ON THE LINEAR INTEGRAL EQUATION

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IN most of the literature in which the linear integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is the subject of consideration, somewhat narrow restrictions are placed upon the nature of the given function $f(s)$ and of the nucleus $K(s, t)$. The nucleus $K(s, t)$ is usually taken to be continuous except at points lying on a finite number of curves with continuously turning tangents, and such that any straight line parallel to either of the axes of s or t intersects these curves in a finite number of points only. The function $f(s)$ is usually taken to be continuous, and the continuous solution of the equation is then obtained by Fredholm's method. In the present communication a wider range is given to the functions $K(s, t)$, $f(s)$. The definite integrals employed are supposed throughout to be defined in accordance with the definition of Lebesgue, and the function $f(s)$ is restricted only to be a summable function. It is shown to be sufficient, in order that the solution of the equation may have only the same discontinuities as $f(s)$, that $K(s, t)$ be restricted in respect of its discontinuities in a much less stringent manner than that referred to above. It is shown that the only summable solution of the integral equation is that given by Fredholm's formula, the nucleus being limited in the square for which it is defined. A large part of the communication is concerned with the cases in which the nucleus is unlimited. Fredholm himself considered the case in which the nucleus and a finite number of the repeated nuclei are unlimited, the remaining repeated nuclei being limited. This case has been treated more fully by Poincaré and others, but is here treated with greater generality than hitherto, at least so far as I am aware. The theory of the canonical sub-groups of the resolvant of a limited nucleus, in the form given by Lalesco, has been here applied to the complete investigation of the case of an unlimited nucleus just referred to. As this case is the one which actually arises in the application of the theory of integral equations to Dirichlet's problem and other problems in the theory of the potential function, it appears to be desirable that the extension of Fredholm's method to such cases should be fully investigated. Certain cases in which all the repeated nuclei are unlimited, but in which Fredholm's method is still applicable, have been here considered.

The Continuity of Integrals with respect to a Parameter.

1. With a view to a discussion of the nature of the solutions of linear integral equations the following theorems will prove useful.

(a) *If $K(s, t)$ is limited and summable in the square defined by $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, and if for any fixed value s_0 of s the set of values of t for which $K(s, t)$ is not continuous with respect to s, for $s = s_0$, forms a set of linear measure zero, then $\int_a^b K(s, \xi)\,\phi(\xi)\,d\xi$ is a continuous function of s at $s = s_0$; where $\phi(t)$ is any summable function, whether limited or unlimited in the interval (a, b) of t.*

Those values of s for which $K(s, \xi)$ is not summable with respect to ξ, if such exist, form a set of linear measure zero. Such values may in the theorem and in the proof be simply disregarded. The theorem holds good whenever the integral exists.

Let M denote the upper limit of $|K(s, t)|$ in the square for which it is defined; and let E_N denote that set of points ξ for which

$$|\phi(\xi)| \geqslant N,$$

where N is some positive number. Let $s_1, s_2, \ldots, s_n, \ldots$ be a sequence of values of s which converges to the limit s_0.

The integral of $[K(s_n, \xi) - K(s_0, \xi)]\,\phi(\xi)$ with respect to ξ, taken over the set of points E_N, does not exceed in absolute value

$$2M \int_{(E_N)} |\phi(\xi)|\,d\xi;$$

and this is less than the arbitrarily chosen positive number $\frac{1}{2}\epsilon$, provided N is chosen so great, and therefore the measure of E_N so small, that

$$\int_{(E_N)} |\phi(\xi)|\,d\xi < \frac{\epsilon}{4M}.$$

To estimate the integral of $[K(s_n, \xi) - K(s_0, \xi)]\,\phi(\xi)$ taken over the set of points $C(E_N)$ complementary to E_N, let $u_n(\xi)$ denote $K(s_n, \xi)\,\phi(\xi)$, and let

$u(\xi)$ denote $K(s_0, \xi)\,\phi(\xi)$. Then $|u_n(\xi)|$ is less than some fixed positive number for all values of n and for all values of ξ that belong to the set $C(E_N)$; hence, in accordance with a known theorem, we have

$$\lim_{n\sim\infty}\int_{C(E_N)} u_n(\xi)\,d\xi = \int_{C(E_N)} u(\xi)\,d\xi,$$

since $u_n(\xi)$ converges to $u(\xi)$ for all values of ξ belonging to $C(E_N)$, with the possible exception of those belonging to a set of which the linear measure is zero. It follows that the absolute value of the integral of $[K(s_n, \xi)-K(s_0, \xi)]\,\phi(\xi)$ taken over the set $C(E_N)$ of values of ξ is less than $\frac{1}{2}\epsilon$, provided n is greater than some fixed integer m.

The numbers N, m being chosen as has been explained, we now see that

$$\left|\int_a^b [K(s_n, \xi)-K(s_0, \xi)]\,\phi(\xi)\,d\xi\right| < \epsilon,$$

provided $n > m$. Since ϵ is arbitrary, we see that

$$\lim_{n\sim\infty}\int_a^b K(s_n, \xi)\,\phi(\xi)\,d\xi = \int_a^b K(s_0, \xi)\,\phi(\xi)\,d\xi.$$

As this holds however the sequence of values of s which converges to s_0 is chosen, it has now been proved that

$$\int_a^b K(s, \xi)\,\phi(\xi)\,d\xi$$

is a function of s that is continuous at $s = s_0$.

2. In the theorem (*a*) we may replace the variable s by a pair of variables (s, t), and no essential change will be needed in the proof of the theorem. The sequence which converges to s_0 will be replaced by sequences $s_1, s_2, \ldots, s_n, \ldots$ converging to s_0, and $t_1, t_2, \ldots, t_n, \ldots$ converging to t_0; $u_n(\xi)$ will now denote $K(s_n, t_n, \xi)$, and $u(\xi)$ will denote $K(s, t, \xi)$. We have now the theorem :—

(*b*) *If $K(s, t, \xi)$ is limited and summable in the domain defined by $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, $a \leqslant \xi \leqslant b$, and if for the values s_0, t_0 of s, t the values of ξ such that $K(s, t, \xi)$ is not continuous with respect to the plane domain (s, t), at (s_0, t_0), form at most a set of which the linear measure is zero, then $\int_a^b K(s, t, \xi)\,\phi(\xi)\,d\xi$ is continuous with respect to (s, t) at the point (s_0, t_0); where $\phi(\xi)$ is any summable function, limited or unlimited in the interval (a, b) of ξ.*

In case $K(s, t, \xi)$ is a function of the form $f(s, \xi)\,g(t, \xi)$, it is sufficient, in order that $K(s, t, \xi)$ may be continuous with respect to (s, t), that $f(s, \xi)$ be continuous with respect to s, and $g(t, \xi)$ be continuous with respect to t. Accordingly we have the theorem :*—

(c) *If $f(s, \xi)$, $g(t, \xi)$ are limited and summable in the two-dimensional domains $a \leqslant s \leqslant b$, $a \leqslant \xi \leqslant b$ and $a \leqslant t \leqslant b$, $a \leqslant \xi \leqslant b$, and if $f(s, \xi)$ is continuous at $s = s_0$ with respect to s for all values of ξ with the possible exception of those belonging to a set of measure zero, and $g(t, \xi)$ is continuous at $t = t_0$ with respect to t for all values of ξ, with a similar exception, then $\int_a^b f(s, \xi)\, g(t, \xi)\, \phi(\xi)\, d\xi$ is continuous with respect to (s, t) at the point (s_0, t_0); where $\phi(\xi)$ is any summable function, whether it be limited or not in the interval (a, b).*

3. The particular case of the theorem (c) which arises when

$$f(s, \xi) = K(s, \xi), \qquad g(t, \xi) = K(\xi, t)$$

will be useful later. It follows from the theorem that

$$\int_a^b K(s, \xi)\, K(\xi, t)\, \phi(\xi)\, d\xi,$$

or, in particular,
$$\int_a^b K(s, \xi)\, K(\xi, t)\, d\xi,$$

is continuous with respect to (s, t) at a point (s_0, t_0), if $K(s, t)$ is continuous with respect to s at $s = s_0$ for all values of t which do not belong to some set of linear measure zero, and if it is also continuous with respect to t at $t = t_0$ for all values of s that do not belong to some set of linear measure zero.

If $K(s, t)$ is a function defined for $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, such that, for each value of s, it is continuous with respect to s for every value of t with the possible exception of a set of linear measure zero (dependent in general on the particular value of s), and also such that a similar condition holds for each value of t as regards continuity with respect to t, the discontinuities of $K(s, t)$ will be said to be *regularly distributed*. We now have, by employing the theorems (a), (c) above :—

(d) *If $K(s, t)$ is limited and summable in the domain $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, and if it have its discontinuities regularly distributed, the*

* The particular case of these theorems (b) and (c) which arises when the function $\phi(\xi)$ is taken to be a constant has been given by Prof. W. H. Young, in an article "On Parametric Integration," see the *Monatshefte für Mathematik und Physik* for 1910, p. 141.

integral $\int_a^b K(s, \xi)\, K(\xi, t)\, d\xi$, *or, more generally,* $\int_a^b K(s, \xi)\, K(\xi, t)\, \phi(\xi)\, d\xi$, *where* $\phi(\xi)$ *is a limited or unlimited summable function, is continuous with respect to the two-dimensional domain* (s, t) *throughout the whole plane domain. Moreover,* $\int_a^b K(s, \xi)\, \phi(\xi)\, d\xi$ *is a continuous function of* s *for all the values of* s, *and* $\int_a^b K(\xi, t)\, \phi(\xi)\, d\xi$ *is a continuous function of* t *for all the values of* t.

As in the case of the former theorems, those values of s for which $K(s, \xi)$ is not summable with respect to ξ, and those of t for which $K(\xi, t)$ is not summable with respect to ξ, if such exist, may, as they form sets of linear measure zero, be disregarded. The theorems hold whenever the integrals exist.

By some writers* the discontinuities of $K(s, t)$ are said to be regularly distributed in the square for which the function is defined if all the discontinuities with respect to (s, t) lie on a finite number of curves with continuously turning tangents, no one of which is met by a line parallel to the axis of s or of t in more than a finite number of points. The theorem (d) shews that the more general definition given above introduces a sufficient restriction on the function for the purposes of the theory of integral equations. It will be observed that the term "regularly distributed," as here used, refers only to the discontinuities of the function with respect to the variables s and t separately, whereas the narrower definition hitherto employed has reference to the discontinuities with respect to (s, t).

In case the discontinuities of $K(s, t)$ with respect to (s, t) form a set of linear measure zero on every straight line parallel to either axis, the above integrals exist without any exception.

The Method of Successive Substitution.

4. The well known method of successive substitution, when applied to the equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt,$$

* See, for example, Bôcher's Tract, *An Introduction to the Study of Integral Equations*, p. 3.

gives us

$$\phi(s) = f(s)+\lambda \int_a^b [K(s, t)+\lambda K_2(s, t)+\ldots+\lambda^{n-1} K_n(s, t)] f(t)\, dt$$

$$+\lambda^{n+1} \int_a^b \int_a^b \ldots K(s, \xi_1) K(\xi_1, \xi_2) \ldots K(\xi_n, t)\, \phi(t)\, dt\, d\xi_1 d\xi_2 \ldots d\xi_n,$$

where $K_2(s, t)$, $K_3(s, t)$, ... denote the successive repeated nuclei corresponding to $K(s, t)$; provided this expression has a definite meaning.

We shall in the first instance suppose that $K(s, t)$ is limited and summable in the square for which it is defined, and we shall suppose that $f(s)$ is summable in the interval (a, b), whether it be limited in that interval or not.

In accordance with a known theorem,* since $K(s, t)$ is summable in the square $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, it is summable with respect to s for each value of t, with the possible exception of a set of values of linear measure zero; and a similar statement holds good as regards summability with respect to t. The function $K(s, t)$ being limited, it follows that $\int_a^b K(s, t) f(t)\, dt$ exists for each value of s, with the possible exception of a linear set of measure zero. The function

$$K_2(s, t) \equiv \int_a^b K(s, \xi) K(\xi, t)\, d\xi,$$

exists for all values of s and t with the possible exception of those belonging to sets of linear measure zero. Moreover $K_2(s, t)$ is a continuous function with respect to (s, t), in case the discontinuities of $K(s, t)$ are regularly distributed. It is easily seen that similar statements apply to $K_3(s, t)$, $K_4(s, t)$, ...

The series $\qquad K(s, t)+\lambda K_2(s, t)+\lambda^2 K_3(s, t)+\ldots$

converges uniformly and absolutely for all values of s, t, λ, such that the terms of the series have a definite meaning, and such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1| < \frac{1}{M(b-a)};$$

where M denotes the upper limit of $|K(s, t)|$ in the square for which it is defined. We denote the sum of the series by $-\bar{K}(s, t)$.

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* See my paper on "Some Fundamental Properties of Lebesgue Integrals," in the *Proceedings*, Ser. 2, Vol. 8 (1909), p. 25.

It follows that, whether $f(t)$ is limited or not, the series

$$\int_a^b K(s, t) f(t)\, dt + \lambda \int_a^b K_2(s, t) f(t)\, dt + \lambda^2 \int_a^b K_3(s, t) f(t)\, dt + \dots$$

converges to $$-\int_a^b \bar{K}(s, t) f(t)\, dt,$$

uniformly for all values of s and λ for which the terms of the series have a definite meaning, such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1|.$$

It will be shewn that the integral equation is satisfied by

$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t) f(t)\, dt,$$

for $$0 \leqslant |\lambda| \leqslant |\lambda_1|,$$

which determines $\phi(s)$ for every value of s, with the possible exception of a set of linear measure zero.

Since

$$\phi(t) = f(t) + \lambda \int_a^b K(t, t') f(t')\, dt' + \lambda^2 \int_a^b K_2(t, t') f(t')\, dt' + \dots,$$

we may, on account of the uniform convergence of the series with respect to (t, λ), apply term by term integration with respect to t, after multiplying both sides by the limited function $K(s, t)$. We have thus

$$\begin{aligned} \int_a^b K(s, t)\, \phi(t)\, dt &= \int_a^b K(s, t) f(t)\, dt + \lambda \int_a^b K_2(s, t) f(t)\, dt \\ &\qquad + \lambda^2 \int_a^b K_3(s, t) f(t)\, dt + \dots \\ &= \frac{\phi(s) - f(s)}{\lambda}; \end{aligned}$$

remembering that the order of repeated integration in a Lebesgue integral is immaterial.

It has thus been shewn that the value of $\phi(s)$ assumed above satisfies the integral equation. The following theorem has now been established:—

(e) *If the nucleus $K(s, t)$ of the integral equation*

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

be limited and summable in the square for which it is defined, then if $f(s)$ *be any summable function, limited or unlimited, the equation is satisfied for every value of* λ, *such that*

$$0 \leqslant |\lambda| \leqslant \frac{1}{M(b-a)},$$

by
$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t) f(t)\, dt,$$

where $-\bar{K}(s, t)$ *is the sum-function of the series*

$$K(s, t) + \lambda K_2(s, t) + \lambda^2 K_3(s, t) + \dots .$$

The value of $\phi(s)$ *is determinate for every value of* s, *with the possible exception of those of a set of linear measure zero. In case* $K(s, t)$ *has its discontinuities regularly distributed,* $\bar{K}(s, t)$ *has the same points of discontinuity with respect to* (s, t), *and* $\int_a^b \bar{K}(s, t) f(t)\, dt$ *is a continuous function; thus in this case* $\phi(s)$, $f(s)$ *have the same points of discontinuity, and if* $f(s)$ *is continuous, so also is* $\phi(s)$.

5. There is one case, in which the nucleus and all the repeated nuclei are unlimited, to which the method may be readily extended. Let us suppose that $K(s, t)$ is of the form $\mu(s)\,\nu(t)\,P(s, t)$, where $P(s, t)$ is a limited summable function, and $\mu(s)$, $\nu(t)$ are unlimited, but such that $\mu(s)\,\nu(s)$ is a summable function in the interval (a, b) of s.

Let
$$\int_a^b |\mu(s)\,\nu(s)|\, ds = A,$$

and let the upper limit of $|P(s, t)|$ be B. We have

$$K_2(s, t) = \mu(s)\,\nu(t) \int_a^b \nu(\xi)\, P(s, \xi)\, \mu(\xi)\, P(\xi, t)\, d\xi$$
$$= \mu(s)\,\nu(t)\, P_2(s, t),$$

where
$$|P_2(s, t)| \leqslant AB^2.$$

Similarly
$$K_3(s, t) = \int_a^b K_2(s, \xi)\, K(\xi, t)\, d\xi$$
$$= \mu(s)\,\nu(t) \int_a^b \nu(\xi)\, P_2(s, \xi)\, \mu(\xi)\, P(\xi, t)\, d\xi$$
$$= \mu(s)\,\nu(t)\, P_3(s, t),$$

where $$|P_3(s, t)| \leqslant A^2B^3.$$

In general $$K_n(s, t) = \mu(s)\,\nu(t)\,P_n(s, t),$$

where $$|P_n(s, t)| \leqslant A^{n-1}B^n.$$

We have $$-\bar{K}(s, t) = \mu(s)\,\nu(t)\,\{P(s, t)+\lambda P_2(s, t)+\lambda^2 P_3(s, t)+\dots\},$$

where the series converges uniformly for all the values of s, t, λ, such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1| < 1/AB.$$

Let it be assumed that $f(s)$, whether it be limited or not, is such that $f(s)\,\nu(s)$ is a summable function. We see then that

$$-\int_a^b \bar{K}(s, t) f(t)\,dt$$
$$= \mu(s)\left\{\int_a^b f(t)\,\nu(t)\,P(s, t)\,dt+\lambda\int_a^b f(t)\,\nu(t)\,P_2(s, t)\,dt\right.$$
$$\left.+\lambda^2\int_a^b f(t)\,\nu(t)\,P_3(s, t)\,dt+\dots\right\},$$

where the series in the bracket on the right-hand side converges uniformly for all values of s and λ, such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1| < 1/AB.$$

The integral equation is satisfied by

$$\phi(s) = f(s)-\lambda\int_a^b \bar{K}(s, t)\,f(t)\,dt$$
$$= f(s)+\lambda\int_a^b K(s, t)\,f(t)\,dt+\lambda^2\int_a^b K_2(s, t)\,f(t)\,dt+\dots,$$

provided $$0 \leqslant |\lambda| < 1/AB;$$

for we have

$$\int_a^b \phi(t)\,K(s, t)\,dt$$
$$= \mu(s)\left\{\int_a^b \nu(t)\,f(t)\,P(s, t)\,dt+\lambda\int_a^b \nu(t)\,f(t)\,P_2(s, t)\,dt\right.$$
$$\left.+\lambda^2\int_a^b \nu(t)\,f(t)\,P_3(s, t)\,dt+\dots\right\},$$

where the series on the right-hand side converges uniformly for all values of s and of λ, such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1| < 1/AB.$$

The expression is equivalent to $\dfrac{\phi(s)-f(s)}{\lambda}$, and therefore the integral equation is satisfied by the value of $\phi(s)$ assumed.

We have thus:—

(*f*) *If the nucleus* $K(s, t)$ *of the integral equation*

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is of the form $\mu(s)\, \nu(t)\, P(s, t)$, *where* $P(s, t)$ *is a limited summable function, and one or both of the functions* $\mu(s)$, $\nu(t)$ *are unlimited but such that* $\mu(s)\, \nu(s)$ *is summable in the interval* (a, b); *then the equation is satisfied by*

$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t)\, f(t)\, dt,$$

for all sufficiently small values of $|\lambda|$, *where* $-\bar{K}(s, t)$ *denotes the sum of the series* $K(s, t)+\lambda K_2(s, t)+\ldots$; *provided* $f(s)$ *is a summable function, and such that* $f(s)\, \nu(s)$ *is also summable in the interval* (a, b) *of* s.

As an example of the application of this theorem we may take

$$K(s, t) = \frac{P(s, t)}{|s-\alpha|^p\, |t-\beta|^q},$$

where p and q are both less than 1, and α, β are both in the interval (a, b), but are unequal. In this case $f(s)\, |s-\beta|^{-q}$ must be summable.

Again, we may take

$$K(s, t) = \frac{P(s, t)}{|s-\alpha|^p\, |t-\alpha|^q},$$

where $p+q<1$, and α is in the interval (a, b). As before $f(s)\, |s-\alpha|^{-q}$ must be summable.

6. Let us suppose that $K(s, t)$, $K_2(s, t)$, ..., $K_{n-1}(s, t)$ are all unlimited in the square for which $K(s, t)$ is defined, but that $K_n(s, t)$ is a limited function. It will further be assumed that $\int_a^b |K(s, t)|\, ds$ is limited for all values of t in (a, b), and that $\int_a^b |K(s, t)|\, dt$ is limited for all values of s in (a, b). Let α, β denote the upper limits of these integrals.

We have $$K_{n+1}(s, t) = \int_a^b K_n(s, \xi)\, K(\xi, t)\, d\xi,$$

and therefore $$|K_{n+1}(s, t)| \leqslant \alpha M_n,$$
where M_n denotes the upper limit of $|K_n(s, t)|$. Again we find that

$$K_{n+2}(s, t) \leqslant \alpha^2 M_n, \qquad K_{n+3}(s, t) \leqslant \alpha^3 M_n, \qquad \ldots .$$

It follows that the series

$$\lambda^{n-1} K_n(s, t) + \lambda^n K_{n+1}(s, t) + \ldots$$

is uniformly convergent with respect to (s, t, λ), provided

$$0 \leqslant |\lambda| \leqslant \lambda_1 < 1/\alpha.$$

If we had expressed the repeated integral $K_{n+1}(s, t)$ in the form

$$\int_a^b K(s, \xi)\, K_n(\xi, t)\, d\xi,$$

and proceeded as before, it would have been shown that the series is uniformly convergent if $0 \leqslant |\lambda| \leqslant \lambda_2 < 1/\beta$. The radius of convergence of the series is therefore not less than the larger of the two numbers $1/\alpha$, $1/\beta$.

We shall assume that $f(s)$ is either limited and summable, or that it is unlimited and summable, and also such that

$$K(s, t) f(t), \; K_2(s, t) f(t), \; \ldots, \; K_{m-1}(s, t) f(t)$$

are all summable in the interval (a, b) of t, for each value of s (with the possible exception of a set of values of linear measure zero).

Subject to these assumptions, it can be verified that, if $\phi(s)$ denote the sum of the series

$$f(s) + \lambda \int_a^b K(s, t) f(t)\, dt + \ldots + \lambda^{n-2} \int_a^b K_{n-1}(s, t) f(t)\, dt$$
$$+ \lambda^{n-1} \int_a^b K_n(s, t)\, f(t)\, dt + \ldots,$$

this value satisfies the integral equation, for every value of s for which the terms have a definite meaning. We have thus obtained the following theorem:—

(g) *If there exists a repeated nucleus $K_n(s, t)$ that is limited, and if $\int_a^b |K(s, t)|\, ds$ exists, and has α for its upper limit for all values of t, and $\int_a^b |K(s, t)|\, dt$ exists, and has β for its upper limit for all values of s,* *then the solution of the integral equation in the form*

$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t) f(t)\, dt,$$

for all values of $|\lambda|$ *which do not exceed the greater of the two numbers* $1/a$, $1/\beta$ *is obtained by taking* $-\bar{K}(s, t)$ *as the sum of the series*

$$K(s, t) + \lambda K_1(s, t) + \ldots;$$

provided $f(s)$, *if it be not limited, is such that*

$$K(s, t) f(t),\ K_2(s, t) f(t),\ \ldots,\ K_{n-1}(s, t) f(t)$$

are all summable with respect to t, *for every value of* s (*with the possible exception of values belonging to a set of linear measure zero*).

As an example of the application of this theorem, the well known case

$$K(s, t) = \frac{P(s, t)}{|s-t|^a},$$

where $a < 1$, may be cited.

7. It has been shewn by E. Schmidt, by a method depending upon the use of Schwarz's inequality, that

$$[K_n(s, t)]^2 \leqslant \left[\int_a^b \int_a^b \{K(s, t)\}^2\, ds\, dt\right]^{n-2} \int_a^b \{K(s, t)\}^2\, ds \int_a^b \{K(s, t)\}^2\, dt,$$

where it must be assumed that the integrals on the right-hand side have a definite meaning. This may be applied to the method of successive substitution in certain cases when $K(s, t)$ is unlimited.

If we assume that $\{K(s, t)\}^2$ is summable in the square for which $K(s, t)$ is defined, and also that $\int_a^b \{K(s, t)\}^2\, ds$ has a finite upper limit for all values of t in the interval (a, b), and further that $\int_a^b \{K(s, t)\}^2\, ds$ has a finite upper limit for all the values of s, we see that the series

$$K(s, t) + \lambda K_2(s, t) + \lambda^2 K_3(s, t) + \ldots$$

converges uniformly for all values of s, t, λ, such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1| < \left\{\int_a^b \int_a^b \{K(s, t)\}^2\, ds\, dt\right\}^{-\frac{1}{2}},$$

and thus that the method is applicable.

This theorem is less general than the theorem (g) of § 6, because it is applicable only in cases in which $K_2(s, t)$ is limited in the fundamental square.

Fredholm's Solution of the Integral Equation.

8. Fredholm's solution of the integral equation is given by

$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t) f(t)\, dt,$$

where

$$-\bar{K}(s, t) = \frac{D\begin{pmatrix}\lambda & s \\ & t\end{pmatrix}}{D(\lambda)},$$

the quotient of two integral functions of λ, expressible in the forms

$$D\begin{pmatrix}\lambda & s \\ & t\end{pmatrix} = K(s, t) - \lambda \int_a^b K\begin{pmatrix} s, & \xi_1 \\ t, & \xi_1 \end{pmatrix} d\xi_1 + \dots$$

$$+ \frac{(-1)^m \lambda^m}{m!} \int_a^b \int_a^b \dots \int_a^b K\begin{pmatrix} s, & \xi_1, & \xi_2, & \dots, & \xi_m \\ t, & \xi_1, & \xi_2, & \dots, & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \dots d\xi_m + \dots,$$

$$D(\lambda) = 1 - \lambda \int_a^b K(\xi_1, \xi_1)\, d\xi_1 + \dots$$

$$+ \frac{(-1)^m \lambda^m}{m!} \int_a^b \int_a^b \dots K\begin{pmatrix} \xi_1, & \xi_2, & \dots, & \xi_m \\ \xi_1, & \xi_2, & \dots, & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \dots d\xi_m + \dots,$$

the notation $K\begin{pmatrix} \xi_1 \xi_2 \dots \xi_m \\ \xi_1' \xi_2' \dots \xi_m' \end{pmatrix}$ being used for the determinant

$$\begin{vmatrix} K(\xi_1, \xi_1'), & K(\xi_2, \xi_1'), & \dots, & K(\xi_m, \xi_1') \\ K(\xi_1, \xi_2'), & K(\xi_2, \xi_2'), & \dots, & K(\xi_m, \xi_2') \\ \dots & \dots & \dots & \dots \\ K(\xi_1, \xi_m'), & K(\xi_2, \xi_m'), & \dots, & K(\xi_m, \xi_m') \end{vmatrix}.$$

It is known to be sufficient to ensure that $D\begin{pmatrix}\lambda & s \\ & t\end{pmatrix}$, $D(\lambda)$ are integral functions, that $K(s, t)$ should be a summable function which is limited in the square for which it is defined. It simplifies the statements to assume that $K(s, t)$ is summable with respect to s for every value of t, and with respect to t for every value of s. This assumption will here be made, though it is not necessary for the essential validity of the results obtained. It is clear, by employing the usual method of verification, that in this case there is a solution of the integral equation, given by Fredholm's formula, provided λ is not a zero of $D(\lambda)$, and provided $f(s)$ is a summable function, limited or unlimited. We have, by taking out the term $K(s, t)$ multiplied by its minor,

$$K\begin{pmatrix} s, \xi_1, \xi_2, \dots, \xi_m \\ t, \xi_1, \xi_2, \dots, \xi_m \end{pmatrix} = K(s, t)\, K\begin{pmatrix} \xi_1, \xi_2, \dots, \xi_m \\ \xi_1, \xi_2, \dots, \xi_m \end{pmatrix} + E_m(s, t),$$

where $E_m(s, t)$ denotes the determinant

$$\begin{vmatrix} 0, & K(\xi_1, t), & K(\xi_2, t), & \dots, & K(\xi_m, t) \\ K(s, \xi_1), & K(\xi_1, \xi_1), & K(\xi_2, \xi_1), & \dots, & K(\xi_m, \xi_1) \\ K(s, \xi_2), & K(\xi_1, \xi_2), & K(\xi_2, \xi_2), & \dots, & K(\xi_m, \xi_2) \\ \dots & \dots & \dots & \dots & \dots \\ K(s, \xi_m), & K(\xi_1, \xi_m), & K(\xi_2, \xi_m), & \dots, & K(\xi_m, \xi_m) \end{vmatrix}$$

On substitution in the expression for $D\left(\lambda \begin{matrix} s \\ t \end{matrix}\right)$, we have

$$D\left(\lambda \begin{matrix} s \\ t \end{matrix}\right) = D(\lambda)\, K(s, t) + \Sigma \frac{(-1)^m \lambda^m}{m!} \int_a^b \int_a^b \dots E_m(s, t)\, d\xi_1 d\xi_2 \dots d\xi_m.$$

Hence the solution of the integral equation takes the form

$$\phi(s) = f(s) + \lambda \int_a^b K(s, t)\, f(t)\, dt$$
$$+ \frac{1}{D(\lambda)} \Sigma \frac{(-1)^m \lambda^m}{m!} \int_a^b \int_a^b \dots E_m(s, t)\, f(t)\, dt\, d\xi_1 d\xi_2 \dots d\xi_m.$$

By applying Hadamard's theorem to the determinant $E_m(s, t)$ it is seen that, for any value of λ which is not a zero of $D(\lambda)$, the series in the last term of the right hand side converges uniformly with respect to (s, t). The determinant $E_m(s, t)$, when expounded, has the form

$$\Sigma \pm K(s, \xi_p)\, K(\xi_q, t)\, K(\xi_{\alpha_1}, \xi_{\beta_1})\, K(\xi_{\alpha_2}, \xi_{\beta_2}) \dots K(\xi_{\alpha_{m-1}}, \xi_{\beta_{m-1}}),$$

where the indices $q, \alpha_1, \alpha_2, \dots, \alpha_{m-1}$ are all different, and also the indices $p, \beta_1, \beta_2, \dots, \beta_{m-1}$ are all different. On substitution in the expression for $\phi(s)$ we see that the coefficient of $\dfrac{1}{D(\lambda)} \dfrac{(-1)^m \lambda^m}{m!}$ consists of two kinds of terms. The terms of the first of these kinds, corresponding to $q = p$, are of the form

$$A \int_a^b \int_a^b K(s, \xi_p)\, K(\xi_p, t)\, f(t)\, d\xi_p\, dt,$$

where A denotes a constant. In case the discontinuities of $K(s, t)$ are regularly distributed, we see by theorem (d), that

$$\int_a^b K(s, \xi_p) K(\xi_p, t) d\xi_p$$

is a continuous function of (s, t), and it then follows that the term is a continuous function of s. The terms of the second kind are of the form

$$A \int_a^b \int_a^b \dots \int_a^b K(s, \xi_p) K(\xi_p, \xi_{p'}) K(\xi_{p'}, \xi_{p''}) \dots K(\xi_{p^{(r)}}, \xi_q) K(\xi_q, t) f(t) \, dt \, d\xi_p \dots d\xi_{p^{(r)}}.$$

On the same assumption, that discontinuities of $K(s, t)$ are regularly distributed, we see that $\int_a^b K(s, \xi_p) K(\xi_p, \xi_{p'}) d\xi_p$ is a continuous function of $(s, \xi_{p'})$, say $C(s, \xi_{p'})$; then $\int_a^b C(s, \xi_{p'}) K(\xi_{p'}, \xi_{p''}) d\xi_{p'}$ is continuous relative to $(s, \xi_{p''})$, and so on. The term ultimately reduces to the form

$$\int_a^b \int_a^b F(s, \xi_q) K(\xi_q, t) f(t) \, dt \, d\xi,$$

where $F(s, \xi_q)$ is continuous with respect to (s, ξ_q); this reduces to

$$\int_a^b G(s, t) f(t) \, dt,$$

where $G(s, t)$ is continuous relative to (s, t), and therefore by the theorem (d) the term is continuous relative to s, for any summable function $f(t)$. Since the series in the expression for $\phi(s)$ converges uniformly with respect to s, for any fixed value of λ, and since its terms are continuous, it follows that its sum-function is continuous. Therefore, when $K(s, t)$ has its discontinuities regularly distributed, $\phi(s)$ and $f(s)$ have only the same points of discontinuity; and in particular $\phi(s)$ is continuous when $f(s)$ is so.

It can be shewn that Fredholm's solution is the only possible solution which is summable in the interval (a, b). For, let $w(s)$ be any summable solution of the integral equation, thus

$$f(t) = w(t) - \lambda \int_a^b K(t, \xi) w(\xi) \, d\xi.$$

Multiplying the equation by $\bar{K}(s, t)$, and integrating with respect to t

through the interval (a, b), we have

$$\int_a^b \bar{K}(s, t) f(t)\, dt = \int_a^b \bar{K}(s, t)\, w(t)\, dt - \lambda \int_a^b \int_a^b \bar{K}(s, t)\, K(t, \xi)\, w(\xi)\, d\xi\, dt$$

$$= \int_a^b w(t) \left\{ \bar{K}(s, t) - \lambda \int_a^b \bar{K}(s, \xi)\, K(\xi, t)\, d\xi \right\} dt$$

$$= -\int_a^b K(s, t)\, w(t)\, dt,$$

in virtue of the fundamental relation

$$K(s, t) + \bar{K}(s, t) = \lambda \int_a^b \bar{K}(s, \xi)\, K(\xi, t)\, d\xi.$$

Since now $\int_a^b K(t, \xi)\, w(\xi)\, d\xi$ has been shewn to be equal to

$$-\int_a^b \bar{K}(t, \xi)\, f(\xi)\, d\xi,$$

we see that
$$w(t) = f(t) - \lambda \int_a^b \bar{K}(t, \xi)\, f(\xi)\, d\xi,$$

or
$$w(s) = f(s) - \lambda \int_a^b \bar{K}(s, t)\, f(t)\, dt,$$

and thus $w(s)$ is identical with Fredholm's solution.

The following results have now been established:—

If the nucleus $K(s, t)$ is limited in the square for which it is defined, and $f(s)$ is any summable function, limited or unlimited in the interval (a, b), then for any value of λ that is not a characteristic value, the only summable solution of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is that of Fredholm. Moreover, in case the discontinuities of $K(s, t)$ are regularly distributed, the solution has only the same points of discontinuity as $f(s)$, and is continuous if $f(s)$ be so.

A non-summable solution of the integral equation may exist which will not be given by Fredholm's method. Assuming that, in the equation, the integral is restricted to be of the Lebesgue type, it may happen that, although $\phi(t)$ is not summable in the interval (a, b), $K(s, t)\, \phi(t)$ is so.

For example, the equation

$$1 = \phi(s) - \int_0^1 K(s, t)\, \phi(t)\, dt,$$

where

$$K(s, t) = t^s - \tfrac{1}{2} t^{\frac{1}{2}}$$

admits of the non-summable solution

$$\phi(s) = 1/s.$$

An example of such a solution has been given by Bôcher for the case of Volterra's equation ;* he states that such solutions must necessarily be non-integrable.

The Integral Equation with Unlimited Nucleus.

9. We proceed to consider cases in which the nucleus $K(s, t)$ of the integral equation is unlimited. Let it be assumed that one of the repeated nuclei $K_n(s, t)$ is limited, $K_{n-1}(s, t)$, $K_{n-2}(s, t)$, ... being all unlimited.

The method of successive substitutions discussed in § 4 shows that

$$\phi(s) = f(s) + \lambda \int_a^b K(s, t) f(t)\, dt + \lambda^2 \int_a^b K_2(s, t) f(t)\, dt + \dots$$
$$+ \lambda^{n-1} \int_a^b K_{n-1}(s, t) f(t)\, dt + \lambda^n \int_a^b K_n(s, t)\, \phi(t)\, dt,$$

it being assumed that $K(s, t)$ and the limited or unlimited summable function $f(t)$ are such that

$$K(s, t) f(t),\ K_2(s, t) f(t),\ \dots,\ K_{n-1}(s, t) f(t)$$

are all summable in the interval (a, b) of t, for all (or almost all) the values of s.

If the integral equation has a solution, that solution must satisfy the equation

$$\phi(s) - \lambda^n \int_a^b K_n(s, t)\, \phi(t)\, dt = f_n(s), \tag{A}$$

where $f_n(s)$ denotes

$$f(s) + \lambda \int_a^b K(s, t) f(t)\, dt + \dots + \lambda^{n-1} \int_a^b K_{n-1}(s, t) f(t)\, dt.$$

* See Bôcher's tract, p. 17. It has been remarked by Prof. W. H. Young that the solutions contemplated by Bôcher are examples rather of unlimited than of discontinuous function ; see his paper "On Integral Equations," *Quarterly Journal of Math.*, Vol. XLI, p. 184.

If it be now assumed that the integrals

$$\int_a^b K(s, t)\,dt, \quad \int_a^b K_2(s, t)\,dt, \quad \ldots, \quad \int_a^b K_{n-1}(s, t)\,dt,$$

$$\int_a^b K(s, t) f(t)\,dt, \quad \int_a^b K_2(s, t) f(t)\,dt, \quad \ldots, \quad \int_a^b K_{n-1}(s, t) f(t)\,dt$$

all exist, and are summable functions of s, it follows that $f_n(s)$ is a summable function of s, and that the equation (A) has a single solution given by Fredholm's expression, provided λ^n is not a characteristic value.

Conversely it will be shewn that this solution $\phi(s)$ of (A) also satisfies the given integral equation.

Let the functions $\phi_1(s)$, $\phi_2(s)$, ..., $\phi_n(s)$ be defined by

$$\phi_1(s) = \lambda \int_a^b K(s, t)\,\phi(t)\,dt + f(s),$$

$$\phi_2(s) = \lambda \int_a^b K(s, t)\,\phi_1(t)\,dt + f(s),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\phi_n(s) = \lambda \int_a^b K(s, t)\,\phi_{n-1}(t)\,dt + f(s).$$

It will be shewn that, in virtue of the hypotheses made as to the nature of the functions $f(t)$, $K(s, t)$, these functions $\phi_1(s)$, $\phi_2(s)$, ..., $\phi_n(s)$ are all summable in the interval (a, b) of s.

The function $\phi(t)$, being given by Fredholm's formula, is equal to $f_n(t) + \chi(t)$, where $\chi(t)$ is a limited summable function of t, for a fixed value of λ. Since

$$\int_a^b K(s, t)\,\phi(t)\,dt = \int_a^b K(s, t) f_n(t)\,dt + \int_a^b K(s, t)\,\chi(t)\,dt,$$

it follows from the hypotheses made that $\int_a^b K(s, t)\,\phi(t)\,dt$ is a summable function of s; therefore $\phi_1(s)$ is a summable function of s.

Again, we have

$$\phi_2(s) = f(s) + \lambda \int_a^b K(s, t) f(t)\,dt + \lambda^2 \int_a^b K_2(s, t)\,\phi(t)\,dt,$$

and it then follows that $\phi_2(s)$ is a summable function of s. Similarly it may be shewn that $\phi_3(s) \ldots \phi_n(s)$ are all summable functions. We have

$$\phi_n(s) = f(s) + \lambda \int_a^b K(s, t) f(t)\, dt + \ldots + \lambda^{n-1} \int_a^b K_{n-1}(s, t) f(t)\, dt$$

$$+ \lambda^n \int_a^b K_n(s, t)\, \phi(t)\, dt$$

$$= f_n(s) + \lambda^n \int_a^b K_n(s, t)\, \phi(t)\, dt;$$

and therefore $$\phi_n(s) = \phi(s).$$

By adding the equations which define $\phi_1(s)$, $\phi_2(s)$, ..., $\phi_n(s)$, we see that

$$\frac{\phi_1(s) + \phi_2(s) + \ldots + \phi_n(s)}{n} = f(s) + \lambda \int_a^b \frac{\phi(t) + \phi_1(t) + \ldots + \phi_{n-1}(t)}{n} K(s, t)\, dt,$$

and since $$\phi(t) = \phi_n(t),$$

it follows that $\dfrac{\phi_1(s) + \phi_2(s) + \ldots + \phi_n(s)}{n}$ satisfies the given integral equation, and it therefore satisfies the integral equation (A). Since (A) has a unique summable solution, it follows that

$$\phi(s) = \frac{\phi_1(s) + \phi_2(s) + \ldots + \phi_n(s)}{n},$$

and thus that $\phi(s)$ satisfies the given integral equation.

The following theorem has now been established:—

If $K(s, t)$, $K_2(s, t)$, ..., $K_{n-1}(s, t)$ are unlimited, and $K_n(s, t)$ is limited in the square $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, and if $\int_a^b K_r(s, t)\, dt$ exists as a summable function of s, for $r = 1, 2, 3, \ldots, n-1$, then if $f(s)$ be any summable function, limited or unlimited, such that $\int_a^b K_r(s, t) f(t)\, dt$ is a summable function of s, for $r = 1, 2, 3, \ldots, n-1$, the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

has a unique summable solution given by

$$\phi(s) = f_n(s) + \lambda^n \int_a^b \frac{D_n\left(\lambda^n \begin{matrix} s \\ t \end{matrix}\right)}{D_n(\lambda^n)} f_n(t)\, dt,$$

where $D_n\left(\lambda^n \begin{matrix} s \\ t \end{matrix}\right)$ *denotes the integral function*

$$K_n(s,\,t)-\lambda^n\int_a^b K_n\begin{pmatrix} s, & \xi_1 \\ t, & \xi_1\end{pmatrix} d\xi_1+\frac{\lambda^{2n}}{2!}\int_a^b\int_a^b K_n\begin{pmatrix} s, & \xi_1, & \xi_2 \\ t, & \xi_1, & \xi_2\end{pmatrix} d\xi_1 d\xi_2-\ldots,$$

and $D_n(\lambda^n)$ *denotes the integral function*

$$1-\lambda^n\int_a^b K_n(\xi,\,\xi_1)\,d\xi_1+\frac{\lambda^{2n}}{2!}\int_a^b\int_a^b K_n\begin{pmatrix} \xi_1, & \xi_2 \\ \xi_1, & \xi_2\end{pmatrix} d\xi_1 d\xi_2-\ldots,$$

and $K_n\begin{pmatrix} \xi_1, & \xi_2, & \ldots, & \xi_r \\ \xi_1', & \xi_2', & \ldots, & \xi_r'\end{pmatrix}$ *denotes the determinant*

$$\begin{vmatrix} K_n(\xi_1,\,\xi_1), & K_n(\xi_1,\,\xi_2), & \ldots, & K_n(\xi_1,\,\xi_r) \\ K_n(\xi_2,\,\xi_1), & K_n(\xi_2,\,\xi_2), & \ldots, & K_n(\xi_2,\,\xi_r) \\ \ldots & \ldots & \ldots & \ldots \\ K_n(\xi_r,\,\xi_1), & K_n(\xi_r,\,\xi_2), & \ldots, & K_n(\xi_r,\,\xi_r)\end{vmatrix}.$$

The function $f_n(s)$ *denotes*

$$f(s)+\lambda\int_a^b K(s,\,t)\,f(t)\,dt+\ldots+\lambda^{n-1}\int_a^b K_{n-1}(s,\,t)\,f(t)\,dt.$$

The value of λ *must not be a zero of* $D_n(\lambda^n)$.

10. Using the notation $\bar{K}_n(s,\,t,\,\lambda^n)$ for the reciprocal function of $K_n(s,\,t)$ when the parameter is λ^n, we have

$$\bar{K}_n(s,\,t,\,\lambda^n)=-\frac{D_n\left(\lambda^n \begin{matrix} s \\ t\end{matrix}\right)}{D_n(\lambda^n)}.$$

Thus the solution of the integral equation

$$F(s)=\phi(s)-\lambda^n\int_a^b K_n(s,\,t)\,\phi(t)\,dt,$$

is
$$\phi(s)=F(s)-\lambda^n\int_a^b \bar{K}_n(s,\,t,\,\lambda^n)\,F(t)\,dt.$$

Writing the solution of the equation

$$f(s)=\phi(s)-\lambda\int_a^b K(s,\,t)\,\phi(t)\,dt$$

in the form
$$\phi(s)=f(s)-\lambda\int_a^b \bar{K}(s,\,t)\,f(t)\,dt,$$

we obtain from the expression obtained in § 9 the relation between $\bar{K}(s, t)$ and $\bar{K}_n(s, t, \lambda^n)$.

It will be convenient to use the notation $U_s\chi(s)$ for $\int_a^b K(s, t)\chi(t)\,dt$, the symbol U_s denoting therefore an operation. We have then also

$$U_s^2\chi(s) = U_s\int_a^b K(s, t)\chi(t)\,dt = \int_a^b K(s, t')\,dt' \int_a^b K(t', t)\chi(t)\,dt$$
$$= \int_a^b K_2(s, t)\chi(t)\,dt,$$

and generally $$U_s^r\chi(s) = \int_a^b K_r(s, t)\chi(t)\,dt.$$

In a similar manner we denote by $V_s\chi(s)$, $\int_a^b K(t, s)\chi(t)\,dt$, and therefore $V_s^r\chi(s)$ denotes $\int_a^b K_n(t, s)\chi(t)\,dt$. With this notation the integral equation can be written in the form

$$f(s) = \phi(s) - \lambda U_s\phi(s),$$

and the associated equation in the form

$$f(s) = \phi(s) - \lambda V_s\phi(s).$$

We have now, from § 9,

$$\bar{K}(s, t) = -\{K(s, t) + \lambda K_2(s, t) + \lambda^2 K_3(s, t) + \ldots + \lambda^{n-2}K_{n-1}(s, t)\}$$
$$+\lambda^{n-1}\{1 + \lambda V_t + \lambda^2 V_t^2 + \ldots + \lambda^{n-1}V_t^{n-1}\}\bar{K}_n(s, t, \lambda^n).$$

If $G_n(s, t, \lambda^n)$ denotes that part of $\bar{K}(s, t, \lambda^n)$ which consists of the sum of terms that contain negative powers of $\lambda^n - \lambda_1^n$, where λ_1^n is a characteristic value, or zero of $D_n(\lambda^n)$, then that part of $\bar{K}(s, t)$ which becomes infinite when λ has any one of the values $\lambda_1, \lambda_1\omega, \lambda_1\omega^2, \ldots, \lambda_1\omega^{n-1}$, where ω is a primitive n-th root of unity, is that part of

$$\lambda^{n-1}\{1 + \lambda V_t + \lambda^2 V_t^2 + \ldots + \lambda^{n-1}V_t^{n-1}\}\, G(s, t, \lambda^n) \text{ or } H(s, t, \lambda),$$

which consists of negative powers of $\lambda - \lambda_1, \lambda - \omega\lambda_1, \ldots, \lambda - \omega^{n-1}\lambda_1$. The remaining part of $\bar{K}(s, t)$ remains finite for the values $\lambda_1, \omega\lambda_1, \omega^2\lambda_1, \ldots$ of λ.

We find easily that

$$\frac{1}{n\lambda^{n-1}}\{H(s,\,t,\,\lambda)+\omega H(s,\,t,\,\omega\lambda)+\omega^2 H(s,\,t,\,\omega^2\lambda)+\ldots+\omega^{n-1}H(s,\,t,\,\omega^{n-1}\lambda)\}$$
$$= G(s,\,t,\,\lambda^n)\,;$$

thus $G(s, t, \lambda^n)$ is expressed in terms of the function $H(s, t, \lambda)$.

In accordance with Lalesco's theory* of the canonical forms of the resolvant $G(s, t, \lambda^n)$, that resolvant is expressible as the sum of a number of canonical sub-groups, each one of which is of the form

$$\frac{C_1(s,\,t)}{\lambda^n-\lambda_1^n}+\frac{C_2(s,\,t)}{(\lambda^n-\lambda_1^n)^2}+\ldots+\frac{C_p(s,\,t)}{(\lambda^n-\lambda_1^n)^p},$$

where
$$\begin{aligned} C_1(s,\,t) &= \phi_1(s)\,\psi_1(t)+\phi_2(s)\,\psi_2(t)+\ldots+\phi_p(s)\,\psi_p(t),\\ C_2(s,\,t) &= a_1\phi_1(s)\,\psi_2(t)+a_2\phi_2(s)\,\psi_3(t)+\ldots+a_{p-1}\phi_{p-1}(s)\,\psi_p(t),\\ C_3(s,\,t) &= a_1a_2\phi_1(s)\,\psi_3(t)+\ldots+a_{p-2}a_{p-1}\phi_{p-2}(s)\,\psi_p(t),\\ &\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ C_p(s,\,t) &= a_1a_2\ldots a_p\phi_1(s)\,\psi_p(t). \end{aligned}$$

The sets of principal functions

$$\phi_1(s),\ \phi_2(s),\ \ldots,\ \phi_p(s)\quad\text{and}\quad\psi_1(t),\ \psi_2(t),\ \ldots,\ \psi_p(t)$$

form a biorthogonal system. Of these only $\phi_1(s)$ is a fundamental function, *i.e.*, a solution of the equation

$$\phi(s)-\lambda_1^n\int_a^b K_n(s,\,t)\,\phi(t)\,dt = 0.$$

Only $\psi_p(s)$ is a solution of the reciprocal equation

$$\psi(s)-\lambda_1^n\int_a^b K(t,\,s)\,\psi(t)\,dt = 0.$$

If, in the expression for $H(s, t, \lambda)$ in terms of $G(s, t, \lambda^n)$, we substitute all the sub-groups of the above form of which $G(s, t, \lambda^n)$ is composed, we obtain an expression for $H(s, t, \lambda)$ which consists of terms each of which involves a negative power of $\lambda^n-\lambda_1^n$. These terms may be expressed by resolution into partial fractions, each as the sum of terms involving negative powers of $\lambda-\lambda_1$, $\lambda-\omega\lambda_1$, ..., $\lambda^n-\omega^{n-1}\lambda_1$.

It thus appears that the part of $\bar{K}(s, t)$ that becomes infinite when λ

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* See his *Introduction à la théorie des équations intégrales*, Chapter II.

has one of the values $\lambda_1, \omega\lambda_1, \omega^2\lambda_1, \ldots, \omega^{n-1}\lambda_1$, is of the form

$$\left[\frac{C_1^{(1)}(s, t)}{\lambda-\lambda_1}+\frac{C_2^{(1)}(s, t)}{(\lambda-\lambda_1)^2}+\ldots+\frac{C_p^{(1)}(s, t)}{(\lambda-\lambda_1)^p}\right]+\left[\frac{C_1^{(2)}(s, t)}{\lambda-\omega\lambda_1}+\frac{C_2^{(2)}(s, t)}{(\lambda-\omega\lambda_1)^2}+\ldots\right]+\ldots$$
$$+\left[\frac{C_1^{(n)}(s, t)}{\lambda-\omega^{n-1}\lambda_1}+\frac{C_2^{(n)}(s, t)}{(\lambda-\omega^{n-1}\lambda_1)^2}+\ldots\right],$$

where all the functions C are expressed in terms of the principal functions $\phi(s)$, $\psi(t)$, of $\bar{K}_n(s, t, \lambda^n)$.

The function $\bar{K}(s, t)$ must satisfy the equation

$$K(s, t)+\bar{K}(s, t) = \lambda \int_a^b \bar{K}(s, t')\, K(t', t)\, dt',$$

which is a necessary condition that

$$\phi(s) = f(s)-\lambda \int_a^b K(s, t) f(t)\, dt,$$

may satisfy the integral equation

$$f(s) = \phi(s)-\lambda \int_a^b K(s, t)\, \phi(t)\, dt.$$

It is clear that the parts of $\bar{K}(s, t)$ which involve negative powers of $\lambda-\lambda_1, \lambda-\omega\lambda_1, \ldots, \lambda-\omega^{n-1}\lambda_1$ must each separately satisfy this equation. It follows that Lalesco's theory of the canonical sub-groups must be applicable to

$$\frac{C_1^{(1)}(s, t)}{\lambda-\lambda_1}+\frac{C_2^{(1)}(s, t)}{(\lambda-\lambda_1)^2}+\ldots+\frac{C_n^{(1)}(s, t)}{(\lambda-\lambda_1)^n},$$

as also to each of the other such portions of $\bar{K}(s, t)$.

This part of $\bar{K}(s, t)$ is therefore expressible as the sum of a number of canonical sub-groups, each one of which is of the form

$$\frac{B_1(s, t)}{\lambda-\lambda_1}+\frac{B_2(s, t)}{(\lambda-\lambda_1)^2}+\ldots+\frac{B_q(s, t)}{(\lambda-\lambda_1)^q},$$

where
$$B_1(s, t) = \Phi_1(s)\, \Psi_1(t)+\Phi_2(s)\, \Psi_2(t)+\ldots+\Phi_q(s)\, \Psi_q(t),$$
$$B_2(s, t) = \beta_1 \Phi_1(s)\, \Psi_2(t)+\ldots+\beta_{q-1}\Phi_{q-1}(s)\, \Psi_q(t),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$B_q(s, t) = \beta_1\beta_2 \ldots \beta_{q-1}\Phi_1(s)\, \Psi_q(t).$$

The sets of functions

$$\Phi_1(s),\ \Phi_2(s),\ \ldots,\ \Phi_q(s) \quad \text{and} \quad \Psi_1(t),\ \Psi_2(t),\ \ldots,\ \Psi_q(t)$$

are principal functions forming a biorthogonal system. The function $\Phi_1(s)$ is the only one of these functions that satisfies the equation

$$\Phi(s)-\lambda_1 U_s \Phi(s) = 0;$$

and $\Psi_p(s)$ is the only one which satisfies the equation

$$\Psi(s)-\lambda_1 V_s \Psi(s) = 0.$$

We proceed to form that part of $\bar{K}_n(s, t, \lambda^n)$ that corresponds to one of the canonical sub-groups of $\bar{K}(s, t)$. Corresponding to

$$\frac{B_1(s, t)}{\lambda-\lambda_1}+\frac{B_2(s, t)}{(\lambda-\lambda_1)^2}+\ldots+\frac{B_q(s, t)}{(\lambda-\lambda_1)^q},$$

the part of $\bar{K}_n(s, t, \lambda^n)$ is

$$\begin{aligned}\frac{1}{n\lambda^{n-1}}\Bigg[\; & B_1(s, t)\left\{\frac{1}{\lambda-\lambda_1}+\frac{\omega}{\omega\lambda-\lambda_1}+\ldots+\frac{\omega^{n-1}}{\omega^{n-1}\lambda-\lambda_1}\right\} \\ & +B_2(s, t)\left\{\frac{1}{(\lambda-\lambda_1)^2}+\frac{\omega}{(\omega\lambda-\lambda_1)^2}+\ldots+\frac{\omega^{n-1}}{(\omega^{n-1}\lambda-\lambda_1)^2}\right\} \\ & +\ldots \\ & +B_q(s, t)\left\{\frac{1}{(\lambda-\lambda_1)^q}+\frac{\omega}{(\omega\lambda-\lambda_1)^q}+\ldots+\frac{\omega^{n-1}}{(\omega^{n-1}\lambda-\lambda_1)^q}\right\}\Bigg].\end{aligned}$$

Employing the identity

$$\frac{1}{(\lambda-\lambda_1)^s}+\frac{\omega}{(\omega\lambda-\lambda_1)^s}+\ldots+\frac{\omega^{n-1}}{(\omega^{n-1}\lambda-\lambda_1)^s}=\frac{n\lambda^{n-1}}{(s-1)!}\frac{d^{s-1}}{d\lambda_1^{s-1}}\frac{1}{\lambda^n-\lambda_1^n},$$

where $s = 1, 2, 3, \ldots, q$, this expression may be written in the form

$$\begin{aligned}\frac{B_1(s, t)}{\lambda^n-\lambda_1^n}+\frac{1}{1!}B_2(s, t)\frac{d}{d\lambda_1}\frac{1}{\lambda^n-\lambda_1^n}+\frac{1}{2!}B_3(s, t)\frac{d^2}{d\lambda_1^2}\frac{1}{\lambda^n-\lambda_1^n}+\ldots \\ +\frac{1}{(q-1)!}B_q(s, t)\frac{d^{q-1}}{d\lambda_1^{q-1}}\frac{1}{\lambda^n-\lambda_1^n}.\end{aligned}$$

It will be observed that the first term in this expression is the only one which involves the first power of $\frac{1}{\lambda^n-\lambda_1^n}$, and that the last term is the only one which involves the q-th power of $\frac{1}{\lambda^n-\lambda_1^n}$; the other terms contain more than one power of $\frac{1}{\lambda^n-\lambda_1^n}$. Each canonical sub-group corresponding to any one of the values $\lambda_1, \omega\lambda_1, \ldots, \omega^{n-1}\lambda_1$ of λ in $\bar{K}(s, t)$ gives rise to a part of $\bar{K}_n(s, t, \lambda^n)$ of the above form. The sum of all the parts of $\bar{K}_n(s, t, \lambda^n)$ so obtained must be equivalent to the expression of the same resolvant as the sum of canonical sub-groups. It will, however, be shewn that

$$\frac{B_1(s, t)}{\lambda^n-\lambda_1^n}+\frac{1}{1!}B_2(s, t)\frac{d}{d\lambda_1}\frac{1}{\lambda^n-\lambda_1^n}+\ldots+\frac{1}{(q-1)!}B_q(s, t)\frac{d^{q-1}}{d\lambda_1^{q-1}}\frac{1}{\lambda^n-\lambda_1^n},$$

is itself equivalent to one of the canonical sub-groups of $\bar{K}_n(s, t, \lambda^n)$. To see this we observe that this expression satisfies the condition that it is the resolvant of the nucleus

$$\frac{B_1(s, t)}{\lambda_1^n}+\frac{1}{1!}B_2(s, t)\frac{d}{d\lambda_1}\frac{1}{\lambda_1^n}+\frac{1}{2!}B_3(s, t)\frac{d^2}{d\lambda_1^2}\frac{1}{\lambda_1^n}+\ldots$$
$$+\frac{1}{(q-1)!}B_q(s, t)\frac{d}{d\lambda_1^{q-1}}\frac{1}{\lambda_1^n},$$

with the parameter λ^n. Denoting this last expression by $k_n(s, t)$, and the former one by $\bar{k}_n(s, t, \lambda^n)$, it is sufficient to shew that

$$k_n(s, t)+\bar{k}_n(s, t, \lambda^n) = \lambda^n\int_a^b k_n(s, t')\,\bar{k}_n(t', t, \lambda^n)\,dt'.$$

In accordance with Lalesco's theory, the functions B satisfy the conditions

$$\int_a^b B_\alpha(s, t')\,B_\beta(t', t)\,dt' = B_{\alpha+\beta-1}(s, t),$$

for $1 \leqslant \alpha+\beta-1 \leqslant q$. Forming the expression for

$$\lambda^n\int_a^b k_n(s, t')\,\bar{k}_n(t', t, \lambda^n)\,dt',$$

the coefficient of $B_p(s, t)$ becomes, in virtue of the relation quoted,

$$\frac{1}{\lambda_1^n}\frac{1}{(p-1)!}\frac{d^{p-1}}{d\lambda_1^{p-1}}\frac{1}{\lambda^n-\lambda_1^n}+\frac{1}{1!}\frac{d}{d\lambda_1}\frac{1}{\lambda_1^n}\frac{d^{p-2}}{d\lambda_1^{p-2}}\frac{1}{\lambda^n-\lambda_1^n}$$
$$+\frac{1}{2!}\frac{d^2}{d\lambda_1^2}\frac{1}{\lambda_1^n}\frac{d^{p-3}}{d\lambda_1^{p-3}}\frac{1}{\lambda^n-\lambda_1^n}+\ldots$$
$$+\frac{1}{(p-1)!}\frac{d^{p-1}}{d\lambda_1^{p-1}}\frac{1}{\lambda_1^n}\frac{1}{\lambda^n-\lambda_1^n},$$

which is equal to
$$\frac{1}{(p-1)!}\frac{d^{p-1}}{d\lambda_1^{p-1}}\frac{1}{\lambda_1^n(\lambda^n-\lambda_1^n)},$$
or to
$$\frac{\lambda^n}{(p-1)!}\frac{d^{p-1}}{d\lambda_1^{p-1}}\left(\frac{1}{\lambda_1^n}+\frac{1}{\lambda^n-\lambda_1^n}\right),$$
and this is the coefficient of $B_p(s, t)$ in $k_n(s, t)+\bar{k}^n(s, t, \lambda^n)$. It has accordingly been verified that
$$\frac{B_1(s, t)}{\lambda^n-\lambda_1^n}+\frac{1}{1!}B_2(s, t)\frac{d}{d\lambda_1}\frac{1}{\lambda^n-\lambda_1^n}+\ldots+\frac{1}{(q-1)!}B_q(s, t)\frac{d^{q-1}}{d\lambda_1^{q-1}}\frac{1}{\lambda^n-\lambda_1^n},$$
is a resolvant, for the parameter λ^n. This resolvant can be expressed as a canonical group, or else as the sum of a number of canonical sub-groups. But the latter case cannot arise, because since the expression contains a term in $\frac{1}{(\lambda^n-\lambda_1^n)^q}$, the order of one of the sub-groups must be q, and if there were other sub-groups the total number of principal functions in s would exceed q, being the sum of the orders of all the sub-groups, and this cannot be the case, because the total number of linearly independent functions of s involved in the expression for the resolvant is equal to q. Therefore the above resolvant is reducible to a single canonical sub-group in $\bar{K}_n(s, t, \lambda^n)$.

It must therefore be reducible to the form
$$\frac{C_1(s, t)}{\lambda^n-\lambda_1^n}+\frac{C_2(s, t)}{(\lambda^n-\lambda_1^n)^2}+\ldots+\frac{C_q(s, t)}{(\lambda^n-\lambda_1^n)^q},$$
where the functions $C_1, C_2, \ldots, C_q$ are expressed in terms of principal functions $\phi_1(s), \ldots, \phi_q(s), \psi_1(s), \ldots, \psi_q(s)$.

On equating the terms in $\frac{1}{\lambda^n-\lambda_1^n}, \frac{1}{(\lambda^n-\lambda_1^n)^q}$, we have
$$\Phi_1(s)\Psi_1(t)+\Phi_2(s)\Psi_2(t)+\ldots+\Phi_q(s)\Psi_q(t)=\phi_1(s)\psi_1(t)+\ldots+\phi_q(s)\psi_q(t),$$
and
$$\beta_1\beta_2\ldots\beta_{q-1}\Phi_1(s)\Psi_q(t)=\alpha_1\alpha_2\ldots\alpha_{q-1}\phi_1(s)\psi_q(t);$$
from which it follows that the fundamental functions $\phi_1(s)$, $\Phi_1(s)$ must be identical, as also the fundamental solutions $\psi_q(s)$, $\Psi_q(s)$ of the reciprocal equations. The other principal functions $\phi_2(s), \ldots, \phi_q(s)$ can be expressed as linear functions of $\Phi_1(s), \ldots, \Phi_p(s)$; and a similar statement applies to $\psi_1(s), \psi_2(s), \ldots, \psi_{q-1}(s)$.

It has now been shewn that:—

To each canonical sub-group in $\bar{K}(s, t)$ there corresponds a single *canonical sub-group in $\bar{K}_n(s, t, \lambda^n)$; the fundamental solutions of the integral equation, and its reciprocal in the one case being identical with the fundamental solutions of the integral equation and its reciprocal in the other case. The corresponding canonical sub-groups have the same order.*

11. Let it now be still assumed that all the repeated nuclei $K_1(s, t)$, $K_2(s, t)$, ..., $K_{n-1}(s, t)$ are unlimited, but that $K_n(s, t)$ is limited in the fundamental square. It will also be assumed, as in § 6, that $\int_a^b |K(s, t)|\, ds$ and $\int_a^b |K(s, t)|\, dt$ are limited functions. The trace $\int_a^b K_r(s, s)\,ds$ corresponding to $K_r(s, t)$ being denoted by k_r, we see that k_n, k_{n+1}, ... are all finite.

Let the function $Q_n(\lambda)$, defined for all values of λ, be such that, for sufficiently small values of $|\lambda|$, it is the sum-function of the series

$$-\frac{k_n}{n}\lambda^n - \frac{k_{n+1}}{n+1}\lambda^{n+1} - \frac{k_{n+2}}{n+2}\lambda^{n+2} - \dots,$$

which has a radius of convergence > 0. The function $Q_n(\lambda)$ outside the circle of convergence is determined as the analytical continuation of the sum-function. It will be shewn that $e^{Q_n(\lambda)}$ and $e^{Q_n(\lambda)} \dfrac{D_n\left(\lambda^n \begin{matrix} s \\ t \end{matrix}\right)}{D_n(\lambda^n)}$ are both integral functions of λ. This theorem has been established by Poincaré* in the special case in which each pole of $\bar{K}_n(s, t, \lambda^n)$ is of the first order, and in which there is only one fundamental function corresponding to each such pole. The theorem will here be established, independently of any such restrictions, by means of the results developed in § 10.

We have, for sufficiently small values of $|\lambda|$,

$$-\frac{dQ_n(\lambda)}{d\lambda} = k_n\lambda^{n-1} + k_{n+1}\lambda^n + \dots$$

$$= \lambda^{n-1}\sum_{p=0}^{\infty} \lambda^{np} k_{n+np} + \lambda^n \sum_{p=0}^{\infty} \lambda^{np} k_{n+np+1} + \dots + \lambda^{2n-2}\sum_{p=0}^{\infty} \lambda^{np} k_{2n-1+np}.$$

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* See "Remarques diverses sur l'équation de Fredholm," in the *Acta Mathematica*, Vol. 33, 1910.

It follows that the value of $-\dfrac{dQ_n(\lambda)}{d\lambda}$, for all values of λ, is given by

$$\lambda^{n-1}\int_a^b \frac{D_n\begin{pmatrix}\lambda^n & s\\ & s\end{pmatrix}}{D_n(\lambda^n)}\,ds+\lambda^n\int_a^b\int_a^b \frac{D_n\begin{pmatrix}\lambda^n & s\\ & t'\end{pmatrix}}{D_n(\lambda^n)}K_1(t', s)\,ds\,dt'$$

$$+\sum_{q=2}^{q=n-1}\lambda^{n-1+q}\int_a^b\int_a^b \frac{D_n\begin{pmatrix}\lambda^n & s\\ & t'\end{pmatrix}}{D_n(\lambda^n)}K_q(t', s)\,ds\,dt',$$

provided it be assumed that

$$\int_a^b |K_1(s, t)|\,ds,\quad \int_a^b |K_2(s, t)|\,ds,\quad \ldots,\quad \int_a^b |K_{n-1}(s, t)|\,ds$$

are all limited functions of t.

Hence we have

$$-\frac{dQ_n(\lambda)}{d\lambda}=\lambda^{n-1}\int_a^b\left[(1+\lambda V_t+\lambda^2V_t^2+\ldots+\lambda^{n-1}V_t^{n-1})\frac{D_n\begin{pmatrix}\lambda^n & s\\ & t\end{pmatrix}}{D_n(\lambda^n)}\right]_{t=s}ds.$$

In accordance with the procedure in § 9 for finding that part of $\bar{K}(s, t)$ which becomes infinite when λ^n has the value λ_1^n, a zero of $D_n(\lambda^n)$, we now see that the part of $\dfrac{dQ_n(\lambda)}{d\lambda}$ that corresponds to a canonical sub-group of $\bar{K}(s, t)$ is

$$\int_a^b\left[\frac{\phi_1(s)\,\psi_1(s)+\ldots+\phi_q(s)\,\psi_q(s)}{\lambda-\lambda_1}\right.$$

$$\left.+\frac{\beta_1\phi_1(s)\,\psi_2(s)+\ldots+\beta_{q-1}\phi_{q-1}(s)\,\psi_q(s)}{(\lambda-\lambda_1)^2}+\ldots\right]ds,$$

and this is equal to $\dfrac{q}{\lambda-\lambda_1}$. Here we have taken λ_1 to be the characteristic value for $\bar{K}(s, t)$ corresponding to the characteristic value λ_1^n for $\bar{K}_n(s, t, \lambda^n)$. A similar result would hold for a canonical sub-group for which $\omega\lambda_1$ or any of the numbers $\omega^2\lambda_1, \ldots, \omega^{n-1}\lambda_1$ is the characteristic value. The sum of all the integers q taken for all the sub-groups which belong to all the characteristic values $\lambda_1, \omega\lambda_1, \ldots, \omega^{n-1}\lambda_1$ is the sum of the orders of all the canonical sub-groups belonging to $\bar{K}_n(s, t, \lambda^n)$ for the characteristic value λ_1^n, and is therefore the degree of multiplicity of the zero λ_1^n of the function $D_n(\lambda^n)$, in accordance with Lalesco's theory.

The complete value of $\dfrac{dQ_n(\lambda)}{d\lambda}$ consists of the sum, taken for all values of λ_1, of terms of the form

$$\frac{p_1}{\lambda-\lambda_1}+\frac{p_2}{\lambda-\omega\lambda_1}+\ldots+\frac{p_n}{\lambda-\omega^{n-1}\lambda_1},$$

where $p_1, p_2, \ldots, p_n$ are integers, any of which may be zero, and such that $p_1+p_2+\ldots+p_n$ is the degree of multiplicity of the zero λ_1^n of $D_n(\lambda^n)$, together with a function which has no singularities for any finite value of λ. It follows that $e^{Q_n(\lambda)}$ is of the form

$$e^{G(\lambda)}\Pi\left(1-\frac{\lambda}{\lambda_1}\right)^{p_1}e^{p_1u(\lambda/\lambda_1)}\left(1-\frac{\lambda}{\omega\lambda_1}\right)^{p_2}e^{p_2u(\lambda/\omega\lambda_1)}\ldots\left(1-\frac{\lambda}{\omega^{n-1}\lambda_1}\right)^{p_n}e^{p_nu(\lambda/\omega^{n-1}\lambda_n)},$$

and it is therefore an integral function of λ.

To show that $e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}$ is an integral function of λ, we observe that if a_1 is the order of infinity of $\lambda-\lambda_1$ in $\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}$, there must be a canonical sub-group in the expression for $\bar{K}(s, t)$ of order a_1, and therefore $p_1 \geqslant a_1$; it follows then that λ_1 is not an infinity of $e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}$, and similarly that $\omega\lambda_1, \omega^2\lambda_1, \ldots, \omega^{n-1}\lambda_1$ are not infinities of that function. Since $e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}$ has no infinities for finite values of λ, and all its zeros are of integral order, it is an integral function of λ.

Since

$$\bar{K}(s, t) = -\left[K(s, t)+\lambda K_2(s, t)+\ldots+\lambda^{n-2}K_{n-1}(s, t)\right]$$

$$-\lambda^{n-1}\{1+\lambda V_t+\lambda^2 V_t^2+\ldots+\lambda^{n-1}V_t^{n-1}\}\frac{e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}}{e^{Q_n(\lambda)}},$$

it now follows that $\bar{K}(s, t)$ can be expressed as the quotient of two integral functions of λ. Hence we have the following theorem :—

If $K_1(s, t), K_2(s, t), \ldots, K_{n-1}(s, t)$ be unlimited, but $K_n(s, t)$ is limited in the fundamental square, and if

$$\int_a^b |K(s, t)|\, dt, \quad \int_a^b |K(s, t)|\, ds, \quad \int_a^b |K_2(s, t)|\, ds, \quad \ldots, \quad \int_a^b |K_{n-1}(s, t)|\, ds$$

are all limited functions, the resolvant $\bar{K}(s, t)$ of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is the quotient of two integral functions of λ.

12. In the ordinary case in which $K_1(s, t), K_2(s, t), \ldots$ are all limited functions, and thus $k_1, k_2, \ldots$ are all finite, we have

$$e^{-k_1\lambda - \frac{1}{2}k_2\lambda^2 - \frac{1}{3}k_3\lambda^3 - \ldots} = 1 - \lambda \int_a^b K(s_1, s_1)\, ds_1 + \frac{\lambda^2}{2!}\int_a^b \int_a^b K\begin{pmatrix} s_1, & s_2 \\ s_1, & s_2 \end{pmatrix} ds_1 ds_2 + \ldots$$
$$+ \frac{(-1)^m \lambda^m}{m!} \int_a^b \ldots \int_a^b K\begin{pmatrix} s_1, & s_2, & \ldots, & s_m \\ s_1, & s_2, & \ldots, & s_m \end{pmatrix} ds_1 ds_2 \ldots ds_m$$
$$+ \ldots .$$

Let $K_0\begin{pmatrix} s_1, & s_2, & \ldots, & s_m \\ s_1, & s_2, & \ldots, & s_m \end{pmatrix}$ denote the determinant

$$\begin{vmatrix} 0, & K(s_2, s_1), & \ldots, & K(s_m, s_1) \\ K(s_1, s_2), & 0, & \ldots, & K(s_m, s_2) \\ \ldots & \ldots & \ldots & \ldots \\ K(s_1, s_m), & \ldots & \ldots, & 0 \end{vmatrix},$$

which is obtained by putting zero for the terms $K(s_1, s_1)$, $K(s_2, s_2)$, ..., $K(s_m, s_m)$ in the diagonal of $K\begin{pmatrix} s_1, & s_2, & \ldots, & s_m \\ s_1, & s_2, & \ldots, & s_m \end{pmatrix}$.

It is easily seen that

$$\int_a^b\int_a^b\cdots\int_a^b K\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2 \dots ds_m$$

$$= \int_a^b\int_a^b\cdots\int_a^b K_0\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2 \dots ds_m$$

$$+mk_1\int_a^b\int_a^b\cdots\int_a^b K_0\begin{pmatrix}s_1, s_2, \dots, s_{m-1}\\ s_1, s_2, \dots, s_{m-1}\end{pmatrix} ds_1 ds_2 \dots ds_{m-1}$$

$$+\frac{m(m-1)}{2!}k_1^2\int_a^b\int_a^b\cdots\int_a^b K_0\begin{pmatrix}s_1, \dots, s_{m-2}\\ s_1, \dots, s_{m-2}\end{pmatrix} ds_1 ds_2 \dots ds_{m-2}$$

$$+\dots.$$

It follows that

$$1-k_1\lambda+\frac{\lambda^2}{2!}\int_a^b\int_a^b K\begin{pmatrix}s_1, s_2\\ s_1, s_2\end{pmatrix} ds_1 ds_2-\dots$$

$$+\frac{(-1)^m\lambda^m}{m!}\int_a^b\cdots\int_a^b K\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2 \dots ds_m$$

$$=\left\{1-k_1\lambda+k_1^2\frac{\lambda_1^2}{2!}-\dots+(-1)^m k_1^m\frac{\lambda^m}{m!}+\dots\right\}$$

$$\times\left\{1+\frac{\lambda^2}{2!}\int_a^b\int_a^b K_0\begin{pmatrix}s_1, s_2\\ s_1, s_2\end{pmatrix} ds_1 ds_2+\dots\right.$$

$$\left.+\frac{(-1)^m\lambda^m}{m!}\int_a^b\int_a^b\cdots K_0\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2 \dots ds_m+\dots\right\}.$$

That the second series on the right-hand side is an integral function follows from the fact that it differs from the expression for $D(\lambda)$ only in the assumption that the value zero is assigned to $K(s, t)$ when $s = t$, the function remaining unaltered otherwise. We thus see that

$$e^{-\frac{1}{2}k_2\lambda^2-\frac{1}{3}k_3\lambda^3-\dots}=1+\frac{\lambda^2}{2!}\int_a^b\int_a^b K_0\begin{pmatrix}s_1, s_2\\ s_1, s_2\end{pmatrix} ds_1 ds_2+\dots$$

$$+\frac{(-1)^m\lambda^m}{m!}\int_a^b\int_a^b\cdots\int_a^b K_0\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2 \dots ds_m$$

$$+\dots,$$

and this holds good for all values of λ. The coefficients in the series on the right-hand side involve powers of k_2, k_3, ..., but are independent of k_1.

$$\text{Thus } \int_a^b \int_a^b \dots \int_a^b K_0 \begin{pmatrix} s_1, s_2, \dots, s_m \\ s_1, s_2, \dots, s_m \end{pmatrix} ds_1 ds_2 \dots ds_m$$

$$= \Sigma \frac{1}{a!\, b!\, c! \dots} \frac{k_\alpha^a k_\beta^a \dots}{\alpha^a \beta^b \dots} (-1)^{a+b+\dots-m},$$

the summation being taken for all integral values of a, b, c, and of $\alpha, \beta, \gamma, \dots, (> 1)$, such that $a\alpha + b\beta + \dots = m$.

Let us now consider the special case of the method of § 11 which arises when $n = 2$, *i.e.*, we suppose $K(s, t)$ to be unlimited, but $K_2(s, t)$, $K_3(s, t)$, ... to be limited. It has been shewn that $e^{-\frac{1}{2}k_2\lambda^2 - \frac{1}{3}k_3\lambda^3 - \dots}$ is then an integral function. In its expression in powers of λ the coefficients involve powers of $k_2, k_3, \dots$, and these coefficients are of the same form as has been obtained above. It follows that in this case the series

$$1 + \frac{\lambda^2}{2!} \int_a^b \int_a^b K_0 \begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} ds_1 ds_2 + \dots$$

$$+ \frac{(-1)^m \lambda^m}{m!} \int_a^b \dots \int_a^b K_0 \begin{pmatrix} s_1, s_2, \dots, s_m \\ s_1, s_2, \dots, s_m \end{pmatrix} ds_1 ds_2 \dots ds_m + \dots$$

is an integral function, and it represents $e^{Q_n(\lambda)}$ the denominator in the expression for $\bar{K}(s, t)$. The numerator $-e^{Q_n(\lambda)} \dfrac{D_n\left(\lambda^n \begin{smallmatrix} s \\ t \end{smallmatrix}\right)}{D_n(\lambda)}$ has been shewn in § 11 to be an integral function.

It can be shewn in the same manner as above that the numerator in the expression for $\bar{K}(s, t)$ differs only from that in Fredholm's formula in having zero in all the diagonal terms of the coefficients, instead of $K(s_1, s_1)$, $K(s_2, s_2)$, ...

We have now established the following theorem :—

If $K(s, t)$ is unlimited, but such that $\int_a^b |K(s, t)|\, ds$, $\int_a^b |K(s, t)|\, dt$ are limited functions, and if $K_2(s, t)$ is limited, then the solution of Fredholm's equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is given by the modified form of Fredholm's expression that arises when zero is substituted for $K(s_1, s_1)$, $K(s_2, s_2)$, ... in the diagonal terms of $K\begin{pmatrix} s, s_1, \dots, s_m \\ t, s_1, \dots, s_m \end{pmatrix}$, $K\begin{pmatrix} s_1, \dots, s_m \\ s_1, \dots, s_m \end{pmatrix}$, which occur in the integrals that express the coefficients of the two integral functions. The function $f(s)$ may be *any summable function, limited or unlimited, such that* $\int_a^b K(s, t) f(t)\, dt$ *is a summable function of s.*

This theorem is a generalization of the well known theorem of Hilbert applicable to the special case

$$K(s, t) = \frac{P(s, t)}{|s-t|^{\alpha}},$$

where $\alpha < \frac{1}{2}$, and $P(s, t)$ is a limited function.

In the more general case in which the order of the first repeated function that is limited is greater than 2, the forms of the integral functions that occur in the expression for $\bar{K}(s, t)$ are of a less simple character. These forms have been investigated by Poincaré (*loc. cit.*).

13. In the case in which $K(s, t)$ is of the form $\mu(s)\nu(t) P(s, t)$, where $\mu(s)\nu(s)$ is summable in the interval (a, b), already considered in § 5, it can be shewn that Fredholm's formula is applicable to obtain the solution of the integral equation. In this case all the successive repeated nuclei are unlimited, containing $\mu(s)\nu(t)$ as factor.

We have $K\begin{pmatrix} s, & t_1, & t_2, & \ldots, & t_m \\ t, & t_1, & t_2, & \ldots, & t_m \end{pmatrix}$ equal to

$$\mu(s)\nu(t)\mu(t_1)\nu(t_1)\mu(t_2)\nu(t_2)\ldots\mu(t_m)\nu(t_m),$$

multiplied by the determinant

$$\begin{vmatrix} P(s, t), & P(s, t_1), & \ldots, & P(s, t_m) \\ P(t_1, t), & P(t_1, t_1), & \ldots, & P(t_1, t_m) \\ \ldots & \ldots & \ldots & \ldots \\ P(t_m, t), & P(t_m, t_1), & \ldots, & P(t_m, t_m) \end{vmatrix}.$$

Hence, since the numerical value of the determinant is, by Hadamard's theorem, not greater than $M^m m^{\frac{1}{2}m}$, where M is the upper limit of $|P(s, t)|$, and $\int_a^b |\mu(t_1)\nu(t_1)|\, dt_1$ has a definite value γ, the series in Fredholm's expression is of the form

$$\mu(s)\nu(t)\left(a_0 - a_1\lambda + a_2\frac{\lambda^2}{2!} - \ldots + (-1)^m a_m \frac{\lambda^m}{m!} + \ldots\right),$$

where $$|a_m| < M^m m^{\frac{1}{2}m}\gamma^m.$$

It follows that the series in the bracket is an integral function of λ. In a similar manner it can be shewn that the denominator in Fredholm's expression is also an integral function of λ; therefore Fredholm's expression is equivalent to $\mu(s)\nu(t)$ multiplied by the quotient of two integral functions. The value of $-\bar{K}(s, t)$ so determined is the analytical continuation of the expression given in § 5, and satisfies the necessary condition for being the resolvant of $K(s, t)$, as may easily be verified.

We have therefore the theorem :—

If $K(s, t)$ is of the form $\mu(s)\nu(t)P(s, t)$, where $P(s, t)$ is limited and $\mu(s)$, $\nu(t)$ are one or both unlimited, but such that $\mu(s)\nu(s)$ is summable, the solution of the integral equation is given by Fredholm's formula, in case $f(s)$ is such that $f(s)\nu(s)$ is summable in the interval (a, b).

Finally, it may be remarked that this theorem could be extended to cases in which $P(s, t)$ is unlimited, for example, to a nucleus of the form $\mu(s)\nu(t)\dfrac{Q(s, t)}{|s-t|^{a}}$, where $a < \frac{1}{2}$, and $Q(s, t)$ is limited, the formula of Fredholm being modified in the manner explained in § 12.