A NEW METHOD OF DESCRIBING A THREE-BAR CURVE

By COL. R. L. HIPPISLEY, C.B.

[Received November 19th, 1917.—Read December 6th, 1917.]

1. The pedal triangle of a point on a three-bar curve, with reference to the triangle of foci, has the property that its vertices are at fixed distances from a variable point.

This can easily be proved. If α, β, and γ are the angles subtended at the moving point by the sides of such a triangle,

$$\alpha+\beta+\gamma = 2\pi,$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma = 1.$$

Denote the sides of the triangle by x, y, z, and the distances of the vertices from the moving point by r_1, r_2, r_3; then

$$\cos\alpha = \frac{r_3^2+r_1^2-x^2}{2r_3r_1}, \quad \cos\beta = \frac{r_1^2+r_2^2-y^2}{2r_1r_2}, \quad \cos\gamma = \frac{r_2^2+r_3^2-z^2}{2r_2r_3};$$

and therefore

$$r_2^2(x^2-r_3^2-r_1^2)^2+r_3^2(y^2-r_1^2-r_2^2)^2+r_1^2(z^2-r_2^2-r_3^2)^2$$
$$+(x^2-r_3^2-r_1^2)(y^2-r_1^2-r_2^2)(z^2-r_2^2-r_3^2) = 4r_1^2r_2^2r_3^2. \quad (1)$$

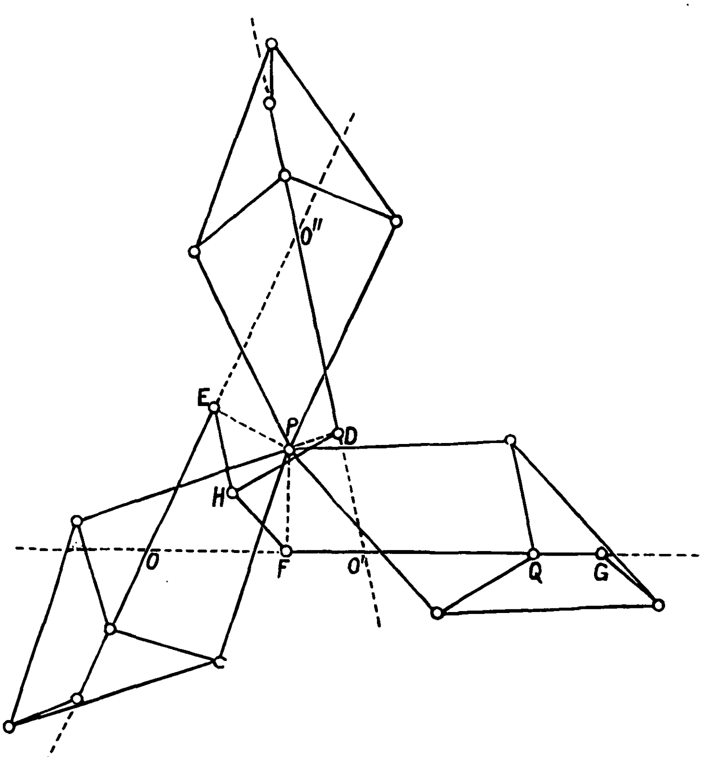
If we denote the focal distances of a point on a three-bar curve by u, v, w, the equation to the curve can be written

$$a_2^2(u^2-b_1^2-c_3^2)^2+b_3^2(v^2-c_2^2-a_1^2)^2+c_1^2(w^2-a_3^2-b_2^2)^2$$
$$+(u^2-b_1^2-c_3^2)(v^2-c_2^2-a_1^2)(w^2-a_3^2-b_2^2) = 4a_1^2b_2^2c_3^2, \quad (2)$$

where a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 are the sides of the three traversing triangles.

If we make $r_1 = a_1\sin B$, $r_2 = b_2\sin C$, $r_3 = c_3\sin A$, *i.e.* the perpendicular distances of the point on the curve from the three traversing links, and write for x, y, z the values $u\sin A$, $v\sin B$, $w\sin C$, which are the lengths of the sides of the pedal triangle for a point (u, v, w), then equation (1) becomes identical with equation (2), showing that such a triangle can be formed for any point on a three-bar curve.

2. This principle may be used for the purpose of constructing a mechanism which will trace out a three-bar curve. If D, E, F are three points, one on each of the sides of the focal triangle $OO'O''$, and we connect them to a variable point H by three bars of lengths l, m, n, then, if they are the vertices of a pedal triangle at all, they will be those of the pedal triangle of a point on a three-bar curve. The only condition that they must observe in order that they should form a pedal triangle is that the perpendiculars through them to the sides of the focal triangle should meet in a point. This can be ensured in the following manner (see figure).



Let the rod FG be made to slide along the side OO'. Attach to it a Peaucellier cell PQG whose fixed pivots are at Q and G, and which is arranged so that the point P describes a straight line through F, perpendicular to FG. Apply a similar mechanism to the other two sides of the focal triangle $OO'O''$, and pivot all three points P together; then P describes a three-bar curve. The links l, m, n may have any lengths, provided that any two are not less than the third, and the focal triangle may have any form. The moving links of the three-bar linkage will be respectively

$$\frac{m}{\sin A}, \quad \frac{n \sin C}{\sin A \sin B}, \quad \frac{l}{\sin B},$$

and the fixed side must be less than the sum of these quantities.

It is recognised that the sliding of the bars FG, &c. between guides would prevent the mechanism being a true linkage. But the rod FG can be made to follow the line OO' by the application to it of two Peaucellier cells both designed to trace out the same straight line OO'. It has not been thought necessary to show these cells in the figure.

3. The equation to the locus of the point H can easily be found.

Let x_0, y_0 be the coordinates of P, and x, y those of H. Take the origin at O and OO' as the axis of x. The point H is the intersection of the three circles described with centres D, E, F and radii $b_2 \sin C$, $c_3 \sin A$, and $a_1 \sin B$. The coordinates of D are

$$c_0 - \{(c_0 - x_0) \cos B + y_0 \sin B\} \cos B, \quad \{(c_0 - x_0) \cos B + y_0 \sin B\} \sin B,$$

those of E are

$$(x_0 \cos A + y_0 \sin A) \cos A, \quad (x_0 \cos A + y_0 \sin A) \sin A,$$

and those of F $\qquad x_0, \quad 0,$

where $\qquad OO' = c_0.$

The D circle is

$$\begin{aligned} x^2 + y^2 - 2x\,[c_0 - \{(c_0 - x_0) \cos B + y_0 \sin B\} \cos B] \\ -2y\,[(c_0 - x_0) \cos B + y_0 \sin B] \sin B \\ +(x_0 \cos B - y_0 \sin B)^2 + (c_0^2 - c_2^2) \sin^2 B = 0. \end{aligned}$$

The E circle is

$$\begin{aligned} x^2 + y^2 - 2x\,(x_0 \cos A + y_0 \sin A) \cos A - 2y\,(x_0 \cos A + y_0 \sin A) \sin A \\ +(x_0 \cos A + y_0 \sin A)^2 - c_3^2 \sin^2 A = 0, \end{aligned}$$

and the F circle is

$$x^2 + y^2 - 2x_0 x + x_0^2 - b_1^2 \sin^2 A = 0.$$

The last circle cuts the first in the straight line

$$2x\,(x_0-c_0+y_0\cot B)+2y\,\{(x_0-c_0)\cot B-y_0\}$$
$$-x_0^2-2x_0y_0\cot B+y_0^2+c_0^2-\mu^2=0, \tag{3}$$

and the second in the straight line

$$2x\,(x_0-y_0\cot A)-2y\,(x_0\cot A+y_0)-x_0^2+2x_0y_0\cot A+y_0^2-\lambda^2=0, \tag{4}$$

in which $\qquad \lambda^2\equiv c_3^2-b_1^2, \quad \mu^2\equiv c_2^2-a_1^2.$

From these two linear equations x and y can be obtained. But we require x_0 and y_0 as functions of x, y. Equations (3) and (4) are

$$(x_0-x)^2+2\,(x_0-x)(y_0-y)\cot B-(y_0-y)^2$$
$$-\{(x-c_0)^2+2\,(x-c_0)\,y\cot B-y^2-\mu^2\}=0,$$
$$(x_0-x)^2-2\,(x_0-x)(y_0-y)\cot A-(y_0-y)^2$$
$$-(x^2-2xy\cot A-y^2-\lambda^2)=0.$$

That is to say, considering x_0 and y_0 as running coordinates, they are both rectangular hyperbolas centred at H, and will have two real and two imaginary intersections. The first is inclined at an angle of $45°-\tfrac{1}{2}B$, and has a semiaxis

$$\{-(x-c_0)^2\sin B-2\,(x-c_0)\,y\cos B+y^2\sin B+\mu^2\sin B\}^{\frac{1}{2}},$$

and the second is inclined at an angle of $45°+\tfrac{1}{2}A$, and has a semiaxis

$$(-x^2\sin A+2xy\cos A+y^2\sin A+\lambda^2\sin A)^{\frac{1}{2}}.$$

Writing x', y' for x_0-x, y_0-y, the equations to the two hyperbolas referred to their centres are

$$x'^2+2x'y'\cot B-y'^2+C_1=0,$$
$$x'^2-2x'y'\cot A-y'^2+C_2=0,$$

where $\qquad C_1=-\{(x-c_0)^2+2\,(x-c_0)\,y\cot B-y^2-\mu^2\},$

$$C_2=-(x^2-2xy\cot A-y^2-\lambda^2)\,;$$

and their intersections are given by

$$x'^2=\frac{-Q\pm\sqrt{P}}{D^2},$$

$$y'^2=\frac{Q\pm\sqrt{P}}{D^2},$$

where
$$Q = C_1 \cot A + C_2 \cot B,$$
$$P = (C_1 - C_2)^2 + (C_1 \cot A + C_2 \cot B)^2,$$
$$D^2 = 2(\cot A + \cot B).$$

We see that P is essentially positive and that $\sqrt{P}$ is greater than Q. Hence we find

$$x_0 = x + \frac{\sqrt{-Q+\sqrt{P}}}{D},\quad x - \frac{\sqrt{-Q+\sqrt{P}}}{D},\quad x + i\frac{\sqrt{Q+\sqrt{P}}}{D},\quad x - i\frac{\sqrt{Q+\sqrt{P}}}{D},$$

$$y_0 = y + \frac{\sqrt{Q+\sqrt{P}}}{D},\quad y - \frac{\sqrt{Q+\sqrt{P}}}{D},\quad y + i\frac{\sqrt{-Q+\sqrt{P}}}{D},\quad y - i\frac{\sqrt{-Q+\sqrt{P}}}{D},$$

and, substituting these values of x_0 and y_0 in the equation to the three-bar curve, we obtain the factors of which the equation to the locus of H is made up. When rationalized it is of the 24th degree.

Since there are two real values of x_0, y_0 for each value of x, y, we see that two real points on the three-bar curve have the same H. Consequently, the point H traverses its orbit twice for each complete circuit of the point P. The equation of the 24th degree may therefore be a perfect square and reducible to one of the 12th degree. The two real values of x_0 and y_0 show that the two points P and P' on the three-bar curve which have a common point H are on the same straight line through H and equidistant from H, the length of HP being given by

$$HP^2 = \frac{2\sqrt{P}}{D},$$

and the inclination of the line by

$$\tan 2\theta = -\frac{C_1 - C_2}{C_1 \cot A + C_2 \cot B}.$$