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**COMMON FIXED POINTS OF RELATIVELY NONEXPANSIVE MAPPINGS BY ITERATION**

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**Abstract**

Let us consider two nonempty closed convex subsets A, B of a strictly convex space and fi : A ∪ B → A ∪ B, i = 1, 2, . . . k be a reltively nonexpansive mappings. ie. fi(A) ⊆ A and fi(B) ⊆ B and ||fix − fiy|| ≤ ||x – y||, for all x ∈ A and y ∈ B. In this paper, we provide the strong convergence of some iteration of the mappings {fi}k1 to a common fixed point of {fi}k1 in strictly convex space setting, which generalizes a result of Kuhfittig [7].

***Key words:*** *Relatively nonexpansive mappings, fixed points*

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1. **INTRODUCTION**

We know that the behaviour of the iterated sequences play an role in fixed point theory. It is well known fact that if an iterated sequence of a continuous mapping T converges, then the limit of it must be a fixed point of T . Also, Banach contraction principle states that every contraction mapping T : A → A, where A is a complete subspace of a metric space X, has unique fixed point in A and every iterated sequence of T starting from any x ∈ A converges to the unique fixed point of T . But the behaviour of the iterated sequences of nonexpansive mappings are completely different from the iterated sequences of contractive type mappings.

Consider a nonexpansive mapping T : A → A, where A is a nonempty closed convex subset of a normed linear space X. In [1], Krasnoselskii proved that in uniformly convex Banach space X, the sequence of successive approximation of the averaged mapping F : A → A given by F (x) := (x + T x)/2, for all x ∈ A, converges to a fixed point of the nonexpansive mappings T . A complete proof of Krasnoselskii’s results in English can be found in [2]. Later, in [3], Edelstein extended Krasnoselskii’s result to strictly convex space setting.

In [4], the authors introduced a class of mappings called relatively nonexpansive defined as follows, which generalizes the notion of nonexpansive mappings.

DEFINITION 1. Let A, B be nonempty subsets of a normed linear space X and T : A ∪ B → A ∪ B be a mapping. Then T is said to be a relatively nonexpansive mapping if and only if

1. T (A) ⊆ A and T (B) ⊆ B,

2. ||T x − T y|| ≤ ||x – y||, for all x ∈ A, y ∈ B.

Define that dist(A, B) = inf{||a – b|| : a ∈ A, b ∈ B} and for any given pair of subsets A, B of a normed linear space X, define A0 = {x ∈ A; ||x – y|| = dist(A, B), for some y ∈ B}. In [5], the authors provided sufficient conditions which ensure the non emptiness of the set A0. In [6], the authors proved that A0 is contained in the boundary of the set A.

In [4], the authors introduced and used the geometric notion called proximal normal structure to prove the existence of the best proximity point. In [8], the authors generalized the results in [4]. In [7], the main result is as follows.

**THEOREM 1.1**. Let C be a convex compact subset of a strictly convex Banach space X and {Ti : i = 1, 2, . . . , k} a family of non-expansive self mappings of C with a nonempty set of common fixed points.

Then for an arbitrary starting point x ∈ C, the sequence {x} converges strongly to a common fixed point of {Ti : i = 1, 2, . . . , k}.

In this article, we generalized the above theorem of [7].

1. **PRELIMINARIES**

In this section, we introduce basic definition and results which we used in our main result. We generalized the iteration of nonexpansive given in [7]

**REMARK 2.1.** Let A, B be two nonempty convex subsets of a Banach space X. Let fi : A∪B → A∪B, i = 1, 2, . . . k, be a reltively nonexpansive mapping. Fix F0 = I. For 0 < α < 1.

Let F1 = (1 − α)I + αf1F0

F2  = (1 − α)I + αf2F1

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Fk = (1 − α)I + αfkFk−1.

xn+1 = (1 − α) xn + α fk Fk-1 xn  (1)

Put k = 1, xn+1 = (1 − α) xn + α f1 F0 xn (2)

= (1 − α) xn + α f1 xn

Let us state an convergence result, which plays a vital role in our main result.

**THEOREM 2.1.** [8]Let A, B be nonempty closed convex subsets of a strictly convex Banach space X such that A0 is nonempty. Let T : A ∪ B → A ∪ B be a relatively nonexpansive mapping. Suppose T (A) is contained in a compact subset A1 of A. Then the Krasnoselskii’s iteration {F n(x)}, where F: A∪B → A∪B given by

F (x) = 1\2(T x+x), converges to a fixed point of T .

1. **MAIN RESULT**

Our main result is as follows.

**THEOREM 3.1.** Let A, B be two nonempty convex, compact subsets of a strictly convex Banach space X with A0 is nonempty. Let fi : A ∪ B → A ∪ B, i = 1, 2, 3 . . .,k be mappings with a non empty set of fixed points ||fi(x) −fi(y)|| ≤ ||x – y||, ∀x ∈ A and ∀y ∈ B ∋ f (A) ⊆ A and f (B) ⊆ B with the condition that f (A) is contained in a compact subset A1 of A. Then {(x)} converges to a fixed point of fi, ∀x ∈ A ∪ B.

**Proof.** We can easily prove that the mappings Fj and fj Fj−1, j = 1, 2, . . . , k. are relatively nonexpansive and map A ∪ B into itself.

Now we are going to prove {F1, F2, . . . , Fk} and {f1, f2, . . . , fk} have the same set of common fixed points.

Let x ∈ A ∪ B with fj (x) = x, j = 1, 2, . . . , k. Then

F1(x) = (1 − α)x + αf1F0(x) = (1 − α)x + αf1(x) = (1 − α)x + αx = x,

F2(x) = (1 − α)x + αf2F1(x) = x

Proceeding like this, we get Fj (x) = x, j = 1, 2, . . . , k.

Now, let Fj (x) = x, j = 1, 2, . . . , k.

x= Fj (x) = (1 − α)x + αfj Fj−1(x) = (1 − α)x + αfj (x)

⇒ αx = αfj (x)

Hence fj (x) = x, j = 1, 2, . . . , k.

Since (1) has the same form as (2), {Fkn(x)} conveges to a fixed point y of fkFk−1. We wish to show next that y is a common fixed point of fk and Fk−1(k ≥ 2). To this we first show that fk−1Fk−2y = y (k ≥ 2). Suppose not, the closed line segment [y, fk−1Fk−2y] has positive length.

Let z = Fk−1y = (1 − α)y + αfk−1Fk−2(y)

By hypothesis, there exists a point w ∈ A ∪ B such that f1w = f2w = · · · = fkw = w. Since fi and Fi have the same common fixed points, it follows that fk−1Fk-2−w = w.

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| By relatively nonexpansive, ||fk−1Fk−2y – w|| ≤ ||y – w|| and ||fkz − w|| ≤ ||z – w|| | (3) |

So w is atleast as close to fkz as to z.

But fkz = fkFk−1y = y. Therefore w is atleast as close to y as to z = (1 − α)y + αfk−1Fk−2y.

Since X is strictly convex, ||y – w|| < ||fk−1Fk−2y – w||, which is a contradiction to (3). Therefore fk−1Fk−2y = y

Now, Fk−1 = (1 − α)y + αy = y and y = fkFk−1y = fky

⇒ y is a common fixed point of fk and Fk−1. Repeating the argument, we conclude that y is a common fixed point of fj and Fj , j = 1, 2, . . . , k

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