On the General Theory of Anharmonics. By Prof. Edgar O. Lovett.

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It is the object of this note to show how Lie's theory of the projective group may be made to yield Clifford's statement* of the general theory of anharmonics.

A geometrical configuration depending on $m.n$ coordinates, say a system of m points in a space of n-dimensions, has at least $mn-r$ absolute invariants under an r-parameter Lie group of point transformations; these $mn-r$ invariant functions, and all others, are solutions of the complete system of simultaneous partial differential equations

$$\sum_1^m i(X_1 f)_i = 0, \quad \sum_1^m i(X_2 f)_i = 0, \quad \ldots, \quad \sum_1^m i(X_r f)_i = 0, \tag{1}$$

where the

$$X_k f = \sum_1^n i\,\xi_{ik}(x_1, \ldots, x_n)\frac{\partial f}{\partial x_i} \quad (k = 1, \ldots, r) \tag{2}$$

are the r independent infinitesimal point transformations which generate the r-parameter continuous group, and $(X_k f)_j$ represents the result of replacing $x_1, \ldots, x_n$ by $x_1^{(j)}, \ldots, x_n^{(j)}$, respectively, in $(X_k f)$,

$$x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)} \quad (j = 1, 2, \ldots, m) \tag{3}$$

being the mn independent coordinates which determine the system of points.

1. Consider the general projective group of the xy-plane, an 8-parameter Lie group generated by the following independent infinitesimal point transformations—

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \quad x\frac{\partial f}{\partial y}, y\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x}, y\frac{\partial f}{\partial y}, \quad x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y}, \quad xy\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y}. \tag{4}$$

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* Clifford: "On the General Theory of Anharmonics," *Proc. Lond. Math. Soc.*, Vol. II.

A plane hexagon regarded as a system of six points depending upon the twelve coordinates, $x_1, y_1, \ldots, x_6, y_6$, possesses at least four independent invariants under the above group. These and all others are found by integrating the complete system of partial differential equations

$$\left.\begin{aligned} &\sum_1^6 i\frac{\partial\phi}{\partial x_i} = 0, \quad \sum_1^6 i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 ix_i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 iy_i\frac{\partial\phi}{\partial x_i} = 0, \\ &\sum_1^6 ix_i\frac{\partial\phi}{\partial x_i} = 0, \quad \sum_1^6 iy_i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 i\left(x_i^2\frac{\partial\phi}{\partial x_i} + x_iy_i\frac{\partial\phi}{\partial y_i}\right) = 0, \\ &\sum_1^6 i\left(x_iy_i\frac{\partial\phi}{\partial x_i} + y_i^2\frac{\partial\phi}{\partial y_i}\right) = 0 \end{aligned}\right\} . \quad (5)$$

That this system of partial differential equations possesses no more than four independent solutions is readily seen by constructing the matrix

$$\left\| \begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & y_1^2 & y_2^2 & y_3^2 & y_4^2 \end{matrix} \right\| , \quad (6)$$

formed by the coefficients of the $\frac{\partial\phi}{\partial x_i}$ and $\frac{\partial\phi}{\partial y_i}$, and observing that all eighth order determinants of this matrix do not vanish; for example, the determinants composed of the first eight columns.

The first and second equations of the system (5) show that the invariant function, ϕ, is a function of the variables

$$u_{i-1}=x_i-x_1, \quad v_i=y_i-y_1 \quad (i=2, 3, \ldots, 6). \quad (7)$$

The introduction of these new variables in the third, fourth, fifth, and sixth equations of the system reduces them to the forms, respectively,

$$\sum_1^5 i u_i \frac{\partial \phi}{\partial v_i}=0, \quad \sum_1^5 i v_i \frac{\partial \phi}{\partial u_i}=0, \quad \sum_1^5 i u_i \frac{\partial \phi}{\partial u_i}=0, \quad \sum_1^5 i v_i \frac{\partial \phi}{\partial v_i}=0. \qquad (8)$$

The first and second of these equations demand that ϕ shall be a function of the determinants

$$w_{ij}=\mid u_i, v_j \mid, \qquad (9)$$

where ij takes successively the values of the ten combinations of the five integers 1, 2, 3, 4, 5, taken two at a time. On introducing the variables (9) in the third and fourth equations of (8), we find the form

$$\Sigma w_{ij} \frac{\partial \phi}{\partial w_{ij}}=0, \qquad (10)$$

which requires that the function ϕ be a function of the ratios of the determinants w_{ij}. Finally, by direct substitution in the transformed forms of the seventh and eighth equations of the system (5), we have the result that the following nine functions are solutions of the system, and hence invariant under the group (4),

$$\frac{w_{\alpha\beta}(w_{\lambda\mu}+w_{\mu\nu}+w_{\nu\lambda})}{w_{ab}(w_{lm}+w_{mn}+w_{nl})}, \qquad (11)$$

where $\qquad (a, b, l, m, n)=(1, 2, 3, 4, 5),$

and $(\alpha\beta\lambda\mu\nu)$ is any other of the permutations of the five integers 1, 2, 3, 4, 5.

These invariants are readily interpreted geometrically; thus, for example, the expression

$$\frac{w_{13}(w_{24}+w_{45}+w_{52})}{w_{ab}(w_{lm}+w_{mn}+w_{nl})}, \qquad (12)$$

when simplified, becomes successively

$$\frac{|u_1, v_3| \{ |u_2, v_4| + |u_4, v_5| + |u_5, v_2| \}}{|u_1, v_2| \{ |u_3, v_4| + |u_4, v_5| + |u_5, v_3| \}}$$

$$= \begin{vmatrix} x_3-x_1 & y_3-y_1 \\ x_4-x_1 & y_4-y_1 \end{vmatrix} . \begin{vmatrix} x_3-x_1 & y_3-y_1 & 1 \\ x_5-x_1 & y_5-y_1 & 1 \\ x_6-x_1 & y_6-y_1 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix} . \begin{vmatrix} x_4-x_1 & y_4-y_1 & 1 \\ x_5-x_1 & y_5-y_1 & 1 \\ x_6-x_1 & y_6-y_1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} . \begin{vmatrix} x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \\ x_6 & y_6 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} . \begin{vmatrix} x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \\ x_6 & y_6 & 1 \end{vmatrix}$$

$$= \frac{124.356}{123.456}, \tag{13}$$

where 123 means the area of the triangle formed by the points 1, 2, 3. The other invariants are composed of corresponding terms formed from the twenty triangles made by the original six points. These nine invariants are not independent, since they may be shown to satisfy five independent identities of the form

$$123.456-124.563+125.634-126.345 \equiv 0. \tag{14}$$

To prove this statement, we first establish the identity (14) by observing that, since $ijk = -kji$, the left-hand member is identical with the expression

$$123.456+124.365+125.634+126.543, \tag{15}$$

which, in terms of the quantities w_{ij}, in virtue of (12) and (13), may be written

$$\begin{aligned} &w_{12}(w_{34}+w_{45}+w_{53})+w_{13}(w_{25}+w_{54}+w_{42}) \\ &+w_{14}(w_{23}+w_{35}+w_{52})+w_{15}(w_{43}+w_{32}+w_{24}), \end{aligned} \tag{16}$$

or

$$\begin{aligned} &(w_{12}w_{34}+w_{23}w_{14}+w_{31}w_{24})+(w_{12}w_{45}+w_{24}w_{15}+w_{41}w_{25}) \\ &+(w_{12}w_{53}+w_{25}w_{13}+w_{51}w_{23})+(w_{13}w_{54}+w_{35}w_{14}+w_{51}w_{34}), \end{aligned} \tag{17}$$

which, by noting the definition (9) of the w_{ij}'s, is zero, by virtue of the identity

$$|\,a_1, b_2\,| \,.\, |\,a_3, b_4\,| + |\,a_2, b_3\,| \,.\, |\,a_1, b_4\,| + |\,a_3, b_1\,| \,.\, |\,a_2, b_4\,| \equiv 0. \tag{18}$$

In the second place, to determine the number of independent identities of the form (14), we remark that the demonstration of (14) just given demonstrates simultaneously the following system of identities derived by permutation:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
123.456-124.563+125.634-126.345=0,\\
124.563-125.634+126.345-123.456=0,\\
125.634-126.345+123.456-124.563=0,\\
126.345-123.456+124.563-125.634=0;
\end{array}\right\}\\
\left.\begin{array}{l}
134.562-135.624+136.245-132.456=0,\\
135.624-136.245+132.456-134.562=0,\\
136.245-132.456+134.562-135.624=0;
\end{array}\right\}\\
\left.\begin{array}{l}
145.623-146.235+142.356-143.562=0,\\
146.235-142.356+143.562-145.623=0;
\end{array}\right\}\\
156.234-152.346+153.462-154.623=0;\\
\left.\begin{array}{l}
234.561-235.614+236.145-231.456=0,\\
235.614-236.145+231.456-234.561=0,\\
236.145-231.456+234.561-235.614=0;
\end{array}\right\}\\
\left.\begin{array}{l}
245.613-246.135+241.356-243.561=0,\\
246.135-241.356+243.561-245.613=0;
\end{array}\right\}\\
256.134-251.346+253.461-254.613=0;\\
\left.\begin{array}{l}
345.612-346.125+341.256-342.561=0,\\
346.125-341.256+342.561-345.612=0;
\end{array}\right\}\\
356.124-351.246+352.461-354.612=0;\\
456.123-451.236+452.361-453.612=0
\end{array}\right\}. \tag{19}$$

By designating the first ten terms in the left-hand column from the top down or the bottom up by $a, b, c, \ldots, j$, respectively, and observing that the first four equations are identical, the next three, and so on, the system (19) reduces to the following system of ten identities:

$$a-b+c-d=0, \tag{20}$$

$$a-e+f-g=0, \tag{21}$$

$$b-e+h-i=0, \tag{22}$$

$$c-f+h-j=0, \tag{23}$$

$$a+h-i+j=0, \tag{24}$$

$$b+f-g+j=0, \tag{25}$$

$$c+e-g+i=0, \tag{26}$$

$$c-d+e+j=0, \tag{27}$$

$$b-d+f+i=0, \tag{28}$$

$$a-d+g+h=0. \tag{29}$$

Of these, the first five are clearly independent; (21), (22), and (24) give (25); (22), (23), and (25) give (26); (20), (21), and (25) give (27); (22), (23), and (27) give (28); (21), (22), and (28) give (29); or, in other words, there exist the following five independent relations connecting the left-hand members of the above identities, viz.,

$$\left.\begin{array}{r} (21)+(22) = (24)+(25) = (28)+(29), \\ (22)+(23) = (25)+(26) = (27)+(28), \\ (20)+(25) = (21)+(27) \end{array}\right\}. \tag{30}$$

Hence, there are five, and but five, independent identities of the form (14) connecting the areas of triangles formed by the six points 1, 2, 3, 4, 5, 6 in a plane. Accordingly, Clifford's statement,* "There

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* *Vide loc. cit.*, § 3.

It is interesting to remark here that, by taking the corrected form (14) of the identity (*a*) and permuting cyclically, we obtain the following six identities:

$$\left.\begin{array}{r} 123.456+124.365+125.634+126.543=0, \\ 234.561+235.416+236.145+231.654=0, \\ 345.612+346.521+341.256+342.165=0, \\ 456.123+451.632+452.361+453.216=0, \\ 561.234+562.143+563.412+564.321=0, \\ 612.345+613.254+614.523+615.432=0 \end{array}\right\}. \tag{b}$$

The sum of the left-hand members of the first, third, and fifth of these identities is are six identical relations connecting the areas of triangles formed by six points, 1, 2, 3, 4, 5, 6, in a plane, viz.,

$$123.456+124.563+125.634+126.345\equiv 0, \qquad (a)$$

with five others obtained from this by permutation," requires modification with regard both to the form of the identities and their number.

To return to the original point, the independent absolute invariants of the system of six points in the plane under the general projective group of the plane are thus determined both in form and number.

2. A system of eight points in ordinary space is determined by twenty-four coordinates. The eight points form seventy different tetrahedra. The volumes of these tetrahedra arrange themselves in pairs in thirty-five different products to satisfy twenty-five independent identical relations of the form

$$1234.5678+1235.6784+1236.7845+1237.8456+1238.4567=0. \quad (31)$$

This identity is established in a manner wholly analogous to the method of proving the identity (14) in the plane: the identity is a direct consequence of the determinantal relation

$$\begin{aligned} &| a_1, b_2, c_3 | \cdot | a_4, b_5, c_6 | - | a_2, b_3, c_4 | \cdot | a_1, b_5, c_6 | \\ &\quad + | a_3, b_4, c_1 | \cdot | a_2, b_5, c_6 | - | a_4, b_1, c_2 | \cdot | a_3, b_5, c_6 | = 0, \quad (32) \end{aligned}$$

which itself results immediately from the identity already used

$$| a_1, b_2 | \cdot | a_3, b_4 | + | a_2, b_3 | \cdot | a_1, b_4 | + | a_3, b_1 | \cdot | a_2, b_4 | = 0.$$

The general projective group of ordinary space is a 15-parameter Lie group generated by the following fifteen independent infinitesimal transformations:

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zero, which is also the sum of the left-hand members of the second, fourth, and sixth; hence but four are independent. They contain all possible products except 135.624; if we write down the identity involving this term, and permute cyclically, the system formed of these six identities and of the six (b) is found to reduce to eight different identities among which three relations appear, leaving but five independent as before.

$$\left.\begin{array}{c} p\equiv\dfrac{\partial f}{\partial x},\quad q\equiv\dfrac{\partial f}{\partial y},\quad r\equiv\dfrac{\partial f}{\partial z}, \\ xp,\ yp,\ zp,\quad xq,\ yq,\ zq,\quad xr,\ yr,\ zr, \\ x(xp+yq+zr),\quad y(xp+yq+zr),\quad z(xp+yq+zr). \end{array}\right\} \tag{33}$$

The nine independent ratios of the terms of the identity (31), and of those identities derived from it by permutation, are solutions of the simultaneous system

$$\sum_1^8 i\frac{\partial\phi}{\partial x_i}=\sum_1^8 i\frac{\partial\phi}{\partial y_i}=\sum_1^8 i\frac{\partial\phi}{\partial z_i}=\sum_1^8 i x_i\frac{\partial\phi}{\partial x_i}$$

$$=\ldots=\sum_1^8 i\left\{z_i\left(x_i\frac{\partial\phi}{\partial x_i}+y_i\frac{\partial\phi}{\partial y_i}+z_i\frac{\partial\phi}{\partial z_i}\right)\right\}=0, \tag{34}$$

and hence are the invariants of the original system of eight points in space of three dimensions by the projective group of that space.

3. The generalization of the identities (14) and (31), together with a study of the Lie group which represents the general projective group of n-dimensional space, furnishes a basis for the complete generalization of Clifford's statement of the theory of anharmonics for two dimensions.

Consider the expression formed by the sum of the following binary products of determinants:

$$\left.\begin{array}{rl} & |\,1, 2, 3, 4, 5, \ldots, n,\,|\cdot|\,n+1, n+2, n+3, \ldots, 2n\,|, \\ (-1)^n & |\,2, 3, 4, 5, 6, \ldots, n, n+1\,|\cdot|\,1, n+2, n+3, \ldots, 2n\,|, \\ & |\,3, 4, 5, 6, \ldots, n, n+1, 1\,|\cdot|\,2, n+2, n+3, \ldots, 2n\,|, \\ (-1)^n & |\,4, 5, 6, \ldots, n, n+1, 1, 2\,|\cdot|\,3, n+2, n+3, \ldots, 2n\,|, \\ & \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots, \\ (-1)^n & |\,n+1, 1, 2, 3, \ldots, n-1\,|\cdot|\,n, n+2, n+3, \ldots, 2n\,| \end{array}\right\}, \tag{35}$$

where, for brevity,

$$|\,1, 2, 3, 4, 5, \ldots, n\,|\equiv|\,a_1^{(1)}, a_2^{(2)}, a_3^{(3)}, \ldots, a_n^{(n)}\,|. \tag{36}$$

By expanding all the determinants with regard to the elements of their left-hand columns, we see that the coefficient of the general term $a_j^{(r)}, a_k^{(s)}$ of the above sum is equal to the sum of determinant products, which sum is of the *form*

$$\begin{aligned}
&|\,1, 2, 3, 4, \ldots, n-1\,| \,.\, |\,n, n+1, n+2, n+3, \ldots, 2(n-1)\,| \\
+(-1)^{n-1}\,&|\,2, 3, 4, \ldots, n-1, n\,| \,.\, |\,1, n+1, n+2, n+3, \ldots, 2(n-1)\,| \\
+\,&|\,3, 4, \ldots, n-1, n, 1\,| \,.\, |\,2, n+1, n+2, n+3, \ldots, 2(n-1)\,| \\
+(-1)^{n-1}\,&|\,4, \ldots, n-1, n, 1, 2\,| \,.\, |\,3, n+1, n+2, n+3, \ldots, 2(n-1)\,| \\
+\,&\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
+(-1)^{n-1}\,&|\,n, 1, 2, 3, \ldots, n-2\,| \,.\, |\,n-1, n+1, n+2, \\
&\qquad\qquad n+3, \ldots, 2(n-1)\,| \,. \qquad (37)
\end{aligned}$$

By treating this sum (37) in the same manner, its general term is found to have a coefficient of the same form in which n is again decreased by unity; carrying this through $n-2$ steps, we arrive finally at a coefficient of the form

$$|\,1, 2\,| \,.\, |\,3, 4\,| + |\,2, 3\,| \,.\, |\,1, 4\,| + |\,3, 1\,| \,.\, |\,2, 4\,|,$$

which is obviously zero; hence the original sum of the determinant products (35) is zero, that is,

$$\Sigma\, |\,\alpha, \beta, \gamma, \delta, \epsilon, \ldots, \mu\,| \,.\, |\,\nu, n+2, n+3, \ldots, 2n\,| \equiv 0, \qquad (38)$$

where $$\alpha\beta\gamma\delta\epsilon \ldots \mu\nu$$

takes successively the values of the sequence

$$1, 2, 3, 4, 5, \ldots, n, n+1$$

and its cyclical permutations, and is always positive, or alternately positive and negative, according as n is even or odd.

A system of $[2(n+1)]$ points in a space of n dimensions depends upon $2n(n+1)$ coordinates and forms $[2(n+1)]!/[(n+1)!]^2$ solids each having $n+1$ vertices. The volumes of these solids satisfy identities of the form

$$\begin{aligned}
&[1, 2, 3, 4, \ldots, n, \quad n+1].[n+2, n+3, n+4, \ldots, 2(n+1)] \\
\pm&[1, 2, 3, 4, \ldots, n, \quad n+2].[n+3, n+4, \ldots, 2(n+1), n+1] \\
+&[1, 2, 3, 4, \ldots, n, \quad n+3].[n+4, \ldots, 2(n+1), n+1, n+2] \\
\pm&[1, 2, 3, 4, \ldots, n, \quad n+4].[n+5, \ldots, 2(n+1), n+1, n+2, n+3] \\
+&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\pm&[1, 2, 3, 4, \ldots, n, 2(n+1)].[n+1, n+2, n+3, \ldots, 2n+1] \equiv 0, \quad (39)
\end{aligned}$$

the upper or lower signs being taken according as n is odd or even. This identity follows directly from (38). The other identities, $[2(n+1)]!/2[(n+1)!]^2-(n^2+2)$ in number, are derived from this one by permutation.

The general projective group of n-dimensional space is an $[n(n+2)]$ parameter group generated by the following $[n(n+2)]$ independent infinitesimal transformations

$$\frac{\partial f}{\partial x_i}, \quad x_i\frac{\partial f}{\partial x_k}, \quad x_i\sum_1^n j x_j\frac{\partial f}{\partial x_j} \quad (i, k = 1, \ldots, n).$$

The n^2 independent ratios of products of the form of those entering (39) are the $[2n(n+1)-n(n+2)]$ absolute invariants of the system of $[2(n+1)]$ points by the general projective group of n-dimensional space.

Zeroes of the Bessel Functions. By H. M. MACDONALD. Communicated (provisionally) April 7th, 1898. Received July 16th, 1898.

The zeroes of any holomorphic function can be approximated to by finding the zeroes of the polynomials obtained by stopping at any term of the power series representing the function. This method indicates the nature of the zeroes, but is unsuitable for the calculation of their values. In the neighbourhood of an essential singularity the function has an infinite number of zeroes, and all these zeroes can be quickly calculated from the expression for the function in that neighbourhood. This is the method applied by Stokes to calculate