These tangents intersect two and two in

$$a/(c^2-a^2),\quad b/(b^2-c^2),\quad c/(a^2-b^2)\ \ldots\ldots\ldots(\text{A, B});$$
$$a/(b^2-c^2),\quad b/(a^2-b^2),\quad c/(c^2-a^2)\ \ldots\ldots\ldots(\text{B, C});$$
$$a/(a^2-b^2),\quad b/(c^2-a^2),\quad c/(b^2-c^2)\ldots\ldots\ldots\ldots(\text{C, A}).$$

15. Since the equations to DF, $D'F'$ are

$$2c^2\cos B\alpha-2ab\cos B\beta+b^2\gamma=0,$$
$$b^2\alpha-2bc\cos B\beta+2a^2\cos B\gamma=0,$$

it follows that the join of B to their intersection, and the like joins for the other angles, cointersect in

$$a/(b^2-c^2),\quad b/(c^2-a^2),\quad c/(a^2-b^2).$$

16. The figure shows that $D'F$ is antiparallel to AC with reference to B; hence the circle DEF circumscribes an in-triangle with two sides EF, ED', parallel to BC, AB, and the third side the antiparallel $D'F$.

The Dioptrics of Gratings. By J. Larmor, F.R.S.

Read March 9th, 1893.

When a beam of light falls upon a ruled or striated surface, a considerable portion of it is inevitably scattered and lost by the inequalities of the surface; and the residue is reflected or refracted in the ordinary manner. But when the striation varies from point to point in a continuous and fairly uniform way, there is sifted out from the incident beam, in addition to the *débris* of scattered light, a series of regular secondary beams, which are propagated onwards in directions inclined to that of the principal one.

The origin of such a diffracted beam, by the union of the diffracted parts from the different striae which arrive at its front in the same phase, was fully explained by Thomas Young, as also was the very perfect separation of the different chromatic constituents of a regular compound beam by a good grating of this kind. In the few pregnant sentences in which Young pointed out the reason of these phenomena,* he, in fact, made the way perfectly clear for that extension of their range which was afterwards worked out experimentally by Fraunhofer, and which has more recently led to the development of the optical grating as the chief instrument of spectral analysis.

The discussion of the action of such gratings, so far as it is usually required for practical purposes, is a simple and well-known matter. But there are questions of some importance, such as the effect of want of perpendicularity of the lines of the grating to the plane of incidence of the light, which are more readily attacked by means of a general theory; while it may also be of interest to formally include general diffracted beams within the domain of dioptrical analysis, and exhibit the rules by which the position of their focal lines, when narrow, and the determination of their caustic surfaces in other cases, is to be accomplished.

The fundamental physical principle is that the existence of a continuous wave-front requires either (i) that the optical path measured up to it of the rays which come from all the striae shall be the same, or (ii) that for successive striae it shall differ by the index multiplied by a multiple of a wave-length of the diffracted beam, say by $n\mu_2\lambda_2$ $(=n\mu_1\lambda_1)$ for the diffracted beam of the nth order. Thus, if m be the number of striations between a selected point on the grating and any origin of reference, the difference of paths for the corresponding rays will be $mn\mu\lambda$. This expression will be a function of the coordinates of the point on the grating; and to obtain the Hamiltonian characteristic function of the diffracted beam we have simply to add this function to the characteristic of the unbroken incident beam.

Let us take the equation of the surface of the grating to be, up to the second order,

$$\zeta = \tfrac{1}{2}\alpha\xi^2 + \tfrac{1}{2}\beta\eta^2 + v\xi\eta + \ldots;$$

and let the lines of the grating be parallel to the axes of η, so that

$$nm\mu\lambda = L\xi + \tfrac{1}{2}\alpha'\xi^2 + \tfrac{1}{2}\beta'\eta^2 + v'\xi\eta + \ldots,$$

where the coefficients α', β', v' represent the effect of any continuous change in the breadths of the striae that may exist. Suppose the characteristic function of an incident beam in the medium of index μ to be

$$V_1 = \mu_1\{l_1\xi_1 + m_1\eta_1 + n_1\zeta_1 + \tfrac{1}{2}A_1\xi_1^2 + \tfrac{1}{2}B_1\eta_1^2 + \tfrac{1}{2}C_1\zeta_1^2$$
$$+ F_1\eta_1\zeta_1 + G_1\zeta_1\xi_1 + H_1\xi_1\eta_1 + \ldots\};$$

* *Phil. Trans.*, 1801.

while the characteristic function of the n'th diffracted beam in the medium of index μ_2 is given by the similar expression with suffix 2, the value of λ_2 above also belonging to this medium; the case of reflexion is obtained by making $\mu_2 = -\mu_1$.

At the surface of the grating we must have

$$V_2 - V_1 = nm\mu\lambda.$$

Hence, considering first the terms of the first degree, we have

$$\mu_2 l_2 - \mu_1 l_1 = L,$$

$$\mu_2 m_2 - \mu_1 m_1 = 0.$$

As these relations are linear, they express that the projection of the incident ray on a normal plane parallel to the lines of striation is bent according to the ordinary law of refraction; while its projection on the normal plane at right angles to these lines is bent in the same manner as an actual ray in this direction would be diffracted, the angles of incidence and diffraction being connected by the relation

$$\mu_2 \sin\phi_2 - \mu_1 \sin\phi_1 = L,$$

where L/n is the value of $\mu\lambda$ divided by the width of a striation at the origin.

The direction of the diffracted beam being thus determined, it remains to find its focal lines. This is done by equating the terms of the second order at the diffracting surface; the equation of the surface must be used to eliminate ζ, and then the two sides of the equation of condition must agree identically. There results

$$(\mu_2 A_2 + n_2 a) - (\mu_1 A_1 + n_1 a) = a',$$

$$(\mu_2 B_2 + n_2 \beta) - (\mu_1 B_1 + n_1 \beta) = \beta',$$

$$(\mu_2 H_2 + n_2 v) - (\mu_1 H_1 + n_1 v) = v';$$

the other coefficients only entering in the third order.

These remaining coefficients are, however, determined by the characteristic equation

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = \mu^2,$$

which requires

$$(l+A\xi+H\eta+G\zeta)^2+(\ldots)^2+(\ldots)^2=1;$$

and this is satisfied up to the first degree of small quantities, provided

$$Al+Hm+Gn=0,$$

$$Hl+Bm+Fn=0,$$

$$Gl+Fm+Cn=0.$$

These equations determine G, F, C in terms of A, B, H.

Now the distances of the focal lines of the beam are the radii of principal curvature, at the origin, of the surface $V=0$. These radii are equal to the squares of the semi-axes of the central section of the surface

$$\tfrac{1}{2}A\xi^2+\ldots+F\eta\zeta+\ldots=1$$

by the plane l, m, n; therefore they are determined by making $R=\xi^2+\eta^2+\zeta^2$ a maximum or minimum, subject to the condition

$$l\xi+m\eta+n\zeta=0.$$

And the analysis may be completed for any special case by means of well-known formulæ in Geometry of Three Dimensions.

A manageable case arises when the incident and diffracted rays are in the same plane, which is therefore normal to the striations. We may now refer each of the beams to its own principal axes. Thus

$$V_1=\mu_1\left\{z_1+\tfrac{1}{2}A_1x_1^2+\tfrac{1}{2}B_1y_1^2+H_1x_1y_1+\ldots\right\},$$

$$V_2=\mu_2\left\{z_2+\tfrac{1}{2}A_2x_2^2+\tfrac{1}{2}B_2y_2^2+H_2x_2y_2+\ldots\right\};$$

and we will gain symmetry by altering the equation of the diffracting surface to

$$0=\zeta+\tfrac{1}{2}\alpha\xi^2+\tfrac{1}{2}\beta\eta^2+v\xi\eta.$$

Change of coordinates is effected by equations of the type

$$\left.\begin{aligned} x&=\xi\cos\phi-\zeta\sin\phi,\\ z&=\xi\sin\phi+\zeta\cos\phi,\\ y&=\eta.\end{aligned}\right\}$$

On eliminating ζ as before, and so identifying at the surface the two sides of the equation of condition, we have

$$\mu_2 \sin \phi_2 - \mu_1 \sin \phi_1 = L_2,$$

$$\mu_2 A_2 \cos^2 \phi_2 - \mu_1 A_1 \cos^2 \phi_1 = a_2 + (\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)\, a,$$

$$\mu_2 B_2 \qquad - \mu_1 B_1 \qquad = \beta_2 + (\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)\, \beta,$$

$$\mu_2 H_2 \cos \phi_2 - \mu_1 H_1 \cos \phi_1 = v_2 + (\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)\, v.$$

In both the general problem and this more special case, it is to be observed that, if a', β', v' are null, *i.e.*, if the striations are symmetrical with respect to the origin, the focal lines are determined by exactly the same formulæ as would apply to simple refraction at the surface, the different direction of the diffracted ray being allowed for. In the case of Rowland's spherical gratings, this result is well known, and is made use of in the instrumental arrangements. The aberration would be expressed by terms of the third degree.

When the incidence is direct, the circumstances of the diffracted beam will be correctly represented by imagining it to be refracted at an ideal surface situated at each point a distance $mn\mu\lambda/(\mu_2-\mu_1)$ in front of the real one. But this rule must be modified when the incidence is oblique. The ideal surface would then vary with the angle of incidence, the distance being now $mn\mu\lambda/(\mu_2 \cos\phi_2 - \mu_1 \cos\phi_1)$; for the interposition of a thickness t of medium of index μ_2 retards the ray by an amount that corresponds to a length

$$(\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)\, t/\mu_1$$

in the medium of index μ_1. The direction of the diffracted ray will be determined by the rule given above; and, once that direction is found, the form of the diffracted beam will be given, when the striation is symmetrical, by the formulæ which belong to ordinary refraction at the surface of the grating.

Another case of some theoretical interest arises when the lines of the grating are closed curves drawn round its centre. If we take

$$nm\mu\lambda = \tfrac{1}{2}a'\xi^2 + \tfrac{1}{2}\beta'\eta^2 + v'\xi\eta + \ldots,$$

that is, if we make L null in the above analysis, these lines will be, in the neighbourhood of the vertex, the system of similar concentric conics $nm\mu\lambda =$ constant, the successive rings enclosed between them being now of equal area. The result indicated by the formulæ is that the diffracted beam follows the same direction as the principal refracted beam, but the equations which give its elements differ by the terms α', β', ν' on their right-hand sides. If the incidence is direct, the grating by itself acts in the same manner as a thin astigmatic lens, whose thickness t is given by

$$(\mu_2-\mu_1)\,t = nm\mu\lambda\,;$$

if it is oblique at an angle ϕ_1, the law of thickness of the equivalent lens is

$$(\mu_2\cos\phi_2-\mu_1\cos\phi_1)\,t = nm\mu\lambda.$$

On a Three-fold Symmetry in the Elements of Heine's Series. By L. J. Rogers. Received March 8th, 1893. Read March 9th, 1893.

In Heine's *Kugelfunctionen*, Vol. I., Appendix to Chap. 2, it is proved that the series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}\,x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}\,x^2+\dots$$

$$\equiv\phi\left[a,\,b,\,c,\,q,\,x\right]$$

$$=\prod_{n=0}^{\infty}\frac{(1-axq^n)(1-bq^n)}{(1-xq^n)(1-cq^n)}\,\phi\left[\frac{c}{b},\;x,\;ax,\;q,\;b\right]\;\dots\dots(1),$$

which, by the symmetry between a and b, and by reapplication of the same formula, leads to other equivalent forms all consisting of infinite products multiplied by a single series of the form

$$\phi\left[a,b,c,q,x\right];$$

for example, $$\prod_{n=0}^{\infty}\frac{(1-bxq^n)\left(1-\frac{c}{b}\,q^n\right)}{(1-xq^n)(1-cq^n)}\,\phi\left[b,\,\frac{abx}{c},\,b,\,x,\,q,\,\frac{c}{b}\right]\;\dots\dots(2),$$

and $$\prod_{n=0}^{\infty}\frac{\left(1-\frac{abx}{c}\,q^n\right)}{(1-xq^n)}\,\phi\left[\frac{c}{a},\,\frac{c}{b},\,c,\,q,\,\frac{abx}{c}\right]\;\dots\dots\dots\dots\dots(3).$$