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# XIV. *A Generalized Hypergeometric Function with* n *parameters.*

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THE behaviour of the series

$$1+\frac{z}{1!\prod\limits_{r=1}^{n-1}(1+\alpha_r)}+\frac{z^2}{2!\prod\limits_{r=1}^{n-1}(1+\alpha_r)(2+\alpha_r)}+\dots$$

as $|z|$ tends to infinity has been investigated in several particular cases. In the case when $n=1$ the series is easily to be derived from the Bessel function, and its behaviour on the circle $|z|=\infty$ is well known. In a recent paper by the present author, the case when $n=3$ was worked out. The asymptotic expansion of the series in the general case has gradually become apparent, and is established in this paper.

We may obtain the *form* of the asymptotic expansion of the series

$${}_0F_{n-1}(1+\alpha_1,\ 1+\alpha_2\dots 1+\alpha_{n-1};\ z)$$
$$=1+\sum_{r=1}^{\infty}\frac{z^v}{v!\prod\limits_{r=1}^{n-1}(1+\alpha_r)(2+\alpha_r)\dots(v+\alpha_r)},$$

when $|z|$ is large after the usual manner.

${}_0F_{n-1}(z)$ satisfies the differential equation

$$\vartheta(\vartheta+\alpha_1)\dots(\vartheta+\alpha_{n-1})y=yz,$$

which becomes

$$\vartheta(\vartheta+n\alpha_1)\dots(\vartheta+n\alpha_{n-1})y=yt^n,$$

if by analogy with the treatment of Bessel functions we put

$$z=(t/n)^n,$$

$t/n$ being any one of the $n$th roots of $z$. Putting $y=e^t t^x y$, the equation becomes

$$(\vartheta+t-x)(\vartheta+t+n\alpha_1-x)\dots(\vartheta+t+n\alpha_{n-1}-x)y_1=t^n y_1.$$

If $x$ be chosen in such a way that the coefficients of the leading power of $t$ on both sides are equal, the asymptotic expansion for ${}_0F_{n-1}\{(t/n)^n\}$ will be

$$\Sigma f(n)e^t t^x\left(1+\frac{\alpha_1}{t}+\frac{\alpha_2}{t^2}\dots\right),$$

* Communicated by the Author.

where $f(n)$ is independent of $t$ and the summation is taken over the various values of $t$, which make

$$(t/n)^n = z,$$

the series $y_1 = 1 + \frac{\alpha_1}{t} + \frac{\alpha_2}{t^2} \dots$ being obtained by solving in descending powers of $t$, the equation resulting from the equation

$$(\vartheta + t - x)(\vartheta + t + n\alpha_1 - x) \dots (\vartheta + t + n\alpha_{n-1} - x) y_1 = t^n y_1,$$

by the substitution of the appropriate value for $x$. It is evident that the coefficients $\alpha_1, \alpha_2 \dots$ are independent of the particular $n$th root of $z$ chosen. Clearly the relation between the coefficients $\alpha_1, \alpha_2$ . will involve more than two consecutive ones ; we shall therefore not attempt to find any but the first two.

The equation may be written in the form

$$\begin{aligned} y_1(t^n + {}_1a_1 t^{n-1} + {}_1a_2 t^{n-1} \dots + {}_1a_{n-1} t + {}_1a_n) + \vartheta(y_1)({}_2a_1 t^{n-1} \\ + {}_2a_2 t^{n-2} \dots + {}_2a_n) + \dots + \vartheta^{n-1}(y_1)({}_{n-1}a_{n-1} t + {}_{n-1}a_n) \\ + \vartheta^n(y_1) = t^n y_1, \end{aligned}$$

and the coefficients ${}_ra_s$ can be found by induction and inspection. In order to find only the first few coefficients in the series of descending powers of $v$, only a selection of the ${}_ra_s$ need be determined.

Equating the coefficients of $t^{n-1}$, $t^{n-2}$, $t^{n-3}$, on either side to each other, we get the equations

$${}_1a_1 = 0,$$

$${}_1a_1 \alpha_1 + {}_1a_2 - {}_2a_1 \alpha_1 = 0,$$

$${}_1a_1 \alpha_2 + {}_1a_2 \alpha_1 + {}_1a_3 - ({}_2a_2 \alpha_1 + 2{}_2a_1 \alpha_2) + {}_3a_1 \alpha_1 = 0.$$

These equations determine $x$, $\alpha_1$ and $\alpha_2$. As to the value of the ${}_ra_s$ coefficients involved, it is evident that

$${}_2a_1 = n, \quad {}_3a_1 = \frac{n(n-1)}{2}, \dots$$

The value of ${}_1a_1$ is $\frac{n(n-1)}{2} = \sum_1$, where $\sum_r$ represents the sum of the terms

$$-x, \quad n\alpha_1 - x, \dots n\alpha_{n-1} - x,$$

taken $r$ at a time. Since ${}_1a_1 = 0$ this gives

$$-x = \sum_{s=1}^{n-1} \alpha_s + \frac{n-1}{2}.$$

The values of the other relevant ${}_r a_s$ coefficients found by induction are as follows:

$$ {}_1a_2 = \sum_2 + \frac{(n-1)(n-2)}{2}\sum_1 + \frac{n(n-1)(n-2)(3n-5)}{24}, $$

$$ {}_1a_3 = \sum_3 + \frac{(n-2)(n-3)}{2}\sum_2 + \frac{(n-1)(n-2)(n-3)(3n-8)}{24}\sum_1 + \sum_{s=1}^{n}\frac{(s-1)(s-2)(s-3)^2(3s-8)}{24}, $$

$$ {}_2a_2 = \frac{n(n-1)(n-2)}{2} + (n-1)\sum_1. $$

$\alpha_1$ and $\alpha_2$ are then found to be

$$ {}_1a_2/{}_2a_1, \quad \left[{}_1a_2{}^2 + ({}_3a_1 - {}_2a_2)\,{}_1a_2 + {}_2a_1 \cdot {}_1a_3\right]/2\,{}_2a_1{}^2, $$

respectively. Subsequent coefficients $\alpha_3, \alpha_4 \ldots$ can be found in the same way, but they will be increasingly cumbrous. They are not, however, of the same practical importance, as $\alpha_1$ and $\alpha_2$ will, in general, alone be required in applications.

The form of the asymptotic expansion of ${}_0F_{n-1}(z)$ is then

$$ |z|^{-\frac{s_n}{n}} \sum_r f_r(n) e^{n\,{}_nz_r}\left(1 + \frac{\alpha_1}{n\,{}_nz_r} + \frac{\alpha_2}{n^2\,{}_nz_r^2} + \ldots\right), $$

where

$$ s_n = \sum_{s=1}^{n-1}\alpha_s + \frac{n-1}{2} $$

and ${}_nz_r$ represents an $n$th root of $z$, the summation being taken over the $n$th roots of $z$.

Now, in the case of a value of ${}_nz_r$ which has a negative real part, whatever the value of $f_r(n)$ the corresponding part of the asymptotic expansion, namely

$$ |z|^{-\frac{s_n}{n}} f_r(n) e^{n\,{}_nz_r}\left(1 + \sum_{v=1}^{\infty}\frac{\alpha_v}{n_v({}_nz_r)^v}\right), $$

is asymptotically equivalent to zero. Hence, although it is true that

$$ {}_0F_{n-1}(z) \sim |z|^{-\frac{s_n}{n}} \sum f_r(n) e^{n\,{}_nz_r}\left(1 + \sum_{v=1}^{\infty}\frac{\alpha_v}{n^v({}_nz_r)^v}\right) \qquad (1) $$

$f_r(n)$ corresponding to a value of ${}_nz_r$ with a negative real part, can take any value whatever, and there is therefore no sense in which any one set of values of $f_r(n)$ corresponding to such values of ${}_nz_r$ gives the asymptotic behaviour of ${}_0F_{n-1}(z)$ when $|z|$ is large rather than any other. It would therefore be misleading to leave (1) as it stands, where certain functions of $n$ are substituted as the values of $f_r(n)$; for the asymptotic equivalence persists, whatever value may be given to those of the set of $f_r(n)$ functions corresponding to $n$th roots of $z$ with negative real parts. There is, indeed, no sense in which one can talk, for example in the case of $n$ even, of a "sudden jump"* in the value of *the* function of $n$ multiplying the exponential term $e^{-n|z|^{1/n}}$ as arg $(z)$ changes from being positive to being negative, since the product of $e^{-n|z|^{1/n}}$ and any function of $n$ whatever is asymptotically equivalent to zero, whatever the value of arg $(z)$. We therefore proceed to find the *unique* set of functions of $n$, whose existence is already plain, which makes the asymptotic expansion of ${}_0F_{n-1}(z)$ equal to

$$|z|^{-s_n/n} \sum_{R({}_nz_r) \geqslant 0} f_r(n)\, e^{n\,{}_nz_r}\left(1+\sum_{v=1}^{\infty} \frac{\alpha_v}{n^v({}_nz_r)^v}\right)$$

when $|z|$ is large, $s_n$ being $\sum_{s=1}^{n-1}\alpha_s + \frac{n-1}{2}$.

If C is a closed contour containing the origin

$${}_0F_n(1+\alpha_1,\ 1+\alpha_2, \dots 1+\alpha_{n-1}\ ;\ z)$$
$$= \frac{1}{2\pi i}\int_C {}_0F_{n-1}(1+\alpha_1 \dots 1+\alpha_{n-1}\ ;\ t)\left[1+\frac{z/t}{1+\alpha_n} + \frac{(z/t)^2}{1+\alpha_n . 2+\alpha_n}\dots\right]\frac{dt}{t} \quad . . (2)$$

Further, if the contour is so chosen that $|t|$ and $|z/t|$ tend to infinity as $|z|$ tends to infinity (and it is plain that such a choice is possible), it will be possible to obtain the leading terms of the asymptotic expansion of ${}_0F_n$ by approximating to the value, as $|z|$ tends to infinity, of the integral in which the leading terms of the asymptotic expansion of ${}_0F_{n-1}(t)$ and of

$$1+\frac{z/t}{1+\alpha_n}+\frac{(z/t)^2}{1+\alpha_n . 2+\alpha_n}+\dots$$

have been put in the place of these series.

* *Cp.* E. W. Barnes, Proc. Camb. Phil. Soc. 1906.

Now it has been established * that

$$1+\frac{u}{1+\alpha}+\frac{u^2}{1+\alpha\,.\,2+\alpha}+\ldots\sim e^u u^{-\alpha}\Gamma(1+\alpha)-\alpha/u-\frac{\alpha(\alpha-1)}{u^2}\ldots$$

$$=\left[e^u u^{-\alpha}-\frac{1}{\Gamma\alpha\,.\,u}-\frac{1}{\Gamma(\alpha-1)\,.\,u^2}-\frac{1}{\Gamma\alpha-2\,.\,u^3}\ldots\right]\Gamma(1+\alpha)$$

if $|\arg u|<\pi$, $u^{-\alpha}$ being interpreted as $e^{-\alpha\log u}$, $\log u$ having its imaginary part between $\pm i\pi$.

It remains to choose the correct hypothesis for the asymptotic expansion of ${}_0F_{n-1}(z)$ and to prove its correctness by induction.

The relevant data are

(1) If $|z|$ is large and $|\arg z|<\pi$,

$$J_n(z)\sim\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left[\cos\left(z-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi\right)\left[1+\frac{\beta_2}{z^2}+\ldots\right]\right.$$

$$\left.-\sin\left(z-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi\right)\left[\frac{\beta_1}{z}-\frac{\beta_3}{z^3}\ldots\right]\right]$$

$$=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left[e^{i\left(z-\frac{1}{2}s_2\right)}\left[1+\frac{\alpha_1}{iz}+\frac{\alpha_2}{i^2z^2}+\ldots\right]\right.$$

$$\left.+e^{-i(z-\frac{1}{2}s_2)}\left[1-\frac{\alpha_1}{iz}+\frac{\alpha_2}{i^2z^2}\ldots\right]\right],$$

where $s_2=n+\frac{1}{2}$.

(2) If $|z|$ is large,

${}_0F_3(1+\alpha_1,\ 1+\alpha_2,\ 1+\alpha_3\ ;\ z)$

$$\sim\frac{\Gamma 1+\alpha_1\,\Gamma 1+\alpha_2\,\Gamma 1+\alpha_3}{2^{5/2}\pi^{3/2}}z^{-\frac{1}{4}(\alpha_1+\alpha_2+\alpha_3+3/2)}\left[e^{4z^{\frac{1}{4}}}\right.$$

$$\left.+e^{+4iz^{\frac{1}{4}}}e^{-\frac{\pi i}{2}(\alpha_1+\alpha_2+\alpha_3+3/2)}+e^{-4iz^{\frac{1}{4}}}e^{+\frac{\pi i}{2}(\alpha_1+\alpha_2+\alpha_3+3/2)}\right]$$

$$=\frac{\Gamma 1+\alpha_1\,\Gamma 1+\alpha_2\,\Gamma 1+\alpha_3}{(2\pi)^{3/2}\sqrt{4}}z^{-\frac{s_4}{4}}\left[e^{4z^{\frac{1}{4}}}+e^{4iz^{\frac{1}{4}}}e^{-\frac{\pi i s_4}{2}}\right.$$

$$\left.+e^{-4iz^{\frac{1}{4}}}e^{+\frac{\pi i s_4}{2}}\right]$$

* Barnes, *loc. cit.*

Both of these results fall under the hypothesis that

$$
{}_0F_{n-1}(1+\alpha_1,\, 1+\alpha_2, \dots 1+\alpha_{n-1};\; z)
$$
$$
\sim \frac{\Pi\Gamma(1+\alpha_s)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}} \sum_{\mathrm{R}({}_nz_r)\geq 0} e^{n\,{}_nz_r}({}_nz_r)^{-s_n}\left[1+\sum_{v=1}^{\infty}\frac{\alpha_v}{n^v({}_nz_r)^v}\right],
$$

where $({}_nz_r)^{-s_n}$ means the complex number whose argument is $-s_n \times \arg({}_nz_r)$, and the summation is taken over those values of ${}_nz_r$, the $n$th root of $z$ whose real parts are not negative.

We may now proceed to discuss the value as $|z| \to \infty$ of the integral in which the integrand of the integral (1) has been replaced by the leading terms of the asymptotic expansions of the series.

We have

$$
2\pi i\, {}_0F_n(z) \sim \frac{\prod_{1}^{n}\Gamma(1+\alpha_s)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}} \int_C \sum_r e^{n\,{}_nt_r}({}_nt_r)^{-s_n}
$$
$$
\times\left(e^{z/t}(z/t)^{-\alpha_n} - \sum_{v=1}^{\infty}\frac{t^v}{z^v\Gamma(\alpha_n-1+v)}\right)\frac{dt}{t}.
$$

The only limitations on C, which is a contour in the $t$-plane, are, that it shall enclose the origin and be such, that on it $|t|$ and $|z/t|$ are large when $|z|$ is large. We may therefore choose as the contour a circle with its centre at the origin and with radius $\delta$, provided $\delta$ and $|z|/\delta$ are large when $|z|$ is large. Then

$$
{}_0F_n(z) \sim \frac{\prod_{1}^{n}\Gamma(1+\alpha_s)}{(2\pi)^{\frac{n+1}{2}}\sqrt{n}} \int_{-\pi}^{\pi} d\theta \sum_s e^{n\delta^{1/n} e^{i\frac{\theta+2s\pi}{n}}}\,\delta^{-\frac{s_n}{n}}\, e^{-i\frac{\theta+2s\pi}{n}s_n}
$$
$$
\times\left(e^{i(\phi-\theta)}\left(\frac{|z|}{\delta}\right)^{-\alpha_n} e^{-i\alpha_n(\phi-\theta)} - \sum_{v=1}^{\infty}\frac{\delta^v e^{iv\theta}}{z^v\Gamma(\alpha_n-1+v)}\right),
$$

the $s$-summation being taken over integer values of $s$, such that

$$
-\frac{\pi}{2} \leq \frac{\theta+2s\pi}{n} \leq \frac{\pi}{2},
$$

$\phi-\theta$ being the argument of $z/t$ (requiring sometimes a different value of $\phi$ in different parts of the range), which lies between $-\pi$ and $+\pi$. [The special case when

$\phi-\theta=+\pi$ or $-\pi$ will not arise.] Let $I_1$ and $I_2$ represent

$$\int_{-\pi}^{\pi}\sum_s e^{n\delta^{1/n}e^{i\frac{\theta+2s\pi}{n}}}\delta^{-\frac{s_n}{n}}e^{-i\frac{\theta+2s\pi}{n}s_n}\times e^{\frac{\delta}{|z|}e^{i(\phi-\theta)}}\left(\frac{|z|}{\delta}\right)^{-\alpha_n}e^{-i\alpha_n(\phi-\theta)}\,d\theta$$

$$\int_{-\pi}^{\pi}\sum_s e^{n\delta^{1/n}e^{i\frac{\theta+2s\pi}{n}}}\delta^{-\frac{s_n}{n}}e^{-i\frac{\theta+2s\pi}{n}s_n}\times\sum_{v=1}^{\infty}\frac{\delta^v e^{iv\theta}}{z^v\Gamma(\alpha_n-1+v)}\,d\theta$$

respectively. $I_1$ is made up of a set of integrals of an oscillatory exponential type and we may approximate to their values by Kelvin's method. "Critical points" of the integrands of these integrals occur where the derivative of the exponent is zero, or

$$\frac{d}{d\theta}\left(n\delta^{1/n}e^{i\frac{\theta+2t\pi}{n}}+\frac{\delta}{|z|}e^{i(\phi-\theta)}\right)=0,$$

and are easily seen to exist when, and only when, $\delta=|z|^{n/n+1}$. Taking this value of $\delta$, we satisfy the condition that on the contour C, $|t|$ and $|z/t|$ are large when $|z|$ is large.

The integrals to be evaluated for $I_1$ are then of the form

$$_tI_1=\int_{-\pi}^{+\pi}e^{|z|^{1/n+1}\left[ne^{i\frac{\theta+2t\pi}{n}}+e^{i(\phi-\theta)}\right]}\times|z|^{-\frac{s_n+\alpha_n}{n+1}}e^{-i\left(\frac{\theta+2t\pi}{n}s_n+(\phi-\theta)\alpha_n\right)}\,d\theta.$$

"Critical points" are those values of $\theta$ for which

$$\frac{\theta+2t\pi}{n}-(\phi-\theta)=2\nu\pi$$

or $$\frac{\theta+2t\pi}{n}-2\nu\pi=\phi-\theta=\frac{\phi+2t\pi}{n+1}-2\nu\pi\cdot\frac{n}{n+1}. \quad (3)$$

Suppose $\theta=\phi-\zeta$ is a critical point of the integrand of $_tI_1$, that is to say $\phi-\zeta$ satisfies (3) and is less than or equal to $\pi$ in absolute value. Then

$$_tI_1\sim\sum_\zeta \operatorname*{lt}_{\epsilon\to 0}|z|^{-\frac{s_n+\alpha_n}{n+1}}e^{-i\zeta(\alpha_n+s_n)}\int_{\phi-\zeta+\pi}^{\phi-\zeta-\pi}e^{|z|^{1/n+1}e^{i\zeta}\left(n+1-\frac{n+1}{2n}\epsilon^2\cdots\right)}\,d\epsilon$$

$$\sim\sum_{-\frac{\pi}{2}\leq\zeta\leq+\frac{\pi}{2}}|z|^{-\frac{s_n+\alpha_n+\frac{1}{2}}{n+1}}e^{-i\zeta(\alpha_n+s_n+\frac{1}{2})}e^{(n+1)|z|^{1/n+1}e^{i\zeta}}\times\sqrt{\frac{2n\pi}{n+1}},$$

since as $x$ tends to infinity

$$\int_0^x e^{-z^2}dz\sim\tfrac{1}{2}\sqrt{\pi},\ \text{if}\ -\frac{\pi}{2}<\arg x<\frac{\pi}{2}.$$

From the equations (3) it is plain that every value of $\zeta$ is such that

$$|z|^{1/n+1}e^{i\zeta}$$

is an $(n+1)$th root of $z$, and the summation is taken over those values of $\zeta$ alone which are less than or equal to $\pi/2$ in absolute value. Thus the asymptotic equivalent for ${}_t\mathrm{I}_1$ is the sum of certain terms of the form

$$\sqrt{\frac{2n\pi}{n+1}}\left({}_{n+1}z_r\right)^{-(s_n+\alpha_n+\frac{1}{2})}e^{n+1}{}_{n+1}{}^{z_r},$$

or since

$$s_{n+1}=s_n+\alpha_n+\tfrac{1}{2}$$

of the terms of the form

$$\sqrt{\frac{2n}{n+1}}\left({}_{n+1}z_r\right)^{-s_{n+1}}e^{n+1}{}_{n+1}{}^{z_r}.$$

The question remains as to whether, when all the integrals of the term ${}_t\mathrm{I}_1$ are taken into account, any of the $(n+1)$th roots of $z$ with a non-negative real part are excluded (it is of no importance whether those with negative real parts are excluded or not) and as to how many times terms corresponding to each of these roots occur. A slight consideration of the equations (3) shows that the first possibility is cut out. As to the second point, it is clear that only values of $\zeta$ less than or equal to $\pi/2$ in absolute value need be considered. In this restricted class of cases, it is plain that the same value of $\zeta$ cannot occur in the case of integrals corresponding to two different values of $t$. Hence

$$\frac{\prod_1^n\Gamma(1+\alpha_s)}{(2\pi)^{\frac{n+1}{2}}\sqrt{n}}\mathrm{I}_1\sim\frac{\prod_1^n\Gamma(1+\alpha_s)}{(2\pi)^{\frac{n}{2}}\sqrt{n+1}}\Sigma\left({}_{n+1}z_r\right)^{-s_{n+1}}e^{(n+1)}{}_{n+1}{}^{z_r}$$

the summation being taken over values of ${}_{n+1}z_r$ satisfying the conditions

$$\left({}_{n+1}z_r\right)^{n+1}=z$$

$$-\frac{\pi}{2}\leq\arg{}_{n+1}z_r\leq\frac{\pi}{2},$$

$\left({}_{n+1}z_r\right)^{-s_{n+1}}$ being interpreted to mean the complex number whose argument is

$$-s_{n+1}\times\arg{}_{n+1}z_r.$$

The integral $\mathrm{I}_2$ has no "critical points" in the Kelvin sense, and therefore by considerations which are sufficiently well known in applications of this method, the important "groups of errors" in the divergent development of the functions all arise from $I_1$. We shall, however, show independently that $I_2$ is negligible compared with $I_1$.

In considering $I_2$ we again choose a circle with centre $t=0$ and radius $\delta$ as the contour, but are free to choose $\delta$ as we please provided only that $\delta$ and $|z|/\delta$ are large when $|z|$ is large. Then

$$\left|\int_c e^{n t_r} {}_n t_r^{-s_n} \times \frac{t^r}{z^r} dt\right|,$$

$$\leq \frac{1}{|z^r|}\int_c e^{n\delta^{1/n}}\left|\delta^{-s_n/n} e^{\frac{2\pi s}{n} i s_n}\right|\delta^r\, dt,$$

$$\leq \frac{2\pi}{|z|^r} e^{n\delta^{1/n}} \delta^{-\frac{R(s_n)}{n}} e^{-\frac{2\pi s}{n} i s_n} \delta^r.$$

But let $\delta$ be so chosen that

$$\delta = |z|^{n/2(n+1)}.$$

Then

$$\left|\int_c e^{n^{t_r}} t_r^{-s_n} \times \frac{t^r}{z^r} dt\right| \leq K e^{n|z|^{\frac{1}{2(n+1)}}} |z|^{\frac{nr}{2(n+1)} - \frac{R(s_n)}{2(n+1)} - r}$$

Whatever the value of $r$, this is negligible compared with any of the terms

$$e^{{}_{n+1}z_r}\left({}_{n+1}z_r\right)^{-s_{n+1}}$$

in which $R(z_r) > 0$. Hence $I_2$ is negligible compared with $I_1$ and the asymptotic expansion for ${}_0F_n$ results in the form

$${}_0F_n(1+\alpha_1, \ldots 1+\alpha_n\,;\, z) \sim \frac{\prod_{s=1}^{n}\Gamma(1+\alpha_s)}{(2\pi)^n \sqrt{n+1}}$$

$$\sum_{|\arg({}_{n+1}z_r)| \leq \frac{\pi}{2}} e^{n+1_{n+1}z_r}\left({}_{n+1}z_r\right)^{-s_{n+1}}.$$

This result is of a similar form to the one assumed for ${}_0F_{n-1}$ and already proved for ${}_0F_3$, and is therefore now established in the general case. The complete form is then

$$ {}_0F_n(1+\alpha_1, \ldots 1+\alpha_n\ ;\ z) \sim \frac{\prod\limits_{s=1}^{n} \Gamma(1+\alpha_s)}{(2\pi)^n \sqrt{n+1}} $$

$$ \times \sum_{\left|\arg {}_{n+1}z_r\right| \leq \frac{\pi}{2}} e^{(n+1)_{n+1}z_r} ({}_{n+1}z_r)^{-s_{n+1}} \left(1+\sum_{v=1}^{\infty} \frac{\alpha_v}{(n+1)^v ({}_{n+1}z_r)^v}\right). $$

It is worth while to notice the forms which result for ${}_0F_{n-1}(z)$ and for ${}_0F_{n-1}(-z)$ when $z$ is real and positive. If $x$ is real, positive, and large, and $s_n = \sum\limits_1^{n-1} \alpha_r + \frac{n-1}{2}$

$$ 1+\frac{x}{1!\prod\limits_1^{n-1}(1+\alpha_r)} + \frac{x^2}{2!\prod\limits_1^{n-1}(1+\alpha_r)(2+\alpha_r)} + \ldots $$

$$ \sim \frac{\prod\limits_1^{n-1} \Gamma(1+\alpha_r)}{(2\pi)^{\frac{n-1}{2}} \sqrt{n}} x^{-s_n/n} \left[ e^{nx^{1/n}} \left[1+\Sigma \frac{\alpha_v}{n^v x^{v/n}}\right]\right. $$

$$ \left. +2\sum_{t=1}^{2t/n \leq \pi/2} e^{nx^{1/n} \cos \frac{2\pi t}{n}} \frac{\alpha_v}{n^v x^{v/n}} \cos\left[nx^{1/n} \sin \frac{2\pi t}{n} - \frac{2\pi t}{n}(s_n+v)\right]\right] $$

As a first approximation we may record the result

$$ 1+\frac{x}{1!\prod\limits_1^{n-1}(1+\alpha_r)} + \ldots \sim \frac{\prod\limits_1^{n-1} \Gamma(1+\alpha_r)}{(2\pi)^{\frac{n-1}{2}} \sqrt{n}} x^{-s_n/n} $$

$$ \times \left\{ e^{nx^{1/n}} + 2 \sum_{t=1}^{\frac{2t}{n} \leq \frac{1}{2}} e^{nx^{1/n} \cos \frac{2\pi t}{n}} \cos\left(nx^{1/n} \sin \frac{2\pi t}{n} - \frac{2\pi t s_n}{n}\right)\right\}, $$

or

$$ 1+\frac{(u/n)^n}{1!\prod\limits_1^{n-1}(1+\alpha_r)} + \ldots \sim \frac{\prod\limits_1^{n-1} \Gamma(1+\alpha_r)}{(2\pi)^{\frac{n-1}{2}} \sqrt{n}} n^{-s_n} $$

$$ \times \left\{ e^u + 2 \sum_{t=1}^{\frac{2t}{n} \leq \frac{1}{2}} e^{u \cos \frac{2\pi t}{n}} \cos\left(u \sin \frac{2\pi t}{n} - \frac{2\pi t s_n}{2}\right)\right\}, $$

giving the familiar result

$$ I_\alpha(u) \sim \frac{e_u}{\sqrt{2\pi u}}, $$

and useful results in other cases such as

$$1+\frac{(u/r)^r}{1!\prod_1^{r-1}1+\alpha_s}+\ldots\sim\frac{\prod_1^{r-1}\Gamma 1+\alpha_s}{(2\pi)^{\frac{r-1}{2}}\sqrt{r}}(u/r)^{-s_r}$$

$$\times\left[e^u+2e^{u\cos\frac{2\pi}{r}}\cos\left(u\sin\frac{2\pi}{r}-\frac{2\pi s_r}{r}\right)\right],$$

when $r$ is 4, 5, 6, or 7, and in the case of $r=8$

$$1+\frac{(u/8)^8}{1!\prod_1^7 1+\alpha_s}+\ldots\sim\frac{\prod_1^7\Gamma 1+\alpha_s}{(2\pi)^{7/2}2\sqrt{2}}(u/8)^{-s_8}$$

$$\times\left[e^u+2e^{u\cos\frac{\pi}{4}}\cos\left(u\sin\frac{\pi}{4}-\frac{\pi s_8}{4}\right)+2\cos\left(u-\frac{\pi s_8}{4}\right)\right],$$

and so on.

Similar results are available in the case when $z$ is negative and real. If $x$ is large, real, and positive

$$1\frac{x}{1!\prod_1^{n-1}1+\alpha_r}+\ldots\sim\frac{\prod_1^{n-1}\Gamma(1+\alpha_r)}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}x^{-s_n/n}$$

$$\times\sum_{t=1}^{\frac{2t-1}{n}\leq\frac{1}{2}}e^{nx^{1/n}\cos\frac{(2t-1)\pi}{n}}\left[\cos nx^{1/n}\sin\frac{2t-1}{n}\pi-\frac{2t-1}{n}\pi s_n\right.$$

$$\left.+\sum_{v=1}^{\infty}\frac{\alpha_v}{n^v x^{v/n}}\cos\left(nx^{1/n}\sin\frac{2t-1}{n}\pi-\frac{2t-1}{n}\pi(s_n+v)\right)\right]$$

and again, as a first approximation, we may put

$$1-\frac{x}{1!\prod 1+\alpha_r}+\ldots\sim 2\frac{\prod_1^{n-1}\Gamma 1+\alpha_r}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}x^{-\frac{s_n}{n}}\sum_{t=1}^{\frac{2t-1}{n}\leq\frac{1}{2}}e^{nx^{1/n}\cos\frac{(2t-1)\pi}{n}}$$

$$\times\cos\left(nx^{1/n}\sin\frac{2t-1}{n}\pi-\frac{2t-1}{n}\pi s_n\right),$$

or

$$1-\frac{(u/n)^n}{1!\prod 1+\alpha_r}\ \ldots\sim 2\frac{\prod_1^{n-1}\Gamma 1+\alpha_r}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}}(u/n)^{-s_n}\sum_{t=1}^{\frac{2t-1}{n}\leq\frac{1}{2}}e^{u\cos\frac{(2t-1)\pi}{n}}$$

$$\times\cos\left(u\sin\frac{2t-1}{n}\pi-\frac{2t-1}{n}\pi s_n\right),$$

which gives the corresponding result in the case when $n=2$ to the one already pointed out for $I_n(v)$, viz.

$$J_n(v) \sim \sqrt{\frac{2}{\pi v}}\cos\left(v-\frac{\alpha}{2}\pi-\tfrac{1}{4}\pi\right),$$

and useful results such as

$$1-\frac{(v/r)^r}{1\,!\prod\limits_1^{r-1} 1+\alpha_s}\cdots \sim \frac{\Pi\Gamma 1+\alpha_s}{(2\pi)^{\frac{r-1}{2}}\sqrt{r}}(v/r)^{-s_r}e^{u\cos\frac{2t-1\,.\,\pi}{n}}$$
$$\times\cos\left(u\sin\frac{\pi}{r}-\frac{\pi s_n}{r}\right),$$

for $r=1, 2, 3, 4, 5$ ; and for $r=6$,

$$1-\frac{(u/6)^6}{1\,!\prod\limits_1^5 1+\alpha_s}\cdots$$

$$\sim\frac{\Pi\,\Gamma 1+\alpha_s}{(2\pi)^{5/2}\sqrt{6}}(u/6)^{-s_6}\Big[e^{u\cos\pi/6}\cos\left(u\sin\pi/6-\pi/6(\Sigma\alpha+5/2)\right)$$
$$+\cos\left(u-\pi/6(\Sigma\alpha+5/2)\right)\Big];$$

and so on.

The other $(n-1)$ solutions of the equation

$$\vartheta(\vartheta+\alpha_1)(\vartheta+\alpha_2)\ldots(\vartheta+\alpha_{n-1})y=zy,$$

satisfied by

$$y={}_0F_{n-1}(1+\alpha_1,\ 1+\alpha_2,\ldots 1+\alpha_{n-1}\ ;\ z),$$

are the series

$$z^{-\alpha_r}{}_0F_{n-1}(1+\alpha_1-\alpha_r,\ldots 1+\alpha_{r-1}-\alpha_r,\ 1-\alpha_r,$$
$$1+\alpha_{r+1}-\alpha_r,\ldots 1+\alpha_{n-1}-\alpha_r\ ;\ z)$$

with $r=1, 2,\ldots n-1$. Calling these solutions $y_1, y_2,\ldots y_{n-1}$, since the sum of the parameters for $y_r$ is $\Sigma\alpha-n\alpha_r$, it is plain that the asymptotic expansion of $y_r$ is of the form

$$\Sigma f_r'(n)z^{-\alpha_r}\,.\,z^{-\frac{\Sigma\alpha-n\alpha_r+\frac{n-1}{2}}{n}}e^{n(z)^{1/n}}$$

or

$$\Sigma f_r'(n)z^{-\left(\Sigma\alpha+\frac{n-1}{2}\right)/n}e^{nz^{1/n}},$$

so that we have

$$y = \Sigma f_r(n) z^{-s_n/n} e^{nz^{1/n}},$$

$$y_1 = \Sigma\, {}_1f_r(n) z^{-s_n/n} e^{nz^{1/n}},$$

$$y_2 = \Sigma\, {}_2f_r(n) z^{-s_n/n} e^{nz^{1/n}},$$

$$y_{n-1} = \Sigma\, {}_{n-1}f_r(n) z^{-s_n/n} e^{nz^{1/n}},$$

where $s_n = \Sigma\alpha + \dfrac{n-1}{2}$.

Linear combinations of $y, y_1, \ldots y_{n-1}$, therefore exist which behave, when $|z|$ is large, like $z^{-s_n/n} e^{nz^{1/n}}$, or like

$$z^{-s_n/n} e^{nz^{1/n} \cos \frac{2\pi t}{n}} \cos\left(nz^{1/n} \sin \frac{2\pi t}{n} - \frac{2\pi t s_n}{n}\right).$$

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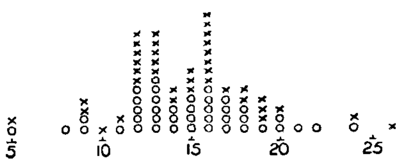
## XV. *A Statistical Survey of the Colour Vision of* 1000 *Students. By* R. A. Houstoun, *D.Sc., Lecturer on Physical Optics, and* Margaret A. Dunlop, *Thomson Experimental Scholar in the University of Glasgow* *.

UNDER the title of "A Statistical Survey of Colour Vision," there appeared a paper by Dr. R. A. Houstoun in the Roy. Soc. Proc. A, vol. xciv. p. 576 (1918). This paper described the results of a test made on the colour vision of 79 observers by means of an apparatus similar to the Edridge-Green colour perception spectrometer. Each observer determined the number of monochromatic patches he saw in a continuous spectrum; this number varied from

Fig. 1.




five in the case of the colour-blind to twenty-six in the case of an observer with exceptionally good colour vision. The results were exhibited in the diagram (fig. 1) where the

* Communicated by the Authors.