side-effect from the original interest in improving education.

Some abstracts

There is a risk in giving abstracts of these books since some subtleties might be lost. The human mind however needs some anchors.

ALOE re-engineers elementary logic. It solves the Liar paradox after 2300 years, and corrects Kurt Gödel by showing that his verbal statements and interpretation do not cover his mathematics. VTFD re-engineers voting theory for democracy, and shows that Kenneth Arrow on his *Impossibility Theorem* for collective decision making gives a verbal interpretation that does not cover his mathematics. COTP re-engineers plane geometry, and shows that derivative and integral are algebraic concepts. This is the fundamental understanding that can be used for highschool and first year math for non-math majors, while it remains an open question what Weierstraß and non-standard analysis add to this. FMNAI re-engineers set theory and number theory w.r.t. infinity. The transfinite by Georg Cantor appear to be based upon ill-defined constructs, comparable to Russell's paradox, so that these can be eliminated. SMOJ / EWVJ suggests that religious concepts on the divine are as abstract as mathematical concepts like line and circle, with similar epistemological questions for existence and current research on mind and brain. It is conceivable that religious differences and perhaps even religious wars relate to misconceptions about Van Hiele levels of abstraction as in mathematics education. EWS and CWNN / KWAG also discuss the mathematics industry and errors in didactics, with a proposal of a parliamentary enquiry to resolve confusion and stagnation.

Parliament

In Holland, the Dijsselbloem *Parliamentary Commission on Education* 2007-2008⁶⁴ made the distinction between *what* and *how*. The government determines *what* will be in the programme (say Dutch and math) and the teaching community will determine *how* the subjects will be taught. It is fine that Parliament has decided that the school programme has to include math. The problem is that "math teachers" present something as "math" which it isn't, see my letter to Parliament Colignatus (2013a). For example, two-and-a-half is written as two-times-a-half ($2\frac{1}{2}$) while it is better to stick to $2 + \frac{1}{2}$, and learn to see that addition as an end-station, in the same manner as $\exp[2]$ can be symbolic and doesn't directly require the use of a calculator. We can understand the use of $2\frac{1}{2}$ from some historical development, but we would all be silly if we were to accept it as decent mathematics. Jan van de Craats solves the issue by mostly using $5 / 2$, but this loses the useful feature of a mixed number. Hence, re-engineer the subject. Hence, let Parliament investigate the mathematics industry.

To repair ages of wrong didactics will be a costly affair. Do we have to wait till the USA is open to my analysis, or might Holland take the lead? I see little other ways of resolution than that members of Parliament look into mathematics education. It suffices that they have had a highschool education with mathematics and then we can explain the current curriculum and the improvements in my books. This will be fun but also necessary for the improvement. I cannot see why mathematicians would be against teaching members of Parliament more about mathematics. If they think that my suggestions for improvement are silly then it should not be difficult to show this. The members of Parliament can also query users of mathematics like economists, physicists and biologists. Finally, note that my target is to establish doubt and to release funds for research. What is an improvement in didactics is an empirical question, and cannot be established by my books only. The

⁶⁴ https://nl.wikipedia.org/wiki/Parlementair_onderzoek_onderwijsvernieuwingen

error of the current math curriculum is that it relies on tradition and it is time that we see that didactics is an empirical issue.

The economic and ecological crises

Econometrics is: to translate economic theories into mathematical format and test these by statistical methods. Another word is “economic engineering”. As an econometrician I have discussions with economists, mathematicians and statisticians. Misunderstandings in one realm may contaminate misunderstandings in another realm. This would not be logical, but it is a human thing to happen. Mathematicians will not be able to judge details in economic theory, but may still think dark thoughts about my economic analyses, given their apprehensions about my books on the education of mathematics. Economists can get insecure if they think that my mathematics would be improper. The best response is to ask everyone to do their job and to be specific about what one can judge about. My position as an econometrician suffers from maltreatment by the various subprofessions, and this better stops. The world is in economic and ecological crises, my econometric analysis would help a resolution, and it doesn't help when the subprofessions malfunction.

As just one example, let me refer to Colignatus (2013b) on bottlenecks against science. Jos de Beus (1952-2013) was a professor of politicology, student of professor or economics J. (Hans) van den Doel (1937-2012), student of Jan Tinbergen (1903-1994), founder of econometrics and winner of the Nobel Prize in Economics, who started as a student of Paul Ehrenfest (1880-1933). Mathematicians have a story about that Prize etcetera. Point is that Van den Doel didn't quite understand Kenneth Arrow's *Impossibility Theorem*.⁶⁵ Jos de Beus was misguided too. These misconceptions have influenced ideas in government circles. It would be such a relief when mathematicians would accept that they contributed to the confusion themselves too.

Conclusion

Mathematics as a profession is not my cup of tea. In my research on key topics in the real world I noticed that mathematics was applied in a wrong manner at key steps in the argumentation. It were especially the mathematicians who advanced such miscomprehension. One might think that people using logic or voting, or the teachers of calculus would be able to correct the misunderstandings of such confused mathematicians, but alas, mathematics also has an aura of authority. If things get complex, follow the specialist, and if a mathematician has a complex paper he or she might be that specialist. However, when I as an econometrician and teacher of mathematics present corrections, I find that mathematicians are not as open to criticism as one would wish.

Let us hope that mathematicians will study my books, learn about the real world, see how important it is to communicate precisely. A sign of success will be when the confusions in voting theory are corrected. A sign of success will be when mathematicians start signing the petition for a parliamentary enquiry into mathematics education, and call on others to help them to save our children from misguided “mathematicians”.

The petition is at <http://www.ipetitions.com/petition/tk-onderzoek-wiskundeonderwijs>.

Unfortunately there is a collective issue with integrity of science, see here ⁶⁶ (in Dutch ⁶⁷), and a particular breach in the area of logic, see Colignatus (2015e)

⁶⁵ In Dutch: See Colignatus & Hulst (2003, PDF page 85). Also *Kennisnet.nl* and booklets of *Epsilon Uitgaven* still report inadequately, see Colignatus (2013c).

⁶⁶ <https://boycottholland.wordpress.com/2014/07/16/integrity-of-science-in-dutch-research-in-didactics-of-mathematics/>

⁶⁷ <http://thomascool.eu/Papers/Math/2014-07-08-Colignatus-aan-KNAW-LOWI.html>

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PM 1. Colignatus is the name of Thomas Cool in science. See <http://thomascool.eu>.

PM 2. Legacy:X means that a paper is superseded by a better version in X.

PM 3. References in footnotes might not be repeated here.

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