There are still three pairs of roots, of which two pairs are now very small when ϵ is very small; and, inasmuch as when ϵ is null these pairs of roots are equal, being zero, they may in the actual case become imaginary for certain values of l, m, n. Thus the instability here attaches to the very slow waves which are outside the limits of physical reality. With a rotatory coefficient of the p type, the trouble would not occur at all.

On the Linear Transformations between Two Quadrics. By HENRY TABER, Clark University, Mass., U.S.A.

Received May 6th, 1893. Read May 11th, 1893.

1. *Introductory.*

In *Crelle's Journal*, Vol. L. (also *Phil. Trans.*, 1858), Cayley gave a representation of the automorphic linear transformation of the unipartite quadric function in the notation of the theory of matrices. In this paper I extend Cayley's method to the determination of the general linear transformation of a given quadric into another given quadric, and apply the results to the determination of the general real linear transformation between two equivalent quadrics and to the reduction of a quadric to a sum of squares. The determination by this method of the general linear transformation between two quadrics depends upon the solution of an algebraic equation of the n^{th} degree; to which the problem, as it originally presents itself, namely, the solution of a system of n^2 quadratic equations in n^2 variables, is thus reducible.

Following Cayley, the matrix of a linear transformation will, in this paper, be regarded as a quantity susceptible of addition. The *sum* or *difference* of two matrices is defined as follows:—

$$\left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{array}\right) \pm \left(\begin{array}{cccc} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{array}\right) = \left(\begin{array}{cccc} a_{11}\pm b_{11} & a_{12}\pm b_{12} & \dots & a_{1n}\pm b_{1n} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} & \dots & a_{2n}\pm b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}\pm b_{n1} & a_{n2}\pm b_{n2} & \dots & a_{nn}\pm b_{nn} \end{array}\right).$$

Products of matrices will be taken as equivalent to compound substitutions. Multiplication is then distributive over addition, *i.e.*,

$$\phi(\psi+\chi) = \phi\psi+\phi\chi,$$
$$(\phi+\psi)\chi = \phi\chi+\psi\chi,$$

for any three matrices ϕ, ψ, and χ. A *scalar** m, regarded as a matrix, has the representation

$$\begin{pmatrix} m & 0 & \dots & 0 \\ 0 & m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & m \end{pmatrix}.$$

An equality between two matrices implies the equality of its corresponding constituents.

If ϕ is the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

the scalars g, for which

$$\begin{vmatrix} a_{11}-g & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-g & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-g \end{vmatrix} = 0,$$

will, following Sylvester, be termed the *latent roots* of ϕ. To denote the determinant of ϕ, the notation $|\,\phi\,|$ will be employed; and, since the left-hand member of the above equation is the determinant of $\phi-g$, it will be denoted by $|\,\phi-g\,|$.

If the determinant of ϕ is not zero, and if

$$\Phi = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix},$$

where A_{rs} denotes the first minor of $|\,\phi\,|$ with respect to a_{rs}, divided by $|\,\phi\,|$, then

$$\phi\Phi = \Phi\phi = 1;$$

i.e., Φ is the *reciprocal* of ϕ. We may denote Φ by ϕ^{-1}. We have

$$(\phi\psi)^{-1} = \psi^{-1}\phi^{-1}.$$

* I employ the term *scalar*, as Hamilton has done, to designate the quantities real and imaginary of ordinary algebra, in order to distinguish these from matrices regarded as quantities; the latter are *non-scalar* quantities.

The *transverse* of ϕ (*i.e.*, the matrix obtained by interchanging the rows and columns of ϕ) will be denoted by $\breve{\phi}$. We have

$$\breve{(\breve{\phi})} = \phi,$$

$$\breve{(\phi+\psi)} = \breve{\phi}+\breve{\psi},$$

$$\breve{(\phi\psi)} = \breve{\psi}\breve{\phi},$$

$$\breve{(\phi^{-1})} = (\breve{\phi})^{-1}.$$

The matrix ϕ is *symmetric* if $a_{rs} = a_{sr}$ for $r, s = 1, 2, \dots n$, where a_{rs} denotes the constituent of ϕ in the r^{th} row and s^{th} column. The necessary and sufficient condition for a symmetric matrix is

$$\breve{\phi} = \phi.$$

The matrix ϕ is skew-symmetric if $a_{rs} = -a_{sr}$ for $r, s = 1, 2, \dots n$; for which the necessary and sufficient condition is

$$\breve{\phi} = -\phi.$$

If μ is any positive integer, I shall take $\phi^{1/\mu}$ to denote the matrix whose μ^{th} power is equal to ϕ. The matrix $\phi^{1/\mu}$ may be termed the μ_{th} root of ϕ. Every matrix whose determinant does not vanish has a μ^{th} root for any index μ. In general, if the order of the matrix is n, the number of the μ^{th} roots is μ^n. In the *Comptes Rendus*, XCIV., Sylvester gave the following expression for the μ^{th} root of any matrix ϕ whose latent roots $g_1, g_2, \dots g_n$, are all distinct, viz.,

$$\sum_{1}^{n}{}_r\, g_r^{1/\mu}\, \frac{(\phi-g_1)(\phi-g_2)\dots(\phi-g_{r-1})(\phi-g_{r+1})\dots(\phi-g_n)}{(g_r-g_1)(g_r-g_2)\dots(g_r-g_{r-1})(g_r-g_{r+1})\dots(g_r-g_n)}.$$

When two or more of the latent roots of ϕ are equal, the expression for $\phi^{1/\mu}$, as stated by Sylvester, may be obtained by expanding this expression in powers of ϕ, and finding the limiting values of the coefficients. If ϕ is real and symmetric, for the case of equal latent roots, $\phi^{1/\mu}$ may be obtained simply by taking the summation for the distinct latent roots.* It is to be observed that the roots of a symmetric matrix given by Sylvester's formula (as they are linear in powers of a symmetric matrix) are themselves symmetric.

* See *Proc. Lond. Math. Soc.*, Vol. XXII., p. 461. The same is true for any matrix ϕ, for which the nullity of $|\phi-g|$, for any multiple latent root g, is equal to the multiplicity of g. If ϕ is of order n, nullity m is equivalent to *rank* (*Rang*) $n-m$.

We may obtain expressions for $(\phi^{-1})^{1/\mu}$ by substituting, in the above summation, $g_r^{-1/\mu}$ for $g_r^{1/\mu}$.* We have

$$(\phi^{-1})^{1/\mu} = (\phi^{1/\mu})^{-1};$$

either of these expressions may, therefore, be denoted simply by $\phi^{-1/\mu}$.

2. *Cayley's expression for the automorphic linear transformation of a quadric.*

Let Ω denote the matrix of the coefficients of the quadric

$$\sum_1^n a_{rs} x_r x_s,$$

where $$a_{rs} = a_{sr}.$$

If the variables x are transformed by the linear substitution or matrix ϕ, so that

$$x_r = \sum_1^n{}_i \, \phi_{ri} \, \xi_i,$$

the matrix of the quadric so obtained will be equal to

$$\breve{\phi} \, \Omega \, \phi.\dagger$$

It may be required that the quadric so obtained shall be identically equal to the quadric $\Sigma b_{rs} \xi_r \xi_s$ (in which $b_{rs} = b_{sr}$), the matrix of whose coefficients is Ω'; for this the necessary and sufficient condition is that ϕ shall satisfy the matrical equation

$$\breve{\phi} \, \Omega \, \phi = \Omega'.$$

It will be assumed that the determinants of the matrices Ω, Ω', and ϕ, do not either of them vanish. Since, by supposition,

$$a_{rs} = a_{sr}, \quad \text{and} \quad b_{rs} = b_{sr},$$

Ω and Ω' are symmetric matrices.

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* This follows from the following more general formula for any function of the matrix ϕ, given by Sylvester in the *Johns Hopkins University Circulars*, Vol. III., viz.,

$$f(\phi) = \sum_1^n{}_r f(g_r) \frac{(\phi - g_1)(\phi - g_2) \dots (\phi - g_{r-1})(\phi - g_{r+1}) \dots (\phi - g_n)}{(g_r - g_1)(g_r - g_2) \dots (g_r - g_{r-1})(g_r - g_{r+1}) \dots (g_r - g_n)}.$$

† Cayley, "Memoir on the Automorphic Linear Transformation of the Bipartite Quadric Function," *Phil. Trans.*, 1858.

For the case of automorphic transformation, *i.e.*, when $\Omega = \Omega'$, Cayley has given the following expression for ϕ, viz.,

$$\phi = \Omega^{-1}(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\Omega,$$

in which Υ is an arbitrary skew-symmetric matrix. From this expression for ϕ, it follows that

$$\begin{aligned}\breve{\phi} &= \breve{\Omega}\,(\breve{\Omega}-\breve{\Upsilon})^{-1}\,(\breve{\Omega}+\breve{\Upsilon})\,\breve{\Omega}^{-1}\\ &= \Omega\,(\Omega+\Upsilon)^{-1}\,(\Omega-\Upsilon)\,\Omega^{-1};\end{aligned}$$

therefore, substituting these expressions in the equation

$$\breve{\phi}\,\Omega\phi = \Omega,$$

we should have

$$\Omega\,(\Omega+\Upsilon)^{-1}\,(\Omega-\Upsilon)\,\Omega^{-1}.\Omega.\Omega^{-1}\,(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\,\Omega = \Omega.$$

But $(\Omega-\Upsilon)\,\Omega^{-1}\,(\Omega+\Upsilon) = \Omega-\Upsilon+\Upsilon-\Upsilon\Omega^{-1}\Upsilon = (\Omega+\Upsilon)\,\Omega^{-1}\,(\Omega-\Upsilon)$;

from which follow, successively,

$$(\Omega+\Upsilon)^{-1}\,(\Omega-\Upsilon)\,\Omega^{-1}\,(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1} = \Omega^{-1},$$

$$\Omega\,(\Omega+\Upsilon)^{-1}(\Omega-\Upsilon)\,\Omega^{-1}.\Omega.\Omega^{-1}\,(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\,\Omega = \Omega.\text{*}$$

Equating the reciprocals of either side of the last equation but one gives

$$(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}.\Omega.(\Omega-\Upsilon)^{-1}\,(\Omega+\Upsilon) = \Omega;$$

but, if $\qquad \phi = (\Omega-\Upsilon)^{-1}\,(\Omega+\Upsilon),$

then $\qquad \begin{aligned}\breve{\phi} &= (\breve{\Omega}+\breve{\Upsilon})(\breve{\Omega}-\breve{\Upsilon})^{-1}\\ &= (\Omega-\Upsilon)(\Omega+\Upsilon)^{-1};\end{aligned}$

therefore this expression for ϕ is also a solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega.$$

This expression is, however, identically equal to Cayley's expression; for from it we derive

$$\breve{\phi}^{-1} = (\Omega+\Upsilon)(\Omega-\Upsilon)^{-1};$$

and from the equation $\qquad \breve{\phi}\,\Omega\,\phi = \Omega$

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* Cayley, *ibid.*

it follows that $$\phi = \Omega^{-1}\breve{\phi}^{-1}\Omega\,;$$

therefore $$(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon) = \Omega^{-1}(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\Omega.^{*}$$

As the expression $(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)$ is somewhat simpler than its equivalent, I shall employ it in what follows.

Cayley's solution fails if it is required that ϕ shall have -1 as a latent root, but not otherwise. For, if ϕ is any solution of the equation
$$\breve{\phi}\,\Omega\,\phi = \Omega,$$
provided -1 is not a latent root of ϕ, we may put
$$\Upsilon = \Omega\,(\phi-1)(\phi+1)^{-1},$$
whence we obtain $$\breve{\Upsilon} = -\Upsilon,$$
since $$\breve{\phi} = \Omega\phi^{-1}\Omega^{-1},^{\dagger}$$
and also $$\phi = (\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)\,;$$
but, since
$$\begin{aligned}\Omega-\Upsilon &= \Omega\left[1-(\phi-1)(\phi+1)^{-1}\right]\\ &= \Omega\,(\overline{\phi+1}-\overline{\phi-1})(\phi+1)^{-1}\\ &= 2\Omega\,(\phi+1)^{-1},\end{aligned}$$

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* The identity between the two expressions may also be shown as follows:—
$$\begin{aligned}\Omega^{-1}(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\Omega &= \Omega^{-1}\,.\,\Omega^{\frac12}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})\,\Omega^{\frac12}\,.\,\Omega^{-\frac12}(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}\Omega^{-\frac12}\,.\,\Omega\\ &= \Omega^{-\frac12}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}\Omega^{\frac12}\\ &= \Omega^{-\frac12}(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})\,\Omega^{\frac12}\\ &= \Omega^{-\frac12}(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}\Omega^{-\frac12}\,.\,\Omega^{\frac12}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})\,\Omega^{\frac12}\\ &= (\Omega-\Upsilon)^{-1}(\Omega+\Upsilon).\end{aligned}$$

† For, if $$\breve{\phi}\,\Omega\,\phi = \Omega,$$
we have $$\breve{\phi} = \Omega\,\phi^{-1}\,\Omega^{-1}\,;$$
therefore, if $$\Upsilon = \Omega\,(\phi-1)(\phi+1)^{-1},$$
$$\begin{aligned}\breve{\Upsilon} &= (\breve{\phi}+1)^{-1}(\breve{\phi}-1)\,\breve{\Omega}\\ &= (\Omega\phi^{-1}\Omega^{-1}+1)^{-1}(\Omega\phi^{-1}\Omega^{-1}-1)\,\Omega\\ &= \left[\Omega\,(\phi^{-1}+1)\,\Omega^{-1}\right]^{-1}.\,\Omega\,(\phi^{-1}-1)\,\Omega^{-1}\,.\,\Omega\\ &= \Omega\,(\phi^{-1}+1)^{-1}\,\Omega^{-1}\,.\,\Omega\,(\phi^{-1}-1)\,\Omega^{-1}\,.\,\Omega\\ &= \Omega\,(\phi^{-1}+1)^{-1}(\phi^{-1}-1)\\ &= \Omega\left[\phi^{-1}(1+\phi)\right]^{-1}.\,\phi^{-1}(1-\phi)\\ &= \Omega\,(1+\phi)^{-1}\phi\,.\,\phi^{-1}(1-\phi)\\ &= -\Omega\,(\phi+1)^{-1}(\phi-1).\end{aligned}$$

it is evident that $\Omega - \Upsilon$ has a reciprocal, and consequently the above expression for ϕ in terms of the skew-symmetric matrix Υ is possible. If, however, -1 is a latent root of ϕ, then $\phi + 1$ has no reciprocal, and we cannot put

$$\Upsilon = \Omega(\varphi - 1)(\phi + 1)^{-1},$$

which is required by Cayley's expression for ϕ in terms of Υ.*

If -1 is a latent root of ϕ, but not $+1$, then $+1$ will be a latent root of $-\phi$, but not -1; for, if

$$|\varphi + 1| = 0, \quad |\phi - 1| \neq 0,$$

then

$$|(-\phi) - 1| = 0, \quad |(-\phi) + 1| \neq 0.$$

Therefore $-\phi$ can be represented as above, giving

$$\phi = -(\Omega - \Upsilon)^{-1}(\Omega + \Upsilon).$$

Thus the expression $\pm(\Omega - \Upsilon)^{-1}(\Omega + \Upsilon)$

gives every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega,$$

except those for which both ± 1 are latent roots.

3. *Determination of the linear transformations of one quadric into another.*

Cayley's solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega$$

gives at once the means of solving the more general equation

$$\phi\,\Omega\,\phi = \Omega',$$

where Ω and Ω' are known symmetric matrices. For this equation may be written

$$\breve{\phi}\,\Omega^{\frac{1}{2}}\,\Omega'^{-\frac{1}{2}}\,.\,\Omega'\,.\,\Omega'^{-\frac{1}{2}}\,\Omega^{\frac{1}{2}}\,\phi = \Omega';$$

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* Moreover, if $\phi = (\Omega - \Upsilon)^{-1}(\Omega + \Upsilon)$,
and if -1 is a latent root of ϕ, then

$$\frac{2^n}{|\Omega - \Upsilon|}\,|\Omega| = |(\Omega - \Upsilon)^{-1}(\overline{\Omega + \Upsilon} + \overline{\Omega - \Upsilon})|$$
$$= |(\Omega - \Upsilon)^{-1}(\Omega + \Upsilon) + 1| = |\phi + 1| = 0,$$

which is impossible, since, by supposition, $|\Omega| \neq 0$.

and, if $\Omega^{\frac{1}{2}}$ and $\Omega'^{-\frac{1}{2}}$ denote symmetric square roots of Ω and Ω'^{-1}, respectively, and

$$\psi = \Omega'^{-\frac{1}{2}}\,\Omega^{\frac{1}{2}}\,\phi,$$

the equation becomes
$$\breve{\psi}\,\Omega'\,\psi = \Omega',$$

of which the general solution is

$$\psi = \pm\,(\Omega' - \Upsilon)^{-1}\,(\Omega' + \Upsilon),$$

where Υ is an arbitrary skew-symmetric matrix. Therefore, *the general expression for the matrix ϕ, satisfying the equation*

$$\breve{\phi}\,\Omega\,\phi = \Omega'$$

(*i.e., the general expression for the linear substitution that will transform a given quadric whose matrix is Ω into another given quadric whose matrix is Ω'*), *is*

$$\pm\,\Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}}\,(\Omega' - \Upsilon)^{-1}\,(\Omega' + \Upsilon),$$

where Υ is an arbitrary skew-symmetric matrix, and $\Omega^{-\frac{1}{2}}$ and $\Omega'^{\frac{1}{2}}$ are symmetric square roots of Ω^{-1} and Ω' respectively. Expressions for $\Omega^{-\frac{1}{2}}$ and $\Omega'^{\frac{1}{2}}$ may be obtained by means of Sylvester's formula.

This solution fails if the condition is imposed that

$$\psi = \Omega'^{-\frac{1}{2}}\,\Omega^{\frac{1}{2}}\,\phi$$

shall have as latent roots both ± 1. If $\left\{\begin{matrix} +1 \\ -1 \end{matrix}\right\}$ is a latent root of ψ, but not $\left\{\begin{matrix} -1 \\ +1 \end{matrix}\right\}$, the $\left\{\begin{matrix} \text{upper} \\ \text{lower} \end{matrix}\right\}$ sign is to be taken.

Another form of the general solution is

$$\phi = \pm\,(\Omega - \Upsilon)^{-1}\,(\Omega + \Upsilon)\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}},$$

where Υ is an arbitrary skew-symmetric matrix. This solution may be verified by substituting for ϕ in $\breve{\phi}\,\Omega\,\phi$, giving

$$\Omega'^{\frac{1}{2}}\,\Omega^{-\frac{1}{2}}\,(\Omega - \Upsilon)(\Omega + \Upsilon)^{-1}\,\Omega\,(\Omega - \Upsilon)^{-1}\,(\Omega + \Upsilon)\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}}$$
$$= \Omega'^{\frac{1}{2}}\,\Omega^{-\frac{1}{2}}\,.\,\Omega\,.\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}} = \Omega'.$$

It may be obtained by writing the equation

$$\breve{\phi}\,\Omega\phi = \Omega'$$

as
$$\breve{\phi}^{-1}\Omega'\,\phi^{-1} = \Omega,$$

and proceeding as in the derivation of the first form; or, it may be derived from that by substituting in it for Υ the skew-symmetric matrix $\Omega'^{\frac{1}{2}}\Omega^{-\frac{1}{2}}\Upsilon\Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}}$, as follows:—

$$\begin{aligned}
&\Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}}(\Omega'-\Omega'^{\frac{1}{2}}\Omega^{-\frac{1}{2}}\Upsilon\Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}})^{-1}(\Omega'+\Omega'^{\frac{1}{2}}\Omega^{-\frac{1}{2}}\Upsilon\Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}}) \\
&\qquad = \Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}}.\Omega'^{-\frac{1}{2}}(1-\Omega^{-\frac{1}{2}}\Upsilon\Omega^{-\frac{1}{2}})^{-1}\Omega'^{-\frac{1}{2}}.\Omega'^{\frac{1}{2}}(1+\Omega^{-\frac{1}{2}}\Upsilon\Omega^{-\frac{1}{2}})\Omega'^{\frac{1}{2}} \\
&\qquad = \Omega^{-\frac{1}{2}}(1-\Omega^{-\frac{1}{2}}\Upsilon\Omega^{-\frac{1}{2}})^{-1}\Omega^{-\frac{1}{2}}.\Omega^{\frac{1}{2}}(1+\Omega^{-\frac{1}{2}}\Upsilon\Omega^{-\frac{1}{2}})\Omega^{\frac{1}{2}}.\Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}} \\
&\qquad = (\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)\Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}}.
\end{aligned}$$

The expression $\quad \pm\Omega^{-\frac{1}{2}}(1-\Upsilon)^{-1}(1+\Upsilon)\Omega'^{\frac{1}{2}}$

is another form of the general solution; it may be derived from the first form by substituting in that for Υ the skew-symmetric matrix $\Omega'^{\frac{1}{2}}\Upsilon\Omega'^{\frac{1}{2}}$.

4. *Determination of the real linear transformations of one quadric into another.*

The expressions for ϕ given in the last section will, in general, be imaginary. If, however, Ω and Ω' are real and have the same number of positive latent roots,* there are real values of ϕ satisfying the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega'.$$

These can be obtained through the representation of Ω and Ω' in what may be termed their canonical or standard forms.

Let Ω be a real symmetric matrix, and let $g_1, g_2, \dots g_m$ denote its positive latent roots, and $-g_{m+1}, -g_{m+2}, \dots g_n$ its negative latent roots;* moreover, let

$$\{c_1, c_2, \dots c_n\}$$

denote a matrix whose constituents are all zero, except those in the principal diagonal, which are severally equal to c_1, c_2, &c. Then, by a well-known theorem, a real orthogonal matrix ϖ_1 can always be found such that

$$\Omega = \breve{\varpi}_1\,\theta\,\varpi_1,$$

where $\quad \theta = \{g_1, g_2, \dots g_m, -g_{m+1}, \dots -g_n\}.$

The right-hand member of the preceding equation I term the canonical form of Ω. Similarly, if Ω' is a real symmetric matrix

* It is assumed that $|\,\Omega\,|$, the determinant of the quadric, is not zero; this is equivalent to the assumption that none of the latent roots of Ω are zero. Since Ω is real and symmetric, its latent roots are all real.

whose positive latent roots are $h_1, h_2, \dots h_m$, and whose negative latent roots are $-h_{m+1}, -h_{m+2}, \dots -h_n$, a real orthogonal matrix ϖ_2 can always be found such that

$$\Omega' = \varpi_2 \eta \breve{\varpi}_2,$$

where $$\eta = \{h_1, h_2, \dots h_m, -h_{m+1}, \dots -h_n\}.$$

Substituting these expressions for Ω and Ω' in the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega',$$

we have $$\breve{\phi}\,\breve{\varpi}_1\,\theta\,\varpi_1\phi = \varpi_2\eta\,\breve{\varpi}_2,$$

i.e., $$\breve{\varpi}_2\breve{\phi}\,\breve{\varpi}_1\theta\varpi_1\phi\,\varpi_2 = \eta.$$

Denoting $\varpi_1\phi\varpi_2$ by ψ, this becomes

$$\breve{\psi}\theta\psi = \eta,$$

of which the general solution, by the preceding section, is

$$\psi = \pm\,\theta^{-\frac{1}{2}}\eta^{\frac{1}{2}}\,(\eta - \Upsilon)^{-1}\,(\eta + \Upsilon),$$

where Υ is an arbitrary skew-symmetric matrix, and we may take

$$\theta^{-\frac{1}{2}} = \left\{\frac{1}{g_1^{\frac{1}{2}}},\ \frac{1}{g_2^{\frac{1}{2}}},\ \dots\ \frac{1}{g_m^{\frac{1}{2}}},\ \frac{1}{g_{m+1}^{\frac{1}{2}}\sqrt{-1}},\ \dots\ \frac{1}{g_n^{\frac{1}{2}}\sqrt{-1}}\right\},$$

$$\eta^{\frac{1}{2}} = \{h_1^{\frac{1}{2}},\ h_2^{\frac{1}{2}},\ \dots\ h_m^{\frac{1}{2}},\ h_{m+1}^{\frac{1}{2}}\sqrt{-1},\ \dots\ h_n^{\frac{1}{2}}\sqrt{-1}\};$$

consequently,

$$\theta^{-\frac{1}{2}}\eta^{\frac{1}{2}} = \left\{\sqrt{\frac{h_1}{g_1}},\ \sqrt{\frac{h_2}{g_2}},\ \dots\ \sqrt{\frac{h_m}{g_m}},\ \sqrt{\frac{h_{m+1}}{g_{m+1}}},\ \dots\ \sqrt{\frac{h_n}{g_n}}\right\}$$

is real.

We have

$$\begin{aligned}\psi &= \pm\,\theta^{-\frac{1}{2}}\eta^{\frac{1}{2}}\breve{\varpi}_2.\varpi_2\,(\eta-\Upsilon)^{-1}\,\breve{\varpi}_2.\varpi_2\,(\eta+\Upsilon)\,\breve{\varpi}_2.\varpi_2 \\ &= \pm\,\theta^{-\frac{1}{2}}\eta^{\frac{1}{2}}\breve{\varpi}_2\,(\varpi_2\eta\,\breve{\varpi}_2 - \varpi_2\Upsilon\breve{\varpi}_2)^{-1}\,(\varpi_2\eta\,\breve{\varpi}_2 + \varpi_2\Upsilon\breve{\varpi}_2)\,\varpi_2 \\ &= \pm\,\theta^{-\frac{1}{2}}\eta^{\frac{1}{2}}\breve{\varpi}_2\,(\Omega' - \Upsilon_1)^{-1}(\Omega' + \Upsilon_1)\,\varpi_2,\end{aligned}$$

where $$\Upsilon_1 = \varpi_2\Upsilon\breve{\varpi}_2$$

is an arbitrary skew-symmetric matrix. Therefore, *the general expression for the real linear substitutions that will transform a given real* *quadric, whose matrix is Ω, into another given real quadric, whose matrix is Ω' (Ω and Ω' having the same number of positive latent roots), is*

$$\breve{\varpi}_1 \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 (\Omega' - \Upsilon)^{-1} (\Omega' + \Upsilon),$$

where $\theta^{-\frac{1}{2}}\eta^{\frac{1}{2}}$, a real matrix, and $\breve{\varpi}_1$ and ϖ_2, also real, have the meanings assigned to them above, and Υ is an arbitrary real skew-symmetric matrix.

Another form in which this expression may appear is

$$\begin{aligned} &\breve{\varpi}_1 (\theta - \Upsilon)^{-1} (\theta + \Upsilon)\, \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 \\ &= \breve{\varpi}_1 (\theta - \Upsilon)^{-1} \varpi_1 . \breve{\varpi}_1 (\theta + \Upsilon)\, \varpi_1 . \breve{\varpi}_1 \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 \\ &= (\breve{\varpi}_1 \theta \varpi_1 - \breve{\varpi}_1 \Upsilon \varpi_1)^{-1} (\breve{\varpi}_1 \theta \varpi_1 + \breve{\varpi}_1 \Upsilon \varpi_1)\, \breve{\varpi}_1 \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 \\ &= (\Omega - \Upsilon_1)^{-1} (\Omega + \Upsilon_1)\, \breve{\varpi}_1 \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2, \end{aligned}$$

where
$$\Upsilon_1 = \breve{\varpi}_1 \Upsilon \varpi_1$$

is an arbitrary real skew-symmetric matrix. The method of obtaining this form from the first is evident from the last section.

5. *Reduction of a quadric to a sum of squares.*

In this case Ω' is of the form $\{c_1, c_2, \dots c_n\}$. Thus, if it is required to transform by a real linear substitution the real quadric $\sum_1^n a_{rs} x_r x_s$, whose matrix is Ω, into the sum of squares $\sum_1^n \pm G_r \xi_r^2$, then, by the principle of inertia of quadratic forms, just so many of the G's must be positive as there are positive latent roots of Ω. As before, let $g_1, g_2, \dots g_m$ be the positive latent roots of Ω, and $-g_{m+1}, -g_{m+2}, \dots -g_n$ the negative latent roots of Ω. Since it is immaterial what subscripts appertain to the ξ's and their coefficients, it may be assumed that the first m of these coefficients are positive, the remainder being negative. If

$$\Omega = \breve{\varpi} \{g_1, g_2, \dots g_m, -g_{m+1} \dots -g_n\} \varpi,$$

ϖ being a real orthogonal matrix, the equation determining the transformation ϕ is

$$\begin{aligned} &\breve{\phi} \breve{\varpi} \{g_1, g_2, \dots g_m, -g_{m+1}, \dots -g_n\} \varpi \phi \\ &\qquad = \{G_1, G_2, \dots G_m, -G_{m+1}, \dots -G_n\}. \end{aligned}$$

Therefore, *the general expression for the real linear substitution that* *will transform a given real quadric into a sum of squares, the coefficients being real, is*

$$\pm\breve{\varpi}\left\{\sqrt{\frac{G_1}{g_1}}, \sqrt{\frac{G_2}{g_2}}, \dots \sqrt{\frac{G_n}{g_n}}\right\}\left(\{G_1, G_2, \dots G_m, -G_{m+1}, \dots -G_n\}-\Upsilon\right)^{-1}$$

$$\times\left(\{G_1, G_2, \dots G_m, -G_{m+1}, \dots -G_n\}+\Upsilon\right),$$

where Υ is an arbitrary real skew-symmetric matrix.

Another form of the general expression is

$$\pm(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)\,\breve{\varpi}\left\{\sqrt{\frac{G_1}{g_1}}, \sqrt{\frac{G_2}{g_2}}, \dots \sqrt{\frac{G_n}{g_n}}\right\},$$

where Υ is an arbitrary real skew-symmetric matrix.

6. *Dependence upon the equation* $\breve{\psi}\psi = 1$ *of the equations* $\breve{\phi}\Omega\phi = \Omega$ *and* $\breve{\phi}\Omega\phi = \Omega'$.

The equation $$\breve{\phi}\,\Omega\,\phi = \Omega$$

may be written $$\Omega^{-\frac{1}{2}}\breve{\phi}\,\Omega^{\frac{1}{2}}\,.\,\Omega^{\frac{1}{2}}\phi\,\Omega^{-\frac{1}{2}} = 1,$$

since, if ϕ satisfies this equation, it satisfies the former, and conversely. This is also true if the two square roots $\Omega^{-\frac{1}{2}}$ and $\Omega^{\frac{1}{2}}$ are taken to be symmetric; and then, if

$$\psi = \Omega^{\frac{1}{2}}\phi\,\Omega^{-\frac{1}{2}},$$

the equation becomes $$\breve{\psi}\,\psi = 1.$$

Therefore $$\phi = \Omega^{-\frac{1}{2}}\,\psi\,\Omega^{\frac{1}{2}}$$

(where $\Omega^{-\frac{1}{2}}$ and $\Omega^{\frac{1}{2}}$ denote symmetric square roots, and ψ is an arbitrary orthogonal matrix) is the most general solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega.$$

Consequently, the problem of the automorphic linear transformation of a quadric resolves itself into that of the representation of an orthogonal matrix.

In this expression for ϕ, no generality is lost by regarding the square root of Ω^{-1}, which appears in this expression, as the reciprocal of that square root of Ω which also enters into this expression for ϕ;

for every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega$$

is a solution of the equation

$$\Omega^{-\frac{1}{2}}\breve{\phi}\,\Omega^{\frac{1}{2}} \,.\, \Omega^{\frac{1}{2}}\phi\,\Omega^{-\frac{1}{2}} = 1,$$

in which the square roots of Ω that enter are all identical.* Similarly, without loss of generality, any square root of Ω may be taken, provided it is symmetric; and then, by a suitable choice of ψ, all other solutions of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega$$

may be obtained.† It will be assumed in what follows that the two square roots of Ω entering into the expression for ϕ are identical, in which case ϕ and ψ have the same latent roots; for then

$$\phi - g = \Omega^{-\frac{1}{2}}\psi\,\Omega^{\frac{1}{2}} - g = \Omega^{-\frac{1}{2}}(\psi - g)\,\Omega^{\frac{1}{2}},$$

and therefore $$|\,\phi - g\,| = |\,\psi - g\,|$$

for all values of g.

The equation $$\breve{\phi}\,\Omega\,\phi = \Omega',$$

may, similarly, be written

$$\Omega'^{-\frac{1}{2}}\breve{\phi}\,\Omega^{\frac{1}{2}} \,.\, \Omega^{\frac{1}{2}}\phi\,\Omega'^{-\frac{1}{2}} = 1,$$

since, if ϕ satisfies this equation, it satisfies the former, and con-

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* Moreover, if $\Omega_1^{\frac{1}{2}}$, $\Omega_2^{\frac{1}{2}}$ denote distinct symmetric square roots of Ω, and if

$$\phi = \Omega_1^{-\frac{1}{2}}\psi\,\Omega_2^{\frac{1}{2}},$$

where ψ is orthogonal, replacing ψ by $\psi\,\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}$, which is also orthogonal, since $\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}$ is orthogonal (for the transverse of $\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}$ is $\Omega_2^{-\frac{1}{2}}\Omega_1^{\frac{1}{2}}$, and

$$\Omega_2^{-\frac{1}{2}}\Omega_1^{\frac{1}{2}} \,.\, \Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}} = \Omega_2^{-\frac{1}{2}} \,.\, \Omega \,.\, \Omega_2^{-\frac{1}{2}} = \Omega_2^{-\frac{1}{2}}\Omega_2^{\frac{1}{2}} \,.\, \Omega_2^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}} = 1),$$

and the product of two orthogonal matrices is orthogonal, the expression for ϕ becomes $\Omega_1^{-\frac{1}{2}}\psi\,\Omega_1^{\frac{1}{2}}$, in which the square roots that enter are reciprocals.

† Thus, if $\Omega_1^{\frac{1}{2}}$ and $\Omega_2^{\frac{1}{2}}$ denote two distinct symmetric square roots of Ω, and if

$$\phi = \Omega_1^{-\frac{1}{2}}\psi\,\Omega_1^{\frac{1}{2}},$$

where ψ is orthogonal, replacing ψ by $\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}\psi\,\Omega_2^{\frac{1}{2}}\Omega_1^{-\frac{1}{2}}$, which is also orthogonal, since $\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}$ and $\Omega_2^{\frac{1}{2}}\Omega_1^{-\frac{1}{2}}$ are orthogonal, ϕ becomes

$$\Omega_2^{-\frac{1}{2}}\psi\,\Omega_2^{\frac{1}{2}}.$$

versely. The matrices $\Omega'^{-\frac{1}{2}}$ and $\Omega^{\frac{1}{2}}$ may be taken to be symmetric; and then, if

$$\psi = \Omega^{\frac{1}{2}}\phi\,\Omega'^{-\frac{1}{2}},$$

the equation becomes $\breve{\psi}\psi = 1.$

Therefore $\phi = \Omega^{-\frac{1}{2}}\psi\,\Omega'^{\frac{1}{2}},$

in which $\Omega^{-\frac{1}{2}}$ and $\Omega'^{\frac{1}{2}}$ are any pair of symmetric square roots of Ω^{-1} and Ω', respectively, and ψ is an arbitrary orthogonal matrix, is the most general solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega'.$$

In what follows I shall give two expressions for orthogonal matrices, and their applications to the equations considered. Of these solutions of the equation

$$\breve{\psi}\psi = 1,$$

the second is absolutely general, and, therefore, gives rise to expressions which contain every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega,$$

and every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega' \quad (\text{see § 7}).$$

The first representation of an orthogonal matrix which will be considered is Cayley's; it gives rise to the solutions already presented in § 2 and § 3.

Thus, if -1 is not a latent root of the orthogonal matrix ψ, we may put

$$\Upsilon = \frac{\psi-1}{\psi+1};$$

whence follows

$$\breve{\Upsilon} = \frac{1}{\breve{\psi}+1}(\breve{\psi}-1) = \frac{1}{\psi^{-1}+1}(\psi^{-1}-1) = \frac{1}{1+\psi}(1-\psi) = \frac{1-\psi}{1+\psi} = -\Upsilon,$$

and $\psi = (1-\Upsilon)^{-1}(1+\Upsilon).$*

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* This expression for ψ is possible, for

$$|\Upsilon-1| = \left|\frac{\psi-1}{\psi+1}-1\right| = \frac{1}{|\psi+1|}\,|\overline{\psi-1}-\overline{\psi+1}| = \frac{|2|}{|\psi+1|} \neq 0;$$

therefore, $\Upsilon-1$ has a reciprocal.

This is Cayley's well-known representation of an orthogonal matrix in terms of a skew-symmetric matrix. If, however, -1 is a latent root of ψ, Υ cannot be thus expressed in terms of ψ.

If -1 is a latent root of ψ, but not $+1$, $-\psi$ will be an orthogonal matrix of which $+1$ is a latent root, but not -1;* therefore $-\psi$ is representable as above, giving

$$\psi = -(1-\Upsilon)^{-1}(1+\Upsilon).$$

Therefore, the expression

$$\pm(1-\Upsilon)^{-1}(1+\Upsilon)$$

will, for a proper choice of the skew-symmetric matrix Υ, give every orthogonal matrix, except those of which both ± 1 are latent roots.

Substituting, for ψ, in the equation

$$\phi = \Omega^{-\frac{1}{2}}\psi\,\Omega^{\frac{1}{2}},$$

Cayley's expression for an orthogonal matrix, we obtain the solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega$$

given in § 2. Thus

$$\begin{aligned} \phi &= \pm\,\Omega^{-\frac{1}{2}}(1-\Upsilon)^{-1}(1+\Upsilon)\,\Omega^{\frac{1}{2}} \\ &= \pm\,\Omega^{-\frac{1}{2}}(1-\Upsilon)^{-1}\,\Omega^{-\frac{1}{2}}\,.\,\Omega^{\frac{1}{2}}(1+\Upsilon)\,\Omega^{\frac{1}{2}} \\ &= \pm\,(\Omega-\Upsilon_1)^{-1}(\Omega+\Upsilon_1), \end{aligned}$$

in which

$$\Upsilon_1 = \Omega^{\frac{1}{2}}\Upsilon\,\Omega^{\frac{1}{2}}$$

is an arbitrary skew-symmetric matrix. And, since the latent roots of ϕ and of ψ are identical, this expression for ϕ (as stated in § 2) contains every solution of this equation, except those which have as latent roots both ± 1.

Similarly, substituting $(1-\Upsilon)^{-1}(1+\Upsilon)$ for ψ in $\Omega^{-\frac{1}{2}}\psi\,\Omega'^{\frac{1}{2}}$, we obtain the solution given in § 3, viz.,

$$\begin{aligned} \phi &= \pm\,\Omega^{-\frac{1}{2}}(1-\Upsilon)^{-1}(1+\Upsilon)\,\Omega'^{\frac{1}{2}} \\ &= \pm\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}}\,.\,\Omega'^{-\frac{1}{2}}(1-\Upsilon)^{-1}\,\Omega'^{-\frac{1}{2}}\,.\,\Omega'^{\frac{1}{2}}(1+\Upsilon)\,\Omega'^{\frac{1}{2}} \\ &= \pm\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}}(\Omega'-\Upsilon_1)^{-1}(\Omega'+\Upsilon_1), \end{aligned}$$

in which

$$\Upsilon_1 = \Omega'^{\frac{1}{2}}\,\Upsilon\,\Omega'^{\frac{1}{2}}$$

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* See p. 296.

is an arbitrary skew-symmetric matrix. This expression contains every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega\,,$$

except those for which $\Omega'^{-\frac{1}{2}}\Omega^{\frac{1}{2}}\phi$ has both ± 1 as latent roots.

7. *Solutions of the equations* $\breve{\phi}\,\Omega\,\phi = \Omega$ *and* $\breve{\phi}\,\Omega\,\phi = {}'\Omega'$, ***based on*** *Tait's representation of an orthogonal matrix.*

Any matrix of non-vanishing determinant is separable into a product of a symmetric matrix and an orthogonal matrix. Thus, if χ is any matrix,

$$\chi = \left(\chi\frac{1}{\sqrt{\breve{\chi}\chi}}\right)\sqrt{\breve{\chi}\chi} = \frac{\chi}{\sqrt{\breve{\chi}\chi}}\sqrt{\breve{\chi}\chi}\,;$$

but $\breve{\chi}\chi$ is symmetric, therefore $\sqrt{\breve{\chi}\chi}$ may be taken to be symmetric ;

and, if $$\psi = \frac{\chi}{\sqrt{\breve{\chi}\chi}},$$

then $$\breve{\psi}\psi = \frac{1}{\sqrt{\breve{\chi}\chi}}\breve{\chi}\cdot\chi\frac{1}{\sqrt{\breve{\chi}\chi}} = \frac{1}{\sqrt{\breve{\chi}\chi}}\sqrt{\breve{\chi}\chi}\sqrt{\breve{\chi}\chi}\frac{1}{\sqrt{\breve{\chi}\chi}} = 1.\text{*}$$

The function $\dfrac{\chi}{\sqrt{\breve{\chi}\chi}}$ will, for a proper choice of χ, be equal to any orthogonal matrix. For, if ψ is any orthogonal matrix, and if we put

$$\chi = \psi\omega,$$

where ω is any symmetric matrix whose determinant does not vanish, one value of this function of χ is ψ; thus,

$$\frac{\chi}{\sqrt{\breve{\chi}\chi}} = \frac{\psi\omega}{\sqrt{\omega\breve{\psi}\,.\,\psi\omega}} = \frac{\psi\omega}{\sqrt{\omega^2}} = \psi.$$

Therefore, substituting for ψ in $\Omega^{-\frac{1}{2}}\psi\,\Omega^{\frac{1}{2}}$, *the most general solution of the matrical equation* $\breve{\phi}\,\Omega\,\phi = \Omega$ *is*

$$\phi = \Omega^{-\frac{1}{2}}\frac{\chi}{\sqrt{\breve{\chi}\chi}}\Omega^{\frac{1}{2}},$$

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* Kelland and Tait's *Quaternions*, Chap. x

in which $\Omega^{-\frac{1}{2}}$ and $\Omega^{\frac{1}{2}}$ are symmetric square roots, and χ is an arbitrary matrix.

Similarly, *the most general solution of the matrical equation* $\breve{\phi}\,\Omega\,\phi = \Omega'$ *is*

$$\phi = \Omega^{-\frac{1}{2}}\,\frac{\chi}{\sqrt{\breve{\chi}\chi}}\,\Omega'^{\frac{1}{2}},$$

in which $\Omega^{-\frac{1}{2}}$ and $\Omega'^{\frac{1}{2}}$ are any pair of symmetric square roots of Ω^{-1} and Ω', respectively, and χ is arbitrary.

Expressions for $\dfrac{\chi}{\sqrt{\breve{\chi}\chi}}$ may be obtained by means of Sylvester's formula.

On a Theorem for Confocal Bicircular Quartics and Cyclides, corresponding to Ivory's Theorem for Confocal Conics and Conicoids. *By* A. L. Dixon, M.A., Fellow of Merton College, Oxford. Received and read May 11th, 1893.

Darboux (*Mémoires de l'Acad. des Sciences de Bordeaux*, t. VIII. and IX.) and Larmor (*Proc. Lond. Math. Soc.*, XVI., p. 198) have discussed Ivory's theorem and its extension, in which $\dfrac{PQ^2}{\phi(P)\,\phi(Q)}$, where $\phi(P)$ is a function of the position of P, is unaltered when for P and Q are substituted the corresponding pair of points, and have shown that, if $\phi(P)$ is a constant, P and Q lie on confocal conicoids, and that, if such a relation hold at all, P and Q lie on confocal cyclides (*i.e.*, quartic surfaces having the imaginary circle at infinity as a double line), or, if all the points are in one plane, on confocal bicircular quartics.

I propose in this paper to find $\phi(P)$, for bicircular quartics and for cyclides. The system of coordinates used has been already investigated by Darboux and Casey.

I have added a proof of the theorem for the particular case