moment of inertia, and k the known change of moment of inertia,

$$K = k\frac{t^2}{t'^2 - t^2}.$$

Now a small error in the determination of either t or t' produces a comparatively large error in the value of K; and it is therefore of considerable importance here that there should be no approach to the previously mentioned synchronisms. It is important also here to notice the fourth source of error which has been mentioned; for it is very difficult to change the moment of inertia without imparting some shock to the wire.

L. *On the Self-induction of Wires.*—Part IV. *By* Oliver Heaviside*.

As mentioned at the close of Part III., it would appear that the only practicable way of making a workable system, which will allow us to introduce the terminal conditions that always occur in practice, in the form of linear differential equations connecting C and V, the current and potential difference at the terminals, is to abolish the very small radial component of current in the conductors. This does not involve the abolition of the radial dielectric current which produces the electric displacement, or alter the equation of continuity to which the total current in the wires is subject. The dielectric current, which is $S\dot{V}$ per unit length of line, and which must be physically continuous with the radial current in the conductors at their boundaries, may, when the latter is abolished, be imagined to be joined on to that part of the longitudinal current in the conductors that goes out of existence by some secret method with which we are not concerned.

We assume, therefore, that the propagation of magnetic induction and electric current into the conductors takes place, at any part of the line, as if it were taking place in the same manner at the same moment at all parts (as when the dielectric displacement is ignored, making it only a question of inertia and resistance), instead of its being in different stages of progress at the same moment in different parts of the line. This requires that a small fraction of its length, along which the change in C is insensible, shall be a large multiple of the radius of the wire. The current may be widely different in strength at places distant, say, a mile, and yet the variation in a few yards be so small that this section, so far as the propa-

* Communicated by the Author.

gation of magnetic induction into it is concerned, may be regarded as independent of the rest of the line; the variation of the boundary magnetic force or of C fully determining the internal state of the conductors, exactly as it would do were there no electrostatic induction.

In a copper wire, in which $\mu=1$, and $k=1/1700$, the value of the quantity $4\pi\mu kp$ is $p/135$. On the other hand, the quantity m in $-s^2=4\pi\mu kp+m^2$ has values 0, π/l, $2\pi/l$, &c., or a similar series, in which l is the length of the line in centimetres, so that $j\pi/l$ is a minute fraction, unless j be excessively large. But then it would correspond to an utterly insignificant normal system. We may therefore take

$$-s^2=4\pi\mu kp.$$

It will be as well to repeat the system that results, from Part II. The line-integral of the radial electric force across the dielectric being V, from the inner to the outer conductor (concentric tubes), and the line-integral of the magnetic force round the inner conductor being 4πC, so that C is the total current in it, accompanied by an oppositely direct current of equal strength in the outer conductor, V and C are connected by two equations, one of continuity of C, the other the equation of electric force, thus:—

$$-\frac{d\mathrm{C}}{dz}=\mathrm{S}\dot{\mathrm{V}}, \quad e-\frac{d\mathrm{V}}{dz}=\mathrm{L}_0\dot{\mathrm{C}}+\mathrm{R}_1''\mathrm{C}+\mathrm{R}_2''\mathrm{C}. \quad . \quad (141)$$

Here e is impressed force, S the electrostatic capacity, and L_0 the electromagnetic capacity, or the inductance, of the dielectric, all per unit length of line; and R_1'' and R_2'' are certain functions of d/dt and constants such that $\mathrm{R}_1''\mathrm{C}$ and $-\mathrm{R}_2''\mathrm{C}$ are the longitudinal electric forces of the field at the inner and outer boundaries of the dielectric, which, when only the first differential coefficient $d\mathrm{C}/dt$ is counted, become

$$\mathrm{R}_1''=\mathrm{R}_1+\mathrm{L}_1\frac{d}{dt}, \quad \mathrm{R}_2''=\mathrm{R}_2+\mathrm{L}_2\frac{d}{dt}$$

respectively, where R_1, L_1, and R_2, L_2 are the steady-flow resistance and inductance of the two conductors.

The forms of R_1'' and R_2'' are known when the conductors are concentric circular tubes, of which the inner may be solid, making it an ordinary round wire. Now if the return conductor be a parallel wire or tube externally placed, it is clear that we may regard R_1'' and R_2'' as known in the same manner, provided their distance apart be sufficiently great to make the departure of the distribution of current in them from symmetry insensible. We have merely to remember that it is now the inner boundary of the return tube that corresponds to the former outer boundary, *i. e.* when it surrounded the inner wire concentrically.

The quantity V will still be the line-integral of the electric force across the dielectric by any path that keeps in one plane perpendicular to the axes of the conductors, in which plane lie the lines of magnetic force. Also, the product VC will still represent the total longitudinal transfer of energy per second in the dielectric at that plane, or, in short, the energy-current. As regards the modified forms of S and L_0, there is, in strictness, some little difficulty, on account of the dielectric being necessarily bounded by other conductors than the pair under consideration, in which others energy is wasted, to a certain extent. This can only be allowed for by the equations of mutual induction of the various conductors, which are not now in question. But if our pair, for instance, be suspended alone at a uniform height above the ground, so that only the very small dissipation of energy in the earth interferes, it would seem, so far as the wire current is concerned, to be an unnecessary refinement to take the earth into consideration. There are, then, two or three practical courses open to us; as to suppose the earth to be a perfect nonconductor and behave as if it were replaced by air, or to treat it as a perfect conductor. In neither case will there be dissipation of energy except in our looped wires, which have no connection with the earth, but there will be a different estimation of the quantities L_0 and S required. For when we suppose the earth is perfectly conducting, we shut it out from the magnetic field as well as from the electric field. The electrostatic capacity S is that of the condenser formed by the two wires and intermediate dielectric, as modified by the presence of the earth (the method of images gives the formula at once), and the value of L_0 is such that $L_0S=\mu c=v^{-2}$, where v is the velocity of undissipated waves through the dielectric; that is, as before, L_0 is simply the inductance of the dielectric, per unit length of line. On the ground there will be both electrification and electric current, due to the discontinuity in the electric displacement and the magnetic force respectively; but with these we have no concern. In the other case, with extension of the magnetic and electric fields, the product L_0S still equals v^{-2}. Neither course is quite satisfactory; perhaps it would be best to sacrifice consistency and let the magnetic field extend unimpeded into the earth, considered as nonconducting, with consequently no electric current and waste of energy, whilst, as regards the external electric field, we treat it as a conductor. We must compromise in some way, unless we take the earth into account fully as an ordinary conductor. Similarly, if the line consist of a single wire whose circuit is completed through the earth, by regarding it as infinitely conducting we replace the true variably distributed return-current by a surface-current, and, terminating the magnetic field there, have $L_0S=v^{-2}$; but if we allow the magnetic field to extend into it, though with insignificant loss of energy by electric current, we shall no longer have this property.

The property is intimately connected with the influence of perfect conductivity on the state of the dielectric. For perfect conductivity will make the lines of electric force normal to the conducting boundaries, will make them cut perpendicularly the magnetic-force lines, which lie in the planes $z=$ const. and are tangential at the boundaries, and will make $L_0S=v^{-2}$, irrespective of the shape of section of the conductors. Now, at the first moment of putting on an impressed force, wires always behave as if they were infinitely conducting, so that, by the above, the initial effect is simply a dielectric disturbance, travelling along the dielectric, guided by the conductors, with velocity v, irrespective of the form of section. Of course dissipation of energy in the conductors immediately begins, and finally completely alters the state of things, which would be, in the absence of dissipation, the to-and-fro passage of a wave through the dielectric for ever. Except the extension to other than round conductors, this does not add to the knowledge already derived from their study. The effect of alternating currents in tending to become mere surface-currents as the frequency is raised (Part I.) may be derived from, or furnish itself a proof of, the property above mentioned—that at the first moment there is merely a dielectric disturbance. For in rapid alternations of impressed force, we are continually stopping the establishment of the steady state at its very commencement and substituting the establishment of a steady state of the opposite kind, to be itself immediately stopped, and so on.

When the dielectric is unbounded, not enclosed within conductors, there is also the outward propagation of disturbances to be considered; but it would appear, by general reasoning, that this is, relatively to the main effect, or propagation parallel to the wires, a secondary phenomenon.

It is clear that the same principles apply to conductors having other forms of section than circular, when V and C are made the variables, provided the functions R_1'' and R_2'' can be properly determined. The quantity VC being in all cases the energy-current, its rate of decrease as we pass along the line is accounted for (as in Part III.), thus, by making use of (141), with $e=0$,

$$-\frac{d}{dz}(\mathrm{VC})=\frac{d}{dt}(\tfrac{1}{2}\mathrm{SV}^2+\tfrac{1}{2}\mathrm{L}_0\mathrm{C}^2)+\mathrm{CR}_1''\mathrm{C}+\mathrm{CR}_2''\mathrm{C}\,; \quad (142)$$

that is, in increasing the electric and magnetic energies in the dielectric, and in transfer of energy into the conductors, to the amounts $\mathrm{CR}_1''\mathrm{C}$ and $\mathrm{CR}_2''\mathrm{C}$ per second respectively, which are, in their turn, accounted for by the rate of increase of the magnetic energy, and the dissipativity, or Joule heat per second in the two conductors; or

$$\mathrm{CR}_1''\mathrm{C}=\mathrm{Q}_1+\dot{\mathrm{T}}_1, \quad \mathrm{CR}_2''\mathrm{C}=\mathrm{Q}_2+\dot{\mathrm{T}}_2, \quad . \quad . \quad (143)$$

Q being the dissipativity and T the magnetic energy per unit length of conductor.

These equations (143) must therefore contain the enlarged definition of the meaning of the functions R_1'' and R_2''. For it is no longer true that $\mathrm{R}_1''\mathrm{C}$ is, as it was in the tubular case, the longitudinal electric force at the boundary of the conductor to which R_1'' belongs. It is a sort of mean value of the longitudinal electric force. Thus, we must have

$$\int \mathrm{EH}/4\pi\,.\,ds=\mathrm{CR}_1''\mathrm{C}, \quad . \quad . \quad . \quad . \quad (144)$$

if E be the longitudinal electric force and H the component of the magnetic force along the line of integration, which is the closed curve boundary of the section of the conductor perpendicular to its length. But no extension of the meaning of V is required from that last stated.

Let us, then, assume that R_1'' and R_2'' can be found, their actual discovery being the subject of independent investigation. We can always fall back upon round wires or tubes if required. They are functions of d/dt and constants, if the line is homogeneous. But, as we have got rid of the radial component of current in the conductors, and its difficulties, the constancy of the constants in R_1'' and R_2'' (as the conductivity and the inductivity, or the steady-flow resistance, or the diameter) need no longer be preserved. Provided the conductors may be regarded as homogeneous along any few yards of length, they may be of widely different resistances &c. at places miles apart. Then R_1'', R_2'' become functions of z as well as of d/dt, and S a function of z. Let our system be

$$-\frac{d\mathrm{C}}{dz}=\mathrm{S}''\mathrm{V}, \quad e-\frac{d\mathrm{V}}{dz}=\mathrm{R}''\mathrm{C}, \quad . \quad . \quad . \quad (145)$$

where both R'' and S'' are functions of d/dt and z. As regards S'', it is simply $\mathrm{S}(d/dt)$ when the dielectric is quite nonconducting. But when leakage is allowed for, it becomes

$K+S(d/dt)$, where K is the conductance, or reciprocal of the resistance, of the dielectric across from one conductor to the other. Then both K and S are functions of z. The conduction current is KV, and the displacement current $S\dot{V}$, whilst their sum, or $S''V$, is the true current across the dielectric per unit length of line. We have now, by (145), with $e=0$,

$$-\frac{d}{dz}(VC) = VS''V+CR''C$$

$$= KV^2+\frac{d}{dt}\tfrac{1}{2}SV^2+CR''C. \quad . \quad . \quad . \quad (146)$$

The additional quantity KV^2 is the dissipativity in the dielectric per unit length, whilst now $CR''C$ includes the whole magnetic energy increase, and the dissipativity (rate of dissipation of energy) in the conductors.

Let V_1, C_1, and V_2, C_2 be two systems satisfying (145) with $e=0$. Then

$$-\frac{d}{dz}V_1C_2=V_1S''V_2+C_2R''C_1,$$

$$-\frac{d}{dz}V_2C_1=V_2S''V_1+C_1R''C_2\,;$$

from which we see that if the systems be normal, d/dt becoming p_1 and p_2 respectively, we shall have

$$\frac{d}{dz}(V_1C_2-V_2C_1)$$

$$=(p_1-p_2)\left\{SV_1V_2-\frac{R_1''-R_2''}{p_1-p_2}C_1C_2\right\}, \quad . \quad (147)$$

R_1'' and R_2'' being what R'' becomes with p_1 and p_2 for d/dt. As the quantity in the $\{\}$ is the $U_{12}-T_{12}$ of Part III., and the first term is U_{12}, we see that the mutual magnetic energy is

$$T_{12}=C_1C_2(R_1''-R_2'')\div(p_1-p_2). \quad . \quad . \quad . \quad (148)$$

The division by p_1-p_2 can be effected, and the right member of (148) put in the form

$$C_1f(p_1)\times C_2f(p_2).$$

When this is done, we can find the mutual magnetic energy of any magnetic field (proper to our system) and a normal field, in terms of the total current in the wire and its differential coefficients with respect to t; so that, in the expansion of an arbitrary initial state, C, $\dot{C}$, $\ddot{C}$, &c., may be the data of the magnetic energy, instead of the magnetic field itself.

We see also, from (148), that if T be the magnetic energy of any normal system per unit length of line, then

$$2\mathrm{T}=\mathrm{C}^2\frac{d\mathrm{R}''}{dp}; \quad \ldots \ldots \quad (149)$$

and therefore, if Q be the dissipativity in the conductors,

$$\mathrm{Q}=\mathrm{R}''\mathrm{C}^2-\dot{\mathrm{T}}=\mathrm{C}^2\left(\mathrm{R}''-p\frac{d\mathrm{R}''}{dp}\right). \quad \ldots \quad (150)$$

Now consider the connection of the two solutions for the normal functions. Since the equation of C in general is, by (145),

$$\frac{d}{dz}\left(\frac{1}{\mathrm{S}''}\frac{d\mathrm{C}}{dz}\right)=\mathrm{R}''\mathrm{C}-e, \quad \ldots \ldots \quad (151)$$

the normal C function, say w, is to be got from

$$\frac{d}{dz}\left(\frac{1}{\mathrm{S}''}\frac{dW}{dz}\right)=\mathrm{R}''w, \quad \ldots \ldots \quad (152)$$

with $d/dt=p$ in R″ and S″, making them functions of z and p. Let X and Y be the two solutions, making

$$w=\mathrm{X}+q\mathrm{Y}, \quad \ldots \ldots \quad (153)$$

where q is a constant. The normal V function, say u, is got from w by the first of (145), giving

$$u=-\frac{1}{\mathrm{S}''}\frac{dw}{dz}=-\frac{1}{\mathrm{S}''}(\mathrm{X}'+q\mathrm{Y}'), \quad \ldots \quad (154)$$

if $\qquad \mathrm{X}'=d\mathrm{X}/dz, \qquad \mathrm{Y}'=d\mathrm{Y}/dz.$

In X and Y, which together make up the w in (153), p has the same value. Therefore, in (147), supposing C_1 to be X and C_2 to be Y, we have disappearance of the right member, making

$$\frac{d}{dz}(\mathrm{V}_1\mathrm{C}_2-\mathrm{V}_2\mathrm{C}_1)=0, \quad \text{or} \quad \mathrm{V}_1\mathrm{C}_2-\mathrm{V}_2\mathrm{C}_1=\text{constant},$$

or $$\mathrm{XY}'-\mathrm{YX}'=\mathrm{S}''\times\text{constant}=h\mathrm{S}'', \text{ say}, \quad \ldots \quad (155)$$

leading to the well-known equation

$$\mathrm{Y}=\mathrm{X}\int\frac{h\mathrm{S}''}{\mathrm{X}^2}dz,$$

connecting the two solutions of the class of equations (152); which we see expresses the reciprocity of the mutual *activities* of the two parts into which we may divide the electromagnetic state represented by a single normal solution.

Also, by (147), integrating with respect to z from 0 to l,

$$\int_0^l \mathrm{S}u_1u_2dz - \int_0^l \frac{\mathrm{R}''_1-\mathrm{R}''_2}{p_1-p_2}w_1w_2dz = \frac{\left[u_1w_2-u_2w_1\right]_0^l}{p_1-p_2}, \quad . \quad (156)$$

either member of which represents the complete $\mathrm{U}_{12}-\mathrm{T}_{12}$ of the line. The negative of this quantity, as in Part III., is the corresponding $\mathrm{U}_{12}-\mathrm{T}_{12}$ in the terminal arrangements; so that the value of $2(\mathrm{U}-\mathrm{T})$ in a complete normal system, including the apparatus, is

$$2(\mathrm{U}-\mathrm{T}) = \int_0^l \mathrm{S}u^2dz - \int_0^l \frac{d\mathrm{R}''}{dp}w^2dz - w_1^2\frac{d\mathrm{Z}_1}{dp} + w_0^2\frac{d\mathrm{Z}_0}{dp}, \quad . \quad (157)$$

if $\mathrm{V}/\mathrm{C}=\mathrm{Z}_1$ and Z_0 at $z=l$ and 0, these being functions of p and constants, and w_1, w_0 are the values of w at $z=l$ and 0. Or, which is the same,

$$2(\mathrm{U}-\mathrm{T}) = \left[w^2\frac{d}{dp}\left(\frac{u}{w}-\mathrm{Z}\right)\right]_0^l, \quad . \quad . \quad . \quad (158)$$

as before used.

There is naturally some difficulty in expressing the state at time t, thus:—

$$\mathrm{V}=\Sigma \mathrm{A}u\epsilon^{pt}, \qquad \mathrm{C}=\Sigma \mathrm{A}w\epsilon^{pt},$$

due to an arbitrary initial state, on account of the difficulty connected with

$$(\mathrm{R}_1''-\mathrm{R}_2'') \div (p_1-p_2),$$

and the unstated form of R''. But when the initial state is such as can be set up by any steadily-acting distribution of longitudinal impressed force (e an arbitrary function of z), so that whilst V is arbitrary, C is only in a very limited sense arbitrary, and $\dot{\mathrm{C}}$, $\ddot{\mathrm{C}}$, &c. are initially zero, and certain definite distributions of electric and magnetic energy in the terminal apparatus are also necessarily involved; in this case we may readily find the full solutions, and therefore also determine the effect of any distribution of e varying anyhow with the time. In fact, by the condenser method of Part III., we shall arrive at the solution (135); we have merely to employ the present u and w, and let M be the value of the right member of (158). The following establishment, however, is quite direct, and less mixed up with physical considerations.

To determine how V and C rise from zero everywhere to the final state due to a steadily-acting arbitrary distribution of e put on at the time $t=0$. Start with e_2 at $z=z_2$ and none elsewhere, and let $(X+q_0Y)A_0$ and $(X+q_1Y)A_1$ be the currents on the left (nearest $z=0$) and right sides of the seat of impressed force. We have to find q_0, q_1, A_0, and A_1. The condition $V=Z_0C$ at $z=0$ gives us, by (153), (154),

$$-(X_0'+q_0Y_0')\div S_0'' = Z_0(X_0+q_0Y_0);$$

therefore

$$q_0=-(X_0'+S_0''Z_0X_0)\div(Y_0'+S_0''Z_0Y_0). \quad . \quad . \quad (159)$$

Similarly, $V=Z_1C$ at $z=l$, gives us

$$q_1=-(X_1'+S_1''Z_1X_1)\div(Y_1'+S_1''Z_1Y_1). \quad . \quad . \quad (160)$$

Here the numbers $_0$ and $_1$ mean that the values of X, &c. and S'' at $z=0$ and at $z=l$ are to be taken.

Now, at the place $z=z_2$ the current is continuous, whilst the V rises by the amount e_2 suddenly in passing through it. These two conditions give us

$$(X_2+q_0Y_2)A_0=(X_2+q_1Y_2)A_1,$$
$$-S_2''e_2+(X_2'+q_0Y_2')A_0=(X_2'+q_1Y_2')A_1,$$

where the $_2$ means that the values at $z=z_2$ are to be taken. These determine A_0 and A_1 to be

$$A_0 \text{ or } A_1=\frac{(X_2+q_1Y_2)e_2 \text{ or } (X_2+q_0Y_2)e_2}{(S_2'')^{-1}(X_2Y_2'-Y_2X_2')(q_0-q_1)}. \quad . \quad . \quad (161)$$

Now use (155), making the denominator in (161) to be $h(q_0-q_1)$. We have then, if C_0 and C_1 are the currents on the left and right sides of the seat of impressed force,

$$\left.\begin{aligned} C_0&=\frac{(X+q_0Y)(X_2+q_1Y_2)}{h(q_0-q_1)}e_2,\\ C_1&=\frac{(X+q_1Y)(X_2+q_0Y_2)}{h(q_0-q_1)}e_2.\end{aligned}\right\} \quad . \quad . \quad . \quad (162)$$

These are, when the p is throughout treated as d/dt, the ordinary differential equations of C_0 and C_1 arising out of the partial differential equation of C by subjecting it to the terminal conditions and to the impressed force discontinuity. Now make use of the algebraical expansion

$$\frac{f(p_0)}{\phi(p_0)}=\Sigma\,\frac{f(p)}{(p_0-p)\dfrac{d\phi}{dp}}, \quad . \quad . \quad . \quad . \quad (163)$$

the summation being with respect to the p's which are the roots of $\phi(p)=0$, without inquiring too curiously into its strict applicability, or bothering about equal roots. Here p_0 has to be d/dt and the p's the roots of

$$\phi=h(q_0-q_1)=0\,;$$

so that (162) expands to

$$\mathrm{C}=\Sigma\frac{(\mathrm{X}+q\mathrm{Y})(\mathrm{X}_2+q\mathrm{Y}_2)}{h\dfrac{d}{dp}(q_0-q_1)}\frac{e_2}{\dfrac{d}{dt}-p}, \quad . \quad . \quad (164)$$

where the single q takes the place of the previous q_0 or q_1, which have now equal values, and C has the same expression on both sides of the seat of impressed force. But e_2 is constant with respect to t, whilst C is initially zero; hence

$$\frac{e_2}{d/dt-p}=\frac{e_2(1-\epsilon^{pt})}{-p},$$

which brings (164) to

$$\mathrm{C}=\Sigma\frac{(\mathrm{X}+q\mathrm{Y})(\mathrm{X}_2+q\mathrm{Y}_2)}{-p\dfrac{d\phi}{dp}}e_2(1-\epsilon^{pt}) \quad . \quad . \quad (165)$$

which is the complete solution. By integration with respect to z we find the effect due to a steady arbitrary distribution of e put on at $t=0$; thus

$$\mathrm{C}=\Sigma\frac{w\displaystyle\int_0^l ewdz}{-p\phi'}(1-\epsilon^{pt}), \quad . \quad . \quad . \quad . \quad (166)$$

where $\phi'=d\phi/dp$, and w is the normal current-function $\mathrm{X}+q\mathrm{Y}$. To express the V solution, turn the first w into u. The extension to e variable with t, as in Part III., is obvious. But as the only practical case of e variable with t is the case of periodic e, whose solution can be got immediately from the equations (162) by putting $p^2=-n^2$, constant, the extension is useless. Note that q_0 and q_1 are not equal in (162), and therefore in the periodic solution obtained from (162) direct they must be both used.

The quantity $-\phi'$ which occurs here is identical with the former complete $2(\mathrm{U}-\mathrm{T})$ of the line and terminal apparatus of (157) or (158).

Let C_0 be the finally reached steady current. By (166) it is

$$C_0=\Sigma\left(-\frac{w}{p\phi'}\right)\int_0^l ewdz. \quad . \quad . \quad . \quad . \quad (167)$$

To this apply (163), with $p_0=0$. Then a finite expression for C_0 is

$$C_0=(w_0/\phi_0)\int_0^l ew_0\,dz, \quad . \quad . \quad . \quad . \quad (168)$$

where w_0 and ϕ_0 are what w and ϕ become when $p=0$ in them. Or, rather, it would be so if q_0 and q_1 taken as identical could be consistent with $p=0$. But this is not generally true, so that (168) is wrong. To suit our present purpose, we must write, by (162),

$$C_0=\Sigma\,\frac{1}{-p\phi'}\left\{(X+q_1Y)\int_0^z e(X+q_0Y)dz+(X+q_0Y)\int_z^l e(X+q_1Y)\,dz\right\}$$

$$=\Sigma(-p\phi')^{-1}\left\{w_1\int_0^z ew_0\,dz+w_0\int_z^l ew_1\,dz\right\}; \quad . \quad . \quad . \quad (169)$$

the q_0 being used in w_0, the q_1 in w_1. Now we can take $p=0$, and get the correct formula to replace (168), viz.

$$C_0=\frac{1}{\phi_0}\left\{w_{10}\int_0^z ew_{00}\,dz+w_{00}\int_z^l ew_{10}\,dz\right\}; \quad . \quad (170)$$

the second $_0$ meaning that $p=0$ in w_0 and w_1.

If there is no leakage ($K=0$ in S''), C_0 becomes a constant, given by

$$C_0=\int_0^l edz\div\left\{\int_0^l Rdz+R_0+R_1\right\}, \quad . \quad . \quad (171)$$

where the numerator is the total impressed force, and the denominator the total steady-flow resistance; R, R_0, and R_1 being what R'', $-Z_0$, and Z_1 become when $p=0$ in them.

But when there is leakage (170) must be used; it would require a very special distribution of impressed force to make C_0 the same everywhere. To find the corresponding distribution of V, say V_0, in the steady state, we have then

$$-dC_0/dz=KV_0,$$

so that a single differentiation applied to (170) finds V_0.

Knowing thus C_0 finitely, we may write (166) thus,

$$C=C_0-\Sigma(-w/p\phi')\int_0^l ew\,dz\,.\,\epsilon^{pt}, \quad . \quad . \quad (172)$$

where C_0 is given in (170). The summation here, with $t=0$, is therefore the expansion of C_0.

The internal state of the wire is to be got by multiplying the first w by such a function of r, distance from the axis, and of whatever other variables may be necessary, as satisfies the conditions relating to inward propagation of magnetic force, and whose value at the boundary is unity. In the simple case of a round solid wire, (172) becomes, by (87), Part II.,

$$C_r=C_{0r}-\Sigma\frac{r}{a_1}\frac{J_1(s_1r)}{J_1(s_1a_1)}\frac{w\int ew\,dz}{(-p\phi')}\epsilon^{pt}. \quad . \quad . \quad (173)$$

This gives C_r the current through the circle of radius r, less than a_1 the radius of the wire, C_{0r} being the final value. The value of s_1 is $(-4\pi\mu_1k_1p)^{\frac{1}{2}}$. Here of course we give to μ_1, k_1, and a_1 their proper values for the particular value of z. As before remarked, they must only vary slowly along z.

In the case of a wire of elliptical section it is naturally suggested that the closed curves taking the place of the concentric circles defined by $r=$constant in (173) are also ellipses; and that in a wire of square section they vary between the square at the boundary and the circle at the axis. The propagation of current into a wire of rectangular section, to be considered later, may easily be investigated by means of Fourier series, at least when the return current closely envelops it.

As an explicit example of the previous, let us, to avoid introducing new functions, choose the electrical data so that the current-functions X and Y are the J_0 and K_0 functions. This can be done by letting R″ be proportional and S″ inversely proportional to the distance from one end of the line. Let there be no leakage, and

$$R''=R_0''z, \quad S=S_0z^{-1};$$

where S_0 is a constant, and R_0'' a function of d/dt, but not of z. The electromagnetic and electrostatic time-constants do not vary from one part of the line to another. The equation of the current-function is

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{dw}{dz}\right)=R_0''S_0pw; \quad . \quad . \quad . \quad (152a)$$

from which we see that

$$X=J_0(fz), \quad Y=K_0(fz),$$

where

$$f=(-R_0''S_0p)^{\frac{1}{2}}.$$

But, owing to the infinite conductivity at the $z=0$ end of the line, making $K_0(fz)=\infty$ there, we shall only be concerned with the J_0 function, that is, on the left side of the impressed force, in the first place. Since V is made permanently zero at $z=0$, the terminal condition there is nugatory. So

$$w=J_0(fz), \qquad \text{and} \quad w=J_0(fz)+q_1K_0(fz)\ ;$$
$$u=(f/Sp)J_1(fz), \quad \text{and} \quad u=(f/Sp)\{J_1(fz)+q_1K_1(fz)\}\ ;$$

on the left and right sides of an impressed force, say at $z=z_2$. The value of q_1, got from the $V=Z_1C$ condition at $z=l$, is

$$q_1=\frac{(fl/S_0p)J_1(fl)-Z_1J_0(fl)}{Z_1K_0(fl)-(fl/S_0p)K_1(fl)}. \quad . \quad . \quad . \quad (160a)$$

We have also

$$\frac{XY'-YX'}{S''p}=\frac{f}{S''p}(J_1K_0-J_0K_1)(fz)=\frac{1}{S_0p}\ ; \quad (155a)$$

and the C solution (166) becomes

$$C=\Sigma(-p\phi')^{-1}J_0(fz)\int_0^l eJ_0(fz)dz\,.\,(1-\epsilon^{pt}), \quad (166a)$$

where $\phi=-q_1/S_0p$, and q_1 is given by (160a).

If we short-circuit at $z=l$, making $Z_1=0$, we introduce peculiarities connected with the presence of the series of p's belonging to $f=0$. The expression of q_1 is then, by (160a), $q_1=-J_1(fl)/K_1(fl)$. It seems rather singular that we should have anything to do with the K_1 function, seeing that C and V are expanded in series of the J_0 and J_1 functions. But on performing the differentiation of ϕ with respect to p it turns out to be all right, the denominator in (166a) becoming

$$-p\phi'=-\tfrac{1}{2}l^2J_0^2(fl)\frac{d}{dp}(R_0''p)$$

in general; whilst in the $f=0$ case, which makes $\phi=\frac{1}{2}R_0''l^2$, we have

$$-p\phi'=-\tfrac{1}{2}pl^2\frac{dR_0''}{dp}.$$

The value of ϕ when $p=0$ in it is, by inspection of the expansions of J_1 and K_1, simply $\frac{1}{2}R_0l^2$, the steady-flow resistance of the line; R_0 being the constant that R_0'' becomes with $p=0$. We may therefore write (166a) thus:—

$$C=\int_0^l edz \div \tfrac{1}{2}R_0 l^2 - \Sigma \int_0^l edz \,.\, \epsilon^{pt} \div \left(-\tfrac{1}{2}pl^2 \frac{dR_0''}{dp}\right)$$

$$-\Sigma_f \frac{J_0(fz)\int_0^l J_0(fz)edz}{\tfrac{1}{2}l^2 J_0^2(fl)} \Sigma_p \frac{\epsilon^{pt}}{-\frac{d}{dp}(pR_0'')}, \quad . \qquad (172a)$$

where the first term is C_0, the finally reached current; the following summation, extending over the p's belonging to $f=0$, is its expansion, and therefore cancels the first term at the first moment; and the third part is a double summation, extending over all the f's except $f=0$, each f term having its following infinite series of p terms. This quantity in the second line is zero initially as well as finally. If there were no elastic displacement permitted ($S_0=0$), the solution would be represented by the first line of (172a), for we should then have C independent of z, and

$$\int_0^l edz = \int_0^l R''dz \,.\, C = \tfrac{1}{2}R_0'' l^2 \,.\, C$$

for the differential equation of C, whose solution is plainly given by the first line. The part in the second line of (172a) is therefore entirely due to the combined action of the electrostatic and electromagnetic induction.

When the impressed force is entirely at $z=l$, and of such strength as to produce the steady current C_0, and if we take $R_0''=R+Lp$, where R and L are constants, there will be only two p's to each f, given by $f^2=-S_0p(R+Lp)$. The subsidence from the steady state, on removal of the impressed force, is represented by

$$C=C_0\epsilon^{-Rt/L} - \Sigma \frac{RC_0}{R+2Lp}\frac{J_0(fz)}{J_0(fl)}\epsilon^{pt},$$

$$V=-\Sigma \frac{RC_0}{R+2Lp}\frac{J_1(fz)}{J_0(fl)}\frac{fz}{S_0p}\epsilon^{pt};$$

where the summations range over the p's, not counting the $p=-R/L$ whose C term is exhibited separately; there is no corresponding V term. A comparatively simple solution of this nature may be of course independently obtained in a more elementary manner. On the other hand, great power is gained by the use of more advanced symbolical methods, which, besides, seem to give us some view of the inner meaning of the expansions and of the operations producing them that is wanting in the treatment of a special problem on its own merits by the easiest way that presents itself.

Leaving, now, the question of variable electrical constants, let the line be homogeneous from beginning to end, so that R'' and S'' are functions of p, but not of z. The normal current-functions are then simply

$$X = \cos mz, \quad Y = \sin mz,$$

where m is the function of p given by $-m^2 = R''S''$, so that

$$\left.\begin{aligned} w &= \cos mz + q \sin mz, \\ u &= (m/S'')(\sin mz - q \cos mz). \end{aligned}\right\} \quad . \quad . \quad (174)$$

Let there be a single impressed force e_2 at $z = z_2$; then the differential equations of the currents on the left and right sides of the same, corresponding to (162), will be

$$\left.\begin{aligned} C_0 &= (\cos mz + q_0 \sin mz)\frac{\cos mz_2 + q_1 \sin mz_2}{(m/S'')(q_0 - q_1)} e_2, \\ C_1 &= (\cos mz + q_1 \sin mz)\frac{\cos mz_2 + q_0 \sin mz_2}{(m/S'')(q_0 - q_1)} e_2, \end{aligned}\right\} \quad (162b)$$

where q_0 and q_1 are given by

$$q_0 = -\frac{S''}{m} Z_0, \quad q_1 = \frac{(m/S'')\sin ml - Z_1 \cos ml}{(m/S'')\cos ml + Z_1 \sin ml}. \quad . \quad (160b)$$

As before, in the case of an arbitrary distribution of e we are led to the solution (165), wherein for w (and for u in the corresponding V formula) use the expressions (174), in which q is to be the common value of the q_0 and q_1 of (160b), and

$$\phi = (m/S'')(q_0 - q_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad (175)$$

is the determinantal equation of the p's.

Use (170) to find the final steady current distribution. Thus, now,

$$C_0 = \left\{(\cos mz + q_1 \sin mz)\int_0^z (\cos mz + q_0 \sin mz) e\,dz\right.$$

$$\left. + (\cos mz + q_0 \sin mz)\int_z^l (\cos mz + q_1 \sin mz) e\,dz\right\} \div \frac{m}{S''}(q_0 - q_1), \quad (176)$$

in which m, q_0, q_1, and S'' have the $p = 0$ values. They are, if $i = (-1)^{\frac{1}{2}}$,

$$S'' = K, \quad m = (-RK)^{\frac{1}{2}} = gi \text{ say},$$

if R is the steady-flow resistance of line (both conductors), and K is the conductance of the insulator, both per unit length of line;

$$q_0=(\mathrm{K}/m)\mathrm{R}_0=-\mathrm{KR}_0 i/g,$$

if R_0 = effective steady-flow resistance at the $z=0$ terminals, and

$$q_1=\frac{gi\sin gli-\mathrm{KR}_1\cos gli}{gi\cos gli+\mathrm{KR}_1\sin gli},$$

if R_1 = effective steady-flow resistance at the $z=l$ terminals.

The expression on the right side of (176) is, of course, real in the exponential form, and the steady distribution of V is got by

$$\mathrm{KV}_0=-d\mathrm{C}_0/dz.$$

Using the thus obtained expressions, we reach the (172) form of C solution, and the corresponding

$$\mathrm{V}=\mathrm{V}_0-\Sigma(-p\phi')^{-1}u\int_0^l\cdot ewdz\,.\,\epsilon^{pt} \quad . \quad . \quad . \quad (176\,\mathrm{A})$$

The value of ϕ' here, got by differentiation with respect to p, may be written in many ways, of which one of the most useful, for expansions in Fourier series, is the following. Let

$$w=(1+q^2)^{\frac{1}{2}}\cos(mz+\theta);$$

then

$$\frac{d\phi}{dp}=\frac{m}{\mathrm{S}''\cos^2\theta}\frac{d}{dp}\left\{\tan^{-1}\left(\frac{m}{\mathrm{S}''}\frac{\mathrm{Z}_1-\mathrm{Z}_0}{(m/\mathrm{S}'')^2+\mathrm{Z}_1\mathrm{Z}_0}\right)-ml\right\}$$

$$=\frac{l}{2\mathrm{S}''\cos^2\theta}\frac{dm^2}{dp}\left\{\cos^2 ml\frac{d}{d(ml)}\left(\frac{m}{\mathrm{S}''}\frac{\mathrm{Z}_1-\mathrm{Z}_0}{(m/\mathrm{S}'')^2+\mathrm{Z}_1\mathrm{Z}_0}\right)-1\right\}. \quad (177)$$

Corresponding to this,

$$\tan ml=\frac{m}{\mathrm{S}''}\frac{\mathrm{Z}_1-\mathrm{Z}_0}{(m/\mathrm{S}'')^2+\mathrm{Z}_1\mathrm{Z}_0} \quad . \quad . \quad . \quad (178)$$

finds the angles ml; it is got by the union of

$$\tan\theta=\mathrm{S}''\mathrm{Z}_0/m, \qquad \tan(ml+\theta)=\mathrm{S}''\mathrm{Z}_1/m, \quad . \quad . \quad (179)$$

which are equivalent to (160 b).

For example, if we take $\mathrm{R}''=\mathrm{R}$, constant, thus abolishing inertia, and $\mathrm{S}''=\mathrm{S}p$, no leakage, and S constant (R and S not containing p, that is to say), the expansion of V_0 an arbitrary function of z is

$$\mathrm{V}_0=\Sigma\sin(mz+\theta)\frac{\int_0^l\mathrm{V}_0\sin(mz+\theta)dz}{\frac{l}{2}\left\{1-\cos^2 ml\frac{d}{d(ml)}\frac{m}{\mathrm{S}p}\frac{\mathrm{Z}_1-\mathrm{Z}_0}{(m/\mathrm{S}p)^2+\mathrm{Z}_1\mathrm{Z}_0}\right\}}, \quad (180)$$

subject to (178). Here $p=-m^2/\mathrm{RS}$, so that the state of the line at time t after it was V_0, when left to itself, is got by multiplying each term in the expansion by $\epsilon^{-m^2t/\mathrm{RS}}$. The corresponding current is given by $\mathrm{RC}=-d\mathrm{V}/dz$. But the solution thus got will usually only be correct, although (180) is correct, when there is, initially, no energy in the terminal apparatus. If there be, additional terms in the numerator of (180) are required, to be found by the energy-difference method of Part III. They will not alter the value of the right member of (180) at all; they only come into effect after the subsidence has commenced. Similar remarks apply whatever be the nature of the line. It is, however, easy to arrange matters so that the energy in the terminal apparatus shall produce no effect in the line. For example, join the two conductors at one end of the line through two equal coils in parallel; if the currents in these coils be equal and similarly directed in the circuit they form by themselves, they will not, in subsiding, affect the line at all.

Returning to (177), or other equivalent expression, it is to be observed that particular attention must be paid to the roots $ml=0$, which may occur, or to the series of roots p belonging to the $m=0$ case, when we are working down from the general to the special, and happen to bring in $m=0$. Take $\mathrm{Z}_1=0$ for instance, making, by (175) and (160 b),

$$\phi=-\mathrm{Z}_0-\frac{m}{\mathrm{S}p}\tan ml,$$

where $m^2=-\mathrm{S}p\mathrm{R}''$. Then

$$\frac{d\phi}{dp}=-\frac{d\mathrm{Z}_0}{dp}-\frac{\tan ml}{2m}\left(\frac{d\mathrm{R}''}{dp}-\frac{\mathrm{R}''}{p}\right)+\frac{l}{2}\sec^2 ml\left(\frac{d\mathrm{R}''}{dp}+\frac{\mathrm{R}''}{p}\right). \quad (181)$$

Now, as long as Z_0 is finite, m cannot vanish; but when Z_0 is zero, giving $ml=$ any integral multiple of π, $m=0$ is one case. Then we have, when m is finite,

$$\frac{d\phi}{dp}=\frac{l}{2}\left(\frac{d\mathrm{R}''}{dp}+\frac{\mathrm{R}''}{p}\right),\quad\text{and}\quad p\frac{d\phi}{dp}=\frac{l}{2}\frac{d}{dp}(p\mathrm{R}''); \quad . \quad . \quad (182)$$

but when m is zero the middle term on the right of the preceding equation becomes finite, making

$$d\phi/dp=l(d\mathrm{R}''/dp).$$

The result is that the current solution contains a term, or infinite series, apparently following a different law to the rest, with no corresponding terms in the V solution. This merely means that the mean current subsides without causing any electric displacement across the dielectric, when the ends are short-circuited ($Z=0$); so that if, in the first place, the current had been steady, and there had been no displacement, there would have been none during the subsidence.

The transition from the combined inertia and elasticity solutions to elasticity alone is very curious. Thus, let $Z=0$ at both ends, and $R''=R+Lp$, where R and L are constants not containing p. The rise of current due to e is shown by

$$C=\frac{\int_0^l edz}{Rl}(1-\epsilon^{-Rt/L})+\frac{2}{l}\Sigma\frac{\cos mz\int_0^l e\cos mz\,dz}{R+2Lp}\epsilon^{pt}, \quad . \quad (183)$$

the m's in the summation being π/l, $2\pi/l$, &c.; and each having two p's, given by

$$0=m^2+RSp+LSp^2.$$

The $m=0$ part is exhibited separately, and is what the solution would be if e were a constant (owing to the constancy of R). But, whatever e be, as a function of z, the summation comes to nothing initially, on account of the doubleness of the p's, just as in (172 a) the part in the second line vanishes by reason of every p summation vanishing when $t=0$.

Now, in (183), let L be exceedingly small. The two p's approximate to $-m^2/RS$, the electrostatic one, and to $-R/L$, the electromagnetic one, which goes up to ∞, the storehouse for roots. The current then rises thus:

$$C=\frac{\int_0^l edz}{Rl}(1-\epsilon^{-Rt/L})+\frac{2}{Rl}\Sigma\cos mz\int_0^l e\cos mz\,dz\,.\,(1-\epsilon^{-Rt/L})$$
$$-\frac{2}{Rl}\Sigma\cos mz\int_0^l e\cos mz\,dz\,.\,(1-\epsilon^{-m^2t/RS}). \quad (184)$$

But the first line on the right side is equivalent to

$$(e/R)(1-\epsilon^{-Rt/L}),$$

and here the exponential term vanishes instantly, on L being made exactly zero, so that (184) becomes

$$C=\frac{e}{R}-\frac{2}{Rl}\Sigma\cos mz\int_0^l e\cos mz\,dz\,.\,(1-\epsilon^{-m^2t/RS}), \quad (185)$$

except at the very first moment, when it gives $C=e/R$, which is quite wrong, although the preceding formula, giving $C=0$ at the first moment, is correct. Or, (185) is equivalent to

$$C=\frac{1}{R}\left(e-\frac{dV}{dz}\right),$$

from which inertia has disappeared. Here V is given by (188) below. The process amounts to taking one half the terms of the summation in (183), and joining them on to the preceding term to make up e/R, which is quite arbitrary. An alternative form of (185) is

$$\mathrm{C}=\frac{\int_0^l e\,dz}{\mathrm{R}l}+\frac{2}{\mathrm{R}l}\Sigma\cos mz\int_0^l e\cos mz\,dz\,.\,\epsilon^{-m^2t/\mathrm{RS}}. \quad . \quad (186)$$

On the other hand, there is no such peculiarity connected with the V solution in the act of abolishing inertia. The $m=0$ term is

$$-\frac{1}{\mathrm{R}l}\left(\frac{m}{\mathrm{S}p}\sin mz\int_0^l edz\right)=0,$$

because m is zero and p finite. Therefore V rises thus,

$$\mathrm{V}=\frac{2}{l}\Sigma\frac{m\sin mz\int_0^l e\cos mz\,dz}{-\mathrm{S}p(\mathrm{R}+2\mathrm{L}p)}(1-\epsilon^{pt}), \quad . \quad (187)$$

before abolition of inertia. But as L is made zero, the denominator becomes m^2 for the electrostatic p, and ∞ for the other; thus one half the terms vanish, leaving

$$\mathrm{V}=\frac{2}{l}\Sigma\frac{\sin mz}{m}\int_0^l e\cos mz\,dz(1-\epsilon^{-m^2t/\mathrm{RS}}), \quad . \quad (188)$$

where $\mathrm{L}=0$, without any of the curious manipulation to which the current formula was subjected.

Next let us consider the transition from the combined elasticity and inertia solution to inertia alone (of course with resistance in both cases, as in the preceding transition). It is usual to wholly ignore electrostatic induction in investigations relating to linear circuits. This is equivalent to taking $\mathrm{S}=0$, stopping elastic displacement, and compelling the current to keep *in* the wires always, *i. e.* when the insulation is perfect, as will be here assumed. We then have, by (145),

$$-\frac{d\mathrm{C}}{dz}=0, \qquad e-\frac{d\mathrm{V}}{dz}=\mathrm{R}''\mathrm{C}. \quad . \quad . \quad . \quad (189)$$

By integrating the second of these with respect to z we get rid of V, and obtain the differential equation of C,

$$\int_0^l edz=\left\{\int_0^l \mathrm{R}''dz+\mathrm{Z}_1-\mathrm{Z}_0\right\}\mathrm{C}=\phi_1\mathrm{C}, \text{ say}, \quad . \quad (190)$$

whence follows this manner of rise of the current, when e is steady, and put on everywhere at the time $t=0$, reaching the final value C_0,

$$C=C_0-\Sigma\left(-p\frac{d\phi_1}{dp}\right)^{-1}\int_0^l edz\,.\,\epsilon^{pt}, \quad . \quad . \quad . \quad (191)$$

$\phi_1=0$ finding the p's. We can find V at distance z by integrating the second of (189) with respect to z from 0 to z; thus

$$V=\int_0^z edz+(Z_0-\int_0^z R''dz)C, \quad . \quad . \quad . \quad (192)$$

wherein C is to be the right member of (191). This finds V by differentiations with respect to t performed on C. In the final state put R_0'' for R'' and $-R_0$ for Z_0, steady-flow resistances. V will usually vary with the time until the steady state is reached; but if the line is homogeneous, with only the two constants R and L, and if also Z_0 and Z_1 are zero, V will be independent of t, and instantly assume its final distribution.

Thus, on these assumptions, we shall have

$$\left.\begin{aligned} C&=(\int_0^l edz/Rl)(1-\epsilon^{-Rt/L}),\\ V&=\int_0^z edz-(z/l)\int_0^l edz,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (193)$$

showing the current to rise independently of the distribution of e, and V to have its final distribution from the first moment, which, when the impressed force is wholly at $z=0$, of amount e_0, is $e_0(1-z/l)$. This infinitely rapid propagation of V is common sense according to the prescribed conditions, but absolute nonsense physically considered, especially in view of the transfer of energy. The question then arises, How does V really set itself up, when the line is so short that the current rises sensibly according to the electromagnetic theory?

To examine this, let the line constants be R, S, L (independent of d/dt), and $Z_1=Z_0=0$. Put on e_0 at $z=0$ at time $t=0$. V and C will rise thus (a special case of (183) and (187)),

$$\left.\begin{aligned} C&=\frac{e_0}{Rl}(1-\epsilon^{-Rt/L})+\frac{2e_0}{Rl}\epsilon^{-Rt/2L}\Sigma\frac{2}{m'}\cos mz\sin\frac{Rm't}{2L},\\ V&=e_0\left(1-\frac{z}{l}\right)-\frac{2e_0}{l}\epsilon^{-Rt/2L}\Sigma\frac{\sin mz}{m}\left(\cos+\frac{\sin}{m'}\right)\frac{Rm't}{2L},\end{aligned}\right\} \quad (194)$$

where m has the values π/l, $2\pi/l$, &c., and

$$m'=(4m^2L/R^2S-1)^{\frac{1}{2}}.$$

It is clear that when S is made to vanish, making $m' = \infty$, the current oscillations wholly vanish, reducing the C solution to the first of (193). But the V oscillations remain in full force, though of infinitely short period, and subside at a definite rate. This means that the mean value of V at any place has to be taken to represent its actual value, and this mean value is its final value. That is, if $\overline{\mathrm{V}}$ denote the mean value, about which V oscillates, we have

$$\overline{\mathrm{V}} = e_0(1 - z/l) = \mathrm{V}_0.$$

Introduce $\mathrm{LS} = v^{-2}$, where v is constant, making

$$m' = 2m\mathrm{L}v/\mathrm{R}$$

very nearly, when the line is short; then the second of (194) becomes

$$\mathrm{V} = e_0\left(1 - \frac{z}{l}\right) - \frac{2e_0}{l}\,\epsilon^{-\mathrm{R}t/2\mathrm{L}}\,\Sigma\,\frac{\sin mz}{m}\cos mvt, \quad . \quad (195)$$

which must very nearly show the subsidence of the oscillations. First ignore the subsidence factor, replacing it by unity, then (195) represents a wave of V travelling to and fro at velocity v, as thus expressed,

$$\left.\begin{array}{l} \mathrm{V} = e_0 \text{ from } z = 0 \text{ to } z = vt, \\ \mathrm{V} = 0 \text{ beyond } z = vt. \end{array}\right\} \text{ when } vt < l.$$

When $vt = l$, the whole line is charged to $\mathrm{V} = e_0$. The wave then moves back in the same manner as it advanced, so that the state of things at time $t = l/v \pm \tau$ is the same, until t reaches $2l/v$, when we have $\mathrm{V} = 0$ as at first. This would be repeated over and over again if there were no resistance, which, through the exponential factor, causes the range of the oscillations of V at any place about the final value to diminish according to the time constant $2\mathrm{L}/\mathrm{R}$. Also, the resistance has the effect of rounding off the abrupt discontinuity in the wave of V.

I have given a fuller description of this case elsewhere (Journal S. T. E. and E. vol. ix., "On Induction between Parallel Wires"), and only bring it in here in connection with the interpretation according to my present views regarding the transfer of energy. As it is clear that this oscillatory phenomenon is, primarily, a dielectric phenomenon, and only affects the conductor secondarily, it is necessary that the L in the above should not at the beginning be the full L of dielectric and wires, but only L_0, that of the dielectric, making v the velocity of undissipated waves, although as the oscillations subside the velocity must diminish, tending towards $v = (\mathrm{LS})^{-2}$, which may, however, be far from being reached, especially in the case of an iron wire. The nature of the dielectric wave is far more simply studied graphically than by means of Fourier series, on the assumption of infinite conductivity, which allows us to represent things by means of two oppositely travelling waves. To this I may return in the next Part.

I will conclude the present Part with a brief outline of the reasoning which guided me six months ago, when my brother's experiments on induction between distant circuits (mentioned in Part II.) in the north of England commenced, to the conclusion that long-distance signalling (*i. e.* hundreds of miles) was possible by induction, a conclusion which has been somewhat supported by results, so far as the experiments have yet gone. Recognizing the great complexity of the problem, and the difficulty of hitting the exact conditions, I made no special calculations but preferred to be guided by general considerations; for, in the endeavour to be precise when the data are uncertain and very variable, one is in great danger of swallowing the camel.

One may be fairly well acquainted with electromagnetism, and also with the capabilities of the telephone, and yet receive the idea of signalling by induction long distances with utter incredulity, or at least in the same way as one might accept the truth of the statement, that when one stamps one's foot the universe is shaken to its foundations. Quite true, but insensible a few yards away. The incredulity will probably be based upon the notion of rapid decrease with distance of inductive effects. This, however, leaves out of consideration an important element, namely the size of the circuits.

The coefficients of electromagnetic induction of linear circuits are proportional to their linear dimensions. If, then, we increase the size of two circuits n times, and also their distance apart n times, the mutual inductance M is increased n times. Let R_1 and R_2 be the resistances of primary and secondary. The induced current (integral) in the secondary due to starting or stopping a current C_1 in the primary is MC_1/R_2, or Me_1/R_1R_2, if e_1 be the impressed force in the primary. Now increasing the linear dimensions, and the distance, in the ratio n (with the same kind of wire) increases M, R_1, and R_2 all n times. So only e_1 remains to be increased n times to get the same secondary-current impulse. We can therefore ensure success in long-distance experiments on the basis of the success of short-distance experiments, with elements of uncertainty arising from new conditions coming into operation at the long distances.

But practically the result must be far more favourable to the long than to the short distances than the above asserts. For no one, when multiplying the distance and size of circuits, say ten times, would think of putting ten telephones in circuit to keep rigidly to the rule. Thus it may be that only a slight increase of e_1 is required, on account of M being multiplied in a far greater ratio than the resistances, or the self-inductances. Thus, it is not uncommon for the R and L of a telephone to be 100 ohms and 12 million centim. These form the principal parts of the R and L of a circuit of moderate size, and of course do not increase when we enlarge the circuit. It is therefore certain that we can signal long distances on the above basis, with a margin in favour of the long distances, which will be large or small according as the circuits are small or large.

Again, if e_1 in the primary be periodic, of frequency $n/2\pi$, the ratio of the amplitude of the current in the secondary to that in the primary will be

$$\mathrm{M}n \div (\mathrm{R}_2^2 + \mathrm{L}_2^2 n^2)^{\frac{1}{2}}.$$

Now, without any statement of the magnitude of the current in the primary, if it be largely in excess of requirements for signalling in the primary, so that $\frac{1}{100}$ part, say, would be sufficient for the purpose, then we shall have enough current in the secondary if the above ratio is only $\frac{1}{100}$. But, without going to precise formulæ, it may be easily seen that the above ratio may be made quite a considerable fraction, in comparison with $\frac{1}{100}$, with closed metallic circuits whose linear dimensions and distance are increased in the same ratio. But we should expect a rapid decrease of effect when the mean distance between the circuits exceeds their diameter, keeping the circuits unchanged. (It should be understood that squares, circles, &c. are referred to.)

The theory seems so very clear (though it is only the first approximation to the theory), that it would be matter for wonder and special inquiry if we found that we could not signal long distances by induction between closed metallic circuits, starting on the basis of a short-distance experiment, and following up the theory.

[As a matter of fact, it was found possible to speak by telephone between two circuits of $\frac{1}{4}$ mile square, $\frac{1}{2}$ mile between centres, using two bichros with the microphone.]

Now coming to metallic lines whose circuits are closed through the earth, the theory is rendered far more difficult on account of there being a conduction-current from the primary to the secondary due to the earth's imperfect conductivity. We therefore have, to say nothing of electrostatic induction, a superposition of effects due to induction and conduction, the latter being far more difficult to theoretically estimate than the former. But the reasoning regarding the electromagnetic induction is not very greatly changed, although not so favourable to long-distance signalling. If the return currents diffused themselves uniformly in all directions from the ends of the line, the same property of n-fold increase of M with n-fold lengthening of the lines and their distance would still be true. But the diffusion is one-sided only, and is even then only partial, especially when exceedingly rapid alternations of current take place. But we have the power of counterbalancing this by the multiplication of the variations of current in the primary that we can get by making and breaking the circuit, with a considerable battery-power if necessary, getting something enormous compared with the feeble variations of current in the microphonic circuit, or that can work a telephone. Electrostatic induction also comes in to assist, as it increases the activity of the battery, and therefore the current in the secondary also.

But, as regards wires connected to earth, this does not profess to be more than the very roughest reasoning, though in my opinion quite plain enough to show that we may ascribe the signalling across 40 miles of country between lines about 50 miles long mainly to induction, as we should be necessitated to do if we carried the experiment further and closed the circuits metallically by roundabout courses, for then the plain argument relating to induction will become valid. Experiments of this kind are of the greatest value from the theoretical point of view, and it is to be hoped that they will be greatly extended.

LI. *Note on the Effect of Stress and Strain on the Electrical Resistance of Carbon.* *By* HERBERT TOMLINSON, *B.A.**

PROFESSOR T. C. MENDENHALL has published in the September number of Silliman's American Journal, and also in the October number of the Philosophical Magazine, an account of some experiments on the effect of pressure on the electrical resistance of carbon. These experiments deal not only with such comparatively hard rods of carbon as are used in the arc lamp, but also with the compressed lampblack seen in Edison's disks. I will first refer to the experiments on the hard carbon. Prof. Mendenhall seems to think that these are in accordance with some experiments made by

* Communicated by the Author.