Philosophical Magazine Series 3

ISSN: 1941-5966 (Print) 1941-5974 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm14

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To cite this article: J.J. Sylvester M.A. F.R.S. (1850) XLVIII. On the solution of a system of equations in which three homogeneous quadratic functions of three unknown quantities are respectively equaled to numerical multiples of a fourth non-homogeneous function of the same , Philosophical Magazine Series 3, 37:251, 370-373, DOI: 10.1080/14786445008646630

To link to this article: http://dx.doi.org/10.1080/14786445008646630

Published online: 30 Apr 2009.

characteristic of the tangential envelope of the conic, x, y, z, t, u, v as the characteristics of the six points of the *circumscribed* hexagon, ϕ the characteristic of the point in which the line x, v meets the line z, t; $ay-\alpha u$ will then be shown to characterize the point in which t, x meets v, z; and thus we see that y, u; t, x; v, z, the three pairs of opposite sides of the hexagon, will meet in one and the same point, which is Brianchon's theorem.

XLVIII. *On the solution of a System of Equations in which three Homogeneous Quadratic Functions of three unknown quantities are respectively equaled to numerical Multiples of a fourth Non-Homogeneous Function of the same. By* J. J. Sylvester, *M.A., F.R.S.**

LET U, V, W be three homogeneous functions of x, y, z, and let ω be any function of x, y, z of the nth degree, and suppose that there is given for solution the system of equations

$$U = A . \omega$$
$$V = B . \omega$$
$$W = C . \omega.$$

Theorem.—The above system can be solved by the solution of a cubic equation, and an equation of the nth degree.

For let D be the determinant in respect to x, y, z of

$$fU + gV + hW,$$

then D is a cubic function of f, g, h. Now make $D=0$

$$Af + Bg + Ch = 0,$$

the ratios of $f : g : h$ which satisfy the last two equations can be determined by the solution of a cubic equation, and there will accordingly be three systems of f, g, h which satisfy the same, as

$$\begin{matrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3. \end{matrix}$$

Now $D=0$ implies that $fU+gV+hW$ breaks up into two linear factors; accordingly we shall find

$$(l_1x + m_1y + x_1z) . (\lambda_1x + \mu_1y + \nu_1z) = 0$$
$$(l_2x + m_2y + x_2z) \ (\lambda_2x + \mu_2y + \nu_2z) = 0$$
$$(l_3x + m_3y + x_3z) \ (\lambda_3x + \mu_3y + \nu_3z) = 0,$$

* Communicated by the Author.

in which the several sets of l, m, n; λ, μ, ν can be expressed without difficulty in terms of the several values of $\sqrt{f}, \sqrt{g}, \sqrt{h}$.

Let the above equations be written under the form

$$PP'=0$$
$$QQ'=0$$
$$RR'=0.$$

Since the given equations are perfectly general, it is readily seen that the equations

$$(P=0 \quad P'=0) \quad (Q=0 \quad Q'=0) \quad (R=0 \quad R'=0)$$

will severally represent pairs of opposite sides of a quadrangle expressed by general coordinates x, y, z; so that one of the two functions R, R′ will be a linear function of P and Q and also of P′ and Q′, and the other will be a linear function of P and Q′ and also of P′ and Q*.

In order to solve the equations, we need only consider two such pairs as $PP'=0$ $QQ'=0$; we then make

$$P=0 \quad Q=0,$$

or

$$P=0 \quad Q'=0,$$

or

$$P'=0 \quad Q=0,$$

or

$$P'=0 \quad Q'=0.$$

Any one of these four systems will give the ratios of $x:y:z$; and then, by substitution in any one of the given equations, we obtain the values of x, y, z by the solution of an ordinary equation of the nth degree. The number of systems x, y, z is therefore always $4n$.

The equations connected with the solution of Malfatti's celebrated problem, "In a given triangle to inscribe three circles such that each circle touches the remaining two circles and also two sides of the triangle," given by Mr. Cayley in the November Number for 1849 of the Cambridge and Dublin Mathematical Journal, to wit,

$$by^2+cz^2+2fyz=\theta^2.\,a(bc-f^2)=A$$
$$cz^2+ax^2+2gzx=\theta^2.\,b(ca-g^2)=B$$
$$ax^2+by^2+2hxy=\theta^2.\,c(ab-h^2)=C,$$

* Were it not for this being the case, the number of solutions would be n times the number of ways of obtaining duads out of three sets of two things, excluding the duads forming the sets, *i. e.* the number of solutions would be $12n$ in place of $4n$, the true number.

come under the general form which has just been solved. It so *happens*, however, that in this particular case

$$\left.\begin{matrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{matrix}\right\}$$

become respectively

$$\left.\begin{matrix} 0 & \frac{1}{B} & -\frac{1}{C} \\ -\frac{1}{B} & 0 & \frac{1}{C} \\ -\frac{1}{C} & \frac{1}{B} & 0 \end{matrix}\right\},$$

and the cubic equation is resolved without extraction of roots.

It follows from my theorem that the eight intersections of three concentric surfaces of the second order can be found by the solution of one cubic and one quadratic equation; and in general, if we have ϕ, ψ, θ any three quadratic functions of x, y, z, and $\phi=0$, $\psi=0$, $\theta=0$ be the system of equations to be solved, provided that we can by linear transformations express ϕ, ψ, θ under the form of

$$\mathrm{U}-aw$$
$$\mathrm{V}-bw$$
$$\mathrm{W}-cw,$$

U, V, W being homogeneous functions, and w a non-homogeneous function of three new variables, x', y', z', we can find the eight points of intersection of the three surfaces, of which U, V, W are the characteristics, by the solution of one cubic and one quadratic. But (as I am indebted to Mr. Cayley for remarking to me) that this may be possible, implies the coincidence of the vertices of one cone of each of the systems of four cones in which the intersections of the three surfaces taken two and two are contained.

I may perhaps enter further hereafter into the discussion of this elegant little theory. At present I shall only remark, that a somewhat analogous mode of solution is applicable to two equations,

$$\mathrm{U}=a\mathrm{P}^2$$
$$\mathrm{V}=b\mathrm{P}^2,$$

in which U, V are homogeneous quadratic functions, and P some non-homogeneous function of x, y.

We have only to make the determinant of $f\mathrm{U}+g\mathrm{V}$ equal to zero, and we shall obtain two systems of values of f, g, wherefrom we derive

$$l_1x+m_1y=\pm\sqrt{af_1+bg_1}\,.\,\mathrm{P}$$
$$l_2x+m_2y=\pm\sqrt{af_2+bg_2}\,.\,\mathrm{P},$$

from which x and y may be determined.

26 Lincoln's-Inn-Fields,
August 28, 1850.

XLIX. *On the Meteorology of England and the South of Scotland during the Quarter ending September* 30, 1850. *By* JAMES GLAISHER, *Esq., F.R.S., Hon. Sec. of the British Meteorological Society, &c.**

THE mean daily temperature of the air was below its average value till July 13; the mean defect was 2°·2. From July 12 to the 24th, the period was warm; the average excess of temperature was 4°·8. From July 25 to August 3, the temperature was below the average; its mean deficiency was 1°. From August 4 to August 18, it was above the average; the mean excess was 2°; this was followed by a long period of fine, clear, dry, but cold weather. The average deficiency of temperature between August 19 and September 17 was 3°·5; and after September 18, the daily temperatures were slightly above their average values. Snow fell on Ben-Lomond on August 23.

The mean temperature of the air at Greenwich for the three months ending August, constituting the three summer months, was 61°·1, being 1°·2 *above* the average of the preceding seventy-nine summers.

For the month of July was 62°·2, *exceeding* that of the average of seventy-nine years by 0°·9, and of nine years by 0°·7.

For the month of August was 60°·2, being 0°·3 *less* than the average of seventy-nine years, and 0°·9 *less* than that of the preceding nine years.

For the month of September was 56°·4, *exceeding* the average from seventy-nine years by 0°·1, and *less* than that of the preceding nine years by 0°·7.

The mean for the quarter was 59°·6, *exceeding* that of the average of seventy-nine summer quarters by 0°·2, and *less* than that of the nine preceding years by 0°·3.

The mean temperature of evaporation at Greenwich—

For the month of July was 58°·6; for August was 56°·6; and for June was 52°·9. These values are 0°·9 *greater*, 0°·2 *greater*, and 1°·6 *less* than those of the averages of the same months in the preceding nine years.

* Communicated by the Author.