

21 January, 1908.

Sir WILLIAM MATTHEWS, K.C.M.G., President,
in the Chair.

(*Paper No. 3674.*)

**“Experimental Investigations of the Stresses in Masonry
Dams subjected to Water-Pressure.”**

By Sir JOHN WALTER OTTLEY, K.C.I.E., M. Inst. C.E., and
ARTHUR WILLIAM BRIGHTMORE, D.Sc., M. Inst. C.E.

THE investigations of which an account is presented in this Paper occupied from first to last a period of about 14 months, and are now submitted as a contribution to existing knowledge on the important subject of the arrangement of stresses in masonry dams.

It is perhaps hardly too much to say that, until comparatively recently, most engineers believed that the question of the correct design of a typical section for a masonry dam to resist water-pressure was one that had long since been settled, and that there was little or nothing new to say as to the stresses that existed in such a structure. It was generally assumed that the resultant stresses on any horizontal plane varied uniformly and acted in a direction parallel to the resultant force on that plane. This assumption, of course, implied a uniformly-varying shearing-stress (i.e. the horizontal component of the resultant stress at any point in a given horizontal plane), as well as a uniformly-varying normal stress (i.e. the vertical component of the resultant stress at that point).

The whole question of the distribution of stress in a dam was, however, reopened by Mr. L. W. Atcherley and Professor Karl Pearson¹ in 1904, when they published a memoir describing the results of certain experiments carried out by them on model dams.

¹ “On Some Disregarded Points in the Stability of Masonry Dams.” [*Drapers’ Company Research Memoir.*] London, 1904. Abstract in Minutes of Proceedings Inst. C.E., vol. clxii, p. 456.

It will be remembered that these models were built up of thin strips of wood—arranged horizontally in one set of experiments, and vertically in another set; the pull representing water-pressure being obtained by means of cords and weights suitably arranged. The stresses were calculated in two entirely different ways, namely, first, on the assumption that the intensity of shearing stress on the base of a dam varies as the ordinates of a parabola, and secondly, on the assumption that the intensity of shearing stress on the base is uniform.

It was claimed that on either of these assumptions the results obtained indicated the presence of considerable tension across vertical planes near the outer toe of the structure.

The late Sir Benjamin Baker, when commenting on these results in the course of the discussion on Mr. Palmer's Paper on "Coolgardie Water-Supply," remarked¹ that "there was no pretence that the investigation was final or exhaustive, but it was the first fruits of an earnest attempt to carry the theory of an elastic solid, in the special case of a dam, a stage farther than it had been carried before."

The publication of the memoir led to considerable discussion in technical journals; and to many engineers it appeared open to question whether models constructed, as these had been, of thin slips of wood were really capable of representing, with any approach to accuracy, the actual conditions obtaining in a masonry dam. In short, there seemed to be reason to think that the experiments ought to be repeated on models constructed of some more suitable material, and this view, which appeared to be somewhat widely held, led to the experiments described in the present Paper.

Meanwhile Sir Benjamin Baker himself experimented on a model dam made of stiff jelly, and when describing² the results obtained by him remarked that the first thing noticeable was that the distribution of shear where the dam met the rock was far more uniform than parabolic. In other words, he showed that the second of the assumptions referred to above was the more nearly correct of the two, a conclusion which the experiments to be described in this Paper have completely confirmed so far as regards the shearing stresses on the base of a dam.

A consideration of the equilibrium of a vertical strip of a dam appeared, however, to point to the fact that the shearing stress could not possibly be uniformly distributed on all horizontal planes. It

¹ Minutes of Proceedings Inst. C.E., vol. clxii, p. 121.

² *Ibid*, vol. clxii, p. 123.

was believed that the fact that the dam is fixed to its base, and the further fact that the water-pressure ceases abruptly at that level, must necessarily introduce conditions at the level of the base that do not exist at higher levels. It was therefore one of the objects of the investigations now reported to ascertain, if possible, the manner in which the shearing stress is distributed at levels higher than the base.

It may be said at once that, although the shearing stress is approximately uniformly distributed over the base, the experiments show that this condition does not hold for higher levels where the old assumption that the intensity of shearing stress varies uniformly from the inner to the outer face of the structure represents very accurately the actual state of affairs.

It will be observed that although both assumptions are in a measure correct, their application in both cases is only partial, because there is a gradual change from a uniform shear on the base to a uniformly-varying shear at higher levels. As in the latter case the maximum shear may amount to twice the average shear, whereas at the base it is only equal to the average, it would seem to follow that the intensity of shear near the outer profile must be greater above the base than at the base, and it will be shown later on—both from theoretical considerations and from the behaviour of the models—that this conclusion is true. It will also be shown that it is near the inner toe rather than near the outer toe that tensions may be anticipated.

The experiments to be described were carried out in the shops of the late Royal Indian Engineering College at Coopers Hill, and were made on models of a dam of typical triangular section under perfect conditions. They were restricted to this type because it was considered that any modifications of this form that might be needed in any particular case to meet special conditions of foundations, etc., would necessarily be matters that would in such case depend on the judgment and skill of the designer, and would consequently be quite outside the scope of the present investigations. Before describing the models used, it appears desirable to say a word as to the limitations which, it was considered, must be imposed on any attempt to solve—as a mathematical problem—the question of the kind and distribution of stress in such structures as are dealt with in this Paper. It is of course necessary to start with certain assumptions which are incorporated in the equations, but it has to be remembered that the results obtained from the solution of these equations are only valuable to the extent that the assumptions themselves accord with facts. It was also considered desirable to apply

mathematics in detail, rather than to attempt to make one complete mathematical problem of the whole question. These, then, were the guiding principles adopted in the present case. An endeavour was made to attack the problem in detail from every point of view that seemed to offer a solid basis of indisputable fact to start from, care being at the same time taken not to push inferences farther than was warranted by experiment, or farther than was believed, from a practical knowledge of the conditions of the problem, to be applicable in actual practice on a large scale.

It was obvious that a model which would reproduce on a smaller scale as many as possible of the conditions existing in a "full-size" structure would be of invaluable assistance in exercising the discrimination above referred to, and the first care, therefore, was to select for the model dam a material whose strength would be in something like the same ratio to the strength of masonry as the size of the model was to the size of the dam it was intended to represent, but which at the same time would be elastic within the limits of the stress to which it would be subjected, because with such precautions the displacements would be of a measurable order, and would give a correct indication of the nature and magnitude of the corresponding stresses. Such a material was fortunately found in "plasticine," a kind of modelling clay.

Its specific gravity is 1.33; its temporary elastic limit in tension was found to be about 2 lbs. per square inch; and its temporary elastic limit in compression was about 3 lbs. per square inch. The word "temporary" is here used because after a time the plasticity of the material obscures its elastic tendency. It was moreover found that plasticine possesses a property which makes it peculiarly valuable for such experiments, namely, that, notwithstanding its plasticity, it will crack quite short along the plane of maximum tension when subjected to relatively small tensions for a sufficiently long period.

DESCRIPTION OF THE MODEL DAM.

The dam was first modelled of triangular section with the vertical face exposed to the pressure of the water, the base being made equal to the height divided by the square root of the specific gravity of the plasticine, so that the resultant of the pressure on the base, due to the weight of the dam itself and the pressure of the water, would act at one-third of the width of the base from the outer toe.¹

¹ See J. H. T. Tudsbury and A. W. Brightmore, "Waterworks Engineering." 3rd edition, p. 214. London, 1905.

The height of the model was taken at 30 inches, which therefore made the base 26 inches. The length was 12 inches.

In subsequent experiments, the width of the base was reduced as in Column 1, Table III, whilst in one case a sloping face was adopted (Fig. 3, Plate 3).

The model was moulded in a frame furnished with thick plate-glass sides, and, in order to permit of accurate measurement of any displacement in the model, corresponding horizontal and vertical lines 2 inches apart were scratched on both the glass and the plasticine. The glass sides were made adjustable, and before the application of water-pressure, care was taken to see that the lines on the glass coincided accurately with the corresponding lines on the plasticine. A clearance was left between the glass sides and the model, so that the latter should receive no support from friction between it and the glass.

Water-pressure was applied to the face of the model by water contained in a thin rectangular india-rubber bag made to fit the frame. In this way the model was subjected to actual water-pressure instead of to a force exerted by a weight or spring equal to that pressure and applied at one-third the height, as it was considered possible that the latter method might not give the true distribution of the water-pressure.

The weight of the model as first made (with a 26-inch base) was 230.0 lbs. and the water-pressure on the vertical face was 195.0 lbs. This gave an average value for the intensity of normal pressure on the base of 0.74 lb. per square inch, and an average intensity of shearing stress over the horizontal base of 0.62 lb. per square inch, thus keeping well within the elastic limit of the plasticine.

DISPLACEMENTS RECORDED AFTER SUBJECTING THE MODEL TO WATER-PRESSURE.

Fig. 1, Plate 3, represents the condition of affairs after the water-pressure had been left on for 1 day. The water was siphoned off after these measurements were taken, and it is worthy of notice that the model entirely recovered its original form in the course of a few days, thus showing that the material was elastic under the pressure to which it had been subjected for the comparatively short period of 1 day.

The horizontal figures on the diagram represent the displacement of the originally vertical lines on the model as measured from the vertical lines on the glass, and the figures written vertically on the diagram denote the altered levels of the originally horizontal

lines measured in a similar manner. The negative sign (—) denotes a rise. It should be observed that the rises shown on the diagram merely denote the reduction of compression caused by the weight of the model previous to the application of water-pressure.

More important results were obtained when the water-pressure was left on for a longer time so as to allow the model to become permanently deformed owing to its plasticity, for it thus became possible to show more definitely the distribution of the stresses in the structure. Fig. 2 shows the state of the model after the water-pressure had been left on for 33 days. The crack shown in the diagram began to appear after 7 days. The displacements of the originally vertical and horizontal lines are indicated as before, by horizontal and vertical figures on the diagram.

An examination of the displacements of the lines which were originally vertical in Fig. 2 shows that near the base they are about equally inclined,¹ thus proving that the intensity of shearing stress on the base is more or less uniform, and not—as had generally been assumed—proportional to the intensity of normal pressure on the base. It will also be observed from Fig. 2 that in the outer portion of the dam, the lines that were originally vertical tend—up to a certain level—to become more inclined as the height from the base increases, thus indicating an increase in the intensity of the shearing stress above the level of the base. Now it is obvious that the mean intensity of the shearing stress on a horizontal plane must decrease as the height of the plane above the base increases, since the water-pressure varies as the square of the depth, whereas the width of the dam only increases as the depth. It follows from these considerations that the shearing stress cannot be uniformly distributed on horizontal planes above the base, but must vary from a small value at the inner face to a maximum near the outer face.

Turning now to the lines that were originally horizontal, it will be seen that near the base they become curved, sloping downwards from the inner face to about two-thirds of the width of the dam, then remaining fairly level to a point near the outer face, when they bend slightly upwards. At higher levels these lines gradually tend to become straight lines sloping uniformly from the inner to the outer face of the dam.

The arrangement of normal stress on the base indicated by the new position of the originally horizontal line is only what might

¹ The presence of the crack at the inner toe magnifies the displacements near that toe.

have been expected, since the tendency of the normal stress to become a maximum at the outer extremity of the base is counteracted by the increase of flexibility due to the decreasing height of the dam as the outer toe is approached. The foregoing arrangement of normal stress has been accepted as the basis of the calculations that follow, and it will first be shown that the curve representing it can be drawn with sufficient accuracy.

The amount of shear in vertical planes will then be deduced and applied to determine the distribution of shearing stress on the base. Lastly, the manner in which this distribution of stress is altered by the fact that the dam is "fixed" at its base will be investigated.

INVESTIGATION OF SHEARING STRESSES ON VERTICAL PLANES IN THE MODEL.

To draw the line representing the intensity of normal reaction at the base the following facts must be borne in mind:—

- (1) The total normal reaction equals the weight of the dam.
- (2) Since the resultant pressure on the base acts at one-third the width from the outer toe, the moment of the reaction stresses about this point must be zero.
- (3) The intensity of the reaction at the outer toe must equal the intensity of the shearing stress in the vertical plane multiplied by the ratio of the height to the base of the dam.

If, therefore, in the diagram (Fig. 4, Plate 3), where AB represents the base of the dam, the line AC is drawn such that BC equals twice the average intensity of normal stress on the base, then the area ABC will equal the total normal stress on the base, i.e. the weight of the dam. From condition (1) above, it follows that the line AE representing the actual intensity of normal reaction must be drawn so that the area Y outside AC must be equal to the sum of the areas X and Z inside that line. From (2) it is obvious that the moments of the areas X, Y, and Z about a vertical line through D (where $DB = \frac{AB}{3}$) must equal zero. From (3) it follows that BE must equal the limiting value of the intensity of shearing stress in a vertical plane near the outer toe, multiplied by the ratio of the height to the base of the dam.

From these considerations the proper shape of the line AE can be found after several trials. The stresses perpendicular to the cross-section are assumed to be balanced, so that only coplanar forces are considered. No doubt such stresses are generally present, but their magnitude is generally indeterminate and depends upon such

circumstances as settlement during construction and variations of temperature, which it is clearly impossible to estimate for in such a calculation as that now presented.

Taking a 1-inch length of the model dam and dividing the breadth of the base into inches, and considering the equilibrium of each vertical strip of 1 square inch in section, it is obvious that the intensity of normal reaction on the base of each strip, minus the weight of the strip, equals the difference of shearing stresses on the vertical planes on either side of that strip.

If therefore in Fig. 4, at the centre of each inch, the weight of the strip (column 2, Table I) be plotted upwards from the curve AE, the ordinates of the curve FE so obtained will represent the differences of the shearing stresses on the vertical planes on either side of the strip concerned.

In column 4 of Table I, the ordinates of the curve FE are

TABLE I.—TRIANGULAR SECTION, 1 INCH LONG.

1	2	3	4	5	6	7
No. of Vertical Section.	Weight of Strip Between Sections.	Normal Reaction on Base of Strip. (See Fig. 4, Plate 3.)	Difference of Shear on the Vertical Sides of Strip.	Total Shear on Vertical Section.	Intensity of Shear on Vertical Section.	Remarks.
	Lbs.	Lbs.	Lbs.	Lbs.	Lbs. per Square Inch.	
0	Outer toe.
1	0.03	1.14	-1.11	1.11	0.97	
2	0.08	1.17	-1.09	2.20	0.93	
3	0.14	1.18	-1.04	3.24	0.92	
4	0.19	1.20	-1.01	4.25	0.91	
5	0.25	1.21	-0.96	5.21	0.90	
6	0.31	1.21	-0.90	6.11	0.88	
7	0.36	1.20	-0.84	6.95	0.86	
8	0.42	1.18	-0.76	7.71	0.84	
9	0.47	1.14	-0.67	8.38	0.81	
10	0.53	1.09	-0.56	8.94	0.78	
11	0.58	1.00	-0.42	9.36	0.74	
12	0.64	0.91	-0.27	9.63	0.70	
13	0.69	0.84	-0.15	9.78	0.65	
14	0.75	0.74	+0.01	9.77	0.61	
15	0.80	0.65	+0.15	9.62	0.56	
16	0.86	0.58	+0.28	9.34	0.51	
17	0.91	0.48	+0.43	8.91	0.45	
18	0.97	0.43	+0.54	8.37	0.40	
19	1.03	0.36	+0.67	7.70	0.35	
20	1.08	0.30	+0.78	6.92	0.30	
21	1.14	0.24	+0.90	6.02	0.25	
22	1.19	0.19	+1.00	5.02	0.20	
23	1.25	0.15	+1.10	3.92	0.15	
24	1.31	0.09	+1.22	2.70	0.10	
25	1.36	0.06	+1.30	1.40	0.05	
26	1.42	0.02	+1.40	Inner toe.

tabulated at each inch in width, and in column 5, the summation of these figures to any vertical section, starting at the foot of the column, gives the total shear in that vertical section. In column 6, the total vertical shears are divided by the heights, thus giving the average intensity of shearing stress in the various sections. The total shears and the average intensities of shearing stresses on the vertical planes are shown by curves in Fig. 4, from which it will be seen that the average intensity of shearing stress on vertical planes in the outer half of the dam is greater than the intensity of shearing stress on the base, and also that near the inner face the average intensity of shear in vertical planes is very small.

Bearing in mind that at any point the intensity of shearing stress on horizontal and vertical planes is equal, and that the intensity of shearing stress on the base is practically constant, it follows that at levels above the base the intensity of shearing stress on horizontal or vertical planes is very small near the inner face, whilst in the outer half of the dam the intensity of shear increases above the base, especially as the outer face is approached. In a horizontal section above the base the intensity of shear therefore increases from zero at the inner face to a maximum at the outer face.

It will be observed that this is the conclusion arrived at independently from observation of the tendency of the displaced vertical lines in the model to increase their inclinations at some little distance above the base.

The foregoing conclusions, and the fact that the originally horizontal lines tend to become straight lines sloping from the inner to the outer face as the distance above the base increases, show that although the old assumption¹ that the shearing stress is proportional to the normal stress does not hold good at the base (because of the "fixing" of the dam at that level), yet at higher levels it is approximately correct, and that therefore the shearing stresses to be provided against are not those at the base, but at rather higher levels near the outer profile.

DETERMINATION OF SHEARING STRESS ON THE BASE NEGLECTING THE "FIXING" AT THAT LEVEL.

It is now proposed, in the first place, to determine the distribution of shearing stress on the base consequent on the distribution of normal stresses already described, and then to examine the reasons for the difference between the results so obtained and the actual distribution as revealed by the model.

¹ "Waterworks Engineering." 3rd edition, p. 210.

If the stress due to the reaction of the foundation on the base, and that due to the weight of each strip, be considered to be constant for each inch of breadth, their resultant will act at the centre of each strip, and (taking moments about the centre of each inch in succession) the difference of the moments of the horizontal pressures on the vertical sides of a strip will equal the sum of the shearing stresses on the vertical sides of that strip multiplied by $\frac{1}{2}$ inch (Column 2, Table II).

Calling P_n the pressure at right angles to a strip n inches from the outer toe, and y_n the height of the point of application of this pressure above the base—

$$P_{26} y_{26} = 162 \cdot 5 \text{ lbs. (the moment of the water-pressure).}$$

$$P_{26} y_{26} - P_{25} y_{25} = \frac{1 \cdot 40}{2} \text{ (from Column 5, Table I).}$$

$$P_{25} y_{25} - P_{24} y_{24} = \frac{4 \cdot 10}{2}.$$

$$P_{24} y_{24} - P_{23} y_{23} = \frac{6 \cdot 62}{2}.$$

$$P_2 y_2 - P_1 y_1 = \frac{3 \cdot 31}{2}.$$

$$P_1 y_1 = \frac{1 \cdot 11}{2}.$$

This enables the moment of the horizontal pressure on each vertical strip to be calculated by adding the figures in Column 2 down to any section as given in Column 3, Table II, and shows incidentally that $P_{26} y_{26}$ is equal to the sum of the shearing-stresses on the vertical planes 1 inch apart.

The horizontal shear on each inch of base is obviously equal to the difference in the horizontal pressure acting on the two sides of that strip. To find the horizontal pressures from their moments, it is necessary to know the heights of their points of application. These can be approximately determined, because at the inner profile the resultant water-pressure acts at right angles to it at one-third the height, whereas at the outer toe the horizontal pressure will act at one-half the height; for, if considering a triangle, say, 1 inch in breadth at the outer toe, the resultant normal stress on its base acts approximately at its centre, but (as there can be no stress on the outer profile itself) the resultant stress on that inch must be parallel to the outer profile (see *Fig. 6*, p. 104).

It therefore follows that the resultant stress on the vertical, 1 inch from the outer toe, acts approximately at the centre of its

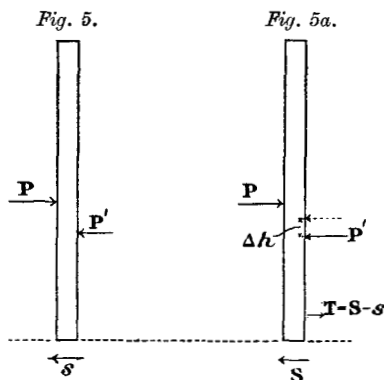
TABLE II.—TRIANGULAR SECTION, 1 INCH LONG.

1	2	3	4	5	6	7
No. of Section.	Difference of Moments of Pressure on the Two Sides of the Strips about the Base.	Moment of Horizontal Pressure on Section.	Height of Point of Application of Pressure at Section.	Horizontal Pressure across Vertical Section.	Shearing Stress on Base of the Strip.	Remarks.
	Inch-lbs.	Inch-lbs.	Inches.	Lbs.	Lbs. per Sq. Inch.	
0	Outer toe.
1	0·6	0·6	0·6	1·00	1·00	
2	1·7	2·3	1·2	1·91	0·91	
3	2·6	4·9	1·8	2·72	0·81	
4	3·7	8·6	2·4	3·58	0·86	
5	4·7	13·3	3·0	4·43	0·85	
6	5·7	19·0	3·6	5·28	0·85	
7	6·5	25·5	4·2	6·07	0·79	
8	7·3	32·8	4·7	6·98	0·91	
9	8·0	40·8	5·2	7·84	0·86	
10	8·7	49·5	5·6	8·84	1·00	
11	9·2	58·7	6·0	9·79	0·95	
12	9·5	68·2	6·4	10·66	0·87	
13	9·7	77·9	6·8	11·46	0·80	
14	9·8	87·7	7·2	12·18	0·72	
15	9·7	97·4	7·6	12·81	0·63	
16	9·5	106·9	8·0	13·36	0·55	
17	9·2	116·1	8·4	13·82	0·46	
18	8·6	124·7	8·7	14·33	0·51	
19	8·0	132·7	9·0	14·74	0·41	
20	7·3	140·0	9·3	15·05	0·31	
21	6·5	146·5	9·5	15·42	0·37	
22	5·5	152·0	9·7	15·67	0·25	
23	4·4	156·4	9·8	15·96	0·29	
24	3·3	159·7	9·9	16·13	0·17	
25	2·1	161·8	9·98	16·21	0·08	
26	0·7	162·5	10·0	16·25	0·04	Inner toe.

height. Now, if a fair curve be drawn tangential to the line at right angles to the inner profile at one-third its height, and also tangential to the line joining the outer toe to the centre of the vertical at 1 inch from it, the ordinates of this curve will represent very approximately the heights of the points of application of the horizontal pressures on the vertical strips. Column 4, Table II, gives these ordinates as approximately measured from Fig. 1. Column 5 of the same Table gives the values of the horizontal pressures deduced from them, and Column 6 gives the difference of these pressures, or, in other words, the horizontal intensities of shearing stress for each inch of the base. It will be noticed that the latter values increase from practically zero at the inner toe to a point near the centre of the base, and then remain fairly constant.

MODIFICATION OF DISTRIBUTION OF SHEARING STRESS ON THE
BASE OWING TO THE DAM BEING "FIXED" AT THAT PLANE.

The pressure of the water on the inner face of the dam exerts a compressive force parallel to the base, and the consequent tendency to displacement in that direction will be a maximum at the inner toe and will diminish to zero at the outer toe. As the dam is fixed to the foundation at the base, this movement is prevented, and the shearing stress thus induced would be a maximum at the inner toe and zero at the outer toe. It has, however, already been shown that the forces acting on the dam would make the shearing stress zero at the inner toe and a maximum at the outer toe. The effect of these two conflicting conditions is to cause the shearing stress to be nearly uniform over the base, as indicated in the models



by the practically constant inclination at that level of the displaced vertical lines to which attention has already been drawn.

It might appear at first sight that the effect of fixing the base would be to cause the first few feet of base near the inner profile to take an altogether undue share of the shear, but a little consideration will show that this is not really the case. If *Fig. 5* represents a vertical strip near the inner profile, neglecting the effect of "fixing," it is evident that it is kept in equilibrium in the horizontal direction by the pressures P and P' and the shear s . Now the effect of "fixing" is to increase the shear s to an amount S (*Fig. 5a*), and, in order that equilibrium should still be maintained, it is obvious that a tension $T = S - s$ must be induced near the base which will reduce the intensity of compression across the vertical plane at this level; in fact, in certain cases the resulting stress across vertical planes at the base near the inner face may be a

tension. The fact of the tension being introduced on the vertical plane necessarily throws the point of application of the resultant pressure higher up the strip by a distance, say, Δh , as indicated by the dotted arrow in *Fig. 5a*. In this way the intensity of shear on the base is automatically prevented from becoming excessive, since the raising of the point of application of the pressure across the vertical plane throws the centre of resistance to that pressure farther back along the base.

Further proof of the uniform distribution of the shear on the base of a dam built of homogeneous material will now be adduced.

FURTHER EVIDENCE OF THE UNIFORM DISTRIBUTION OF THE SHEARING STRESS ON THE BASE OF A DAM.

It was thought desirable to obtain further experimental evidence as to the uniformity or otherwise of the distribution of the shearing stress over the base of the dam, and for this purpose advantage was taken of the property of "plasticine" to crack quite short along the plane of maximum tension. Model dams were constructed of the same height (30 inches) as before, but with widths of base equal to 26 inches, 23 inches, 20 inches and 17 inches respectively. When these models had been subjected to water-pressure for a sufficient time, cracks invariably appeared at the inner toe, and the angles (of inclination to the horizontal) of these cracks were found to steadily diminish as the base decreased; from 45° in the case of the widest base to about 25° in the case of the narrowest one.

Table III gives the measured values of the cotangents of these angles as compared with the calculated values. The figures in the last column are the mean of the measurements on the two sides of the models. Column 2 was calculated on the assumption of "uniformly varying" normal stress. Columns 4 and 5 were calculated from the following well-known and readily deduced formulas (see Appendix) for the amount and direction of the principal stresses at a point, when p and p' are the intensities of normal stresses on the horizontal and vertical planes (compressions being taken as positive and tensions as negative), q is the intensity of shear on the two planes, and is taken as equal to the average intensity of shear on the base, f is the principal stress, and θ the inclination of its plane of application to the horizontal:—

$$f = \frac{p + p' \pm \sqrt{(p + p')^2 - 4(p p' - q^2)}}{2} \quad . \quad . \quad . \quad (1)$$

$$\text{Cot } \theta = \frac{f - p'}{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

TABLE III.

1	2	3	4	5	6
Width of Base of Model Dam.	Calculated Tension Normal to the Base at the Inner Toe.	Average Intensity of Shearing Stress on the Base.	Calculated Intensity of Maximum Tension at the Inner Toe.	Calculated Cotangent of Inclination of Plane of Maximum Tension.	Measured Value of Cotangent of Inclination of Crack in Model Dam.
Inches.	Lbs. per Square Inch.	Lbs. per Square Inch.	Lbs. per Square Inch.		
26	0.0	0.62	0.62	1.0	1.0
23	0.32	0.71	0.88	1.24	1.3
20	0.93	0.81	1.40	1.74	1.8
17	1.83	0.95	2.23	2.35	2.1

In the case of the first three models, p' was practically zero, because the plasticine could draw away from the frame at the front of the base. In the last case, namely, that of the 17-inch-base dam, the plasticine below the base adhered to the wooden frame at the front, and this, according to formula (2), would cause the crack to be steeper—a result that was actually observed (see Column 4 of the Table). It is also probable that the limit of temporary elasticity in tension was exceeded in this particular experiment.

It will be observed from the figures in Columns 5 and 6 of Table III that the inclination of the crack formed at the inner toe accords very closely in each case with its direction as calculated on the assumption of a uniformly distributed shearing stress over the base, and therefore these experiments strongly support the inference that the shearing stress on the base is practically uniformly distributed, not only in dams of the proper theoretical breadth, but even when this breadth is considerably reduced.

OCCURRENCE OF TENSIONS NEAR THE INNER TOE.

Although a dam may be designed so as to have no tension on the horizontal plane at the inner toe when subjected to water-pressure, there are still tensions on other planes passing through that point—a fact that has been often overlooked.

To emphasize this, a model dam was constructed with sloping face (Fig. 3, Plate 3), the slope being 1 in 15 and the width of base 26 inches (the same as in Figs. 1 and 2).

With this design, when the water-pressure is applied there should be a slight compression on the horizontal plane at the inner toe. Notwithstanding this (as will be seen from Fig. 3), a crack was produced. The inclination of this crack was rather steeper than equation 2 would account for if p' be taken equal to zero, but this appears to be due to the fact that, in this case also, the plasticine below the level of the base¹ adhered to the wood of the frame at the front. That this is the correct explanation would appear to be proved by the fact that a vertical crack was formed in the plasticine below the base-line in front of the inner face (Fig. 3). This would cause a tension across the vertical plane at the inner toe. In other words, p' in equation 2 is not equal to zero (as assumed), but has a negative value which reduces the numerical value of the cotangent (see Appendix).

It is evident that the plane of maximum tension at the inner toe can be made more vertical if the front part of the dam below the level of the base be tied horizontally into the foundation, as, for instance, by embedding horizontal steel bars into that foundation, because such a device would introduce a tension across the vertical plane as soon as water-pressure was applied, and it is hardly necessary to point out that in the event of the formation of a crack, the more vertical this crack is the better for the work, so that any "uplift" due to water getting into it may be avoided. The Authors do not advocate the use of steel bars in this position, but consider that it is desirable to get all the bond possible with the rock in front of the dam below the base-level.

IMPOSSIBILITY OF TENSIONS OCCURRING ACROSS VERTICAL PLANES NEAR THE OUTER TOE.

Equation (1) shows that if at any point

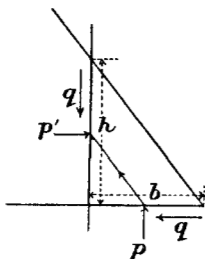
$$pp' > q^2$$

there is no tension on any plane passing through that point, because in that case both principal stresses (and consequently the stresses on all other planes through that point) are compressions. It can be shown that this condition is satisfied near the outer toe, and that consequently there cannot be a tension across any vertical plane in that position.

¹ The base may here be defined as the horizontal line at its level, at which the water-pressure ceases to act.

To make this clear, it is only necessary to consider the equilibrium of a wedge of unit length at right angles to the paper cut off by a vertical plane infinitely near to the outer toe. If p' (Fig. 6) is the intensity of pressure normal to the vertical plane at the base, and p the intensity of reaction normal to the base, and q the intensity of shearing stress, then obviously $p' \times h = q \times b$

Fig. 6.



$$\text{or } p' = q \frac{b}{h} \quad \dots \quad (1)$$

Now the weight of the triangle becomes negligible because it varies as h^2 , and since the resultant stress must be parallel to the outer profile it follows that

$$p = q \frac{h}{b} \quad \dots \quad (2)$$

Multiplying (1) and (2), $pp' = q^2$ at the outer toe. Now it will be observed from an inspection of the curve A E (Fig. 4, Plate 3) that the value of p increases for some distance from the outer toe; and that, as shown by the dotted curve in Fig. 1, the point of application of the resultant pressure, normal to the vertical plane, becomes relatively lower as the section under consideration recedes from the outer toe, and, since the average intensity of the pressure remains constant, it follows that p' increases as the distance from the outer toe increases, and consequently in the neighbourhood of the outer toe pp' must be greater than q^2 , and consequently there can be no tension in this region.

It may be added that this absence of tension is confirmed by the behaviour of the models, as will be seen from the recorded measurements.

PERMISSIBLE VALUES OF COMPRESSIVE AND SHEARING STRESSES IN DAMS.

With respect to the magnitude of the permissible compressive and shearing stresses in a masonry dam, it has been suggested¹ that for rubble masonry 10 tons per square foot is a suitable value for the maximum intensity of compressive stress, i.e. the principal stress near the outer toe. If the specific gravity of the masonry be taken as $2\frac{1}{4}$, the ratio of base to height of a triangular dam would be 2 : 3,

¹ "Waterworks Engineering." 3rd edition, p. 230.

therefore calling α the angle subtended by the profiles of the dam at the apex, the maximum intensity of normal stress on the horizontal planes at the outer toe would then be less than $10 \times \cos^2 \alpha = 10 \times \frac{9}{13}$ = rather less than 7 tons per square foot, and the maximum intensity of shearing stress on either horizontal or vertical planes would be less than $10 \sin \alpha \cos \alpha = 10 \times \frac{6}{13} = 4.6$ tons per square foot—stresses which it is believed will be generally conceded to be safe values for good rubble masonry.

It may be pointed out incidentally that only the maximum normal intensity of compression on the horizontal plane is usually mentioned when stating the working intensity of stress for which a dam is designed, whereas the facts to which attention has been drawn above point to the conclusion that the maximum intensity of compressive stress in the dam (which is about 50 per cent. greater) is really the stress which should not be allowed to exceed the safe working intensity of stress permissible for the material employed in the construction.

CONCLUSIONS.

From the foregoing investigation it is submitted that the following conclusions may be drawn :—

(1) If a masonry dam be designed on the assumption that the stresses on the base are “uniformly varying,” and that these stresses are parallel to the resultant force acting on the base, the actual normal and shearing stresses, on both horizontal and vertical planes, would (in the absence of stresses due to such factors as changes in temperature, unequal settlement, etc.) be less than those provided for.

(2) There can be no tension on any plane at points near the outer toe.

(3) There will be tension on certain planes other than the horizontal plane near the inner toe; the maximum intensity of such tension in the foundation being generally equal to the average intensity of shearing stress on the base, and the inclination of its plane of action being about 45° ; and its maximum intensity in the dam, above the base, about one-half the above amount and acting on a plane less inclined to the horizontal.

The Paper is accompanied by diagrams from which Plate 3 and the Figures in the text have been prepared.

[APPENDIX.

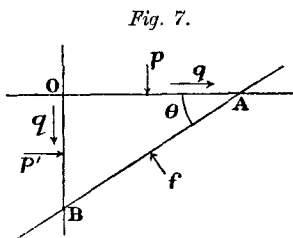
APPENDIX.

FORMULA FOR AMOUNT AND DIRECTION OF "PRINCIPAL" STRESSES AT A POINT.

Consider the equilibrium of a small triangular wedge OAB (*Fig. 7*) at the point O, of unit length at right angles to the paper.

p and q are the normal and shearing intensities of stress on OA.
and p' „ q „ „ similar intensities of stress on OB.

Let AB be one plane of "principal" stress, and f its intensity. Resolving vertically and horizontally:—



$$f \cos \theta = p \cos \theta + q \sin \theta$$

$$f \sin \theta = q \cos \theta + p' \sin \theta;$$

$$\therefore f - p = q \tan \theta \quad \dots \quad (1)$$

$$\text{and } f - p' = q \cot \theta \quad \dots \quad (2)$$

Multiplying (1) by (2)

$$(f - p)(f - p') = q^2$$

$$\text{or } f^2 - (p + p')f + pp' - q^2 = 0;$$

$$\therefore f = \frac{p + p' \pm \sqrt{(p + p')^2 - 4(pp' - q^2)}}{2} \quad \dots \quad (3)$$

$$\text{or it may be written } f = \frac{p + p' \pm \sqrt{(p - p')^2 + 4q^2}}{2}$$

$$\text{and from (2)} \quad \cot \theta = \frac{f - p'}{q} \quad \dots \quad (4)$$

In the special case when $p' = 0$, equation (3) becomes

$$f = \frac{p \pm \sqrt{p^2 + 4q^2}}{2} \quad \dots \quad (5)$$

$$\text{and (4) becomes} \quad \cot \theta = \frac{f}{q}.$$

From equation (5) it is obvious that, in this special case, the larger of the principal stresses is a compression, and the smaller a tension.

In the particular case when there is a tension on the plane OA and also a tension across the vertical plane OB, the plane of maximum tension becomes more vertical than if the latter stress = 0; because in the latter case

$$\cot \theta = \frac{p - \sqrt{p^2 + 4q^2}}{2q}, \text{ when } p \text{ is negative;}$$

and in the former case

$$\cot \theta = \frac{p - \sqrt{(p - p')^2 + 4q^2}}{2q},$$

which is numerically less than $\cot \theta$ when $p' = 0$, because p and p' are both negative; therefore the plane is more vertical in the latter case.

Finally, it is evident from equation (3) that if $pp' > q^2$, both principal stresses are of the same sign, and, as this is the case near the outer toe, and p and p' are compressions there, the principal stresses are compressions, and consequently there can be no tension on any plane passing through such points.

Fig. 1.

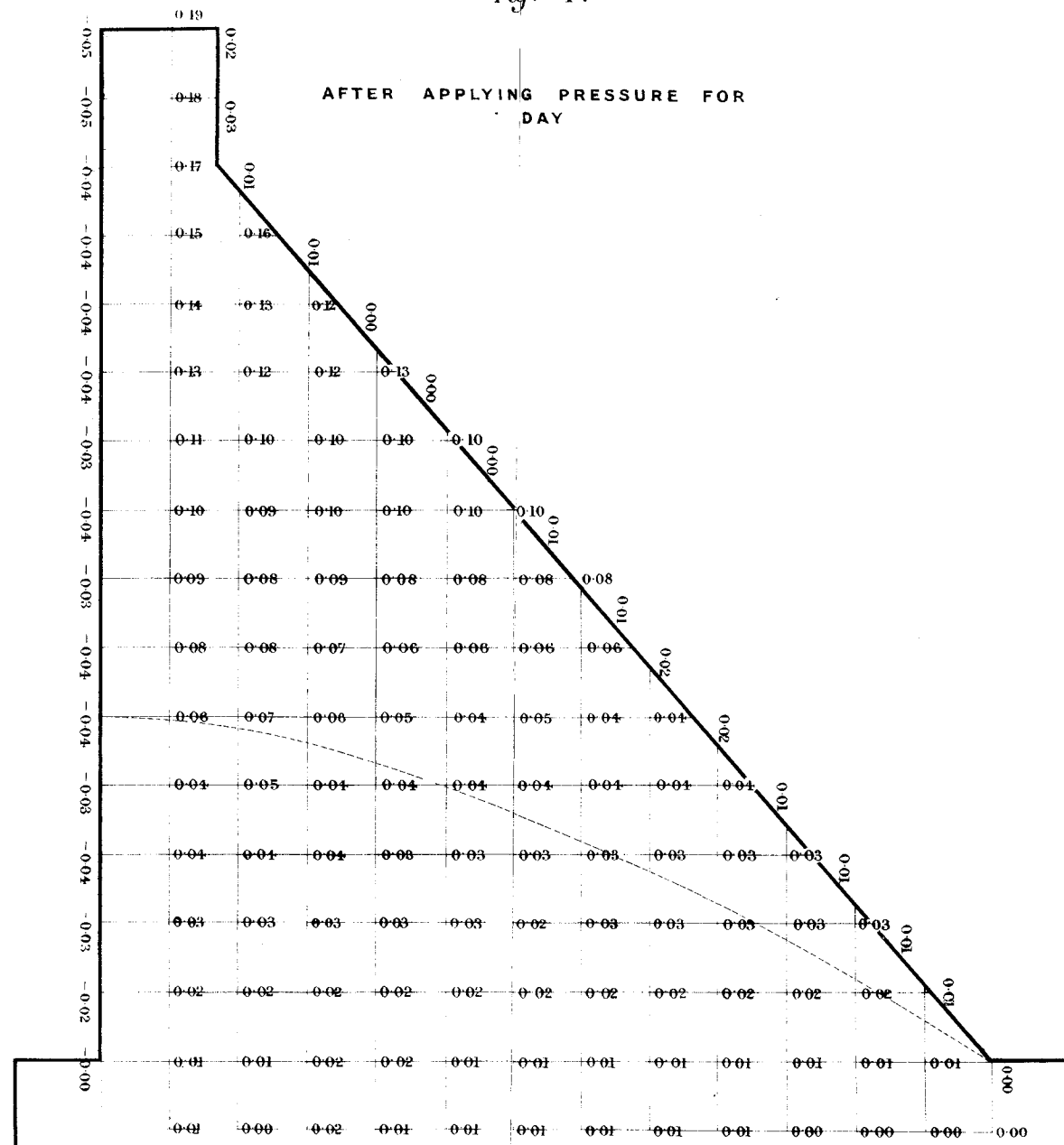


Fig. 2.

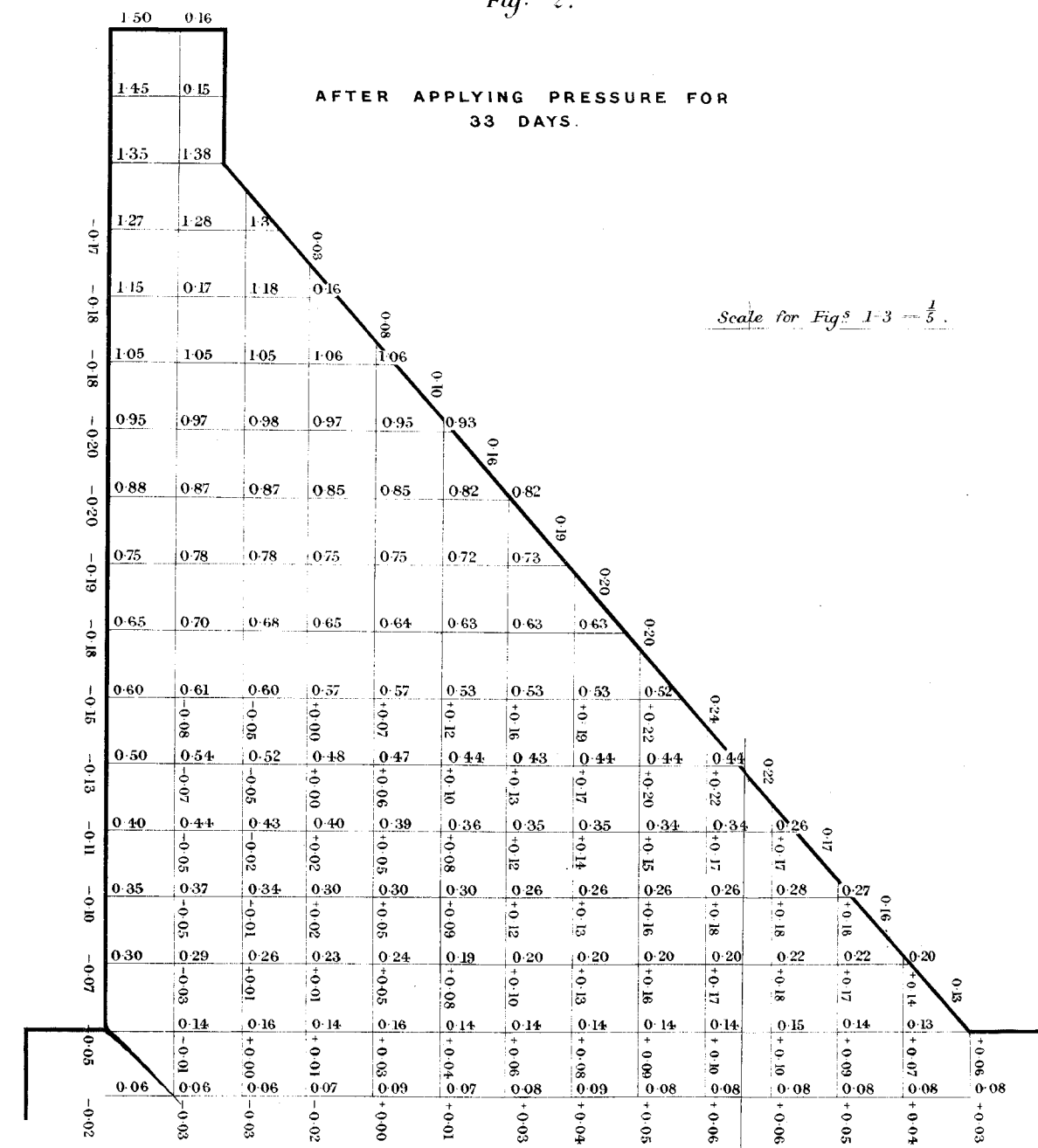


Fig. 3.

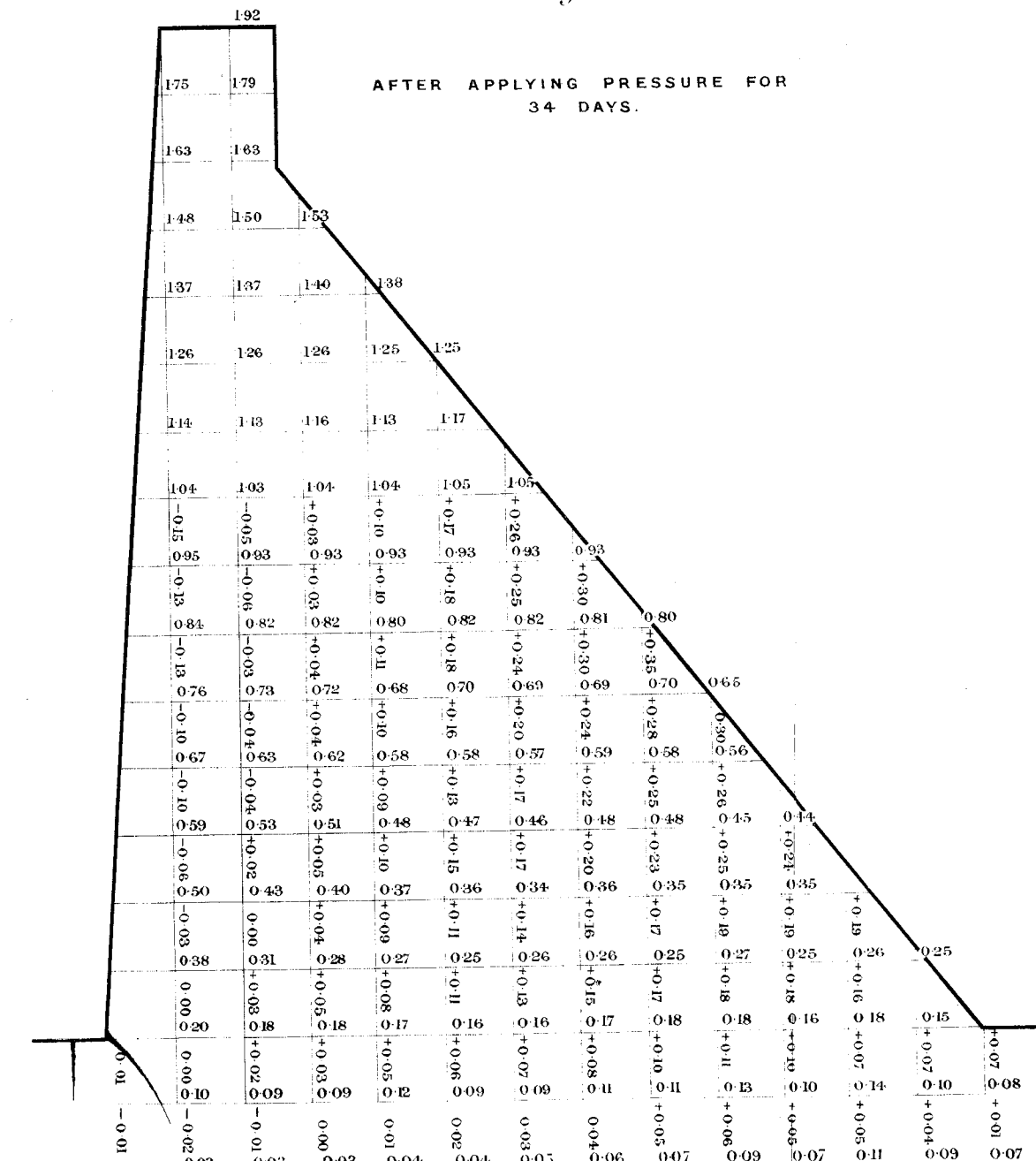
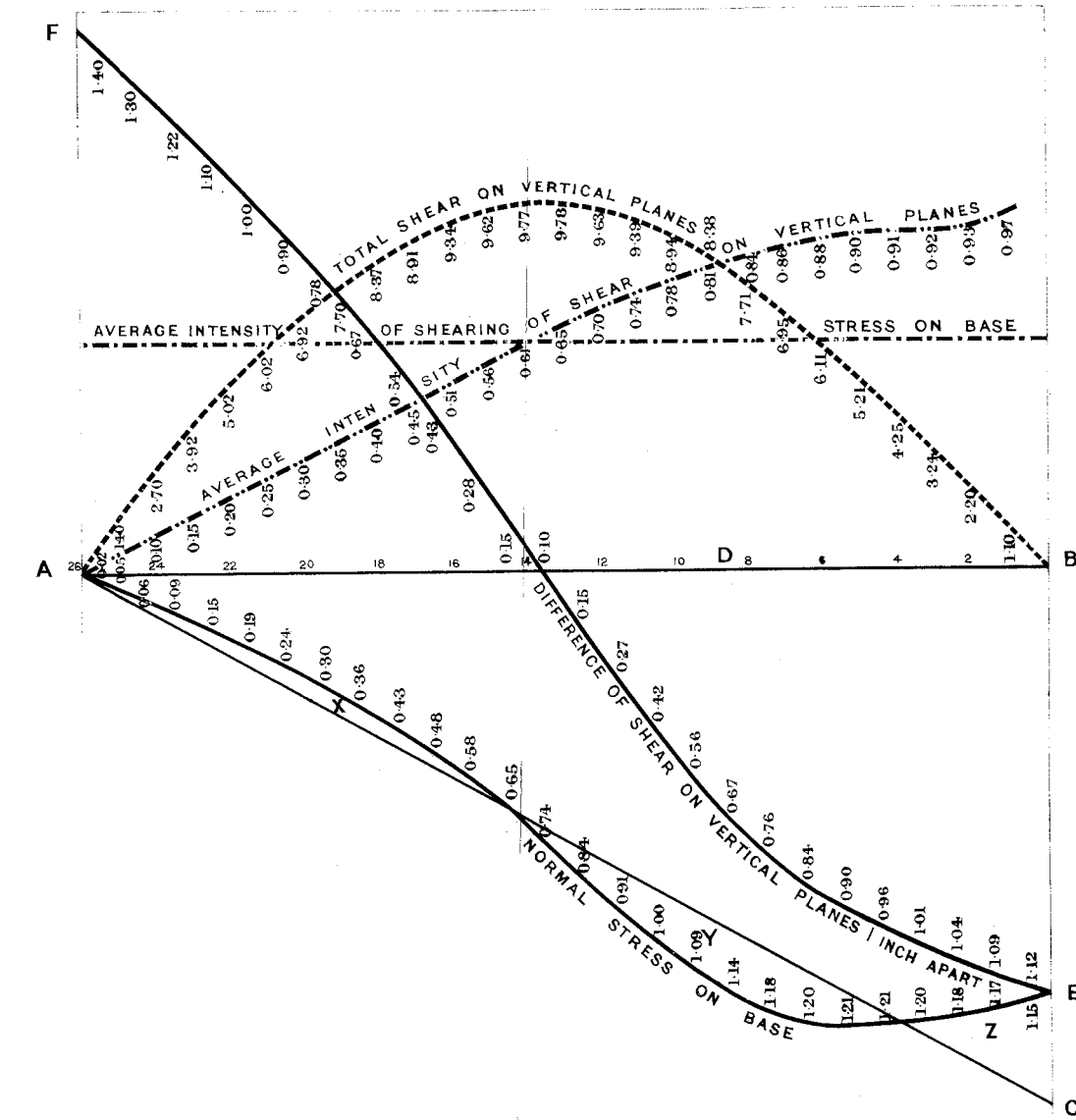
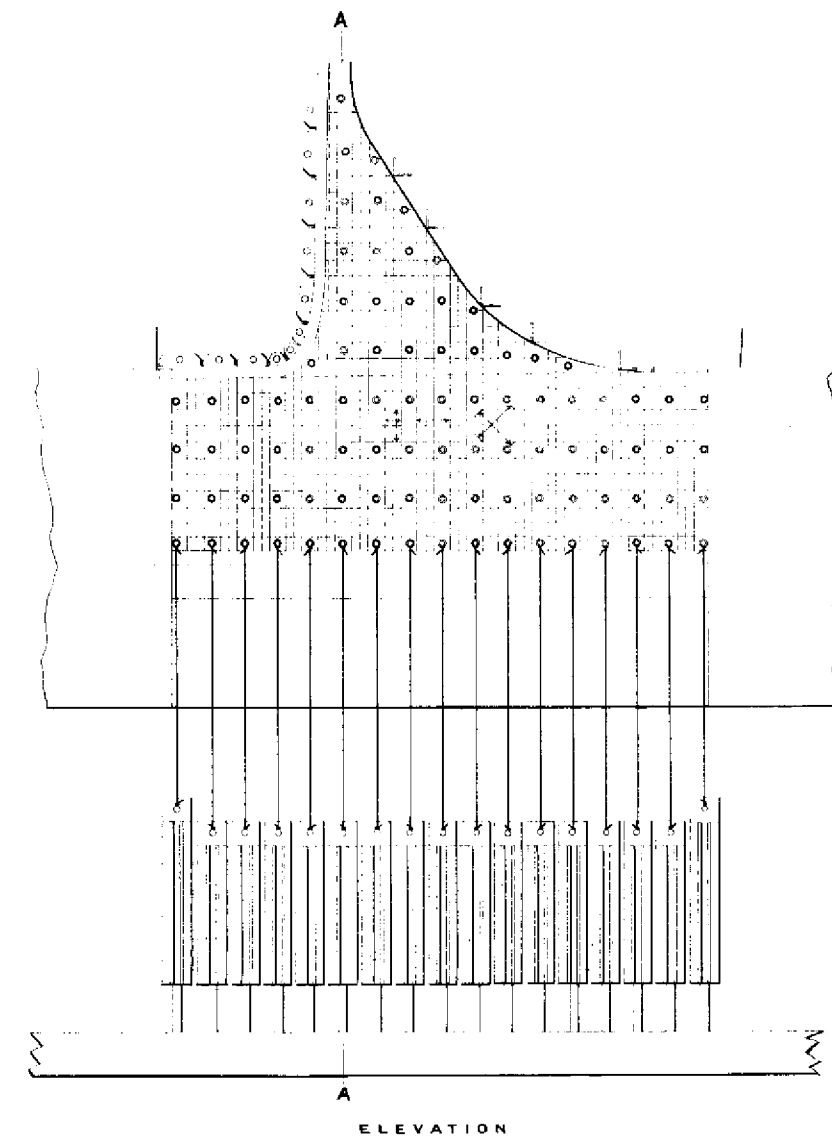


Fig. 4.



Figs. 1.



Figs. 2.

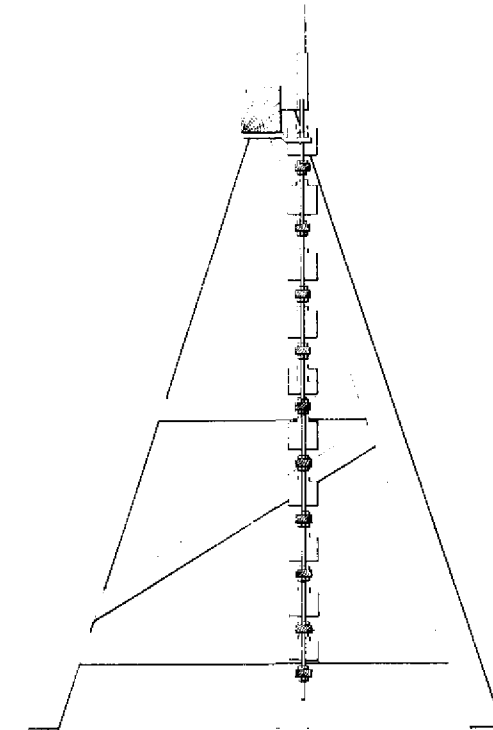


Table I.
SHOWING VERTICAL STRESSES AND DIMENSIONS AT UP-STREAM FACE

VERTICAL STRESSES		DIMENSIONS	
RESERVOIR FULL	RESERVOIR EMPTY	HORIZONTAL DISTANCE FROM FACE TO VERTICAL LINE	VERTICAL DISTANCE FROM FLOOD-LEVEL
0.41	0.0	1.75	1.12
0.2	0.05	0.41	0.0

0.51	0.56	1.78	1.97	0.7	0.18	21.0	6.4
0.43	0.47	3.24	3.74	1.1	0.34	41.7	12.67
0.20	0.22	4.11	4.82	1.6	0.47	62.0	19.46
0.32	0.37	5.50	6.01	2.3	0.70	84.4	25.70
0.78	0.86	6.36	6.96	4.3	1.31	105.4	32.40
1.61	1.76	6.90	7.57	7.8	2.34	126.10	38.70

Figs. 3.

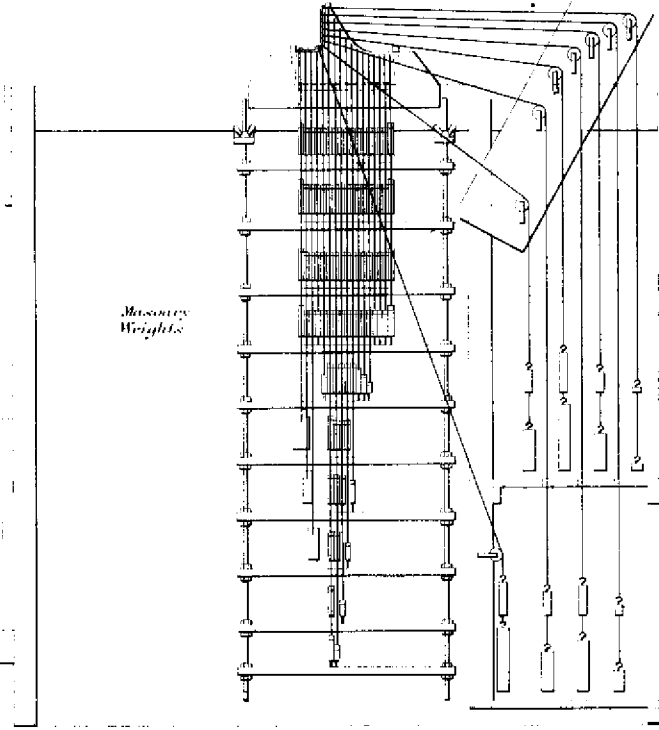
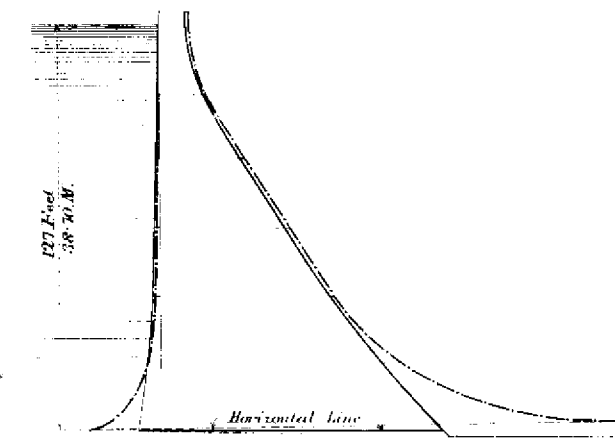


Table II.
SHOWING DIMENSIONS AND VERTICAL STRESSES AT DOWN-STREAM FACE

DIMENSIONS		VERTICAL STRESSES	
HORIZONTAL DISTANCE FROM FACE TO VERTICAL LINE	VERTICAL DISTANCE FROM FLOOD-LEVEL	RESERVOIR FULL	RESERVOIR EMPTY
0.41	0.0	1.75	1.12
0.2	0.05	0.41	0.0

13.6	4.12	0.55	0.60	1.86	2.03
25.7	7.81	0.03	0.04	2.82	3.00
38.8	11.78	0.07	0.08	4.19	4.59
52.2	16.25	0.03	0.04	5.24	5.71
69.2	21.40	0.16	0.18	5.95	6.51
88.6	26.97	0.45	0.49	6.12	6.70

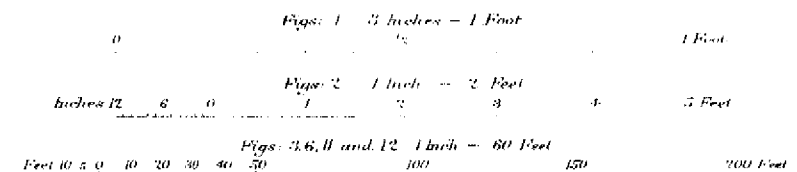
Fig. 6.



COMPARISON OF SECTIONS

Section B
Section C

SCALES OF DIMENSIONS



SCALES OF STRESSES AND STRAINS

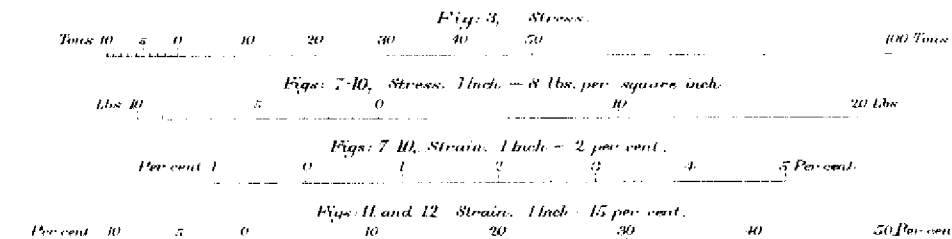
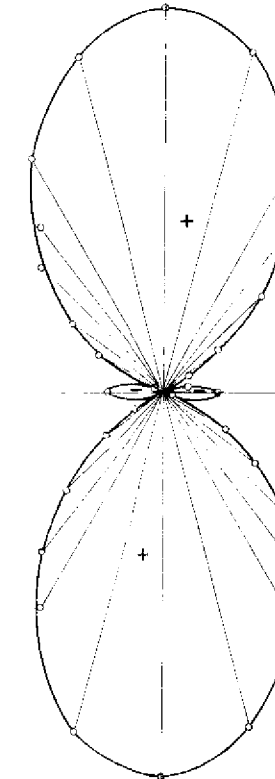
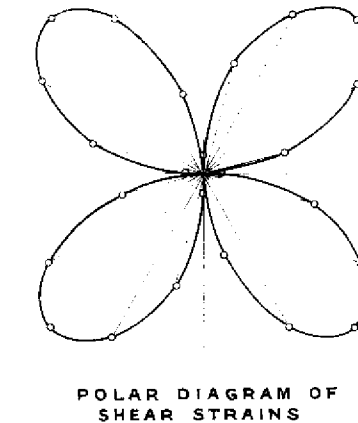


Fig. 7.



POLAR DIAGRAM OF DIRECT STRAINS

Fig. 8.



POLAR DIAGRAM OF SHEAR STRAINS

Fig. 11.

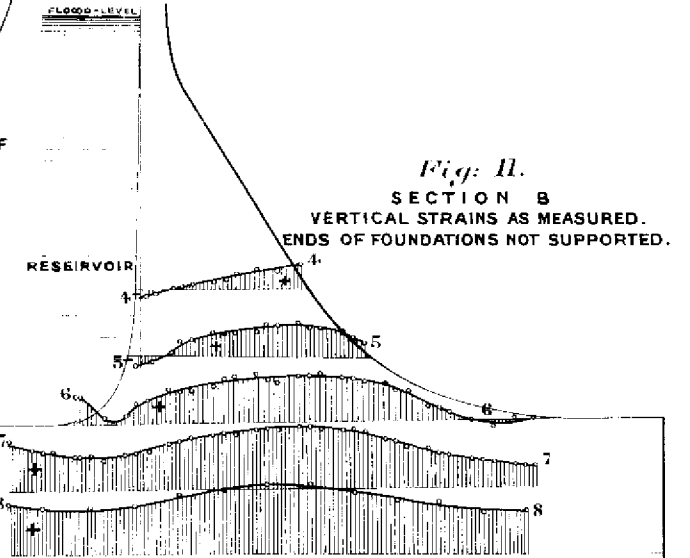
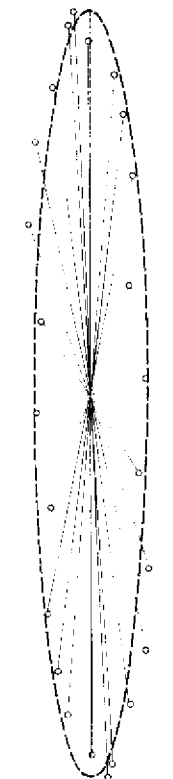
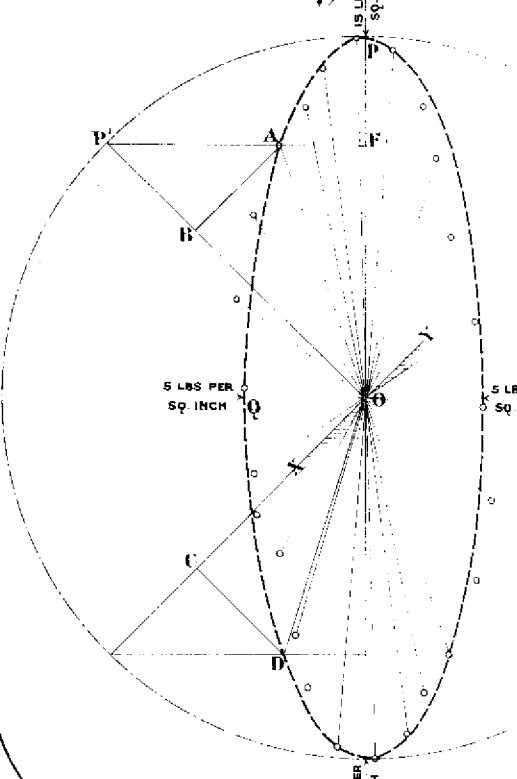
SECTION B
VERTICAL STRAINS AS MEASURED.
ENDS OF FOUNDATIONS NOT SUPPORTED.

Fig. 9.



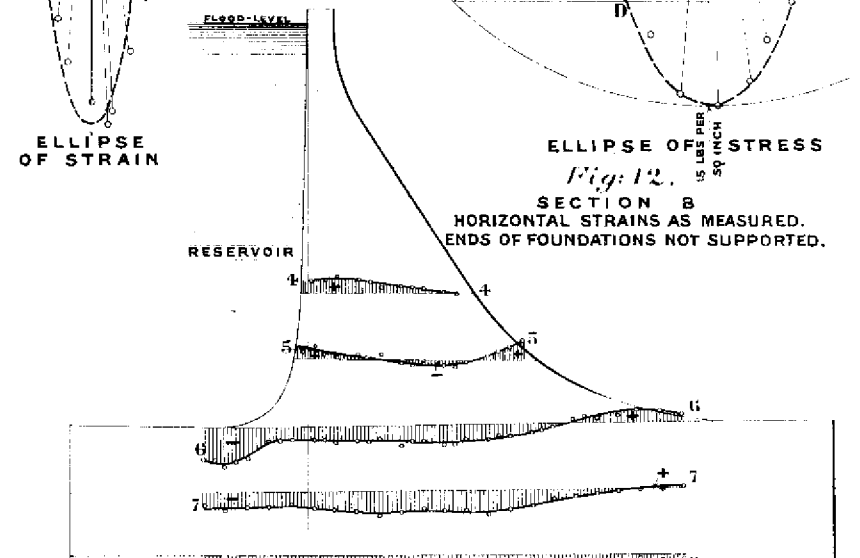
ELLIPSE OF STRAIN

Fig. 10.



ELLIPSE OF STRESS

Fig. 12.

SECTION B
HORIZONTAL STRAINS AS MEASURED.
ENDS OF FOUNDATIONS NOT SUPPORTED.

STRESSES IN DAMS.

PLATE 5.

