

Perfect Square Sum Magic Squares¹

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Abstract

This paper shows how to create magic squares with a perfect square number for the total sum of their entries. This has been done in two ways: Firstly, by using the sum of consecutive odd numbers, and secondly, by using consecutive natural numbers. In the first case, for all orders of magic squares, one can always obtain a perfect square sum. In the second case, magic squares with perfect square magic sums do exist, but only for odd order magic squares. For the even order magic squares, such as 4, 6, 8, etc. it is not possible to write consecutive natural number magic squares with perfect square sums of their entries. A simplified idea is introduced to check when it is possible to obtain minimum perfect square sums. Also, a uniform method is presented so that, if k is the order of a magic square, then the magic sum of the square is k^3 , and the sum of all entries of the magic square is k^4 . Examples are given for the magic squares of orders 3 to 25.

Contents

1	Introduction	3
2	Series and Magic Square Sums	5
2.1	Natural Number Series Sums	6
2.2	Odd Number Series Sums	6
2.3	Magic Square Sums and Sums of the Entries	6
2.4	Perfect Square Sums with Consecutive Odd Numbers	6
2.5	Procedure	7
2.5.1	Uniformity	8
2.5.2	General Case	8
3	Minimum Perfect Square Sums and Uniformity	9
4	Perfect Square Sum Magic Squares: Examples	16
4.1	Magic Squares of Order 3	16
4.1.1	First Approach: Consecutive Odd Numbers	16
4.1.2	Second Approach: Consecutive Natural Numbers	16
4.1.3	Third Approach: Minimum Perfect Square Sum	17
4.2	Magic Squares of Order 4	17
4.3	Magic Squares of Order 5	18
4.3.1	First Approach: Consecutive Odd Numbers	18
4.3.2	Second Approach: Uniformity	18
4.3.3	Third Approach: Minimum Perfect Square Sum	19
4.4	Magic Squares of Order 6	19

¹This is the reorganized version of author's work <http://rgmia.org/papers/v20/v20a136.pdf> [27] done in 2017.

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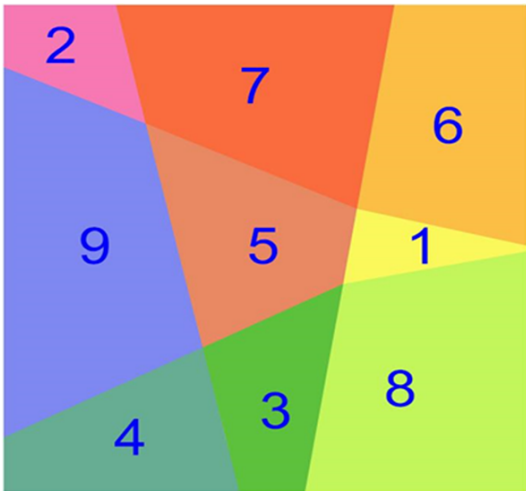
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4.5	Magic Squares of Order 7	20
4.5.1	First Approach: Consecutive Odd Numbers	20
4.5.2	Second Approach: Uniformity	20
4.5.3	Third Approach: Minimum Perfect Square Sum	21
4.6	Bimagic Squares of Order 8	22
4.7	Bimagic Squares of Order 9	22
4.7.1	First Approach: Consecutive Odd Numbers	23
4.7.2	Second Approach: Uniformity	23
4.7.3	Third Approach: Minimum Perfect Square Sum	24
4.8	Magic Squares of Order 10	25
4.9	Magic Squares of Order 11	25
4.9.1	First Approach: Consecutive Odd Numbers	26
4.9.2	Second Approach: Uniformity	26
4.9.3	Third Approach: Minimum Perfect Square Sum	27
4.10	Magic Squares of Order 12	28
4.11	Magic Squares of Order 13	28
4.11.1	First Approach: Consecutive Odd Numbers	28
4.11.2	Second Approach: Uniformity	29
4.11.3	Third Approach: Minimum Perfect Square Sum	30
4.12	Magic Squares of Order 14	31
4.13	Magic Squares of Order 15	32
4.13.1	First Approach: Consecutive Odd Numbers	32
4.13.2	Second Approach: Uniformity	33
4.13.3	Third Approach: Minimum Perfect Square Sum	34
4.14	Bimagic Squares of Order 16	35
4.15	Magic Squares of Order 17	36
4.15.1	First Approach: Consecutive Odd Numbers	36
4.15.2	Second Approach: Uniformity	37
4.15.3	Third Approach: Minimum Perfect Square Sum	38
4.16	Magic Squares of Order 18	39
4.17	Magic Squares of Order 19	42
4.17.1	First Approach: Consecutive Odd Numbers	42
4.17.2	Second Approach: Uniformity	43
4.17.3	Third Approach: Minimum Perfect Square Sum	44
4.18	Magic Squares of Order 20	45
4.19	Magic Squares of Order 21	47
4.19.1	First Approach: Consecutive Odd Order	47
4.19.2	Second Approach: Uniformity	50
4.19.3	Third Approach: Minimum Perfect Square Sum	51
4.20	Magic Squares of Order 22	53
4.21	Magic Squares of Order 23	54
4.21.1	First Approach: Consecutive Odd Numbers	54
4.21.2	Second Approach: Uniformity	54
4.21.3	Third Approach: Minimum Perfect Square Sum	55
4.22	Magic Squares of Order 24	56
4.23	Bimagic Squares of Order 25	59
4.23.1	First Approach: Consecutive Odd Numbers	59
4.23.2	Second Approach: Uniformity	59

4.23.3	Third Approach: Minimum Perfect Square Sum	60
5	Final Comments	62
6	Author’s Contributions to Magic Squares	62

1 Introduction

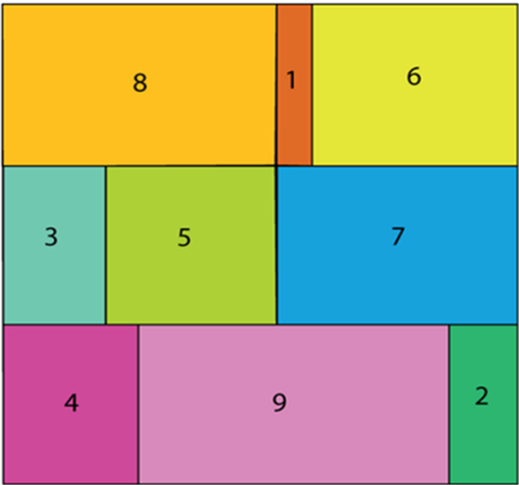
Recently, William Walkington [8] started an interesting discussion as to how to create magic squares with cells that had the same areas as their numbers. Below is a graphic design for a 2017 seasonal greetings card, showing a magic square with approximate areas that was constructed by William Walkington (2016):



William Walkington - 2016

Figure 1

Lee Sallows (2017) [5] also constructed another magic square representing the areas as rectangles. See below:



Lee Sallows - 2017

Figure 2

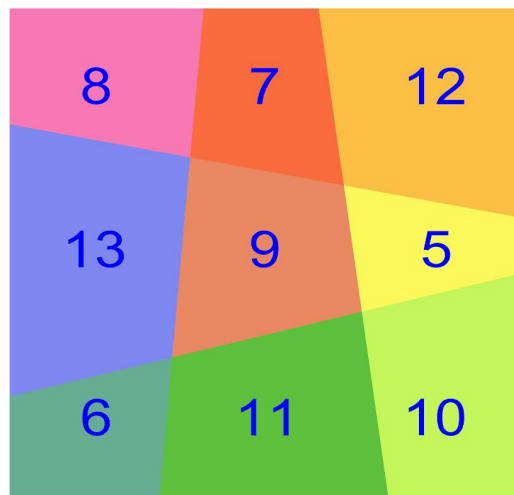
The sum of all the numbers is given by

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45. \quad (1)$$

The number 45 is not a perfect square. If we make a slight change, then we can transform the sum into a perfect square:

$$5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = 81 = 9^2 \quad (2)$$

Using this number sequence, Walter Trump (2017) [7] was able to construct the following area magic square:



Walter Trump, 2017-01-06, based on ideas of William Walkington and Inder Taneja

Figure 3

Adding 4 to each number in (1), we obtain the numbers given in (2). Observing area-wise the Figures 1 and 3, there is a considerable difference: For example, from numbers 1 to 2, the cell area is doubled, while from numbers 5 to 6, there is proportionally less increase between the cell areas.

In order to construct a magic square with cell areas that are in proportion to their numbers it is not necessary that the numbers always sum to a perfect square. Below is another example constructed by William Walkington (2017) [8] with sequential numbers from 3 to 11:

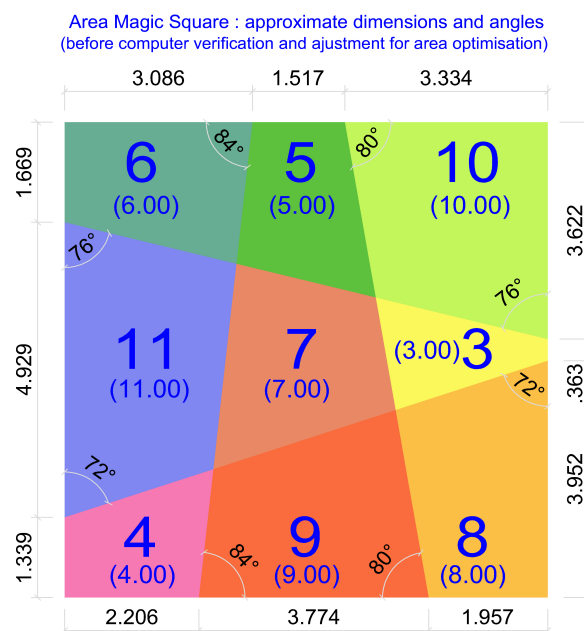
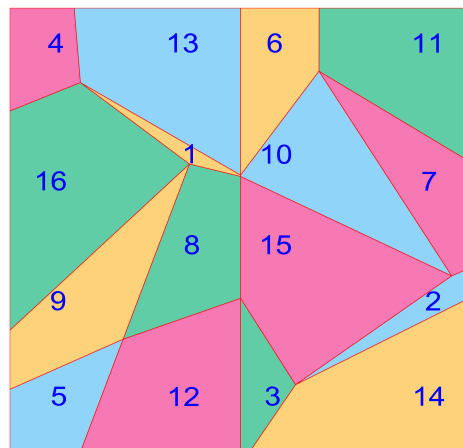


Figure 4

We observe that in Figures 1, 3 and 4, the number cell areas are proportional and aligned in both directions. In Figure 2, the proportionality of the areas is only present in one direction, which is horizontal.

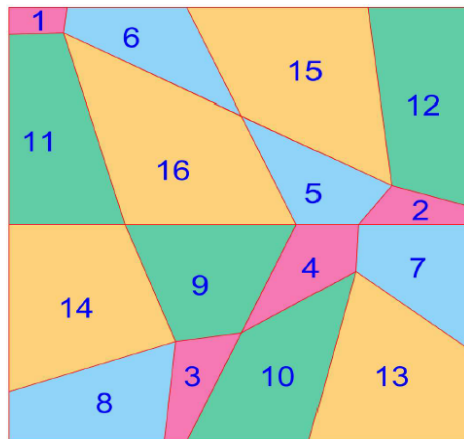
Below are two examples of classical order 4 magic squares with cell areas that are proportional to their numbers:



William Walkington, 2017-02-03 : AMS interpretation of Frénicle's square n°730

Figure 5

First Area Magic Square made of a Classical Magic 4x4-Square



Walter Trump, 2017-01-26 : AMS interpretation of Frénicle's square n°63

Graphic made by William Walkington who had the idea of Area Magic Squares.

Figure 6

More examples of similar kinds of order 4 area magic squares, together with order 6 area magic squares, can be seen in William Walkington's pages [8]. From equation (2), the question arises, how to create higher order magic squares such that the sum of numbers is always a perfect square. **Based on this idea, in this paper we establish some formulas, so that sum of the numbers in magic squares always becomes a perfect square.**

2 Series and Magic Square Sums

This section presents some basic ideas of series and magic square sums.

2.1 Natural Number Series Sums

It is well-known that the positive natural number series sum is given by

$$T_n := 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}, \quad n \geq 1 \quad (3)$$

The sequence T_n is also famous for showing **Pascal's triangle values**

2.2 Odd Number Series Sums

It is well-known that sum of **odd-number series** is given by

$$F_n := 1 + 3 + 5 + \dots + (2n - 1) = n^2, \quad n \geq 1, \quad (4)$$

Since we are working with magic squares, let us write $n = k^2$, then we have

$$F_{k^2} := 1 + 3 + 5 + \dots + (2k^2 - 1) = k^4, \quad k \geq 1, \quad (5)$$

The number k is the order of a magic square. Thus for all $k \geq 3$ we can always have odd numbers for the entries of a magic square with the sum of all numbers being a perfect square.

2.3 Magic Square Sums and Sums of the Entries

It is well known that the magic sum of a magic square of order k with elements $1, 2, 3, \dots, k^2$ is given by

$$S_{k \times k} := \frac{k(k^2 + 1)}{2}, \quad k \geq 1, \quad (6)$$

The sum of all of the entries of a magic square is given by

$$F_{k^2} := \frac{k^2(k^2 + 1)}{2}, \quad k \geq 1, \quad (7)$$

2.4 Perfect Square Sums with Consecutive Odd Numbers

Below is shown a general formula for writing **perfect square sum magic squares** with consecutive natural numbers.

Let us consider

$$G := T(n) - T(n-1) = k \left(n - \frac{k-1}{2} \right), \quad n \geq k, \quad (8)$$

a) For even order entries magic squares $k = 2p$:

$$G := k \left(n - \frac{2p-1}{2} \right). \quad (9)$$

In this case, the expression $n - \frac{2p-1}{2}$ is never a natural number.

Thus, there is no magic square of consecutive even numbers with a perfect square sum for all of its entries.

b) For odd order magic squares, $k = 2p + 1$

$$G := T(n) - T(n - k) = k \left(n - \frac{2p + 1 - 1}{2} \right) = k(n - p), \quad (10)$$

In this case, we can always find a natural number, such that $k(n - p)$ is a perfect square with $n - p \geq k$. By considering $n - p = k$, we obtain $G := k^2$, and the sum of all the numbers is $F_{k^2} := k^4$.

Result 1. *Thus, from equations (5) and (10), we conclude that there are at least two ways of writing magic squares with all entries summing to a perfect square:*

(i) *Magic squares with consecutive odd numbers starting from 1;*

(ii) *Magic squares with consecutive natural numbers.*

For the first case (i), the author [26] has recently calculated magic squares with odd numbers not starting from 1 using the idea of Pythagorean triples to obtain a perfect square sum magic squares.

For the second case (ii), the subsection below gives a procedure that can be used to obtain a perfect square sum of all the entries of a magic square.

2.5 Procedure

Before considering examples, let us analyse some simple formulas. After simplifying equation (8), one can write

$$\begin{aligned} G(n, k) &= T(n) - T(n - k) \\ &= (n - k + 1) + (n - k + 3) + \cdots + n \\ &= k \left(n - \frac{k - 1}{2} \right), \quad n \geq k, \\ &= \frac{k}{2} (2n - k + 1), \quad n \geq k \end{aligned} \quad (11)$$

Since we are working with magic squares and k is the total number of elements in a magic square it should also be the square of the order of the magic square. Therefore if p is the order of magic square then $k = p^2$. In order to obtain a perfect square for $G(n, k)$ we should have

$$\sqrt{G(n, k)} = t, \quad t \in \mathbf{N}_+, \quad n \geq k \quad (12)$$

For each case, we find a natural number t .

2.5.1 Uniformity

From equation (10), we observe that the first element of the sequence is $n - k + 1$ and the last is n . In case of uniformity always choose

$$n = k + \frac{k-1}{2} = \frac{3k-1}{2}$$

This gives

$$G = T\left(\frac{3k-1}{2}\right) - T\left(\frac{k-1}{2}\right) = k^2, \quad n \geq k$$

Since k is the total number of elements, let us consider p as the order of a magic square, and then the first and last members of the sequence are $F_1 := \frac{p^2+1}{2}$ and $F_{p^2} := \frac{3p^2-1}{2}$ respectively. In this case, the sum of the sequence is given by

$$F_1 + F_2 + \dots + F_{p^2} := p^4.$$

For simplicity, let's represent the **uniformity** as,

$$\langle p, p^2, p^3, p^4 \rangle \quad (13)$$

where

$$\begin{aligned} p &\rightarrow \text{order of a magic square;} \\ p^2 &\rightarrow \text{total number of entries;} \\ p^3 &\rightarrow \text{magic square sum;} \\ p^4 &\rightarrow \text{sum of all the entries of a magic square.} \end{aligned}$$

The above process gives the results in a **uniform way, but it does not guarantee a minimum perfect square sum for the entries of a magic square.**

2.5.2 General Case

Let p be an order of a magic square, i.e., $k = p^2$. Then from equations (11) or (12), we have

$$p\sqrt{\frac{2n - p^2 + 1}{2}} = p\sqrt{n - \frac{p^2 - 1}{2}} = t, \quad t \in \mathbb{N}_+, \quad n \geq p^2,$$

The sum of all entries of a magic square is given by

$$F_1 + F_1 + \dots + F_{p^2}$$

where $F_1 = n - p^2 + 1$ and $F_{p^2} := n$.

Again, let's consider, $n = m^2 + \frac{p^2-1}{2}$, and then we have

$$G(m, p) := p\sqrt{\left(m^2 + \frac{p^2-1}{2}\right) - \left(\frac{p^2-1}{2}\right)} = t, \quad t \in \mathbb{N}_+, \quad n \geq p^2. \quad (14)$$

In this case, how do we find the minimum value of m ? For simplicity, let's write,

$$L(n, k) := \left(\sqrt{k}, n - k + 1, n, \frac{G(n, k)^2}{\sqrt{k}}, G(n, k)^2, \sqrt{G(n, k)} \right), \quad (15)$$

where $n = m^2 + \frac{p^2-1}{2}$, $k = p^2$ and p is the order of a magic square.

From above we observe that, if $\sqrt{G(n, k)}$ equals the order of a magic square then, we have a **uniformity case**. This only happens when $m = p$. And, first positive entry obtained from m give a minimum perfect square sum (see the results given in 3).

For simplicity, let's write $L(a, b, c, d, e, f)$, for the representations given in expression (15), where

- $a \rightarrow$ Order of magic square;
- $b \rightarrow$ First member of a sequence;
- $c \rightarrow$ Last member of a sequence;
- $d \rightarrow$ Sum of magic square;
- $e \rightarrow$ Sum of all entries of a magic square;
- $f \rightarrow$ Uniformity, if $a = f$.

... (16)

3 Minimum Perfect Square Sums and Uniformity

The examples that follow are of **odd order magic squares** written according to the notations (16) and are based on the equation (15). First example in each case is the **minimum perfect square sum**. The **uniformity cases** are represented as

$$L(a, b, c, d, e, f) \Rightarrow \langle p, p^2, p^3, p^4 \rangle,$$

where $\langle p, p^2, p^3, p^4 \rangle$ is expressed as in (13).

Remark 1. In all cases, we use consecutive natural numbers. Amongst the magic squares of orders 3 to 51, magic square of orders 7 and 41 are the only ones starting with the number 1 that yield a minimum perfect square sum. According to Sloane [6], the next magic square with consecutive natural numbers starting with the number 1 that yields a minimum perfect square sum is of order 239. The magic square of order 3 is the only one where the first value of m gives a minimum perfect square sum and has the uniformity property. It should be noted that we are only examining positive entries for the magic squares. We can also obtain perfect square sum magic squares with some negative entries, but these cases are not studied here.

• Order 3

1. $L\left(2^2 + \frac{3^2-1}{2}, 3^2\right) \rightarrow (3, 0, 8, 12, 36, 2\sqrt{6})$
2. $L\left(3^2 + \frac{3^2-1}{2}, 3^2\right) \rightarrow (3, 5, 13, 27, 81, 3) \Rightarrow \langle 3, 3^2, 3^3, 3^4 \rangle$
3. $L\left(4^2 + \frac{3^2-1}{2}, 3^2\right) \rightarrow (3, 12, 20, 48, 144, 2\sqrt{3})$

• **Order 5**

1. $L\left(4^2 + \frac{5^2 - 1}{2}, 5^2\right) \rightarrow (5, 4, 28, 80, 400, 2\sqrt{5})$
2. $L\left(5^2 + \frac{5^2 - 1}{2}, 5^2\right) \rightarrow (5, 13, 37, 125, 625, 5) \Rightarrow \langle 5, 5^2, 5^3, 5^4 \rangle$
3. $L\left(6^2 + \frac{5^2 - 1}{2}, 5^2\right) \rightarrow (5, 24, 48, 180, 900, \sqrt{30})$

• **Order 7**

1. $L\left(5^2 + \frac{7^2 - 1}{2}, 7^2\right) \rightarrow (7, 1, 49, 175, 1225, \sqrt{35})$
2. $L\left(6^2 + \frac{7^2 - 1}{2}, 7^2\right) \rightarrow (7, 12, 60, 252, 1764, \sqrt{42})$
3. $L\left(7^2 + \frac{7^2 - 1}{2}, 7^2\right) \rightarrow (7, 25, 73, 343, 2401, 7) \Rightarrow \langle 7, 7^2, 7^3, 7^4 \rangle$

• **Order 9**

1. $L\left(7^2 + \frac{9^2 - 1}{2}, 9^2\right) \rightarrow (9, 9, 89, 441, 3969, 3\sqrt{7})$
2. $L\left(8^2 + \frac{9^2 - 1}{2}, 9^2\right) \rightarrow (9, 24, 104, 576, 5184, 6\sqrt{2})$
3. $L\left(9^2 + \frac{9^2 - 1}{2}, 9^2\right) \rightarrow (9, 41, 121, 729, 6561, 9) \Rightarrow \langle 9, 9^2, 9^3, 9^4 \rangle$

• **Order 11**

1. $L\left(8^2 + \frac{11^2 - 1}{2}, 11^2\right) \rightarrow (11, 4, 124, 704, 7744, 2\sqrt{22})$
2. $L\left(9^2 + \frac{11^2 - 1}{2}, 11^2\right) \rightarrow (11, 21, 141, 891, 9801, 3\sqrt{11})$
3. $L\left(10^2 + \frac{11^2 - 1}{2}, 11^2\right) \rightarrow (11, 40, 160, 1100, 12100, \sqrt{110})$
4. $L\left(11^2 + \frac{11^2 - 1}{2}, 11^2\right) \rightarrow (11, 61, 181, 1331, 14641, 11) \Rightarrow \langle 11, 11^2, 11^3, 11^4 \rangle$

• **Order 13**

1. $L\left(10^2 + \frac{13^2 - 1}{2}, 13^2\right) \rightarrow (13, 16, 184, 1300, 16900, \sqrt{130})$
2. $L\left(11^2 + \frac{13^2 - 1}{2}, 13^2\right) \rightarrow (13, 37, 205, 1573, 20449, \sqrt{143})$
3. $L\left(12^2 + \frac{13^2 - 1}{2}, 13^2\right) \rightarrow (13, 60, 228, 1872, 24336, 2\sqrt{39})$
4. $L\left(13^2 + \frac{13^2 - 1}{2}, 13^2\right) \rightarrow (13, 85, 253, 2197, 28561, 13) \Rightarrow \langle 13, 13^2, 13^3, 13^4 \rangle$

• **Order 15**

1. $L\left(11^2 + \frac{15^2 - 1}{2}, 15^2\right) \rightarrow (15, 9, 233, 1815, 27225, \sqrt{165})$
2. $L\left(12^2 + \frac{15^2 - 1}{2}, 15^2\right) \rightarrow (15, 57, 281, 2535, 38025, \sqrt{195})$
-
5. $L\left(15^2 + \frac{15^2 - 1}{2}, 15^2\right) \rightarrow (15, 113, 337, 3375, 50625, 15) \Rightarrow \langle 15, 15^2, 15^3, 15^4 \rangle$

• **Order 17**

1. $L\left(13^2 + \frac{17^2 - 1}{2}, 17^2\right) \rightarrow (17, 25, 313, 2873, 48841, \sqrt{221})$
2. $L\left(14^2 + \frac{17^2 - 1}{2}, 17^2\right) \rightarrow (17, 52, 340, 3332, 56644, \sqrt{238})$
-
5. $L\left(17^2 + \frac{17^2 - 1}{2}, 17^2\right) \rightarrow (145, 433, 289 = 17^2, 83521 = 17^4) \Rightarrow \langle 17, 17^2, 17^3, 17^4 \rangle$

• **Order 19**

1. $L\left(14^2 + \frac{19^2 - 1}{2}, 19^2\right) \rightarrow (19, 16, 376, 3724, 70756, \sqrt{266})$
2. $L\left(15^2 + \frac{19^2 - 1}{2}, 19^2\right) \rightarrow (19, 45, 405, 4275, 81225, \sqrt{285})$
-
6. $L\left(19^2 + \frac{19^2 - 1}{2}, 19^2\right) \rightarrow (19, 181, 541, 6859, 130321, 19) \Rightarrow \langle 19, 19^2, 19^3, 19^4 \rangle$

• **Order 21**

$$\begin{aligned}
 1. \quad & L \left(15^2 + \frac{21^2 - 1}{2}, 21^2 \right) \rightarrow (21, 5, 445, 4725, 99225, 3\sqrt{35}) \\
 2. \quad & L \left(16^2 + \frac{21^2 - 1}{2}, 21^2 \right) \rightarrow (21, 36, 476, 5376, 112896, 4\sqrt{21}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 7. \quad & L \left(21^2 + \frac{21^2 - 1}{2}, 21^2 \right) \rightarrow (21, 221, 661, 9261, 194481, 21) \Rightarrow \langle 21, 21^2, 21^3, 21^4 \rangle
 \end{aligned}$$

• **Order 23**

$$\begin{aligned}
 1. \quad & L \left(17^2 + \frac{23^2 - 1}{2}, 23^2 \right) \rightarrow (23, 25, 553, 6647, 152881, \sqrt{391}) \\
 2. \quad & L \left(18^2 + \frac{23^2 - 1}{2}, 23^2 \right) \rightarrow (23, 60, 588, 7452, 171396, 3\sqrt{46}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 7. \quad & L \left(23^2 + \frac{23^2 - 1}{2}, 23^2 \right) \rightarrow (23, 265, 793, 12167, 279841, 23) \Rightarrow \langle 23, 23^2, 23^3, 23^4 \rangle
 \end{aligned}$$

• **Order 25**

$$\begin{aligned}
 1. \quad & L \left(18^2 + \frac{25^2 - 1}{2}, 25^2 \right) \rightarrow (25, 12, 636, 8100, 202500, 15\sqrt{2}) \\
 2. \quad & L \left(19^2 + \frac{25^2 - 1}{2}, 25^2 \right) \rightarrow (25, 49, 673, 9025, 225625, 5\sqrt{19}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 8. \quad & L \left(25^2 + \frac{25^2 - 1}{2}, 25^2 \right) \rightarrow (25, 313, 937, 15625, 390625, 25) \Rightarrow \langle 25, 25^2, 25^3, 25^4 \rangle
 \end{aligned}$$

• **Order 27**

$$\begin{aligned}
 1. \quad & L \left(20^2 + \frac{27^2 - 1}{2}, 27^2 \right) \rightarrow (27, 36, 764, 10800, 291600, 6\sqrt{15}) \\
 2. \quad & L \left(21^2 + \frac{27^2 - 1}{2}, 27^2 \right) \rightarrow (27, 77, 805, 11907, 321489, 9\sqrt{7}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 8. \quad & L \left(27^2 + \frac{27^2 - 1}{2}, 27^2 \right) \rightarrow (27, 365, 1093, 19683, 531441, 27) \Rightarrow \langle 27, 27^2, 27^3, 27^4 \rangle
 \end{aligned}$$

• **Order 29**

$$\begin{aligned}
 1. \quad & L \left(21^2 + \frac{29^2 - 1}{2}, 29^2 \right) \rightarrow (29, 21, 861, 12789, 370881, \sqrt{609}) \\
 2. \quad & L \left(22^2 + \frac{29^2 - 1}{2}, 29^2 \right) \rightarrow (29, 64, 904, 14036, 407044, \sqrt{638}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 9. \quad & L \left(29^2 + \frac{29^2 - 1}{2}, 29^2 \right) \rightarrow (29, 421, 1261, 24389, 707281, 29) \Rightarrow \langle 29, 29^2, 29^3, 29^4 \rangle
 \end{aligned}$$

• **Order 31**

$$\begin{aligned}
 1. \quad & L \left(22^2 + \frac{31^2 - 1}{2}, 31^2 \right) \rightarrow (31, 4, 964, 15004, 465124, \sqrt{682}) \\
 2. \quad & L \left(23^2 + \frac{31^2 - 1}{2}, 31^2 \right) \rightarrow (31, 49, 1009, 16399, 508369, \sqrt{713}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 10. \quad & L \left(31^2 + \frac{31^2 - 1}{2}, 31^2 \right) \rightarrow (31, 481, 1441, 29791, 923521, 31) \Rightarrow \langle 31, 31^2, 31^3, 31^4 \rangle
 \end{aligned}$$

• **Order 33**

$$\begin{aligned}
 1. \quad & L \left(24^2 + \frac{33^2 - 1}{2}, 33^2 \right) \rightarrow (33, 32, 1120, 19008, 627264, 6\sqrt{22}) \\
 2. \quad & L \left(25^2 + \frac{33^2 - 1}{2}, 33^2 \right) \rightarrow (33, 81, 1169, 20625, 680625, 5\sqrt{33}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 10. \quad & L \left(33^2 + \frac{33^2 - 1}{2}, 33^2 \right) \rightarrow (33, 545, 1633, 35937, 1185921, 33) \Rightarrow \langle 33, 33^2, 33^3, 33^4 \rangle
 \end{aligned}$$

• **Order 35**

$$\begin{aligned}
 1. \quad & L \left(25^2 + \frac{35^2 - 1}{2}, 35^2 \right) \rightarrow (35, 13, 1237, 21875, 765625, 5\sqrt{35}) \\
 2. \quad & L \left(26^2 + \frac{35^2 - 1}{2}, 35^2 \right) \rightarrow (35, 64, 1288, 23660, 828100, \sqrt{910}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 11. \quad & L \left(35^2 + \frac{35^2 - 1}{2}, 35^2 \right) \rightarrow (35, 613, 1837, 42875, 1500625, 35) \Rightarrow \langle 35, 35^2, 35^3, 35^4 \rangle
 \end{aligned}$$

• **Order 37**

$$\begin{aligned}
 1. \quad & L \left(27^2 + \frac{37^2 - 1}{2}, 37^2 \right) \rightarrow (37, 45, 1413, 26973, 998001, 3 \sqrt{111}) \\
 2. \quad & L \left(28^2 + \frac{37^2 - 1}{2}, 37^2 \right) \rightarrow (37, 100, 1468, 29008, 1073296, 2 \sqrt{259}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 11. \quad & L \left(37^2 + \frac{37^2 - 1}{2}, 37^2 \right) \rightarrow (37, 685, 2053, 50653, 1874161, 37) \Rightarrow \langle 37, 37^2, 37^3, 37^4 \rangle
 \end{aligned}$$

• **Order 39**

$$\begin{aligned}
 1. \quad & L \left(28^2 + \frac{39^2 - 1}{2}, 39^2 \right) \rightarrow (39, 24, 1544, 30576, 1192464, 2 \sqrt{273}) \\
 2. \quad & L \left(29^2 + \frac{39^2 - 1}{2}, 39^2 \right) \rightarrow (39, 81, 1601, 32799, 1279161, \sqrt{1131}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 12. \quad & L \left(39^2 + \frac{39^2 - 1}{2}, 39^2 \right) \rightarrow (39, 761, 2281, 59319, 2313441, 39) \Rightarrow \langle 39, 39^2, 39^3, 39^4 \rangle
 \end{aligned}$$

• **Order 41**

$$\begin{aligned}
 1. \quad & L \left(29^2 + \frac{41^2 - 1}{2}, 41^2 \right) \rightarrow (41, 1, 1681, 34481, 1413721, \sqrt{1189}) \\
 2. \quad & L \left(30^2 + \frac{41^2 - 1}{2}, 41^2 \right) \rightarrow (41, 60, 1740, 36900, 1512900, \sqrt{1230}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 13. \quad & L \left(41^2 + \frac{41^2 - 1}{2}, 41^2 \right) \rightarrow (41, 841, 2521, 68921, 2825761, 41) \Rightarrow \langle 41, 41^2, 41^3, 41^4 \rangle
 \end{aligned}$$

• **Order 43**

$$\begin{aligned}
 1. \quad & L \left(31^2 + \frac{43^2 - 1}{2}, 43^2 \right) \rightarrow (43, 37, 1885, 41323, 1776889, \sqrt{1333}) \\
 2. \quad & L \left(32^2 + \frac{43^2 - 1}{2}, 43^2 \right) \rightarrow (43, 100, 1948, 44032, 1893376, 4 \sqrt{86}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 13. \quad & L \left(43^2 + \frac{43^2 - 1}{2}, 43^2 \right) \rightarrow (43, 925, 2773, 79507, 3418801, 43) \Rightarrow \langle 43, 43^2, 43^3, 43^4 \rangle
 \end{aligned}$$

• **Order 45**

$$\begin{aligned}
 1. \quad & L \left(32^2 + \frac{45^2 - 1}{2}, 45^2 \right) \rightarrow (45, 12, 2036, 46080, 2073600, 12 \sqrt{10}) \\
 2. \quad & L \left(33^2 + \frac{45^2 - 1}{2}, 45^2 \right) \rightarrow (45, 144, 2168, 52020, 2340900, 3 \sqrt{170}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 14. \quad & L \left(45^2 + \frac{45^2 - 1}{2}, 45^2 \right) \rightarrow (45, 1013, 3037, 91125, 4100625, 45) \Rightarrow \langle 45, 45^2, 45^3, 45^4 \rangle
 \end{aligned}$$

• **Order 47**

$$\begin{aligned}
 1. \quad & L \left(34^2 + \frac{47^2 - 1}{2}, 47^2 \right) \rightarrow (47, 52, 2260, 54332, 2553604, \sqrt{1598}) \\
 2. \quad & L \left(35^2 + \frac{47^2 - 1}{2}, 47^2 \right) \rightarrow (47, 121, 2329, 57575, 2706025, \sqrt{1645}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 14. \quad & L \left(47^2 + \frac{47^2 - 1}{2}, 47^2 \right) \rightarrow (47, 1105, 3313, 103823, 4879681, 47) \Rightarrow \langle 47, 47^2, 47^3, 47^4 \rangle
 \end{aligned}$$

• **Order 49**

$$\begin{aligned}
 1. \quad & L \left(35^2 + \frac{49^2 - 1}{2}, 49^2 \right) \rightarrow (49, 25, 2425, 60025, 2941225, 7 \sqrt{35}) \\
 2. \quad & L \left(36^2 + \frac{49^2 - 1}{2}, 49^2 \right) \rightarrow (49, 96, 2496, 63504, 3111696, 42) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 15. \quad & L \left(49^2 + \frac{49^2 - 1}{2}, 49^2 \right) \rightarrow (49, 1201, 3601, 117649, 5764801, 49) \Rightarrow \langle 49, 49^2, 49^3, 49^4 \rangle
 \end{aligned}$$

• **Order 51**

$$\begin{aligned}
 1. \quad & L \left(37^2 + \frac{51^2 - 1}{2}, 51^2 \right) \rightarrow (51, 69, 2669, 69819, 3560769, \sqrt{1887}) \\
 2. \quad & L \left(38^2 + \frac{51^2 - 1}{2}, 51^2 \right) \rightarrow (51, 144, 2744, 73644, 3755844, \sqrt{1938}) \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \\
 15. \quad & L \left(51^2 + \frac{51^2 - 1}{2}, 51^2 \right) \rightarrow (51, 1301, 3901, 132651, 6765201, 51) \Rightarrow \langle 51, 51^2, 51^3, 51^4 \rangle
 \end{aligned}$$

4 Perfect Square Sum Magic Squares: Examples

Magic squares are constructed following two different methods. The first method uses the formula given in equation (5). In this case, all the magic squares have odd number entries starting from 1, and there are always perfect square sum solutions. The second method, uniquely for odd-orders, uses the expressions given in section 3. In these cases, the construction is presented in two ways: The first of these is for the uniformity cases written section 3, and the second of these is for minimum perfect square sum, i.e., the first listed in each case in section 3. The examples below are for magic squares of orders 3 to 25.

4.1 Magic Squares of Order 3

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 3 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. Below, examples of both types are presented.

4.1.1 First Approach: Consecutive Odd Numbers

Taking $k = 3$ in (7), we get

$$\begin{aligned} 1 + 3 + 5 + \dots + (2 \times 3^2 - 1) &= 3^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 17 &= 9^2 = 3^4 \end{aligned} \quad (17)$$

Example 1. According to the numbers given in equation (17), we obtain following magic square of order 3:

			27
3	13	11	27
17	9	1	27
7	5	15	27
27	27	27	27

In this case, we have a magic sum $S_{3 \times 3} = 27 = 3^3$ and the sum of all entries $F_9 = 3 \times 27 = 81 = 9^2 = 3^4$. This satisfies the property 13 of uniformity 2.5.1, i.e., $\langle 3, 3^2, 3^3, 3^4 \rangle$.

4.1.2 Second Approach: Consecutive Natural Numbers

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-9)(n-8)}{2} = 9(n-4)$$

Taking $n = 13$, we get a perfect square, i.e.,

$$G := T(13) - T(4) = \frac{13 \times 14}{2} - \frac{4 \times 5}{2} = 9 \times 8 = 81$$

Simplifying, we get

$$5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = 81$$

Example 2. According to the values given above, there is a perfect sum magic square of order 3 given by

			27
6	11	10	27
13	9	5	27
8	7	12	27
27	27	27	27

In this case also, we have a magic sum $S_{3 \times 3} = 27 = 3^3$ and the sum of all entries $F_9 = 3 \times 27 = 81 = 9^2 = 3^4$. The perfect square sum obtained above satisfy the uniformity property 13 given in subsection 2.5.1, but it is not minimum. See following subsection for the minimum perfect square sum magic square.

4.1.3 Third Approach: Minimum Perfect Square Sum

Let's observe the first example of order 3 magic square given in section 3:

$$L\left(2^2 + \frac{3^2 - 1}{2}, 3^2\right) \rightarrow (3, 0, 8, 12, 36, 2\sqrt{6})$$

In this case, we have magic square of order 3 with a perfect square sum $36 = 6^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. See below a magic square of order 3, where sum of all entries is a minimum perfect square.

Example 3. A magic square of order 3 with perfect square sum is given by

			12
1	6	5	12
8	4	0	12
3	2	7	12
12	12	12	12

In this case the magic sum is $S_{3 \times 3} = 12$, and the sum of all entries is $F_9 := 36 = 6^2$.

Remark 2. By choosing $m = 2$ in equation (15), we get a minimum perfect square sum for a magic square given in example 3. The reason is if we choose $m = 1$, we get some entries as negative numbers, see below:

$$L\left(1^2 + \frac{3^2 - 1}{2}, 3^2\right) \rightarrow (3, -3, 5, 3, 9, 2\sqrt{3})$$

This magic square is formed by entries $-3, -2, -1, 0, 1, 2, 3, 4, 5$ giving a perfect square sum as 9. Since it is formed by negative entries, it is excluded from our study. Our work is limited only to **natural numbers**.

4.2 Magic Squares of Order 4

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 4 with odd number entries. Taking $k = 4$ in equation (5), we get

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 = 256 = 16^2$$

Example 4. According to the values given above, the **pan diagonal magic square of order 4** is given by

		64	64	64	64
	13	23	1	47	64
64	3	25	15	21	64
64	31	5	19	9	64
64	17	11	29	7	64
	64	64	64	64	64

In this case, the magic sum is $S_{4 \times 4} := 64 = 4^3$, and the sum of the entries is $F_{16} := 256 = 16^2 = 4^4$. This satisfies the property 13 of uniformity 2.5.1, i.e., $\langle 4, 4^2, 4^3, 4^4 \rangle$.

4.3 Magic Squares of Order 5

According to equations (5) and (10), one can obtain **perfect square sum magic squares** of order 5 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.3.1 First Approach: Consecutive Odd Numbers

Taking $k = 5$ in equation (5), we get

$$\begin{aligned}
 1 + 3 + 5 + \dots + (2 \times 5^2 - 1) &= 5^4 \\
 \Rightarrow 1 + 3 + 5 + \dots + 47 + 49 &= 625 = 25^2 = 5^4
 \end{aligned} \tag{18}$$

Example 5. According to the 25 values given in equation (18), the *pan diagonal magic square of order 5* is given by

		125	125	125	125	125
	1	17	23	39	45	125
125	33	49	5	11	27	125
125	15	21	37	43	9	125
125	47	3	19	25	31	125
125	29	35	41	7	13	125
	125	125	125	125	125	125

In this case, the magic sum is $S_{5 \times 5} := 125 = 5^3$, and the sum of the entries is $F_{25} := 625 = 25^2 = 5^4$.

4.3.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-25)(n-24)}{2} = 25(n-12)$$

Taking $n = 37$, we get a perfect square, i.e.,

$$G := T(37) - T(12) = \frac{37 \times 38}{2} - \frac{12 \times 13}{2} = 37 \times 19 - 6 \times 13 \\ = 703 - 78 = 625 = 25^2 = 5^4.$$

Simplifying, we get

$$13 + 14 + 15 + \dots + 35 + 37 = 625 = 25^2 = 5^4$$

Example 6. Below is a *perfect square sum pan diagonal magic square of order 5* for the entries 13 to 37:

		125	125	125	125	125
	13	21	24	32	35	125
125	29	37	15	18	26	125
125	20	23	31	34	17	125
125	36	14	22	25	18	125
125	27	30	33	16	19	125
	125	125	125	125	125	125

In this case also, the magic sum is $S_{5 \times 5} = 125 = 5^3$, and the sum of all the entries is $F_{25} := 625 = 5^4$.

Both the examples 5 and 6 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 5, 5^2, 5^3, 5^4 \rangle$.

4.3.3 Third Approach: Minimum Perfect Square Sum

Let's observe the first example of the order 5 magic square given in section 3:

$$L\left(4^2 + \frac{5^2 - 1}{2}, 5^2\right) \rightarrow (5, 4, 28, 80, 400, 2\sqrt{5})$$

In this case, we have magic square of order 5 with a perfect square sum of all entries $400 = 20^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. See below a **pan diagonal magic square of order 5**, where there is a minimum perfect square sum of all the entries.

Example 7. A *pan diagonal magic square of order 5 with perfect square sum* is given by

		80	80	80	80	80
	4	10	16	22	28	80
80	21	27	8	9	15	80
80	13	14	20	26	7	80
80	25	6	12	18	19	80
80	17	23	24	5	11	80
	80	80	80	80	80	80

In this case the magic sum is $S_{5 \times 5} = 80$, and the sum of all entries is $F_{25} := 400 = 20^2$.

4.4 Magic Squares of Order 6

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 6 with odd number entries. Taking $k = 6$ in equation (5), we get

$$1 + 3 + 5 + \dots + (2 \times 6^2 - 1) = 6^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 69 + 71 = 1296 = 36^2 = 6^4. \quad (19)$$

Example 8. According to the values given in equation (19), the magic square of order 6 is given by

						216
1	45	55	67	33	15	216
57	13	69	27	41	9	216
23	11	25	53	61	43	216
63	31	7	47	19	49	216
37	65	21	5	59	29	216
35	51	39	17	3	71	216
216	216	216	216	216	216	216

In this case, the magic sum is $S_{6 \times 6} := 216 = 6^3$, and the sum of the entries is $F_{36} := 1296 = 36^2 = 6^4$. This satisfies the property 13 of uniformity 2.5.1, i.e., $\langle 6, 6^2, 6^3, 6^4 \rangle$.

4.5 Magic Squares of Order 7

According to equations (5) and (10), one can obtain **perfect square sum magic squares** of order 7 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.5.1 First Approach: Consecutive Odd Numbers

Taking $k = 7$ in equation (5), we get

$$\begin{aligned}
 1 + 3 + 5 + \dots + (2 \times 7^2 - 1) &= 7^4 \\
 \Rightarrow 1 + 3 + 5 + \dots + 95 + 97 &= 2401 = 49^2 = 7^4
 \end{aligned} \tag{20}$$

Example 9. According to the 49 values given in equation (20), the **pan diagonal magic square** of order 7 is given by

		343	343	343	343	343	343	343
	1	17	33	49	65	81	97	343
343	79	95	13	15	31	47	63	343
343	45	61	77	93	11	27	29	343
343	25	41	43	59	75	91	9	343
343	89	7	23	39	55	57	73	343
343	69	71	87	5	21	37	53	343
343	35	51	67	83	85	3	19	343
	343	343	343	343	343	343	343	343

4.5.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-49)(n-48)}{2} = 49(n-24)$$

Taking $n = 73$, we get a perfect square, i.e.,

$$G := T(73) - T(24) = \frac{73 \times 74}{2} - \frac{24 \times 25}{2} = 73 \times 37 - 12 \times 25 \\ = 2701 - 300 = 2401 = 49^2 = 7^4.$$

Simplifying, we get

$$25 + 26 + 27 + \dots + 72 + 73 = 2401 = 49^2 = 7^4$$

This gives a perfect square sum for the 49 consecutive natural numbers from 25 to 73.

Example 10. *The pan diagonal magic square of order 7 is given by*

		343	343	343	343	343	343	343
	25	33	41	49	57	65	73	343
343	64	72	31	32	40	48	56	343
343	47	55	63	71	30	38	39	343
343	37	45	46	54	62	70	29	343
343	69	28	36	44	52	53	61	343
343	59	60	68	27	35	43	51	343
343	42	50	58	66	67	26	34	343
	343	343	343	343	343	343	343	343

In both the examples given above the magic sum is $S_{7 \times 7} = 343 = 7^3$, and the sum of all the entries is $F_{49} := 2401 = 49^2 = 7^4$.

Both the examples 9 and 10 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 7, 7^2, 7^3, 7^4 \rangle$.

4.5.3 Third Approach: Minimum Perfect Square Sum

Let's observe the first example of the order 7 magic square given in section 3:

$$L \left(5^2 + \frac{7^2 - 1}{2}, 7^2 \right) \rightarrow (7, 1, 49, 175, 1225, \sqrt{35}).$$

In this case, we have a magic square of order 7 with a perfect square sum of all entries $1225 = 175^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. See below **pan diagonal magic square** of order 7, where there is a minimum perfect square sum of all the entries:

Example 11. *The pan diagonal magic square of order 7 is given by*

		175	175	175	175	175	175	175
	1	9	17	25	33	41	49	175
175	40	48	7	8	16	24	32	175
175	23	31	39	47	6	14	15	175
175	13	21	22	30	38	46	5	175
175	45	4	12	20	28	29	37	175
175	35	36	44	3	11	19	27	175
175	18	26	34	42	43	2	10	175
	175	175	175	175	175	175	175	175

In this case the magic sum is $S_{7 \times 7} = 175$, and the sum of all entries is $F_{49} := 7 \times 175 = 1225 = 35^2$. This is the only example (amongst the orders 3 to 51) of a minimum perfect square sum with all entries starting from the number 1.

4.6 Bimagic Squares of Order 8

According to equation (9), we cannot obtain a consecutive natural number magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 8 with this property if we use odd number entries. Taking $k = 8$ in equation (5), we get

$$1 + 3 + 5 + \dots + (2 \times 8^2 - 1) = 8^4$$

$$\Rightarrow 1 + 3 + 5 + \dots + 123 + 125 + 127 = 4096 = 64^2 = 8^4. \quad (21)$$

Example 12. According to the values given in equation (21), the *pan diagonal magic square* of order 8 is given by

		512	512	512	512	512	512	512	512
	31	81	71	9	53	123	109	35	512
512	51	125	107	37	25	87	65	15	512
512	1	79	89	23	43	101	115	61	512
512	45	99	117	59	7	73	95	17	512
512	75	5	19	93	97	47	57	119	512
512	103	41	63	113	77	3	21	91	512
512	85	27	13	67	127	49	39	105	512
512	121	55	33	111	83	29	11	69	512
	512	512	512	512	512	512	512	512	512

In this case the magic sum is $S_{8 \times 8} = 512$, and the sum of the entries is $F_{64} = 4096 = 64^2 = 8^4$. This magic square is also **bimagic** and has a bimagic sum $Sb_{8 \times 8} = 43688$.

Example 13. The bimagic square of order 8 using the entries of example 12 is given by

								43688
31^2	81^2	71^2	9^2	53^2	123^2	109^2	35^2	43688
51^2	125^2	107^2	37^2	25^2	87^2	65^2	15^2	43688
1^2	79^2	89^2	23^2	43^2	101^2	115^2	61^2	43688
45^2	99^2	117^2	59^2	7^2	73^2	95^2	17^2	43688
75^2	5^2	19^2	93^2	97^2	47^2	57^2	119^2	43688
103^2	41^2	63^2	113^2	77^2	3^2	21^2	91^2	43688
85^2	27^2	13^2	67^2	127^2	49^2	39^2	105^2	43688
121^2	55^2	33^2	111^2	83^2	29^2	11^2	69^2	43688
43688	43688	43688	43688	43688	43688	43688	43688	43688

4.7 Bimagic Squares of Order 9

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 9 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.7.1 First Approach: Consecutive Odd Numbers

Taking $k = 9$ in equation (5), we get

$$\begin{aligned} 1 + 3 + 5 + \dots + (2 \times 9^2 - 1) &= 9^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 159 + 161 &= 6561 = 81^2 = 9^4. \end{aligned} \quad (22)$$

Example 14. According to the 81 values given in equation (22), the magic square of order 9 is given by

									729
1	35	45	69	79	95	119	129	157	729
65	75	103	109	143	153	15	25	41	729
123	133	149	11	21	49	55	89	99	729
53	9	19	97	59	87	147	121	137	729
93	67	83	161	117	127	43	5	33	729
151	113	141	39	13	29	107	63	73	729
27	37	17	77	105	61	139	155	111	729
85	101	57	135	145	125	23	51	7	729
131	159	115	31	47	3	81	91	71	729
729	729	729	729	729	729	729	729	729	729

In this case the magic sum is $S_{9 \times 9} = 729$, and the sum of the entries is $F_{81} = 6561 = 81^2 = 9^4$. This magic square is also **bimagic** and has a bimagic sum $Sb_{9 \times 9} = 78729$. See below:

Example 15. The bimagic square of order 9 using the entries of example 14 is given by

									78729
1^2	35^2	45^2	69^2	79^2	95^2	119^2	129^2	157^2	78729
65^2	75^2	103^2	109^2	143^2	153^2	15^2	25^2	41^2	78729
123^2	133^2	149^2	11^2	21^2	49^2	55^2	89^2	99^2	78729
53^2	9^2	19^2	97^2	59^2	87^2	147^2	121^2	137^2	78729
93^2	67^2	83^2	161^2	117^2	127^2	43^2	5^2	33^2	78729
151^2	113^2	141^2	39^2	13^2	29^2	107^2	63^2	73^2	78729
27^2	37^2	17^2	77^2	105^2	61^2	139^2	155^2	111^2	78729
85^2	101^2	57^2	135^2	145^2	125^2	23^2	51^2	7^2	78729
131^2	159^2	115^2	31^2	47^2	3^2	81^2	91^2	71^2	78729
78729	78729	78729	78729	78729	78729	78729	78729	78729	78729

4.7.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-81)(n-80)}{2} = 81(n-40)$$

Taking $n = 121$, we get a perfect square, i.e.,

$$\begin{aligned} G &:= T(121) - T(40) = \frac{121 \times 122}{2} - \frac{40 \times 42}{2} = 121 \times 61 - 20 \times 41 \\ &= 7381 - 820 = 6561 = 81^2 = 9^4. \end{aligned}$$

Simplifying, we get

$$41 + 42 + 43 + \dots + 120 + 121 = 6561 = 81^2 = 9^4$$

This gives a perfect square sum for the 81 entries using consecutive natural numbers from 41 to 121.

Example 16. The magic square of order 9 using the above numbers from 41 to 121 is given by

									729
41	58	63	75	80	88	100	105	119	729
73	78	92	95	112	117	48	53	61	729
102	107	115	46	51	65	68	85	90	729
67	45	50	89	70	84	114	101	109	729
87	74	82	121	99	104	62	43	57	729
116	97	111	60	47	55	94	72	77	729
54	59	49	79	93	71	110	118	96	729
83	91	69	108	113	103	52	66	44	729
106	120	98	56	64	42	81	86	76	729
729	729	729	729	729	729	729	729	729	729

In both the examples 14 and 16, the magic sum is $S_{9 \times 9} = 729 = 9^3$, and the sum of all entries is $F_{81} := 6561 = 81^2 = 9^4$. This magic square shown above is **bimagic** with a bimagic sum $Sb_{9 \times 9} = 63969$. We observe that although the examples 14 and 16 show the same magic sums, their bimagic sums are different.

Both the examples 14 and 16 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 9, 9^2, 9^3, 9^4 \rangle$.

4.7.3 Third Approach: Minimum Perfect Square Sum

The examples given in 14 and 16 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 9, 9^2, 9^3, 9^4 \rangle$. However, the total entries of the magic square example 16 do not yield a minimum perfect square sum. In order to get, minimum perfect square sum. In order to obtain a minimum perfect square sum, let's consider the first example of the magic square of order 9 given in section 3:

$$L \left(7^2 + \frac{9^2 - 1}{2}, 9^2 \right) \rightarrow (9, 9, 89, 441, 3969, 3\sqrt{7})$$

In this case, we have magic square of order 9 with a perfect square sum of all entries $3969 = 63^2$, but it does not satisfy the property (13) of uniformity given in subsection 2.5.1. See below the magic square of order 9 constructed according to details given above.

Example 17. The magic square of order 9 with 81 consecutive entries from 9 to 89 is given by

		441	441	441	441	441	441	441	441	441
	10	26	30	41	45	61	69	76	83	441
441	42	49	56	64	80	84	14	18	34	441
441	68	72	88	15	22	29	37	53	57	441
441	28	17	21	59	36	52	87	67	74	441
441	60	40	47	82	71	75	32	9	25	441
441	86	63	79	33	13	20	55	44	48	441
441	19	35	12	50	54	43	78	85	65	441
441	51	58	38	73	89	66	23	27	16	441
441	77	81	70	24	31	11	46	62	39	441
	441	441	441	441	441	441	441	441	441	441

The above example 17 has a with magic sum $S_{9 \times 9} = 441$, and the sum of all entries is $F_{81} := 9 \times 441 = 3969 = 63^2$. It is **pan diagonal**, but **not bimagic**. It should be noted that the sum of each 3×3 block is the same for each of the three magic squares of examples 14, 16 and 17.

4.8 Magic Squares of Order 10

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 10 with this property if we use odd number entries. Taking $k = 10$ in equation (5), we get

$$1 + 3 + 5 + \dots + (2 \times 10^2 - 1) = 10^4$$

$$\Rightarrow 1 + 3 + 5 + \dots + 197 + 199 = 10000 = 100^2 = 10^4. \quad (23)$$

Example 18. According to the values given in equation (23), the magic square of order 10 is given by

										1000
1	159	129	193	77	43	95	171	105	27	1000
195	23	17	131	179	147	109	65	81	53	1000
93	161	45	157	31	69	187	119	123	15	1000
139	113	175	67	3	181	57	29	151	85	1000
167	197	103	21	89	135	145	13	59	71	1000
25	75	87	19	153	111	163	41	189	137	1000
149	91	79	165	55	37	133	183	7	101	1000
117	47	191	83	121	5	39	155	73	169	1000
51	9	33	115	185	99	61	127	177	143	1000
63	125	141	49	107	173	11	97	35	199	1000
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

The above example 18 has a magic sum $S_{10 \times 10} = 1000$, and the sum of all entries is $F_{100} := 10 \times 1000 = 10000 = 100^2 = 10^4$. It satisfies the property (13) of uniformity 2.5.1, i.e., $\langle 10, 10^2, 10^3, 10^4 \rangle$.

4.9 Magic Squares of Order 11

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 11 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods;

the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.9.1 First Approach: Consecutive Odd Numbers

Taking $k = 11$ in equation (5), we get

$$\begin{aligned} 1 + 3 + 5 + \dots + (2 \times 11^2 - 1) &= 11^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 239 + 241 &= 14641 = 121^2 = 11^4 \end{aligned} \quad (24)$$

Example 19. According to the 121 values given in equation (24), the *pan diagonal magic square* of order 11 is given by

		1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331
	1	41	59	77	95	113	153	171	189	207	225	1331
1331	201	241	17	35	53	71	89	129	147	165	183	1331
1331	159	177	217	235	11	29	47	87	105	123	141	1331
1331	117	135	175	193	211	229	5	23	63	81	99	1331
1331	75	93	111	151	169	187	205	223	21	39	57	1331
1331	33	51	69	109	127	145	163	181	199	239	15	1331
1331	233	9	27	45	85	103	121	139	157	197	215	1331
1331	191	209	227	3	43	61	79	97	115	133	173	1331
1331	149	167	185	203	221	19	37	55	73	91	131	1331
1331	107	125	143	161	179	219	237	13	31	49	67	1331
1331	65	83	101	119	137	155	195	213	231	7	25	1331
	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331

4.9.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-121)(n-120)}{2} = 121(n-60)$$

Taking $n = 181$, we get a perfect square, i.e.,

$$\begin{aligned} G &:= T(181) - T(60) = \frac{181 \times 182}{2} - \frac{60 \times 61}{2} = 181 \times 91 - 30 \times 61 \\ &= 16471 - 1830 = 14641 = 121^2 = 11^4. \end{aligned}$$

Simplifying, we get

$$61 + 62 + 63 + \dots + 180 + 181 = 14641 = 121^2 = 11^4.$$

This gives a perfect square sum for the 121 consecutive natural numbers from 61 to 181.

Example 20. A *pan diagonal magic square* of order 11 for 121 entries from 61 to 181 is given by

		1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331
	61	81	90	99	108	117	137	146	155	164	173	1331
1331	161	181	69	78	87	96	105	125	134	143	152	1331
1331	140	149	169	178	66	75	84	104	113	122	131	1331
1331	119	128	148	157	166	175	63	72	92	101	110	1331
1331	98	107	116	136	145	154	163	172	71	80	89	1331
1331	77	86	95	115	124	133	142	151	160	180	68	1331
1331	177	65	74	83	103	112	121	130	139	159	168	1331
1331	156	165	174	62	82	91	100	109	118	127	147	1331
1331	135	144	153	162	171	70	79	88	97	106	126	1331
1331	114	123	132	141	150	170	179	67	76	85	94	1331
1331	93	102	111	120	129	138	158	167	176	64	73	1331
	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331

In both the examples 19 and 20 given above the magic sum is $S_{11 \times 11} = 1331 = 11^3$, and the sum of all numbers is $F_{121} := 14641 = 121^2 = 11^4$.

Both of the examples 19 and 20 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 11, 11^2, 11^3, 11^4 \rangle$.

4.9.3 Third Approach: Minimum Perfect Square Sum

The examples given in 19 and 20 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 11, 11^2, 11^3, 11^4 \rangle$. However, the sum of the magic square entries of 20 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 11 given in section 3:

$$L\left(8^2 + \frac{11^2 - 1}{2}, 11^2\right) \rightarrow (11, 4, 124, 704, 7744, 2\sqrt{22})$$

In this case, we have a magic square of order 11 with sum of all entries $7744 = 88^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

Example 21. A *pan diagonal magic square* of order 11 with 121 entries starting from 4 to 124 is given by

		704	704	704	704	704	704	704	704	704	704	704
	4	24	33	42	51	60	80	89	98	107	116	704
704	104	124	12	21	30	39	48	68	77	86	95	704
704	83	92	112	121	9	18	27	47	56	65	74	704
704	62	71	91	100	109	118	6	15	35	44	53	704
704	41	50	59	79	88	97	106	115	14	23	32	704
704	20	29	38	58	67	76	85	94	103	123	11	704
704	120	8	17	26	46	55	64	73	82	102	111	704
704	99	108	117	5	25	34	43	52	61	70	90	704
704	78	87	96	105	114	13	22	31	40	49	69	704
704	57	66	75	84	93	113	122	10	19	28	37	704
704	36	45	54	63	72	81	101	110	119	7	16	704
	704	704	704	704	704	704	704	704	704	704	704	704

The above example 21 has a magic sum $S_{11 \times 11} = 704$, and the sum of all entries is $F_{121} := 11 \times 704 = 7744 = 88^2$. Moreover, it is **pan diagonal**.

4.10 Magic Squares of Order 12

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 12 with this property if we use odd number entries. Taking $k = 12$ in equation (5), we get

$$\begin{aligned} 1 + 3 + 5 + \dots + (2 \times 12^2 - 1) &= 12^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 285 + 287 &= 20736 = 144^2 = 12^4. \end{aligned} \quad (25)$$

Example 22. According to the values given in equation (25), the magic square of order 12 is given by

												1728
135	165	3	273	89	211	53	223	109	191	25	251	1728
9	267	141	159	55	221	91	209	35	241	119	181	1728
285	15	153	123	235	65	199	77	263	37	179	97	1728
147	129	279	21	197	79	233	67	169	107	253	47	1728
85	215	49	227	111	189	27	249	137	163	5	271	1728
59	217	95	205	33	243	117	183	7	269	139	161	1728
239	61	203	73	261	39	177	99	283	17	151	125	1728
193	83	229	71	171	105	255	45	149	127	281	19	1728
113	187	29	247	133	167	1	275	87	213	51	225	1728
31	245	115	185	11	265	143	157	57	219	93	207	1728
259	41	175	101	287	13	155	121	237	63	201	75	1728
173	103	257	43	145	131	277	23	195	81	231	69	1728
1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728

In the example 22 given above the magic sum is $S_{12 \times 12} = 1728 = 12^3$, and the sum of all entries is $F_{144} := 20736 = 144^2 = 12^4$. Thus, the example 22 satisfies the property (13) of uniformity 2.5.1, i.e., $(12, 12^2, 12^3, 12^4)$. Moreover, this is a **block-wise magic square**, i.e., each 4×4 block is a **pan diagonal magic square** with magic sum 576.

This **block-wise** construction of a magic square of order 12 is a combination of the construction procedures for magic squares of order 4 given in subsection 4.2 and for those of order 3 given in subsection 4.1.

4.11 Magic Squares of Order 13

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 11 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.11.1 First Approach: Consecutive Odd Numbers

Taking $k = 11$ in equation (5), we get

$$\begin{aligned} 1 + 3 + 5 + \dots + (2 \times 13^2 - 1) &= 13^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 335 + 337 &= 28561 = 169^2 = 13^4 \end{aligned} \quad (26)$$

Example 23. According to the 169 values given in equation (26), the *pan diagonal magic square* of order 13 is given by

		2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197
	1	49	71	93	115	137	159	207	229	251	273	295	317	2197
2197	289	337	21	43	65	87	109	131	179	201	223	245	267	2197
2197	239	261	309	331	15	37	59	81	129	151	173	195	217	2197
2197	189	211	259	281	303	325	9	31	53	101	123	145	167	2197
2197	139	161	183	231	253	275	297	319	3	51	73	95	117	2197
2197	89	111	133	181	203	225	247	269	291	313	23	45	67	2197
2197	39	61	83	105	153	175	197	219	241	263	311	333	17	2197
2197	327	11	33	55	103	125	147	169	191	213	235	283	305	2197
2197	277	299	321	5	27	75	97	119	141	163	185	233	255	2197
2197	227	249	271	293	315	25	47	69	91	113	135	157	205	2197
2197	177	199	221	243	265	287	335	19	41	63	85	107	155	2197
2197	127	149	171	193	215	237	285	307	329	13	35	57	79	2197
2197	77	99	121	143	165	187	209	257	279	301	323	7	29	2197
	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197

4.11.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-169)(n-168)}{2} = 169(n-84)$$

Taking $n = 253$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(253) - T(84) &= \frac{253 \times 254}{2} - \frac{84 \times 85}{2} = 253 \times 127 - 42 \times 85 \\ &= 28561 = 169^2 = 13^4. \end{aligned}$$

Simplifying, we get

$$85 + 86 + 87 + \dots + 252 + 253 = 28561 = 169^2 = 13^4.$$

This gives a perfect square sum for the 169 consecutive natural numbers from 85 to 253.

Example 24. The *pan diagonal magic square* of order 13 for the 169 entries from 85 to 253 is given by

		2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197
	85	109	120	131	142	153	164	188	199	210	221	232	243	2197
2197	229	253	95	106	117	128	139	150	174	185	196	207	218	2197
2197	204	215	239	250	92	103	114	125	149	160	171	182	193	2197
2197	179	190	214	225	236	247	89	100	111	135	146	157	168	2197
2197	154	165	176	200	211	222	233	244	86	110	121	132	143	2197
2197	129	140	151	175	186	197	208	219	230	241	96	107	118	2197
2197	104	115	126	137	161	172	183	194	205	216	240	251	93	2197
2197	248	90	101	112	136	147	158	169	180	191	202	226	237	2197
2197	223	234	245	87	98	122	133	144	155	166	177	201	212	2197
2197	198	209	220	231	242	97	108	119	130	141	152	163	187	2197
2197	173	184	195	206	217	228	252	94	105	116	127	138	162	2197
2197	148	159	170	181	192	203	227	238	249	91	102	113	124	2197
2197	123	134	145	156	167	178	189	213	224	235	246	88	99	2197
	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197	2197

In both the examples 23 and 24 given above the magic sum is $S_{13 \times 13} = 2197 = 13^3$, and the sum of all numbers is $F_{169} := 28561 = 169^2 = 13^4$.

Both the examples 23 and 24 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 13, 13^2, 13^3, 13^4 \rangle$.

4.11.3 Third Approach: Minimum Perfect Square Sum

The examples given in 23 and 24 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 13, 13^2, 13^3, 13^4 \rangle$. However, the sum of the magic square entries of 24 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 13 given in section 3:

$$L \left(10^2 + \frac{13^2 - 1}{2}, 13^2 \right) \rightarrow (13, 16, 184, 1300, 16900, \sqrt{130})$$

In this case, we have a magic square of order 13 with the sum of all entries $16900 = 130^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

Example 25. A *pan diagonal* magic square of order 13 with the 169 entries starting from 16 to 184 is given by

		1300	1300	1300	1300	1300	1300	1300	1300	1300	1300	1300	1300	1300
	16	40	51	62	73	84	95	119	130	141	152	163	174	1300
1300	160	184	26	37	48	59	70	81	105	116	127	138	149	1300
1300	135	146	170	181	23	34	45	56	80	91	102	113	124	1300
1300	110	121	145	156	167	178	20	31	42	66	77	88	99	1300
1300	85	96	107	131	142	153	164	175	17	41	52	63	74	1300
1300	60	71	82	106	117	128	139	150	161	172	27	38	49	1300
1300	35	46	57	68	92	103	114	125	136	147	171	182	24	1300
1300	179	21	32	43	67	78	89	100	111	122	133	157	168	1300
1300	154	165	176	18	29	53	64	75	86	97	108	132	143	1300
1300	129	140	151	162	173	28	39	50	61	72	83	94	118	1300
1300	104	115	126	137	148	159	183	25	36	47	58	69	93	1300
1300	79	90	101	112	123	134	158	169	180	22	33	44	55	1300
1300	54	65	76	87	98	109	120	144	155	166	177	19	30	1300
	1300	1300	1300	1300	1300	1300	1300	1300	1300	1300	1300	1300	1300	1300

The above example 25 has a magic sum $S_{11 \times 11} = 704$, and the sum of all entries is $F_{121} := 11 \times 704 = 7744 = 88^2$. Moreover, it is **pan diagonal**.

4.12 Magic Squares of Order 14

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 14 with this property if we use odd number entries. Taking $k = 14$ in equation (5), we get

$$\begin{aligned}
 1 + 3 + 5 + \dots + (2 \times 14^2 - 1) &= 14^4 \\
 \Rightarrow 1 + 3 + 5 + \dots + 389 + 391 &= 38416 = 96^2 = 14^4.
 \end{aligned} \tag{27}$$

Example 26. According to the values given in equation (27), the magic square of order 14 is given by

														2744
1	189	381	223	275	155	115	63	33	317	291	355	249	97	2744
293	31	219	351	375	253	173	233	7	105	139	165	75	325	2744
251	323	61	273	343	125	159	11	93	51	193	197	297	367	2744
379	221	299	91	309	229	9	153	67	339	45	191	279	133	2744
327	167	109	23	241	171	57	201	265	123	345	49	371	295	2744
207	19	177	73	41	331	357	307	389	147	87	267	117	225	2744
37	69	263	113	5	305	391	329	359	183	243	101	143	203	2744
89	129	141	179	71	385	303	361	335	237	259	3	205	47	2744
59	85	119	145	187	363	333	387	301	213	13	235	43	261	2744
135	287	311	53	149	95	209	185	239	271	365	83	21	341	2744
161	373	349	227	137	35	269	103	169	27	211	285	319	79	2744
277	347	15	321	283	215	231	29	157	369	77	121	107	195	2744
353	257	245	383	111	65	39	127	199	281	163	315	181	25	2744
175	247	55	289	217	17	99	255	131	81	313	377	337	151	2744
2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744

In the example 26 given above, the magic sum is $S_{14 \times 14} = 2744 = 14^3$, and the sum of all entries is $F_{196} := 38416 = 196^2 = 14^4$. Thus, the example 26 satisfies the property (13) of uniformity 2.5.1, i.e., $\langle 14, 14^2, 14^3, 14^4 \rangle$. Moreover, the inner 4×4 blocks are magic squares with magic sum $S_{4 \times 4} = 1384$.

4.13 Magic Squares of Order 15

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 15 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.13.1 First Approach: Consecutive Odd Numbers

Taking $k = 15$ in equation (5), we get

$$\begin{aligned} 1 + 3 + 5 + \dots + (2 \times 15^2 - 1) &= 15^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 447 + 449 &= 50625 = 225^2 = 15^4 \end{aligned} \quad (28)$$

Example 27. According to the 225 values given in equation (28), the magic square of order 15 is given by

															3375
3	171	189	357	375	65	109	247	299	437	31	143	221	325	403	3375
339	387	15	153	201	277	449	77	95	259	311	415	43	121	233	3375
165	183	351	369	27	107	245	289	427	89	133	211	323	401	55	3375
381	9	177	195	333	439	67	119	257	275	413	41	145	223	301	3375
207	345	363	21	159	269	287	425	79	97	235	313	391	53	131	3375
61	113	251	295	433	33	141	219	327	405	5	169	187	359	377	3375
281	445	73	91	263	309	417	45	123	231	337	389	17	155	199	3375
103	241	293	431	85	135	213	321	399	57	167	185	349	367	29	3375
443	71	115	253	271	411	39	147	225	303	379	7	179	197	335	3375
265	283	421	83	101	237	315	393	51	129	209	347	365	19	157	3375
35	139	217	329	407	1	173	191	355	373	63	111	249	297	435	3375
307	419	47	125	229	341	385	13	151	203	279	447	75	93	261	3375
137	215	319	397	59	163	181	353	371	25	105	243	291	429	87	3375
409	37	149	227	305	383	11	175	193	331	441	69	117	255	273	3375
239	317	395	49	127	205	343	361	23	161	267	285	423	81	99	3375
3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375

In this case the magic sum is $S_{15 \times 15} = 3375$, and the sum of all entries is $F_{225} := 15 \times 3375 = 50625 = 225^2 = 15^4$. Moreover, each 5×5 block is a **pan diagonal magic square** which in turn produces another magic square of order 3. See below:

Example 28. The magic square of order 3 arising due to 9 pan diagonal magic squares of order 5 in example 27 is given by

			3375
1095	1157	1123	3375
1153	1125	1097	3375
1127	1093	1155	3375
3375	3375	3375	3375

4.13.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-225)(n-224)}{2} = 225(n-112)$$

Taking $n = 337$, we get a perfect square, i.e.,

$$G := T(337) - T(112) = \frac{337 \times 338}{2} - \frac{112 \times 113}{2} = 50625 = 225^2 = 15^4.$$

Simplifying, we get

$$113 + 114 + 115 + \dots + 336 + 337 = 50625 = 225^2 = 15^4.$$

This gives a perfect square sum for the 225 entries of consecutive natural numbers from 113 to 337.

Example 29. The magic square of order 15 for 225 entries from 113 to 337 is given by

															3375
114	198	207	291	300	145	167	236	262	331	128	184	223	275	314	3375
282	306	120	189	213	251	337	151	160	242	268	320	134	173	229	3375
195	204	288	297	126	166	235	257	326	157	179	218	274	313	140	3375
303	117	201	210	279	332	146	172	241	250	319	133	185	224	263	3375
216	285	294	123	192	247	256	325	152	161	230	269	308	139	178	3375
143	169	238	260	329	129	183	222	276	315	115	197	206	292	301	3375
253	335	149	158	244	267	321	135	174	228	281	307	121	190	212	3375
164	233	259	328	155	180	219	273	312	141	196	205	287	296	127	3375
334	148	170	239	248	318	132	186	225	264	302	116	202	211	280	3375
245	254	323	154	163	231	270	309	138	177	217	286	295	122	191	3375
130	182	221	277	316	113	199	208	290	299	144	168	237	261	330	3375
266	322	136	175	227	283	305	119	188	214	252	336	150	159	243	3375
181	220	272	311	142	194	203	289	298	125	165	234	258	327	156	3375
317	131	187	226	265	304	118	200	209	278	333	147	171	240	249	3375
232	271	310	137	176	215	284	293	124	193	246	255	324	153	162	3375
3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375

In the examples 27 and 29 given above the magic sum is $S_{15 \times 15} = 3375 = 15^3$, and the sum of all entries $F_{225} := 50625 = 225^2 = 15^4$. What is more, each 5×5 block is a **pan diagonal magic square** and when these are combined they produce another magic square of order 3:

Example 30. The magic square of order 3 arising due to 9 pan diagonal magic squares of order 5 in example 29 is given by

			3375
1110	1141	1124	3375
1139	1125	1111	3375
1126	1109	1140	3375
3375	3375	3375	3375

The examples given in 27 and 29 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 15, 15^2, 15^3, 15^4 \rangle$.

4.13.3 Third Approach: Minimum Perfect Square Sum

The examples given in 27 and 29 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 15, 15^2, 15^3, 15^4 \rangle$. However, the magic square example 29 does not have a minimum perfect square sum of its entries. Let's observe the first example of a magic square of order 15 given in section 3:

$$L\left(11^2 + \frac{15^2 - 1}{2}, 15^2\right) \rightarrow (15, 9, 233, 1815, 27225, \sqrt{165})$$

In this case, we have a magic square of order 15 with a perfect square sum of all its entries $27225 = 165^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square that follows is constructed according to the details given above.

Example 31. The magic square of order 15 with 225 entries from 9 to 233 is given by

															1815
114	198	207	291	300	145	167	236	262	331	128	184	223	275	314	1815
282	306	120	189	213	251	337	151	160	242	268	320	134	173	229	1815
195	204	288	297	126	166	235	257	326	157	179	218	274	313	140	1815
303	117	201	210	279	332	146	172	241	250	319	133	185	224	263	1815
216	285	294	123	192	247	256	325	152	161	230	269	308	139	178	1815
143	169	238	260	329	129	183	222	276	315	115	197	206	292	301	1815
253	335	149	158	244	267	321	135	174	228	281	307	121	190	212	1815
164	233	259	328	155	180	219	273	312	141	196	205	287	296	127	1815
334	148	170	239	248	318	132	186	225	264	302	116	202	211	280	1815
245	254	323	154	163	231	270	309	138	177	217	286	295	122	191	1815
130	182	221	277	316	113	199	208	290	299	144	168	237	261	330	1815
266	322	136	175	227	283	305	119	188	214	252	336	150	159	243	1815
181	220	272	311	142	194	203	289	298	125	165	234	258	327	156	1815
317	131	187	226	265	304	118	200	209	278	333	147	171	240	249	1815
232	271	310	137	176	215	284	293	124	193	246	255	324	153	162	1815
1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815	1815

The above example 31 has a magic sum $S_{15 \times 15} = 1815$, and the sum of all entries is $F_{225} := 15 \times 1815 = 27225 = 165^2$. The magic square of order 15 constructed with the values given in example 31 has a minimum perfect square sum of all its entries. Similar to the other examples 27 and 29, this magic square also has 9 pan diagonal magic squares of order 5, which in turn form a magic square of order 3. See the example below:

Example 32. The 9 blocks of *pan diagonal magic squares* of order 5 given in example 31 in turn form a magic square of order 3 given by

			1815
590	621	604	1815
619	605	591	1815
606	589	620	1815
1815	1815	1815	1815

The block-wise constructions of magic squares of order 15 given in examples 27, 29 and 31 are the compositions of pan diagonal magic squares of order 5 given in subsection 4.3, together with magic squares of order 3 given in subsection 4.1.

4.14 Bimagic Squares of Order 16

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 16 with this property if we use odd number entries. Taking $k = 16$ in equation (5), we get

$$1 + 3 + 5 + \dots + (2 \times 16^2 - 1) = 16^4$$

$$\Rightarrow 1 + 3 + 5 + \dots + 509 + 511 = 65536 = 256^2 = 16^4. \quad (29)$$

Example 33. According to the values given in equation (29), the magic square of order 16 is given by

																4096
1	307	477	239	45	287	497	195	87	357	395	185	123	329	423	149	4096
463	253	19	289	483	209	63	269	409	171	69	375	437	135	105	347	4096
243	449	303	29	223	493	259	49	165	407	377	75	137	443	341	103	4096
317	15	225	467	273	35	205	511	363	89	183	389	327	117	155	425	4096
91	361	391	181	119	325	427	153	13	319	465	227	33	275	509	207	4096
405	167	73	379	441	139	101	343	451	241	31	301	495	221	51	257	4096
169	411	373	71	133	439	345	107	255	461	291	17	211	481	271	61	4096
359	85	187	393	331	121	151	421	305	3	237	479	285	47	193	499	4096
109	351	433	131	65	371	413	175	59	265	487	213	23	293	459	249	4096
419	145	127	333	399	189	83	353	501	199	41	283	473	235	5	311	4096
159	429	323	113	179	385	367	93	201	507	277	39	229	471	313	11	4096
337	99	141	447	381	79	161	403	263	53	219	489	299	25	247	453	4096
55	261	491	217	27	297	455	245	97	339	445	143	77	383	401	163	4096
505	203	37	279	469	231	9	315	431	157	115	321	387	177	95	365	4096
197	503	281	43	233	475	309	7	147	417	335	125	191	397	355	81	4096
267	57	215	485	295	21	251	457	349	111	129	435	369	67	173	415	4096
4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096

In the example 33 given above, the magic sum is $S_{16 \times 16} = 4096 = 16^3$, and the sum of all entries is $F_{256} := 65536 = 256^2 = 16^4$. Thus, the example 33 satisfies the property (13) of uniformity 2.5.1, i.e., $\langle 16, 16^2, 16^3, 16^4 \rangle$. Moreover, 16 blocks of order 4 are magic squares with magic sum $S_{4 \times 4} = 1024$. The example shown above is also a **bimagic square** with bimagic sum $Sb_{16 \times 16} = 1398096$.

Example 34. The bimagic transformation of example 33 is given by

																1398096
1 ²	307 ²	477 ²	239 ²	45 ²	287 ²	497 ²	195 ²	87 ²	357 ²	395 ²	185 ²	123 ²	329 ²	423 ²	149 ²	1398096
463 ²	253 ²	19 ²	289 ²	483 ²	209 ²	63 ²	269 ²	409 ²	171 ²	69 ²	375 ²	437 ²	135 ²	105 ²	347 ²	1398096
243 ²	449 ²	303 ²	29 ²	223 ²	493 ²	259 ²	49 ²	165 ²	407 ²	377 ²	75 ²	137 ²	443 ²	341 ²	103 ²	1398096
317 ²	15 ²	225 ²	467 ²	273 ²	35 ²	205 ²	511 ²	363 ²	89 ²	183 ²	389 ²	327 ²	117 ²	155 ²	425 ²	1398096
91 ²	361 ²	391 ²	181 ²	119 ²	325 ²	427 ²	153 ²	13 ²	319 ²	465 ²	227 ²	33 ²	275 ²	509 ²	207 ²	1398096
405 ²	167 ²	73 ²	379 ²	441 ²	139 ²	101 ²	343 ²	451 ²	241 ²	31 ²	301 ²	495 ²	221 ²	51 ²	257 ²	1398096
169 ²	411 ²	373 ²	71 ²	133 ²	439 ²	345 ²	107 ²	255 ²	461 ²	291 ²	17 ²	211 ²	481 ²	271 ²	61 ²	1398096
359 ²	85 ²	187 ²	393 ²	331 ²	121 ²	151 ²	421 ²	305 ²	3 ²	237 ²	479 ²	285 ²	47 ²	193 ²	499 ²	1398096
109 ²	351 ²	433 ²	131 ²	65 ²	371 ²	413 ²	175 ²	59 ²	265 ²	487 ²	213 ²	23 ²	293 ²	459 ²	249 ²	1398096
419 ²	145 ²	127 ²	333 ²	399 ²	189 ²	83 ²	353 ²	501 ²	199 ²	41 ²	283 ²	473 ²	235 ²	5 ²	311 ²	1398096
159 ²	429 ²	323 ²	113 ²	179 ²	385 ²	367 ²	93 ²	201 ²	507 ²	277 ²	39 ²	229 ²	471 ²	313 ²	11 ²	1398096
337 ²	99 ²	141 ²	447 ²	381 ²	79 ²	161 ²	403 ²	263 ²	53 ²	219 ²	489 ²	299 ²	25 ²	247 ²	453 ²	1398096
55 ²	261 ²	491 ²	217 ²	27 ²	297 ²	455 ²	245 ²	97 ²	339 ²	445 ²	143 ²	77 ²	383 ²	401 ²	163 ²	1398096
505 ²	203 ²	37 ²	279 ²	469 ²	231 ²	9 ²	315 ²	431 ²	157 ²	115 ²	321 ²	387 ²	177 ²	95 ²	365 ²	1398096
197 ²	503 ²	281 ²	43 ²	233 ²	475 ²	309 ²	7 ²	147 ²	417 ²	335 ²	125 ²	191 ²	397 ²	355 ²	81 ²	1398096
267 ²	57 ²	215 ²	485 ²	295 ²	21 ²	251 ²	457 ²	349 ²	111 ²	129 ²	435 ²	369 ²	67 ²	173 ²	415 ²	1398096
1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096	1398096

4.15 Magic Squares of Order 17

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 17 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.15.1 First Approach: Consecutive Odd Numbers

Taking $k = 11$ in equation (5), we get

$$\begin{aligned}
 1 + 3 + 5 + \dots + (2 \times 17^2 - 1) &= 17^4 \\
 \Rightarrow 1 + 3 + 5 + \dots + 575 + 577 &= 83521 = 289^2 = 17^4
 \end{aligned} \tag{30}$$

Example 35. According to the 289 values given in equation (30), the *pan diagonal magic square* of order 17 is given by

		4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913
	1	65	95	125	155	185	215	245	275	339	369	399	429	459	489	519	549	4913
4913	513	577	29	59	89	119	149	179	209	239	303	333	363	393	423	453	483	4913
4913	447	477	541	571	23	53	83	113	143	173	237	267	297	327	357	387	417	4913
4913	381	411	475	505	535	565	17	47	77	107	137	201	231	261	291	321	351	4913
4913	315	345	375	439	469	499	529	559	11	41	71	135	165	195	225	255	285	4913
4913	249	279	309	373	403	433	463	493	523	553	5	35	99	129	159	189	219	4913
4913	183	213	243	273	337	367	397	427	457	487	517	547	33	63	93	123	153	4913
4913	117	147	177	207	271	301	331	361	391	421	451	481	511	575	27	57	87	4913
4913	51	81	111	141	171	235	265	295	325	355	385	415	445	509	539	569	21	4913
4913	563	15	45	75	105	169	199	229	259	289	319	349	379	409	473	503	533	4913
4913	497	527	557	9	39	69	133	163	193	223	253	283	313	343	407	437	467	4913
4913	431	461	491	521	551	3	67	97	127	157	187	217	247	277	307	371	401	4913
4913	365	395	425	455	485	515	545	31	61	91	121	151	181	211	241	305	335	4913
4913	299	329	359	389	419	449	479	543	573	25	55	85	115	145	175	205	269	4913
4913	233	263	293	323	353	383	413	443	507	537	567	19	49	79	109	139	203	4913
4913	167	197	227	257	287	317	347	377	441	471	501	531	561	13	43	73	103	4913
4913	101	131	161	191	221	251	281	311	341	405	435	465	495	525	555	7	37	4913
	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913

4.15.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-289)(n-288)}{2} = 289(n-144)$$

Taking $n = 433$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(433) - T(144) &= \frac{433 \times 434}{2} - \frac{144 \times 145}{2} = 433 \times 217 - 72 \times 145 \\ &= 83521 = 289^2 = 17^4. \end{aligned}$$

Simplifying, we get

$$145 + 146 + 147 + \dots + 432 + 433 = 83521 = 289^2 = 17^4.$$

This gives a perfect square sum for the 289 consecutive natural numbers from 145 to 433.

Example 36. The *pan diagonal magic square* of order 17 for the 289 entries from 145 to 433 is given by

		4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913
	145	177	192	207	222	237	252	267	282	314	329	344	359	374	389	404	419	4913
4913	401	433	159	174	189	204	219	234	249	264	296	311	326	341	356	371	386	4913
4913	368	383	415	430	156	171	186	201	216	231	263	278	293	308	323	338	353	4913
4913	335	350	382	397	412	427	153	168	183	198	213	245	260	275	290	305	320	4913
4913	302	317	332	364	379	394	409	424	150	165	180	212	227	242	257	272	287	4913
4913	269	284	299	331	346	361	376	391	406	421	147	162	194	209	224	239	254	4913
4913	236	251	266	281	313	328	343	358	373	388	403	418	161	176	191	206	221	4913
4913	203	218	233	248	280	295	310	325	340	355	370	385	400	432	158	173	188	4913
4913	170	185	200	215	230	262	277	292	307	322	337	352	367	399	414	429	155	4913
4913	426	152	167	182	197	229	244	259	274	289	304	319	334	349	381	396	411	4913
4913	393	408	423	149	164	179	211	226	241	256	271	286	301	316	348	363	378	4913
4913	360	375	390	405	420	146	178	193	208	223	238	253	268	283	298	330	345	4913
4913	327	342	357	372	387	402	417	160	175	190	205	220	235	250	265	297	312	4913
4913	294	309	324	339	354	369	384	416	431	157	172	187	202	217	232	247	279	4913
4913	261	276	291	306	321	336	351	366	398	413	428	154	169	184	199	214	246	4913
4913	228	243	258	273	288	303	318	333	365	380	395	410	425	151	166	181	196	4913
4913	195	210	225	240	255	270	285	300	315	347	362	377	392	407	422	148	163	4913
	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913	4913

In both the examples 35 and 36 given above the magic sum is $S_{17 \times 17} = 4913 = 17^3$, and the sum of all numbers is $F_{289} := 83521 = 289^2 = 17^4$.

Both the examples 35 and 36 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 17, 17^2, 17^3, 17^4 \rangle$.

4.15.3 Third Approach: Minimum Perfect Square Sum

The examples given in 35 and 36 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 17, 17^2, 17^3, 17^4 \rangle$. However, the sum of the magic square entries of 36 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 17 given in section 3:

$$L \left(13^2 + \frac{17^2 - 1}{2}, 17^2 \right) \rightarrow (17, 25, 313, 2873, 48841, \sqrt{221})$$

In this case, we have magic square of order 17 with sum of all entries $48841 = 221^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

Example 37. The *pan diagonal magic square* of order 17 with 289 entries starting from 25 to 313 is given by

		2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873
	25	57	72	87	102	117	132	147	162	194	209	224	239	254	269	284	299	2873
2873	281	313	39	54	69	84	99	114	129	144	176	191	206	221	236	251	266	2873
2873	248	263	295	310	36	51	66	81	96	111	143	158	173	188	203	218	233	2873
2873	215	230	262	277	292	307	33	48	63	78	93	125	140	155	170	185	200	2873
2873	182	197	212	244	259	274	289	304	30	45	60	92	107	122	137	152	167	2873
2873	149	164	179	211	226	241	256	271	286	301	27	42	74	89	104	119	134	2873
2873	116	131	146	161	193	208	223	238	253	268	283	298	41	56	71	86	101	2873
2873	83	98	113	128	160	175	190	205	220	235	250	265	280	312	38	53	68	2873
2873	50	65	80	95	110	142	157	172	187	202	217	232	247	279	294	309	35	2873
2873	306	32	47	62	77	109	124	139	154	169	184	199	214	229	261	276	291	2873
2873	273	288	303	29	44	59	91	106	121	136	151	166	181	196	228	243	258	2873
2873	240	255	270	285	300	26	58	73	88	103	118	133	148	163	178	210	225	2873
2873	207	222	237	252	267	282	297	40	55	70	85	100	115	130	145	177	192	2873
2873	174	189	204	219	234	249	264	296	311	37	52	67	82	97	112	127	159	2873
2873	141	156	171	186	201	216	231	246	278	293	308	34	49	64	79	94	126	2873
2873	108	123	138	153	168	183	198	213	245	260	275	290	305	31	46	61	76	2873
2873	75	90	105	120	135	150	165	180	195	227	242	257	272	287	302	28	43	2873
	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873	2873

The above example 37 has a magic sum $S_{17 \times 17} = 2873$, and the sum of all entries is $F_{289} := 17 \times 2873 = 48841 = 221^2$.

4.16 Magic Squares of Order 18

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 18 with this property if we use odd number entries. Taking $k = 18$ in equation (5), we get

$$\begin{aligned}
 1 + 3 + 5 + \dots + (2 \times 18^2 - 1) &= 18^4 \\
 \Rightarrow 1 + 3 + 5 + \dots + 645 + 647 &= 104976 = 324^2 = 18^4.
 \end{aligned} \tag{31}$$

Below are three different ways of writing a magic square of order 18 with the entries given in (31).

• First Way

Example 38. According to the values given in equation (31), the magic square of order 18 is given by

																		5832
217	261	271	283	249	231	577	621	631	643	609	591	73	117	127	139	105	87	5832
273	229	285	243	257	225	633	589	645	603	617	585	129	85	141	99	113	81	5832
239	227	241	269	277	259	599	587	601	629	637	619	95	83	97	125	133	115	5832
279	247	223	263	235	265	639	607	583	623	595	625	135	103	79	119	91	121	5832
253	281	237	221	275	245	613	641	597	581	635	605	109	137	93	77	131	101	5832
251	267	255	233	219	287	611	627	615	593	579	647	107	123	111	89	75	143	5832
145	189	199	211	177	159	289	333	343	355	321	303	433	477	487	499	465	447	5832
201	157	213	171	185	153	345	301	357	315	329	297	489	445	501	459	473	441	5832
167	155	169	197	205	187	311	299	313	341	349	331	455	443	457	485	493	475	5832
207	175	151	191	163	193	351	319	295	335	307	337	495	463	439	479	451	481	5832
181	209	165	149	203	173	325	353	309	293	347	317	469	497	453	437	491	461	5832
179	195	183	161	147	215	323	339	327	305	291	359	467	483	471	449	435	503	5832
505	549	559	571	537	519	1	45	55	67	33	15	361	405	415	427	393	375	5832
561	517	573	531	545	513	57	13	69	27	41	9	417	373	429	387	401	369	5832
527	515	529	557	565	547	23	11	25	53	61	43	383	371	385	413	421	403	5832
567	535	511	551	523	553	63	31	7	47	19	49	423	391	367	407	379	409	5832
541	569	525	509	563	533	37	65	21	5	59	29	397	425	381	365	419	389	5832
539	555	543	521	507	575	35	51	39	17	3	71	395	411	399	377	363	431	5832
5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832

In this case the magic sum is $S_{18 \times 18} = 5832 = 18^3$, and the sum of all entries is $F_{324} := 18 \times 5832 = 104976 = 324^2 = 18^4$. In this case each 6×6 block is a **magic square of order 6**, and these blocks in turn form a magic square of order 3:

Example 39. Below is a magic square of order 3 formed by each block of order 6 from example 38:

			5832
1512	3672	648	5832
1080	1944	2808	5832
3240	216	2376	5832
5832	5832	5832	5832

$$= 6^3 \times$$

			27
7	17	3	27
5	19	13	27
15	1	11	27
27	27	27	27

Interestingly, the same happens with the sums of all entries of each block of order 6 which again form a magic square of order 3.

Example 40. Below is a magic square of order 3 formed by the sums of all entries of each block of order 6:

			34992
9072	22032	3888	34992
6480	11664	16848	34992
9072	22032	3888	34992
34992	34992	34992	34992

$$= 6^4 \times$$

			27
7	17	3	27
5	19	13	27
15	1	11	27
27	27	27	27

• Second Way

By redistributing the total entries in a slightly different way, we get another method for writing a magic square of order 18 with the same values given in (34). See below:

Example 41. According to the values given in equation (31), the magic square of order 18 is given by

																			5832
7	403	493	601	295	133	17	413	503	611	305	143	3	399	489	597	291	129		5832
511	115	619	241	367	79	521	125	629	251	377	89	507	111	615	237	363	75		5832
205	97	223	475	547	385	215	107	233	485	557	395	201	93	219	471	543	381		5832
565	277	61	421	169	439	575	287	71	431	179	449	561	273	57	417	165	435		5832
331	583	187	43	529	259	341	593	197	53	539	269	327	579	183	39	525	255		5832
313	457	349	151	25	637	323	467	359	161	35	647	309	453	345	147	21	633		5832
5	401	491	599	293	131	9	405	495	603	297	135	13	409	499	607	301	139		5832
509	113	617	239	365	77	513	117	621	243	369	81	517	121	625	247	373	85		5832
203	95	221	473	545	383	207	99	225	477	549	387	211	103	229	481	553	391		5832
563	275	59	419	167	437	567	279	63	423	171	441	571	283	67	427	175	445		5832
329	581	185	41	527	257	333	585	189	45	531	261	337	589	193	49	535	265		5832
311	455	347	149	23	635	315	459	351	153	27	639	319	463	355	157	31	643		5832
15	411	501	609	303	141	1	397	487	595	289	127	11	407	497	605	299	137		5832
519	123	627	249	375	87	505	109	613	235	361	73	515	119	623	245	371	83		5832
213	105	231	483	555	393	199	91	217	469	541	379	209	101	227	479	551	389		5832
573	285	69	429	177	447	559	271	55	415	163	433	569	281	65	425	173	443		5832
339	591	195	51	537	267	325	577	181	37	523	253	335	587	191	47	533	263		5832
321	465	357	159	33	645	307	451	343	145	19	631	317	461	353	155	29	641		5832
5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832		5832

In this case also the magic sum is $S_{18 \times 18} = 5832 = 18^3$, and the sum of all entries is $F_{324} := 18 \times 5832 = 104976 = 324^2 = 18^4$. Each 6×6 block is a **magic square of order 6** and these in turn form a magic square of order 3:

Example 42. Below is a magic square of order 3 formed by each block of order 6:

			5832
1932	1992	1908	5832
1920	1944	1968	5832
1980	1896	1956	5832
5832	5832	5832	5832

 $= 12 \times$

			486
161	166	159	486
160	162	164	486
165	158	163	486
486	486	486	486

Interestingly, the same happens with the sums of all entries of each block of order 6 which again form a magic square of order 3.

Example 43. Below is a magic square of order 3 formed by sums of all entries of each block of order 6:

			34992
11592	11952	11448	34992
11520	11664	11808	34992
11880	11376	11736	34992
34992	34992	34992	34992

 $= 72 \times$

			486
161	166	159	486
160	162	164	486
165	158	163	486
486	486	486	486

The example 42 has a more uniform distribution of entries than that of the example 34.

• Third Way

There is yet another way of writing a magic square of order 18 with the same entries given in equation (37). In this case the 6×6 blocks are not magic squares but the middle 4×4 block is a magic square of order 4

Example 44. According to the values given in equation (37), the magic square of order 18 is given by

																			5832
1	411	449	71	479	317	277	85	573	167	111	583	387	355	621	199	237	509		5832
275	39	627	413	105	445	315	109	535	215	149	385	353	587	235	21	475	549		5832
313	287	77	591	629	139	397	147	497	249	187	351	553	19	57	441	515	383		5832
615	311	285	115	555	593	173	185	459	31	225	505	55	93	407	481	395	349		5832
207	581	323	283	153	519	557	223	421	65	11	91	129	625	433	393	347	471		5832
521	241	547	321	281	191	483	9	635	99	49	165	577	399	391	359	437	127		5832
447	485	23	513	319	279	229	47	611	133	73	543	617	389	357	403	163	201		5832
221	7	45	83	121	145	183	267	379	309	341	465	503	527	565	603	641	427		5832
607	569	531	493	455	431	645	345	305	375	271	41	3	217	193	155	117	79		5832
405	623	589	541	507	473	439	377	273	343	303	245	211	177	143	95	61	27		5832
43	81	119	157	181	219	5	307	339	269	381	643	429	467	491	529	567	605		5832
141	453	487	29	329	369	67	601	75	517	575	419	263	295	171	613	197	231		5832
489	523	247	327	367	33	107	639	37	551	599	195	457	261	293	137	409	161		5832
559	213	325	365	251	59	525	425	13	585	637	125	159	495	259	291	103	443		5832
179	337	363	203	25	561	595	463	227	619	423	477	89	123	533	257	289	69		5832
335	361	169	243	597	631	131	501	189	401	461	35	511	53	87	571	255	301		5832
373	135	209	633	415	97	333	539	151	435	499	299	239	545	17	51	609	253		5832
101	175	417	451	63	331	371	563	113	469	537	265	297	205	579	233	15	647		5832
5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832

The middle block is a magic square of order 4 with magic sum $S_{4 \times 4} = 1296$.

The **block-wise** constructions of magic squares of order 18 given in examples 38 and 41 are the compositions of magic squares of order 6 given in subsection 4.4 together with magic squares of order 3 given in subsection 4.1. The construction of the magic square given in example 44 is based on a procedure given by White [10].

4.17 Magic Squares of Order 19

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 11 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.17.1 First Approach: Consecutive Odd Numbers

Taking $k = 19$ in equation (5), we get

$$\begin{aligned}
 1 + 3 + 5 + \dots + (2 \times 19^2 - 1) &= 19^4 \\
 \Rightarrow 1 + 3 + 5 + \dots + 719 + 721 &= 130321 = 361^2 = 19^4
 \end{aligned} \tag{32}$$

Example 45. According to the 361 values given in equation (32), the *pan diagonal magic square* of order 19 is given by

		6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859
	1	73	107	141	175	209	243	277	311	345	417	451	485	519	553	587	621	655	689
6859	649	721	33	67	101	135	169	203	237	271	305	377	411	445	479	513	547	581	615
6859	575	609	681	715	27	61	95	129	163	197	231	303	337	371	405	439	473	507	541
6859	501	535	607	641	675	709	21	55	89	123	157	191	263	297	331	365	399	433	467
6859	427	461	495	567	601	635	669	703	15	49	83	117	189	223	257	291	325	359	393
6859	353	387	421	493	527	561	595	629	663	697	9	43	77	149	183	217	251	285	319
6859	279	313	347	381	453	487	521	555	589	623	657	691	3	75	109	143	177	211	245
6859	205	239	273	307	379	413	447	481	515	549	583	617	651	685	35	69	103	137	171
6859	131	165	199	233	267	339	373	407	441	475	509	543	577	611	683	717	29	63	97
6859	57	91	125	159	193	265	299	333	367	401	435	469	503	537	571	643	677	711	23
6859	705	17	51	85	119	153	225	259	293	327	361	395	429	463	497	569	603	637	671
6859	631	665	699	11	45	79	151	185	219	253	287	321	355	389	423	457	529	563	597
6859	557	591	625	659	693	5	39	111	145	179	213	247	281	315	349	383	455	489	523
6859	483	517	551	585	619	653	687	37	71	105	139	173	207	241	275	309	343	415	449
6859	409	443	477	511	545	579	613	647	719	31	65	99	133	167	201	235	269	341	375
6859	335	369	403	437	471	505	539	573	645	679	713	25	59	93	127	161	195	229	301
6859	261	295	329	363	397	431	465	499	533	605	639	673	707	19	53	87	121	155	227
6859	187	221	255	289	323	357	391	425	459	531	565	599	633	667	701	13	47	81	115
6859	113	147	181	215	249	283	317	351	385	419	491	525	559	593	627	661	695	7	41
	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859

4.17.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-361)(n-360)}{2} = 361(n-180)$$

Taking $n = 541$, we get a perfect square, i.e.,

$$\begin{aligned} G &:= T(541) - T(180) = \frac{541 \times 542}{2} - \frac{180 \times 181}{2} = 541 \times 271 - 90 \times 181 \\ &= 130321 = 361^2 = 19^4. \end{aligned}$$

Simplifying, we get

$$181 + 182 + 183 + \dots + 540 + 541 = 130321 = 361^2 = 19^4.$$

This gives a perfect square sum for the 121 consecutive natural numbers from to 61 to 181.

Example 46. The *pan diagonal magic square* of order 19 for 361 entries from 181 to 541 is given by

		6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859
	181	217	234	251	268	285	302	319	336	353	389	406	423	440	457	474	491	508	525	6859
6859	505	541	197	214	231	248	265	282	299	316	333	369	386	403	420	437	454	471	488	6859
6859	468	485	521	538	194	211	228	245	262	279	296	332	349	366	383	400	417	434	451	6859
6859	431	448	484	501	518	535	191	208	225	242	259	276	312	329	346	363	380	397	414	6859
6859	394	411	428	464	481	498	515	532	188	205	222	239	275	292	309	326	343	360	377	6859
6859	357	374	391	427	444	461	478	495	512	529	185	202	219	255	272	289	306	323	340	6859
6859	320	337	354	371	407	424	441	458	475	492	509	526	182	218	235	252	269	286	303	6859
6859	283	300	317	334	370	387	404	421	438	455	472	489	506	523	198	215	232	249	266	6859
6859	246	263	280	297	314	350	367	384	401	418	435	452	469	486	522	539	195	212	229	6859
6859	209	226	243	260	277	313	330	347	364	381	398	415	432	449	466	502	519	536	192	6859
6859	533	189	206	223	240	257	293	310	327	344	361	378	395	412	429	465	482	499	516	6859
6859	496	513	530	186	203	220	256	273	290	307	324	341	358	375	392	409	445	462	479	6859
6859	459	476	493	510	527	183	200	236	253	270	287	304	321	338	355	372	408	425	442	6859
6859	422	439	456	473	490	507	524	199	216	233	250	267	284	301	318	335	352	388	405	6859
6859	385	402	419	436	453	470	487	504	540	196	213	230	247	264	281	298	315	351	368	6859
6859	348	365	382	399	416	433	450	467	503	520	537	193	210	227	244	261	278	295	331	6859
6859	311	328	345	362	379	396	413	430	447	483	500	517	534	190	207	224	241	258	294	6859
6859	274	291	308	325	342	359	376	393	410	446	463	480	497	514	531	187	204	221	238	6859
6859	237	254	271	288	305	322	339	356	373	390	426	443	460	477	494	511	528	184	201	6859
	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859	6859

In both the examples 45 and 46 given above the magic sum is $S_{19 \times 19} = 6859$, and the sum of all entries is $F_{361} := 19 \times 6859 = 130321 = 361^2 = 19^4$.

Both the examples 45 and 46 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 19, 19^2, 19^3, 19^4 \rangle$.

4.17.3 Third Approach: Minimum Perfect Square Sum

The examples given in 45 and 46 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 19, 19^2, 19^3, 19^4 \rangle$. However, the sum of the magic square entries of 46 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 19 given in section 3:

$$L \left(14^2 + \frac{19^2 - 1}{2}, 19^2 \right) \rightarrow (19, 16, 376, 3724, 70756, \sqrt{266})$$

In this case, we have magic square of order 19 with sum of all entries $70756 = 266^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

Example 47. The *pan diagonal magic square* of order 19 with 361 entries starting from 16 to 376 is given by

		3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724
	181	217	234	251	268	285	302	319	336	353	389	406	423	440	457	474	491	508	525	3724
3724	505	541	197	214	231	248	265	282	299	316	333	369	386	403	420	437	454	471	488	3724
3724	468	485	521	538	194	211	228	245	262	279	296	332	349	366	383	400	417	434	451	3724
3724	16	52	69	86	103	120	137	154	171	188	224	241	258	275	292	309	326	343	360	3724
3724	340	376	32	49	66	83	100	117	134	151	168	204	221	238	255	272	289	306	323	3724
3724	303	320	356	373	29	46	63	80	97	114	131	167	184	201	218	235	252	269	286	3724
3724	266	283	319	336	353	370	26	43	60	77	94	111	147	164	181	198	215	232	249	3724
3724	229	246	263	299	316	333	350	367	23	40	57	74	110	127	144	161	178	195	212	3724
3724	192	209	226	262	279	296	313	330	347	364	20	37	54	90	107	124	141	158	175	3724
3724	155	172	189	206	242	259	276	293	310	327	344	361	17	53	70	87	104	121	138	3724
3724	118	135	152	169	205	222	239	256	273	290	307	324	341	358	33	50	67	84	101	3724
3724	81	98	115	132	149	185	202	219	236	253	270	287	304	321	357	374	30	47	64	3724
3724	44	61	78	95	112	148	165	182	199	216	233	250	267	284	301	337	354	371	27	3724
3724	368	24	41	58	75	92	128	145	162	179	196	213	230	247	264	300	317	334	351	3724
3724	331	348	365	21	38	55	91	108	125	142	159	176	193	210	227	244	280	297	314	3724
3724	294	311	328	345	362	18	35	71	88	105	122	139	156	173	190	207	243	260	277	3724
3724	257	274	291	308	325	342	359	34	51	68	85	102	119	136	153	170	187	223	240	3724
3724	220	237	254	271	288	305	322	339	375	31	48	65	82	99	116	133	150	186	203	3724
3724	183	200	217	234	251	268	285	302	338	355	372	28	45	62	79	96	113	130	166	3724
3724	146	163	180	197	214	231	248	265	282	318	335	352	369	25	42	59	76	93	129	3724
3724	109	126	143	160	177	194	211	228	245	281	298	315	332	349	366	22	39	56	73	3724
3724	72	89	106	123	140	157	174	191	208	225	261	278	295	312	329	346	363	19	36	3724
	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724	3724

The above example 47 has a magic sum $S_{19 \times 19} = 3724$, and the sum of all entries is $F_{361} := 19 \times 3724 = 70756 = 266^2$. Moreover, it is **pan diagonal**.

4.18 Magic Squares of Order 20

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 20 with this property if we use odd number entries. Taking $k = 20$ in equation (5), we get

$$\begin{aligned}
 1 + 3 + 5 + \dots + (2 \times 20^2 - 1) &= 20^4 \\
 \Rightarrow 1 + 3 + 5 + \dots + 797 + 799 &= 160000 = 400^2 = 20^4.
 \end{aligned} \tag{33}$$

We shall construct magic squares of order 20 with odd numbers in two different ways: firstly by using a 5×4 format and secondly by using a 4×5 format.

• First Way: 5×4

Example 48. According to the values given in equation (33), the magic square of order 20 with odd numbers is given by

		8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000
	43	269	373	595	699	81	231	415	553	737	7	305	329	639	663	125	187	451	517	781	8000
8000	573	715	59	243	389	535	753	97	201	431	609	679	23	287	345	491	797	141	165	467	8000
8000	259	363	589	693	75	217	401	551	735	113	303	327	625	649	39	181	445	507	771	157	8000
8000	709	53	275	379	563	751	95	233	417	521	665	9	319	343	607	787	131	197	461	485	8000
8000	395	579	683	69	253	433	537	721	111	215	359	623	647	25	289	477	501	765	147	171	8000
8000	5	307	331	637	661	127	185	449	519	783	41	271	375	593	697	83	229	413	555	739	8000
8000	611	677	21	285	347	489	799	143	167	465	575	713	57	241	391	533	755	99	203	429	8000
8000	301	325	627	651	37	183	447	505	769	159	257	361	591	695	73	219	403	549	733	115	8000
8000	667	11	317	341	605	785	129	199	463	487	711	55	273	377	561	749	93	235	419	523	8000
8000	357	621	645	27	291	479	503	767	145	169	393	577	681	71	255	435	539	723	109	213	8000
8000	121	191	455	513	777	3	309	333	635	659	85	227	411	557	741	47	265	369	599	703	8000
8000	495	793	137	161	471	613	675	19	283	349	531	757	101	205	427	569	719	63	247	385	8000
8000	177	441	511	775	153	299	323	629	653	35	221	405	547	731	117	263	367	585	689	79	8000
8000	791	135	193	457	481	669	13	315	339	603	747	91	237	421	525	705	49	279	383	567	8000
8000	473	497	761	151	175	355	619	643	29	293	437	541	725	107	211	399	583	687	65	249	8000
8000	87	225	409	559	743	45	267	371	597	701	123	189	453	515	779	1	311	335	633	657	8000
8000	529	759	103	207	425	571	717	61	245	387	493	795	139	163	469	615	673	17	281	351	8000
8000	223	407	545	729	119	261	365	587	691	77	179	443	509	773	155	297	321	631	655	33	8000
8000	745	89	239	423	527	707	51	277	381	565	789	133	195	459	483	671	15	313	337	601	8000
8000	439	543	727	105	209	397	581	685	67	251	475	499	763	149	173	353	617	641	31	295	8000
	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000

In this case the magic sum is $S_{20 \times 20} = 8000 = 20^3$, and the sum of all entries is $F_{400} := 20 \times 8000 = 160000 = 400^2 = 20^4$. Moreover, each 5×5 block is a **pan diagonal magic square** that has a different magic sum, and when these sums are combined, they form another **pan diagonal magic square** of order 4. See the example below:

Example 49. Each 5×5 block of a magic square of order 20 given in 48 is a **pan diagonal magic square** with a different magic sum, and when these sums are combined they form another **pan diagonal magic square** of order 4:

		8000	8000	8000	8000
	1979	2017	1943	2061	8000
8000	1941	2063	1977	2019	8000
8000	2057	1939	2021	1983	8000
8000	2023	1981	2059	1937	8000
	8000	8000	8000	8000	8000

• Second Way: 4×5

Example 50. According to the values given in equation (33), the magic square of order 20 with odd numbers is given by

		8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000
	379	401	39	781	333	447	73	747	297	483	117	703	251	529	151	669	215	565	195	625	8000
8000	21	799	361	419	67	753	327	453	103	717	283	497	149	671	249	531	185	635	205	575	8000
8000	761	19	421	399	727	53	467	353	683	97	503	317	649	131	549	271	605	175	585	235	8000
8000	439	381	779	1	473	347	733	47	517	303	697	83	551	269	651	129	595	225	615	165	8000
8000	257	523	157	663	211	569	191	629	375	405	35	785	339	441	79	741	293	487	113	707	8000
8000	143	677	243	537	189	631	209	571	25	795	365	415	61	759	321	459	107	713	287	493	8000
8000	643	137	543	277	609	171	589	231	765	15	425	395	721	59	461	359	687	93	507	313	8000
8000	557	263	657	123	591	229	611	169	435	385	775	5	479	341	739	41	513	307	693	87	8000
8000	335	445	75	745	299	481	119	701	253	527	153	667	217	563	197	623	371	409	31	789	8000
8000	65	755	325	455	101	719	281	499	147	673	247	533	183	637	203	577	29	791	369	411	8000
8000	725	55	465	355	681	99	501	319	647	133	547	273	603	177	583	237	769	11	429	391	8000
8000	475	345	735	45	519	301	699	81	553	267	653	127	597	223	617	163	431	389	771	9	8000
8000	213	567	193	627	377	403	37	783	331	449	71	749	295	485	115	705	259	521	159	661	8000
8000	187	633	207	573	23	797	363	417	69	751	329	451	105	715	285	495	141	679	241	539	8000
8000	607	173	587	233	763	17	423	397	729	51	469	351	685	95	505	315	641	139	541	279	8000
8000	593	227	613	167	437	383	777	3	471	349	731	49	515	305	695	85	559	261	659	121	8000
8000	291	489	111	709	255	525	155	665	219	561	199	621	373	407	33	787	337	443	77	743	8000
8000	109	711	289	491	145	675	245	535	181	639	201	579	27	793	367	413	63	757	323	457	8000
8000	689	91	509	311	645	135	545	275	601	179	581	239	767	13	427	393	723	57	463	357	8000
8000	511	309	691	89	555	265	655	125	599	221	619	161	433	387	773	7	477	343	737	43	8000
	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000

In this case the magic sum is $S_{20 \times 20} = 8000 = 20^3$, and the sum of all entries is $F_{400} := 20 \times 8000 = 160000 = 400^2 = 20^4$. Each 4×4 block is a **pan diagonal magic square** of order 4 with the same magic sum $S_{4 \times 4} = 1600$. These blocks also have the perfect square sum property, i.e., the sum of all entries in each block is $F_{16} := 4 \times 1600 = 6400 = 80^2$.

4.19 Magic Squares of Order 21

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 21 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.19.1 First Approach: Consecutive Odd Order

Taking $k = 21$ in equation (5), we get

$$\begin{aligned}
 1 + 3 + 5 + \dots + (2 \times 21^2 - 1) &= 21^4 \\
 \Rightarrow 1 + 3 + 5 + \dots + 879 + 881 &= 194481 = 441^2 = 21^4.
 \end{aligned} \tag{34}$$

According to the values given in equation (34), we shall construct the order 21 magic squares in two different ways.

• First Way

Example 51. According to the values given in equation (34), the magic square of order 21 is given by

																					9261
295	311	327	343	359	375	391	785	801	817	833	849	865	881	99	115	131	147	163	179	195	9261
373	389	307	309	325	341	357	863	879	797	799	815	831	847	177	193	111	113	129	145	161	9261
339	355	371	387	305	321	323	829	845	861	877	795	811	813	143	159	175	191	109	125	127	9261
319	335	337	353	369	385	303	809	825	827	843	859	875	793	123	139	141	157	173	189	107	9261
383	301	317	333	349	351	367	873	791	807	823	839	841	857	187	105	121	137	153	155	171	9261
363	365	381	299	315	331	347	853	855	871	789	805	821	837	167	169	185	103	119	135	151	9261
329	345	361	377	379	297	313	819	835	851	867	869	787	803	133	149	165	181	183	101	117	9261
197	213	229	245	261	277	293	393	409	425	441	457	473	489	589	605	621	637	653	669	685	9261
275	291	209	211	227	243	259	471	487	405	407	423	439	455	667	683	601	603	619	635	651	9261
241	257	273	289	207	223	225	437	453	469	485	403	419	421	633	649	665	681	599	615	617	9261
221	237	239	255	271	287	205	417	433	435	451	467	483	401	613	629	631	647	663	679	597	9261
285	203	219	235	251	253	269	481	399	415	431	447	449	465	677	595	611	627	643	645	661	9261
265	267	283	201	217	233	249	461	463	479	397	413	429	445	657	659	675	593	609	625	641	9261
231	247	263	279	281	199	215	427	443	459	475	477	395	411	623	639	655	671	673	591	607	9261
687	703	719	735	751	767	783	1	17	33	49	65	81	97	491	507	523	539	555	571	587	9261
765	781	699	701	717	733	749	79	95	13	15	31	47	63	569	585	503	505	521	537	553	9261
731	747	763	779	697	713	715	45	61	77	93	11	27	29	535	551	567	583	501	517	519	9261
711	727	729	745	761	777	695	25	41	43	59	75	91	9	515	531	533	549	565	581	499	9261
775	693	709	725	741	743	759	89	7	23	39	55	57	73	579	497	513	529	545	547	563	9261
755	757	773	691	707	723	739	69	71	87	5	21	37	53	559	561	577	495	511	527	543	9261
721	737	753	769	771	689	705	35	51	67	83	85	3	19	525	541	557	573	575	493	509	9261
9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261

In this case the magic sum is $S_{21 \times 21} = 9261 = 21^3$, and the sum of all entries is $F_{441} := 21 \times 9261 = 194481 = 441^2 = 21^4$. Moreover, each 7×7 block is a **pan diagonal magic square** with a different magic sum, and the combination of these sums forms another magic square of order 3. See below

Example 52. Each 7×7 blok of the magic square of order 21 given in 51 is a **pan diagonal magic square** with a different magic sum, and the combination of these sums forms another magic square of order 3

$$\begin{array}{|c|c|c|c|} \hline & & & 9261 \\ \hline 2401 & 5831 & 1029 & 9261 \\ \hline 1715 & 3087 & 4459 & 9261 \\ \hline 5145 & 343 & 3773 & 9261 \\ \hline 9261 & 9261 & 9261 & 9261 \\ \hline \end{array} = 7^3 \times \begin{array}{|c|c|c|c|} \hline & & & 27 \\ \hline 7 & 17 & 3 & 27 \\ \hline 5 & 9 & 13 & 27 \\ \hline 15 & 1 & 11 & 27 \\ \hline 27 & 27 & 27 & 27 \\ \hline \end{array}$$

• Second Way

Example 53. According to the values given in equation (34), the magic square of order 21 is given by

																					9261
7	151	295	439	583	727	871	17	161	305	449	593	737	881	3	147	291	435	579	723	867	9261
709	853	115	133	277	421	565	719	863	125	143	287	431	575	705	849	111	129	273	417	561	9261
403	547	691	835	97	241	259	413	557	701	845	107	251	269	399	543	687	831	93	237	255	9261
223	367	385	529	673	817	79	233	377	395	539	683	827	89	219	363	381	525	669	813	75	9261
799	61	205	349	493	511	655	809	71	215	359	503	521	665	795	57	201	345	489	507	651	9261
619	637	781	43	187	331	475	629	647	791	53	197	341	485	615	633	777	39	183	327	471	9261
313	457	601	745	763	25	169	323	467	611	755	773	35	179	309	453	597	741	759	21	165	9261
5	149	293	437	581	725	869	9	153	297	441	585	729	873	13	157	301	445	589	733	877	9261
707	851	113	131	275	419	563	711	855	117	135	279	423	567	715	859	121	139	283	427	571	9261
401	545	689	833	95	239	257	405	549	693	837	99	243	261	409	553	697	841	103	247	265	9261
221	365	383	527	671	815	77	225	369	387	531	675	819	81	229	373	391	535	679	823	85	9261
797	59	203	347	491	509	653	801	63	207	351	495	513	657	805	67	211	355	499	517	661	9261
617	635	779	41	185	329	473	621	639	783	45	189	333	477	625	643	787	49	193	337	481	9261
311	455	599	743	761	23	167	315	459	603	747	765	27	171	319	463	607	751	769	31	175	9261
15	159	303	447	591	735	879	1	145	289	433	577	721	865	11	155	299	443	587	731	875	9261
717	861	123	141	285	429	573	703	847	109	127	271	415	559	713	857	119	137	281	425	569	9261
411	555	699	843	105	249	267	397	541	685	829	91	235	253	407	551	695	839	101	245	263	9261
231	375	393	537	681	825	87	217	361	379	523	667	811	73	227	371	389	533	677	821	83	9261
807	69	213	357	501	519	663	793	55	199	343	487	505	649	803	65	209	353	497	515	659	9261
627	645	789	51	195	339	483	613	631	775	37	181	325	469	623	641	785	47	191	335	479	9261
321	465	609	753	771	33	177	307	451	595	739	757	19	163	317	461	605	749	767	29	173	9261
9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261

In this case the magic sum is $S_{21 \times 21} = 9261 = 21^3$, and the sum of all entries is $F_{441} := 21 \times 9261 = 194481 = 441^2 = 21^4$. Moreover, each 7×7 block is a **pan diagonal magic square** with a different magic sum, and the combination of these sums forms another magic square of order 3. See below

Example 54. Each 7×7 block of the magic square of order 21 given in 53 is a **pan diagonal magic square** with a different magic sum, and the combination of these sums forms another magic square of order 3:

			9261
3073	3143	3045	9261
3059	3087	3115	9261
3129	3031	3101	9261
9261	9261	9261	9261

 $= 7 \times$

			1323
439	449	435	1323
437	441	445	1323
447	433	443	1323
1323	1323	1323	1323

Interestingly, the same happens with the sums of all entries of each block of order 7, which also form a magic square of order 3:

Example 55. The combination of the sums of all the entries of each 7×7 block of the magic square of order 21 given in 53 is once again a magic square of order 3:

			64827
21511	22001	21315	64827
21413	21609	21805	64827
21903	21217	21707	64827
64827	64827	64827	64827

$$= 7^2 \times \begin{array}{|c|c|c|c|} \hline & & & 1323 \\ \hline 439 & 449 & 435 & 1323 \\ \hline 437 & 441 & 445 & 1323 \\ \hline 447 & 433 & 443 & 1323 \\ \hline 1323 & 1323 & 1323 & 1323 \\ \hline \end{array}$$

4.19.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-441)(n-440)}{2} = 225(n-220)$$

Taking $n = 661$, we get a perfect square, i.e.,

$$G := T(661) - T(220) = 194481 = 441^2 = 21^4.$$

Simplifying, we get

$$221 + 222 + 223 + \dots + 660 + 661 = 194481 = 441^2 = 21^4.$$

This gives a perfect square sum for the 441 entries of consecutive natural numbers from 221 to 661.

Example 56. According to the values from 221 to 661 given above, we can easily construct a magic square of order 21 of consecutive natural numbers with a perfect square sum of all entries:

																					9261
368	376	384	392	400	408	416	613	621	629	637	645	653	661	270	278	286	294	302	310	318	9261
407	415	374	375	383	391	399	652	660	619	620	628	636	644	309	317	276	277	285	293	301	9261
390	398	406	414	373	381	382	635	643	651	659	618	626	627	292	300	308	316	275	283	284	9261
380	388	389	397	405	413	372	625	633	634	642	650	658	617	282	290	291	299	307	315	274	9261
412	371	379	387	395	396	404	657	616	624	632	640	641	649	314	273	281	289	297	298	306	9261
402	403	411	370	378	386	394	647	648	656	615	623	631	639	304	305	313	272	280	288	296	9261
385	393	401	409	410	369	377	630	638	646	654	655	614	622	287	295	303	311	312	271	279	9261
319	327	335	343	351	359	367	417	425	433	441	449	457	465	515	523	531	539	547	555	563	9261
358	366	325	326	334	342	350	456	464	423	424	432	440	448	554	562	521	522	530	538	546	9261
341	349	357	365	324	332	333	439	447	455	463	422	430	431	537	545	553	561	520	528	529	9261
331	339	340	348	356	364	323	429	437	438	446	454	462	421	527	535	536	544	552	560	519	9261
363	322	330	338	346	347	355	461	420	428	436	444	445	453	559	518	526	534	542	543	551	9261
353	354	362	321	329	337	345	451	452	460	419	427	435	443	549	550	558	517	525	533	541	9261
336	344	352	360	361	320	328	434	442	450	458	459	418	426	532	540	548	556	557	516	524	9261
564	572	580	588	596	604	612	221	229	237	245	253	261	269	466	474	482	490	498	506	514	9261
603	611	570	571	579	587	595	260	268	227	228	236	244	252	505	513	472	473	481	489	497	9261
586	594	602	610	569	577	578	243	251	259	267	226	234	235	488	496	504	512	471	479	480	9261
576	584	585	593	601	609	568	233	241	242	250	258	266	225	478	486	487	495	503	511	470	9261
608	567	575	583	591	592	600	265	224	232	240	248	249	257	510	469	477	485	493	494	502	9261
598	599	607	566	574	582	590	255	256	264	223	231	239	247	500	501	509	468	476	484	492	9261
581	589	597	605	606	565	573	238	246	254	262	263	222	230	483	491	499	507	508	467	475	9261
9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261

In the examples given above the magic sum is $S_{21 \times 21} = 9261 = 21^3$, and the sum of all entries is $F_{441} := 194481 = 441^2 = 21^4$. In this case, each 7×7 block is a **pan diagonal magic square** that has a different magic sum, and the combination of these sums forms another magic square of order 3:

Example 57. The following magic square of order 3 results from the combination of the different magic sums of each 7×7 **pan diagonal magic square** block of the order 21 magic square in example 56:

$$\begin{array}{|c|c|c|c|} \hline & & & 9261 \\ \hline 2744 & 4459 & 2058 & 9261 \\ \hline 2401 & 3087 & 3773 & 9261 \\ \hline 4116 & 1715 & 3430 & 9261 \\ \hline 9261 & 9261 & 9261 & 9261 \\ \hline \end{array} = 7^3 \times \begin{array}{|c|c|c|c|} \hline & & & 27 \\ \hline 8 & 13 & 6 & 27 \\ \hline 7 & 9 & 11 & 27 \\ \hline 12 & 5 & 10 & 27 \\ \hline 27 & 27 & 27 & 27 \\ \hline \end{array}$$

Interestingly, the same happens with the sums of all entries of each 7×7 block which again form a magic square of order 3.

Example 58. The following magic square of order 3 results from the combination of the different sums of the entries of each 7×7 **pan diagonal magic square** block of the order 21 magic square in example 56:

$$\begin{array}{|c|c|c|c|} \hline & & & 64827 \\ \hline 19208 & 31213 & 14406 & 64827 \\ \hline 16807 & 21609 & 26411 & 64827 \\ \hline 28812 & 12005 & 24010 & 64827 \\ \hline 64827 & 64827 & 64827 & 64827 \\ \hline \end{array} = 7^4 \times \begin{array}{|c|c|c|c|} \hline & & & 27 \\ \hline 8 & 13 & 6 & 27 \\ \hline 7 & 9 & 11 & 27 \\ \hline 12 & 5 & 10 & 27 \\ \hline 27 & 27 & 27 & 27 \\ \hline \end{array}$$

4.19.3 Third Approach: Minimum Perfect Square Sum

The examples given in 51, 53 and 56 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 21, 21^2, 21^3, 21^4 \rangle$. However, the example 56 with consecutive natural numbers does not have a minimum perfect square sum. Let's see the first example of the magic square of order 21 given in section 3:

$$L \left(15^2 + \frac{21^2 - 1}{2}, 21^2 \right) \rightarrow (21, 5, 445, 4725, 99225, 3\sqrt{35}).$$

In this case, we have magic square of order 21 with a perfect square sum of all entries $99225 = 315^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

Example 59. The minimum perfect square sum magic square of order 21 is given by

																					4725
8	80	152	224	296	368	440	13	85	157	229	301	373	445	6	78	150	222	294	366	438	4725
359	431	62	71	143	215	287	364	436	67	76	148	220	292	357	429	60	69	141	213	285	4725
206	278	350	422	53	125	134	211	283	355	427	58	130	139	204	276	348	420	51	123	132	4725
116	188	197	269	341	413	44	121	193	202	274	346	418	49	114	186	195	267	339	411	42	4725
404	35	107	179	251	260	332	409	40	112	184	256	265	337	402	33	105	177	249	258	330	4725
314	323	395	26	98	170	242	319	328	400	31	103	175	247	312	321	393	24	96	168	240	4725
161	233	305	377	386	17	89	166	238	310	382	391	22	94	159	231	303	375	384	15	87	4725
7	79	151	223	295	367	439	9	81	153	225	297	369	441	11	83	155	227	299	371	443	4725
358	430	61	70	142	214	286	360	432	63	72	144	216	288	362	434	65	74	146	218	290	4725
205	277	349	421	52	124	133	207	279	351	423	54	126	135	209	281	353	425	56	128	137	4725
115	187	196	268	340	412	43	117	189	198	270	342	414	45	119	191	200	272	344	416	47	4725
403	34	106	178	250	259	331	405	36	108	180	252	261	333	407	38	110	182	254	263	335	4725
313	322	394	25	97	169	241	315	324	396	27	99	171	243	317	326	398	29	101	173	245	4725
160	232	304	376	385	16	88	162	234	306	378	387	18	90	164	236	308	380	389	20	92	4725
12	84	156	228	300	372	444	5	77	149	221	293	365	437	10	82	154	226	298	370	442	4725
363	435	66	75	147	219	291	356	428	59	68	140	212	284	361	433	64	73	145	217	289	4725
210	282	354	426	57	129	138	203	275	347	419	50	122	131	208	280	352	424	55	127	136	4725
120	192	201	273	345	417	48	113	185	194	266	338	410	41	118	190	199	271	343	415	46	4725
408	39	111	183	255	264	336	401	32	104	176	248	257	329	406	37	109	181	253	262	334	4725
318	327	399	30	102	174	246	311	320	392	23	95	167	239	316	325	397	28	100	172	244	4725
165	237	309	381	390	21	93	158	230	302	374	383	14	86	163	235	307	379	388	19	91	4725
4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725	4725

The above example 59 is with magic sum $S_{21 \times 21} = 4725$, and the sum of all entries is $F_{441} := 21 \times 1815 = 99225 = 315^2$. In this case each 7×7 block is a **pan diagonal magic square** with a different magic sum, and the combination of these sums forms another magic square of order 3:

Example 60. The following magic square of order 3 results from the combination of the different magic sums of each 7×7 **pan diagonal magic square** block of the order 21 magic square:

			4725
1568	1603	1554	4725
1561	1575	1589	4725
1596	1547	1582	4725
4725	4725	4725	4725

 $= 7 \times$

			675
224	229	222	675
223	225	227	675
228	221	226	675
675	675	675	675

Interestingly, the same happens with the sums of all entries of each 7×7 block which again form a magic square of order 3.

Example 61. The following magic square of order 3 results from the combination of the different sums of the entries of each 7×7 **pan diagonal magic square** block of the order 21 magic square in example 59:

59 is given by

			33075
10976	11221	10878	33075
10927	11025	11123	33075
11172	10829	11074	33075
33075	33075	33075	33075

$$= 7^2 \times \begin{array}{|c|c|c|c|} \hline & & & 675 \\ \hline 224 & 229 & 222 & 675 \\ \hline 223 & 225 & 227 & 675 \\ \hline 228 & 221 & 226 & 675 \\ \hline 675 & 675 & 675 & 675 \\ \hline \end{array}$$

The **block-wise** constructions of magic squares of order 21 given in examples 51, 53, 56 and 59 are the compositions of magic squares of order 7 given in subsection 4.5 together with magic squares of order 3 given in subsection 4.1.

4.20 Magic Squares of Order 22

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 22 with this property if we use odd number entries. Taking $k = 22$ in equation (5), we get

$$\begin{aligned} 1 + 3 + 5 + \dots + (2 \times 22^2 - 1) &= 22^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 965 + 967 &= 234256 = 484^2 = 22^4. \end{aligned} \quad (35)$$

Example 62. According to the values given in equation (35), the magic square of order 22 for odd numbers is given by

1	187	151	399	227	773	953	913	863	823	49	97	475	523	395	345	305	549	719	683	629	589	10648
229	47	11	195	401	869	819	779	949	909	95	133	351	301	481	519	391	639	585	545	725	675	10648
403	221	93	55	19	955	905	865	825	775	141	179	525	387	347	307	477	721	681	631	595	541	10648
63	405	223	139	99	821	781	951	911	861	177	5	303	483	521	393	343	587	551	717	677	637	10648
143	107	397	225	185	907	867	817	777	957	3	51	389	349	299	479	527	673	633	593	543	727	10648
563	743	703	653	613	231	499	459	419	365	325	285	749	929	889	839	799	25	205	165	115	75	10648
659	609	569	739	699	331	277	237	505	451	411	371	845	795	755	925	885	121	71	31	201	161	10648
745	695	655	615	565	417	363	323	283	243	497	457	931	881	841	801	751	207	157	117	77	27	10648
611	571	741	701	651	503	463	409	369	329	275	235	797	757	927	887	837	73	33	203	163	113	10648
697	657	607	567	747	281	241	495	455	415	375	321	883	843	793	753	933	159	119	69	29	209	10648
91	137	183	9	45	367	327	287	233	501	461	407	643	689	735	561	597	873	919	965	791	827	10648
181	7	53	89	135	453	413	373	319	279	239	507	733	559	605	641	687	963	789	835	871	917	10648
769	939	903	849	809	35	215	175	125	85	601	649	553	715	679	423	251	441	489	361	311	271	10648
859	805	765	945	895	131	81	41	211	171	647	685	253	599	539	723	425	317	267	447	485	357	10648
941	901	851	815	761	217	167	127	87	37	693	731	427	245	645	583	547	491	353	313	273	443	10648
807	771	937	897	857	83	43	213	173	123	729	557	591	429	247	691	627	269	449	487	359	309	10648
893	853	813	763	947	169	129	79	39	219	555	603	671	635	421	249	737	355	315	265	445	493	10648
465	513	385	335	295	529	709	669	619	579	831	879	21	191	155	101	61	783	935	899	433	261	10648
341	291	471	509	381	625	575	535	705	665	877	915	111	57	17	197	147	263	829	759	943	435	10648
515	377	337	297	467	711	661	621	581	531	923	961	193	153	103	67	13	437	255	875	803	767	10648
293	473	511	383	333	577	537	707	667	617	959	787	59	23	189	149	109	811	439	257	921	847	10648
379	339	289	469	517	663	623	573	533	713	785	833	145	105	65	15	199	891	855	431	259	967	10648

This magic square example has a magic sum $S_{22 \times 22} := 10648 = 22^3$ and the sum of all 484 entries is $F_{484} := 234256 = 484^2 = 22^4$. There are 13 blocks of order 4 and 1 of order 7 which are magic squares with their entries written in **bold face letters**. The approach applied to construct this magic square is based on the procedure of H. White [10].

4.21 Magic Squares of Order 23

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 23 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.21.1 First Approach: Consecutive Odd Numbers

Taking $k = 23$ in equation (5), we get

$$1 + 3 + 5 + \dots + (2 \times 23^2 - 1) = 23^4$$

$$\Rightarrow 1 + 3 + 5 + \dots + 1055 + 1057 = 279841 = 529^2 = 23^4 \quad (36)$$

Example 63. According to the 529 values given in equation (36), the *pan diagonal magic square* of order 23 is given by

	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	
	1	89	131	173	215	257	299	341	383	425	467	509	597	639	681	723	765	807	849	891	933	975	1017	12167
12167	969	1057	41	83	125	167	209	251	293	335	377	419	461	549	591	633	675	717	759	801	843	885	927	12167
12167	879	921	1009	1051	35	77	119	161	203	245	287	329	371	459	501	543	585	627	669	711	753	795	837	12167
12167	789	831	919	961	1003	1045	29	71	113	155	197	239	281	323	411	453	495	537	579	621	663	705	747	12167
12167	699	741	783	871	913	955	997	1039	23	65	107	149	191	233	321	363	405	447	489	531	573	615	657	12167
12167	609	651	693	781	823	865	907	949	991	1033	17	59	101	143	185	273	315	357	399	441	483	525	567	12167
12167	519	561	603	645	733	775	817	859	901	943	985	1027	11	53	95	183	225	267	309	351	393	435	477	12167
12167	429	471	513	555	643	685	727	769	811	853	895	937	979	1021	5	47	135	177	219	261	303	345	387	12167
12167	339	381	423	465	507	595	637	679	721	763	805	847	889	931	973	1015	45	87	129	171	213	255	297	12167
12167	249	291	333	375	417	505	547	589	631	673	715	757	799	841	883	925	967	1055	39	81	123	165	207	12167
12167	159	201	243	285	327	369	457	499	541	583	625	667	709	751	793	835	877	965	1007	1049	33	75	117	12167
12167	69	111	153	195	237	279	367	409	451	493	535	577	619	661	703	745	787	829	917	959	1001	1043	27	12167
12167	1037	21	63	105	147	189	231	319	361	403	445	487	529	571	613	655	697	739	827	869	911	953	995	12167
12167	947	989	1031	15	57	99	141	229	271	313	355	397	439	481	523	565	607	649	691	779	821	863	905	12167
12167	857	899	941	983	1025	9	51	93	181	223	265	307	349	391	433	475	517	559	601	689	731	773	815	12167
12167	767	809	851	893	935	977	1019	3	91	133	175	217	259	301	343	385	427	469	511	553	641	683	725	12167
12167	677	719	761	803	845	887	929	971	1013	43	85	127	169	211	253	295	337	379	421	463	551	593	635	12167
12167	587	629	671	713	755	797	839	881	923	1011	1053	37	79	121	163	205	247	289	331	373	415	503	545	12167
12167	497	539	581	623	665	707	749	791	833	875	963	1005	1047	31	73	115	157	199	241	283	325	413	455	12167
12167	407	449	491	533	575	617	659	701	743	785	873	915	957	999	1041	25	67	109	151	193	235	277	365	12167
12167	317	359	401	443	485	527	569	611	653	695	737	825	867	909	951	993	1035	19	61	103	145	187	275	12167
12167	227	269	311	353	395	437	479	521	563	605	647	735	777	819	861	903	945	987	1029	13	55	97	139	12167
12167	137	179	221	263	305	347	389	431	473	515	557	599	687	729	771	813	855	897	939	981	1023	7	49	12167
12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167

4.21.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-529)(n-528)}{2} = 529(n-264)$$

Taking $n = 793$, we get a perfect square, i.e.,

$$G := T(793) - T(264) = \frac{793 \times 794}{2} - \frac{264 \times 265}{2} = 793 \times 393 - 132 \times 265$$

$$= 16471 - 1830 = 279841 = 529^2 = 23^4.$$

Simplifying, we get

$$265 + 266 + 267 + \dots + 792 + 793 = 279841 = 529^2 = 23^4.$$

This gives a perfect square sum for the 529 consecutive natural numbers from 265 to 793.

Example 64. The *pan diagonal magic square* of order 23 for the 529 entries from 265 to 793 is given by

	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167
	265	309	330	351	372	393	414	435	456	477	498	519	563	584	605	626	647	668	689	710	731	752	773	12167
12167	749	793	285	306	327	348	369	390	411	432	453	474	495	539	560	581	602	623	644	665	686	707	728	12167
12167	704	725	769	790	282	303	324	345	366	387	408	429	450	494	515	536	557	578	599	620	641	662	683	12167
12167	659	680	724	745	766	787	279	300	321	342	363	384	405	426	470	491	512	533	554	575	596	617	638	12167
12167	614	635	656	700	721	742	763	784	276	297	318	339	360	381	425	446	467	488	509	530	551	572	593	12167
12167	569	590	611	655	676	697	718	739	760	781	273	294	315	336	357	401	422	443	464	485	506	527	548	12167
12167	524	545	566	587	631	652	673	694	715	736	757	778	270	291	312	356	377	398	419	440	461	482	503	12167
12167	479	500	521	542	586	607	628	649	670	691	712	733	754	775	267	288	332	353	374	395	416	437	458	12167
12167	434	455	476	497	518	562	583	604	625	646	667	688	709	730	751	772	287	308	329	350	371	392	413	12167
12167	389	410	431	452	473	517	538	559	580	601	622	643	664	685	706	727	748	792	284	305	326	347	368	12167
12167	344	365	386	407	428	449	493	514	535	556	577	598	619	640	661	682	703	747	768	789	281	302	323	12167
12167	299	320	341	362	383	404	448	469	490	511	532	553	574	595	616	637	658	679	723	744	765	786	278	12167
12167	783	275	296	317	338	359	380	424	445	466	487	508	529	550	571	592	613	634	678	699	720	741	762	12167
12167	738	759	780	272	293	314	335	379	400	421	442	463	484	505	526	547	568	589	610	654	675	696	717	12167
12167	693	714	735	756	777	269	290	311	355	376	397	418	439	460	481	502	523	544	565	609	630	651	672	12167
12167	648	669	690	711	732	753	774	266	310	331	352	373	394	415	436	457	478	499	520	541	585	606	627	12167
12167	603	624	645	666	687	708	729	750	771	286	307	328	349	370	391	412	433	454	475	496	540	561	582	12167
12167	558	579	600	621	642	663	684	705	726	770	791	283	304	325	346	367	388	409	430	451	472	516	537	12167
12167	513	534	555	576	597	618	639	660	681	702	746	767	788	280	301	322	343	364	385	406	427	471	492	12167
12167	468	489	510	531	552	573	594	615	636	657	701	722	743	764	785	277	298	319	340	361	382	403	447	12167
12167	423	444	465	486	507	528	549	570	591	612	633	677	698	719	740	761	782	274	295	316	337	358	402	12167
12167	378	399	420	441	462	483	504	525	546	567	588	632	653	674	695	716	737	758	779	271	292	313	334	12167
12167	333	354	375	396	417	438	459	480	501	522	543	564	608	629	650	671	692	713	734	755	776	268	289	12167
	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167	12167

In both the examples 63 and 64 the magic sum is $S_{23 \times 23} = 12167 = 23^3$, and the sum of all numbers is $F_{529} := 279841 = 529^2 = 23^4$.

Both of the examples 63 and 64 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 23, 23^2, 23^3, 23^4 \rangle$.

4.21.3 Third Approach: Minimum Perfect Square Sum

The examples given in 63 and 64 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 23, 23^2, 23^3, 23^4 \rangle$. However, the sum of the magic square entries of 64 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 23 given in section 3:

$$L \left(17^2 + \frac{23^2 - 1}{2}, 23^2 \right) \rightarrow (23, 25, 553, 6647, 152881, \sqrt{391})$$

In this case, we have magic square of order 23 with sum of all entries $152881 = 391^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

Example 65. The *pan diagonal magic square* of order 23 with 529 entries starting from 25 to 553 is given by

	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647
6647	25	69	90	111	132	153	174	195	216	237	258	279	323	344	365	386	407	428	449	470	491	512	533	6647
	509	553	45	66	87	108	129	150	171	192	213	234	255	299	320	341	362	383	404	425	446	467	488	6647
6647	464	485	529	550	42	63	84	105	126	147	168	189	210	254	275	296	317	338	359	380	401	422	443	6647
6647	419	440	484	505	526	547	39	60	81	102	123	144	165	186	230	251	272	293	314	335	356	377	398	6647
6647	374	395	416	460	481	502	523	544	36	57	78	99	120	141	185	206	227	248	269	290	311	332	353	6647
6647	329	350	371	415	436	457	478	499	520	541	33	54	75	96	117	161	182	203	224	245	266	287	308	6647
6647	284	305	326	347	391	412	433	454	475	496	517	538	30	51	72	116	137	158	179	200	221	242	263	6647
6647	239	260	281	302	346	367	388	409	430	451	472	493	514	535	27	48	92	113	134	155	176	197	218	6647
6647	194	215	236	257	278	322	343	364	385	406	427	448	469	490	511	532	47	68	89	110	131	152	173	6647
6647	149	170	191	212	233	277	298	319	340	361	382	403	424	445	466	487	508	552	44	65	86	107	128	6647
6647	104	125	146	167	188	209	253	274	295	316	337	358	379	400	421	442	463	507	528	549	41	62	83	6647
6647	59	80	101	122	143	164	208	229	250	271	292	313	334	355	376	397	418	439	483	504	525	546	38	6647
6647	543	35	56	77	98	119	140	184	205	226	247	268	289	310	331	352	373	394	438	459	480	501	522	6647
6647	498	519	540	32	53	74	95	139	160	181	202	223	244	265	286	307	328	349	370	414	435	456	477	6647
6647	453	474	495	516	537	29	50	71	115	136	157	178	199	220	241	262	283	304	325	369	390	411	432	6647
6647	408	429	450	471	492	513	534	26	70	91	112	133	154	175	196	217	238	259	280	301	345	366	387	6647
6647	363	384	405	426	447	468	489	510	531	46	67	88	109	130	151	172	193	214	235	256	300	321	342	6647
6647	318	339	360	381	402	423	444	465	486	530	551	43	64	85	106	127	148	169	190	211	232	276	297	6647
6647	273	294	315	336	357	378	399	420	441	462	506	527	548	40	61	82	103	124	145	166	187	231	252	6647
6647	228	249	270	291	312	333	354	375	396	417	461	482	503	524	545	37	58	79	100	121	142	163	207	6647
6647	183	204	225	246	267	288	309	330	351	372	393	437	458	479	500	521	542	34	55	76	97	118	162	6647
6647	138	159	180	201	222	243	264	285	306	327	348	392	413	434	455	476	497	518	539	31	52	73	94	6647
6647	93	114	135	156	177	198	219	240	261	282	303	324	368	389	410	431	452	473	494	515	536	28	49	6647
	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647	6647

The above example 65 has a magic sum $S_{23 \times 23} = 6647$, and the sum of all entries is $F_{529} := 23 \times 6647 = 152881 = 391^2$. Moreover, it is **pan diagonal**.

4.22 Magic Squares of Order 24

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 24 with this property if we use odd number entries. Taking $k = 24$ in equation (5), we get

$$\begin{aligned} 1 + 3 + 5 + \dots + (2 \times 24^2 - 1) &= 24^4 \\ \Rightarrow 1 + 3 + 5 + \dots + 1149 + 1151 &= 331776 = 576^2 = 24^4. \end{aligned} \quad (37)$$

There are many ways of constructing magic squares of order 24 with odd numbers and these include for example, 8×3 , 3×8 , 4×6 or 6×4 . Instead of using all of these four ways, we shall use only two. Using the 8×3 format, we can work with the bimagic square of order 8 as given in subsection 4.6.

- **First Way:** 8×3

Example 66. According to the values given in equation (35), the magic square of order 24 with odd numbers is given by

277	727	637	79	475	1105	979	313	287	737	647	89	485	1115	989	323	273	723	633	75	471	1101	975	309	13824
457	1123	961	331	223	781	583	133	467	1133	971	341	233	791	593	143	453	1119	957	327	219	777	579	129	13824
7	709	799	205	385	907	1033	547	17	719	809	215	395	917	1043	557	3	705	795	201	381	903	1029	543	13824
403	889	1051	529	61	655	853	151	413	899	1061	539	71	665	863	161	399	885	1047	525	57	651	849	147	13824
673	43	169	835	871	421	511	1069	683	53	179	845	881	431	521	1079	669	39	165	831	867	417	507	1065	13824
925	367	565	1015	691	25	187	817	935	377	575	1025	701	35	197	827	921	363	561	1011	687	21	183	813	13824
763	241	115	601	1141	439	349	943	773	251	125	611	1151	449	359	953	759	237	111	597	1137	435	345	939	13824
1087	493	295	997	745	259	97	619	1097	503	305	1007	755	269	107	629	1083	489	291	993	741	255	93	615	13824
275	725	635	77	473	1103	977	311	279	729	639	81	477	1107	981	315	283	733	643	85	481	1111	985	319	13824
455	1121	959	329	221	779	581	131	459	1125	963	333	225	783	585	135	463	1129	967	337	229	787	589	139	13824
5	707	797	203	383	905	1031	545	9	711	801	207	387	909	1035	549	13	715	805	211	391	913	1039	553	13824
401	887	1049	527	59	653	851	149	405	891	1053	531	63	657	855	153	409	895	1057	535	67	661	859	157	13824
671	41	167	833	869	419	509	1067	675	45	171	837	873	423	513	1071	679	49	175	841	877	427	517	1075	13824
923	365	563	1013	689	23	185	815	927	369	567	1017	693	27	189	819	931	373	571	1021	697	31	193	823	13824
761	239	113	599	1139	437	347	941	765	243	117	603	1143	441	351	945	769	247	121	607	1147	445	355	949	13824
1085	491	293	995	743	257	95	617	1089	495	297	999	747	261	99	621	1093	499	301	1003	751	265	103	625	13824
285	735	645	87	483	1113	987	321	271	721	631	73	469	1099	973	307	281	731	641	83	479	1109	983	317	13824
465	1131	969	339	231	789	591	141	451	1117	955	325	217	775	577	127	461	1127	965	335	227	785	587	137	13824
15	717	807	213	393	915	1041	555	1	703	793	199	379	901	1027	541	11	713	803	209	389	911	1037	551	13824
411	897	1059	537	69	663	861	159	397	883	1045	523	55	649	847	145	407	893	1055	533	65	659	857	155	13824
681	51	177	843	879	429	519	1077	667	37	163	829	865	415	505	1063	677	47	173	839	875	425	515	1073	13824
933	375	573	1023	699	33	195	825	919	361	559	1009	685	19	181	811	929	371	569	1019	695	29	191	821	13824
771	249	123	609	1149	447	357	951	757	235	109	595	1135	433	343	937	767	245	119	605	1145	443	353	947	13824
1095	501	303	1005	753	267	105	627	1081	487	289	991	739	253	91	613	1091	497	299	1001	749	263	101	623	13824
13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824

The above example 66 is with magic sum $S_{24 \times 24} = 13824 = 24^3$ and the sum of all entries is $F_{576} := 24 \times 13824 = 331776 = 576^2 = 24^4$.

The block-wise construction of the magic square of order 24 in example 66 is the composition of **bimagic** squares of order 8 given in subsection 4.6 with a magic square of order 3 given in subsection 4.1.

The magic square of order 3 presented below is formed by the combination of the different **bimagic** squares of order 8 which compose the magic square of order 24

Example 67. Each 8×8 block is a bimagic square forms again a magic square of order 3 given by

$$\begin{array}{|c|c|c|c|} \hline & & & 13824 \\ \hline 4592 & 4672 & 4560 & 13824 \\ \hline 4576 & 4608 & 4640 & 13824 \\ \hline 4656 & 4544 & 4624 & 13824 \\ \hline 13824 & 13824 & 13824 & 13824 \\ \hline \end{array} = 2^4 \times \begin{array}{|c|c|c|c|} \hline & & & 864 \\ \hline 287 & 292 & 285 & 864 \\ \hline 286 & 288 & 290 & 864 \\ \hline 291 & 284 & 289 & 864 \\ \hline 864 & 864 & 864 & 864 \\ \hline \end{array}$$

• **Second Way:** 4×6

Example 68. A magic square of order 24 formed by 24 blocks of *perfect square pan diagonal magic squares* of order 4:

433	793	1	937	477	837	45	981	487	847	55	991	499	859	67	1003	465	825	33	969	447	807	15	951	13824
73	865	505	721	117	909	549	765	127	919	559	775	139	931	571	787	105	897	537	753	87	879	519	735	13824
1081	145	649	289	1125	189	693	333	1135	199	703	343	1147	211	715	355	1113	177	681	321	1095	159	663	303	13824
577	361	1009	217	621	405	1053	261	631	415	1063	271	643	427	1075	283	609	393	1041	249	591	375	1023	231	13824
489	849	57	993	445	805	13	949	501	861	69	1005	459	819	27	963	473	833	41	977	441	801	9	945	13824
129	921	561	777	85	877	517	733	141	933	573	789	99	891	531	747	113	905	545	761	81	873	513	729	13824
1137	201	705	345	1093	157	661	301	1149	213	717	357	1107	171	675	315	1121	185	689	329	1089	153	657	297	13824
633	417	1065	273	589	373	1021	229	645	429	1077	285	603	387	1035	243	617	401	1049	257	585	369	1017	225	13824
455	815	23	959	443	803	11	947	457	817	25	961	485	845	53	989	493	853	61	997	475	835	43	979	13824
95	887	527	743	83	875	515	731	97	889	529	745	125	917	557	773	133	925	565	781	115	907	547	763	13824
1103	167	671	311	1091	155	659	299	1105	169	673	313	1133	197	701	341	1141	205	709	349	1123	187	691	331	13824
599	383	1031	239	587	371	1019	227	601	385	1033	241	629	413	1061	269	637	421	1069	277	619	403	1051	259	13824
495	855	63	999	463	823	31	967	439	799	7	943	479	839	47	983	451	811	19	955	481	841	49	985	13824
135	927	567	783	103	895	535	751	79	871	511	727	119	911	551	767	91	883	523	739	121	913	553	769	13824
1143	207	711	351	1111	175	679	319	1087	151	655	295	1127	191	695	335	1099	163	667	307	1129	193	697	337	13824
639	423	1071	279	607	391	1039	247	583	367	1015	223	623	407	1055	263	595	379	1027	235	625	409	1057	265	13824
469	829	37	973	497	857	65	1001	453	813	21	957	437	797	5	941	491	851	59	995	461	821	29	965	13824
109	901	541	757	137	929	569	785	93	885	525	741	77	869	509	725	131	923	563	779	101	893	533	749	13824
1117	181	685	325	1145	209	713	353	1101	165	669	309	1085	149	653	293	1139	203	707	347	1109	173	677	317	13824
613	397	1045	253	641	425	1073	281	597	381	1029	237	581	365	1013	221	635	419	1067	275	605	389	1037	245	13824
467	827	35	971	483	843	51	987	471	831	39	975	449	809	17	953	435	795	3	939	503	863	71	1007	13824
107	899	539	755	123	915	555	771	111	903	543	759	89	881	521	737	75	867	507	723	143	935	575	791	13824
1115	179	683	323	1131	195	699	339	1119	183	687	327	1097	161	665	305	1083	147	651	291	1151	215	719	359	13824
611	395	1043	251	627	411	1059	267	615	399	1047	255	593	377	1025	233	579	363	1011	219	647	431	1079	287	13824
13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824	13824

The above example 68 has a magic sum $S_{24 \times 24} = 13824 = 24^3$, and the sum of all entries is $F_{576} := 24 \times 13824 = 331776 = 576^2 = 24^4$.

The block-wise construction of the magic square of order 24 in example 68 is the composition of magic squares of order 4 given in subsection 4.2 with magic square of order 6 given in subsection 4.4. Each 4×4 block is a perfect square pan diagonal magic square of order 4 which has a different magic sum. In turn, when combined, these magic sums make another magic square of order 6:

Example 69. According to the values given in equation (19), the magic square of order 6 is given by

						13824
2164	2340	2380	2428	2292	2220	13824
2388	2212	2436	2268	2324	2196	13824
2252	2204	2260	2372	2404	2332	13824
2412	2284	2188	2348	2236	2356	13824
2308	2420	2244	2180	2396	2276	13824
2300	2364	2316	2228	2172	2444	13824
13824	13824	13824	13824	13824	13824	13824

						3456
541	585	595	607	573	555	3456
597	553	609	567	581	549	3456
563	551	565	593	601	583	3456
603	571	547	587	559	589	3456
577	605	561	545	599	569	3456
575	591	579	557	543	611	3456
3456	3456	3456	3456	3456	3456	3456

= 4 ×

4.23 Bimagic Squares of Order 25

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 25 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

4.23.1 First Approach: Consecutive Odd Numbers

We will obtain an odd numbered **bimagic square** of order 25. Taking $k = 25$ in equation (5), we get

$$1 + 3 + 5 + \dots + (2 \times 25^2 - 1) = 25^4$$

$$\Rightarrow 1 + 3 + 5 + \dots + 1247 + 1249 = 390625 = 625^2 = 25^4. \quad (38)$$

Example 70. According to values given in (38), the magic square of order 25 is given by

1	337	613	949	1225	885	1161	247	273	559	469	545	821	1107	183	1093	129	405	731	767	677	953	1039	65	391
913	1249	25	301	637	297	573	859	1185	211	1121	157	483	519	845	705	781	1067	143	429	89	365	691	977	1003
325	601	937	1213	49	1159	235	261	597	873	533	819	1145	171	457	117	443	729	755	1081	991	1027	53	389	665
1237	13	349	625	901	561	897	1173	209	285	195	471	507	833	1119	779	1055	131	417	743	353	689	965	1041	77
649	925	1201	37	313	223	259	585	861	1197	807	1133	169	495	521	431	717	793	1079	105	1015	91	377	653	989
1069	145	421	707	783	693	979	1005	81	367	27	303	639	915	1241	851	1187	213	299	575	485	511	847	1123	159
721	757	1083	119	445	55	381	667	993	1029	939	1215	41	327	603	263	599	875	1151	237	1147	173	459	535	811
133	419	745	771	1057	967	1043	79	355	681	341	627	903	1239	15	1175	201	287	563	899	509	835	1111	197	473
795	1071	107	433	719	379	655	981	1017	93	1203	39	315	641	927	587	863	1199	225	251	161	497	523	809	1135
407	733	769	1095	121	1031	67	393	679	955	615	941	1227	3	339	249	275	551	887	1163	823	1109	185	461	547
877	1153	239	265	591	451	537	813	1149	175	1085	111	447	723	759	669	995	1021	57	383	43	329	605	931	1217
289	565	891	1177	203	1113	199	475	501	837	747	773	1059	135	411	71	357	683	969	1045	905	1231	17	343	629
1191	227	253	589	865	525	801	1137	163	499	109	435	711	797	1073	983	1019	95	371	657	317	643	929	1205	31
553	889	1165	241	277	187	463	549	825	1101	761	1097	123	409	735	395	671	957	1033	69	1229	5	331	617	943
215	291	577	853	1189	849	1125	151	487	513	423	709	785	1061	147	1007	83	369	695	971	631	917	1243	29	305
685	961	1047	73	359	19	345	621	907	1233	893	1179	205	281	567	477	503	839	1115	191	1051	137	413	749	775
97	373	659	985	1011	921	1207	33	319	645	255	581	867	1193	229	1139	165	491	527	803	713	799	1075	101	437
959	1035	61	397	673	333	619	945	1221	7	1167	243	279	555	881	541	827	1103	189	465	125	401	737	763	1099
361	697	973	1009	85	1245	21	307	633	919	579	855	1181	217	293	153	489	515	841	1127	787	1063	149	425	701
1023	59	385	661	997	607	933	1219	45	321	231	267	593	879	1155	815	1141	177	453	539	449	725	751	1087	113
493	529	805	1131	167	1077	103	439	715	791	651	987	1013	99	375	35	311	647	923	1209	869	1195	221	257	583
1105	181	467	543	829	739	765	1091	127	403	63	399	675	951	1037	947	1223	9	335	611	271	557	883	1169	245
517	843	1129	155	481	141	427	703	789	1065	975	1001	87	363	699	309	635	911	1247	23	1183	219	295	571	857
179	455	531	817	1143	753	1089	115	441	727	387	663	999	1025	51	1211	47	323	609	935	595	871	1157	233	269
831	1117	193	479	505	415	741	777	1053	139	1049	75	351	687	963	623	909	1235	11	347	207	283	569	895	1171

In this example the magic sum is $S_{25 \times 25} := 15625 = 25^3$ and the sum of all 625 entries is $F_{625} := 390625 = 625^2 = 25^4$. The magic square is both **pan diagonal** and **bimagic** with a bimagic sum $Sb_{25 \times 25} := 13020825$. Moreover, each 5×5 block is a **pan diagonal magic square** with a same magic sum $S_{5 \times 5} := 3125 = 5^3$ and with a same perfect square sum $S_{25} := 15625 = 5^4$ for their entries.

4.23.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-625)(n-624)}{2} = 625(n-312)$$

Taking $n = 937$, we get a perfect square, i.e.,

$$G := T(937) - T(312) = \frac{937 \times 938}{2} - \frac{312 \times 313}{2} = 390625 = 625^2 = 25^4.$$

Simplifying, we get

$$313 + 314 + 115 + \dots + 936 + 937 = 390625 = 625^2 = 25^4.$$

This gives a perfect square sum for the 625 entries of consecutive natural numbers from 313 to 937.

Example 71. According to the values given above, using consecutive natural numbers we can construct a magic square of order 25 with a perfect square sum of all entries:

313	481	619	787	925	755	893	436	449	592	547	585	723	866	404	859	377	515	678	696	651	789	832	345	508
769	937	325	463	631	461	599	742	905	418	873	391	554	572	735	665	703	846	384	527	357	495	658	801	814
475	613	781	919	337	892	430	443	611	749	579	722	885	398	541	371	534	677	690	853	808	826	339	507	645
931	319	487	625	763	593	761	899	417	455	410	548	566	729	872	702	840	378	521	684	489	657	795	833	351
637	775	913	331	469	424	442	605	743	911	716	879	397	560	573	528	671	709	852	365	820	358	501	639	807
847	385	523	666	704	659	802	815	353	496	326	464	632	770	933	738	906	419	462	600	555	568	736	874	392
673	691	854	372	535	340	503	646	809	827	782	920	333	476	614	444	612	750	888	431	886	399	542	580	718
379	522	685	698	841	796	834	352	490	653	483	626	764	932	320	900	413	456	594	762	567	730	868	411	549
710	848	366	529	672	502	640	803	821	359	914	332	470	633	776	606	744	912	425	438	393	561	574	717	880
516	679	697	860	373	828	346	509	652	790	620	783	926	314	482	437	450	588	756	894	724	867	405	543	586
751	889	432	445	608	538	581	719	887	400	855	368	536	674	692	647	810	823	341	504	334	477	615	778	921
457	595	758	901	414	869	412	550	563	731	686	699	842	380	518	348	491	654	797	835	765	928	321	484	627
908	426	439	607	745	575	713	881	394	562	367	530	668	711	849	804	822	360	498	641	471	634	777	915	328
589	757	895	433	451	406	544	587	725	863	693	861	374	517	680	510	648	791	829	347	927	315	478	621	784
420	458	601	739	907	737	875	388	556	569	524	667	705	843	386	816	354	497	660	798	628	771	934	327	465
655	793	836	349	492	322	485	623	766	929	759	902	415	453	596	551	564	732	870	408	838	381	519	687	700
361	499	642	805	818	773	916	329	472	635	440	603	746	909	427	882	395	558	576	714	669	712	850	363	531
792	830	343	511	649	479	622	785	923	316	896	434	452	590	753	583	726	864	407	545	375	513	681	694	862
493	661	799	817	355	935	323	466	629	772	602	740	903	421	459	389	557	570	733	876	706	844	387	525	663
824	342	505	643	811	616	779	922	335	473	428	446	609	752	890	720	883	401	539	582	537	675	688	856	369
559	577	715	878	396	851	364	532	670	708	638	806	819	362	500	330	468	636	774	917	747	910	423	441	604
865	403	546	584	727	682	695	858	376	514	344	512	650	788	831	786	924	317	480	618	448	591	754	897	435
571	734	877	390	553	383	526	664	707	845	800	813	356	494	662	467	630	768	936	324	904	422	460	598	741
402	540	578	721	884	689	857	370	533	676	506	644	812	825	338	918	336	474	617	780	610	748	891	429	447
728	871	409	552	565	520	683	701	839	382	837	350	488	656	794	624	767	930	318	486	416	454	597	760	898

In this example the magic sum is $S_{25 \times 25} := 15625 = 25^3$ and the sum of all 625 entries is $F_{625} := 390625 = 625^2 = 25^4$. The magic square is **pan diagonal and bimagic** with a imagic sum $Sb_{25 \times 25} := 10579425$. Moreover, each of the 25 5×5 blocks is a pan diagonal magic square with the same magic sum $S_{5 \times 5} := 3125 = 5^3$ and with the same perfect square sum $S_{25} := 15625 = 5^4$ for their entries. Apart from their contrasting consecutive odd or natural number constructions, the main difference between the examples 70 and 71 is the **bimagic sum**. What is more, both 70 and 71 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 25, 25^2, 25^3, 25^4 \rangle$.

4.23.3 Third Approach: Minimum Perfect Square Sum

The examples given in 70 and 71 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 25, 25^2, 25^3, 25^4 \rangle$. Moreover each block of order 5 also has the same property, $\langle 5, 5^2, 5^3, 5^4 \rangle$. However, the example 71

does not have a minimum perfect square sum for its entries. Let's consider the first example of magic square order 25 given in section 3:

$$L\left(18^2 + \frac{25^2 - 1}{2}, 25^2\right) \rightarrow (25, 12, 636, 8100, 202500, 15\sqrt{2})$$

In this case, we obtain a magic square of order 25 with a perfect square sum $202500 = 450^2$ of all entries, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries.

Example 72. According to the values given above, we can construct magic square of order 25 of consecutive natural numbers from 12 to 636 with a perfect square sum of all entries:

12	180	318	486	624	454	592	135	148	291	246	284	422	565	103	558	76	214	377	395	350	488	531	44	207
468	636	24	162	330	160	298	441	604	117	572	90	253	271	434	364	402	545	83	226	56	194	357	500	513
174	312	480	618	36	591	129	142	310	448	278	421	584	97	240	70	233	376	389	552	507	525	38	206	344
630	18	186	324	462	292	460	598	116	154	109	247	265	428	571	401	539	77	220	383	188	356	494	532	50
336	474	612	30	168	123	141	304	442	610	415	578	96	259	272	227	370	408	551	64	519	57	200	338	506
546	84	222	365	403	358	501	514	52	195	25	163	331	469	632	437	605	118	161	299	254	267	435	573	91
372	390	553	71	234	39	202	345	508	526	481	619	32	175	313	143	311	449	587	130	585	98	241	279	417
78	221	384	397	540	495	533	51	189	352	182	325	463	631	19	599	112	155	293	461	266	429	567	110	248
409	547	65	228	371	201	339	502	520	58	613	31	169	332	475	305	443	611	124	137	92	260	273	416	579
215	378	396	559	72	527	45	208	351	489	319	482	625	13	181	136	149	287	455	593	423	566	104	242	285
450	588	131	144	307	237	280	418	586	99	554	67	235	373	391	346	509	522	40	203	33	176	314	477	620
156	294	457	600	113	568	111	249	262	430	385	398	541	79	217	47	190	353	496	534	464	627	20	183	326
607	125	138	306	444	274	412	580	93	261	66	229	367	410	548	503	521	59	197	340	170	333	476	614	27
288	456	594	132	150	105	243	286	424	562	392	560	73	216	379	209	347	490	528	46	626	14	177	320	483
119	157	300	438	606	436	574	87	255	268	223	366	404	542	85	515	53	196	359	497	327	470	633	26	164
354	492	535	48	191	21	184	322	465	628	458	601	114	152	295	250	263	431	569	107	537	80	218	386	399
60	198	341	504	517	472	615	28	171	334	139	302	445	608	126	581	94	257	275	413	368	411	549	62	230
491	529	42	210	348	178	321	484	622	15	595	133	151	289	452	282	425	563	106	244	74	212	380	393	561
192	360	498	516	54	634	22	165	328	471	301	439	602	120	158	88	256	269	432	575	405	543	86	224	362
523	41	204	342	510	315	478	621	34	172	127	145	308	451	589	419	582	100	238	281	236	374	387	555	68
258	276	414	577	95	550	63	231	369	407	337	505	518	61	199	29	167	335	473	616	446	609	122	140	303
564	102	245	283	426	381	394	557	75	213	43	211	349	487	530	485	623	16	179	317	147	290	453	596	134
270	433	576	89	252	82	225	363	406	544	499	512	55	193	361	166	329	467	635	23	603	121	159	297	440
101	239	277	420	583	388	556	69	232	375	205	343	511	524	37	617	35	173	316	479	309	447	590	128	146
427	570	108	251	264	219	382	400	538	81	536	49	187	355	493	323	466	629	17	185	115	153	296	459	597

In this example the magic sum is $S_{25 \times 25} := 8100$ and the sum of all 625 entries is $F_{625} := 202500 = 450^2$. The magic square is **pan diagonal and bimagic** with a bimagic sum $Sb_{25 \times 25} := 3438200$. Moreover, each of the 25 5×5 blocks is a **pan diagonal magic square** with the same magic sum, $S_{5 \times 5} := 1620$ and with the same perfect square sum $S_{25} := 8100 = 90^2$ for their entries.

The block-wise constructions of the three examples 70, 71 and 72 of magic squares of order 25 are the composition of 25 magic squares of order 5 with another magic square of order 5 from subsection 4.3. These constructions are based on mutually orthogonal diagonal Latin squares. For details, refer [18], [23].

5 Final Comments

In this work we constructed magic squares of orders 3 to 25 in such a way that all of them have a perfect square sum of their entries. This has been done in two ways, firstly by using consecutive odd numbers starting from 1, and secondly by using consecutive natural numbers which have a minimum perfect square sum for their entries and/or satisfy the uniformity property given in subsection 2.5.1. This work is a **revised and generalized** version of author's previous work [25]. The examples of magic squares considered above are the adaptations of the author's work [21, 22, 23] on **intervally distributed magic squares**. All the magic square in this work are constructed without using computer programming, except two given in examples 44 and 62 for magic square of orders 18 and 22 respectively. These two are based on White's [10] software. The **block-wise** construction of examples **bimagic squares** of orders 8, 9, 16 and 25 given in in subsections 4.6, 4.7, 4.14, 4.23, are based on author's work [19]. The idea of **mutual orthogonal diagonal Latin squares** [9, 11] is applied to magic squares of orders 10 and 14. White's [10] approach, based on **self-orthogonal diagonal Latin squares**, is applied to magic squares of orders 18 and 22 (see examples 44 and 62). Moreover, the **software** created by White [10] allows us to construct doubles of prime order magic squares, in orders 10, 14, 22, 26, etc. The **block-wise** constructions given in [21, 22, 23] are helpful for constructing magic squares of orders, such as, 12, 15, 18, 21, 24, etc. More on magic and bimagic squares refers to [2, 3, 4].

6 Author's Contributions to Magic Squares

The **item-wise** author's work on magic squares is as follows:

1. *Digital Numbers Magic Squares* - [12, 13, 14, 15, 16, 17];
2. *Block-Wise Construction of Bimagic Squares* - [18];
3. *Connections with Genetic Tables and Shannon's entropy* - [19];
4. *Selfie and Palindromic-type Magic Squares* - [20, 36];
5. *Intervally Distributed and Block-Wise Magic Squares* - [21, 22, 23, 36];
6. *Multi-digits and Number Patterns Magic Squares* - [24, 35];
7. *Perfect Square Sum Magic Squares with Uniformity, Minimum Sum and Pythagorean Triples* - [25, 26];
8. *Block-Wise Constructions of Magic and Bimagic Squares* - [28, 29, 30, 31, 34, 37];
9. *Magic Crosses: Repeated and Non Repeated Entries* - [32];
10. *Letters and Numbers With Equal Sums Magic Squares of Orders 4 and 6* - [33].

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