Perturbation Method for the Analysis and Optimization of Microwave Devices

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***Abstract*—A perturbation method for determining the parameters of microwave devices, such as admittance, impedance, and scattering parameters, is presented. Several types of perturbations are considered: variation of component parameters, differentiable variation of medium properties, thin film growth, and boundary shifting. The concept of power waves is generalized to nondiagonal and infinite-dimensional complex source impedance operators. Based on this concept, the radial waveguide concept is developed, and parameters that incorporate phase information are introduced for antenna characterization. These concepts and parameters enable the application of the presented method for solving a wide range of problems, including matching complex impedances and antenna optimization.**

***Index Terms*—Antenna radiation patterns, energy storage, optimization, perturbation methods, scattering parameters.**

I. Introduction

PERTURBATION methods are methods for finding an approximate solution based on a known solution to a similar problem. A perturbation is a small change parametrized by a variable called a perturbation parameter. The result is given as a Taylor series in powers of that variable [1].

In the context of microwave devices, a perturbation is a change in the properties of the medium inside a device. Such a change leads to a change in the electromagnetic field. Perturbation methods are based on approximating the perturbed field [2, Sec. 17]. Three approaches are used to approximate a perturbed field: the change in a certain physical quantity can be neglected, the boundary conditions are satisfied, or the perturbing object distorts the field in a manner similar to that in some other problem.

Most of the perturbation methods are designed for solving eigenvalue problems, namely, resonant frequency problems and waveguide propagation coefficient problems [2, Sec. 16], [3, Ch. 7], [4, Ch. 6], [5]. The main practical application of resonant frequency problems is to measure the properties of materials.

Apart from that, there are methods that involve reflection and transmission coefficients. Perturbation methods for measuring electric and magnetic field intensities are developed in [6], [7], [8], [9]. In these methods, the electric and magnetic field intensities are determined by measuring the variation in a reflection coefficient caused by a perturbing object. In [2, Sec. 16.5], the scattering parameters of a perturbed waveguide are determined. In [10], the variations in the impedance parameters of a two-port are determined. In [11], the variation in the electromagnetic field at a given point is determined.

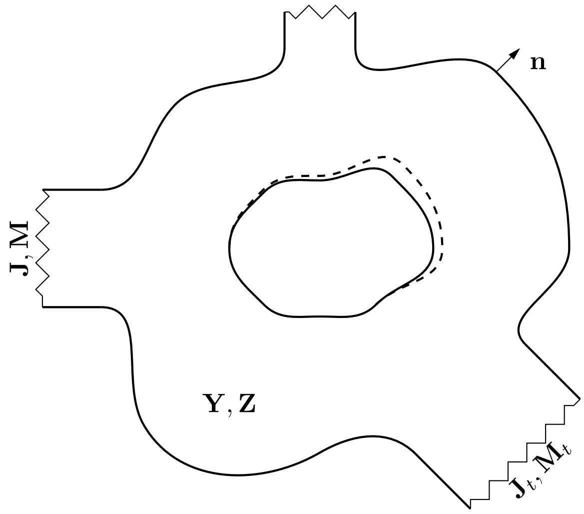

Fig. 1. Perturbation in the form of a boundary shift.

The novelty of the present work lies in the generalization and systematization of perturbation methods. The obtained formulas are suitable for perturbations of anisotropic dispersive media (a printed circuit board substrate is an example of such a medium [12]) and have an explicit form. The presented method can be directly applied to determine admittance, impedance, and scattering parameters. In the paper, a theory is developed that allows antennas and antenna arrays to be characterized in terms of scattering parameters [13, Ch. 9]. This theory enables the application of the presented method for antenna optimization. In addition, this theory results in a straightforward concept of the energy stored in a lossless antenna [14].

The design process for a microwave device usually consists of finding an initial approximation, parametrizing its geometrical dimensions and performing manual or automatic optimization. To determine how the characteristics of the device will change when some parameter is slightly varied, a perturbed problem is often solved in full. However, solving a three-dimensional problem via numerical methods can require a considerable amount of time.

The data that are obtained when a problem is solved via numerical methods are sufficient to estimate the electric and magnetic field intensities at every point in the problem domain. However, most of these data are often not used. This article shows that knowledge of the electric and magnetic field intensities at a certain location allows one to predict how the characteristics of a device will change due to a small modification of the device at that location. The information obtained via the presented method can be used to find the gradient of a goal function during automatic optimization. In addition, this information can be visualized such that it is possible to see where and how the device should be modified in order to improve performance.

The remainder of the paper is organized as follows: the basic equations are obtained in Section II; various types of perturbations are considered in Section III; the variation of the medium properties with frequency is considered from the presented theoretical perspective in Section IV; applications of the presented method are exemplified in Sections V, VI, and VII; the main conclusions are given in Section VIII; the concepts of power waves and a radial waveguide are developed in Appendixes A and B; and formulas for the stored energy in terms of voltages and currents are obtained in Appendix C.

II. Basic equations

This article considers steady-state harmonic oscillations and uses complex notation [3, Sec. 1.7] to describe them. All vectors and operators are expressed as matrices. In particular, complex vectors in three-dimensional space are expressed as complex column vectors with three rows. The word "complex" is omitted henceforth.

Maxwell's equations are as follows [3, Sec. 1.9]:

$$\begin{cases}\nabla_\times \mathbf{E} + \mathbf{Z}\,\mathbf{H} = -\mathbf{M},\\ \nabla_\times \mathbf{H} - \mathbf{Y}\mathbf{E} = \mathbf{J},\end{cases} \tag{1}$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$, and $\mathbf{M}$ are the electric field intensity vector, the magnetic field intensity vector, the impressed electric current density vector, and the impressed magnetic current density vector, respectively;

$$\nabla = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}; \quad \nabla_\times = \begin{pmatrix}0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x}\\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\end{pmatrix}; \tag{2}$$

and $\mathbf{Y}$ and $\mathbf{Z}$ are the admittivity and impedivity matrices, respectively.

For a homogeneous isotropic medium with propagation coefficient γ and wave impedance W, the admittivity and impedivity matrices are scalar matrices:

$$\mathbf{Y} = \gamma W^{-1}\mathbf{U}, \tag{3}$$

$$\mathbf{Z} = \gamma W\mathbf{U}, \tag{4}$$

where $\mathbf{U}$ is the identity matrix.

The field intensities in the problem defined by (1) and the field intensities in another problem with the same admittivity and impedivity matrices but possibly different impressed currents are assumed to be known. The latter problem is used to define a parameter whose variation is of interest. The currents and field intensities in the latter problem are denoted by a subscript t (from the word "test"):

$$\begin{cases}\nabla_\times \mathbf{E}_t + \mathbf{Z}\,\mathbf{H}_t = -\mathbf{M}_t,\\ \nabla_\times \mathbf{H}_t - \mathbf{Y}\mathbf{E}_t = \mathbf{J}_t.\end{cases} \tag{5}$$

The quantities related to the perturbed problem are denoted by a prime:

$$\begin{cases}\nabla_\times \mathbf{E}' + \mathbf{Z}'\,\mathbf{H}' = -\mathbf{M}',\\ \nabla_\times \mathbf{H}' - \mathbf{Y}'\mathbf{E}' = \mathbf{J}'.\end{cases} \tag{6}$$

From (5) and (6), two basic identities are obtained using the Ostrogradsky-Gauss theorem (details are presented in [3, Sec. 3.8], [15]), and the quantities $\mathcal{Q}$ and $\mathcal{R}$ are introduced:

$$\begin{aligned}\mathcal{Q} &= \int_s \left(\mathbf{E}_t^H \mathbf{n}_\times \mathbf{H}' - \mathbf{H}_t^H \mathbf{n}_\times \mathbf{E}'\right) ds\\ &\quad - \int_v \left(\mathbf{E}_t^H \mathbf{J}' + \mathbf{H}_t^H \mathbf{M}' + \mathbf{J}_t^H \mathbf{E}' + \mathbf{M}_t^H \mathbf{H}'\right) dv\\ &= \int_v \left(\mathbf{E}_t^H \left(\mathbf{Y}' + \mathbf{Y}^H\right)\mathbf{E}' + \mathbf{H}_t^H \left(\mathbf{Z}' + \mathbf{Z}^H\right)\mathbf{H}'\right) dv,\end{aligned} \tag{7}$$

$$\begin{aligned}\mathcal{R} &= \int_s \left(\mathbf{E}_t^T \mathbf{n}_\times \mathbf{H}' + \mathbf{H}_t^T \mathbf{n}_\times \mathbf{E}'\right) ds\\ &\quad - \int_v \left(\mathbf{E}_t^T \mathbf{J}' - \mathbf{H}_t^T \mathbf{M}' - \mathbf{J}_t^T \mathbf{E}' + \mathbf{M}_t^T \mathbf{H}'\right) dv\\ &= \int_v \left(\mathbf{E}_t^T \left(\mathbf{Y}' - \mathbf{Y}^T\right)\mathbf{E}' - \mathbf{H}_t^T \left(\mathbf{Z}' - \mathbf{Z}^T\right)\mathbf{H}'\right) dv,\end{aligned} \tag{8}$$

where the superscripts T and H denote transposition and Hermitian conjugation, respectively; v is the region occupied by the studied device; s is the boundary of v; $\mathbf{n}$ is the external normal; and $\mathbf{n}_\times$ is the skew-symmetric matrix that corresponds to $\mathbf{n}$ in the same way that $\nabla_\times$ corresponds to ∇ in (2).

The purpose of introducing the quantities $\mathcal{Q}$ and $\mathcal{R}$ is to enable the analysis of a perturbation regardless of which global parameter is of interest. Formulas for $\mathcal{Q}$ and $\mathcal{R}$ in terms of circuit theory are obtained below. By virtue of the chosen terminology, these formulas are similar to (7) and (8), which makes it easy to switch from electrodynamics to circuit theory and back.

Next, impressed currents are assumed to be absent inside the studied device (as shown in Fig. 1).

By decomposing the components of the electric and magnetic field intensities that are tangent to the surface s into Fourier series (details are presented in [3, Sec. 8.2] and Appendix B), the following equation is obtained for $\mathcal{Q}$ in terms of the voltage vectors V' and V_t and the current vectors I' and I_t:

$$\mathcal{Q} = V_t^H I' + I_t^H V'. \tag{9}$$

When perturbation is absent, $\mathbf{J} = \mathbf{J}_t$, and $\mathbf{M} = \mathbf{M}_t$, the quantity $\mathcal{Q}/2$ is equal to the mean dissipated power.

For cases in which the source impedance operator (the reader is referred to Appendix A) does not change due to perturbation, a simple formula for $\mathcal{Q}$ in terms of the power wave vectors can be obtained by expressing the voltage and current vectors in terms of the power wave vectors:

$$\frac{1}{2}\mathcal{Q} = a_t^H a' - b_t^H b'. \tag{10}$$

Hereafter, when using (10) and the results obtained using (10), the source impedance operator is assumed not to change due to perturbation.

Equations (9) and (10) can be rewritten as follows:

$$\mathcal{Q} = V_t^H \left(Y' + Y^H\right) V' = I_t^H \left(Z' + Z^H\right) I', \tag{11}$$

$$\frac{1}{2}\mathcal{Q} = a_t^H \left(U - S^H S'\right) a', \tag{12}$$

where Y and Y' are admittance operators, Z and Z' are impedance operators, S and S' are scattering operators, and U is the identity operator.

It follows from (7), (11), and (12) that if the admittivity and impedivity matrices at every point in a device are anti-Hermitian, then $\mathcal{Q}$ is equal to zero, the device is lossless, the admittance and impedance operators of the device are anti-Hermitian, and the scattering operator of the device is unitary. The converse does not hold in general, but for brevity, the admittivity and impedivity matrices at every point in a lossless device are assumed to be anti-Hermitian.

For devices that are lossless in the unperturbed state, conversion to infinitesimals yields

$$\frac{d\mathcal{Q}}{d\xi} = V_t^H \frac{dY}{d\xi} V = I_t^H \frac{dZ}{d\xi} I, \tag{13}$$

$$\frac{1}{2}\frac{d\mathcal{Q}}{d\xi} = -a_t^H S^H \frac{dS}{d\xi} a, \tag{14}$$

where ξ is a perturbation parameter.

Similar results are obtained for the quantity $\mathcal{R}$. Decomposing the components of the electric and magnetic field intensities that are tangent to the surface s into Fourier series yields

$$\mathcal{R} = V_t^T I' - I_t^T V'. \tag{15}$$

When perturbation is absent, $\mathbf{J} = \mathbf{J}_t$, and $\mathbf{M} = \mathbf{M}_t$, the quantity $\mathcal{R}$ is equal to zero (because a transposed scalar is equal to the original scalar).

For cases in which the source impedance operator is symmetric and does not change due to perturbation, a simple formula for $\mathcal{R}$ in terms of the power wave vectors can be obtained by expressing the voltage and current vectors in terms of the power wave vectors:

$$\frac{1}{2}\mathcal{R} = -a_t^T b' + b_t^T a'. \tag{16}$$

Hereafter, when using (16) and the results obtained using (16), the source impedance operator is assumed to be symmetric and not to change due to perturbation.

Equations (15) and (16) can be rewritten as follows:

$$\mathcal{R} = V_t^T \left(Y' - Y^T\right) V' = -I_t^T \left(Z' - Z^T\right) I', \tag{17}$$

$$\frac{1}{2}\mathcal{R} = -a_t^T \left(S' - S^T\right) a'. \tag{18}$$

Devices with symmetric admittance, impedance, and scattering operators are said to be reciprocal. It follows from (8), (17), and (18) that if the admittivity and impedivity matrices at every point in a device are symmetric, then $\mathcal{R}$ is equal to zero and the device is reciprocal (this is the reciprocity theorem [16]). The converse does not hold in general, but for brevity, the admittivity and impedivity matrices at every point in a reciprocal device are assumed to be symmetric.

For devices that are reciprocal in the unperturbed state, conversion to infinitesimals yields

$$\frac{d\mathcal{R}}{d\xi} = V_t^T \frac{dY}{d\xi} V = -I_t^T \frac{dZ}{d\xi} I, \tag{19}$$

$$\frac{1}{2}\frac{d\mathcal{R}}{d\xi} = -a_t^T \frac{dS}{d\xi} a. \tag{20}$$

By choosing appropriate stimuli, it is possible to distinguish a device parameter whose variation is of interest. Next, u and u_t are assumed to be constants in the linear space of the power wave vectors. For the scattering operator of a device that is lossless in the unperturbed state, it follows from (14) and the unitarity condition of the scattering operator that

$$\frac{d\left(u_t^H S u\right)}{d\xi} = -\frac{1}{2}\frac{d\mathcal{Q}}{d\xi}\bigg|_{\substack{a=u \\ a_t^H = u_t^H S}}. \tag{21}$$

For the scattering operator of a device that is reciprocal in the unperturbed state, it follows from (20) that

$$\frac{d\left(u_t^H S u\right)}{d\xi} = -\frac{1}{2}\frac{d\mathcal{R}}{d\xi}\bigg|_{\substack{a=u \\ a_t^T = u_t^H}}. \tag{22}$$

Problems corresponding to the vector u can be called "forward" problems, whereas problems corresponding to the vector u_t can be called "backward" problems.

The presented method can be applied to solve electrostatic problems by letting

$$Y = i\omega C, \tag{23}$$

where ω is the angular frequency and C is the capacitance operator, and then canceling out $i\omega$. A similar approach can be taken for magnetostatic problems.

III. Various types of perturbations

A. *Variation of component parameters*

From (7), it follows that $\mathcal{Q}$ is an additive quantity. Consequently, for a device that is lossless in the unperturbed state, the derivative (with respect to the perturbation parameter) of $\mathcal{Q}$ of the whole device is equal to the sum of the corresponding derivatives of $\mathcal{Q}$ of the varying components. Similar statements hold true for $\mathcal{R}$.

B. *Differentiable variation of the admittivity and impedivity matrices*

For cases in which the admittivity and impedivity matrices are differentiable with respect to the perturbation parameter ξ, it follows from (7) for devices that are lossless in the unperturbed state and from (8) for devices that are reciprocal in the unperturbed state that

$$\frac{d\mathcal{Q}}{d\xi} = \int_v \left(\mathbf{E}_t^H \frac{d\mathbf{Y}}{d\xi}\mathbf{E} + \mathbf{H}_t^H \frac{d\mathbf{Z}}{d\xi}\mathbf{H}\right) dv, \tag{24}$$

$$\frac{d\mathcal{R}}{d\xi} = \int_v \left(\mathbf{E}_t^T \frac{d\mathbf{Y}}{d\xi}\mathbf{E} - \mathbf{H}_t^T \frac{d\mathbf{Z}}{d\xi}\mathbf{H}\right) dv. \tag{25}$$

C. Thin film growth and boundary shifting

In this subsection, a perturbation in the form of a thin film of a material with finite admittivity and impedivity matrices is considered. This thin film is defined by a surface s, a thickness h, an admittivity matrix $\mathbf{Y}'$, and an impedivity matrix $\mathbf{Z}'$. h is assumed to be differentiable with respect to ξ, while $\mathbf{Y}'$ and $\mathbf{Z}'$ are assumed not to depend on ξ.

The shifting of a boundary between two materials with finite admittivity and impedivity matrices is treated as thin film growth.

The considered perturbation is singular in the sense [1, Sec. 1.2] that the limit field intensities inside the film, in general, differ from the field intensities in the unperturbed problem.

Based on the boundary conditions, the tangent components of the vectors $\mathbf{E}$ and $\mathbf{H}$ and the normal components of the vectors $\mathbf{YE}$ and $\mathbf{ZH}$ are assumed to remain almost unchanged inside the thin film:

$$\mathbf{E}' - \mathbf{n}\mathbf{n}^T\mathbf{E}' \approx \mathbf{E} - \mathbf{n}\mathbf{n}^T\mathbf{E}, \tag{26}$$

$$\mathbf{H}' - \mathbf{n}\mathbf{n}^T\mathbf{H}' \approx \mathbf{H} - \mathbf{n}\mathbf{n}^T\mathbf{H}, \tag{27}$$

$$\mathbf{n}\mathbf{n}^T\mathbf{Y}'\mathbf{E}' \approx \mathbf{n}\mathbf{n}^T\mathbf{Y}\mathbf{E}, \tag{28}$$

$$\mathbf{n}\mathbf{n}^T\mathbf{Z}'\mathbf{H}' \approx \mathbf{n}\mathbf{n}^T\mathbf{Z}\mathbf{H}, \tag{29}$$

where $\mathbf{n}$ is the normal with respect to s.

The electric field intensity vector inside the film can be found in explicit form ($\mathbf{Y}$ and $\mathbf{Y}'$ can be nondiagonal) as follows:

$$\begin{aligned}\mathbf{E}' &= \mathbf{E}' - \mathbf{n}\mathbf{n}^T\mathbf{E}' + \mathbf{n}\mathbf{n}^T\mathbf{E}' \\ &= \mathbf{E}' - \mathbf{n}\mathbf{n}^T\mathbf{E}' + \frac{\mathbf{n}(\mathbf{n}^T\mathbf{Y}'\mathbf{n})\mathbf{n}^T\mathbf{E}'}{\mathbf{n}^T\mathbf{Y}'\mathbf{n}} \\ &= \mathbf{E}' - \mathbf{n}\mathbf{n}^T\mathbf{E}' - \frac{\mathbf{n}\mathbf{n}^T\mathbf{Y}'\left(\mathbf{E}' - \mathbf{n}\mathbf{n}^T\mathbf{E}'\right) - \mathbf{n}\mathbf{n}^T\mathbf{Y}'\mathbf{E}'}{\mathbf{n}^T\mathbf{Y}'\mathbf{n}} \\ &\approx \mathbf{E} - \mathbf{n}\mathbf{n}^T\mathbf{E} - \frac{\mathbf{n}\mathbf{n}^T\mathbf{Y}'\left(\mathbf{E} - \mathbf{n}\mathbf{n}^T\mathbf{E}\right) - \mathbf{n}\mathbf{n}^T\mathbf{Y}\mathbf{E}}{\mathbf{n}^T\mathbf{Y}'\mathbf{n}} \\ &= \mathbf{E} - \frac{\mathbf{n}\mathbf{n}^T(\mathbf{Y}' - \mathbf{Y})}{\mathbf{n}^T\mathbf{Y}'\mathbf{n}}\mathbf{E}.\end{aligned} \tag{30}$$

Similarly, for the magnetic field intensity vector,

$$\mathbf{H}' \approx \mathbf{H} - \frac{\mathbf{n}\mathbf{n}^T(\mathbf{Z}' - \mathbf{Z})}{\mathbf{n}^T\mathbf{Z}'\mathbf{n}}\mathbf{H}. \tag{31}$$

Finally, it follows from (7) for devices that are lossless in the unperturbed state and from (8) for devices that are reciprocal in the unperturbed state that

$$\begin{aligned}\frac{d\mathcal{Q}}{d\xi} = &\int_s \mathbf{E}_t^H \left(\mathbf{Y}' - \mathbf{Y}\right)\mathbf{E}\frac{dh}{d\xi}ds \\ &+ \int_s \mathbf{H}_t^H \left(\mathbf{Z}' - \mathbf{Z}\right)\mathbf{H}\frac{dh}{d\xi}ds \\ &- \int_s \mathbf{E}_t^H \frac{\left(\mathbf{Y}' - \mathbf{Y}\right)\mathbf{n}\mathbf{n}^T\left(\mathbf{Y}' - \mathbf{Y}\right)}{\mathbf{n}^T\mathbf{Y}'\mathbf{n}}\mathbf{E}\frac{dh}{d\xi}ds \\ &- \int_s \mathbf{H}_t^H \frac{\left(\mathbf{Z}' - \mathbf{Z}\right)\mathbf{n}\mathbf{n}^T\left(\mathbf{Z}' - \mathbf{Z}\right)}{\mathbf{n}^T\mathbf{Z}'\mathbf{n}}\mathbf{H}\frac{dh}{d\xi}ds,\end{aligned} \tag{32}$$

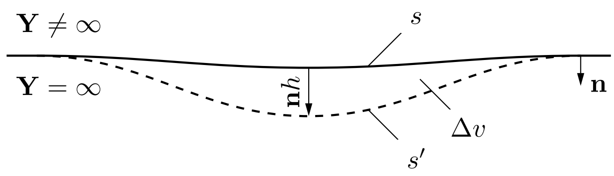

Fig. 2. Shifting of a perfect electric conductor boundary.

$$\begin{aligned}\frac{d\mathcal{R}}{d\xi} = &\int_s \mathbf{E}_t^T \left(\mathbf{Y}' - \mathbf{Y}\right)\mathbf{E}\frac{dh}{d\xi}ds \\ &- \int_s \mathbf{H}_t^T \left(\mathbf{Z}' - \mathbf{Z}\right)\mathbf{H}\frac{dh}{d\xi}ds \\ &- \int_s \mathbf{E}_t^T \frac{\left(\mathbf{Y}' - \mathbf{Y}\right)\mathbf{n}\mathbf{n}^T\left(\mathbf{Y}' - \mathbf{Y}\right)}{\mathbf{n}^T\mathbf{Y}'\mathbf{n}}\mathbf{E}\frac{dh}{d\xi}ds \\ &+ \int_s \mathbf{H}_t^T \frac{\left(\mathbf{Z}' - \mathbf{Z}\right)\mathbf{n}\mathbf{n}^T\left(\mathbf{Z}' - \mathbf{Z}\right)}{\mathbf{n}^T\mathbf{Z}'\mathbf{n}}\mathbf{H}\frac{dh}{d\xi}ds.\end{aligned} \tag{33}$$

D. Shifting of a perfect electric conductor boundary

This perturbation is depicted in Fig. 2, where s and s' are the unperturbed and perturbed boundaries of the perfect electric conductor, respectively; Δv is the region bounded by s and s'; and h is the shift in the boundary measured in the direction of the external normal $\mathbf{n}$. It is assumed that s and s' are sufficiently smooth and that h is differentiable with respect to ξ.

For a lossless device, it follows from the additivity of $\mathcal{Q}$ that

$$\begin{aligned}\mathcal{Q} &= \int_s \mathbf{H}_t^H \mathbf{n}_\times \mathbf{E}' ds \approx \int_s \mathbf{H}_t'^H \mathbf{n}_\times \mathbf{E}' ds \\ &= \int_{\Delta v} \left(\left(\nabla_\times \mathbf{H}_t'\right)^H \mathbf{E}' - \mathbf{H}_t'^H \left(\nabla_\times \mathbf{E}'\right) \right) dv \\ &= \int_{\Delta v} \left(\mathbf{E}_t'^H \mathbf{Y}^H \mathbf{E}' + \mathbf{H}_t'^H \mathbf{Z}\mathbf{H}' \right) dv\end{aligned} \tag{34}$$

for a nonnegative h and

$$\begin{aligned}\mathcal{Q} &= -\int_{s'} \mathbf{E}_t^H \mathbf{n}_\times \mathbf{H}' ds \approx -\int_{s'} \mathbf{E}_t^H \mathbf{n}_\times \mathbf{H} ds \\ &= \int_{\Delta v} \left(\mathbf{E}_t^H \left(\nabla_\times \mathbf{H}\right) - \left(\nabla_\times \mathbf{E}_t\right)^H \mathbf{H} \right) dv \\ &= \int_{\Delta v} \left(\mathbf{E}_t^H \mathbf{Y}\mathbf{E} + \mathbf{H}_t^H \mathbf{Z}^H \mathbf{H} \right) dv\end{aligned} \tag{35}$$

for a nonpositive h.

Hence, for a lossless device, the derivative of $\mathcal{Q}$ with respect to the perturbation parameter ξ is given by

$$\frac{d\mathcal{Q}}{d\xi} = -\int_s \left(\mathbf{E}_t^H \mathbf{Y}\mathbf{E} - \mathbf{H}_t^H \mathbf{Z}\mathbf{H} \right) \frac{dh}{d\xi} ds. \tag{36}$$

For a reciprocal device, it similarly follows from the additivity of $\mathcal{R}$ that

$$\frac{d\mathcal{R}}{d\xi} = -\int_s \left(\mathbf{E}_t^T \mathbf{Y}\mathbf{E} + \mathbf{H}_t^T \mathbf{Z}\mathbf{H} \right) \frac{dh}{d\xi} ds. \tag{37}$$

IV. Frequency Variation

A differentiable variation of the admittivity and impedivity matrices with respect to the frequency can be treated as a perturbation of the type analyzed in Section III-B.

For lossless devices, it follows from (13), (14), and (24) that

$$\frac{1}{2}V_t^H\frac{dY}{d(i\omega)}V=\frac{1}{2}I_t^H\frac{dZ}{d(i\omega)}I=-a_t^HS^H\frac{dS}{d(i\omega)}a$$
$$=\frac{1}{2}\int_v\left(\mathbf{E}_t^H\frac{d\mathbf{Y}}{d(i\omega)}\mathbf{E}+\mathbf{H}_t^H\frac{d\mathbf{Z}}{d(i\omega)}\mathbf{H}\right)dv. \quad (38)$$

For reciprocal devices, it follows from (19), (20), and (25) that

$$\frac{1}{2}V_t^T\frac{dY}{d(i\omega)}V=-\frac{1}{2}I_t^T\frac{dZ}{d(i\omega)}I=-a_t^T\frac{dS}{d(i\omega)}a$$
$$=\frac{1}{2}\int_v\left(\mathbf{E}_t^T\frac{d\mathbf{Y}}{d(i\omega)}\mathbf{E}-\mathbf{H}_t^T\frac{d\mathbf{Z}}{d(i\omega)}\mathbf{H}\right)dv. \quad (39)$$

Formulas (38) and (39) may be used for the interpolation and extrapolation of parameters calculated at discrete frequencies.

From (38) and Appendix C, one can obtain the following formulas for the mean energy stored in a lossless device:

$$\mathcal{W}=\frac{1}{2}V^H\frac{dY}{d(i\omega)}V=\frac{1}{2}I^H\frac{dZ}{d(i\omega)}I=-a^HS^H\frac{dS}{d(i\omega)}a$$
$$=\frac{1}{2}\int_v\left(\mathbf{E}^H\frac{d\mathbf{Y}}{d(i\omega)}\mathbf{E}+\mathbf{H}^H\frac{d\mathbf{Z}}{d(i\omega)}\mathbf{H}\right)dv. \quad (40)$$

For a lossless transmitting antenna, it can be found from (40) and Appendix B that the energy $\mathcal{W}$ stored in the volume bounded by the generator and a sphere of radius r centered at the origin is given by

$$\lim_{r\to\infty}\left(\mathcal{W}-\mathcal{W}_r\right)$$
$$=-a_{fed}^H\left(S_{fed,fed}^H\frac{dS_{fed,fed}}{d(i\omega)}+\mathbf{T}^H\frac{d\mathbf{T}}{d(i\omega)}\right)a_{fed}, \quad (41)$$

where

$$\mathcal{W}_r=-a_{fed}^H\mathbf{T}^He^{\gamma r}\frac{de^{-\gamma r}}{d(i\omega)}\mathbf{T}a_{fed}=||\mathbf{T}a_{fed}||^2\frac{d(\gamma r)}{d(i\omega)}. \quad (42)$$

The quantity $\mathcal{W}_r$ is equal to the product of the radiated power and the group delay, which is determined by the propagation constant γ and the distance r.

Clearly, the problem of determining the energy stored in an antenna [14] is equivalent to the problem of determining the energy stored in a waveguide junction.

V. Electric Field Intensity at the End of the Inner Conductor of a Coaxial Line

This section considers the problem of finding the electric field intensity at the end of the inner conductor of the coaxial line shown in Fig. 3, which has perfectly electrically conducting walls and contains a homogeneous isotropic dielectric with a purely imaginary γ and a purely real W. In practice, this problem is of interest for the design of coaxial resonators [17, Ch. 17, pp. 9-11].

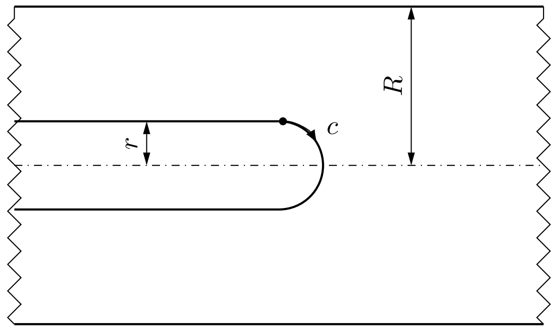

Fig. 3. End of a coaxial line.

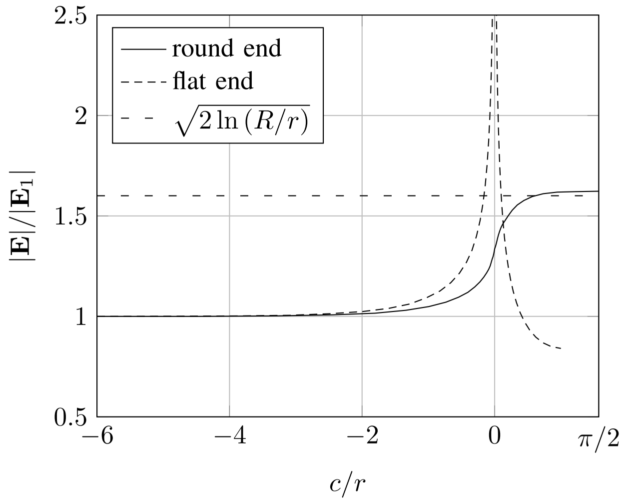

Fig. 4. Electric field intensity on the inner conductor of a coaxial line.

The reflection coefficient S_{11} (where the index 1 corresponds to the fundamental mode) depends on the distance l from the reference plane to the end of the coaxial line in the following way:

$$S_{11}(l)=S_{11}(0)e^{-2\gamma l}. \quad (43)$$

Consequently,

$$\frac{dS_{11}}{dl}=-2\gamma S_{11}. \quad (44)$$

From (44), (20), and (37), it follows that

$$\frac{1}{2}\int_s\left(\mathbf{E}^T\gamma W^{-1}\mathbf{E}+\mathbf{H}^T\gamma W\mathbf{H}\right)ds_\perp=2\gamma S_{11}a_1^2, \quad (45)$$

where s is the surface of the end of the inner conductor and $ds_\perp$ is the differential of the cross-sectional area.

Using the well-known results for a regular coaxial line [18, Sec. 6.9], it can be shown that the following equation holds:

$$\frac{1}{2}\left(\mathbf{E}_1^T\gamma W^{-1}\mathbf{E}_1-\mathbf{H}_1^T\gamma W\mathbf{H}_1\right)\pi r^2=\frac{\gamma S_{11}a_1^2}{\ln\left(\frac{R}{r}\right)}, \quad (46)$$

where $\mathbf{E}_1$ and $\mathbf{H}_1$ are the electric and magnetic field intensity vectors, respectively, of the fundamental mode on the inner conductor of the regular coaxial line.

By comparing (45) and (46), it can be concluded that in the low-frequency approximation, the square of the magnitude of

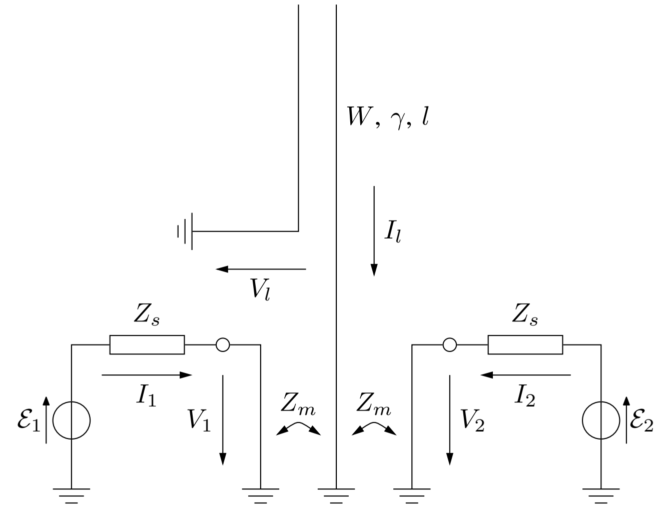

Fig. 5. Filter schematic.

the electric field intensity at the end of the inner conductor averaged over the cross-sectional area is

$$2\ln\left(\frac{R}{r}\right) \tag{47}$$

times greater than the square of the magnitude of the electric field intensity of the fundamental mode on the inner conductor of the regular line.

Graphs of the magnitude of the electric field intensity at zero frequency on the inner conductor of a coaxial line with $R/r = 3.6$, as found via the finite element method [19] for round and flat ends of the inner conductor, are shown in Fig. 4.

VI. Filter

In this section, a filter that is a prototype for a quarter-wave coaxial-line-based filter with inductive coupling loops [17, Ch. 17, pp. 9-11] is considered. The presented method is applied to find the resonator length that maximizes the transmission coefficient (this length is not quite one-quarter of the wavelength).

The filter schematic is shown in Fig. 5. The resonant transmission line has a characteristic impedance of W, a propagation coefficient of γ, and a length of l. The impedance of this line is

$$Z_l = \frac{V_l}{-I_l} = W\coth(\gamma l) \approx W\left(\gamma l - i\frac{\pi}{2}\right). \tag{48}$$

The impedance Z_s determines the parameters of the sources and the parameters of the input circuits, such as the self-inductance of the coupling loops. The impedance Z_m determines the coupling between the input circuits and the resonator. For simplicity, the coupling between the input circuits that bypasses the resonator is neglected. The following equation describes the problem:

$$\begin{pmatrix} Z_s & 0 & Z_m \\ 0 & Z_s & Z_m \\ Z_m & Z_m & Z_l \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \\ I_l \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ 0 \end{pmatrix}. \tag{49}$$

The solution to this equation is

$$\begin{pmatrix} I_1 \\ I_2 \\ I_l \end{pmatrix} = \begin{pmatrix} 2-\eta \\ -\eta \\ \frac{Z_s}{Z_m}\eta \end{pmatrix}\frac{\mathcal{E}_1}{2Z_s} + \begin{pmatrix} -\eta \\ 2-\eta \\ \frac{Z_s}{Z_m}\eta \end{pmatrix}\frac{\mathcal{E}_2}{2Z_s}, \tag{50}$$

where

$$\eta = \frac{1}{1 - \frac{1}{2}\frac{Z_s Z_l}{Z_m^2}}. \tag{51}$$

Other parameters can be found from the known currents:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_l \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -\frac{Z_l}{Z_m} \end{pmatrix}\eta\frac{\mathcal{E}_1 + \mathcal{E}_2}{2}, \tag{52}$$

$$a_1 = \frac{\mathcal{E}_1}{2\sqrt{\mathrm{Re}(Z_s)}},\ a_2 = \frac{\mathcal{E}_2}{2\sqrt{\mathrm{Re}(Z_s)}}, \tag{53}$$

$$S_{21} = \frac{\mathrm{Re}(Z_s)}{Z_s}\eta,\ S_{11} = S_{21} - \frac{\overline{Z_s}}{Z_s}, \tag{54}$$

where the overline denotes complex conjugation.

Increasing the length l by an infinitesimally small value dl is equivalent to connecting to the end of the transmission line a two-terminal circuit with admittance

$$dY = \gamma W^{-1} dl. \tag{55}$$

Following Section III-A, the derivative of the transmission coefficient can be found as follows:

$$\frac{dS_{21}}{dl} = -\frac{1}{2}\frac{V_{le,t}V_{le}}{a_{2,t}a_1}\frac{dY}{dl}, \tag{56}$$

where V_{le} and $V_{le,t}$ are the voltages at the end of the transmission line,

$$V_{le} = \frac{V_l}{\cosh(\gamma l)},\ V_{le,t} = \frac{V_{l,t}}{\cosh(\gamma l)}. \tag{57}$$

The optimality condition for the length of the resonant transmission line is given by

$$\frac{d\left|S_{21}\right|^2}{dl} = -\mathrm{Re}(\mathcal{F}) = 0, \tag{58}$$

where

$$\begin{aligned}\mathcal{F} &= -2\overline{S_{21}}\frac{dS_{21}}{dl} = \overline{S_{21}}\frac{V_{le,t}V_{le}}{a_{2,t}a_1}\gamma W^{-1} \\ &= \frac{2|S_{21}|^2\left(-\operatorname{csch}^2(\gamma l)\right)}{\frac{1}{\gamma}\coth(\gamma l) - \frac{2Z_m^2}{\gamma W Z_s}} \approx \frac{2|S_{21}|^2}{l - \frac{i}{\gamma}\frac{\pi}{2} - \frac{2Z_m^2}{\gamma W Z_s}}.\end{aligned} \tag{59}$$

The optimal length is found by setting the real part of (59) equal to zero:

$$l_{opt} \approx \mathrm{Re}\left(\frac{i}{\gamma}\frac{\pi}{2} + \frac{2Z_m^2}{\gamma W Z_s}\right). \tag{60}$$

It is instructive to find a formula for determining the optimal length from the known voltages at the end of the resonator and the known external properties of the slightly detuned filter.

It can be observed that

$$\frac{\mathrm{Re}(\mathcal{F})}{\mathrm{Im}(\mathcal{F})} \approx \frac{l - l_{opt}}{\mathrm{Im}\left(\frac{i}{\gamma}\frac{\pi}{2} + \frac{2Z_m^2}{\gamma W Z_s}\right)}. \tag{61}$$

In the approximation in which the propagation coefficient γ is proportional to the frequency and the impedances Z_m, Z_s, and W do not depend on the frequency (a good approximation for narrowband filters), the pole of the function η (expressed as a function of the Laplace transform variable) is given by

$$p \approx \frac{i\omega}{l}\left(\frac{i}{\gamma}\frac{\pi}{2} + \frac{2Z_m^2}{\gamma W Z_s}\right). \tag{62}$$

The quality factor of this pole

$$Q = -\frac{|p|}{2\,\mathrm{Re}(p)} \approx \frac{\pi}{4|\gamma|\,\mathrm{Im}\left(\frac{i}{\gamma}\frac{\pi}{2} + \frac{2Z_m^2}{\gamma W Z_s}\right)} \tag{63}$$

is equal to the loaded quality factor of the filter.

Combining (61) and (63) yields the desired formula:

$$l - l_{opt} \approx \frac{\pi}{4Q|\gamma|}\frac{\mathrm{Re}(\mathcal{F})}{\mathrm{Im}(\mathcal{F})}. \tag{64}$$

VII. Optimization

A. Common provisions

In this section, the presented method is applied to spatial problems in which

$$\frac{d}{dz} = 0,\ E_z = 0,\ H_x = 0,\ H_y = 0, \tag{65}$$

and the space of interest is a flat layer of thickness τ, perpendicular to the z-axis, filled with either a homogeneous isotropic medium with a purely real wave impedance W and a purely imaginary propagation coefficient γ or a perfect electric conductor. These problems are essentially two-dimensional in nature. The thickness τ is used only to allow the term "power" to be used.

The electromagnetic modeling in this section was mainly performed using the method of moments (MOM) [20]. To realize waveguide ports, along with equivalent surface electric currents, equivalent surface magnetic currents were used. Piecewise linear approximations of the waveguide mode fields were chosen as the basis functions at the waveguide ports. Triangular basis functions were chosen for the surfaces of the perfect electric conductor.

B. Discontinuity in a waveguide

In this subsection, the reflection coefficient S_{11} of the dented waveguide shown in Fig. 6 is investigated (here, the index 1 corresponds to the fundamental mode of the left waveguide port). In the present case, the "forward" problem coincides with the "backward" problem. From (22) and (37), it follows that

$$\frac{dS_{11}}{d\xi} = \frac{1}{2a_1^2}\int_s \left(\mathbf{E}^T\gamma W^{-1}\mathbf{E} + \mathbf{H}^T\gamma W\mathbf{H}\right)\frac{dh}{d\xi}ds. \tag{66}$$

Consequently,

$$\frac{d}{d\xi}|S_{11}|^2 = 2\,\mathrm{Re}\left(S_{11}^H\frac{dS_{11}}{d\xi}\right) = \int_s \mathrm{Re}(\mathcal{F})\frac{dh}{d\xi}ds, \tag{67}$$

where

$$\mathcal{F} = \frac{\overline{S_{11}}}{a_1^2}\left(\mathbf{E}^T\gamma W^{-1}\mathbf{E} + \mathbf{H}^T\gamma W\mathbf{H}\right). \tag{68}$$

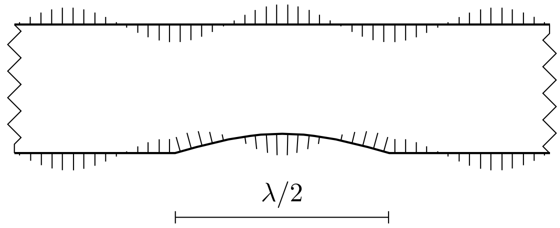

Fig. 6. Dented waveguide.

Thus, if the boundary of the waveguide is shifted in the direction of the external normal at locations where $\mathrm{Re}(\mathcal{F})$ is negative and in the direction of the internal normal at locations where $\mathrm{Re}(\mathcal{F})$ is positive, then the magnitude of S_{11} will decrease. The function $\mathrm{Re}(\mathcal{F})$ is shown in Fig. 6 as line segments perpendicular to the boundary, indicating the direction in which the boundary should be shifted to decrease the reflection coefficient.

C. Antenna

In this subsection, an example of antenna optimization is considered. The goal is to design an antenna with the following directivity characteristic (up to a phase factor, which may vary with frequency but must not depend on direction) in the frequency band of $[\omega_1, 2\omega_1]$ (the notion of the directivity characteristic is given in Appendix B):

$$\mathbf{f}_g(\varphi) = \mathbf{1}_\varphi \begin{cases} 0, & \text{if } \varphi \in \left[-\pi, -\frac{\pi}{3}\right] \\ \sqrt{6}\cos\left(\frac{3}{2}\varphi\right), & \text{if } \varphi \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \\ 0, & \text{if } \varphi \in \left[\frac{\pi}{3}, \pi\right] \end{cases}, \tag{69}$$

where φ is the azimuth angle, $\mathbf{1}_\varphi$ is the unit vector in the direction of increasing φ, and the cosine multiplier is chosen such that

$$\int_{-\pi}^{\pi} \mathbf{f}_g^H(\varphi)\mathbf{f}_g(\varphi)\frac{d\varphi}{2\pi} = 1. \tag{70}$$

The antenna is fed by a waveguide with a transverse size of $\lambda_1/10$, where λ_1 is the wavelength corresponding to the frequency ω_1.

The transmission coefficient S_{21} is given by

$$S_{21} = \int_{-\pi}^{\pi} \mathbf{f}_g^H(\varphi)\mathbf{f}(\varphi)\frac{d\varphi}{2\pi}, \tag{71}$$

where the index 1 corresponds to the fundamental mode of the feeder, the index 2 corresponds to the desired directivity characteristic, and $\mathbf{f}$ is the directivity characteristic corresponding to the fundamental mode of the feeder.

The goal function that is maximized during the optimization process is

$$g = \frac{1}{N}\sum_n |S_{21}(\omega_n)|^2, \tag{72}$$

where

$$\omega_n = \omega_1\left(1 + \frac{n-1}{N-1}\right), \tag{73}$$

$$n = 1, 2, \ldots, N,\ N = 11. \tag{74}$$

Fig. 7. Antenna – initial approximation.

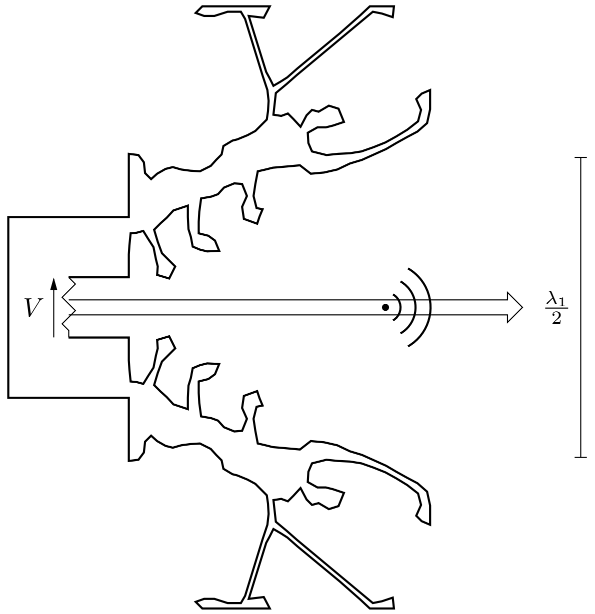

Fig. 8. Antenna – optimization result.

The dependence on ω_n is omitted henceforth. It follows from (22) and (37) that

$$\frac{dg}{d\xi} = \int_s \mathrm{Re}(\mathcal{F}) \frac{dh}{d\xi} ds, \tag{75}$$

where

$$\mathcal{F} = \frac{1}{N} \sum_n \frac{\overline{S_{21}}}{a_{2,t} a_1} \left(\mathbf{E}_t^T \gamma W^{-1} \mathbf{E} + \mathbf{H}_t^T \gamma W \mathbf{H} \right). \tag{76}$$

In the present case, the "backward" problem is a diffraction problem for the incident plane wave spectrum (three-dimensional analogs of the following formulas and explanations are presented in Appendix B):

$$\mathbf{E}_{inc,t}(\mathbf{r}) = -a_{2,t} \sqrt{\frac{\gamma W}{\tau}} \int_{-\pi}^{\pi} e^{\gamma \mathbf{v}^T \mathbf{r}} \overline{\mathbf{f}_g(\varphi)} \frac{d\varphi}{2\pi}, \tag{77}$$

$$\mathbf{H}_{inc,t}(\mathbf{r}) = -a_{2,t} \sqrt{\frac{\gamma}{\tau W}} \int_{-\pi}^{\pi} e^{\gamma \mathbf{v}^T \mathbf{r}} \mathbf{v}_\times^T \overline{\mathbf{f}_g(\varphi)} \frac{d\varphi}{2\pi}. \tag{78}$$

For the optimization of the antenna, an iterative algorithm that involves assigning certain initial weights to points on the boundary (∞ for fixed points and 1.0 for others) and executing the following steps in every iteration was designed:

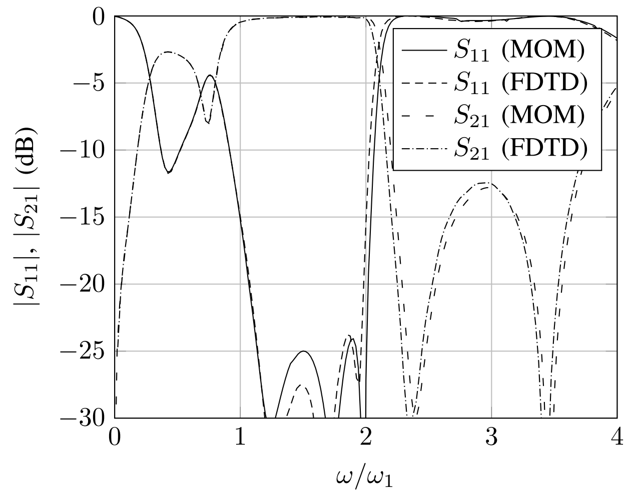

Fig. 9. Scattering parameters.

- Splitting excessively long segments ($> \lambda_1/20$) and combining excessively short segments ($< \lambda_1/60$).
- Solving the "forward" problem.
- Finding the phase center (by maximizing the goal function).
- Solving the "backward" problem (with the phase center as the origin of the coordinate system).
- Multiplying the weight of each point where $\mathrm{Re}(\mathcal{F})$ has changed sign by some value greater than one (1.05) and dividing the weight of each point where $\mathrm{Re}(\mathcal{F})$ has not changed sign by the same value.
- Shifting boundary points in proportion to the ratio between $\mathrm{Re}(\mathcal{F})$ and the weight such that the maximum shift distance is equal to some constant value ($3\lambda_1/400$).
- Removing points that are vertices of angles that are less than some minimal value ($45°$).
- Removing points that are closer than some minimum distance ($3\lambda_1/200$) to a boundary segment (the two boundary segments before the considered point and the two segments after it are not considered).

The initial approximation is shown in Fig. 7. The optimization result (iteration 579) is shown in Fig. 8. In these figures, the symbol consisting of concentric arcs and a point denotes the phase center, and the arrow denotes the reference plane of the waveguide port (this plane is shifted by the maximum distance such that the phase of the φ component of the directivity characteristic in the direction of $\varphi = 0$ does not increase with increasing n). The extent of the shifting of the boundary was limited by a rectangle with side lengths of $\lambda_1/2$ and λ_1 (as can be observed in Fig. 8). Graphs of the resulting characteristics are shown in Fig. 9, Fig. 10, and Fig. 11. The finite-difference time-domain method (FDTD) [21] was used to verify the obtained results. In Fig. 9, the corresponding results obtained via the FDTD method are provided for comparison.

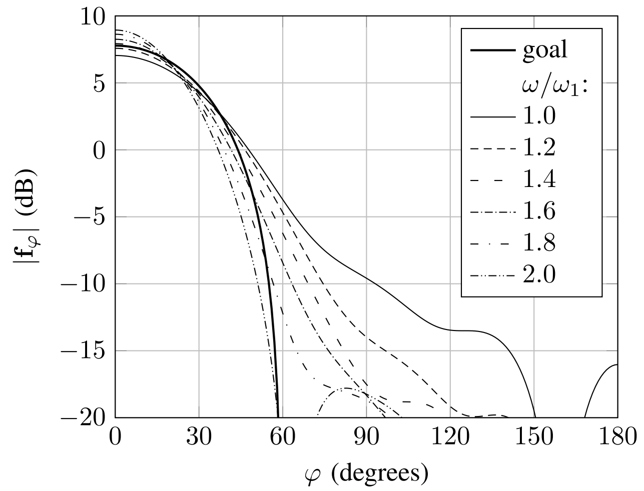

Fig. 10. Absolute value of the φ component of the directivity characteristic.

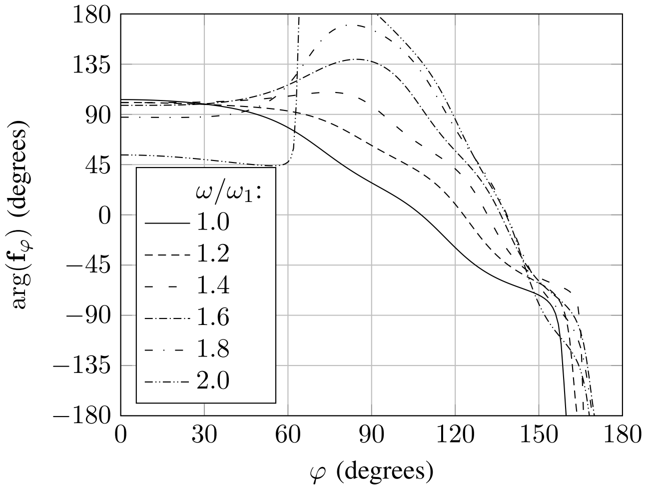

Fig. 11. Phase of the φ component of the directivity characteristic.

VIII. Conclusion

In this work, a perturbation method that produces a first-order approximation of the variations in the characteristics of a microwave device due to a small change in that device was designed. According to this method, to determine how a certain transmission coefficient will change due to a perturbation, the "forward" and "backward" problems must be solved. The result is a bilinear or sesquilinear form, the arguments of which are the electric and magnetic field intensities of the "forward" and "backward" problems at the location of the perturbation. In developing this method, the concepts of power waves and a radial waveguide have been developed, allowing the same formulas to be used for electrical circuits, microwave multiports, and antennas.

Appendix A
Power waves

In this appendix, the concept of power waves [22], [23], [24], [13, Ch. 5] is generalized to nondiagonal and infinite-dimensional source impedance operators. The overall space is assumed to consist of two regions: the source, which contains impressed currents, and the analyzed device, which does not. The given problem is described by the following system of equations:

$$\begin{cases} V + Z_s I = \mathcal{E}, \\ V - ZI = 0, \end{cases} \tag{79}$$

where $\mathcal{E}$ is the electromotive force vector of the source, Z_s is the source impedance operator, and Z is the impedance operator of the analyzed device.

The solution to this system is

$$\begin{cases} I = (Z + Z_s)^{-1}\mathcal{E}, \\ V = ZI. \end{cases} \tag{80}$$

The mean power dissipated in the analyzed device is

$$P = \frac{1}{2}\left(V^H I + I^H V\right) = \frac{1}{2} I^H (Z^H + Z) I = \frac{1}{2}\mathcal{E}^H (Z + Z_s)^{-1H} (Z^H + Z)(Z + Z_s)^{-1}\mathcal{E}. \tag{81}$$

The following relation holds for the differential of an inverse operator:

$$d\left(A^{-1}\right) = -A^{-1}(dA)A^{-1}. \tag{82}$$

Using this relation, it can be shown that the differential of (81) is equal to zero when $Z = Z_s^H$. Hence, substituting $Z = Z_s^H$ into (81) yields the available mean power:

$$P_{av} = \frac{1}{2}\mathcal{E}^H (Z_s^H + Z_s)^{-1}\mathcal{E}. \tag{83}$$

Based on physical considerations, the operator $2\left(Z_s + Z_s^H\right)^{-1}$ is assumed to exist and to be positive. Consequently, a unique positive Hermitian root of this operator exists [25]:

$$\kappa = \sqrt{2(Z_s^H + Z_s)^{-1}}. \tag{84}$$

The incident power wave vector a, where $P_{av} = a^H a$, is introduced as follows:

$$a = \frac{1}{2}\kappa(V + Z_s I) = \frac{1}{2}\kappa\mathcal{E}. \tag{85}$$

The reflected power wave vector b, which is zero when $Z = Z_s^H$, is introduced as follows:

$$b = \frac{1}{2}\kappa(V - Z_s^H I). \tag{86}$$

Using the identity

$$Z_s^H \left(Z_s^H + Z_s\right)^{-1} Z_s - Z_s \left(Z_s^H + Z_s\right)^{-1} Z_s^H = \left(Z_s^H + Z_s\right)\left(Z_s^H + Z_s\right)^{-1} Z_s - Z_s \left(Z_s^H + Z_s\right)^{-1}\left(Z_s^H + Z_s\right) = 0, \tag{87}$$

the following can be verified:

$$a^H a - b^H b = \frac{1}{2}\left(V^H I + I^H V\right) = P. \tag{88}$$

The scattering operator S is introduced as follows:

$$b = Sa. \tag{89}$$

Formulas for finding the voltage and current vectors from the known power wave vectors can be obtained from (85) and (86):

$$V = Z_s^H \kappa a + Z_s \kappa b, \tag{90}$$

$$I = \kappa a - \kappa b. \tag{91}$$

Formulas for transforming the impedance operator into the scattering operator can be obtained from (90) and (91):

$$\begin{aligned} S &= \kappa^{-1}(Z + Z_s)^{-1}(Z - Z_s^H)\kappa \\ &= \kappa^{-1}(U - (Z + Z_s)^{-1}2\kappa^{-2})\kappa \\ &= \kappa(Z - Z_s^H)(Z + Z_s)^{-1}\kappa^{-1} \\ &= U - 2\left(\kappa(Z + Z_s)\kappa\right)^{-1}. \end{aligned} \tag{92}$$

A formula for transforming the scattering operator into the impedance operator can be obtained from (92):

$$Z = 2\left(\kappa(U - S)\kappa\right)^{-1} - Z_s. \tag{93}$$

Appendix B
Radial waveguide

In this appendix, parameters for characterizing an antenna located in a homogeneous isotropic medium with a purely imaginary propagation coefficient γ and a purely real wave impedance W are introduced.

A spherical coordinate system (r, θ, φ) is introduced. The unit vector in the direction from the origin to the viewpoint is given by

$$\mathbf{v} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}. \tag{94}$$

The linear space of the complex vector functions of $\mathbf{v}$, the outputs of which are orthogonal to $\mathbf{v}$, is denoted by $\mathbf{O}$. For any two elements $\mathbf{f}_1$ and $\mathbf{f}_2$ in the space $\mathbf{O}$, their scalar product is defined as

$$\mathbf{f}_2^H \mathbf{f}_1 = \int_\Omega \mathbf{f}_2^H(\mathbf{v})\mathbf{f}_1(\mathbf{v})\frac{d\Omega}{4\pi}, \tag{95}$$

where

$$d\Omega = \sin(\theta)d\theta d\varphi. \tag{96}$$

The space $\mathbf{O}$ is a Hilbert space. It is assumed that some orthonormal basis consisting of real vector functions is adopted in $\mathbf{O}$. The set of normalized vector spherical harmonics [3, Sec. 6.3] can be taken as this basis. However, for antenna optimization, for example, it may be sufficient to use only a vector function corresponding to a desired directivity characteristic as a basis vector while not considering any other basis vectors. The coordinate space of $\mathbf{O}$ is denoted by O. Vectors in the space O are expressed as column vectors. The Fourier operator [2, Sec. 5.1] that maps $\mathbf{O}$ to O is denoted by $\mathbf{F}$. Because the Fourier operator is unitary (by Plancherel's theorem), the inverse operator is equal to the Hermitian-conjugated operator. In addition, because the adopted basis consists only of real vectors, the Hermitian-conjugated operator is equal to the transposed operator:

$$\mathbf{F}^{-1} = \mathbf{F}^H = \mathbf{F}^T. \tag{97}$$

A sphere with its center at the origin of the coordinate system and of a radius such that the sphere contains the entire antenna can be treated as a waveguide port with the following voltage and current vectors:

$$V_{sph} = \mathbf{F}\sqrt{4\pi r^2}\mathbf{v}_\times \mathbf{v}_\times^T \mathbf{E}(\mathbf{v}r), \tag{98}$$

$$I_{sph} = \mathbf{F}\sqrt{4\pi r^2}\mathbf{v}_\times \mathbf{H}(\mathbf{v}r), \tag{99}$$

where $\mathbf{v}_\times$ is the skew-symmetric matrix corresponding to $\mathbf{v}$ (analogous to (2)). Indeed, (98) and (99) are consistent with (9) and (15).

If normalized vector spherical harmonics are taken as the basis, then the impedance operator of the region outside the sphere is diagonal [3, Sec. 6.4]. For any basis, as the radius of the sphere approaches infinity, this operator approaches the scalar operator WU.

Once the voltage and current vectors and the impedance operator of the region outside the sphere are defined, the incident power wave vector a_{sph} and the reflected power wave vector b_{sph} are also defined. The limits

$$a_{far} = \lim_{r\to\infty} \frac{a_{sph}}{e^{\gamma r}}, \tag{100}$$

$$b_{far} = \lim_{r\to\infty} \frac{b_{sph}}{e^{-\gamma r}} \tag{101}$$

exist and have a finite norm for any linear combination of spherical wave functions [3, Ch. 6]. The vectors a_{far} and b_{far} can be taken as the incident power wave vector and the reflected power wave vector, respectively, of a spherical waveguide port of infinite radius whose "reference sphere" is of zero radius.

Now, the antenna can be characterized in terms of scattering operators:

$$\begin{pmatrix} b_{fed} \\ b_{far} \end{pmatrix} = \begin{pmatrix} S_{fed,fed} & S_{fed,far} \\ S_{far,fed} & S_{far,far} \end{pmatrix} \begin{pmatrix} a_{fed} \\ a_{far} \end{pmatrix}, \tag{102}$$

where a_{fed} and b_{fed} are the incident power wave vector and the reflected power wave vector, respectively, of the antenna feeder.

In cases in which γ and W do not change due to perturbation, $\mathcal{Q}$ and $\mathcal{R}$ do not depend on the radius of the spherical waveguide port:

$$\begin{aligned} \mathcal{Q} &= a_{fed,t}^H a'_{fed} - b_{fed,t}^H b'_{fed} + \frac{a_{sph,t}^H}{e^{-\gamma r}}\frac{a'_{sph}}{e^{\gamma r}} - \frac{b_{sph,t}^H}{e^{\gamma r}}\frac{b'_{sph}}{e^{-\gamma r}} \\ &= a_{fed,t}^H a'_{fed} - b_{fed,t}^H b'_{fed} + a_{far,t}^H a'_{far} - b_{far,t}^H b'_{far}, \end{aligned} \tag{103}$$

$$\mathcal{R} = -a_{fed,t}^T b'_{fed} + b_{fed,t}^T a'_{fed} - a_{far,t}^T b'_{far} + b_{far,t}^T a'_{far}. \tag{104}$$

Accordingly, in these cases, the presented method can deal with the scattering operators introduced in (102) in the same way as with the scattering operators introduced in Appendix A.

From (85), (86), (98), and (99), equations can be obtained for finding the electric and magnetic field intensities at an infinite distance from the antenna when a_{far} and b_{far} are known:

$$(\mathbf{F}^H a_{far})(\mathbf{v}) = \lim_{r\to\infty} \sqrt{\frac{4\pi r^2}{W}} \frac{\mathbf{v}_\times \mathbf{v}_\times^T \mathbf{E}(\mathbf{v}r) + W\mathbf{v}_\times \mathbf{H}(\mathbf{v}r)}{2e^{\gamma r}}, \tag{105}$$

$$(\mathbf{F}^H b_{far})(\mathbf{v}) = \lim_{r\to\infty} \sqrt{\frac{4\pi r^2}{W}} \frac{\mathbf{v}_\times \mathbf{v}_\times^T \mathbf{E}(\mathbf{v}r) - W\mathbf{v}_\times \mathbf{H}(\mathbf{v}r)}{2e^{-\gamma r}}. \tag{106}$$

The electric and magnetic field intensities near the origin in the absence of the antenna can be found from (105) and (106) using the equivalence principle:

$$\mathbf{E}_{inc}(\mathbf{r}) = \gamma\sqrt{\frac{W}{\pi}} \int_\Omega e^{\gamma \mathbf{v}^T \mathbf{r}} (\mathbf{F}^H a_{far})(\mathbf{v}) \frac{d\Omega}{4\pi}, \tag{107}$$

$$\mathbf{H}_{inc}(\mathbf{r}) = \gamma\sqrt{\frac{1}{\pi W}} \int_\Omega e^{\gamma \mathbf{v}^T \mathbf{r}} \mathbf{v}_\times^T (\mathbf{F}^H a_{far})(\mathbf{v}) \frac{d\Omega}{4\pi}, \tag{108}$$

where $\mathbf{r}$ is the position vector of the viewpoint.

Next, $a_{fed} \neq 0$ and $a_{far} = 0$ are assumed. The transmission operator $\mathbf{T}$ is introduced as follows:

$$\mathbf{T}(\mathbf{v}) a_{fed} = \lim_{r\to\infty} \sqrt{\frac{4\pi r^2}{W}} \frac{\mathbf{v}_\times \mathbf{v}_\times^T \mathbf{E}(\mathbf{v}r)}{e^{-\gamma r}}. \tag{109}$$

The coefficients in (109) are chosen such that

$$G = \frac{|\mathbf{T}(\mathbf{v}) a_{fed}|^2}{||a_{fed}||^2} \tag{110}$$

is the realized gain. Formula (109) uniquely defines the electric and magnetic field intensities at an infinitely large distance from the origin. Substituting these intensities into (106) yields

$$\mathbf{F}^H b_{far} = \mathbf{T} a_{fed}. \tag{111}$$

Consequently,

$$S_{far,fed} = \mathbf{FT}. \tag{112}$$

Now, it is assumed that $a_{fed} = 0$ and that an incident plane wave is incoming from direction $\mathbf{v}_{inc}$ with an electric field intensity of

$$\mathbf{E}_{inc}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{0}) e^{\gamma \mathbf{v}_{inc}^T \mathbf{r}}. \tag{113}$$

The reception operator $\mathbf{R}$ is introduced as follows:

$$b_{fed} = \frac{1}{\gamma}\sqrt{\frac{\pi}{W}} \mathbf{R}(\mathbf{v}_{inc}) \mathbf{E}_{inc}(\mathbf{0}). \tag{114}$$

Substituting $\mathbf{E}_{inc}$ from (107) into (114) yields

$$b_{fed} = \int_\Omega \mathbf{R}(\mathbf{v}) (\mathbf{F}^H a_{far})(\mathbf{v}) \frac{d\Omega}{4\pi} = \mathbf{RF}^H a_{far}. \tag{115}$$

Consequently,

$$S_{fed,far} = \mathbf{RF}^H. \tag{116}$$

It follows from (114) that the effective area of the antenna is

$$A = \frac{\pi}{|\gamma|^2} \frac{|\mathbf{R}(\mathbf{v}_{inc}) \mathbf{E}_{inc}(\mathbf{0})|^2}{|\mathbf{E}_{inc}(\mathbf{0})|^2}. \tag{117}$$

The scattering operator $\mathbf{S}$ is introduced as follows:

$$\frac{1}{\gamma}\sqrt{\frac{\pi}{W}} \mathbf{S}(\mathbf{v}, \mathbf{v}_{inc}) \mathbf{E}_{inc}(\mathbf{0}) = \lim_{r\to\infty} \sqrt{\frac{4\pi r^2}{W}} \frac{\mathbf{v}_\times \mathbf{v}_\times^T (\mathbf{E}(\mathbf{v}r) - \mathbf{E}_{inc}(\mathbf{v}r))}{e^{-\gamma r}}. \tag{118}$$

Analogous to (112) and (116), it is found that

$$S_{far,far} - S_{far,far}^{\varnothing} = \mathbf{FSF}^H, \tag{119}$$

where $S_{far,far}^{\varnothing}$ is the scattering operator of empty space.

It follows from (118) that the bistatic radar cross section [18, Sec. 9.2] is

$$\sigma = \frac{\pi}{|\gamma|^2} \frac{|\mathbf{S}(\mathbf{v}, \mathbf{v}_{inc}) \mathbf{E}_{inc}(\mathbf{0})|^2}{|\mathbf{E}_{inc}(\mathbf{0})|^2}. \tag{120}$$

For antenna 1 and antenna 2 located at an infinitely far distance r_{21} from each other, it is found from (109) and (114) that the S_{21} operator, which transforms the incident power wave vector of the feeder of antenna 1 into the reflected power wave vector of the feeder of antenna 2, is

$$S_{21} = \frac{1}{2} \frac{e^{-\gamma r_{21}}}{\gamma r_{21}} \mathbf{R}_2(-\mathbf{v}_{21}) \mathbf{T}_1(\mathbf{v}_{21}), \tag{121}$$

where $\mathbf{v}_{21}$ is the unit vector in the direction from antenna 1 to antenna 2, $\mathbf{T}_1$ is the transmission operator of antenna 1, and $\mathbf{R}_2$ is the reception operator of antenna 2.

From the general reciprocity condition

$$S^T = S, \tag{122}$$

it follows that

$$\mathbf{R}^T = \mathbf{T}. \tag{123}$$

The Friis transmission equation [26] follows from (121) and (123).

From the general no-loss condition

$$S^H S = U, \tag{124}$$

one obtains the no-loss condition for a transmitting antenna array [27]:

$$S_{fed,fed}^H S_{fed,fed} + \mathbf{T}^H \mathbf{T} = U. \tag{125}$$

The notion of the directivity characteristic is introduced as follows. For some vector a_{fed} in the linear space of the power wave vectors of the feeder, the function $\mathbf{T} a_{fed}/||a_{fed}||$ is called the transmission directivity characteristic and the function $\mathbf{R}^T a_{fed}/||a_{fed}||$ is called the reception directivity characteristic. For reciprocal antennas, the words “transmission” and “reception” may be omitted.

Appendix C
Stored energy

In this appendix, formulas for the mean energy stored in a lossless device are derived. The derivation is similar to that in [28, Sec. 80] but is performed in terms of the voltage and current vectors and is applicable to lossless nonreciprocal multiports.

$I(t)$ (where t is time) is assumed to be some narrowband complex signal whose Fourier transform $I(\omega)$ is concentrated near the frequency ω_0. A Taylor series expansion of the impedance operator around $\omega = \omega_0$ yields

$$V(\omega) \approx \left(Z|_{\omega=\omega_0} + \left.\frac{dZ}{d\omega}\right|_{\omega=\omega_0} (\omega - \omega_0) \right) I(\omega). \quad (126)$$

Since multiplication by frequency in the frequency domain corresponds to a time derivative in the time domain,

$$V(t) \approx \left.\left(\left(Z - \frac{dZ}{d\omega}\omega \right) I(t) + \frac{dZ}{d(i\omega)} \frac{dI(t)}{dt} \right)\right|_{\omega=\omega_0}. \quad (127)$$

The mean energy accumulation rate (mean power) is given by

$$\frac{d\mathcal{W}(t)}{dt} = \frac{1}{2} V^H(t) I(t) + \frac{1}{2} I^H(t) V(t). \quad (128)$$

By substituting the voltage vector from (127) into (128) and using the anti-Hermiticity of the impedance operator, the following can be obtained:

$$\begin{aligned} \frac{d\mathcal{W}(t)}{dt} &\approx -\frac{1}{2} \frac{dI^H(t)}{dt} \left.\frac{dZ^H}{d(i\omega)}\right|_{\omega=\omega_0} I(t) \\ &\quad + \frac{1}{2} I^H(t) \left.\frac{dZ}{d(i\omega)}\right|_{\omega=\omega_0} \frac{dI(t)}{dt} \\ &= \frac{1}{2} \frac{d}{dt} \left(I^H(t) \left.\frac{dZ}{d(i\omega)}\right|_{\omega=\omega_0} I(t) \right). \quad (129) \end{aligned}$$

Therefore, if the stored energy was zero at the initial instant of time, then

$$\mathcal{W}(t) \approx \frac{1}{2} I^H(t) \left.\frac{dZ}{d(i\omega)}\right|_{\omega=\omega_0} I(t). \quad (130)$$

Substituting

$$I(t) = I e^{i\omega_0 t} \quad (131)$$

into (130) results in the formula for the mean energy stored in a lossless device under harmonic excitation:

$$\mathcal{W} = \frac{1}{2} I^H \frac{dZ}{d(i\omega)} I. \quad (132)$$

The formula for the mean stored energy in terms of the voltage vector and admittance operator can be similarly obtained:

$$\mathcal{W} = \frac{1}{2} V^H \frac{dY}{d(i\omega)} V. \quad (133)$$

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