A NEW GYROSCOPIC CONSTRUCTION IN SIMPLE VECTORS.

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Synopsis.

Simple Vector Representation of the Steady Precession of a Gyroscope.—By means of vector angular momentum diagrams, the quadratic for steady precession is derived graphically, and the variation in the steady precession at constant angle with change of applied couple is represented so as to be easily visualized. The couple required to prevent precession due to the earth's rotation is also given in a vector diagram.

IN all cases of the free precession of a rotating body, the looped or oscillatory path of the axis is an essential part of the motion. Yet, whenever the angular momentum of spin is high enough, these oscillations become vanishingly small. As most of the cases of interest to the average student are of this type, *i.e.*, the approximately steady precession, the following geometrical constructions are of interest.

Following the usual plan, resolve the precessional velocity, $\dot{\psi}$, into (1) a compoment $\dot{\psi}\cos\theta$ about OC (Fig. 1, *a* and *b*) and (2) $\dot{\psi}\sin\theta$ about OE. Take C, A, A, as the moments of inertia of the top about the axes OC, OE, OD, and let ω be the angular velocity of spin about OC. Thus the components of angular momentum are $C(\omega + \dot{\psi}\cos\theta)$, or Cn, about OC and $A\dot{\psi}\sin\theta$ about OE. Represent these at a given instant by the vector diagram of Fig. 2 in which $OS = Cn$, and $SQ = A\dot{\psi}\sin\theta$. Thus

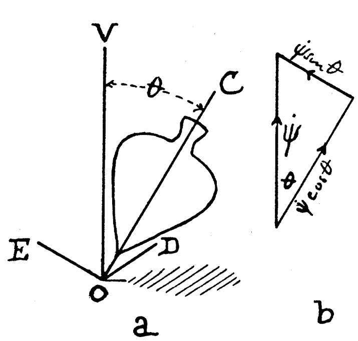


Fig. 1.

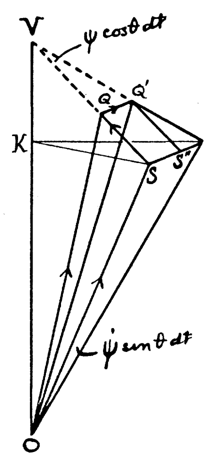


Fig. 2.

OQ is the resultant angular momentum at the instant when OS is parallel to the axis of the top. Now, instead of attacking this plane figure, $OVQS$, algebraically as in the well-known construction of Sir George Greenhill,[1] let us represent on the same diagram (in three dimensions) the position of these vectors after the lapse of a small time dt.[2] The plane defined by OSV will have rotated through the small angle $\dot{\psi} dt$ (given by SKS' on the figure, and as the final angular momentum OQ' is equal to the initial value OQ plus the vector QQ', the latter must be the angular impulse $(Mgh \sin \theta)dt$, received by the system during the interval. Here M is the mass, and h the distance from the pivot to the center of mass of the top. Thus we have:

$$\begin{aligned} QQ' &= Mgh \sin \theta dt, \\ OS &= OS' = Cn, \\ SQ &= S'Q' = A\dot{\psi} \sin \theta, \\ \angle\, SKS' &= \dot{\psi} dt, \\ \angle\, SVS' &= \dot{\psi} dt \cos \theta, \\ \angle\, SOS' &= \dot{\psi} dt \sin \theta. \end{aligned}$$

Also if $Q'S''$ be drawn parallel to QS,

$$\begin{aligned} SS' = SS'' + S''S' &= QQ' + Q'S' \sin S'Q'S''. \\ &= (Mgh \sin \theta)dt + (A\dot{\psi} \sin \theta)(\dot{\psi} \cos \theta dt). \end{aligned}$$

But $SS' = OS \angle SOS' = Cn\dot{\psi} \sin \theta dt.$

Whence, equating the two values of SS', and cancelling the common factors we find

$$(Cn - A\dot{\psi} \cos \theta)\dot{\psi} = Mgh,$$

the well-known quadratic for steady precession, usually obtained in less simple ways.

Consider now the relations involved in the elementary triangles VQQ' and VSS' remembering that the angles at V are all infinitesimal; and make a new diagram (Fig. 3, *a* and *b*) in which the plane of these figures is the plane of the paper, the same letters being used for the same points

[1] Greenhill, Notes on Dynamics. Greenhill, Report on Gyroscopic Theory, Chap. I., §§ 3 et sqq. Gray and Gray, Treatise on Dynamics, § 271.

[2] Some have condemned this use of the element of time dt, apparently on the ground that, as the motion is strictly oscillatory, the value of θ must, in general, change during this time. Perhaps these writers would consent to the employment of the symbol Δt with the proviso that it denote an integral number of the periods of the motion. This would still be a small time as the period is very short in the cases to which the above considerations are applicable. The vectors representing the change in the angular momentum still give the integral impulse over the time taken. This surely is an unnecessary refinement, for when the precession is approximately steady the period is very small and the amplitude negligible.

as in Fig. 2. Take A as the mid point of VS, and denote the distance AQ by x and VS by L

$$QQ' = VQ \angle QVQ' = (VA + AQ) \angle QVQ'$$

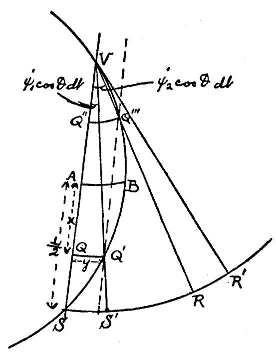


Fig. 3, *a*.

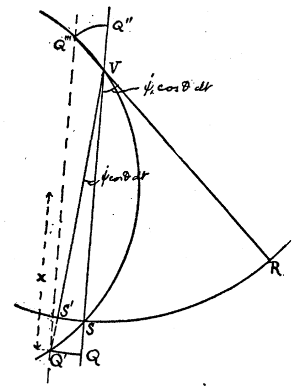


Fig. 3, *b*.

$$= \left(\frac{L}{2} + x\right)(\dot{\psi} \cos \theta dt)$$

$$= (Mgh \sin \theta)dt, \text{ as before.}$$

Hence

$$\left(\frac{L}{2} + x\right)(\dot{\psi} \cos \theta) = Mgh \sin \theta \tag{1}$$

Also

$$QS = \left(\frac{L}{2} - x\right) = A\dot{\psi} \sin \theta \tag{2}$$

From these, by multiplication,

$$\frac{L^2}{4} - x^2 = AMgh \tan \theta \sin \theta \tag{3}$$

$$= A \tan \theta \frac{QQ'}{dt}.$$

Hence

$$\frac{QQ'}{dt} = y = \frac{\frac{L^2}{4} - x^2.}{A \tan \theta}$$

Or, substituting (see Fig. 2) $L = Cn \tan \theta$,

$$\frac{C^2n^2 \tan \theta}{4A} - y = \frac{x^2}{A \tan \theta}.$$

In view of the fact that, due to the short time dt, the arcs QQ', etc., may be treated as short straight lines perpendicular to VS, this equation exhibits the locus of Q', for constant θ, as a parabola passing through the points V and S, with its vertex opposite A, ($x = 0$) at a distance ($C^2n^2 \tan\theta/4A$). Thus for a given value of the applied couple ($Mgh \sin\theta$, represented by the distance of the dotted line from VS) we find two solutions corresponding to the two points common to the curve and the dotted line. These give the two possible steady precessions at angle θ for the same applied couple; $\dot{\psi}_1$ proportional to the angle SVS', and $\dot{\psi}_2$ proportional to the angle SVR, as either of these angles is swept out in the same small time dt.

Further it is obvious, from the nature of the curve, that an increase in the applied couple brings both Q' and Q''' towards the vertex B, increasing the lower value of the precession, *i.e.*, increasing the angle SVS', and decreasing the value of the higher precession, *i.e.*, decreasing the angle SVR. With still greater increase in the applied couple these points (Q', Q''') finally meet at the vertex B for which condition, by putting x equal to zero in equation 1 we find the limiting value

$$Mgh \sin\theta_{x=0} = \frac{L}{2}\dot{\psi}\cos\theta\,,$$

or inserting $L = VS = Cn\tan\theta$. (See Fig. 1.)

$$= \frac{Cn}{2}\dot{\psi}\tan\theta,\ \cos\theta.$$

Or from equation 3 in the same way

$$Mgh \sin\theta_{x=0} = \frac{C^2n^2\tan\theta}{4A}.$$

For values of the applied couple greater than this there is obviously no real solution and therefore no possible value for *steady* precession.

Now, following the changes in the opposite direction (of decreasing values of the applied couple) we are led, for zero couple, to zero precession for the lower value (Q' coinciding with S) and for the upper value (Q''' coinciding with V) to a precession given by

$$\dot{\psi}_V = \frac{\angle SVR'}{\cos\theta dt}$$

(VR' being tangent to the curve at V) which is the free precession of Euler or the adynamic precession of Kelvin.

If the applied couple now becomes negative (Fig. 3b) we deal with points on the curve below S and above V. The lower precession changes sign, having passed through zero, while the upper one continues to increase, having passed through the adynamic value.

From these characteristics of the change of the precession with change in the applied couple, the rise or fall of the axis of the top with a forced change in the magnitude of the free precession, follows in the usual way.

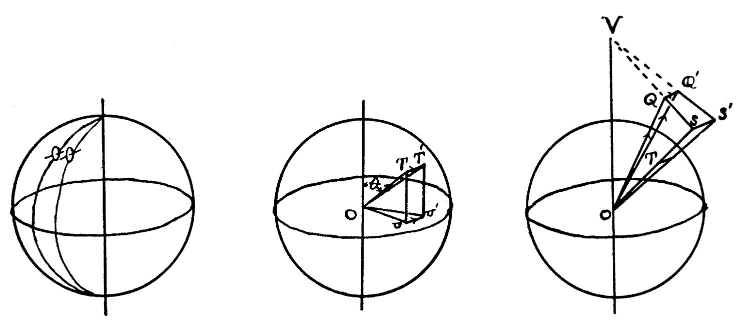
There is another way of indicating these changes, which while it gives a less vivid picture of the relations between them, yet gives exact values in place of the merely relative values of the other method. One notes from equation 2 that SQ and SQ'' represent the two values of $A\dot{\psi}\sin\theta$. Thus

$$\dot{\psi}_1 = \frac{SQ}{A\sin\theta}$$

and

$$\dot{\psi}_2 = \frac{SQ''}{A\sin\theta},$$

so the values of the two precessions are given by the distances of Q and Q'' from S, whether these be positive or negative (*i.e.*, whether these points fall between S and V or are external to that segment. Further we see that since $x = AQ = AQ''$ that the sum of $SQ + SQ''$ is always SV, *i.e.*, the sum of the two velocities for steady precession is always that of the free or adynamic precession, $Cn/(A\cos\theta)$.



a c b

Fig. 4.

In the case of a rotating body in precession along a predetermined path, as, for instance, a flywheel with axis fixed with regard to a rotating earth, the general construction is the same. We should have a position diagram (Fig. 4, *a*) showing the positions of the rotating body at an interval of *dt*. Then the vector diagram (Fig. 4, *b*) where all points are lettered to correspond to those of Fig. 1, and where θ is the angle between the axes of the earth and of the flywheel. QQ' is the angular impulse required to change the resultant angular momentum OQ to the value

(OQ') it has dt later and the calculation follows as before. However as the angular velocity of the earth about its axis, $\dot{\psi}_e$ is so very small with respect to ω in the cases of chief interest (effort of flywheel to precess, couple on gyro-compass, etc.) the vectors TS and SQ, *i.e.*, $C\dot{\psi}_e \cos\theta$ and $A\dot{\psi}_e \sin\theta$, are ignored and the figure becomes that of Fig. 4 *c*, where $OT = OT' = C\omega$ and TT' (or UU') is the impulse $(C\omega \sin\theta\dot{\psi}_e)dt$. Hence the couple required to prevent precession towards the pole of the earth is $C\omega\dot{\psi}_e \sin\theta$; the result usually obtained by more complex analysis. The couple on a gyro-compass is simply the vertical component of this couple at the place in question.

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