PROOFS OF THE ADDITION AND SUBTRACTION FORMULAS BY MEANS OF PTOLEMY'S THEOREM.

By Albert Babbitt,
University of Nebraska, Lincoln.

Ptolemy's theorem states that "*the rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides.*"[1] By means of this theorem and the Law of Sines, the addition and subtraction formulas of trigonometry can be easily derived.

I. $\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$.

Proof. Draw the diameter AB, construct at the point A

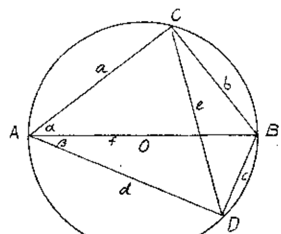

FIG. 1

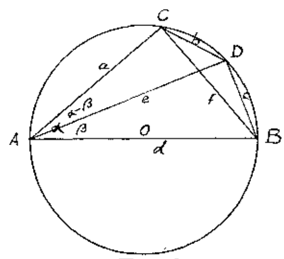

FIG. 2

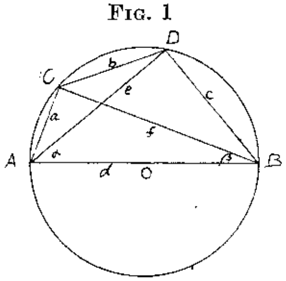

FIG. 3

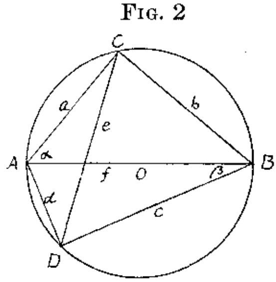

FIG. 4

$\angle$s α and β and complete the construction as shown in Fig. I.

From Ptolemy's theorem it follows that

$$ef = ac+bd.$$

Dividing through by f^2, we get

$$\frac{e}{f} = \frac{ac}{f^2}+\frac{bd}{f^2}\text{, or}$$

$$\frac{e}{f} = \frac{a}{f}\cdot\frac{c}{f}+\frac{b}{f}\cdot\frac{d}{f}. \qquad (1)$$

[1] Cajori, *A History of Mathematics* (1909), p. 57.
"It is believed that the elegant theorem generally known as Ptolemy's Theorem is due to Hipparchus and was copied from him by Ptolemy." Ball, *A Short History of Mathematics* (1909), p. 88.

From the Law of Sines[2] it follows that

$$\frac{e}{f} = \sin(\alpha+\beta).$$

Also from the definitions of the trigonometric functions,

$$\frac{a}{f} = \cos\alpha.$$

$$\frac{c}{f} = \sin\beta.$$

$$\frac{b}{f} = \sin\alpha.$$

$$\frac{d}{f} = \cos\beta.$$

Hence, by substitution in (1) we get

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta+\sin\alpha\cos\beta.$$

II. $\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta.$

Proof. Perform the constructions shown in Fig. 2.

Now,

$$ef = ac+bd \text{ (Ptolemy's Theorem).}$$

Dividing through by d^2, we get

$$\frac{e}{d}\,\frac{f}{d} = \frac{a}{d}\,\frac{c}{d}+\frac{b}{d}. \qquad (1)$$

It is easily seen that

$$\frac{e}{d} = \cos\beta. \qquad \frac{c}{d} = \sin\beta, \text{ and}$$

$$\frac{f}{d} = \sin\alpha. \qquad \frac{b}{d} = \sin(\alpha-\beta)$$

$$\frac{a}{d} = \cos\alpha.$$

Substituting into (1) and transposing we have

[2] $a/\sin\alpha = b/\sin\beta = c/\sin\gamma = 2r$, or $a/2r = \sin\alpha$; $b/2r = \sin\beta$; $c/2r = \sin\gamma$.

$\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta.$

III. $\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta.$

Proof. Construct the angles α and β at the points A and B respectively and complete the construction as shown in Fig. 3.

Now,

$ef = ac+bd$ (Ptolemy's Theorem),

$$\frac{e}{d}\cdot\frac{f}{d} = \frac{a}{d}\cdot\frac{c}{d}+\frac{b}{d},$$

$$\cos\alpha\cos\beta = \sin\beta\sin\alpha+\sin \text{CBD}. \qquad (1)$$

But, $\sin\angle\text{CBD} = \sin[90^\circ-(\alpha+\beta)] = \cos(\alpha+\beta)$

Hence, $\cos\alpha\cos\beta = \sin\beta\sin\alpha+\cos(\alpha+\beta)$, or

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta.$$

IV. $\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta.$

Proof. Perform the constructions as shown in Fig. 4.

Now, $ef = ac+bd$. (Ptolemy's Theorem),

$$\frac{e}{f} = \frac{a}{f}\,\frac{c}{f}+\frac{b}{f}\,\frac{d}{f}. \qquad (1)$$

But, $\frac{e}{f} = \sin\angle\text{CBD} = \sin(\text{CBA}+\beta)$ (Law of Sines)

and, $\angle\text{CBA} = 90^\circ-\alpha.$

Hence, $\frac{e}{f} = \sin(90^\circ-\alpha+\beta) = \cos(\alpha-\beta)$

Since, $\frac{a}{f} = \cos\alpha,\quad \frac{c}{f} = \cos\beta,$

$\frac{b}{f} = \sin\alpha,\quad \frac{d}{f} = \sin\beta,$ we have

from (1)

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

NOTE: Carnot showed how the whole of elementary plane trigonometry could be deduced from the addition and subtraction formulas. (Ball, *A Short History of Mathematics*, p. 88.)