A Theory of the Synchronous Motor

1894 Proc. Phys. Soc. London 13 502

(http://iopscience.iop.org/1478-7814/13/1/344)

XLIII. *A Theory of the Synchronous Motor.* *By* W. G. Rhodes, *M.Sc.**

1. Several foreign writers, notably Steinmetz †, have given theories of the synchronous motor, but most of them, by failing to see how the analysis could be simplified, add to the difficulties of the theory by mathematical intricacies which are apparently quite unnecessary. The author offers the following attempt to present a theory of the synchronous motor in as simple a way as possible, and as the mathematics for the most part consists of simple algebra, the difficulties are reduced to a physical conception of the subject. Many of the results have already been obtained, and the part for which the author chiefly claims originality is the method of attacking the problem.

2. We consider the case of an alternating-current machine whose field is excited by a direct current, while a simple alternating current passes round the armature.

Let p = output of motor ;
c = virtual value of armature current ;
R = resistance of armature ;
E = virtual value of impressed E.M.F. ;
e = " " counter E.M.F. ;
L = coefficient of self-induction of armature ;
n = frequency of armature current ;
I = impedance of armature $= \{\mathrm{R}^2 + (2\pi n\mathrm{L})^2\}^{\frac{1}{2}}$;
S = reactance $= 2\pi n\mathrm{L}$;
ψ = phase-difference between c and E ;
ϕ = " " c and e ;
θ = " " c and $\mathrm{I}c$.

Then the input $= p + c^2\mathrm{R}$;

and also $= c\mathrm{E}\cos\psi$;

therefore $$p + c^2\mathrm{R} = c\mathrm{E}\cos\psi.$$

* Read April 26, 1895.

† Trans. Am. Inst. Elec. Eng., December 1894.

Solving for c we get

$$c=\frac{\mathrm{E}}{2\mathrm{R}}\cos\psi\pm\frac{1}{2\mathrm{R}}\sqrt{\{\mathrm{E}^2\cos^2\psi-4p\mathrm{R}\}}. \quad . \quad . \quad (1)$$

Since c is always real, we must have

$$\mathrm{E}^2\cos^2\psi \geqq 4p\mathrm{R}\,;$$

therefore the maximum value of p is

$$p=\frac{\mathrm{E}^2}{4\mathrm{R}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

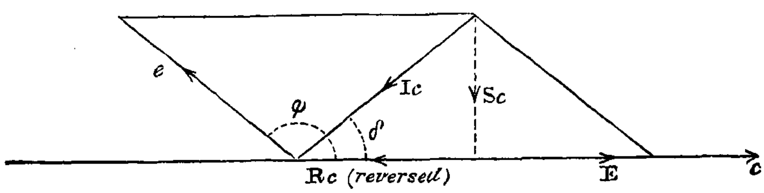
This occurs when $\psi=0$; that is, when the current and the impressed E.M.F. are in phase with each other.

3. We notice that the maximum output is the same as the maximum energy which can be given to an external circuit by a generator of constant E.M.F., E. From (1) we get the corresponding value of the current

$$c=\frac{\mathrm{E}}{2\mathrm{R}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To find the corresponding value of e we notice that E, e, and Ic (the resultant of Sc and Rc reversed) are in equilibrium

Fig. 1.



amongst themselves; so that taking components of these along and at right angles to the direction of E, we have

$$\left.\begin{aligned} -e\cos\phi &= \mathrm{E}-\mathrm{R}c \\ \text{and} \qquad e\sin\phi &= \mathrm{S}c \end{aligned}\right\};$$

$$\left.\begin{aligned} \text{therefore} \qquad e\cos\phi &= 2\mathrm{R}c-\mathrm{R}c=\mathrm{R}c \\ \text{and} \qquad e\sin\phi &= \mathrm{S}c \end{aligned}\right\}\text{from (3).}$$

Squaring and adding, we get

$$e^2=(\mathrm{R}^2+\mathrm{S}^2)c^2=\mathrm{I}^2c^2,$$

therefore

$$e=\mathrm{I}c=\frac{\mathrm{IE}}{2\mathrm{R}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Also, by division,

$$-\tan\phi=\frac{\mathrm{S}}{\mathrm{R}}=\tan\theta \text{ (see fig. 1).}$$

4. We thus find that when working at maximum output :—

(1) The impressed E.M.F. is in phase with the current in the armature.

(2) The maximum output is $p=\dfrac{\mathrm{E}^2}{4\mathrm{R}}$.

(3) The corresponding current in armature is $c=\dfrac{\mathrm{E}}{2\mathrm{R}}$.

(4) The corresponding counter E.M.F. is $e=\dfrac{\mathrm{IE}}{2\mathrm{R}}$.

(5) The angle of phase between the armature-current and the E.M.F. necessary to overcome the resistance and self-induction of the armature is equal and opposite to the angle between the current and the counter E.M.F.

(6) Also from (4) we see that

$$e \gtreqless \mathrm{E} \text{ according as } \mathrm{I} \gtreqless 2\mathrm{R},$$

$$\text{that is according as } (2\pi n\mathrm{L})^2+\mathrm{R}^2 \gtreqless 4\mathrm{R}^2,$$

$$\text{,,} \qquad \text{,,} \qquad \mathrm{L}^2 \gtreqless \frac{3\mathrm{R}^2}{4\pi^2n^2},$$

$$\text{,,} \qquad \text{,,} \qquad \mathrm{L} \gtreqless \frac{\mathrm{R}\sqrt{3}}{2\pi n}.$$

Running Light.

5. We have

$$\left.\begin{aligned} p+c^2\mathrm{R}&=c\mathrm{E}\cos\psi \\ \text{and} \quad p&=ce\cos\phi \end{aligned}\right\}.$$

If we neglect the friction of the bearings &c., we may, in this case, put $p=0$; we then have

$$\phi=\pm\frac{\pi}{2}$$

and

$$cR = E \cos \psi.$$

Hence the maximum value of c is (putting $\psi = 0$)

$$c = \frac{E}{R},$$

the same as would be produced by a constant E.M.F. E in a non-inductive circuit of resistance R.

Also putting $\psi = \pm \frac{\pi}{2}$, the minimum current is zero.

Now, from fig. 1, we get (of course E and c are not now in phase)

$$E^2 = e^2 + I^2c^2 + 2Ice \cos (\theta - \phi);$$

when $\phi = \pm \frac{\pi}{2}$, this becomes

$$E^2 = e^2 + I^2c^2 \pm 2Ice \sin \theta$$

$$= e^2 + I^2c^2 \pm 2Sce, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

since $\sin \theta = \frac{S}{I}$.

The upper sign in (5) corresponds to the machine running as a generator, and the lower sign as a motor.

We also see from (5) that corresponding to $c = \frac{E}{R}$ we have

$$e = \mp \frac{ES}{R},$$

and, corresponding to $c = 0$, we have

$$e = \pm E.$$

Now, solving equation (5) as a quadratic in c we get

$$c = \mp \frac{eS}{I^2} \pm \frac{1}{I^2} \sqrt{(I^2E^2 - R^2e^2)}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and, as c is real, we must have

$$I^2E^2 \geqq R^2e^2$$

or

$$IE \geqq Re;$$

therefore the maximum value of e is given by

$$e = \pm \frac{IE}{R}$$

and the corresponding current is

$$c = \mp \frac{eS}{I^2}$$

$$= \mp \frac{SE}{RI}.$$

The equation

$$e^2 - 2Sce + I^2c^2 = E^2$$

is the characteristic curve of the motor running light. It may be written

$$e^2 - 2Sce + S^2c^2 + R^2c^2 = E^2$$

or

$$(e - Sc)^2 + R^2c^2 = E^2,$$

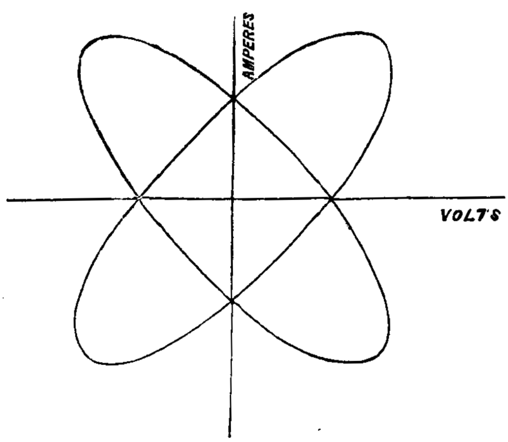
or

$$\frac{(e-Sc)^2}{E^2} + \frac{c^2}{\frac{E^2}{R^2}} = 1,$$

which is the equation to an ellipse having as conjugate diameters the lines

$$e - Sc = 0 \text{ and } c = 0.$$

Fig. 2.



Similarly, the equation

$$e^2 + 2Sce + I^2c^2 = E^2,$$

which refers to the generator, may be written

$$\frac{(e+\mathrm{S}c)^2}{\mathrm{E}^2}+\frac{c^2}{\frac{\mathrm{E}^2}{\mathrm{R}^2}}=1,$$

also the equation to an ellipse. These ellipses are represented in fig. 2.

Minimum Current at Given Power.

6. We have

$$p+c^2\mathrm{R}=\mathrm{E}c\cos\psi.$$

The current is a minimum when $\frac{dc}{d\psi}=0$. Now, differentiating with respect to ψ,

$$(2c\mathrm{R}-\mathrm{E}\cos\psi)\frac{dc}{d\psi}+\mathrm{E}\,c\sin\psi=0\,;$$

therefore, when $\frac{dc}{d\psi}=0$, we have

$$\sin\psi=0,$$

or $$\psi=0\,;$$

that is, the current is a minimum when in phase with the impressed E.M.F., as is otherwise obvious. Putting, therefore, $\psi=0$, we have

$$p+c^2\mathrm{R}=\mathrm{E}c. \qquad (7)$$

This curve is of the second degree in c and p, and is satisfied by the following system of points:—

$$(a)\ \left.\begin{array}{l}c=0,\\ p=0,\end{array}\right\}\qquad (b)\ \left.\begin{array}{l}c=\frac{\mathrm{E}}{2\mathrm{R}},\\ p=\frac{\mathrm{E}^2}{4\mathrm{R}}\end{array}\right\}\qquad (c)\ \left.\begin{array}{l}c=\frac{\mathrm{E}}{\mathrm{R}},\\ p=0.\end{array}\right\}$$

That the equation is satisfied by (a) is obvious, and we see that it is satisfied by the points (b) and (c) by writing it in the form

$$\left(c-\frac{\mathrm{E}}{2\mathrm{R}}\right)^2+\frac{p}{\mathrm{R}}=\frac{\mathrm{E}^2}{4\mathrm{R}^2}.$$

7. Thus we see that the curve of minimum current at given power passes through the points of

(*a*) zero current and zero power;
(*b*) maximum power;
(*c*) maximum current and zero power.

We notice that the maximum current at no load is $c = \frac{E}{R}$, whereas if the motor were at rest the current would be $c = \frac{E}{I}$; that is, the maximum current at no load is in all cases greater than the maximum current if the armature is at rest.

8. Again, from the equation

$$p + c^2 r = cE \cos\psi$$

we have

$$\psi = \cos^{-1}\frac{p + c^2 r}{cE},$$

therefore

$$\frac{d\psi}{dc} = \frac{c^2 r - p}{c\sqrt{\{c^2E^2 - (p + c^2 r)^2\}}}$$

$$= 0$$

when

$$p = c^2 r.$$

ψ is then a maximum, and we see that the maximum difference of phase between the current and the impressed E.M.F. takes place when the electrical efficiency is $\frac{1}{2}$.

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9. *Example.*—Suppose we have a 50 kilowatt motor driven by a 1000 volt generator, and suppose that R=3 ohms and S=4 ohms, so that I=5 ohms.

Then maximum output . . $= \frac{10^6}{12} =$ 83·3 kilowatts.

Corresponding current . . $= \frac{1000}{6} =$ 166·7 amperes.

" counter E.M.F. $= \frac{5000}{6} =$ 833·3 volts.

Maximum current running light $= \frac{1000}{3} =$ 333·3 amperes.

Corresponding counter E.M.F. $= \frac{4000}{3} =$ 1333·3 volts.

&c.

10. To find the characteristic of the motor at any given load we must eliminate θ and ϕ between the equations

$$\left.\begin{aligned} p&=ce\cos\phi,\\ \sin\theta&=\frac{S}{I},\\ \text{and}\qquad E^2&=e^2+I^2c^2+2Ice\cos(\theta-\phi). \end{aligned}\right\}$$

The eliminant is

$$E^2-e^2-I^2c^2-2Rp=2S\sqrt{c^2e^2-p^2}, \quad . \quad . \quad . \quad (8)$$

which is the required general relation between e and c.

11. In the paper referred to above, Steinmetz calls equation (8) the Fundamental Equation of the Synchronous Motor: the equation is there developed and plotted; results are obtained directly from equation (8), but the development is so cumbrous that the writer thinks that his simple treatment may benefit those interested in the subject of Alternate Current Motors.

Discussion.

Prof. S. P. Thompson said that the mathematical part of the paper was much simpler than that in previous investigations on this subject, and the method of arriving at the results by rejecting imaginary roots of the equations was particularly neat and instructive. The part of the paper relating to armature reactions and phase relationships was quite new. Two results deserved special attention; first, that the maximum current of zero power was the same as if the circuit was non-inductive; second, that the maximum current at zero power was double the current corresponding to maximum output.

Mr. Blakesley said that the paper did not consider the stability of the system, and he thought some of the results corresponded to regions of instability.

XLIV. *A Theory of the Synchronous Motor.* *By* W. G. Rhodes, *M.Sc.**

[Continued from p. 509.]

Armature Reaction in a Single Phase Alternate Current Machine.

12. In the foregoing analytical treatment the magnitudes only of θ, ϕ, and ψ are considered. We proceed to investigate the signs of these angles in order to determine the lags or leads of the E.M.F.'s over the current and to apply the results to find out whether the field excitation of generator and motor is strengthened or weakened by the reaction of the armature currents.

13. It will now be convenient to slightly alter the meanings of the symbols.

Let E = virtual value of generator E.M.F.
  e = ,, ,, counter E.M.F. of motor (as before);
  R = total resistance in circuit consisting of generator, motor, and line:
  L = sum of coefficients of self-induction of generator and motor armatures; or, if the line possesses self-induction, the coefficient of self-induction of the whole circuit;
  I = impedance of the complete circuit;
  S = reactance of the complete circuit.

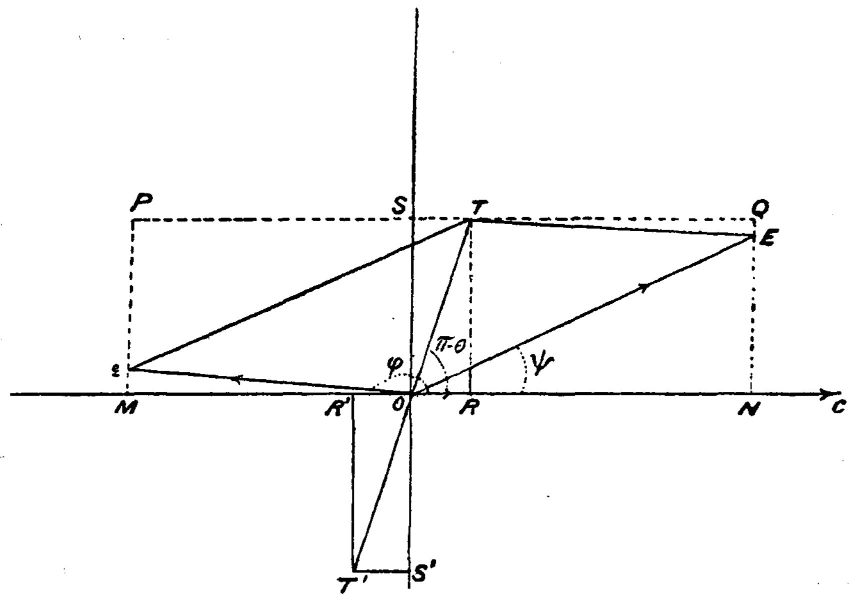
14. The E.M.F. Rc which drives the current is the resultant of E, e, and Sc; so that E, e, Sc, and Rc reversed form a system of E.M.F.'s in equilibrium.

In fig. 3 let the positive direction of rotation be counter-clockwise, and let Oc be the direction of the current. The instantaneous value of the current does not concern us at present. Take OR′ equal to Rc reversed, and consequently opposing the current; let OS′ = Sc, lagging behind the current by a quarter of a period. The resultant, OT′, of OR′ and OS′ will then be equal and opposite to the resultant of E and e. If,

* Read June 28, 1895.

therefore, we produce T′O to T and make OT = OT′, OT will represent in magnitude and direction the resultant of E and e. If, now, we are given the magnitudes of E and e we can find their directions by the parallelogram law. Now two parallelograms can be constructed having OT as diagonal and E, e as adjacent sides, but, since E is the E.M.F. of the generator, we take that which gives the component of E along

Fig. 3.



Oc in the same sense as the current. The other parallelogram would make e the generator. We may notice that the possibility of constructing these two parallelograms affords a proof of the fact that, in general, either of two alternate current machines may be driven as a motor by the other, irrespective of their relative E.M.F.'s. An analytical proof of this is given by the energy equation

$$p + c^2\mathrm{R} = c\mathrm{E}\cos\psi,$$

remembering that

$$p = ce\cos\phi,$$

and that ϕ and ψ are independent.

The condition that E represents the generator E.M.F. limits our choice of the two parallelograms to OETe (fig. 3):

We then have

$$\text{angle } c\text{OE} = \psi,$$
$$\text{angle } c\text{OT}' = \theta,$$
and
$$\text{angle } c\text{O}e = \phi.$$

15. Now through T draw PSTQ parallel to the line of current, and draw PM, SO, TR, and QN through e, O, T, and Q respectively, at right angles to the line of current.

We have

$$\tan\theta = -\frac{\mathrm{S}c}{\mathrm{R}c} = -\frac{\mathrm{S}}{\mathrm{R}} = -\frac{2\pi n\mathrm{L}}{\mathrm{R}}, \quad . \quad . \quad . \quad (9)$$

that is, θ is independent of the current and OT is a fixed direction relative to Oc so long as the speeds of the machines are kept constant, and L is considered constant.

16. In fig. 3, OS (or PM) is proportional to the current, $[=2\pi n\mathrm{L}c]$;

OM is the component of e directly opposing c,

OR is the E.M.F. required to overcome resistance, $[=\mathrm{R}c]$;

and ON is the component of E in the direction of c;

hence, rectangle PSOM is proportional to the output of the motor $[p]$;

rectangle OSTR is proportional to the c^2R losses ;

and rectangle OSQN ,, ,, output of generator $[c\mathrm{E}\cos\psi]$.

From this, and the equation

$$p + c^2\mathrm{R} = c\mathrm{E}\cos\psi,$$

it follows that the efficiency of transformation

$$= \frac{\mathrm{OM}}{\mathrm{ON}} = \frac{\mathrm{OM}}{\mathrm{MR}}.$$

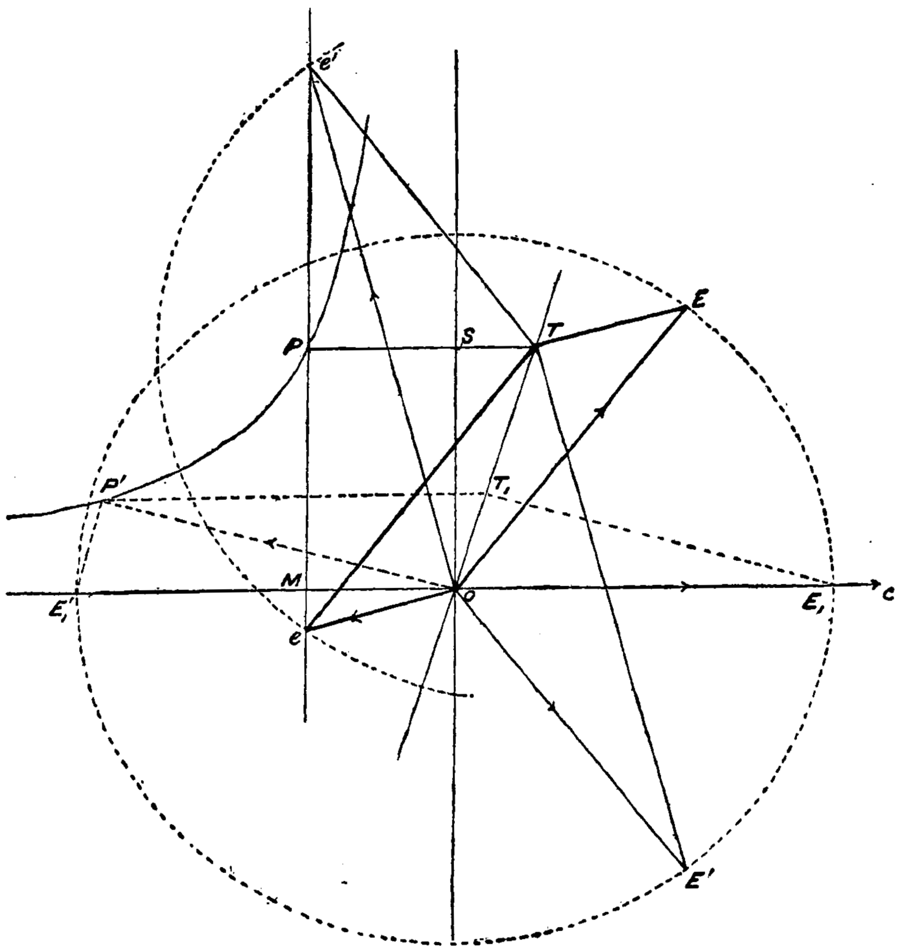
17. If the output of the motor is kept constant, we have

$$\text{rectangle PSOM} = \text{constant},$$

and the locus of P is a rectangular hyperbola having OM and OS as asymptotes (fig. 4).

Take any point P on this hyperbola. We have seen (9) that OT has a fixed direction relative to Oc ; and the point T (fig. 3) on this direction is found by drawing through P a line parallel to Oc. Again, e lies on the line through P parallel to OS and eT$=$E, in magnitude. Let the E.M.F. of the generator be kept constant and equal to E. With centre T and radius E describe a circle cutting PM in e and e'; then the corresponding counter E.M.F. of the motor may be

Fig. 4.



either Oe or Oe', and the current is represented in magnitude by PM: that is corresponding to given values of E and c there are two values of e. The relative phases in the two cases are shown in the parallelograms OeTE and Oe'TE' (fig. 4).

18. To find the point P' on the hyperbola corresponding to

minimum current we have to bring the points e and e' into coincidence. The point P′ is obviously got by taking OE_1' equal to E and through E_1' drawing $E_1'P$ parallel to OT. The resulting parallelogram $OP'T_1E_1$ shows that the generator E.M.F. is in phase with the current.

19. Suppose now that the excitation of the generator field is kept constant while that of the motor is varied.

When the motor excitation is small, as Oe (fig. 4), we see that the current leads over the motor E.M.F. and lags behind the generator E.M.F.

When e exactly opposes c, the latter lags behind E.

When E and c are in phase (minimum current), c lags behind e.

When e is still further increased, as Oe', c leads before E and lags behind e.

This explains the condenser action of an over-excited synchronous motor noticed by Professor S. P. Thompson and others.

20. It is known that a leading current strengthens the field of a generator and weakens that of a motor, while with a lagging current the reverse is the case. We therefore conclude that when the excitation of the motor field is small, armature reaction weakens the fields of both generator and motor, and when the motor is over-excited both machines have their fields strengthened. When working at minimum current, armature reaction strengthens the motor field and does not affect the field of the generator.

When the motor field is unaffected, the generator field is weakened.

21. Now the field of the motor is, under ordinary working conditions, excited to a somewhat greater extent than is required to obtain minimum current; for, though the c^2R losses are a minimum and the efficiency a maximum when the current is a minimum, it is advisable to increase the counter E.M.F. to a certain extent in order to cope with accidental variations of the load. Under ordinary working conditions, therefore, the effect of armature reaction is to strengthen the field of the motor and also of the generator, but to a less extent.

22. We now proceed to obtain an expression for the altera-

tion, in ampere turns, of the field excitation due to armature reaction.

Let ϕ be the displacement of phase of the current over the E.M.F. of the machine; n the number of turns of wire in one section of the armature; i the virtual current, and $i_0 \sin pt$ the instantaneous value of the current, so that i_0 is its maximum value.

It has hitherto been customary to assume that the alteration in the excitation is given by the expression

$$in\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where ϕ is expressed in circular measure.

Now it is not the *virtual* current to which the armature reaction is due, but the *mean value of the current* through an angle ϕ on each side of its maximum value; that is, the proper value of the current is given by

$$\mathrm{I} = \frac{p}{2\phi}\int_{\left(\frac{\pi}{2}-\phi\right)\frac{1}{p}}^{\left(\frac{\pi}{2}+\phi\right)\frac{1}{p}} i_0 \sin pt\, dt$$

$$= \frac{i_0}{\phi} \sin \phi.$$

And, remembering that $i_0 = \sqrt{2} \,.\, i$, we get

$$\mathrm{I} = \frac{\sqrt{2} \,.\, i}{\phi} \sin \phi.$$

Thus the proper expression for the alteration of the field excitation in ampere turns is

$$2\mathrm{I}n\phi$$

or

$$2\sqrt{2}\, i n \sin \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

To find the total excitation of the field we must add expression (11) to or subtract it from the ampere turns on the field according as the current leads or lags in a generator, and lags or leads in a motor.

Discussion.

Mr. Tunzelmann expressed a hope that the author would amplify parts of his paper.

Mr. Blakesley said the conclusion of the author, that "either of two alternate current machines may be driven as a motor by the other, irrespective of their relative E.M.F.'s," is not invariably correct. The facts of the case were these:—The E.M.F. of the motor may exceed that of the other machine to a certain extent; but that E.M.F. multiplied by the cosine of the angle of electric lag must yield a product not greater than the E.M.F. of the generator,—*i. e.*, using Mr. Rhodes's symbols, $e \cos \theta$ must not be greater than E. Mr. Blakesley gave a geometrical proof of this; but the same proposition had been given by him some ten years ago, in the course of investigating the subject generally. This was at a time when Dr. John Hopkinson was, with less than his usual perspicuity, teaching that synchronous alternate current machines could not be run in series with stability, both doing work. Referring to the author's diagrams, Mr. Blakesley said that in a problem involving so many elements as that under consideration, it was impossible, with the limited dimensions of space, to represent the results with the complete generality of a formula. Some elements had to be taken as the independent, others as the dependent, variables. The author had considered the power transmitted to the motor, the E.M.F. of the generator, and the angle of electric lag, as independent; the E.M.F. of the motor as dependent. In Mr. Blakesley's original diagrams the E.M.F.'s were both considered independent, as well as the electric lag, and the powers applied or transmitted as dependent variables. In any case the formulæ properly derived from such diagrams became perfectly general, and it did not appear to him that the change of method indicated could properly be called a new theory on the subject. As a matter of fact, diagrams based on the independence of the E.M.F.'s and the electric lag would furnish a better means of discussing the question of the stability of the motion than Mr. Rhodes's plan, and this might account for the entire omission from the paper of this important matter.

Prof. S. P. Thompson said it was impossible to discuss the question of stability till the subject of armature reaction had been thoroughly investigated. The terms lag and lead had been used by Mr. Rhodes in a consistent manner; but this was not always done, and he recommended that the phase of the current which was common to both generator and motor be taken as the standard.

The Author, in his reply, said he agreed with Mr. Blakesley that there was a limit to the extent to which the motor might be excited, and this upper limit could easily be obtained from the figure given in the paper. The question of armature reaction was, however, most important, as it might excite the field two or three times more than the original excitation. Since motors were designed to do a certain amount of work, and not the work to fit the motor, it was most natural to take the output of the motor as fixed.

XLV. *On the Use of an Iodine Voltameter for the Measurement of Small Currents.* *By* E. F. Herroun, *F.I.C., Professor of Natural Philosophy, Queen's College, London*.*

The instruments in general use for the direct measurement of currents by electrolysis, as in the determination of reduction factors of galvanometers, comprise the hydrogen, silver, and copper voltameters: each of these possesses certain advantages, but none is free from defect. When occasion arises to make a large number of determinations of relatively small currents, the difficulties attending their use become more manifest.

Some of the defects of the ordinary types may be briefly stated. The hydrogen voltameter, consisting of platinum plates immersed in dilute sulphuric acid, has the following disadvantages:—

1. It acquires a large counter electromotive force of polarization, so that the current, on first joining up, is very inconstant.

2. Its internal resistance is subject to considerable fluctuations, according to the rate and mode of disengagement of the electrolytic gases.

3. Oxygen present in the dilute acid, or migrating from

* Read May 10, 1895.