XXXV.—*On the Theory of Rolling Curves.* By Mr JAMES CLERK MAXWELL. Communicated by the Rev. Professor KELLAND.

(Read, 19th February 1849.)

There is an important geometrical problem which proposes to find a curve having a given relation to a series of curves described according to a given law. This is the problem of Trajectories in its general form.

The series of curves is obtained from the general equation to a curve by the variation of its parameters. In the general case, this variation may change the form of the curve, but, in the case which we are about to consider, the curve is changed only in position.

This change of position takes place partly by rotation, and partly by transference through space. The rolling of one curve on another is an example of this compound motion.

As examples of the way in which the new curve may be related to the series of curves, we may take the following:—

1. The new curve may cut the series of curves at a given angle. When this angle becomes zero, the curve is the envelope of the series of curves.

2. It may pass through corresponding points in the series of curves. There are many other relations which may be imagined, but we shall confine our attention to this, partly because it affords the means of tracing various curves, and partly on account of the connection which it has with many geometrical problems.

Therefore the subject of this paper will be the consideration of the relations of three curves, one of which is fixed, while the second rolls upon it and traces the third. The subject of rolling curves is by no means a new one. The first idea of the cycloid is attributed to ARISTOTLE, and involutes and evolutes have been long known.

In the "History of the Royal Academy of Sciences" for 1704, page 97, there is a memoir entitled "Nouvelle formation des Spirales," by M. VARIGNON, in which he shews how to construct a polar curve from a curve referred to rectangular co-ordinates by substituting the radius vector for the abscissa, and a circular arc for the ordinate. After each curve, he gives the curve into which it is "unrolled," by which he means the curve which the spiral must be rolled upon in order that its pole may trace a straight line; but as this is not the principal subject of his paper, he does not discuss it very fully.

There is also a memoir by M. DE LA HIRE, in the volume for 1706, Part II., page 489, entitled,—"Methode generale pour réduire toutes les Lignes courbes à des Roulettes, leur generatrice ou leur base étant donnée telle qu'on voudra."

M. DE LA HIRE treats curves as if they were polygons, and gives geometrical constructions for finding the fixed curve or the rolling curve, the other two being given; but he does not work any examples.

In the volume for 1707, page 79, there is a paper entitled,—"Methode generale pour déterminer la nature des Courbes formées par le roulement de toutes sortes de Courbes sur une autre Courbe quelconque." Par M. NICOLE.

M. NICOLE takes the equations of the three curves referred to rectangular co-ordinates, and finds three general equations to connect them. He takes the tracing-point either at the origin of the co-ordinates of the rolled curve or not. He then shews how these equations may be simplified in several particular cases. These cases are,—

1st, When the tracing-point is the origin of the rolled curve.
2d, When the fixed curve is the same as the rolling curve.
3d, When both of these conditions are satisfied.
4th, When the fixed line is straight.

He then says, that if we roll a geometric curve on itself, we obtain a new geometric curve, and that we may thus obtain an infinite number of geometric curves.

The examples which he gives of the application of his method are all taken from the cycloid and epicycloid, except one which relates to a parabola, rolling on itself, and tracing a cissoid with its vertex. The reason of so small a number of examples being worked may be, that it is not easy to eliminate the co-ordinates of the fixed and rolling curves from his equations.

The case in which one curve rolling on another produces a circle is treated of in WILLIS's *Principles of Mechanism.* Class C. *Rolling Contact.*

He employs the same method of finding the one curve from the other which is used here, and he attributes it to EULER (see the *Acta Petropolitana,* vol. v.).

Thus, nearly all the simple cases have been treated of by different authors; but the subject is still far from being exhausted, for the equations have been applied to very few curves, and we may easily obtain new and elegant properties from any curve we please.

Almost all the more notable curves may be thus linked together in a great variety of ways, so that there are scarcely two curves, however dissimilar, between which we cannot form a chain of connected curves.

This will appear in the list of examples given at the end of this paper.

Let there be a curve KAS, whose pole is at C.

Let the angle DCA$=\theta_1$ and CA$=r_1$ and let

$$\theta_1=\varphi_1\,(r_1).$$

Let this curve remain fixed to the paper.

Let there be another curve BAT, whose pole is B.

Let the angle MBA$=\theta_2$, and BA$=r_2$, and let

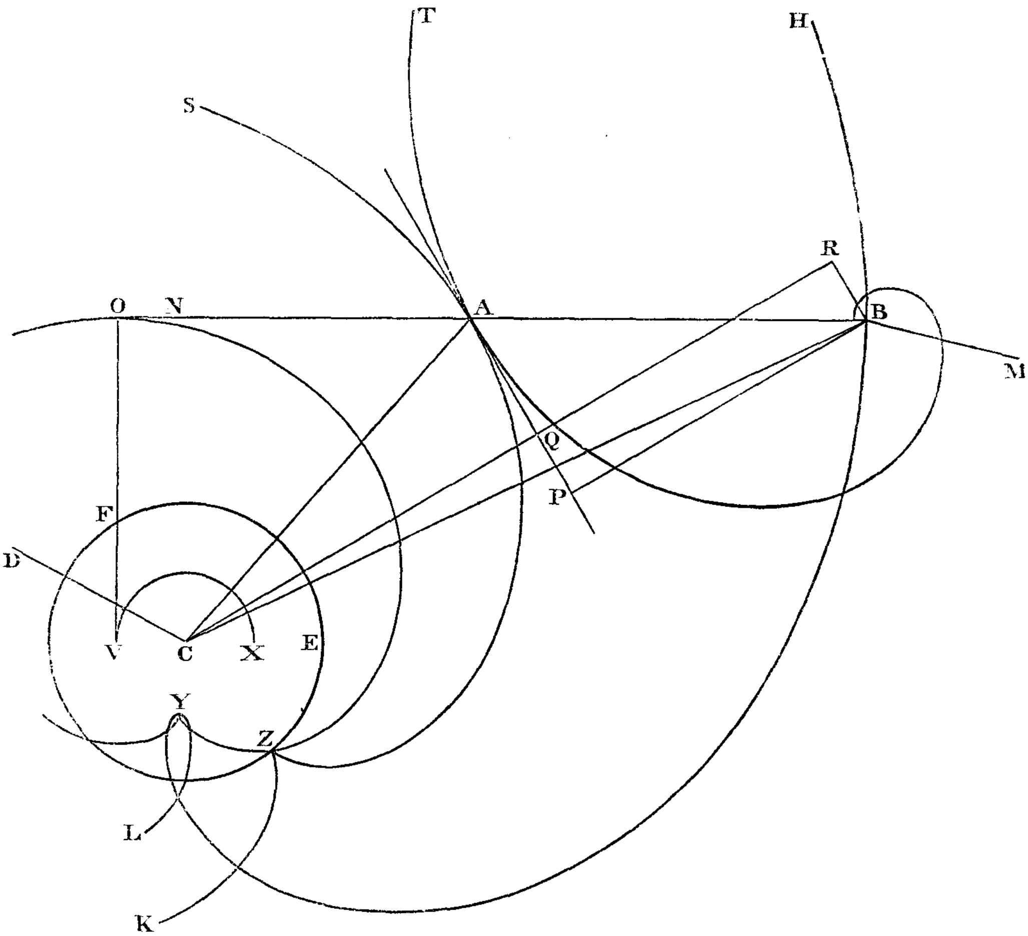
$$\theta_2=\varphi_2\,(r_2).$$

Let this curve roll along the curve KAS without slipping.

Then the pole B will describe a third curve, whose pole is C.

Let the angle DCB$=\theta_3$, and CB$=r_3$, and let

$$\theta_3=\varphi_3\ (r_3).$$



We have here six unknown quantities, $\theta_1\ \theta_2\ \theta_3\ r_1\ r_2\ r_3$; but we have only three equations given to connect them, therefore the other three must be sought for in the enunciation.

But before proceeding to the investigation of these three equations, we must premise that the three curves will be denominated as follows:—

The Fixed Curve, Equation, $\theta_1=\varphi_1\ (r_1)$
The Rolled Curve, Equation, $\theta_2=\varphi_2\ (r_2)$
The Traced Curve, Equation, $\theta_3=\varphi_3\ (r_3)$

When it is more convenient to make use of equations between rectangular co-ordinates, we shall use the letters $x_1\ y_1$, $x_2\ y_2$, $x_3\ y_3$. We shall always employ the letters s_1 s_2 s_3 to denote the length of the curve from the pole, p_1 p_2 p_3 for the perpendiculars from the pole on the tangent, and q_1 q_2 q_3 for the intercepted part of the tangent.

Between these quantities, we have the following equations :—

$$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$s = \int\sqrt{r^2+\left(\frac{d\,r}{d\,\theta}\right)^2}\,d\,\theta \qquad s = \int\sqrt{1+\left(\frac{d\,y}{d\,x}\right)^2}\,d\,x$$

$$p = \frac{r^2}{\sqrt{r^2+\left(\frac{d\,r}{d\,\theta}\right)^2}} \qquad p = \frac{y\,d\,x - x\,d\,y}{\sqrt{(d\,x)^2+(d\,y)^2}}$$

$$q = \frac{\frac{r\,d\,r}{d\,\theta}}{\sqrt{r^2+\left(\frac{d\,r}{d\,\theta}\right)^2}} \qquad q = \frac{x\,d\,x + y\,d\,y}{\sqrt{(d\,x)^2+(d\,y)^2}}$$

$$\mathrm{R} = \frac{\left(r^2+\left(\frac{d\,r}{d\,\theta}\right)^2\right)^{\frac{3}{2}}}{r^2+2\left(\frac{d\,r}{d\,\theta}\right)^2 - r\frac{d^2\,r}{d\,\theta^2}} \qquad \mathrm{R} = \frac{\left(1+\left(\frac{d\,y}{d\,x}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2\,y}{d\,x^2}}$$

We come now to consider the three equations of rolling which are involved in the enunciation. Since the second curve rolls upon the first *without slipping*, the length of the fixed curve at the point of contact is the measure of the length of the rolled curve, therefore we have the following equation to connect the fixed curve and the rolled curve,—

$$s_1 = s_2$$

Now, by combining this equation with the two equations

$$\left\{\begin{matrix}\theta_1 = \varphi_1\,(r_1)\\ \theta_2 = \varphi_2\,(r_2)\end{matrix}\right\} \text{ or } \left\{\begin{matrix}x_1 = \psi_1\,(y_1)\\ x_2 = \psi_2\,(y_2)\end{matrix}\right\},$$

it is evident that from any of the four quantities θ_1 r_1 θ_2 r_2 or x_1 y_1 x_2 y_2, we can obtain the other three, therefore we may consider these quantities as known functions of each other.

Since the curve *rolls* on the fixed curve, they must have a common tangent.

Let PA be this tangent, draw BP, CQ perpendicular to PA, produce CQ, and draw BR perpendicular to it, then we have CA$=r_1$, BA$=r_2$, and CB$=r_3$; CQ$=p_1$, PB$=p_2$, and BN$=p_3$; AQ$=q_1$, AP$=q_2$ and CN$=q_3$.

Also,
$$r_3^2 = CB^2 = CR^2 + RB^2 = (CQ + PB)^2 + (AP - AQ)^2$$
$$= (p_1 + p_2)^2 + (q_2 - q_1)^2$$
$$= p_1^2 + 2\,p_1\,p_2 + p_2^2 + r_2^2 - p_2^2 - 2\,q_1\,q_2 + r^2{}_1 - p^2{}_1$$
$$r_3^2 = r_1^2 + r_2^2 + 2\,p_1\,p_2 - 2\,q_1\,q_2$$

Since the first curve is fixed to the paper, we may find the angle θ_3

Thus
$$\theta_3 = DCB = DCA + ACQ + RCB$$
$$= \theta_1 + \tan^{-1}\frac{q_1}{p_1} + \tan^{-1}\frac{RB}{RC}$$
$$\theta_3 = \theta_1 + \tan^{-1}\frac{d\,r_1}{r_1\,d\,\theta_1} + \tan^{-1}\frac{q_2-q_1}{p_2+p_1}$$

Thus we have found three independent equations, which, together with the equations of the curves, make up six equations, of which each may be deduced from the others. There is an equation connecting the radii of curvature of the three curves which is sometimes of use.

The angle through which the rolled curve revolves during the description of the element $d\,s_3$, is equal to the angle of contact of the fixed curve and the rolling curve, or to the sum of their curvatures,

$$\therefore\ \frac{d\,s_3}{r_2} = \frac{d\,s_1}{R_1} + \frac{d\,s_2}{R_2}$$

But the radius of the rolled curve has revolved in the opposite direction through an angle equal to $d\,\theta_2$, therefore the angle between two successive positions of r_2 is equal to $\frac{d\,s_3}{r_2} - d\,\theta_2$. Now this angle is the angle between two successive positions of the normal to the traced curve, therefore, if O be the centre of curvature of the traced curve, it is the angle which $d\,s_3$ or $d\,s_1$ subtends at O. Let OA=T, then

$$\frac{d\,s_3}{R_3} = \frac{r_2\,d\,\theta_2}{T} = \frac{d\,s_3}{r_2} - d\,\theta_2 = \frac{d\,s_2}{R_1} + \frac{d\,s_2}{R_2} - d\,\theta_2$$
$$\therefore\ r^2\frac{d\,\theta_2}{d\,s_2}\,\frac{1}{T} = \frac{1}{R_1} + \frac{1}{R_2} - r_2\left(\frac{d\,\theta_2}{d\,s_2}\right)^2$$
$$\therefore\ \frac{p_2}{r_2}\left(\frac{1}{T} + \frac{1}{r_2}\right) = \frac{1}{R_1} + \frac{1}{R_2}$$

As an example of the use of this equation, we may examine a property of the logarithmic spiral.

In this curve, $p = m\,r$, and $R = \frac{r}{m}$, therefore if the rolled curve be the logarithmic spiral

$$m\left(\frac{1}{T}+\frac{1}{r_2}\right)=\frac{1}{R_1}+\frac{m}{r_2}$$

$$\frac{m}{T}=\frac{1}{R_1}$$

therefore AO in the figure $= m\ R_1$, and $\frac{AO}{R_1} = m$.

Let the locus of O, or the evolute of the traced curve LYBH, be the curve OZY, and let the evolute of the fixed curve KZAS be FEZ, and let us consider FEZ as the fixed curve, and OZY as the traced curve.

Then in the triangles BPA, AOF, we have OAF = PBA, and $\frac{OA}{AF} = m = \frac{BP}{AB}$, therefore the triangles are similar, and FOA = APB $= \frac{\pi}{2}$, therefore OF is perpendicular to OA, the tangent to the curve OZY, therefore OF is the radius of the curve which when rolled on FEZ traces OZY, and the angle which the curve makes with this radius is OFA = PAB $= \sin^{-1} m$, which is constant, therefore the curve, which, when rolled on FEZ, traces OZY, is the logarithmic spiral. Thus we have proved the following proposition: "The involute of the curve traced by the pole of a logarithmic spiral which rolls upon any curve, is the curve traced by the pole of the same logarithmic spiral when rolled on the involute of the primary curve."

It follows from this, that if we roll on any curve a curve having the property $p_1 = m_1\ r_1$, and roll another curve having $p_2 = m_2\ r_2$ on the curve traced, and so on, it is immaterial in what order we roll these curves. Thus, if we roll a logarithmic spiral, in which $p = m\ r$, on the nth involute of a circle whose radius is a, the curve traced is the $n+1$th involute of a circle whose radius is $\sqrt{1-m^2}$.

Or, if we roll successively m logarithmic spirals, the resulting curve is the $n+m$th involute of a circle, whose radius is

$$a\sqrt{1-m_1^2}\sqrt{1-m_2^2}\sqrt{\text{etc.}}$$

We now proceed to the cases in which the solution of the problem may be simplified. This simplification is generally effected by the consideration that the radius vector of the rolled curve is the normal drawn from the traced curve to the fixed curve.

In the case in which the curve is rolled on a straight line, the perpendicular on the tangent of the rolled curve is the distance of the tracing point from the straight line; therefore, if the traced curve be defined by an equation in x_3 and y_3.

$$x_3 = p_2 = \frac{r_2^2}{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}} \quad . \quad . \quad . \qquad (1.)$$

and $$r_2 = x\sqrt{\left(\frac{d\,x}{d\,y}\right)^2 + 1} \quad . \quad . \quad . \quad . \quad (2.)$$

By substituting for r_2 in the first equation, its value, as derived from the second, we obtain

$$x_3^2 \left(\frac{d\,x_3}{d\,y_3}\right)^2 \left[\left(\frac{d\,x_3}{d\,y_3}\right)^2 + 1\right] = \left(\frac{d\,r_2}{d\,\theta_2}\right)^2$$

If we know the equation to the rolled curve, we may find $\left(\frac{d\,r_2}{d\,\theta_2}\right)^2$ in terms of r_2, then by substituting for r_2 its value in the second equation, we have an equation containing x_3 and $\frac{d\,x_3}{d\,y_3}$, from which we find the value of $\frac{d\,x_3}{d\,y_3}$ in terms of x_3, the integration of this gives the equation of the traced curve.

As an example, we may find the curve traced by the pole of a hyperbolic spiral which rolls on a straight line.

The equation of the rolled curve is $\theta_2 = \frac{a}{r_2}$

$$\therefore \quad \left(\frac{d\,r_2}{d\,\theta_2}\right)^2 = \frac{r_2^4}{a^2}$$

$$= x_3^2 \left(\frac{d\,x_3}{d\,y_3}\right)^2 \left[\left(\frac{d\,x_3}{d\,y_3}\right)^2 + 1\right] = \frac{x_3^4}{a^2}\left[\left(\frac{d\,x_3}{d\,y_3}\right)^2 + 1\right]^2$$

$$\therefore \quad a^2\left(\frac{d\,x_3}{d\,y_3}\right)^2 = x_3^2\left[\left(\frac{d\,x_3}{d\,y_3}\right)^2 + 1\right]$$

$$\therefore \quad \frac{d\,x_3}{d\,y_3} = \frac{x_3}{\sqrt{a^2 - x_3^2}}$$

This is the differential equation of the tractory of the straight line, which is the curve traced by the pole of the hyperbolic spiral.

By eliminating x_3 in the two equations, we obtain

$$\frac{d\,r_2}{d\,\theta_2} = r_2\left(\frac{d\,x_3}{d\,y_3}\right)$$

This equation serves to determine the rolled curve when the traced curve is given.

As an example we shall find the curve, which being rolled on a straight line, traces a common catenary.

Let the equation to the catenary be

$$x = \frac{a}{2}\left(e^{\frac{a}{y}} + e^{-\frac{a}{y}}\right)$$

Then $$\frac{d\,x_3}{d\,y_3} = \sqrt{\frac{x_3^2}{a^2} - 1}$$

$$\therefore \quad \left(\frac{d\,r_2}{d\,\theta_2}\right)^2 = \frac{r_2^{\,2}}{a^2}\,\frac{r^4}{\left(\frac{d\,r_2}{d\,\theta_2}\right)^2 + r_2^{\,2}} - r_2^{\,2}$$

$$\therefore \quad \left[\left(\frac{d\,r_2}{d\,\theta_2}\right)^2 + r_2^{\,2}\right]^2 = \left(\frac{r_2^{\,2}}{a}\right)$$

$$\therefore \quad \left(\frac{d\,r_2}{d\,\theta_2}\right)^2 = \frac{r_2^{\,2}}{a}\,(r-a)$$

$$\therefore \quad \frac{d\,\theta}{d\,r} = \frac{1}{r\sqrt{\frac{r}{a}-1}} \quad \text{then by integration}$$

$$\theta = \cos^{-1}\left(\frac{2\,a}{r} - 1\right)$$

$$r = \frac{2\,a}{1 + \cos\theta}$$

This is the polar equation of the parabola, the focus being the pole, therefore, if we roll a parabola on a straight line, its focus will trace a catenary.

The rectangular equation of this parabola is $x^2 = 4\,a\,y$, and we shall now consider what curve must be rolled along the axis of y to trace the parabola.

By the second equation (2.),

$$r_2 = x_3\sqrt{\frac{4\,a^2}{x_3^{\,2}} + 1} \qquad \text{but } x_3 = p_2$$

$$\therefore \quad r_2 = \sqrt{4\,a^2 + p_2^{\,2}}$$

$$\therefore \quad r_2^{\,2} = p_2^{\,2} = 4\,a^2$$

$$\therefore \quad 2\,a = \sqrt{r_2^{\,2} - p_2^{\,2}} = q_2$$

but q_2 is the perpendicular on the normal, therefore the normal to the curve always touches a circle whose radius is Q a, therefore the curve is the involute of this circle.

Therefore we have the following method of describing a catenary by continued motion.

Describe a circle whose radius is twice the parameter of the catenary ; roll a straight line on this circle, then any point in the line will describe an involute of the circle ; roll this curve on a straight line, and the centre of the circle will describe a parabola ; roll this parabola on a straight line, and its focus will trace the catenary required.

We come now to the case in which a straight line rolls on a curve.

When the tracing-point is in the straight line, the problem becomes that of involutes and evolutes, which we need not enter upon, and when the tracing-point is not in the straight line, the calculation is somewhat complex, we shall therefore consider only the relations between the curves described in the first and second cases.

Definition.—The curve which cuts at a given angle all the circles of a given radius whose centres are in a given curve, is called a tractory of the given curve.

Let a straight line roll on a curve A, and let a point in the straight line describe a curve B, and let another point, whose distance from the first point is b, and from the straight line a, describe a curve C, then it is evident that the curve B cuts the circle whose centre is in C, and whose radius is b, at an angle whose sine is equal to $\frac{a}{b}$, therefore the curve B is a tractory of the curve C.

When $a = b$, the curve B is the orthogonal tractory of the curve C. If tangents equal to a be drawn to the curve B, they will be terminated in the curve C; and if one end of a thread be carried along the curve C, the other end will trace the curve B.

When $a = 0$, the curves B and C are both involutes of the curve A, they are always equidistant from each other, and if a circle, whose radius is b, be rolled on the one, its centre will trace the other.

If the curve A is such that, if the distance between two points measured along the curve is equal to b, the two points are similarly situate, then the curve B is the same with the curve C. Thus, the curve A may be a re-entrant curve, the circumference of which is equal to b.

When the curve A is a circle, the curves B and C are always the same.

The equations between the radii of curvature become

$$\frac{1}{\mathrm{T}} + \frac{1}{r_2} = \frac{r}{a\,\mathrm{R}_1}$$

When $a = 0$, $\mathrm{T} = 0$, or the centre of curvature of the curve B is at the point of contact. Now, the normal to the curve C passes through this point, therefore—

"The normal to any curve passes through the centre of curvature of its tractory."

In the next case, one curve, by rolling on another, produces a straight line. Let this straight line be the axis of y, then, since the radius of the rolled curve is perpendicular to it, and terminates in the fixed curve, and since these curves have a common tangent, we have these equations,

$$r_2 = x_1 \quad \frac{d\,y_1}{d\,x_1} = r_2 \frac{d\,\theta_2}{d\,r_2}$$

If the equation of the rolled curve be given, find $\frac{d\,\theta_2}{d\,r_2}$ in terms of r_2, substitute

x_1 for r_2, and multiply by x_1, equate the result to $\frac{dy}{dx}$, and integrate.

Thus, if the equation of the rolled curve be

$$\theta = A r^{-n} + \text{etc.} + K r^{-2} + L r^{-1} + M \log r + N r + \text{etc.} + Z r^n$$

$$\frac{d\theta}{dr} = - n A r^{-(n+1)} - \text{etc.} - 2 K r^{-3} - L r^{-2} + M r^{-1} + N + \text{etc.} + n Z r^{n-1}$$

$$\frac{dy}{dx} = - n A x^{-n} - \text{etc.} - 2 K x^{-2} - L x^{-1} + M + N x + \text{etc.} + n Z x^n$$

$$y = \frac{n}{n-1} A x^{1-n} + \text{etc.} + 2 K x^{-1} - L \log x + M x + \tfrac{1}{2} N x^2 + \text{etc.} + \frac{n}{n+1} Z x^{n+1}$$

which is the equation of the fixed curve.

If the equation of the fixed curve be given, find $\frac{dy}{dx}$ in terms of x, substitute r for x, and divide by r, equate the result to $\frac{d\theta}{dr}$, and integrate.

Thus, if the fixed curve be the orthogonal tractory of the straight line, whose equation is

$$y = a \log \frac{x}{a + \sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2}$$

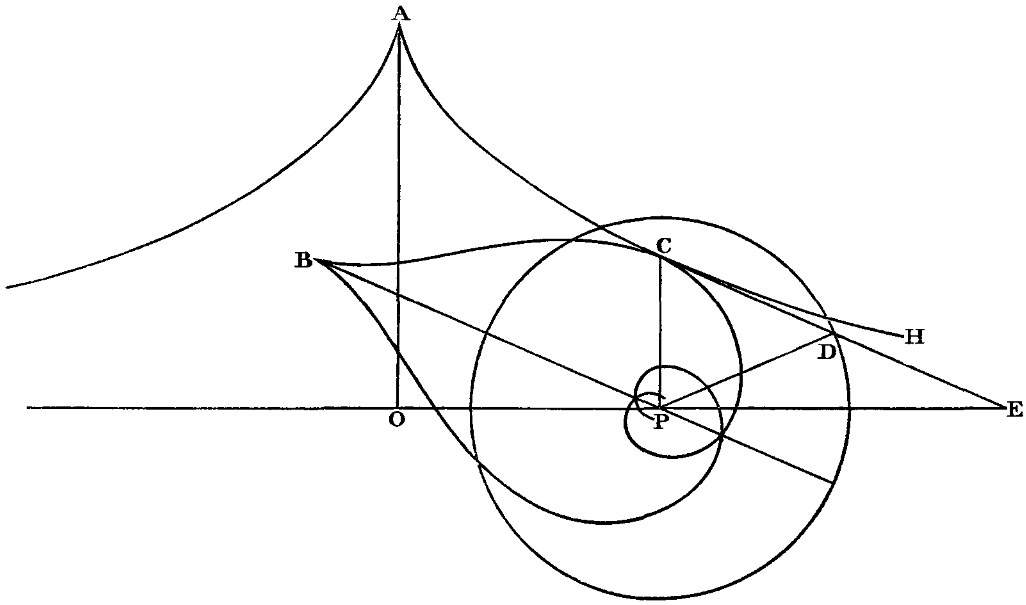
$$\frac{dy}{dx} = \frac{\sqrt{a^2 - x^2}}{x}$$

$$\frac{d\theta}{dr} = \frac{\sqrt{a^2 - r^2}}{r^2}$$

$$\theta = \cos^{-1} \frac{r}{a} - \sqrt{\frac{a^2}{r^2} - 1}$$

this is the equation to the orthogonal tractory of a circle whose diameter is equal to the constant tangent of the fixed curve, and its constant tangent equal to half that of the fixed curve.

This property of the tractory of the circle may be proved geometrically, thus— Let P be the centre of a circle whose radius is PD, and let CD be a line constantly equal to the radius. Let BCP be the curve described by the point C when the point D is moved along the circumference of the circle, then if tangents equal to CD be drawn to the curve, their extremities will be in the circle. Let ACH be the curve on which BCP rolls, and let OPE be the straight line traced by the pole, let CDE be the common tangent, let it cut the circle in D, and the straight line in E.



Then $CD = PD \therefore \angle DCP = \angle DPC$, and CP is perpendicular to OE, $\therefore \angle CPE = \angle DCP + \angle DEP$. Take away $\angle DCP = \angle DPC$, and there remains $DPE = DEP \therefore PD = DE \because CE = 2\,PD$.

Therefore the curve ACH has a constant tangent equal to the diameter of the circle, therefore ACH is the orthogonal tractory of the straight line, which is the tractrix or equitangential curve.

The operation of finding the fixed curve from the rolled curve is what Sir JOHN LESLIE calls " divesting a curve of its radiated structure."

The method of finding the curve which must be rolled on a circle to trace a given curve is mentioned here because it generally leads to a double result, for the normal to the traced curve cuts the circle in two points, either of which may be a point in the rolled curve.

Thus, if the traced curve be the involute of a circle concentric with the given circle, the rolled curve is one of two similar logarithmic spirals.

If the line traced be a tangent to the circle, the rolled curve is either of the parts of the polar catenary.

If the curve traced be the spiral of ARCHIMEDES, the rolled curve may be either the hyperbolic spiral or the straight line.

In the next case, one curve rolls on another and traces a circle.

Since the curve traced is a circle, the distance between the poles of the fixed curve and the rolled curve is always the same; therefore, if we fix the rolled curve and roll the fixed curve, the curve traced will still be a circle, and, if we fix the poles of both the curves, we may roll them on each other without friction.

Let a be the radius of the traced circle, then the sum or difference of the radii of the other curves is equal to a, and the angles which they make with the radius at the point of contact are equal,

$$\therefore\ r_1 = \pm(a \pm r_2) \text{ and } r_1 \frac{d\theta_1}{dr_1} = r_2 \frac{d\theta_2}{dr_2}$$

$$\therefore\ \frac{d\theta_2}{dr_2} = \frac{\pm(a \pm r_2)}{r_2} \frac{d\theta_1}{dr_1}$$

If we know the equation between θ_1 and r_1, we may find $\frac{d\theta_1}{dr_1}$ in terms of r_1, substitute $\pm(a \pm r_2)$ for r_1, multiply by $\frac{\pm(a \pm r_2)}{r_2}$, and integrate.

Thus, if the equation between θ_1 and r_1 be

$$r_1 = a \sec \theta_1$$

which is the polar equation of a straight line touching the traced circle whose equation is $r = a$,

then

$$\frac{d\theta}{dr_1} = \frac{a}{r_1 \sqrt{r_1^2 - a^2}}$$

$$= \frac{a}{(r_2 \pm a)\sqrt{r_2^2 \pm 2 r_2 a}}$$

$$\frac{d\theta_2}{dr_2} = \frac{r_2 \pm a}{r_2} \frac{a}{(r_2 \pm a)\sqrt{r_2^2 \pm 2 r_2 a}}$$

$$= \frac{a}{r_2 \sqrt{r_2^2 \pm 2 a r_2}}$$

$$\theta_2 = \pm \sqrt{1 \pm 2\frac{a}{r}}$$

$$r_2 = \frac{2a}{\theta_2^2 - 1} = \frac{2a}{\theta^2 - 1}$$

Now, since the rolling curve is a straight line, and the tracing point is not in its direction, we may apply to this example the observations which have been made upon tractories.

Let, therefore, the curve $r = \frac{2a}{\theta^2 - 1}$ be denoted by A, its involute by B, and the circle traced by C, then B is the tractory of C; therefore the involute of the curve $r = \frac{2a}{\theta^2 - 1}$ is the tractory of the circle, the equation of which is $\theta = \cos^{-1}\frac{r}{a} - \sqrt{\frac{a^2}{r^2} - 1}$. The curve whose equation is $r = \frac{2a}{\theta^2 - 1}$ seems to be among spirals what the catenary is among curves whose equations are between rectangular co-ordinates; for, if we represent the vertical direction by the radius vector, the tangent of the angle which the curve makes with this line is proportional to the length of the curve reckoned from the origin; the point at the distance a from a straight line rolled on this curve generates a circle, and when rolled on the catenary produces a straight line; the involute of this curve is the tractory of the circle, and that of the catenary is the tractory of the straight line, and the tractory of the circle rolled on that of the straight line traces the straight line; if this curve is rolled on the catenary, it produces the straight line touching the catenary at its vertex; the method of drawing tangents is the same as in the catenary, namely, by describing a circle whose radius is a on the production of the radius vector, and drawing a tangent to the circle from the given point.

In the next case, the rolled curve is the same as the fixed curve. It is evident that the traced curve will be similar to the locus of the intersection of the tangent with the perpendicular from the pole; the magnitude, however, of the traced curve will be double that of the other curve; therefore, if we call $r_0 = \phi_0\, \theta_0$ the equation to the fixed curve, $r_1 = \phi_1\, \theta_1$ that of the traced curve, we have,

$$r_1 = 2p_0 \qquad \theta_1 = \theta_0 - \cos^{-1}\frac{p_0}{r_0} = \theta_0 - \frac{\pi}{2} + \sin^{-1}\frac{p_0}{r_0}$$

$$\text{also, } \frac{p_1}{r_1} = \frac{p_0}{r_0}$$

$$\text{Similarly, } r_2 = 2p_1 = 2r_1\frac{p_0}{r_0} = 4\frac{p_0^2}{r_0}\, 4r_0\left(\frac{p_0}{r_0}\right)^2, \quad \theta^2 = \theta_0 - 2\cos^{-1}\frac{p_0}{r_0}$$

$$\text{Similarly, } r_n = 2p_{n-1} = 2r_{n-1}\frac{p_0}{r_0} \text{ etc.} = 2^n r_0\left(\frac{p_0}{r_0}\right)^n$$

$$\text{and } \frac{p_n}{r_n} = \frac{p_0}{r_0}$$

$$\theta_n = \theta_0 - n\cos^{-1}\frac{p_0}{r_0}$$

$$\theta_n = \theta_0 - n\cos^{-1}\frac{p_n}{r_n}$$

Let θ_n become $\theta^1{}_n$; θ_0, $\theta_0{}^1$ and $\frac{p_0}{r_0}, \frac{p^1{}_0}{r_0{}^1}$. Let $\theta^1{}_n - \theta_n = \alpha$

$$\theta^1{}_n = \theta^1{}_0 - n\cos^{-1}\frac{p^1}{r^1{}_n}$$

$$\alpha = \theta_n{}^1 - \theta_n = \theta^1{}_0 - \theta_0 - n\cos^{-1}\frac{p_n{}^1}{r_n{}^1} + n\cos^{-1}\frac{p_n}{r_n}$$

$$\therefore \cos^{-1}\frac{p_n}{r_n} - \cos^{-1}\frac{p^1{}_n}{r_n} = \frac{\alpha}{n} + \frac{\theta_0 - \theta^1{}_0}{n}$$

Now, $\cos^{-1}\frac{p_n}{r_n}$ is the complement of the angle at which the curve cuts the radius vector, and $\cos^{-1}\frac{p_n}{r_n} - \cos^{-1}\frac{p^1{}_n}{r^1{}_n}$ is the variation of this angle when θ_n varies by an angle equal to α. Let this variation $= \phi$; then if $\theta_0 - \theta_0{}^1 = \beta$

$$\phi = \frac{\alpha}{n} + \frac{\beta}{n}$$

Now, if n increases, ϕ will diminish; and if n become infinite,

$$\phi = \frac{\alpha}{\infty} + \frac{\beta}{\infty} = 0 \text{ when } \alpha \text{ and } \beta \text{ are finite.}$$

Therefore, when n is infinite, ϕ vanishes; therefore, the curve cuts the radius vector at a constant angle; therefore the curve is the logarithmic spiral.

Therefore, if any curve be rolled on itself, and the operation repeated an infinite number of times, the resulting curve is the logarithmic spiral.

Hence we may find, analytically, the curve which, being rolled on itself, traces itself.

For the curve which has this property, if rolled on itself, and the operation repeated an infinite number of times, will still trace itself.

But, by this proposition, the resulting curve is the logarithmic spiral; therefore the curve required is the logarithmic spiral. As an example of a curve rolling on itself, we will take the curve whose equation is

$$r_0 = 2^n a \left(\cos\frac{\theta_0}{n}\right)^n$$

$$\text{Here} - \frac{d r_0}{d \theta^0} = 2^n\ a \left(\sin\frac{\theta_0}{n}\right)\left(\cos\frac{\theta_0}{n}\right)^{n-1}$$

$$\therefore r_1 = 2\ p_0 = 2\frac{2^{2n} a^2 \left(\cos\frac{\theta_0}{n}\right)^{2n}}{\sqrt{2^{2n} a^2 \left(\cos\frac{\theta_0}{n}\right)^{2n} + 2^{2n} a^2 \left(\sin\frac{\theta_0}{n}\right)^2 \left(\cos\frac{\theta_0}{n}\right)^{2n-2}}}$$

$$r_1 = 2\frac{2^n a \left(\cos\frac{\theta_0}{n}\right)^{n+1}}{\sqrt{\left(\cos\frac{\theta_0}{n}\right)^2 + \left(\sin\frac{\theta_0}{n}\right)^2}} = 2^{n+1}\ a \left(\cos\frac{\theta_0}{n}\right)^{n+1}$$

$$\text{Now } \theta_1 - \theta_0 = -\cos^{-1}\frac{p_0}{r_0} = -\cos^{-1}\cos\frac{\theta_0}{n} = \frac{\theta_0}{n}$$

$$\therefore\ \theta_0 = \theta_1 \frac{n}{n+1}$$

substituting this value of θ_0 in the expression for r_1

$$r_1 = 2^{n+1}\ a \left(\cos\frac{\theta_1}{n+1}\right)^{n+1}$$

similarly if the operation be repeated m times, the resulting curve is

$$r_m = 2^{n+m} a \left(\cos \frac{\theta_m}{n+m} \right)^{n+m}$$

When $n = 1$, the curve is

$$r = 2\, a \cos \theta$$

the equation to a circle, the pole being in the circumference.
When $n = 2$, it is the equation to the cardioid.

$$r = 4\, a \left(\cos \frac{\theta}{2} \right)^2$$

In order to obtain the cardioid from the circle, we roll the circle upon itself, and thus obtain it by one operation; but there is an operation which, being performed on a circle, and again on the resulting curve, will produce a cardioid, and the intermediate curve between the circle and cardioid is

$$r = 2^{\frac{3}{2}} a \left(\cos \frac{2\,\theta}{3} \right)^{\frac{3}{2}}$$

As the operation of rolling a curve on itself is represented by changing n into $\overline{n+1}$ in the equation, so this operation may be represented by changing n into $\overline{n+\frac{1}{2}}$.

Similarly there may be many other fractional operations performed upon the curves comprehended under the equation

$$r = 2^n a \left(\cos \frac{\theta}{n} \right)^n$$

We may also find the curve, which, being rolled on itself, will produce a given curve, by making $n = -1$.

We may likewise prove by the same method as before, that the result of performing this inverse operation an infinite number of times is the logarithmic spiral.

As an example of the inverse method, let the traced line be straight, let its equation be

$$r_0 = 2\, a \sec \theta_0$$

$$\text{then } \frac{p_{-1}}{r_{-1}} = \frac{p_0}{r_0} = \frac{2\,a}{r_0} = \frac{2\,a}{2\,p_{-1}}$$

$$\therefore p^2_{-1} = a\, r_{-1}$$

therefore suppressing the suffix,

$$\frac{r^4}{r^2 + \frac{d\,r^2}{d\,\theta^2}} = a\,r$$

$$\therefore \quad \left(\frac{d\,r}{d\,\theta}\right)^2 = \frac{r^3}{a} - r^2$$

$$\therefore \quad \frac{d\,\theta}{d\,n} = \frac{1}{r\sqrt{\frac{r}{a} - 1}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{2\,a}{r} - 1\right)$$

$$r = \frac{2\,a}{1 - \cos\theta}$$

the polar equation of the parabola whose parameter is 4 a.

The last case which we shall here consider, affords the means of constructing two wheels whose centres are fixed, and which shall roll on each other, so that the angle described by the first shall be a given function of the angle described by the second.

$$\text{Let } \theta_2 = \varphi\,\theta_1 \text{ then } r_1 + r_2 = a, \text{ and } \frac{d\,\theta_2}{d\,\theta_1} = \frac{r}{r_2}$$

$$\therefore \quad \frac{d\,(\varphi\,\theta_1)}{d\,\theta_1} = \frac{r_1}{a - r_1}$$

Let us take as an example, the pair of wheels which will represent the angular motion of a comet in a parabola.

$$\text{Here } \quad \theta_2 = \tan\frac{\theta_1}{2} \quad \therefore \quad \frac{d\,\theta^2}{d\,\theta_1} = \frac{1}{\cos^2\frac{\theta_1}{2}} = \frac{r}{a - r_1}$$

$$\therefore \quad \frac{r_1}{a} = \frac{1}{2 + \cos\theta_1}$$

therefore the first wheel is an ellipse, whose major axis is equal to $\frac{4}{3}$ of the distance between the centres of the wheels, and in which the distance between the foci is half the major axis.

Now, since $\qquad \theta_1 = 2\tan^{-1}\theta_2$ and $r_1 = a - r_2$

$$\frac{r}{a} = 1 + \frac{1}{2\,(2 - \theta^4)}$$

$$\theta^4 = 2 - \frac{1}{2\,\frac{r}{a} - 2}$$

which is the equation to the wheel which revolves with constant angular velocity.

Before proceeding to give a list of examples of rolling curves, we shall state a theorem which is almost self-evident after what has been shewn previously.

Let there be three curves, A, B, and C. Let the curve A, when rolled on itself, produce the curve B, and when rolled on a straight line let it produce the curve C, then, if the dimensions of C be doubled, and B be rolled on it, it will trace a straight line.

A Collection of Examples of Rolling Curves.

1*st*. Examples of a curve rolling on a straight line.

Ex. 1. When the rolling curve is a circle whose tracing-point is in the circumference, the curve traced is a cycloid, and when the point is not in the circumference, the cycloid becomes a trochoid.

Ex. 2. When the rolling curve is the involute of the circle whose radius is $2a$, the traced curve is a parabola whose parameter is $4a$.

Ex. 3. When the rolled curve is the parabola whose parameter is $4a$, the traced curve is a catenary whose parameter is a, and whose vertex is distant a from the straight line.

Ex. 4. When the rolled curve is a logarithmic spiral, the pole traces a straight line which cuts the fixed line at the same angle as the spiral cuts the radius vector.

Ex. 5. When the rolled curve is the hyperbolic spiral, the traced curve is the tractory of the straight line.

Ex. 6. When the rolled curve is the polar catenary

$$\theta = \pm \sqrt{1 \pm \frac{2a}{r}}$$

the traced curve is a circle whose radius is a, and which touches the straight line.

Ex. 7. When the equation of the rolled curve is

$$\theta = \log\left(\sqrt{\frac{r^4}{a^4} - 1} + \frac{r^2}{a^2}\right) - \log\left(\sqrt{\frac{a^4}{r^4} + 1} - \frac{a^2}{r^2}\right)$$

the traced curve is the hyperbola whose equation is

$$y^2 = a^2 + x^2$$

2*d*. In the examples of a straight line rolling on a curve, we shall use the letters A, B, and C to denote the three curves treated of in page 555.

Ex. 1. When the curve A is a circle whose radius is a, then the curve B is the involute of that circle, and the curve C is the spiral of Archimedes, $r = a\theta$.

Ex. 2. When the curve A is a catenary whose equation is

$$x = \frac{a}{2}\left(e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right)$$

the curve B is the tractory of the straight line, whose equation is

$$y = a \log \frac{x}{a + \sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2}$$

and C is a straight line at a distance a from the vertex of the catenary.

Ex. 3. When the curve A is the polar catenary

$$\theta = \pm \sqrt{1 \pm \frac{2\,a}{r}}$$

the curve B is the tractory of the circle

$$\theta = \cos^{-1} \frac{r}{a} - \sqrt{\frac{a^2}{r^2} - 1}$$

and the curve C is a circle of which the radius is $\frac{a}{2}$.

3*d*. Examples of one curve rolling on another, and tracing a straight line.

Ex. 1. The curve whose equation is

$$\theta = \mathrm{A}\, r^{-n} + \text{etc.} + \mathrm{K}\, r^{-2} + \mathrm{L}\, r^{-1} + \mathrm{M} \log r + \mathrm{N}\, r + \text{etc.} + \mathrm{Z}\, r^{n}$$

when rolled on the curve whose equation is

$$y = \frac{n}{n-1} \mathrm{A}\, x^{1-n} + \text{etc.} + 2\,\mathrm{K}\, x^{-1} - \mathrm{L} \log x + \mathrm{M}\, x + \tfrac{1}{2} \mathrm{N}\, x^2 + \text{etc.} + \frac{n}{n+1} \mathrm{Z}\, x^{n+1}$$

traces the axis of y.

Ex. 2. The circle whose equation is $r = a \cos \theta$ rolled on the circle whose radius is a traces a diameter of the circle.

Ex. 3. The curve whose equation is

$$\theta = \sqrt{\frac{2\,a}{r} - 1} - \text{versin}^{-1} \frac{r}{a}$$

rolled on the circle whose radius is a traces the tangent to the circle.

Ex. 4. If the fixed curve be a parabola whose parameter is $4\,a$, and if we roll on it the spiral of Archimedes $r = a\,\theta$, the pole will trace the axis of the parabola.

Ex. 5. If we roll an equal parabola on it, the focus will trace the directrix of the first parabola.

Ex. 6. If we roll on it the curve $r = \frac{a}{4\,\theta^2}$ the pole will trace the tangent at the vertex of the parabola.

Ex. 7. If we roll the curve whose equation is

$$r = a \cos \left(\frac{a}{b}\, \theta \right)$$

on the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

the pole will trace the axis b.

Ex. 8. If we roll the curve whose equation is

$$r = \frac{a}{2}\left(e^{\frac{a\theta}{b}} - e^{-\frac{a\theta}{b}}\right)$$

on the hyperbola whose equation is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

the pole will trace the axis b.

Ex. 9. If we roll the lituus, whose equation is

$$r^2 = \frac{a^2}{3\theta}$$

on the hyperbola whose equation is

$$x\,y = a^2$$

the pole will trace the asymptote.

Ex. 10. The cardioid whose equation is

$$r = a\,(1 + \cos\theta)$$

rolled on the cycloid whose equation is

$$y = a \operatorname{versin}^{-1}\frac{x}{a} + \sqrt{2\,a\,x - x^2}$$

traces the base of the cycloid.

Ex. 11. The curve whose equation is

$$\theta = \operatorname{versin}^{-1}\frac{r}{a} + 2\sqrt{\frac{2\,a}{r} - 1}$$

rolled on the cycloid traces the tangent at the vertex.

Ex. 12. The straight line whose equation is

$$r = a \sec\theta$$

rolled on a catenary whose parameter is a, traces a line whose distance from the vertex is a.

Ex. 13. The part of the polar catenary whose equation is

$$\theta = \pm\sqrt{1 + \frac{2\,a}{r}}$$

rolled on the catenary traces the tangent at the vertex.

Ex. 14. The other part of the polar catenary whose equation is

$$\theta = \pm\sqrt{1 - \frac{2\,a}{r}}$$

rolled on the catenary traces a line whose distance from the vertex is equal to $2\,a$.

Ex. 15. The tractory of the circle whose diameter is a, rolled on the tractory of the straight line whose constant tangent is a, produces the straight line.

Ex. 16. The hyperbolic spiral whose equation is

$$r = \frac{a}{\theta}$$

rolled on the logarithmic curve whose equation is

$$y = a \log \frac{x}{a}$$

traces the axis of y or the asymptote.

Ex. 17. The involute of the circle whose radius is a, rolled on an orthogonal trajectory of the catenary whose equation is

$$y = \frac{x}{2a} \sqrt{x^2 - a^2} + \frac{a}{2} \log \left(\sqrt{\frac{x^2}{a^2} - 1} + \frac{x}{a} \right)$$

traces the axis of y.

Ex. 18. The curve whose equation is

$$\theta = \left(\frac{a}{x} + 1 \right) \sqrt{2 \frac{a}{x} + 1}$$

rolled on the witch, whose equation is

$$y = 2a \sqrt{\frac{2a}{x} - 1}$$

traces the asymptote.

Ex. 19. The curve whose equation is

$$r = a \tan \theta$$

rolled on the curve whose equation is

$$y = \frac{a}{2} \log \left(\frac{x^2}{a^2} - 1 \right)$$

traces the axis of y.

Ex. 20. The curve whose equation is

$$\theta = \frac{2r}{\sqrt{a^2 - r^2}}$$

rolled on the curve whose equation is

$$y = \frac{x^2}{\sqrt{a^2 - x^2}} \qquad \text{or } r = a \tan \theta$$

traces the axis of y.

Ex. 21. The curve whose equation is

$$r = a (\sec \theta - \tan \theta)$$

rolled on the curve whose equation is

$$y = a \log \left(\frac{x^2}{a^2} + 1 \right)$$

traces the axis of y.

4th. Examples of pairs of rolling curves which have their poles at a fixed distance $= a$

Ex. 1. { The straight line whose equation is $\theta = \sec^{-1}\frac{r}{a}$
The polar catenary whose equation is $\theta = \pm\sqrt{1 \pm \frac{2a}{r}}$

Ex. 2. Two equal ellipses or hyperbolas centred at the foci.

Ex. 3. Two equal logarithmic spirals.

Ex. 4. { Circle whose equation is $r = 2a\cos\theta.$
Curve whose equation is $\theta = \sqrt{2\frac{a}{r} - 1} + \text{versin}^{-1}\frac{r}{a}$

Ex. 5. { Cardioid whose equation is $r = 2a(1 + \cos\theta.)$
Curve whose equation is $\theta = \sin^{-1}\frac{r}{a} + \log\frac{r}{\sqrt{a^2 - r^2} + a}$

Ex. 6. { Conchoid, $r = a(\sec\theta - 1.)$
Curve, $\theta = \sqrt{1 - \frac{a^2}{r^2}} + \sec^{-1}\frac{r}{a}$

Ex. 7. { Spiral of Archimedes, $r = a\theta$
Curve, $\theta = \frac{r}{a} + \log\frac{r}{a}$

Ex. 8. { Hyperbolic spiral, $r = \frac{a}{\theta}$
Curve, $r = \frac{a}{e^\theta + 1}$

Ex. 9. { Ellipse whose equation is, $r = a\frac{1}{2 + \cos\theta}$
Curve, $r = a\left(1 + \frac{1}{2(2 - \theta^4)}\right)$

Ex. 10. { Involute of circle, $\theta = \sqrt{\frac{r^2}{a^2} - 1}\sec^{-1}\frac{r}{a}$
Curve, $\theta = \sqrt{\frac{r^2}{a^2} \pm 2\frac{r}{a}} \pm \log\left(\frac{r}{a} \pm 1 + \sqrt{\frac{r^2}{a^2} \pm 2\frac{r}{a}}\right)$

5th. Examples of curves rolling on themselves.

Ex. 1. When the curve which rolls on itself is a circle, equation

$$r = a\cos\theta$$

the traced curve is a cardioid, equation $r = a(1 + \cos\theta)$.

Ex. 2. When it is the curve whose equation is

$$r = 2^n a\left(\cos\frac{\theta}{r}\right)^n$$

the equation of the traced curve is

$$r = 2^{n+1} a\left(\cos\frac{\theta}{n+1}\right)^{n+1}$$

Ex. 3. When it is the involute of the circle, the traced curve is the spiral of Archimedes.

Ex. 4. When it is a parabola, the focus traces the directrix, and the vertex traces the cissoid.

Ex. 5. When it is the hyperbolic spiral, the traced curve is the tractory of the circle.

Ex. 6. When it is the polar catenary, the equation of the traced curve is

$$\theta = \sqrt{\frac{2a}{r} - 1} - \text{versin}^{-1}\frac{r}{a}$$

Ex. 7. When it is the curve whose equation is

$$\theta = \log\left(\sqrt{\frac{r^4}{a^4} - 1} + \frac{r^2}{a^2}\right) - \log\left(\sqrt{1 + \frac{a^4}{r^4}} - \frac{a^2}{r^2}\right)$$

the equation of the traced curve is, $r = a\,(e^{\theta} - e^{-\theta})$.