THE

MATHEMATICAL GAZETTE.

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LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. VI. | MAY, 1911. | No. 92.

NOTICE.

THE MATHEMATICAL ASSOCIATION TEACHING COMMITTEE wishes to draw the attention of examiners to the following points about GEOMETRY EXAMINATIONS:

Many examinations already follow the practices suggested.

I. When a CONSTRUCTION is asked for, it should be clearly stated, either in the question or as a general heading,

(i) What instruments may be used.

(ii) Whether the construction is to be fully described in words, or only those parts of it which are not obvious from the figure.

(iii) Whether a theoretical proof is required.

II. When the proof of a THEOREM is asked for, if the examiner wishes for an accurate figure drawn with instruments, this should be stated in the question.

III. The Committee is of opinion that, in general, it is inadvisable to require that the figures for the proofs of theorems should be drawn accurately with instruments; the Committee considers that such a requirement tends to waste of time, and that a neatly drawn freehand figure should suffice.—Signed on behalf of the Committee,

E. W. HOBSON,
April, 1911. *President of the Mathematical Association.*

[The above has been sent out to Examining Bodies by the Secretary of the Committee.]

ON TERTIAL, QUINTAL, ETC., FRACTIONS.

BY LT.-COL. ALLAN CUNNINGHAM, R.E., FELL. OF KING'S COLL. LOND.

1. DEF. The reciprocal $(1/N)$ of any number (N) will be termed a *Binal*,* *Tertial*, *Quintal*, ... *Decimal*, ... r-*mal* fraction, when expressed in

*The subject of Binal Fractions was treated of in the author's previous paper on *Binal Fractions* (*Math. Gazette*, vol. iv., 1908, pp. 259-267). For facility of comparison, the present paper is written in the *same style* as the former, and the corresponding Articles and Results bear the same reference numbers. [The author's acknowledgments are due to Mr. M. J. Woodall, A.R.C.Sc., for help in reading the proof sheets.]

the scale whose radix is $r=2, 3, 4, \dots 10, \dots r$ respectively. In this paper the developments will be given for the most part in a general form in the r-scale; the examples will be in the scales of $r=3$ and 5. The notation is similar to that of ordinary *decimals*. Thus (see Tables at end):

1°. A "point" (the tertial, quintal, etc., *point*) marks the beginning of the (tertial, quintal, etc.) fraction.

2°. The tertial, quintal, etc. $(1/N)$ consists of a certain sequence of ciphers and other digits ($<3, 5$, etc., r) to the right of the "point"; partly non-recurrent, and partly *recurrent* (all recurrent in the Tables at end).

3°. The beginning and end of the recurrent portion (or *repetend*) are marked by a point placed over the first and last figure of the repetend.

2. *N a power of r.* Let $N=r^a$; then

$$1/N = \cdot 000 \dots 01, \ [(a-1) \text{ ciphers, one unit}], \quad \dots\dots (1)$$

and, note that there is *no recurrence*. ......(1*a*)

Simple non-repetends. A number of these (non-repetend) forms may be written down at sight.

$$\frac{1}{r} \cdot \frac{r^a - 1}{r-1} = \cdot 111 \dots 1, \ [a \text{ units}]. \quad \dots\dots (1b)$$

$$\frac{1}{r^{2a}} \cdot \frac{r^{2a}-1}{r^2-1} = \cdot 0101 \dots 01, \ [a \text{ repetitions}]. \quad \dots\dots (1c)$$

$$\frac{1}{r^{3a}} \cdot \frac{r^{3a}-1}{r-1} = \cdot 001001 \dots 001, \ [a \text{ repetitions}], \quad \dots\dots (1d)$$

and so on.

Also, writing $q=(r-1)$, $q_2=(r-2)$, $q_3=(r-3)$, etc., as *digits*: ......(1*e*)

$$\frac{1}{r}(r-1) = \cdot q, \ \frac{1}{r^2} \cdot (r^2-1) = \cdot qq. \quad \dots\dots (1f)$$

$$\frac{1}{r^a} \cdot (r^a - 1) = \cdot qqq \dots q, \ [a \text{ digits}]. \quad \dots\dots (1g)$$

$$\frac{1}{r^q} \cdot \frac{r^q - 1}{q^2} = \cdot 0123 \dots q_4 q_3 q, \ [q \text{ digits, no } q_2]. \quad \dots\dots (1h)$$

Also,

$$\frac{1}{r^2} \cdot \frac{(r^3+1)}{r+1} = \cdot q1, \ \frac{1}{r^4} \cdot \frac{q^5+1}{r+1} = \cdot q0q1. \quad \dots\dots (1i)$$

$$\frac{1}{r^{2a}} \cdot \frac{r^{2a+1}+1}{r+1} = \cdot q0q0 \dots q1, \ [a \text{ digits} = q]. \quad \dots\dots (1j)$$

2a. *Pure repetends.* Let $N=A$, an *integer* prime to r.

$1/N$ consists *solely of a repeating cycle*. ......(2)

2b. *Multiples of powers of radix.* Let $N=r^a . A$, where A is an integer prime to r, and let P be the period of $1/A$ which is *wholly repetend*; then

$1/N$ consists of a group of a ciphers following the r-mal "point" non-recurrent, due to the factor r^a, followed by the period P of $1/A$, which is the repetend. ...(3)

Hence, it is unnecessary to consider multiples of r^a any further, as their reciprocals can be at once written down by the above rule, when the repetend cycle of $1/A$ is known.

3. *Notation.*

Let N be an integer prime to r, and $>r^a$, but $<r^{a+1}$.

" P be the repetend cycle of $1/N$ in the r-scale.

" n = the number of digits in P.

" c = number of leading ciphers (*i.e.* following the "point") in P.

Let v = number of digits in P following the leading ciphers.

" ξ be the Haupt-Exponent* of r to mod N, *i.e.* the *least* exponent giving $r^\xi \equiv +1 \pmod{N}$.

" $q = r - 1$.

3a. *Number of figures* (n) *in period* (P). By the above,

$$c + v = n = \xi. \quad \text{.......} \quad (4)$$

$$c = a. \quad \text{.......} \quad (5)$$

Thus, in all cases, the leading a figures (next to the "point") are ciphers, followed by a digit >0; there are $(\xi - a)$ digits following those leading ciphers, and none of them can exceed $q = (r-1)$.

4. *Fractional form of* $1/N$. By definition, P is one period of the repetend of $1/N$, expressed as an integer in the r-scale, and contains ξ figures. Hence $P \div r^\xi$ is the real value of the first period of $1/N$, and therefore

$$\frac{1}{N} = \frac{P}{r^\xi} + \frac{P}{r^{2\xi}} + \frac{P}{r^{3\xi}} + \text{etc. } ad.\ inf. \quad \text{.......} \quad (7)$$

$$\therefore\ r^\xi \cdot \frac{1}{N} = P + \left(\frac{P}{r^\xi} + \frac{P}{r^{2\xi}} + \dots\ ad.\ inf.\right) = P + \frac{1}{N}.$$

$$\therefore\ P = \frac{r^\xi - 1}{N}. \quad \text{.......} \quad (8)$$

$\therefore$ P is the quotient of $(r^\xi - 1) \div N$ in the r-scale. ....(8a)

The quantity (P), thus found, will be an integer, containing only $a = (\xi - a)$ digits (the first and last of which are >0). A group of a ciphers with the "points" on the left should be prefixed to form the r-mal fraction P/r^ξ, which is one period of $1/N$. If a point be now placed over the first and last figures, to show that the whole group is recurrent, the result will be the complete or real fractional form of $1/N$.

4a. *Periods of co-factors of* $(r^\xi - 1)$. Let N_1, N_2 be a pair of co-factors of $(r^\xi - 1)$, and let P_1, P_2 be the periods of $1/N_1$, $1/N_2$ respectively; then

$$P_1 = \frac{r^\xi - 1}{N_1} = N_2; \text{ and } P_2 = \frac{r^\xi - 1}{N_2} = N_1, \text{ [all in } r\text{-scale]}. \quad \text{.......} \quad (9)$$

Ex.† Scale of $r = 3$. Take $n = 6$; then $3^6 - 1 = 26 \cdot 28 = N_1 \cdot N_2$.

P_1 of $26 = \frac{1}{26}(3^6 - 1) = 28 = (1001 \text{ in 3-scale})$,

P_2 of $28 = \frac{1}{28}(3^6 - 1) = 26 = (222 \text{ in 3-scale})$.

$\therefore\ \frac{1}{26} = \cdot\dot{0}0100\dot{1} = \ddagger \cdot\dot{0}0\dot{1},\ \frac{1}{28} = \cdot\dot{0}0022\dot{2}$ (in 3-scale).

Ex.† Scale of 5. Take $n = 5$; then $5^5 - 1 = 44 \cdot 71 = N_1 \cdot N_2$.

P_1 of $44 = \frac{1}{44}(5^5 - 1) = 71 = (241 \text{ in 5-scale})$,

P_2 of $71 = \frac{1}{71}(5^5 - 1) = 44 = (134 \text{ in 5-scale})$.

$\therefore\ \frac{1}{44} = \dot{0}024\dot{1},\ \frac{1}{71} = \cdot\dot{0}013\dot{4}$ (in 5-scale).

Hence, to find the *least* number (N_n) such that the period (P_2) of $1/N_2$ shall be a *given* number (N_1) in the r-scale, preceded only by ciphers; it suffices to find the Haupt-Exponent (ξ) of r to mod. N_1: thus, by (9), taking

$$N_2 = \frac{1}{N_1} \cdot (r^\xi - 1) \text{ gives } P_2 = 1/N_1 \text{ (as required)}. \quad \text{.......} \quad (9a)$$

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* In what follows ξ is always supposed *known*. For prime, and powers of prime, moduli ($N = p$ or p^n) < 1000, it can be easily found from Jacobi's *Canon Arithmeticus*.

† Compare Tables at end.

‡ The reduction of $\cdot\dot{0}0100\dot{1}$ to $\cdot\dot{0}0\dot{1}$ is due to the fact that $N_1 = (3^3 - 1)$, for which $\xi = 3$.

Although the Rule (9) just given suffices for computing the complete period (P) of the r-mal fraction of the reciprocal ($1/N$) of any integer (N), it is thought that an investigation of the properties of these fractions, with examples for the radices $r=3, 5$, will be interesting: this will be found to give in many cases simpler methods for finding $1/N$.

5. *Simple Cases* $[N=(r^a \mp 1)]$. The periods (P) can here be written down *at sight.*

$$N=r^a-1;\quad 1/N=\cdot\dot{0}00\ldots0\dot{1},\quad [(a-1) \text{ ciphers, 1 unit}]. \quad\ldots\ldots(10a)$$
$$N=r^a+1;\quad 1/N=\cdot\dot{0}00\ldots00qq\ldots\dot{q},\quad [a \text{ ciphers, } a \text{ digits}=q]. \quad\ldots\ldots(10b)$$
$$N=r^{a+1}-1;\quad 1/N=\cdot\dot{0}00\ldots00\dot{1},\quad [a \text{ ciphers, 1 unit}]. \quad\ldots\ldots(10c)$$

Hence the reciprocal ($1/N$) of all numbers (N) from $N=(r^a+1)$ to $N=(r^{a+1}-1)$ have their periods (P) preceded by a group of a ciphers, whilst P itself gradually decreases from $P=qqq\ldots q$ (a digits) when $N=(r^a+1)$ to $P=1$ when $N=(r^{a+1}-1)$.

Another simple form is

$$N=q^2, \text{ then } 1/N=\dot{0}123\ldots q_4q_3\dot{q}, \ [q \text{ digits, no } q_2]. \quad\ldots\ldots(10d)$$

Compare with this the non-repetend ($1h$) with *same period* (P).

6. *Factors of* $(r^{xa}-1)$. Let $R=r^a$, and let N_x be defined by the succession-formula

$$N_{x+1}=R\,.\,N_x+1=r^a\,.\,N_x+1, \ [\text{with } N_0=1]. \quad\ldots\ldots(11)$$

Then
$$N_x=R^x+R^{x-1}+\ldots+R^2+R+1 \quad\ldots\ldots(11a)$$
$$=111\ldots11, \ [(x+1) \text{ units in } R\text{-scale}]$$
$$=100\ldots0100\ldots0100\ldots0100\ldots01, \ [\text{in } r\text{-scale}; \quad\ldots\ldots(13)$$
$$(x+1)a \text{ digits, } (x+1) \text{ units; } (a-1) \text{ ciphers in sub-period}]$$
$$=\frac{R^{x+1}-1}{R-1}=\frac{r^{(x+1)a}-1}{r^a-1}. \quad\ldots\ldots(12)$$

Hence N_x, (r^a-1) are co-factors of $(r^{(x+1)a}-1)$.

Hence $\quad P_x$ (of $1/N_x$) $=(r^a-1)=qqq\ldots q$, [a digits in r-scale]. $\ldots\ldots(14)$

And $\quad 1/N_x=\cdot\dot{0}00\ldots0qq\ldots\dot{q}$, [$xa$ ciphers, a digits$=q$]. $\ldots\ldots(14a)$

A few of these numbers, arising in the scales of $r=3, 5$, are shown below with their equivalents in the denary scale.

a	N (denary).				N (in r-scale). [$(x+1)$ units, $(a-1)$ ciphers.]	Period of $1/N$. [xa ciphers, a of q-digits.]
	$r=3$, $R=3^a$.		$r=5$, $R=5^a$.			
	R	$x=0$, 1, 2, 3	R	$x=0$, 1, 2, 3		
1	3	1, 4, 13, 40	5	1, 6, 31, 156	111 1	$\cdot\dot{0}00\ldots0\dot{q}$
2	9	1, 10, 91, 820	25	1, 26, 656, —	10101 101	$\cdot\dot{0}00\ldots0q\dot{q}$
3	27	1, 28, 757, —	125	1, 126, —, —	1001001..1001	$\cdot\dot{0}00\ldots0qq\dot{q}$
4	81	1, 82, —, —	625	1, 626, —, —	100010001 ... 1	$\cdot\dot{0}00\ldots0qqq\dot{q}$

It will be seen that the top line has $N=(r^{x+1}-1)\div(r-1)$, and the 2nd column has $N=(r^a+1)$.

6a. $N=2(r^a \mp 1)$, [r *odd*]. The periods can in these cases also be written down at sight.

Let $\quad N'=(r^a-1),\ N''=(r^a+1)$.

Then, writing $\quad b=\tfrac{1}{2}(r-1),\quad a=\tfrac{1}{2}(r+1)$,

$$1/2N'=\cdot\dot{0}00\ldots0bb\ldots b\dot{a}, \ [a \text{ ciphers, } (a-1) \text{ digits}=b]. \quad\ldots\ldots(14b)$$
$$1/2N''=\cdot\dot{0}00\ldots0bb\ldots b\dot{b}, \ [a \text{ ciphers, } a \text{ digits}=b]. \quad\ldots\ldots(14c)$$

6b. $N=\frac{1}{2^k}\cdot(r^a \mp 1)$, [$r$ *odd*]. The periods can in these cases also be written down at sight. [N', N'' as in Art. 6a.]

$1\Big/\frac{1}{2}N'=\cdot\dot{0}00\ldots00\dot{2}$, [$(a-1)$ ciphers, one 2; $r>2$]. ....(14d)

$1\Big/\frac{1}{4}N'=\cdot\dot{0}00\ldots00\dot{4}$, [$(a-1)$ ciphers, one 4; $r>4$]. ....(14e)

$\vdots$

$1\Big/\frac{1}{2^\kappa}\cdot N'=\cdot\dot{0}00\ldots00\dot{k}$, [$(a-1)$ ciphers, one $k=2^\kappa$, $r>k$]. ....(14f)

$1\Big/\frac{1}{2^\kappa}\cdot N'=\cdot\dot{0}00\ldots0\,[\dot{2^\kappa}]$, [$a$ figures in P, $[2^\kappa]$ means 2^κ expressed in r-scale]. (14g)

$1\Big/\frac{1}{2}N''=\cdot\dot{0}00\ldots01qqq\ldots q\dot{q}_2$, [$(a-1)$ ciphers, $(a-1)$ digits$=q$]. ....(14h)

Ex. Here follow examples arising in the scales of $r=3, 5$; the equivalents of $2N'$, $2N''$; $\frac{1}{2^\kappa}\cdot N'$, $\frac{1}{2^\kappa}N''$ are given in the denary scale.

	Integers (denary). $r=3$.				Integers (denary). $r=5$.				Reciprocals [in r-scale.]	Reference.
$a=$	1,	2,	3,	4	1,	2,	3,	4		
$N'=$	2,	8,	26,	80	4,	24,	124,	624	$\cdot\dot{0}00\ldots0\dot{1}$	See (10a)
$2N'=$	4,	16,	52,	160	8,	48,	248,	—	$\cdot\dot{0}00\ldots00bbb\ldots b\dot{a}$	See (14b)
$\frac{1}{2}N'=$	1,	4,	13,	40	2,	12,	62,	312	$\cdot\dot{0}00\ldots0\dot{2}$	See (14d)
$\frac{1}{4}N'=$	.,	2,	.,	.	1,	6,	31,	78	$\cdot\dot{0}00\ldots0\dot{4}$ [$r>4$]	See (14c)
$\frac{1}{4}N'=$	.,	2,	.,	20	.,	.,	.,	—	$\cdot\dot{0}00\ldots0[\dot{4}]$	See (14g)
$\frac{1}{8}N'=$	.,	1,	.,	10	.,	3,	.,	39	$\cdot\dot{0}00\ldots0[\dot{8}]$	See (14g)
$N''=$	4,	10,	28,	82	6,	26,	126,	626	$\cdot\dot{0}00\ldots00qqq\ldots q\dot{q}$	See (10b)
$2N''=$	8,	20,	56,	164	12,	52,	252,	—	$\cdot\dot{0}00\ldots00bbb\ldots b\dot{b}$	See (14c)
$\frac{1}{2}N''=$	2,	5,	14,	41	3,	13,	63,	313	$\cdot\dot{0}00\ldots qqqqq\ldots q\dot{q}_2$	See (14h)

7. *End Figures of P.*

Let $N=r^a+B$, where $B<r^a$.
Let P, Q be the periods of $1/N$ and $1/B$.
Then, $P\equiv Q \pmod{r^a}$, ...(15)
so that

The end group of a figures of $1/N$, $(N>r^a)$, is the *same* as the end group of a figures of $1/B=1/(N-r^a)$. ....(15a)

Hence $2a$ figures of the period (P) of $1/N$ are known for all numbers $N>r^a$, viz. a ciphers at the beginning (Art. 3a), and the end group of a figures as above (if that group be known for the period of $1/B$); this leaves only $(\xi-2a)$ figures undetermined. Hence also the *complete period* of $1/N$ may be written down at sight for all numbers $N>r^a$ but $<r^{a+1}$, when $\xi \not> 2a$, if the end group of a figures of the period of $1/B$ is known.

The Table following gives the mod 3-, or 4-figure groups of the periods of $1/N$ for all numbers (N) as follows:

4-figure endings for $r=3$; 3-figure endings for $r=5$.

This Table enables the *complete periods* of $1/N$ to be written down as follows:

In 3-scale, $N \not> 122$, $\xi \not> 8$; in 5-scale, $N \not> 185$, $\xi \not> 6$,

and part of the periods when ξ exceeds 8 or 6 respectively.

(*To be continued.*)

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