Integrating Graded Knowledge and Temporal Change in a Modal Fragment of OWL

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Abstract. Natural language statements uttered in diagnosis, but more general in daily life are usually *graded*, i.e., are associated with a degree of *uncertainty* about the validity of an assessment and is often expressed through specific words in natural language. In this paper, we look into a *representation* of such graded statements by presenting a simple non-standard modal logic which comes with a set of modal operators, directly associated with the words indicating the uncertainty and interpreted through confidence intervals in the model theory. We complement the model theory by a set of RDFS-/OWL 2 RL-like entailment (*if-then*) rules, acting on the syntactic representation of modalized statements. After that, we extend the modal statements by *transaction time*, in order to implement a notion of temporal change. Our interest in such a formalization is related to the use of OWL as the *de facto* language in today's ontologies and its weakness to represent and reason about assertional knowledge that is *uncertain* and that changes over time.

1 Introduction

Medical natural language statements uttered by physicians or other health professionals and found in medical examination letters are usually *graded*, i.e., are associated with a degree of uncertainty about the validity of a medical assessment. This uncertainty is often expressed through specific *verbs, adverbs, adjectives*, or even *phrases* in natural language which we will call *gradation words* (related to *linguistic hedges*); e.g., *Dr. X suspects that Y suffers from Hepatitis* or *The patient probably has Hepatitis* or *(The diagnosis of) Hepatitis is confirmed*. Our approach is clearly not restricted to medical statements, but is applicable to graded statements in general, e.g., in technical diagnosis (*the engine is probably overheated*) or in everyday conversation (*I'm pretty sure that Joe has signed a contract with Foo Inc.*), involving *trust* (*I'm not an expert, but ...*) which can be seen as the common case (contrary to true *universal* statements).

In this paper, we look into a representation of such graded statements by presenting a simple *non-standard modal logic* which comes with a small set of *partially-ordered modal operators*, directly associated with the words indicating the uncertainty and interpreted through *confidence intervals* in the model theory. Our interest in such a formalization is related to the use of OWL in our projects as the *de facto standard* for ontologies today and its *weakness* to represent and reason about assertional knowledge that is uncertain [16] or that changes over time [10]. There are two principled ways to address such a restriction: *either* by sticking with the existing formalism (viz., OWL) and trying to find an encoding that still enables some useful forms of reasoning [16]; *or* by deviating from a defined standard in order to arrive, at best, at an easier, intuitive, and less error-prone representation [10].

Here, we follow the latter avenue, but employ and extend the standard entailment rules from [7, 18, 15] for positive binary relation instances in RDFS and OWL towards modalized n-ary relation instances, including transaction time and negation. These entailment rules talk about, e.g., subsumption, class membership, or transitivity, and have been found useful in many applications. The proposed solution has been implemented for the binary relation case (extended triples: quintuples) in *HFC* [11], a forward chaining engine that builds Herbrand models which are compatible with the open-world view underlying OWL.

This paper extends [12, 14] by new material, addressing the *temporal change of graded statements*. We will introduce a special notion of *transaction time* [17] (the time period in which a database entry is valid), contrary to *valid time* which we have investigated in [10] for the non-modal case. Due to space restrictions, we let the interested reader refer to [12, 14] for more material that we can not cover here, viz., (i) more on implementing modal entailments in *HFC*, (ii) specialized custom entailments, (iii) further kinds of modals (dual, in-the-middle), and (iv) related work, including the relation to the normal modal logic **K** and to *Subjective Logic* [8].

2 OWL vs. Modal Representation

We note here that the names of our *initial* modal operators were inspired by the *qualitative information parts* of diagnostic statements from [16] as shown in Figure 1.

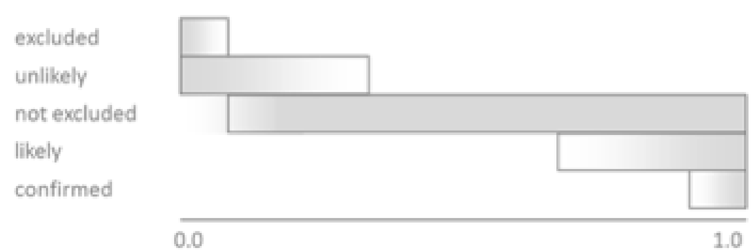


Fig. 1. Schematic mappings of the qualitative information parts *excluded* (E), *unlikely* (U), *not excluded* (N), *likely* (L), and *confirmed* (C) to *confidence intervals*. Picture taken from [16].

These qualitative parts were used in medical statements about, e.g., liver inflammation with varying levels of detail [16] in order to infer, e.g., **if** *Hepatitis* *is confirmed* **then** *Hepatitis is likely* but **not** *Hepatitis is unlikely*. And **if** *Viral Hepatitis B is confirmed*, **then** both *Viral Hepatitis is confirmed* **and** *Hepatitis is confirmed* (generalization). Things "turn around" when we look at the adjectival modifiers *excluded* and *unlikely*: **if** *Hepatitis is excluded* **then** *Hepatitis is unlikely*, but **not** *Hepatitis is not excluded*. Furthermore, **if** *Hepatitis is excluded*, **then** both *Viral Hepatitis is excluded* **and** *Viral Hepatitis B is excluded* (specialization).

[16] consider five OWL encodings, from which only two were able to fully reproduce the *plausible* inferences for the above Hepatitis use case. The encodings in [16] were quite *cumbersome* as the primary interest was to stay within the limits of the underlying calculus. Besides coming up with complex encodings, only minor forms of reasoning were possible, viz., subsumption reasoning. Furthermore, each combination of disease and qualitative information part required a *new* OWL class definition/new class name, and there exist a lot of them! These disadvantages are a result of two conscious decisions: OWL only provides unary and binary relations (concepts and roles) and comes up with a (mostly) fixed set of entailment/tableaux rules.

In our approach, however, the *qualitative information parts* from Figure 1 are first class citizens of the object language (the modal operators) and *diagnostic statements* from the Hepatitis use case are expressed through the binary property $\mathsf{suffersForm}$ between p (patients, people) and d (diseases, diagnoses). The plausible inferences are then simply a *byproduct* of the *instantiation* of the entailment rule schemas (G) from Section 5.1, and (S1) and (S0) from Section 5.2 for property $\mathsf{suffersForm}$ (the rule variables are universally quantified; $\top$ = *universal truth*; C = *confirmed*; L = *likely*), e.g.,

(S1) $\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \wedge \mathsf{ViralHepatitisB}(d)$
$\quad\rightarrow \top\mathsf{ViralHepatitis}(d)$

(G) $C\mathsf{suffersFrom}(p,d) \rightarrow L\mathsf{suffersFrom}(p,d)$

Two things are worth mentioning here. *Firstly*, not only OWL properties can be graded, such as $C\mathsf{suffersFrom}(p,d)$ (= *it is confirmed that p suffers from d*), but also class membership, e.g., $C\mathsf{ViralHepatitisB}(d)$ (= *it is confirmed that d is of type Viral Hepatitis B*). As the original OWL example from [16] can not make use of any modals, we employ the special modal $\top$ here: $\top\mathsf{ViralHepatitisB}(d)$. *Secondly*, modal operators are only applied to assertional knowledge (the ABox in OWL)—neither TBox nor RBox axioms are being affected by modals in our approach, as they are supposed to express universal truth.

3 Confidence and Confidence Intervals

We address the *confidence* of an asserted (medical) statement [16] through *graded* modalities applied to propositional formulae: E (*excluded*), U (*unlikely*), N (*not excluded*), L (*likely*), and C (*confirmed*). For various (technical) reasons, we add a *wildcard* modality ? (*unknown*), a complementary *failure* modality ! (*error*), plus two further modalities to syntactically state definite truth and falsity: $\top$

($true,$ or top) and $\perp$ ($false$ or $bottom$).[1] Let $\triangle$ now denotes the set of all modalities: $\triangle := \{?, !, \top, \perp, E, U, N, L, C\}$.

A *measure function* $\mu : \triangle \mapsto [0,1] \times [0,1]$ is a mapping which returns the associated *confidence interval* $\mu(\delta) = [l,h]$ for a modality from $\delta \in \triangle$ ($l \leq h$). We write $||\delta|| = h - l$ to denote the *length* of the confidence interval and presuppose that $\mu(?) = [0,1]$, $\mu(\top) = [1,1]$, $\mu(\perp) = [0,0]$, and $\mu(!) = \emptyset$.[2]

In addition, we define two disjoint subsets of $\triangle$, called $\underline{1} := \{\top, C, L, N\}$ and $\underline{0} := \{\perp, E, U\}$ and again make a presupposition: the confidence intervals for modals from $\underline{1}$ *end* in 1, whereas the confidence intervals for $\underline{0}$ modals always *start* with 0. It is worth noting that we do *not* make use of μ in the syntax of the modal language (for which we employ the modalities from $\triangle$), but in the semantics when dealing with the satisfaction relation of the model theory (see Section 4).

We have talked about *confidence intervals* now several times without saying what we actually mean by this. Suppose that a physician says that it is *confirmed* ($= C$) that patient p suffers from disease d, for a set of observed symptoms (or evidence) $S = \{S_1, \ldots, S_k\}$: $C\,\mathit{suffersFrom}(p, d)$.
Assuming that a different patient p' shows the same symptoms S (and only S, and perhaps further symptoms which are, however, *independent* from S), we would assume that the same doctor would diagnose $C\,\mathit{suffersFrom}(p', d)$.
Even an other, but similar trained physician is supposed to grade the two patients *similarly*. This similarity which originates from patients showing the same symptoms and from physicians being taught at the same medical school is addressed by confidence *intervals* and not through a *single* (posterior) probability, as there are still variations in diagnostic capacity and daily mental state of the physician. By using intervals (instead of single values), we can usually reach a consensus among people upon the *meaning* of gradation words, even though the low/high values of the confidence interval for, e.g., *confirmed* might depend on the context.

Being a bit more theoretic, we define a *confidence interval* as follows. Assume a *Bernoulli experiment* [9] that involves a large set of n patients P, sharing the same symptoms S. W.r.t. our example, we would like to know whether $\mathit{suffersFrom}(p, d)$ or $\neg\mathit{suffersFrom}(p, d)$ is the case for every patient $p \in P$, sharing S. Given a Bernoulli trials sequence $\boldsymbol{X} = (X_1, \ldots, X_n)$ with indicator random variables $X_i \in \{0, 1\}$ for a patient sequence $(p_1, \ldots, p_n)$, we can approximate the *expected value* E for *suffersFrom* being *true*, given disease d and background symptoms S by the *arithmetic mean* A: $\mathrm{E}[\boldsymbol{X}] \approx \mathrm{A}[\boldsymbol{X}] = \frac{\sum_{i=1}^{n} X_i}{n}$.

[1] We also call $\top$ and $\perp$ *propositional* modals as they lift propositional statements to the modal domain. We refer to ? and ! as *completion* modals since they complete the modal hierarchy by adding unique most general and most specific elements (see Section 4.3).

[2] Recall that intervals are (usually infinite) sets of real numbers, together with an ordering relations (e.g., $<$ or $\leq$) over the elements, thus $\emptyset$ is a perfect, although degraded interval.

Due to the *law of large numbers*, we expect that if the number of elements in a trials sequence goes to infinity, the arithmetic mean will coincide with the expected value: $\mathrm{E}[\boldsymbol{X}] = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n}$.

Clearly, the arithmetic mean for each new *finite* trials sequence is different, but we can try to *locate* the expected value within an interval around the arithmetic mean: $\mathrm{E}[\boldsymbol{X}] \in [\mathrm{A}[\boldsymbol{X}] - \epsilon_1, \mathrm{A}[\boldsymbol{X}] + \epsilon_2]$. For the moment, we assume $\epsilon_1 = \epsilon_2$, so that $\mathrm{A}[\boldsymbol{X}]$ is in the center of this interval which we will call from now on *confidence interval*.

Coming back to our example and assuming $\mu(C) = [0.9, 1]$, $C\,\mathit{suffersFrom}(p, d)$ can be read as being true in 95% of all cases *known* to the physician, involving patients p potentially having disease d and sharing the same prior symptoms (evidence) $S_1, \ldots, S_k$: $(\sum_{p\in P} \mathrm{Prob}(\mathit{suffersFrom}(p, d)|S))/n \approx 0.95$.

The variance of $\pm 5\%$ is related to varying diagnostic capabilities between (comparative) physicians, daily mental form, undiscovered important symptoms or examinations which have not been carried out (e.g., lab values), or perhaps even by the physical stature of the patient (crooked vs. upright) which unconsciously affects the final diagnosis, etc, as elaborated above. Thus the individual modals from $\triangle$ express (via μ) different forms of the physician's *confidence*, depending on the set of already acquired symptoms as (potential) explanations for a specific disease.

4 Normal Form and Model Theory

Let $\mathcal{C}$ denote the set of constants that serve as the arguments of a relation instance. For instance, in an RDF/OWL setting, $\mathcal{C}$ would exclusively consist of XSD atoms, blank nodes, and URIs/IRIs. In order to define basic n-ary propositional formulae (ground atoms), let $p(\boldsymbol{c})$ abbreviates $p(c_1, \ldots, c_n)$, for $c_1, \ldots, c_n \in C$, given $\mathit{length}(\boldsymbol{c}) = n$. In case the number of arguments does not matter, we sometimes simply write p, instead of, e.g., $p(c, d)$ or $p(\boldsymbol{c})$. As before, we assume $\triangle = \{?, !, \top, \bot, E, U, N, L, C\}$. We inductively define the set of *well-formed formulae* ϕ of our modal language as follows:

$$\phi ::= p(\boldsymbol{c}) \mid \neg\phi \mid \phi \wedge \phi' \mid \phi \vee \phi' \mid \triangle\phi$$

4.1 Simplification and Normal Form

We now syntactically *simplify* the set Φ of well-formed formulae ϕ by restricting the uses of *negation* and *modalities* to the level of propositional letters π:

- $\pi ::= p(\boldsymbol{c}) \mid \neg p(\boldsymbol{c})$
- $\phi ::= \pi \mid \triangle\pi \mid \phi \wedge \phi' \mid \phi \vee \phi'$

The design of this language is driven by two main reasons: *firstly*, we want to effectively implement the logic (in our case, in *HFC*), and *secondly*, the application of the below semantic-preserving simplification rules in an offline pre-processing step makes the implementation easier and guarantees a more efficient runtime system. To address negation, we first need the notion of a *complement* modal δ^{C} for every $\delta \in \triangle$, where

$$\mu(\delta^{\mathsf{C}}) := \mu(\delta)^{\mathsf{C}} = \mu(?) \setminus \mu(\delta) = [0,1] \setminus \mu(\delta)$$

I.e., $\mu(\delta^{\mathsf{C}})$ is defined as the complementary interval of $\mu(\delta)$ (within the bounds of $[0,1]$, of course). For example, E and N (*excluded, not excluded*) or ? and ! (*unknown, error*) are already existing complementary modals.

We also require *mirror* modals δ^{M} for every $\delta \in \triangle$ whose confidence interval $\mu(\delta^{\mathsf{M}})$ is derived by "mirroring" $\mu(\delta)$ to the opposite side of the confidence interval, either to the left or to the right:[3]

$$\textbf{if } \mu(\delta) = [l,h] \textbf{ then } \mu(\delta^{\mathsf{M}}) := [1-h, 1-l]$$

For example, E and C (*excluded, confirmed*) or $\top$ and $\bot$ (*top, bottom*) are mirror modals. In order to transform ϕ into its *negation normal form*, we need to apply simplification rules a finite number of times (until rules are no longer applicable). We depict those rules by using the $\vdash$ relation, read as *formula* $\vdash$ *simplified formula* (ϵ = empty word):

1. $?\phi \vdash \epsilon$ ($?\phi$ is not informative at all)
2. $\neg\neg\phi \vdash \phi$
3. $\neg(\phi \wedge \phi') \vdash \neg\phi \vee \neg\phi'$
4. $\neg(\phi \vee \phi') \vdash \neg\phi \wedge \neg\phi'$
5. $\neg\triangle\phi \vdash \triangle^{\mathsf{C}}\phi$ (example: $\neg E\phi = E^{\mathsf{C}}\phi = N\phi$)
6. $\triangle\neg\phi \vdash \triangle^{\mathsf{M}}\phi$ (example: $E\neg\phi = E^{\mathsf{M}}\phi = C\phi$)

Clearly, the mirror modals δ^{M} ($\delta \in \triangle$) are not necessary as long as we explicitly allow for negated statements (which we do), and thus case 6 can, in principle, be dropped.

What is the result of simplifying $\triangle(\phi \wedge \phi')$ and $\triangle(\phi \vee \phi')$? Let us start with the former case and consider as an example the statement about an engine that *a mechanical failure m <u>and</u> an electrical failure e is <u>confirmed</u>*: $C(m \wedge e)$. It seems *plausible* to simplify this expression to $Cm \wedge Ce$. Commonsense tells us furthermore that neither Em nor Ee is compatible with this description (we should be alarmed if, e.g., both Cm and Em happen to be the case).

Now consider the "opposite" statement $E(m \wedge e)$ which must *not* be rewritten to $Em \wedge Ee$, as *either Cm or Ce* is well *compatible* with $E(m \wedge e)$. Instead, we rewrite this kind of "negated" statement as $Em \vee Ee$, and this works fine with either Cm or Ce.

In order to address the other modal operators, we generalize these *plausible* inferences by making a distinction between $\underline{\mathsf{0}}$ and $\underline{\mathsf{1}}$ modals (cf. Section 3):

7a. $\underline{\mathsf{0}}(\phi \wedge \phi') \vdash \underline{\mathsf{0}}\phi \vee \underline{\mathsf{0}}\phi'$
7b. $\underline{\mathsf{1}}(\phi \wedge \phi') \vdash \underline{\mathsf{1}}\phi \wedge \underline{\mathsf{1}}\phi'$

[3] This construction procedure comes in handy when dealing with *in-the-middle* modals, such as *fifty-fifty* or *perhaps*, whose confidence intervals neither touch 0 nor 1. Such modals have a *real* background in (medical) diagnosis.

Let us now focus on disjunction inside the scope of a modal operator. As we do allow for the full set of Boolean operators, we are allowed to deduce

8. $\triangle(\phi \vee \phi') \vdash \triangle(\neg(\neg(\phi \vee \phi'))) \vdash \triangle(\neg(\neg\phi \wedge \neg\phi')) \vdash \triangle^{\mathsf{M}}(\neg\phi \wedge \neg\phi')$

This is, again, a conjunction, so we apply schemas 7a and 7b, giving us

8a. $\underline{0}(\phi \vee \phi') \vdash \underline{0}^{\mathsf{M}}(\neg\phi \wedge \neg\phi') \vdash \underline{1}(\neg\phi \wedge \neg\phi') \vdash \underline{1}\neg\phi \wedge \underline{1}\neg\phi' \vdash \underline{1}^{\mathsf{M}}\phi \wedge \underline{1}^{\mathsf{M}}\phi' \vdash \underline{0}\phi \wedge \underline{0}\phi'$
8b. $\underline{1}(\phi \vee \phi') \vdash \underline{1}^{\mathsf{M}}(\neg\phi \wedge \neg\phi') \vdash \underline{0}(\neg\phi \wedge \neg\phi') \vdash \underline{0}\neg\phi \vee \underline{0}\neg\phi' \vdash \underline{0}^{\mathsf{M}}\phi \vee \underline{0}^{\mathsf{M}}\phi' \vdash \underline{1}\phi \vee \underline{1}\phi'$

Note how the modals from $\underline{0}$ in 7a and 8a act as a kind of *negation* operator to turn the logical operators into their counterparts, similar to de *Morgan's law*.

The final case considers two consecutive modals:

9. $\delta_1\delta_2\phi \vdash (\delta_1 \circ \delta_2)\phi$

We interpret the $\circ$ operator as a kind of *function composition*, leading to a new modal δ which is the result of $\delta_1 \circ \delta_2$. We take a liberal stance here of what the result is, but indicate that it depends on the domain and, again, plausible inferences we like to capture. The $\circ$ operator will probably be different from the related operation $\odot$ which is used in Section 5.3.

4.2 Model Theory

In the following, we extend the standard definition of modal (Kripke) frames and models [3] for *graded* modal operators from $\triangle$ by employing the confidence function μ and focussing on the minimal definition for ϕ. A *frame* $\mathcal{F}$ for the probabilistic modal language is a pair $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}_\triangle \rangle$ where $\mathcal{W}$ is a non-empty set of *worlds* (or *situations, states, points, vertices, etc.*) and $\mathcal{R}_\triangle$ a family of binary relations over $\mathcal{W} \times \mathcal{W}$, called *accessibility relations*. In the following, we write R_δ to depict the accessibility relation for modal $\delta \in \triangle$.

A *model* $\mathcal{M}$ for the probabilistic modal language is a triple $\mathcal{M} = \langle \mathcal{F}, \mathcal{V}, \mu \rangle$, such that $\mathcal{F}$ is a *frame*, $\mathcal{V} : \varPhi \mapsto 2^{\mathcal{W}}$ is a *valuation*, assigning each proposition $\phi \in \varPhi$ a subset of $\mathcal{W}$, viz., the set of worlds in which ϕ holds, and μ is a mapping, returning the confidence interval for a given modality from $\triangle$. Note that we only require a definition for μ in $\mathcal{M}$ (the model, but *not* in the frame), as $\mathcal{F}$ represents the relational structure without interpreting the edge labelling R_δ of the graph.

The *satisfaction relation* $\models$, given a model $\mathcal{M}$ and a specific world w is inductively defined over the set of well-formed formulae in *negation normal form* (remember $\pi ::= p(\boldsymbol{c}) \mid \neg p(\boldsymbol{c})$):

1. $\mathcal{M}, w \models p(\boldsymbol{c})$ **iff** $w \in \mathcal{V}(p(\boldsymbol{c}))$ **and** $w \notin \mathcal{V}(\neg p(\boldsymbol{c}))$
2. $\mathcal{M}, w \models \neg p(\boldsymbol{c})$ **iff** $w \in \mathcal{V}(\neg p(\boldsymbol{c}))$ **and** $w \notin \mathcal{V}(p(\boldsymbol{c}))$
3. $\mathcal{M}, w \models \phi \wedge \phi'$ **iff** $\mathcal{M}, w \models \phi$ **and** $\mathcal{M}, w \models \phi'$
4. $\mathcal{M}, w \models \phi \vee \phi'$ **iff** $\mathcal{M}, w \models \phi$ **or** $\mathcal{M}, w \models \phi'$
5. **for all** $\delta \in \underline{1}$: $\mathcal{M}, w \models \delta\pi$ **iff** $\frac{\#\{u \mid (w,u) \in R_\delta \textbf{ and } \mathcal{M},u \models \pi\}}{\# \cup_{\delta' \in \triangle} \{v \mid (w,v) \in R_{\delta'}\}} \in \mu(\delta)$

6. **for all** $\delta \in \underline{0}$: $\mathcal{M}, w \models \delta\pi$ **iff** $1 - \frac{\#\{u|(w,u)\in R_\delta \textbf{ and } \mathcal{M},u\models\pi\}}{\#\cup_{\delta'\in\triangle}\{v|(w,v)\in R_{\delta'}\}} \in \mu(\delta)$

The last two cases of the satisfaction relation addresses the modals: for a world w, we look for the successor states u that are directly reachable via R_δ and in which π holds, and divide the number of such states $(\#\cdot)$ by the number of all worlds that are reachable from w by an arbitrary $R_{\delta'}$ in the denominator. This number, lying between 0 and 1, is then required to be an element of the confidence interval $\mu(\delta)$ of δ, in case $\delta \in \underline{1}$. For the modals whose confidence intervals start at 0, we clearly need to subtract this number from 1.
It is worth noting that the satisfaction relation above differs from the standard definition in its handling of $\mathcal{M}, w \models \neg p(\boldsymbol{c})$, as negation is *not* interpreted through the *absence* of $p(\boldsymbol{c})$ ($\mathcal{M}, w \not\models p(\boldsymbol{c})$), but through the *existence* of $\neg p(\boldsymbol{c})$. This treatment addresses the *open-world* nature in OWL and the evolvement of a (medical) domain over time.
We also note that the definition of the satisfaction relation for modalities (last clause) is related to the *possibility operators* $M_k\cdot$ ($= \Diamond^{\geq k}\cdot$; $k \in \mathbb{N}$) introduced by [5] and *counting modalities* $\cdot \geq n$ [1], used in modal logic characterizations of *description logics* with *cardinality* restrictions.

4.3 Two Constraints: Well-Behaved Frames

The definition of the satisfaction relation $\models$ above makes no assumptions about the underlying frame $\mathcal{F}$. For various reasons described below, we will now impose two constraints $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ on $\mathcal{F}$.

As we will see later, it is handy to assume that the graded modals are arranged in a kind of hierarchy—the more we move along the arrows in the hierarchy, the more a statement ϕ in the scope of a modal $\delta \in \triangle$ becomes *uncertain*. In order to address this, we slightly extend the notion of a *frame* by a third component $\preceq \subseteq \triangle \times \triangle$, a partial order (i.e., a reflexive, antisymmetric, and transitive binary relation) between modalities: $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}_\triangle, \preceq \rangle$.

Let us consider the following modal hierarchy that we build from the set $\triangle$ of already introduced modals (cf. Figure 1):

$$\begin{tikzcd} & \top \arrow[r] & C \arrow[r] & L \arrow[r] & N \arrow[dr] & \\ ! \arrow[ur] \arrow[dr] & & & & & ? \\ & \bot \arrow[r] & E \arrow[r] & U \arrow[urr] & & \end{tikzcd}$$

This graphical representation is just a compact way to specify a set of 33 binary relation instances over $\triangle \times \triangle$, such as $\top \preceq \top$, $\top \preceq N$, $C \preceq N$, $\bot \preceq ?$, or $! \preceq ?$.
The above mentioned form of uncertainty is expressed by the measure function μ in that the associated confidence intervals become larger:

$$\textbf{if } \delta \preceq \delta' \textbf{ then } \mu(\delta) \subseteq \mu(\delta')$$

In order to arrive at a proper and intuitive model-theoretic semantics which mirrors intuitions such as **if** ϕ *is confirmed* ($C\phi$) **then** ϕ *is likely* ($L\phi$), we will focus here on *well-behaved* frames $\mathcal{F}$ which enforce the existence of edges in $\mathcal{W}$, given $\preceq$ and $\delta, \delta^{\uparrow} \in \triangle$:

$$(\mathcal{C}_1)\ \textbf{if}\ (w,u) \in R_\delta\ \textbf{and}\ \delta \preceq \delta^{\uparrow}$$
$$\textbf{then}\ (w,u) \in R_{\delta^{\uparrow}}$$

However, by imposing this constraint, we also need to adapt the last two cases of the satisfiability relation from Section 4.2:

5. **for all** $\delta \in \underline{1}$: $\mathcal{M}, w \models \delta\pi$ **iff** $\frac{\# \cup_{\delta^{\uparrow} \succeq \delta} \{u \mid (w,u) \in R_{\delta^{\uparrow}} \textbf{ and } \mathcal{M}, u \models \pi\}}{\# \cup_{\delta' \in \triangle} \{v \mid (w,v) \in R_{\delta'}\}} \in \mu(\delta)$
6. **for all** $\delta \in \underline{0}$: $\mathcal{M}, w \models \delta\pi$ **iff** $1 - \frac{\# \cup_{\delta^{\uparrow} \succeq \delta} \{u \mid (w,u) \in R_{\delta^{\uparrow}} \textbf{ and } \mathcal{M}, u \models \pi\}}{\# \cup_{\delta' \in \triangle} \{v \mid (w,v) \in R_{\delta'}\}} \in \mu(\delta)$

Not only are we scanning for edges (w,u) labeled with R_δ and for successor states u of w in which π holds in the numerator (original definition), but also take into account edges $R_{\delta^{\uparrow}}$ marked with more general modals $\delta^{\uparrow}$, given $\delta^{\uparrow} \succeq \delta$. This mechanism implements a kind of *built-in model completion* that is not necessary in ordinary modal logics as they deal with only a *single* relation (viz., unlabelled arcs).
We have also seen that negated propositions inside the scope of a modal can be formulated equivalently by using the mirror modal: $\delta\neg\phi \equiv \delta^{\mathsf{M}}\phi$. Since $\mathcal{F}$ is only constrained by $(\mathcal{C}_1)$ so far, we impose a further restriction to guarantee that the satisfaction relation works properly for the interplay between negation and mirror modals as otherwise the fraction in case (5.) will yield wrong numbers. In order to capture both the left-to-right and the right-to-left direction of the equivalence, we use π here for abbreviating the propositional letters $\pi ::= p(\boldsymbol{c}) \mid \neg p(\boldsymbol{c})$ (see Section 4.1):

$$(\mathcal{C}_2)\ \textbf{if}\ (w,u) \in R_\delta\ \textbf{s.t.}\ u \in \mathcal{V}(\neg\pi)$$
$$\textbf{then}\ \exists u' \in \mathcal{W}\ \textbf{s.t.}\ (w,u') \in R_{\delta^{\mathsf{M}}}\ \textbf{and}\ u' \in \mathcal{V}(\pi)$$

5 Entailment Rules

We now turn our attention, again, to the syntax of our language and to the syntactic consequence relation. This section addresses a restricted subset of entailment rules which will unveil new (or implicit) knowledge from already existing graded statements. Recall that these kind of statements (in negation normal form) are a consequence of the application of simplification rules as depicted in Section 4.1. Thus, we assume a *pre-processing step* here that "massages" more complex statements that arise from a representation of graded (medical) statements in *natural language*. The entailments which we will present in a moment can either be *directly* implemented in a *tuple*-based reasoner, such as *HFC* [11], or in *triple*-based engines (e.g., *Jena* [4] or *OWLIM* [2]) which need to *reify* the medical statements in order to be compliant with the RDF triple model.

5.1 Modal Entailments

The entailments presented in this section deal with *plausible* inference centered around modals $\delta, \delta' \in \triangle$ which are, in part, also addressed in [16] in a pure

OWL setting. We use the implication sign $\rightarrow$ to depict the entailment rules *lhs* $\rightarrow$ *rhs* which act as *completion* (or *materialization*) rules the way as described in, e.g., [7] and [18], and used in today's *semantic repositories* (e.g., *OWLIM*). We sometimes even use the biconditional $\leftrightarrow$ to address that the LHS and the RHS are semantically equivalent, but will indicate the direction that should be used in a practical setting. As before, we define $\pi ::= p(\boldsymbol{c}) \mid \neg p(\boldsymbol{c})$. We furthermore assume that for every modal $\delta \in \triangle$, a *complement* modal δ^{C} and a *mirror* modal δ^{M} exist (cf. Section 4.1).

Lift (L) $\boldsymbol{\pi \leftrightarrow \top\pi}$**.** This rule interprets propositional statements as special modal formulae. It might be dropped and can be seen as a pre-processing step. We have used it in the Hepatitis example above. Usage: left-to-right direction.

Generalize (G) $\boldsymbol{\delta\pi \wedge \delta \preceq \delta' \rightarrow \delta'\pi}$**.** This rule schema can be instantiated in various ways, using the modal hierarchy from Section 4.3, e.g., $\top\pi \rightarrow C\pi$, $C\pi \rightarrow L\pi$, or $E\pi \rightarrow U\pi$. It has been used in the Hepatitis example.

Complement (C) $\boldsymbol{\neg\delta\pi \leftrightarrow \delta^{\mathsf{C}}\pi}$**.** In principle, (C) is not needed in case the statement is already in negation normal form. This schema might be useful for natural language paraphrasing (explanation). Given $\triangle$, there are four possible instantiations: $E\pi \leftrightarrow \neg N\pi$, $N\pi \leftrightarrow \neg E\pi$, $?\pi \leftrightarrow \neg!\pi$, and $!\pi \leftrightarrow \neg?\pi$.

Mirror (M) $\boldsymbol{\delta\neg\pi \leftrightarrow \delta^{\mathsf{M}}\pi}$**.** Again, (M) is in principle not needed as long as the modal proposition is in negation normal form, since we do allow for negated propositional statements $\neg p(\boldsymbol{c})$. This schema might be useful for natural language paraphrasing (explanation). For $\triangle$, there are six possible instantiations: $E\pi \leftrightarrow C\neg\pi$, $C\pi \leftrightarrow E\neg\pi$, $L\pi \leftrightarrow U\neg\pi$, $U\pi \leftrightarrow L\neg\pi$, $\top\pi \leftrightarrow \bot\neg\pi$, and $\bot\pi \leftrightarrow \top\neg\pi$.

Uncertainty (U) $\boldsymbol{\delta\pi \wedge \neg\delta\pi \leftrightarrow \delta\pi \wedge \delta^{\mathsf{C}}\pi \leftrightarrow ?\pi}$**.** The *co-occurrence* of $\delta\pi$ and $\neg\delta\pi$ does *not* imply logical *inconsistency* (propositional case: $\pi \wedge \neg\pi$), but leads to complete *uncertainty* about the validity of π. Usage: left-to-right direction. Remember that $\mu(?) = \mu(\delta) \uplus \mu(\delta^{\mathsf{C}}) = [0, 1]$:

$$\begin{array}{ll} & 0 \qquad\qquad\qquad\quad 1 \\ \mu : & |\!-\!\delta^{\mathsf{C}}\!-\!|\!-\!\!-\!\!-\!\delta\!-\!\!-\!\!-\!| \\ & \quad\ \pi \qquad\quad\ \ \pi \end{array}$$

Negation (N) $\boldsymbol{\delta(\pi \wedge \neg\pi) \leftrightarrow \delta^{\mathsf{M}}(\pi \wedge \neg\pi)}$**.** (N) can be easily shown by applying the simplification rules from Section 4.1. $\delta(\pi \wedge \neg\pi)$ can be formulated equivalently by using the mirror modal δ^{M}:

$$\begin{array}{ll} & 0 \qquad\qquad\qquad\qquad\quad 1 \\ \mu : & |\!-\!\delta^{\mathsf{M}}\!-\!|\!-\!\!-\!\!-\!|\!-\!\delta\!-\!| \\ & \ \ \pi \wedge \neg\pi \qquad\ \pi \wedge \neg\pi \end{array}$$

In general, (N) is *not* the modal counterpart of the *law of non-contradiction*, as $\pi \wedge \neg\pi$ is usually afflicted by uncertainty, meaning that from $\delta(\pi \wedge \neg\pi)$, we can *not* infer that $\pi \wedge \neg\pi$ is the case for the concrete example in question (recall the intention behind the confidence intervals; cf. Section 3). There is one notable exception, involving the $\top$ and $\bot$ modals. This is formulated by the next entailment rule.

Error (E) $\boldsymbol{\top(\pi \wedge \neg\pi) \leftrightarrow \bot(\pi \wedge \neg\pi) \rightarrow !(\pi \wedge \neg\pi) \leftrightarrow !\pi}$**.** (E) *is* the modal counterpart of the *law of non-contradiction* (note: $\bot^{\mathsf{M}} = \top, \top^{\mathsf{M}} = \bot, !^{\mathsf{M}} = !$). For this reason and *by definition*, the *error* (or *failure*) modal ! from Section 3 comes into play here. The modal ! can serve as a hint to either stop a computation the first time it occurs, or to continue reasoning and to syntactically memorize the ground literal π. Usage: left-to-right direction.

5.2 Subsumption Entailments

As before, we define two subsets of $\triangle$, called $\underline{1} = \{\top, C, L, N\}$ and $\underline{0} = \{\bot, E, U\}$, thus effectively become $\underline{1} = \{\top, C, L, N, U^{\mathsf{C}}\}$ and $\underline{0} = \{\bot, U, E, C^{\mathsf{C}}, L^{\mathsf{C}}, N^{\mathsf{M}}\}$ due to the use of complement modals δ^{C} and mirror modals δ^{M} for every base modal $\delta \in \triangle$ and by assuming that $E = N^{\mathsf{C}}$, $E = C^{\mathsf{M}}$, $U = L^{\mathsf{M}}$, and $\bot = \top^{\mathsf{M}}$, together with the four "opposite" cases.

Now, let $\sqsubseteq$ abbreviate relation subsumption as known from description logics and realized through `subClassOf` and `subPropertyOf` in RDFS. Given this, we define two further very practical and plausible modal entailments which can be seen as the modal extension of the entailment rules (rdfs9) and (rdfs7) for classes and properties in RDFS [7]:

$$(\mathsf{S1})\ \ \underline{1}p(\boldsymbol{c}) \wedge p \sqsubseteq q \rightarrow \underline{1}q(\boldsymbol{c}) \qquad (\mathsf{S0})\ \ \underline{0}q(\boldsymbol{c}) \wedge p \sqsubseteq q \rightarrow \underline{0}p(\boldsymbol{c})$$

Note how the use of p and q switches in the antecedent and the consequent, even though $p \sqsubseteq q$ holds in both cases. Note further that propositional statements π are restricted to the positive case $p(\boldsymbol{c})$ and $q(\boldsymbol{c})$, as their negation in the antecedent will not lead to any valid entailments.

Here are two *instantiations* of $(\mathsf{S0})$ and $(\mathsf{S1})$ for the unary and binary case (remember, $E \in \underline{0}$ and $C \in \underline{1}$):

$$\mathsf{ViralHepatitis} \sqsubseteq \mathsf{Hepatitis} \wedge E\mathsf{Hepatitis}(x) \rightarrow E\mathsf{ViralHepatitis}(x)$$
$$\mathsf{deeplyEnclosedIn} \sqsubseteq \mathsf{containedIn} \wedge C\mathsf{deeplyEnclosedIn}(x, y) \rightarrow C\mathsf{containedIn}(x, y)$$

5.3 Extended RDFS and OWL Entailments

In this section, we will consider further entailment rules for RDFS [7] and a restricted subset of OWL [18, 15]. Remember that modals only head positive and negative propositional letters π, not TBox or RBox axioms. Concerning the original entailment rules, we will distinguish *four principal cases* to which the extended rules belong (we will only consider the unary and binary case here as used in description logics/OWL):

1. TBox and RBox axiom schemas will not undergo a modal extension;
2. rules get extended in the antecedent;
3. rules take over modals from the antecedent to the consequent;
4. rules aggregate several modals from the antecedent in the consequent.

We will illustrate the individual cases in the following with examples by using a kind of description logic rule syntax. Clearly, the set of extended entailments depicted here is *not complete*.

Case-1: No Modals. Entailment rule (rdfs11) from [7] deals with class subsumption: $\mathsf{C} \sqsubseteq \mathsf{D} \wedge \mathsf{D} \sqsubseteq \mathsf{E} \rightarrow \mathsf{C} \sqsubseteq \mathsf{E}$. As this is a terminological axiom schema, the rule stays *constant* in the modal domain. Example rule instantiation:

$\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \wedge \mathsf{ViralHepatitis} \sqsubseteq \mathsf{Hepatitis}$
$\rightarrow \mathsf{ViralHepatitisB} \sqsubseteq \mathsf{Hepatitis}$

Case-2: Modals on LHS, No Modals on RHS. The following original rule (rdfs3) from [7] imposes a range restriction on objects of binary ABox relation instances: $\forall \mathsf{P}.\mathsf{C} \wedge \mathsf{P}(\mathsf{x},\mathsf{y}) \rightarrow \mathsf{C}(\mathsf{y})$. The extended version needs to address the ABox proposition in the antecedent (*don't care* modal δ), but must not change the consequent (even though we always use the $\top$ modality here—the range restriction $\mathsf{C}(\mathsf{y})$ is always true, independent of the uncertainty of $\mathsf{P}(\mathsf{x},\ \mathsf{y})$; cf. Section 2 example):

$$(\mathsf{Mrdfs3}) \quad \forall \mathsf{P}.\mathsf{C} \wedge \delta\mathsf{P}(\mathsf{x},\mathsf{y}) \rightarrow \top\mathsf{C}(\mathsf{y})$$

Example rule instantiation:

$\forall \mathsf{suffersFrom}.\mathsf{Disease} \wedge L\mathsf{suffersFrom}(x,y) \rightarrow \top\mathsf{Disease}(y)$

Case-3: Keeping LHS Modals on RHS. Inverse properties switch their arguments [18] as described by (rdfp8): $\mathsf{P} \equiv \mathsf{Q}^{-} \wedge \mathsf{P}(\mathsf{x},\mathsf{y}) \rightarrow \mathsf{Q}(\mathsf{y},\mathsf{x})$. The extended version simply keeps the modal operator:

$$(\mathsf{Mrdfp8}) \quad \mathsf{P} \equiv \mathsf{Q}^{-} \wedge \delta\mathsf{P}(\mathsf{x},\mathsf{y}) \rightarrow \delta\mathsf{Q}(\mathsf{y},\mathsf{x})$$

Example rule instantiation:

$\mathsf{containedIn} \equiv \mathsf{contains}^{-} \wedge C\mathsf{containedIn}(x,y) \rightarrow C\mathsf{contains}(y,x)$

Case-4: Aggregating LHS Modals on RHS. Now comes the most interesting case of modalized RDFS & OWL entailment rules, that offers several possibilities on a varying scale between *skeptical* and *credulous* entailments, depending on the degree of uncertainty, as expressed by the measuring function μ of the modal operator. Consider the original rule (rdfp4) from [18] for transitive properties: $\mathsf{P}^{+} \sqsubseteq \mathsf{P} \wedge \mathsf{P}(\mathsf{x},\mathsf{y}) \wedge \mathsf{P}(\mathsf{y},\mathsf{z}) \rightarrow \mathsf{P}(\mathsf{x},\mathsf{z})$.

Now, how does the modal on the RHS of the extended rule look like, depending on the two LHS modals? There are several possibilities. By operating directly on the *modal hierarchy*, we are allowed to talk about, e.g., the *least upper bound* or the *greatest lower bound* of δ_1 and δ_2. When taking the associated *confidence intervals* into account, we might play with the low and high numbers of the intervals, say, by applying min/max, the *arithmetic mean* or even by *multiplying* the corresponding numbers. Let us first consider the general rule from which more specialized versions can be derived, simply by instantiating the combination operator $\odot$:

$$(\mathsf{Mrdfp4}) \quad \mathsf{P}^{+} \sqsubseteq \mathsf{P} \wedge \delta_1\mathsf{P}(\mathsf{x},\mathsf{y}) \wedge \delta_2\mathsf{P}(\mathsf{y},\mathsf{z}) \rightarrow (\delta_1 \odot \delta_2)\mathsf{P}(\mathsf{x},\mathsf{z})$$

Here is an instantiation of (Mrdfp4) as used in *HFC*, dealing with the transitive relation contains from above, assuming that $\odot$ reduces to the *least upper bound* (i.e., $C \odot L = L$):

$$C\mathsf{contains}(x, y) \wedge L\mathsf{contains}(y, z) \rightarrow L\mathsf{contains}(x, z)$$

What is the general result of $\delta_1 \odot \delta_2$? It depends, probably both on the application domain and the *epistemic commitment* one is willing to accept about the "meaning" of gradation words/modal operators. To enforce that $\odot$ is at least both *commutative* and *associative* (as is the least upper bound) is probably a good idea, making the sequence of modal clauses *order independent*. And to work on the modal hierarchy instead of combining low/high numbers of the corresponding intervals is probably a good decision for forward chaining engines, as the latter strategy might introduce *new* individuals through operations such as multiplication, thus posing a problem for the implementation of the generalization schema (G) (see Section 5.1).

6 Adding Time

Temporal databases [17] distinguish between (at least) two different notions of time and the representation of temporal change: *valid time*, the temporal interval in which a statement about the world is valid, and *transaction time*, the temporal duration during which a statement has been stored in a database (or ontology, in our case). Valid time is able to add information about the past, present, and future, given a moment in time, whereas transaction time add present time (= *now*) when a statement is entered to the database. At the end of this section, we will have established a transaction time extension for the modal fragment of OWL derived so far, including a set of entailment rules and a corresponding extended model theory.

6.1 Metric Linear Time

In the following, we assume that the temporal measuring system is based on a one-dimensional *metric linear time* $\langle \mathcal{T}, \leq \rangle$, so that we can compare starting/ending points, using operators, such as $\leq$, or pick out input arguments in aggregates, using *min* or *max*. We require, for reasons which will become clear, that time is *discrete* and represented by natural or rational numbers.
The implementation of *HFC* employs 8-byte long integers (XSD datatype `long`) to encode *milli* or even *nano* seconds w.r.t. a fixed starting point (Unix Epoch time, starting with 1 January 1970, 00:00:00). Alternatively, the XSD `dateTime` format can be used which provides an arbitrarily fine precision, if needed.
As a consequence, given a time point $t \in \mathcal{T}$, the next smallest or successor time point would then be $t + 1$ (after a potential normalization). We often use this kind of notation to derive the ending time of a valid proposition $\top\phi@t$ from the time it gets invalidated: $\bot\phi@t{+}1$; see Section 6.3.

6.2 Valid Time

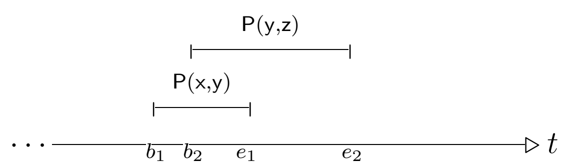
Valid time is a useful concept when representing, e.g., biographical knowledge which has been obtained from the Web. Various forms of OWL representations involving time have been investigated [6, 13, 19]. However, reasoning and querying with such representations is extremely complex, expensive, and error-prone and standard OWL reasoning is no longer applicable. In [10], we have investigated *valid time* for a non-graded extension of RDFS and OWL (triples, binary relation instances), representing the time period of an atemporal statement by two further argument, giving us quintuples instead of triples in the end. This extension is a pure syntactic calculus, defined as a set of tableaux-like entailment rules à la [7, 18], able to derive useful new information in a temporal environment. For instance, the standard entailment rule (rdfp4) in OWL for *transitive properties* (see Section 5.3)

$$\mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \mathsf{P}(\mathsf{x},\mathsf{y}) \wedge \mathsf{P}(\mathsf{y},\mathsf{z}) \rightarrow \mathsf{P}(\mathsf{x},\mathsf{z})$$

then becomes

$$\mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \mathsf{P}(\mathsf{x},\mathsf{y},b_1,e_1) \wedge \mathsf{P}(\mathsf{y},\mathsf{z},b_2,e_2) \wedge [b_1,e_1] \cap [b_2,e_2] \neq \emptyset \rightarrow \mathsf{P}(\mathsf{x},\mathsf{z},b,e)$$

where $[b_1, e_1]$ and $[b_2, e_2]$ are the temporal intervals during which P(x,y) and P(y,z) are valid, given $b = max(b_1, b_2)$ and $e = min(e_1, e_2)$. I.e., P(x,z) is only valid during the proper intersection of $[b_1, e_1]$ and $[b_2, e_2]$. This is depicted in the following figure:



Note that P(x,z) is definitely the case for $[b, e]$, but we do not know if it holds before b or after e. This inference harmonizes well with the open-world assumption underlying OWL.

In *HFC*, the meaning of the *original* entailment rule (left) and the *extension* for valid time (right) can be straightforwardly derived from the abstract syntax above:

```
?p rdf:type owl:TransitiveProperty
?x ?p ?y
?y ?p ?z
->
?x ?p ?z
```

```
?p rdf:type owl:TransitiveProperty
?x ?p ?y ?b1 ?e1
?y ?p ?z ?b2 ?e2
->
?x ?p ?z ?b ?e
@test
IntersectionNotEmpty ?b1 ?e1 ?b2 ?e2
@action
?b = Max ?b1 ?b2
?e = Min ?e1 ?e2
```

In *HFC*, `IntersectionNotEmpty` refers to the (Java) implementation of a specific method of the corresponding class which realizes the above intersection of the corresponding temporal intervals (pseudo code):

IntersectionNotEmpty start1 end1 start2 end2 $\equiv$
  start := max(start1, start2)
  end := min(end1, end2)
  return (start $\leq$ end)

This computationally cheap left-hand side test (cf. the `@test` section in the above *HFC* rule) is applied after LHS matching and before right-hand side instantiation. The RHS generation of the resulting interval $[b, e]$ is achieved by the two aggregates `Max` and `Min` whose return values are bound to the RHS-only rule variables `?b` and `?e`, resp. (cf. the `@action` section above). It is worth noting that these two aggregates do *not* generate brand-new individuals (contrary to addition, for example), thus a terminating rule set and so a finite model is guaranteed overall.

The interesting observation when adding *valid time* to the RDFS & ter Horst subset of OWL is that only an additional test (cf. `IntersectionNotEmpty`) and two aggregates (cf. `Max`, `Min`) are needed [10]. Almost the same is true when adding *transaction time* to the modal extension of RDFS & OWL that we have investigated so far in the first part of this article. The additional test in *HFC* is called `ValidInBetween` and the aggregates are `Min` and `Max`, as before.

6.3 Transaction Time

Like valid time, the original approach to transaction time makes use of temporal intervals in order to represent the time during which a fact is stored in the database, even though the ending time is not known in advance. This is indicated by the wildcard **?** which will later be *overwritten* by the concrete ending time. We *deviate* here from the interval view by specifying both the starting time when an ABox statement is entered to an ontology, and, via a *separate* statement, the ending time when the statement is *invalidated*.[4] For this, we exploit the propositional modals $\top$ and $\bot$ from before. This idea is shown in the following figure for a binary relation P. We write `P(c,d,b,e)` to denote the row `<c,d,b,e>` in the database table `P` for relation P.

Time	Database View	Ontology View
$\vdots$	$\vdots$	$\vdots$
t_1	add: P(c,d,t_1,?)	add: $\top$P(c,d)@t_1
$\vdots$	$\vdots$	$\vdots$
t_2	**overwrite**: P(c,d,t_1,t_2)	——
$t_2 + 1$	——	**add**: $\bot$P(c,d)@t_2+1
$\vdots$	$\vdots$	$\vdots$

[4] When *we* say *transaction time* we usually mean the time a statement is *added* to the *ontology*, say t_1 or t_2+1 in the figure.

As we see from this picture, the invalidation in the ontology happens at t_2+1, whereas $[\mathtt{t}_1, \mathtt{t}_2]$ specifies the transaction time in the database. Clearly, the same transaction time interval for $\mathsf{P(c,d)}$ in the ontology can be derived from the two statements $\top\mathsf{P(c,d)}$`@`t_1 and $\bot\mathsf{P(c,d)}$`@`t_2+1, assuming that there does *not* exist a $\bot\mathsf{P(c,d)}$`@`t, such that $t_1 \leq t \leq t_2$ (we can effectively query for this by employing the `ValidInBetween` test).
Extending ontologies by transaction time the way we proceed here gives us a means to easily encode *time series data*, i.e., allows us to record the *history* of data that changes over time, and so simulating imperative *variables* in a declarative environment.

6.4 Entailment Rules for Graded Modals and Transaction Time

We have almost introduced the *abstract* syntax for graded propositions with transaction time ($\delta \in \triangle$)

$$\delta\phi@t$$

Here, we focus on the binary relation case in order to address the RDFS [7] and ter Horst extension of OWL [18] from above. For this, we will then write

$$\delta\mathsf{P}(\mathsf{c},\mathsf{d})@t$$

The corresponding *quintuple* representation in *HFC* then becomes

$$\delta \;\; \mathtt{c} \;\; \mathtt{P} \;\; \mathtt{d} \;\; \mathtt{t}$$

We opt for a *uniform* representation, thus *axiomatic triples* need to be extended by two further arguments; for instance,

```
owl:sameAs rdf:type owl:TransitiveProperty
```

becomes

```
logic:true owl:sameAs rdf:type owl:TransitiveProperty "0"^^xsd:long
```

We read the above statement as *being true* ($\top$ = `logic:true`) *from the beginning of time* (long int 0 = `"0"^^xsd:long`). We are now ready to distinguish, again, between the *four principled cases* from Section 5.3, where we compared the original rules from [7, 18] to the graded modal extension, but now extend them further by a transaction time argument.

Case-1: Top Modals Only, Zero Time. We have already seen that the entailment rule (rdfs11) from [7] deals with class subsumption: $\mathsf{C} \sqsubseteq \mathsf{D} \wedge \mathsf{D} \sqsubseteq \mathsf{E} \rightarrow \mathsf{C} \sqsubseteq \mathsf{E}$. As this rule concerns only terminological knowledge (TBox), we decided not to change it in the modal domain. Since we argued above for a uniform quaternary relation or quintuple representation, this rule leads us quite naturally to the extended version of (rdfs11):

$$(\mathsf{TMrdfs11}) \quad \top\mathsf{C} \sqsubseteq \mathsf{D}@0 \wedge \top\mathsf{D} \sqsubseteq \mathsf{E}@0 \rightarrow \top\mathsf{C} \sqsubseteq \mathsf{E}@0$$

This notation simply highlights that the original class subsumption entailment *is true at every time*, i.e., expresses an universal truth (remember the meaning of $\top$ and transaction time 0, and compare this to the axiomatic triple from above).

Case-2: Modals on LHS, Top Modals on RHS, Keeping Time. The original rule (rdfs3) from [7] imposes a range restriction on P: $\forall \mathsf{P.C} \wedge \mathsf{P}(\mathsf{x},\mathsf{y}) \rightarrow \mathsf{C}(\mathsf{y})$. Adding modals gave us (Mrdfs3): $\forall \mathsf{P.C} \wedge \delta\mathsf{P}(\mathsf{x},\mathsf{y}) \rightarrow \top\mathsf{C}(\mathsf{y})$. Extending this rule with transaction time is easy:

$$(\mathsf{TMrdfs3}) \quad \top\forall \mathsf{P.C}@0 \wedge \delta\mathsf{P}(\mathsf{x},\mathsf{y})@t \rightarrow \top\mathsf{C}(\mathsf{y})@t$$

The range restriction is a universal RBox statement (thus $\top$ and 0). P(x,y) is graded (δ) and happens at a specific time t. Thus, the class prediction C(y) of the range argument y at time t is true ($\top$).

Case-3: Keeping LHS Modals on RHS, Keeping Time. Inverse properties are described in [18] by (rdfp8): $\mathsf{P} \equiv \mathsf{Q}^- \wedge \mathsf{P}(\mathsf{x},\mathsf{y}) \rightarrow \mathsf{Q}(\mathsf{y},\mathsf{x})$. The modalized version simply kept the modal operator (Mrdfp8): $\mathsf{P} \equiv \mathsf{Q}^- \wedge \delta\mathsf{P}(\mathsf{x},\mathsf{y}) \rightarrow \delta\mathsf{Q}(\mathsf{y},\mathsf{x})$. The transaction time version furthermore takes over the temporal argument:

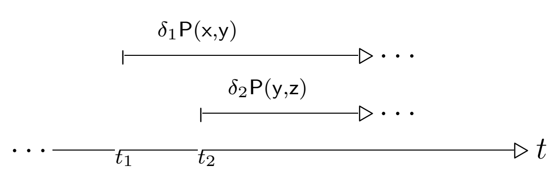
$$(\mathsf{TMrdfp8}) \quad \top\mathsf{P} \equiv \mathsf{Q}^-@0 \wedge \delta\mathsf{P}(\mathsf{x},\mathsf{y})@t \rightarrow \delta\mathsf{Q}(\mathsf{y},\mathsf{x})@t$$

Again, $\mathsf{P} \equiv \mathsf{Q}^-$ is a universal RBox statement (use $\top$ and0) and both the grading of $\mathsf{P}(\mathsf{x},\mathsf{y})$ and time t is consequently transferred to $\mathsf{Q}(\mathsf{y},\mathsf{x})$.

Case-4: Aggregating Modals, Aggregating Time. This case is the most challenging and computationally expensive one. The concrete implementation in *HFC* employs the above-mentioned test `ValidInBetween` (two times use in lines 2 and 3 below in a different form) and the aggregates `Min` and `Max`. Again, we will focus on one specific rule here, viz., (rdfp4) from [18] for transitive properties: $\mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \mathsf{P}(\mathsf{x},\mathsf{y}) \wedge \mathsf{P}(\mathsf{y},\mathsf{z}) \rightarrow \mathsf{P}(\mathsf{x},\mathsf{z})$. The modal extension led us to (Mrdfp4): $\mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \delta_1\mathsf{P}(\mathsf{x},\mathsf{y}) \wedge \delta_2\mathsf{P}(\mathsf{y},\mathsf{z}) \rightarrow (\delta_1 \odot \delta_2)\mathsf{P}(\mathsf{x},\mathsf{z})$. This blueprint can be utilized to derive the final transaction time version:

$$\begin{aligned}
(\mathsf{TMrdfp4}) \quad & \top\mathsf{P}^+ \sqsubseteq \mathsf{P}@0 \wedge \delta_1\mathsf{P}(\mathsf{x},\mathsf{y})@t_1 \wedge \delta_2\mathsf{P}(\mathsf{y},\mathsf{z})@t_2 \wedge \\
\texttt{ValidInBetween}: \quad & \nexists\, \delta_1^{\mathsf{M}}\mathsf{P}(\mathsf{x},\mathsf{y})@t \text{ s.t. } min(t_1,t_2) \leq t \leq max(t_1,t_2) \wedge \\
\texttt{ValidInBetween}: \quad & \nexists\, \delta_1^{\mathsf{M}}\mathsf{P}(\mathsf{y},\mathsf{z})@t' \text{ s.t. } min(t_1,t_2) \leq t' \leq max(t_1,t_2) \wedge \\
& \rightarrow (\delta_1 \odot \delta_2)\mathsf{P}(\mathsf{x},\mathsf{z})@max(t_1,t_2)
\end{aligned}$$

Lines 1 and 4 of (TMrdfp4) are easy to grasp when compared to the plain modal extension (Mrdfp4) and the fact that the transaction time for the consequent $(\delta_1 \odot \delta_2)\mathsf{P}(\mathsf{x,z})$ is based on the time when *both* $\delta_1\mathsf{P}(\mathsf{x,y})$ and $\delta_2\mathsf{P}(\mathsf{y,z})$ are the case, i.e., $max(t_1,t_2)$:



Furthermore, lines 2 and 3 guarantee that graded *contradictory* information with an equal or *less* degree of uncertainty $||\delta_{1,2}^{\mathsf{M}}||$ (i.e., equal or *more* trustworthiness) does *not* exist as it would argue too strongly against the graded entailment of $\mathsf{P(x,\ z)}$. Here, it is important to understand the interplay between $(\mathsf{TMrdfp4})$ and the extension of the binary generalization schema (G) from Section 5.1:

$$(\mathsf{TG}) \quad \delta\mathsf{P}(\mathsf{x},\mathsf{y})@\mathsf{t} \wedge \top\delta \preceq \delta'@0 \rightarrow \delta'\mathsf{P}(\mathsf{x},\mathsf{y})@t$$

Consider, for example, that $\delta_1\mathsf{P}(\mathsf{x},\mathsf{y})@t_1$ matches $C\mathsf{P}(\mathsf{x},\mathsf{y})@30$ and $\delta_2\mathsf{P}(\mathsf{y},\mathsf{z})@t_2$ matches $L\mathsf{P}(\mathsf{y},\mathsf{z})@42$ in $(\mathsf{TMrdfp4})$. Given statement $E\mathsf{P}(\mathsf{x},\mathsf{y})@40$ ($30 \leq 40 \leq 42$), we are thus *not* allowed to derive the instantiation of the antecedent of $(\mathsf{TMrdfp4})$. The more *certain* statement $\bot\mathsf{P}(\mathsf{x},\mathsf{y})@40$ does also *not* support the rule as (TG) would allow us to derive $E\mathsf{P}(\mathsf{x},\mathsf{y})@40$ again. Only a *more uncertain* modal than E will do the trick, e.g., U (recall that $||\bot|| < ||E|| < ||U||$). Thus, $U\mathsf{P}(\mathsf{x},\mathsf{y})@40$ is a necessary requirement for finally deriving $(C \odot L)\mathsf{P}(\mathsf{x},\mathsf{z})@42$.

6.5 Model Theory for Graded Modals and Transaction Time

The model theory for graded modals including transaction time will not differ much from what we already introduced in Sections 4.2 and 4.3. Time points $t \in \mathcal{T}$, indicated by the notation $@t$ relative to a proposition $\mathsf{P(x,y)}$, are related to what *hybrid logics* [3] call *nominals—handles to worlds* which are indexed by t and which are made available in the syntax of the modal language via $@t$. In our setting and contrary to hybrid logics, t does *not* refer to a single world, but to multiple ones.

For transaction time, we still keep the notion of a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}_\triangle, \preceq \rangle$ and, in principle, that of a model $\mathcal{M} = \langle \mathcal{F}, \mathcal{V}, \mu \rangle$ (see Section 4.2). However, we will *modify* the valuation function $\mathcal{V} : \Phi \mapsto 2^{\mathcal{W}}$ in that its domain now also takes time points from $\mathcal{T}$ into account; i.e., $\mathcal{V} : \Phi \times \mathcal{T} \mapsto 2^{\mathcal{W}}$ returns those worlds at which $\phi@t$ is valid, given $\phi \in \Phi$ and $t \in \mathcal{T}$. This directly leads us to the extension of the six cases for the satisfaction relation $\models$ from Sections 4.2 and 4.3:

1. $\mathcal{M}, w \models p(\boldsymbol{c})@t$ **iff** $w \in \mathcal{V}(p(\boldsymbol{c}), t)$ **and** $w \notin \mathcal{V}(\neg p(\boldsymbol{c}), t)$
2. $\mathcal{M}, w \models \neg p(\boldsymbol{c})@t$ **iff** $w \in \mathcal{V}(\neg p(\boldsymbol{c}), t)$ **and** $w \notin \mathcal{V}(p(\boldsymbol{c}), t)$
3. $\mathcal{M}, w \models \phi@t \wedge \phi'@t'$ **iff** $\mathcal{M}, w \models \phi@t$ **and** $\mathcal{M}, w \models \phi'@t'$
4. $\mathcal{M}, w \models \phi@t \vee \phi'@t'$ **iff** $\mathcal{M}, w \models \phi@t$ **or** $\mathcal{M}, w \models \phi'@t'$
5. **for all** $\delta \in \underline{1}$: $\mathcal{M}, w \models \delta\pi@t$ **iff** $\frac{\# \cup_{\delta\uparrow \succeq \delta} \{u | (w,u) \in R_{\delta\uparrow} \textbf{ and } \mathcal{M}, u \models \pi@t\}}{\# \cup_{\delta' \in \triangle} \{v | (w,v) \in R_{\delta'}\}} \in \mu(\delta)$
6. **for all** $\delta \in \underline{0}$: $\mathcal{M}, w \models \delta\pi@t$ **iff** $1 - \frac{\# \cup_{\delta\uparrow \succeq \delta} \{u | (w,u) \in R_{\delta\uparrow} \textbf{ and } \mathcal{M}, u \models \pi@t\}}{\# \cup_{\delta' \in \triangle} \{v | (w,v) \in R_{\delta'}\}} \in \mu(\delta)$

We also keep constraint $(\mathcal{C}_1)$ for well-behaved frames, but need to modify constraint $(\mathcal{C}_2)$ to incorporate transaction time (cf. Section 4.3):

$$\begin{aligned}(\mathcal{C}_2)\ &\textbf{if } (w,u) \in R_\delta \textbf{ s.t. } u \in \mathcal{V}(\neg\phi, t)\\ &\textbf{then } \exists u' \in \mathcal{W} \textbf{ s.t. } (w,u') \in R_{\delta^{\mathsf{M}}} \textbf{ and } u' \in \mathcal{V}(\phi, t)\end{aligned}$$

Furthermore, we impose a third constraint on the relational structure $\mathcal{F}$ which models the intuition *if* ϕ *is valid at time* t*, so is* ϕ *at* $t+1$, in case nothing *argues* *heavily against* ϕ (compare this to a similar argumentation expressed by lines 2 and 3 in (TMrdfp4) of case 4 in Section 6.4):

$$(\mathcal{C}_3)\ \textbf{if}\ (w,u) \in R_\delta\ \textbf{s.t.}\ u \in \mathcal{V}(\phi,t)\ \textbf{and}\ \nexists (w,v) \in R_{\delta^{\mathsf{M}}}\ \textbf{s.t.}\ v \in \mathcal{V}(\phi,t)$$
$$\textbf{then}\ \exists (w,x) \in R_\delta\ \textbf{s.t.}\ x \in \mathcal{V}(\phi,t+1)$$

Here, however, we do not need to check for proposition $\delta^{\mathsf{M}}\phi$ between t and $t+1$, as time is discrete and normalized, so that $t+1$ is the *immediate* successor of t. Constraint $(\mathcal{C}_3)$ can be seen as a kind of *forward monotonicity* in that valid propositions at time t will always hold at time $t+1$. As a consequence, this will give us an *infinite* frame (cf. the *existential* variable x in the consequent), i.e., an infinite number of worlds.

To implement such a kind of model behaviour in the syntax through a *finite* number of propositions, we make the following assumption. Propositions will *never* be brought to the *temporal forefront* (never being updated), i.e., there is *no* rule such as $\delta\phi@t \rightarrow \delta\phi@(t+1)$. Only if $\delta\phi$ needs to be *invalidated* at t', we will add the further statement $\perp\phi@t'$. Thus, through the use of the test `ValidInBetween` from above, we are then able to query whether $\delta\phi$ is still valid at a different time $t'' > t$.

7 Summary

In this paper, we have explored a fragment of a non-standard modal logic, being able to represent graded statements about the world. The modal operators in the syntax of the modal language were derived from gradation words and were further extended through mirror and complement operations. The operators were interpreted through confidence intervals in the model theory for expressing the uncertainty about the validity of a proposition. The model theory was complemented by a set of RDFS-/OWL-like entailment rules, acting on the syntactic representation of modalized statements. Finally, we extended the framework by transaction time in order to implement a notion of temporal change. The framework has been implemented in *HFC* for the case of binary propositions.

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