A Conic Can Be Drawn through Any Five Points

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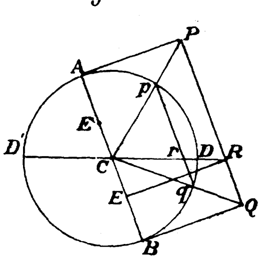
A CONIC CAN BE DRAWN THROUGH ANY FIVE POINTS.

WITH the aid of the construction (I.) given below, I am able to give an elementary geometrical proof of the above theorem, analogous to the algebraical proof, viz., by reduction to centre and principal axes. It is customary to assume, without proof, in geometrical conics that the curves represented algebraically by $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$, and $y^2=px$ are conics whether the axes be rectangular or not. It is quite legitimate when the axes are rectangular, as we can construct vertices, foci, and eccentricity; but when they are not, we ought to show how to reduce the curves to their principal axes, which I have done in II. and III. In the latter part of the proof I have followed the order and method adopted in Dr. Macaulay's *Geometrical Conics*, p. 202, though the proof might have been slightly shortened by use of the cross-ratio property of conics.

I. CONSTRUCTION. *To place a chord in a circle, parallel to one diameter, and divided by another internally or externally in a given ratio* μ.



To place in the circle ADB, centre C, a chord parallel to AB, and divided by CD in a given ratio μ.

Divide AB, internally or externally as the case may be, at E in the ratio

$$EA : EB = \mu.$$

At A, B, E draw perpendiculars AP, BQ, ER, the latter meeting CD in R; draw RPQ parallel to AB; join CP, CQ, cutting the circle in p, q; join pq cutting DC in r.

Then evidently $CP=CQ$, and $Cp=Cq$;

$$\therefore\ pq \text{ is parallel to } PQ;$$

$$\therefore\ rp:rq=RP:RQ=EA:EB=\mu.$$

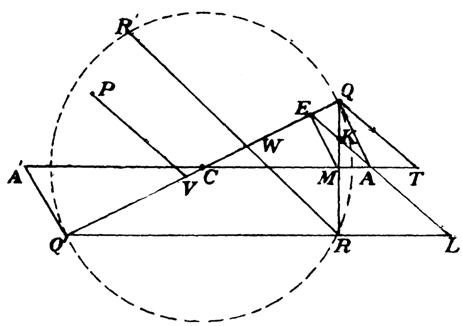
Again, no other chord $p'q'r'$ can be drawn divided in the same ratio in the same order; for since pqr, $p'q'r'$ would be similarly divided, pp', qq', rr' would be concurrent; but pp' qq' meet on the diameter at right angles to AB, since they join the extremities of chords parallel to AB; and r, r' are on a different diameter.

It is evident, however, that if we divide AB at E' so that $E'B:E'A=EA:EB=\mu$, we shall get, on the other side of AB, a second chord, similarly divided, but with corresponding points in reversed order.

Also, when the diameters CD and AB are at right angles, the construction gives chords of zero length at D, D', all chords of finite length parallel to AB being in this case, of course, bisected by CD.

In what follows, all equalities are true in sign as well as in magnitude, and all constructions are real.

II. Theorem. *The curve $PV^2:VQ.VQ'=\mu$, a constant in magnitude and sign, QQ' being fixed real points, and PV parallel to a fixed direction meeting QQ' in V, is a hyperbola or an ellipse, according as μ is positive or negative.*



Let P be a point on the curve, PV parallel to the fixed direction. Bisect QQ' at C; with centre C draw the circle QRQ', and place in it, on the Q side of C, a chord RWR' parallel to PV, and divided by QQ' (internally or externally as μ is negative or positive), so that $WR:WR'=\mu$;

$$\therefore\ \frac{RW^2}{WQ.WQ'}=\frac{WR.WR}{WR.WR'}, \text{ since } QRQ'R' \text{ are concyclic,}$$

$$=\mu;$$

$$\therefore\ R \text{ is a point on the curve.}$$

Draw QT, CT parallel to PV, $Q'R$.

Then since $Q'RQ$, in a semicircle, is a right angle, CT bisects QR at right angles in M.

Determine A, A' so that $CM.CT^*=CA^2=CA'^2$.

* The points M, T are on the same side of C, for the parallels WR, QT are on the same side of C, by hypothesis, and M is midway between the parallels.

Join AQ, and draw $EKAL$ parallel to PV, cutting QQ, QR, $Q'R$ in E, K, L; and join EM.

Then $$\frac{CM}{CA}=\frac{CA}{CT}=\frac{CE}{CQ}, \text{ by similar triangles,}$$

$$\therefore\ EM \text{ is parallel to } QA;$$

$$\therefore \text{ by similar triangles, } \frac{EK}{EA}=\frac{EM}{EM+QA}=\frac{CE}{CE+CQ}$$
$$=\frac{CE}{Q'E}=\frac{EA}{EL};$$
$$\therefore\ EK.EL=EA^2=AE^2;$$
$$\therefore\ \frac{AE^2}{EQ.EQ'}=\frac{EK.EL}{EQ.EQ'}=\frac{WR.WR}{WQ.WQ'}=\mu;$$

$\therefore$ A and also A' are points on the curve.

Construct the conic with major axis AA', and semi-minor axis CB given by the relation
$$\frac{BC^2}{AC^2}=-\frac{QM^2}{MA.MA'}.$$

This passes through Q, and since $CM.CT=CA^2$, QT is the tangent at Q, and therefore conjugate in direction to CQ.

Therefore all points P on this conic satisfy the relation
$$\frac{PV^2}{VQ.VQ'}=\frac{AE^2}{EQ.EQ'}=\mu;$$
and other points P, on the same side of the tangents at QQ' as the conic, will have a different value of PV^2 for the same value of $VQ.VQ'$; and all points on the other side of these tangents will have a different sign for $PV^2 : VQ.VQ'$;

$\therefore$ the curve and the conic are identical.

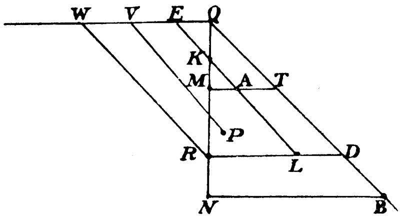
If PV, QQ' are originally at right angles, the conic through P with QQ' as major axis, and that alone, coincides with the curve.

Lastly, the conic is a hyperbola or an ellipse according as M is without or within AA', i.e. according as E is without or within QQ', since AQ, ME, $A'Q'$ are parallel, i.e. according as $\mu\left(=\frac{AE^2}{EQ.EQ'}\right)$ is positive or negative.

III. THEOREM. *The curve $PV^2=\mu.QV$, where μ is a constant positive length, and PV, parallel to a fixed direction, meets a fixed line through a fixed point Q in V, is one or other of two parabolas, according to the direction of QV which is considered positive.*

Draw QN at right angles to QV, and $QB=\mu$ parallel to PV, in the direction such that the perpendicular BN upon QN is positive.

Make $QD=BN$, along QB; draw DR perpendicular to QN; bisect QR at right angles by MT; bisect MT at A; and draw $EKAL$ and RW each parallel to PV.



Then

$$\frac{RW}{QW}=\frac{DQ}{DR}=\frac{BQ}{BN}=\frac{QB}{DQ}=\frac{\mu}{RW};$$

$$\therefore \quad RW^2=\mu \,.\, QW,$$

and hence R is on the curve.

Again, $\quad EQ=AT=MA$;

$\therefore \quad EK=KA$, and $EA=AL$;

$\therefore \quad EK\,.\,EL=EA^2=AE^2$;

$$\therefore \quad \frac{AE^2}{RW^2}=\frac{EK\,.\,EL}{RW\,.\,RW}=\frac{KE}{RW}=\frac{QE}{QW};$$

$\therefore \quad AE^2=\mu\,.\,QE$, and hence A is on the curve.

Draw the parabola with vertex A, axis AM, and latus rectum l, given by $QM^2=l\,.\,AM$.

This passes through Q, and since $MA=AT$, QT is the tangent at Q, hence all points P on the parabola satisfy the relation

$$\frac{PV^2}{AE^2}=\frac{QV}{QE}, \text{ or } PV^2=\mu\,.\,QV,$$

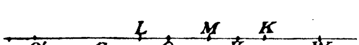
i.e. all points P on this parabola, and as in the previous case no points not on it, are on the curve;

$\therefore$ the curve and the parabola are identical.

If QV be measured in the opposite sense, then measuring off QB also in the opposite sense, and proceeding as before, we shall get another parabola equal to the first but lying on the other side of the tangent QT.

IV. Construction. *Three points C, V, W being in a line, to determine on the line two others Q, Q' equidistant from C, such that $VQ\,.\,VQ' : WQ\,.\,WQ' = a^2 : b^2$, a and b being given lines.*

Firstly. CVW being in the order written, determine the unique point M, which will be without VW, so that



$$MV : MW = a^2 : b^2.$$

Determine the real points K (between V and W) and L such that

$$MV\,.\,MW=MK^2=ML^2, \; M \text{ bisecting } KL. \quad \ldots\ldots\ldots(1)$$

Determine Q, Q' so that

$$CQ^2=CQ'^2=CL\,.\,CK;$$

then $$CQ^2 = CM^2 - ML^2 = CM^2 - MV \,.\, MW;$$
and $$\begin{aligned} VQ \,.\, VQ' &= CV^2 - CQ^2 = CV^2 - CM^2 + MV \,.\, MW \\ &= MV(CV + CM) + MV \,.\, MW \\ &= MV(CV + CW). \end{aligned}$$

Similarly $WQ \,.\, WQ' = MW(CV + CW)$;

$$\therefore\ VQ \,.\, VQ' : WQ \,.\, WQ' = MV : MW = a^2 : b^2.$$

Again from (1)

$$\frac{MV}{MW} = \text{duplicate ratio of } \frac{MV}{MK};$$

$$\therefore\ \frac{a}{b} = \frac{MV}{MK} = \frac{MK}{MW} = \frac{MK - MV}{MW - MK} = \frac{VK}{KW} = \text{similarly } \frac{LV}{LW}.$$

Hence L and K are the points dividing VW externally and internally in the ratio $a : b$. Hence the construction,

Divide VW externally and internally at the points L, K, in the ratio $a : b$, and find Q, Q' so that $CQ^2 = CQ'^2 = CL \,.\, CK$.

Q, Q' are real or imaginary according as $CL \,.\, CK$ is positive or negative.

Secondly. If C be between V and W ($CV < CW$), make $CV' = CV$, and construct for $CV'W$; then since

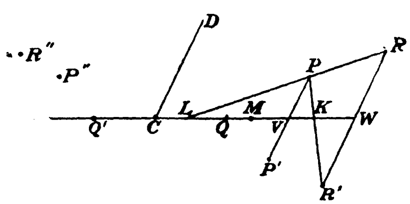
$$V'Q \,.\, V'Q' = VQ \,.\, VQ',$$

the points Q, Q' satisfy the given conditions.

V. Theorem. *One only conic can be drawn having any two given parallel chords and its centre at any point on the line bisecting the chords; and the conic is a parabola when, and only when, the centre is at infinity.*

Let PP', RR' be the given chords, C the centre on VW bisecting them, CD parallel to PV.

Then taking all possible cases in order,



(i.) Let PP', RR' be on the same side of CD; P, R in the same quadrant DCV, then PR must cut either CV or CD in this quadrant.

(a) Firstly. Let PR cut CV at L in the quadrant DCV; join PKR', and determine the real points Q, Q' such that $CQ^2 = CQ'^2 = CL \,.\, CK$, which is positive.

Now $\dfrac{LV}{LW} = \dfrac{PV}{RW} = \dfrac{PV}{WR'} = \dfrac{VK}{KW}$, by similar triangles;

$\therefore$ L, K are the points cutting VW externally and internally in the ratio $PV : RW$; and $CQ^2 = CQ'^2 = CL \,.\, CK$, by construction;

$$\therefore \text{ by IV., } \frac{VQ \,.\, VQ'}{WQ \,.\, WQ'} = \frac{PV^2}{RW^2};$$

$\therefore$ by II., the conic $\mu = PV^2 : VQ . VQ'$, and that only, passes through PP', RR', having C for centre.

(b) Secondly. Let PR cut CD and not CV in the quadrant DCV. Draw PP'', RR'' parallel to CVW to be bisected by CD.

Then the conic $PP''RR''$, centre C, passes through P', R', since CD, CV are conjugate.

(ii.) If RR' is not on the same side of CD as PP', draw RS, $R'S'$ parallel to CVW to be bisected by CD; then the conic $PP'SS'$, centre C, passes through RR', since CD, CV are conjugate.

(iii.) Since M, the mid-point of LK, uniquely satisfies the condition (see IV.), $MV : MW = PV^2 : RW^2$, the parabola $PV^2 = \mu . MV$, and that alone, passes through PP', RR'.

And if the diameter MV meet this again in M',

$$\frac{VM . VM'}{WM . WM'} = \frac{PV^2}{RW^2} = \frac{VM}{WM};$$

$$\therefore \frac{VM'}{WM'} = 1, \text{ and } M' \text{ is at infinity;}$$

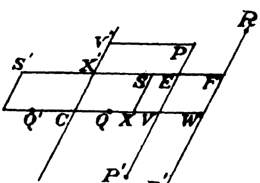
$\therefore$ the centre bisecting MM' is also at infinity.

With regard to degenerate conics, it will be readily seen that the above construction includes as special cases,

(i.) When the centre is at K or L, two right lines through centre;

(ii.) When PP' and RR' are equal, either (centre at mid-point of VW) any number of ellipses and hyperbolas, or (centre elsewhere) two parallel lines.

VI. Theorem. *One only conic can be drawn through the extremities of any two parallel chords, to pass through a given point.*



Let PP', RR' be the given chords bisected by V, W, and S the given point. Draw SX, SEF parallel to PP', VW.

$$\text{Firstly, if } \frac{ES}{FS} = \frac{EP . EP'}{FR . FR'}, \qquad \text{............(1)}$$

draw the parabola through P, P', R, R', cutting VW in Q; and let $PV^2 = \mu . QV$,

$$\therefore \frac{PV^2}{\mu . QV} = \frac{RW^2}{\mu . QW} = \frac{RW^2 - PV^2}{\mu . VW};$$

and (1) may be written

$$\frac{PV^2 - SX^2}{\mu . XV} = \frac{RW^2 - SX^2}{\mu . XW} = \frac{RW^2 - PV^2}{\mu . VW};$$

$$\therefore \frac{PV^2}{\mu . QV} = \frac{PV^2 - SX^2}{\mu . XV} = \frac{SX^2}{\mu . QX}.$$

Hence S is on this parabola.

Secondly, if (1) be not true, determine S' on SEF so that

$$\frac{ES \,.\, ES'}{FS \,.\, FS'} = \frac{EP \,.\, EP'}{FR \,.\, FR'}. \qquad \text{(2)}$$

This gives a single value for the ratio $ES' : FS'$, and therefore a single position for S', which cannot be at infinity since then $ES' : FS' =$ unity, and relation (1) is true.

Draw $X'C$ parallel to PP', bisecting SS' in X', and meeting VW in C, which cannot be at infinity. Draw PV' parallel to CV.

Draw the conic, centre C, through P, P', R, R', meeting CV, suppose, in real points Q, Q';

$$\therefore \frac{PV^2}{VQ \,.\, VQ'} = \frac{RW^2}{WQ \,.\, WQ'} = \frac{RW^2 - PV^2}{CW^2 - CV^2};$$

and (2) may be written

$$\frac{PV^2 - SX^2}{CV^2 - CX^2} = \frac{RW^2 - SX^2}{CW^2 - CX^2} = \frac{RW^2 - PV^2}{CW^2 - CV^2};$$

$$\therefore \frac{PV^2}{VQ \,.\, VQ'} = \frac{PV^2 - SX^2}{CV^2 - CX^2} = \frac{SX^2}{XQ \,.\, XQ'}.$$

Hence S is on this conic.

And if the conic do not meet CV in real points, it must meet its conjugate CX' in real points D, D'.

Then we can prove, as before,

$$\frac{PV'^2}{V'D \,.\, V'D'} = \frac{SX'^2}{X'D \,.\, X'D'}.$$

Hence S is on the conic.

VII. THEOREM. *One only conic can be drawn through any five points.*

Let the points (see previous figure) be R, R', S, S', P; determine P' in the parallel through P to RR' so that

$$\frac{ES \,.\, ES'}{FS \,.\, FS'} = \frac{EP \,.\, EP'}{FR \,.\, FR'}, \qquad \text{(1)}$$

This gives a single value for EP', and therefore a single position for P'. It should be noticed, however, that in general the line bisecting PP', RR' will not be, as in the figure, parallel to SS'.

Then the conic through P, P', R, R', S meets ES in a point S_1, such that $\dfrac{ES \,.\, ES_1}{FS \,.\, FS_1} = \dfrac{EP \,.\, EP'}{FR \,.\, FR'}$;

$$\therefore \text{ by (1) } \frac{ES_1}{FS_1} = \frac{ES'}{FS'}, \text{ and } S_1 \text{ coincides with } S';$$

hence the conic passes through S'. Conversely, a conic through R, R', S, S', P must pass through P'. Hence there is only one conic.

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