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# III. On the inverse calculus of definite integrals

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same purposes as this cotton, and if it be notoriously known that what is now called pyroxyloïdine was not brought before the French Academy and the scientific world until towards the middle of last November, the idea of attributing to France the discovery of gun-cotton cannot be seriously entertained, or of assigning to me merely a practical application of that which another would have discovered.

I appeal to the justice of Frenchmen, to decide the point to whom belongs the honour of not only being the first to apply the new substance in question, but also of having first prepared it—to MM. Braconnot and Pelouze, or myself. I must, moreover, add expressly, that it was not xyloïdine even which led to my discovery, however intimate may be its relation with gun-cotton; it was theoretical ideas, possibly very erroneous ones, but which are peculiarly my own, as well as some facts which I was also the first to discover. *Suum cuique* is a principle of morality on which society at large rests; why should it not be strictly respected in the republic of science? M. Pelouze is a distinguished chemist, and already possesses a sufficiently high reputation not to require to elevate his pretensions on the merits of others; and I am fully persuaded that this estimable chemist, of well-known truth of character, will, appreciating with impartiality the circumstances which have occurred, freely render me the justice to which I consider myself entitled.

Bâle, Dec. 28, 1846.

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## III. *On the Inverse Calculus of Definite Integrals.* *By the* Rev. Brice Bronwin*.

THIS paper contains several very simple and easy methods in the inverse calculus of definite integrals; and they show that the function under the sign of integration may have more than one form. The exponents $n$ and $p$ are always positive, and $n+p=i$ an integer.

First, let $\phi(x)=\Sigma \mathrm{A}_m x^m$, an ascending series. Then

$$\int_0^a x^{n-1}dx\phi(a-x)=\Sigma \mathrm{A}_m\int_0^a x^{n-1}dx(a-x)^m$$

$$=\Sigma \mathrm{A}_m a^{m+n}\int_0^1 x^{n-1}dx(1-x)^m$$

$$=\Gamma(n)\Sigma \mathrm{A}_m a^{m+n}\frac{\Gamma(m+1)}{\Gamma(m+n+1)}=\psi(a) \text{ suppose.}$$

* Communicated by the Author.

Then also

$$\Gamma(n)\Sigma A_m (a-x)^{m+n} \frac{\Gamma(m+1)}{\Gamma(m+n+1)} = \psi(a-x);$$

and

$$\Gamma(n)\Sigma A_m \frac{\Gamma(m+1)}{\Gamma(m+n+1)} \int_0^a x^{p-1} dx (a-x)^{m+n}$$

$$= \int_0^a x^{p-1} dx \psi(a-x),$$

or

$$\Gamma(n)\Gamma(p)\Sigma A_m a^{m+i} \frac{\Gamma(m+1)}{\Gamma(m+i+1)} = \int_0^a x^{p-1} dx \psi(a-x).$$

Operate with $\left(\frac{d}{da}\right)^i$ on both members, and we have

$$\Gamma(n)\Gamma(p)\Sigma A_m a^m = \Gamma(n)\Gamma(p)\phi(a) = \left(\frac{d}{da}\right)^i \int_0^a x^{p-1} dx \psi(a-x)$$

$$= \left(\frac{d}{da}\right)^i a^p \int_0^1 dv (1-v)^{p-1} \psi(av),$$

by making $a-x=av$. Therefore

$$\left.\begin{aligned} &\int_0^a x^{n-1} dx \phi(a-x) = \psi(a),\ \phi(a) = \frac{1}{\Gamma(n)\Gamma(p)} \\ &\left(\frac{d}{da}\right)^i a^p \int_0^1 dv (1-v)^{p-1} \psi(av). \end{aligned}\right\} \quad . \quad . \quad . \quad (1.)$$

Next, let $\phi(x) = \Sigma \frac{A_m}{x^m}$, a descending series. Then

$$\phi(a+x) = \Sigma \frac{A_m}{(a+x)^m},$$

and $$\int_0^\infty x^{n-1} dx \phi(a+x) = \Sigma A_m \int_0^\infty \frac{x^{n-1} dx}{(a+x)^m}$$

$$= \Sigma \frac{A_m}{a^{m-n}} \int_0^\infty \frac{x^{n-1} dx}{(1+x)^m} = \Gamma(n)\Sigma \frac{A_m}{a^{m-n}} \frac{\Gamma(m-n)}{\Gamma(m)} = \psi(a)$$

suppose. Therefore

$$\Gamma(n)\Sigma \frac{A_m}{(a+x)^{m-n}} \frac{\Gamma(m-n)}{\Gamma(m)} = \psi(a+x);$$

and

$$\Gamma(n)\Sigma A_m \frac{\Gamma(m-n)}{\Gamma(m)} \int_0^\infty \frac{x^{p-1} dx}{(a+x)^{m-n}} = \int_0^\infty x^{p-1} dx \psi(a+x);$$

or

$$\Gamma(n)\Gamma(p)\Sigma \frac{A_m}{a^{m-i}} \frac{\Gamma(m-i)}{\Gamma(m)} = \int_0^\infty x^{p-1} dx \psi(a+x).$$

Operate with $\left(\frac{d}{da}\right)^i$ on both members; then

$$(-1)^i\Gamma(n)\Gamma(p)\Sigma\frac{A_m}{a^m}=\left(\frac{d}{da}\right)^i\int_0^\infty x^{p-1}dx\psi(a+x);$$

or

$$\left.\begin{aligned}\phi(a)=\frac{(-1)^i}{\Gamma(n)\Gamma(p)}\left(\frac{d}{da}\right)^i\int_0^\infty x^{p-1}dx\psi(a+x),\\ \int_0^\infty x^{n-1}dx\phi(a+x)=\psi(a).\end{aligned}\right\}\quad . \quad . \quad (2.)$$

We may put $\phi(a)$ under a different form by making $a+x=\frac{a}{v}$. The forms of $\phi(a)$ obtained in (1.) and (2.) differ from those given by Mr. Boole in the Cambridge Mathematical Journal, No. 20; but by varying the process a little, we might obtain his results. We may observe that the least value of $m$ in (1.) must be greater than $(-1)$, and in (2.) greater than $n+p$ or $i$.

In $\phi(x)=\varepsilon^{xD}\phi(0)$, which is Taylor's theorem (D standing for $\frac{d}{do}$), change $\phi(x)$ into $\phi(\varepsilon^x)$, and then $x$ into $\log x$; we have

$$\phi(x)+x^D\phi(\varepsilon^0).\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a.)$$

Therefore, also,

$$\phi(a-x)=(a-x)^D\phi(\varepsilon^0),$$

and

$$\int_0^a x^{n-1}dx\phi(a-x)=\left\{\int_0^a x^{n-1}dx(a-x)^D\right\}\phi(\varepsilon^0)$$

$$=\frac{\Gamma(n)\Gamma(D+1)}{\Gamma(D+n+1)}a^{D+n}\phi(\varepsilon^0)=\psi(a).$$

Consequently

$$\frac{\Gamma(n)\Gamma(D+1)}{\Gamma(D+n+1)}(a-x)^{D+n}\phi(\varepsilon^0)=\psi(a-x),$$

and

$$\frac{\Gamma(n)\Gamma(D+1)}{\Gamma(D+n+1)}\left\{\int_0^a x^{p-1}dx(a-x)^{D+n}\right\}\phi(\varepsilon^0)$$

$$=\int_0^a x^{p-1}dx\psi(a-x);$$

or

$$\Gamma(n)\Gamma(p)\frac{\Gamma(D+1)}{\Gamma(D+i+1)}a^{D+i}\phi(\varepsilon^0)=\int_0^a x^{p-1}dx\psi(a-x).$$

Operating with $\left(\frac{d}{da}\right)^i$ on both members, we find

$$\Gamma(n)\Gamma(p)a^{\mathrm{D}}\phi(\varepsilon^{0})=\Gamma(n)\Gamma(p)\phi(a)=\left(\frac{d}{da}\right)^{i}\int_{0}^{a}x^{p-1}dx\psi(a-x),$$

the same result as in (1.).

In (*a*.) change $\phi\ (x)$ into $\phi\left(\frac{1}{x}\right)$, and then $x$ into $\frac{1}{x}$; we have

$$\phi(x)=x^{-\mathrm{D}}\phi(\varepsilon^{-0})\,; \quad \ldots\ldots \quad (b.)$$

and therefore

$$\phi(a+x)=(a+x)^{-\mathrm{D}}\phi(\varepsilon^{-0}),$$

and

$$\int_{0}^{\infty}x^{n-1}dx\phi(a+x)=\left\{\int_{0}^{\infty}x^{n-1}dx(a+x)^{-\mathrm{D}}\right\}\phi(\varepsilon^{-0})$$

$$=\frac{\Gamma(n)\Gamma(\mathrm{D}-n)}{\Gamma(\mathrm{D})}a^{n-\mathrm{D}}\phi\ (\varepsilon^{-0})=\psi(a).$$

Hence

$$\frac{\Gamma(n)\Gamma(\mathrm{D}-n)}{\Gamma(\mathrm{D})}(a+x)^{n-\mathrm{D}}\phi(\varepsilon^{-0})=\psi(a+x),$$

and

$$\frac{\Gamma(n)\Gamma(\mathrm{D}-n)}{\Gamma(\mathrm{D})}\left\{\int_{0}^{\infty}x^{p-1}dx(a+x)^{n-\mathrm{D}}\right\}\phi(\varepsilon^{-0}).$$

$$=\int_{0}^{\infty}x^{p-1}dx\psi(a+x)\,;$$

or

$$\Gamma(n)\Gamma(p)\frac{\Gamma(\mathrm{D}-i)}{\Gamma(\mathrm{D})}a^{i-\mathrm{D}}\phi(\varepsilon^{-0})=\int_{0}^{\infty}x^{p-1}dx\psi(a+x).$$

And, as before,

$$(-1)^{i}\Gamma(n)\Gamma(p)a^{-\mathrm{D}}\phi(\varepsilon^{-0})=(-1)^{i}\Gamma(n)\Gamma(p)\phi(a)$$

$$=\left(\frac{d}{da}\right)^{i}\int_{0}^{\infty}x^{p-1}dx\psi(a+x),$$

the same result as in (2.).

We might by this method derive the forms of $\phi\ (a)$ given by Mr. Boole; but my object is merely to show one use out of many which may be made of the formulæ (*a*.) and (*b*.)

If $\Delta r=1$, and $\mathrm{E}=1+\Delta$; $\mathrm{E}^{k}r=r+k$, $\mathrm{E}^{k}x^{r}=x^{r}x^{k}$. Giving to $k$ an infinity of different values, multiplying the results by any constants, and taking the sum, we have

$$x^{r}\phi(x)=\phi(\mathrm{E})x^{r}. \quad \ldots\ldots \quad (c.)$$

It is plain that we may give to $k$, not only integer values, but fractional ones also, and any values whatever, and negative as well as positive ones; for the operation $\mathrm{E}^{k}$ performed on $r$, or on $x^{r}$, merely changes them into $r+k$, and $x^{r+k}$ respectively. The function $\phi(x)$ is therefore very general.

Change $x$ into $a-x$, and we have

$$(a-x)^r\phi(a-x)=\phi(\mathrm{E})(a-x)^r.$$

Therefore

$$\int_0^a x^{n-1}\,dx(a-x)^r\phi(a-x)=\phi(\mathrm{E})\int_0^a x^{n-1}\,dx(a-x)^r$$

$$=\phi(\mathrm{E})a^{r+n}\int_0^1 x^{n-1}dx(1-x)^r=\Gamma(n)\phi(\mathrm{E})a^{r+n}\frac{\Gamma(r+1)}{\Gamma(r+n+1)}=\psi(a)$$

suppose. Change $a$ into $a-x$, and we have

$$\Gamma(n)\phi(\mathrm{E})(a-x)^{r+n}\frac{\Gamma(r+1)}{\Gamma(r+n+1)}=\psi(a-x),$$

and

$$\Gamma(n)\phi(\mathrm{E})\frac{\Gamma(r+1)}{\Gamma(r+n+1)}\int_0^a x^{p-1}dx(a-x)^{r+n}=\int_0^a x^{p-1}dx\psi(a-x);$$

or

$$\Gamma(n)\Gamma(p)\phi(\mathrm{E})a^{r+i}\frac{\Gamma(r+1)}{\Gamma(r+i+1)}=\int_0^a x^{p-1}dx\psi(a-x),$$

and

$$\Gamma(n)\Gamma(p)\phi(\mathrm{E})a^r=\Gamma(n)\Gamma(p)a^r\phi(a)=\left(\frac{d}{da}\right)^i\int_0^a x^{p-1}dx\psi(a-x).$$

Change $x^r\phi(x)$ into $\phi(x)$, and transform the second member; then

$$\int_0^a x^{n-1}\,dx\phi(a-x)=\psi(a),\ \ \phi(a)=\frac{1}{\Gamma(n)\Gamma(p)}\left(\frac{d}{da}\right)^i a^p\int_0^1 dv$$

$$(1-v)^{p-1}\psi(av),$$

as before.

Resuming the equation

$$\Gamma(n)\phi(\mathrm{E})(a-x)^{r+n}\frac{\Gamma(r+1)}{\Gamma(r+n+1)}=\psi(a-x),$$

we have

$$\Gamma(n)\phi(\mathrm{E})\frac{\Gamma(r+1)}{\Gamma(r+n+1)}\int_0^a x^{p-1}dx(a-x)^{r-p}=\int_0^a\frac{x^{p-1}dx}{(a-x)^i}\psi(a-x);$$

or

$$\Gamma(n)\Gamma(p)\phi(\mathrm{E})a^r\frac{\Gamma(r-p+1)}{\Gamma(r+n+1)}=\int_0^a\frac{x^{p-1}dx}{(a-x)^i}\psi(a-x)$$

$$=a^{p-i}\int_0^1 v^{-i}dv(1-v)^{p-1}\psi(av).$$

Multiply by $a^{i-p}=a^n$, and operate with $\left(\frac{d}{da}\right)^i$; there results

$$\Gamma(n)\Gamma(p)\phi(\mathrm{E})a^{r-p}=\left(\frac{d}{da}\right)^i\int_0^1 v^{-i}dv(1-v)^{p-1}\psi(av).$$

If therefore we change $x^r\phi(x)$ into $\phi(x)$, we now have

$$\left.\begin{array}{l}\displaystyle\int_0^a x^{n-1}\,dx\phi(a-x)=\psi(a),\ \ \phi(a)=\frac{a^p}{\Gamma(n)\Gamma(p)}\left(\frac{d}{da}\right)^i \\ \displaystyle\qquad\int_0^1 v^{-i}dv(1-v)^{p-1}\psi(av).\end{array}\right\}\quad .\ .\quad (3.)$$

If we put D for $\dfrac{d}{dr}$, then

$$\mathrm{D}^k\varepsilon^{rx}=x^k\varepsilon^{rx},\ \ (-\mathrm{D})^k\varepsilon^{-rx}=x^k\varepsilon^{-rx};$$

and, as in (*c.*), we have

$$\varepsilon^{rx}\phi(x)=\phi(\mathrm{D})\varepsilon^{rx},\quad \varepsilon^{-rx}\phi(x)=\phi(-\mathrm{D})\varepsilon^{-rx}\quad .\ .\quad (d.)$$

Or, if we put $\rho^k$ for the operation which converts $\varepsilon^{rx}$ into $x^k\varepsilon^{rx}$, and $\theta^k$ for that which changes $\varepsilon^{-rx}$ into $x^k\varepsilon^{-rx}$, then

$$\varepsilon^{rx}\phi(x)=\phi(\rho)\varepsilon^{rx},\quad \varepsilon^{-rx}\phi(x)=\phi(\theta)\varepsilon^{-rx};\quad .\ .\quad (e.)$$

and $k$ may be positive or negative, integer or fractional, or any quantity whatever. I believe these formulæ are new, and they admit of many uses.

Changing $x$ into $a+x$, we have

$$\varepsilon^{-r(a+x)}\phi(a+x)=\phi(\theta)\varepsilon^{-r(a+x)};$$

and

$$\int_0^\infty x^{n-1}\,dx\varepsilon^{-r(a+x)}\phi(a+x)=\phi(\theta)\varepsilon^{-ra}\int_0^\infty x^{n-1}\,dx\varepsilon^{-rx}$$

$$=\Gamma(n)\phi(\theta)\,\frac{1}{r^n}\,\varepsilon^{-ra}=\psi(a)\text{ suppose.}$$

Changing $a$ into $a+x$, this gives

$$\Gamma(n)\phi(\theta)\,\frac{1}{r^n}\,\varepsilon^{-r(a+x)}=\psi(a+x),$$

and

$$\Gamma(n)\phi(\theta)\,\frac{1}{r^n}\,\varepsilon^{-ra}\int_0^\infty x^{p-1}\,dx\varepsilon^{-rx}=\Gamma(n)\Gamma(p)\phi\,(\theta)\,\frac{1}{r^i}\,\varepsilon^{-ra}$$

$$=\int_0^\infty x^{p-1}\,dx\psi(a+x);$$

or

$$(-1)^i\Gamma(n)\Gamma(p)\phi(\theta)\varepsilon^{-ra}=(-1)^i\Gamma(n)\Gamma(p)\varepsilon^{-ra}\phi(a)$$

$$=\left(\frac{d}{da}\right)^i\int_0^\infty x^{p-1}dx\psi(a+x).$$

Change the function $\varepsilon^{-rx}\phi(x)$ into $\phi(x)$, and we have

$$\int_0^\infty x^{n-1}\,dx\phi(a+x)=\psi(a),\quad \phi(a)=\frac{(-1)^i}{\Gamma(n)\Gamma(p)}\left(\frac{d}{da}\right)^i$$

$$\int_0^\infty x^{p-1}dx\psi(a+x)$$

as heretofore.

The equation

$$\Gamma(n)\phi(\theta)\,\frac{1}{r^n}\,\varepsilon^{-ra}=\psi(a),$$

found in this investigation, gives

$$\Gamma(n)\phi(\theta)\,\frac{1}{r^n}\,\varepsilon^{-r\left(\frac{a}{v}\right)}=\psi\left(\frac{a}{v}\right),$$

and

$$\Gamma(n)\phi(\theta)\frac{1}{r^n}\int_0^1 v^{-p}\,dv(1-v)^{p-1}\varepsilon^{-r\left(\frac{a}{v}\right)}=\int_0^1 v^{-p}\,dv\,(1-v)^{p-1}\psi\left(\frac{a}{v}\right).$$

Differentiate this for $a$, then

$$-\Gamma(n)\phi(\theta)\,\frac{1}{r^{n-1}}\int_0^1 v^{-p-1}\,dv(1-v)^{p-1}\varepsilon^{-r\left(\frac{a}{v}\right)}=\frac{d}{da}\int_0^1 v^{-p}\,dv\,(1-v)^{p-1}\psi\left(\frac{a}{v}\right).$$

Make $\frac{1}{v}-1=x$, and

$$\int_0^1 v^{-p-1}\,dv(1-v)^{p-1}\varepsilon^{-r\left(\frac{a}{v}\right)}$$

will be transformed into

$$\varepsilon^{-ra}\int_0^\infty x^{p-1}\,dx\varepsilon^{-rax}=\Gamma(p)\,\frac{a^{-p}}{r^p}\,\varepsilon^{-ra};$$

and the preceding will become

$$-\Gamma(n)\Gamma(p)\phi(\theta)\frac{a^{-p}}{r^{i-1}}\,\varepsilon^{-ra}=\frac{d}{da}\int_0^1 v^{-p}\,dv(1-v)^{p-1}\psi\left(\frac{a}{v}\right).$$

Multiply by $a^p$, and operate with $\left(\frac{d}{da}\right)^{i-1}$, there results

$$(-1)^i\Gamma(n)\Gamma(p)\phi(\theta)\varepsilon^{-ra}=(-1)^i\Gamma(n)\Gamma(p)\varepsilon^{-ra}\phi(a)$$
$$=\left(\frac{d}{da}\right)^{i-1}a^p\frac{d}{da}\int_0^1 v^{-p}\,dv(1-v)^{p-1}\psi\left(\frac{a}{v}\right).$$

After changing the function $\phi$, as before, we now have

$$\left.\begin{aligned}&\int_0^\infty x^{n-1}\,dx\phi(a+x)=\psi(a),\quad \phi(a)=\frac{(-1)^i}{\Gamma(n)\Gamma(p)}\\&\left(\frac{d}{da}\right)^{i-1}a^p\frac{d}{da}\int_0^1 v^{-p}\,dv(1-v)^{p-1}\psi\left(\frac{a}{v}\right).\end{aligned}\right\}\quad .\ .\quad (4.)$$

The formulæ (3.) and (4 ) are the same as those given by Mr. Boole in the paper before referred to. From the last method of investigation, it appears that the functions $\phi$ and $\psi$ may be any whatever, consistent with the required relation between them. But if we are obliged to integrate by series, they will in general be subject to the restrictions mentioned in (1.) and (2.); I say, in general, for infinite quantities may vanish by the operations $\left(\frac{d}{da}\right)^i$.

To give an example in each of the theorems: in (1.) let

$$n=\frac{1}{2},\ p=\frac{1}{2},\ \psi(x)=\sqrt{x}.$$

We find

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2\phi(a)=\frac{d}{da}\left(\frac{\pi a}{2}\right),$$

and

$$\phi(a)=\frac{1}{2};$$

then

$$\psi(a)=\sqrt{a},$$

as it should be.

In (2.) let

$$n=\frac{1}{2},\ p=\frac{1}{2},\ \psi(x)=\frac{1}{\sqrt{x}}$$

We find

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2\phi(a)=\frac{d}{da}(\log a-\log o),\ \ \phi(a)=\frac{1}{\pi a};$$

and then

$$\psi(a)=\frac{1}{\sqrt{a}}.$$

In the last example $n$ and $p$ are not conformed to the restrictions, but the infinite quantity goes out by differentiation. The theorems (3.) and (4.) are likewise satisfied by these examples. It must not be supposed that the values of $\phi\ (a)$, given in (1.) and (3.), or in (2.) and (4.), are necessarily equal; for they will not reduce the one to the other. Yet we may have

$$\int_0^\infty x^{n-1}\,dx\phi(a+x)=\psi(a),\quad \int_0^a x^{n-1}\,dx\phi(a-x)=\psi(a)$$

in both cases; since we know from examples that the integrals of different functions may be of the same form.

Gunthwaite Hall, near Barnsley,
May 24, 1847.