On Some Cases of Wave-Motion on Deep Water.

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This paper treats of some problems of wave-motion on deep water. For simplicity the motion is supposed restricted to two dimensions, one vertical, the other horizontal.

The first section contains a somewhat simplified demonstration of the results of Cauchy and Poisson relating to the waves due to a local initial disturbance of the surface. The method leads to one or two novel formulae.

The next problem is concerned with the effect of a periodic expansion (a « simple source ») at an internal point. From this we easily derive the wave-system due to a sudden explosive action.

Finally, I consider the waves produced by the motion of a submerged horizontal cylinder, advancing steadily at right angles to its length, through still water. A somewhat remarkable expression for the wave-making resistance experienced by the cylinder is deduced.

1. The Cauchy-Poisson Wave Problem.

Let the axis of x be taken horizontal, that of y vertically downwards, the origin being in the free surface. The component velocities of the fluid satisfy the conditions

$$u = -\frac{\partial \varphi}{\partial x}, \quad v = -\frac{\partial \varphi}{\partial y}, \tag{1}$$

with

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0. \tag{2}$$

The motion being assumed to be small, the variable part of the pressure is given by

$$\frac{p}{\rho} = \frac{\partial \varphi}{\partial t} + g y, \tag{3}$$

where ρ is the density. The surfaces of equal pressure at any instant will be, approximately, horizontal planes; and the vertical displacement upwards at any point of such a surface will be

$$\eta = \frac{1}{g} \frac{\partial \varphi}{\partial t}, \tag{4}$$

to the first order. In particular, the form of the free surface is obtained by putting $y = 0$ on the right hand of this equation.

The constancy of pressure at the free surface requires that

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} = 0, \tag{5}$$

for

$$y = \eta.$$

This gives

$$\frac{\partial^2 \varphi}{\partial t^2} - g \frac{\partial \varphi}{\partial y} = 0, \tag{6}$$

to be satisfied for $y = 0$, the terms neglected being of the second order.

The typical solution of the preceding equations, corresponding to a state of initial rest, is

$$\eta = \cos \sigma t \cos k x, \tag{7}$$

$$\varphi = g \frac{\sin \sigma t}{\sigma} e^{-ky} \cos k x, \tag{8}$$

provided

$$\sigma^2 = g k. \tag{9}$$

This represents a simple-harmonic train of standing oscillations, of wave-length $2\pi/k$.

By superposition of such trains we derive the case where the initial form of the free surface is any whatever, say

$$\eta = f(x), \tag{10}$$

for $t = 0$. In particular, if the initial elevation is confined to the immediate neighbourhood of the origin, and is such that

$$\int_{-\infty}^{\infty} f(x)\, d\, x = 1\,, \tag{11}$$

we have

$$\varphi = \frac{g}{\pi} \int_0^{\infty} \frac{\sin \sigma\, t}{\sigma}\, e^{-ky} \cos k\, x\, d\, k \tag{12}$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \sigma\, t\, e^{-\sigma^2 y/g} \cos (\sigma^2\, x/g)\, d\, \sigma. \tag{13}$$

The surface-elevation at time t is then given by (4); thus

$$\eta = \lim_{y=0} \frac{2}{\pi\, g} \int_0^{\infty} \cos \sigma\, t\, e^{-\sigma^2 y/g} \cos (\sigma^2\, x/g)\, \sigma\, d\, \sigma. \tag{14}$$

This solution was discussed at length by CAUCHY, but the investigation can be simplified in various ways. One fairly direct method is to take advantage of the principle that, owing to the symmetry with respect to the axis $O\,y$, the value of φ throughout the fluid is determined by its values along this axis. Now when $x = 0$ we have

$$\varphi = \frac{2}{\pi} \int_0^{\infty} \sin \sigma\, t\, e^{-\sigma^2 y/g}\, d\, \sigma, \tag{15}$$

or, in virtue of the formula (*),

$$\int_0^{\infty} e^{-x^2} \sin 2\, b\, x\, d\, x = e^{-b^2} \int_0^{b} e^{x^2}\, d\, x, \tag{16}$$

$$\varphi = \frac{2\, g^{\frac{1}{2}}}{\pi\, y^{\frac{1}{2}}}\, e^{-\beta^2} \int_0^{\beta} e^{z}\, d\, z \tag{17}$$

if

$$\beta^2 = g\, t^2/4\, y. \tag{18}$$

(*) This formula occurs as a subsidiary result in the familiar process of evaluating Laplace's integral

$$\int_0^{\infty} e^{-x^2} \cos 2\, b\, x\, d\, x$$

by means of a contour integration. It has of course long been known.

To find the general value of φ, we have only to write $y - i x$ for y, and take the real part, the square roots being interpreted in accordance with the requirements of continuity. In particular, to obtain the value of φ at the free surface $y = 0$, we put, for positive values of x,

$$(-i x)^{\frac{1}{2}} = x^{\frac{1}{2}} e^{-\frac{1}{4} i\pi}, \qquad \beta = \omega e^{\frac{1}{4} i\pi} \tag{19}$$

where

$$\omega = (g t^2/4 x). \tag{20}$$

Thus, from (17)

$$\left.\begin{aligned} \varphi_0 &= \frac{2 g^{\frac{1}{2}}}{\pi x^{\frac{1}{2}}} e^{\frac{1}{4} i\pi} e^{-i\omega^2} \int_0^\beta e^{z^2} d z \\ &= \frac{2 i g^{\frac{1}{2}}}{\pi x^{\frac{1}{2}}} e^{-i\omega^2} \int_0^\omega e^{i\rho^2} d\rho \end{aligned}\right\} \tag{21}$$

where ρ has been written for $z e^{\frac{1}{4} i\pi}$ The surface-elevation is then given by (4), viz. we find, taking the real part,

$$\eta = \frac{g^{\frac{1}{2}} t}{2\pi x^{\frac{1}{2}}} \int_0^\omega \cos(\rho^2 - \omega^2)\, d\rho. \tag{22}$$

This is the result obtained, in different ways, by Cauchy and Poisson. When $g t^2/4 x$ is sufficiently great, the upper limit of integration may be replaced by ∞, and we have

$$\eta = \frac{g^{\frac{1}{2}} t}{2 \pi^{\frac{1}{2}} x^{\frac{3}{2}}} \cos\left(\frac{g t^2}{4 x} - \frac{1}{4}\pi\right), \tag{23}$$

in virtue of the formula

$$\int_0^\infty \cos\rho^2\, d\rho = \int_0^\infty \sin\rho^2\, d\rho = \frac{\sqrt{\pi}}{2\sqrt{2}}. \tag{24}$$

The general value of φ can be expressed in a similar manner. We write

$$y - i\,x = r\,e^{-i\theta}, \qquad \beta = \omega\,e^{\frac{1}{2}i\theta} \tag{25}$$

where

$$\omega = \sqrt{(g\,t^2/4\,r)}. \tag{26}$$

We thus find

$$\varphi = \frac{2\,g^{\frac{1}{2}}}{\pi\,r^{\frac{1}{2}}}\,e^{i\theta}\,.\,e^{-\omega^2 e^{i\theta}}\int_0^{\omega} e^{\rho^2 e^{i\theta}}\,d\,\rho, \tag{27}$$

or, taking the real part,

$$\varphi = \frac{2\,g^{\frac{1}{2}}}{\pi\,r^{\frac{1}{2}}}\int_0^{\omega} e^{(\rho^2-\omega^2)\cos\theta}\cos\left\{(\rho^2-\omega^2)\sin\theta + \theta\right\}d\,\rho. \tag{28}$$

The interpretation of the expressions (22) and (23) has been fully dis cussed on a former occasion (*), with special reference to the later stages of the motion, when the hypothesis of an initial concentration of an elevation of finite volume on a mathematical line of the surface begins to betray its artificial character. The indefinite increase in the amplitude of the waves passing any particular point, as time goes on, which is indicated by the above formulae, is replaced by a more or less gradual extinction when the initial elevation is diffused over a band of finite breadth.

A convenient representation of a diffused initial elevation is furnished by the assumption

$$f(x) = \frac{1}{\pi}\,\frac{\alpha}{x^2+\alpha^2} = \frac{1}{\pi}\int_0^{\infty} e^{-k\alpha}\cos k\,x\,d\,k. \tag{29}$$

This admits of any degree of concentration, by diminishing the value of α, the integral amount being still given by (11). I wish to point out that this case is already covered by our formulae, provided these be suitably interpreted.

It was remarked by Poisson that the condition (5) of constancy of pressure for a moving particle is satisfied (in the case of infinite depth) throughout

(*) *Proc. Lond. Math. Soc.*, vol. 2, p. 371 (1904); *Hydrodynamics*, 3rd Ed. Art. 238.

the fluid, and not merely at the surface. This is readily seen to be the case with the typical solution (8), and its truth in the case of (28) follows by superposition, the equations being linear. Let us now fix our attention on the particles which initially lie in the plane $y = \alpha$. The formula (28) will relate to these provided

$$r = \sqrt{(x^2 + \alpha^2)}, \qquad \tan\theta = x/\alpha\,, \tag{30}$$

so that

$$\omega^2 = g\,t^2/4\,(x^2 + \alpha^2)^{\frac{1}{2}}. \tag{31}$$

The vertical displacement is given by (4), or

$$\eta = \frac{\omega}{g\,t}\,\frac{\partial\,\varphi}{\partial\,\omega}\,. \tag{32}$$

Hence, differentiating (27), and taking the real part,

$$\eta = \frac{1}{\pi\,r}\left[\cos\theta - 2\,\omega\int_0^\omega e^{(\rho^2-\omega^2)\cos\theta}\cos\left\{(\rho^2 - \omega^2)\sin\theta + 2\,\theta\right\}d\,\rho\right]. \tag{33}$$

Since the pressure is constant for the particles in question, we may imagine them to form a free surface, the fluid above being annihilated. The initial elevation at this surface is got by putting $\omega = 0$, viz. it is

$$\eta = \frac{\cos\theta}{\pi\,r} = \frac{1}{\pi}\,\frac{\alpha}{x^2 + \alpha^2}\,, \tag{34}$$

which is the law above referred to.

At points of the surface whose distance from the origin is large compared with α, θ is nearly equal to $\frac{1}{2}\pi$. Hence provided $\omega^2\cos\theta$, or $g\,t^2\,\alpha/4\,r^2$, is small, the formula (33) reduces approximately to the shape (22), which relates to the case of a concentrated elevation. Moreover, if at the same time $g\,t^2/r$ be large, we have the simplified expression (23).

When, however, as t increases, $\frac{1}{4}g\,t^2$ becomes comparable with r^2/α, the case is altered. To examine the limiting form which (33) assumes when $\omega^2\cos\theta$ is large, we write

$$2\,\omega\,e^{2i\theta}\int_0^\omega e^{(\rho^2-\omega^2)e^{i\theta}}\,d\,\rho = a\;e^{i\theta}\int_0^1 e^{-as}\,\frac{d\,s}{\sqrt{(1-s)}}\,, \tag{35}$$

where

$$a = \omega^2 e^{i\theta}. \tag{36}$$

Now

$$\left.\begin{aligned}\int_0^1 e^{-as}\frac{ds}{\sqrt{(1-s)}} &= 2 - 2a\int_0^1 e^{-as}\sqrt{(1-s)}\,ds \\ &= 2 - 2a\int_0^1 e^{-as}\left(1 - \frac{1}{2}s - \frac{1.1}{2.4}s^2 - \dots\right)ds.\end{aligned}\right\} \tag{36$_a$}$$

The series in brackets is uniformly convergent, and the integration can be effected term by term. The result may be written symbolically

$$2 - 2a\left(1 + \frac{1}{2}\frac{d}{da} - \frac{1.1}{2.4}\frac{d^2}{da^2} + \dots\right)\frac{1}{a}(1 - e^{-a}); \tag{36$_b$}$$

and the limiting form of this when a is large is

$$\frac{1}{a} + \frac{1}{2a^2} + \dots$$

Hence

$$e^{i\theta} - 2\omega e^{2i\theta}\int_0^\omega e^{(\varrho^2-\omega^2)e^{i\theta}}\,d\varrho = -\frac{1}{2\omega^2}, \tag{36$_c$}$$

ultimately. Taking the real part, we have from (33), on taking account of (31),

$$\eta = -\frac{2}{\pi g t^2}, \tag{37}$$

as the asymptotic value of the surface-elevation.

The result is independent of x; but this peculiarity appears to be special to the particular type of initial disturbance. In the general case the later stages of the disturbance are marked by the recurrence of *groups* of waves, of gradually diminishing amplitude, following one another at intervals.

2. Waves due to an Internal Source.

I propose now to investigate the surface waves due to a source of disturbance at any given depth (f) below the free surface. The source is supposed in the first instance to be periodic, its velocity-potential being

$$\varphi = -\frac{1}{2\pi}\log r \,.\, e^{i\sigma t} \tag{38}$$

where r denotes distance from the point $(0, f)$, and σ is prescribed. The time-factor $e^{i\sigma t}$ is, in the sequel, temporarily omitted. We assume, for the total disturbance produced by the source,

$$\varphi = -\frac{1}{2\pi}\log\frac{r}{r_1} + \chi, \tag{39}$$

where r_1 denotes distance from the point $(0, -f)$, and χ is to be determined. Hence, at the free surface ($y = 0$) we have $\varphi = \chi$, and

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial y} &= -\frac{f}{\pi r^2} + \frac{\partial\chi}{\partial y}\\ &= -\frac{1}{\pi}\int_0^\infty e^{-kf}\cos kx\,dk + \frac{\partial\chi}{\partial y}.\end{aligned}\right\} \tag{39$_a$}$$

The surface-condition of constancy of pressure therefore reduces to

$$\frac{\partial^2\chi}{\partial t^2} - g\frac{\partial\chi}{\partial y} = -\frac{g}{\pi}\int_0^\infty e^{-kf}\cos kx\,dk, \tag{40}$$

to be fulfilled for $y = 0$. This is satisfied by

$$\chi = \frac{g}{\pi}\int_0^\infty \frac{e^{-k(y+f)}\cos kx}{\sigma^2 - gk}\,dk, \tag{41}$$

which also satisfies (2), and fulfils the condition of zero velocity for $y = \infty$. The time-factor $e^{i\sigma t}$ is of course understood. If we put

$$\sigma^2 = g\varkappa, \tag{42}$$

so that $2\pi/\varkappa$ is the wave-length corresponding to the prescribed period $2\pi/\sigma$, we have

$$\chi = -\frac{1}{\pi}\int_0^\infty \frac{e^{-k(y+f)}\cos kx}{k-\varkappa}\,dk. \tag{43}$$

This integral is of course indeterminate, but we contemplate, for a moment, its « principal value » only. Now if x be positive, the principal value of the integral

$$\int_0^\infty \frac{e^{-k(y+f)+ikx}}{k-\varkappa}\,dk \tag{44}$$

is found by a contour integration to be

$$i\varkappa\, e^{-\varkappa(y+f)+i\varkappa x} + \int_0^\infty \frac{e^{-mx-im(y+f)}}{m-i\varkappa}\,dm. \tag{45}$$

Hence, taking the principal value in (43), we have

$$\left.\begin{aligned}\chi = \pm\, e^{-\varkappa(y+f)} \sin \varkappa x \\ -\frac{1}{\pi}\int_0^{\infty} \frac{m \cos m(y+f) - \varkappa \sin m(y+f)}{m^2+\varkappa^2}\, e^{\mp m x}\, dm,\end{aligned}\right\} \qquad (46)$$

where the upper or the lower signs are to be taken according as x is positive or negative. The last term diminishes indefinitely as x increases, and will be disregarded.

The expression (46), when multiplied by $e^{i\sigma t}$, satisfies all the mathematical conditions of the problem, but it does not fulfil the physical requirement that the surface disturbance, at a distance from the origin on either side, must consist of waves travelling outwards only. The conditions are, however, still satisfied if we superpose the value of χ corresponding to any arbitrary system of free waves; in particular we may add the term

$$i\, e^{-\varkappa(y+f)} \cos \varkappa x. \qquad (47)$$

This makes

$$\chi = i\, e^{-\varkappa(y+f)}\, e^{i(\sigma t \mp k x)} + \text{etc.}, \qquad (48)$$

and therefore

$$\eta = -\frac{\sigma}{g}\, e^{-\varkappa f}\, e^{i(\sigma t \mp k x)} + \text{etc.}, \qquad (49)$$

which is in accordance with the physical principle (*).

In real form, we may say that a simple source whose velocity potential is

$$\varphi = -\frac{1}{2\pi} \log r \,.\, \cos \sigma t \qquad (50)$$

will generate a wave-system whose form is given, at a distance from the origin, by

$$\eta = -\frac{\sigma}{g}\, e^{-\varkappa f} \cos(\sigma t \mp \varkappa x). \qquad (51)$$

Since $\sigma^2 = g\varkappa$, the amplitude of this wave-system is a maximum, for a given amplitude of the source, if $2k\varkappa = 1$; that is, when the wave-length is $4\pi f$.

(*) The somewhat artificial procedure is avoided if we introduce slight frictional forces, as in the author's *Hydrodynamics*, Arts. 240, 241.

The case where the time-variation of the source follows any arbitrary law, say

$$\varphi = -\frac{1}{2\pi} \log r \,.\, F(t), \tag{52}$$

can be derived by Fourier's theorem. Thus in the case of an instantaneous impulse about the instant $t = 0$, such that

$$\int_{-\infty}^{\infty} F(t)\, dt = 1, \tag{53}$$

we have, neglecting the terms which are of least importance when x is very large,

$$\eta = -\frac{1}{g} \int_0^{\infty} e^{-\varkappa f} \cos(\sigma t \mp \varkappa x)\, \sigma\, d\sigma \tag{54}$$

where $\varkappa = \sigma^2/g$. It would not be difficult to transform this integral into a more intelligible shape, but as we are already committed to an approximation it may be sufficient to apply the method of approximate evaluation given by Kelvin (*). For large positive values of this gives

$$\eta = -\frac{1}{4} \frac{\pi^{\frac{1}{2}} g^{\frac{1}{2}} t}{x^{\frac{3}{2}}} e^{-\frac{1}{4} g t^2 f/x^2} \cos\left(\frac{g}{4}\frac{t^2}{x} - \frac{1}{4}\pi\right). \tag{55}$$

The amplitude at any point diminishes indefinitely as t increases. It becomes insensible when the wave-length falls below (say) double the depth of the source.

If a cylinder whose radius a is small compared with f be placed horizontally in a liquid at a depth f, and made to oscillate horizontally at right angles to its length, with the velocity

$$U \cos \sigma t,$$

the velocity-potential in the immediate neighbourhood is given by

$$\varphi = U a^2 \frac{\partial}{\partial x} \log r \,.\, \cos \sigma t, \tag{56}$$

(*) *Phil. Mag.* (6), vol. 23, 1887; *Papers*, vol. 4, p. 303. See also Rayleigh, *Phil. Mag.* (6), vol. 21, 1911. The method had been employed by Stokes (1850); see his *Papers*, vol. 2, p. 341.

approximately. It is evident that the results appropriate to this case will be obtained from the foregoing by differentiating with respect to x. Thus, for the surface elevation, we find from (51)

$$\eta = \frac{2\pi U a^2}{g} k \sigma e^{-\varkappa f} \cos(\sigma t - \varkappa x), \tag{57}$$

for large positive values of x. From this we could derive the case where the cylinder receives a sudden shift parallel to x, of small amount.

3. Waves due to the Motion of a Submerged Cylinder.

Finally, we may consider the disturbance produced in the flow of a uniform stream by a submerged cylindrical obstacle of small radius. The cylinder is supposed placed horizontally at right angles to the stream. This problem could be deduced from the one last-mentioned, the relative motion being the same if the cylinder be supposed to advance through the fluid, which is otherwise at rest. As independent treatment is however, here adopted, the cylinder being assumed to be at rest, and the motion « steady ».

Let c denote the general velocity of the stream in the direction of x-positive, and let us write

$$\varphi = -c x \left(1 + \frac{a^2}{r^2}\right) + \chi, \tag{58}$$

$$\chi = \int_0^\infty e^{-ky} \sin kx \, \alpha(k) \, dk, \tag{59}$$

where

$$r = \sqrt{\left\{ x^2 + (y - f)^2 \right\}}, \tag{60}$$

the origin being in the free surface as before, and f denoting the depth of the cylinder. The function $\alpha(k)$ is yet to be determined. For the equation of the steady free surface we assume

$$\eta = \int_0^\infty \cos kx \, \beta(k) \, dk, \tag{61}$$

where η denotes elevation *above* the mean level.

The geometrical condition at the free surface is

$$\frac{\partial \varphi}{\partial y} = c \frac{d \eta}{d x}, \tag{62}$$

to be satisfied for $y = 0$. Since (58) may be written in the form

$$\varphi = -c x - a^2 c \int_0^\infty e^{k(y-f)} \sin k x \, d k + \chi, \tag{63}$$

this condition gives

$$k a^2 c e^{-kf} + k \alpha(k) = k c \beta(k). \tag{64}$$

Again, we have, at the free surface,

$$\left.\begin{aligned}\frac{p}{\rho} &= g y - \frac{1}{2}(\text{vel.})^2 = g y - \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 \\ &= g y + k^2 a^2 c^2 \int_0^\infty e^{k(y-f)} \cos k x \, d k - c \frac{\partial \chi}{\partial x} + \text{const.} \\ &= g \eta + k^2 a^2 c^2 \int_0^\infty e^{-kf} \cos k x \, d k - k c \int_0^\infty \cos k x \, \alpha(k) \, d k + \text{const.}\end{aligned}\right\} \tag{65}$$

This will be independent of x, provided

$$g \beta(k) + k^2 a^2 c^2 e^{-kf} - k c \alpha(k) = 0. \tag{66}$$

Combining this with (64) we find

$$\beta(k) = \frac{2 k a^2 c^2 e^{-kf}}{k c^2 - g}, \tag{67}$$

or, if we put

$$\varkappa = g/c^2, \tag{68}$$

$$\beta(k) = 2 a^2 \frac{k e^{-kf}}{k - \varkappa}. \tag{69}$$

Hence

$$\left.\begin{aligned}\eta &= 2 a^2 \int_0^\infty \frac{k e^{-kf} \cos k x}{k - \varkappa} d k \\ &= 2 a^2 \int_0^\infty e^{-kf} \cos k x \, d k + 2 \varkappa a^2 \int_0^\infty \frac{e^{-kf} \cos k x}{k - \varkappa} d k \\ &= \frac{2 a^2 f}{x^2 + f^2} + 2 \varkappa a^2 \int_0^\infty \frac{e^{-kf} \cos k x}{k - \varkappa} d k.\end{aligned}\right\} \tag{70}$$

Now, if $x > 0$, the principal value of the integral

$$\int_0^\infty \frac{e^{-kf+ikx}}{k-\varkappa}\, d\,k \tag{71}$$

is

$$-\,i\,\pi\,e^{-\varkappa f+i\varkappa x} - i\int_0^\infty \frac{e^{-imf-mx}}{i\,m-\varkappa}\, d\,m. \tag{72}$$

Hence, taking the principal value in (70), we find

$$\left.\begin{aligned}\eta = \frac{2\,a^2 f}{x^2+f^2} + 2\,\pi\,\varkappa\,a^2\,e^{-\varkappa f}\sin\varkappa\,x \\ + \int_0^\infty \frac{(\varkappa \sin m\,f - m\cos m\,f)\,e^{-mx}}{m^2+\varkappa^2}\, d\,m.\end{aligned}\right\} \tag{73}$$

The last term becomes insensible for large values of x. Since the expression in (70) is an even function of x, we must have, for large negative values of x,

$$\eta = \frac{2\,a^2 f}{x^2+f^2} - 2\,\pi\,\varkappa\,a^2\,e^{-\varkappa f}\sin\varkappa\,x + \text{ etc.} \tag{74}$$

On the disturbance represented by these formulae, we can superpose any system of waves of length $2\,\pi/\varkappa$, since these could maintain their position in space, in spite of the motion of the stream; and if we choose as our additional system

$$\eta = 2\,\pi\,\varkappa\,a^2\,e^{-\varkappa f}\sin\varkappa\,x, \tag{75}$$

we obtain, finally,

$$\eta = \frac{2\,a^2 f}{x^2+f^2} + 4\,\pi\,\varkappa\,a^2\,e^{-\varkappa f}\sin\varkappa\,x + \text{ etc.}, \tag{76}$$

on the downstream side ($x > 0$), and

$$\eta = \frac{2\,a^2 f}{x^2+f^2} + \text{ etc.} \tag{77}$$

on the upstream side ($x < 0$). We have now annulled the disturbance, at a distance, on the upstream side, as is required for a physical solution (*).

(*) As in the former problem, the procedure is improved, at the cost of some increased complexity in the formulae, if we introduce frictional forces.

As this is one of the few cases where the « wave-making » resistance can be calculated, it may be worth while to give the result. The mean energy, per unit area of the water-surface, of the waves represented by the second term in (76) is (*)

$$\frac{1}{4} g \rho + (4 \pi \varkappa a^2 e^{-\varkappa f})^2;$$

and the addition made to the area occupied by the waves, in unit time, is c, per unit length of the cylinder. Again, if R be the resistance experienced by the cylinder, the additional wave-energy is $R c$ per unit length. Hence

$$R = 4 \pi^2 g \rho a^4 . \varkappa^2 e^{-2\varkappa f}. \tag{78}$$

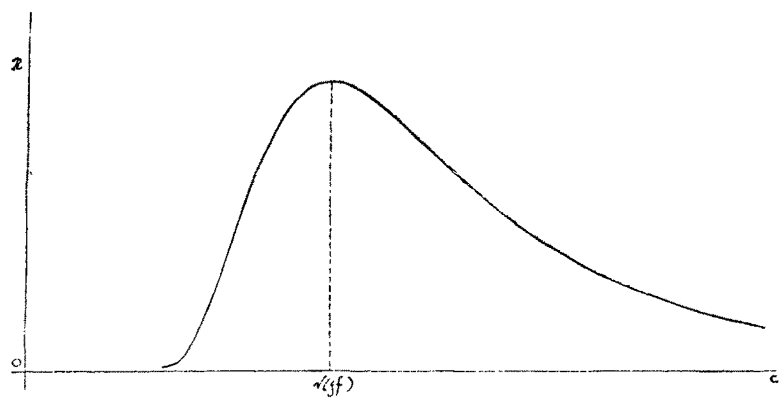
For a given depth of immersion (f), this is greatest when $\varkappa f = 1$, or

$$c = \sqrt{(g f)}. \tag{79}$$

The formula (78) also gives, of course, the resistance when the cylinder advances with the velocity c through still water. In terms of this velocity we have

$$R = 4 \pi^2 g^3 \rho a^4 . c^{-4} e^{-2gf/c^2}. \tag{80}$$

The graph of this function of the velocity is appended.



Manchester, November 18th, 1912.

(*) *Hydrodynamics*, Art. 229.