The Elastic Curve, under uniform normal pressure.

By

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The mathematical discussion of this problem, due originally to Maurice Lévy (*Comptes Rendus*, XCVII), is given by Halphen in his *Fonctions elliptiques* (F. E.) t. II, Chap. V, as affording an interesting application of Elliptic Functions to a mechanical problem.

The subject is of practical importance to the engineer, in the consideration of the stability of boiler tubes and flues, with the high pressure and temperature now employed, and also in its bearing on the buckling tendency of the steel plates of a ship, when required to be very thin, as in a torpedo boat.

In the present article the mathematical treatment of Halphen is resumed, as an exercise on the theory of the Elliptic Integral of the Third Kind; and some cases are worked out in which this integral is of the simplest *pseudo-elliptic* character, so that the form of this Elastic Curve can be calculated numerically by means of existing mathematical tables, including those given by Legendre of the Elliptic Integrals.

The reader is referred to articles in the *Proceedings of the London Mathematical Society* vol. XXV and XXVII (designated in the sequel by the abbreviations L. M. S. XXV, XXVII) for the development of the analysis of the results quoted in this article, which is intended to provide an additional mechanical application, in which the results are reduced to a shape in which numbers can be substituted immediately, as illustrated by the diagrams.

1. We start with Halphen's notation and formulas for Lévy's Elastic Curve (F. E. II, Chap. V)

$$(1) \qquad \frac{1}{\varrho} = 4Ar^2 + 2B,$$

connecting ϱ the radius of curvature, and r the central distance; and denoting the length of the perpendicular from the origin on the tangent by p,

(2) $$\frac{1}{r}\frac{dp}{dr} = 4Ar^2 + 2B,$$

so that, integrating,

(3) $$p = Ar^4 + Br^2 + C.$$

Then

(4) $$\frac{1}{4}\left(\frac{dr^2}{ds}\right)^2 = r^2 - p^2$$
$$= r^2 - (Ar^4 + Br^2 + C)^2 = R,$$

suppose; so that

(5) $$s = \frac{1}{2}\int \frac{dr^2}{\sqrt{R}},$$

an elliptic integral, of the first kind.

Again

(6) $$\frac{d\theta}{ds} = \frac{p}{r^2} = \frac{Ar^4 + Br^2 + C}{r^2},$$

so that

(7) $$\theta = \frac{1}{2}\int \frac{Ar^4 + Br^2 + C}{r^2\sqrt{R}}\, dr^2,$$

introducing Elliptic Integrals of the Third Kind.

2. But now, in the inversion of these elliptic integrals, we shall follow a different procedure from Halphen, and employ as our Standard Elliptic Integral of the Third Kind the form (circular)

(1) $$I = \frac{1}{2}\int^{\infty} \frac{P(s-\sigma) - \sqrt{-\Sigma}}{s-\sigma}\,\frac{ds}{\sqrt{S}},$$

where s is a new variable (which must, not be confounded with the s employed above to denote the arc of the curve, distinguished in the sequel by an accent).

Also

(2) $$S = 4s(s+x)^2 - \{(1+y)s + xy\}^2,$$
$$= 4(s-s_1)(s-s_2)(s-s_3),$$

where x and y are the quantities defined by Halphen (F. E. I, p. 103); Σ is the value of S when s is replaced by σ; and P is a certain constant at our disposal, so chosen as to cancel the secular term depending on the elliptic integral of the first kind, when (1) is made *pseudo-elliptic* by selecting as a parameter a fraction $f\omega'$ of the imaginary period ω'.

The elliptic argument u is given by

(3) $$u = \int^{\infty} \frac{ds}{\sqrt{S}},$$

so that s is a one-valued elliptic function of u.

This may be distinguished as $s(u)$, and then $\sigma = s(v)$, v denoting the parameter of I the elliptic integral of the third kind in (1); and now, in Weierstrass' notation,

(4) $$\sqrt{} - \Sigma = + i\wp' v, \text{ when } v = f\omega'.$$

Replacing I and P in (1) by $I(v)$ and $P(v)$,

(5) $$iI(v) = \log \sqrt{\frac{\mathfrak{S}(u-v)}{\mathfrak{S}(u+v)}}\, e^{\left[\frac{1}{2} iP(v) + \zeta v\right] u}$$

(6) $$e^{2iI(v)} = \frac{\mathfrak{S}(u-v)}{\mathfrak{S}(u+v)}\, e^{[iP(v)+2\zeta v]u}$$

$$= \frac{\mathfrak{S}(u-v)}{\mathfrak{S}(u+v)}\, e^{2\frac{\eta}{\omega} vu},$$

(Halphen, F. E. I, p. 224), on taking

(7) $$\frac{1}{2} iP(v) = \frac{\eta}{\omega} v - \zeta v.$$

3. Changing Halphen's u into $u - v$, and his v into $2v$ (F. E. II, p. 197),

(1) $$\frac{r^2}{a^2} = [\wp(u+v) - \wp 2v][\wp(u-v) - \wp 2v],$$

(2) $$\frac{p}{a} = \wp(u+v) + \wp(u-v) - 2\wp 2v,$$

(3) $$\frac{\sqrt{R}}{a} = \frac{1}{2}\wp(u-v) - \frac{1}{2}\wp(u+v)$$

$$= \frac{1}{2}\frac{\wp' u \wp' v}{(\wp u - \wp v)^2},$$

(4) $$\frac{a}{\varrho} = -\frac{1}{\wp' 2v}\frac{\wp'(u-v) - \wp'(u+v)}{\wp(u-v) - \wp(u+v)}.$$

This shows that $r = \infty$ for $u = v$, and $r = 0$ for $u = 3v$, thus suggesting the new substitution

(5) $$\frac{r^2}{a^2} = \frac{\wp u - \wp 3v}{\wp u - \wp v}.$$

In the notation of L. M. S. XXV, we write

(6) $\wp u - \wp v = M^2(s+x)$, $\quad i\wp' v = M^3 x$;

(7) $\wp u - \wp 2v = M^2 s$, $\quad i\wp' 2v = M^3 xy$;

(8) $\wp u - \wp 3v = M^2(s + x - y)$, $\quad i\wp' 3v = M^3(-x + y - y^2)$;

(9) $\wp u - \wp 4v = M^2\left(s + x\frac{x-y+y^2}{y}\right)$, $\quad i\wp' 4v = M^3 x \frac{x(x-y+y^2)+(x-y)^2}{y^3}$;

. . . .

Now to reduce the integrals in (7) § 1 to the standard form (1) § 2, substitute

(10) $$\frac{r^2}{a^2} = \frac{s+x-y}{s+x},$$

and

(11) $$\frac{R}{a^2} = \frac{\frac{1}{4}S}{(s+x)^4};$$

and writing t for $s+x$, and T for the corresponding value of S,

(12) $$T = 4t^2(t-x) - \{(1+y)t - x\}^2,$$

this makes

(13) $$\frac{p^2}{a^2} = \frac{r^2-R}{a^2} = \frac{t-y}{t} - \frac{\frac{1}{4}T}{t^4}$$
$$= \frac{\{2t^2-(1+y)t+x\}^2}{4t^4},$$

(14) $$\frac{p}{a} = 1 - \frac{1+y}{2t} + \frac{x}{2t^2}.$$

As

(15) $$\frac{1}{t} = \frac{1}{y}\left(1-\frac{r^2}{a^2}\right),$$

therefore

(16) $$\frac{p}{a} = Aa^3\left(\frac{r}{a}\right)^4 + Ba\left(\frac{r}{a}\right)^2 + \frac{C}{a}$$
$$= 1 - \frac{1+y}{2y}\left(1-\frac{r^2}{a^2}\right) + \frac{x}{2y^2}\left(1-\frac{r^2}{a^2}\right)^2,$$

so that

(17) $$Aa^3 = \frac{x}{2y^2}, \quad Ba = -\frac{2x-y-y^2}{2y^2}, \quad \frac{C}{a} = \frac{x-y+y^2}{2y^2};$$

and

(18) $$\frac{p}{a} = \frac{x}{2y^2}\left(\frac{r^4}{a^4} - \frac{2x-y-y^2}{x}\frac{r^2}{a^2} + \frac{x-y+y^2}{x}\right),$$

(19) $$\frac{R}{a^2} = \frac{x^2}{4y^4}\left(1-\frac{r^2}{a^2}\right)\left[\frac{r^6}{a^6} - \frac{3x-2y-2y^2}{x}\frac{r^4}{a^4} + \frac{(x-y-y^2)(3x-y-y^2)}{x^2}\frac{r^2}{a^2} - \frac{(x-y+y^2)^2}{x^2}\right]$$
$$= \frac{x^2}{4y^4}\left(1-\frac{r^2}{a^2}\right)\left[\frac{r^2}{a^2}\left(\frac{r^2}{a^2} - \frac{x-y-y^2}{x}\right)^2 - \left(\frac{r^2}{a^2} - \frac{x-y+y^2}{x}\right)^2\right].$$

Thus $p=0$, when

(20) $$\frac{r^2}{a^2} = \frac{2x-y-y^2 \pm y\sqrt{\{(1+y)^2-8x\}}}{2x}$$

and then $R = r^2$.

At a point of inflexion, $\rho = \infty$, and

(21) $$\frac{r^2}{a^2} = -\frac{Ba}{2Aa^3} = \frac{2x-y-y^2}{2x},$$

(22) $$t = s + x = \frac{2x}{1+y}.$$

Now, from (10),

$$\frac{dr^2}{a^2}=\frac{y\,ds}{(s+x)^2}, \tag{23}$$

so that dr^2 and ds are of the same sign if y is positive;

$$\frac{dr^2}{\sqrt{R}}=2ay\,\frac{ds}{\sqrt{S}}, \tag{24}$$

$$\frac{s'}{a}=\frac{1}{2a}\int^{a^2}\frac{dr^2}{\sqrt{R}}=y\int^{\infty}\frac{ds}{\sqrt{S}}=yu, \tag{25}$$

$$\theta=\frac{1}{2}\int^{\infty}\frac{2t^2-(1+y)t+x}{t(t-y)}\,\frac{y\,ds}{\sqrt{S}} \tag{26}$$

$$=\int^{\infty}\left(y-\frac{1}{2}\,\frac{x}{s+x}-\frac{1}{2}\,\frac{-x+y-y^2}{s+x-y}\right)\frac{ds}{\sqrt{S}},$$

and now θ has been resolved into three elliptic integrals, one of the first kind proportional to the arc s', and two of the third kind, with parameters v and $3v$.

4. Written in the standard form (1) § 2,

$$I(v)=\frac{1}{2}\int^{\infty}\frac{P(v)(s+x)-x}{s+x}\,\frac{ds}{\sqrt{S}}, \tag{1}$$

$$I(2v)=\frac{1}{2}\int\frac{P(2v)s-xy}{s}\,\frac{ds}{\sqrt{S}}, \tag{2}$$

$$I(3v)=\frac{1}{2}\int\frac{P(3v)(s+x-y)-(-x+y-y^2)}{s+x-y}\,\frac{ds}{\sqrt{S}}, \tag{3}$$

. . . .

so that, in (26) § 3,

$$\theta=I(v)+I(3v)+\left[y-\frac{1}{2}P(v)-\frac{1}{2}P(3v)\right]u \tag{4}$$

$$=I(v)+I(3v)-uP(2v),$$

in consequence of the formula

$$P(v)-2P(2v)+P(3v) \tag{5}$$

$$=2i(\zeta v-2\zeta 2v+\zeta 3v)$$

$$=\frac{2i\wp' 2v}{\wp 2v-\wp v}=\frac{2xy}{x}=2y.$$

The formula for the addition of the parameters of two elliptic integrals of the third kind may now be employed for the simplification of $I(v)+I(3v)$ in the expression for θ in (4).

5. In the first place it is easily verified by a differentiation that

(1) $$I(3v) - 3I(v) = \sin^{-1} \frac{\sqrt{S}}{2(s+x)^{\frac{3}{2}}(s+x-y)^{\frac{1}{2}}}$$
$$= \cos^{-1} \frac{2(s+x)^2 - (1+y)(s+x) + x}{2(s+x)^{\frac{3}{2}}(s+x-y)^{\frac{1}{2}}}$$
$$= \cos^{-1} \frac{p}{r} = \lambda,$$

in Halphen's notation (F. E. II, p. 196); so that

(2) $$\omega = \theta - \lambda = 4I(v) - uP(2v)$$
$$= 4I(v) - \frac{P(2v)}{y}\frac{s'}{a}.$$

This is otherwise evident from the intrinsic equation of the curve in (1) § 1, which gives

(3) $$\frac{d\omega}{ds'} = \frac{1}{\varrho} = 4Ar^2 + 2B,$$

(4) $$d\omega = (2Ar^2 + B)\frac{dr^2}{\sqrt{R}}$$
$$= \left(1 + y - \frac{2x}{s+x}\right)\frac{ds}{\sqrt{S}},$$

so that

(5) $$\omega = 4I(v) - uP(2v),$$

because

(6) $$P(2v) - 2P(v) = 2i(\zeta 2v - 2\zeta v)$$
$$= i\frac{\wp'' v}{\wp' v} = i\frac{x(1+y)}{-ix} = -1 - y.$$

6. By another method of addition of the parameters, we find

(1) $$I(v) + I(3v) = I(4v) + \Phi,$$

where

(2) $$\Phi = \sin^{-1} \frac{\sqrt{S}}{2\sqrt{\left\{(s+x)(s+x-y)\left(s+x\frac{x-y+y^2}{y}\right)\right\}}}$$
$$= \cos^{-1} \frac{\frac{2x-y+y^2}{y}(s+x) - x}{2\sqrt{\left\{(s+x)(s+x-y)\left(s+x\frac{x-y+y^2}{y}\right)\right\}}},$$

which can be verified by a differentiation.

In fact, the general formula for the addition of parameters can be written

(3) $$I(mv) + I(nv) = I(m+n)v + \Phi$$

where

(4) $$\Phi = \sin^{-1} \frac{\sqrt{S}}{2\sqrt{\{s - s(mv) \,.\, s - s(nv) \,.\, s - s(m+n)v\}}}$$

$$= \cos^{-1} \frac{\sqrt{-\Sigma_m} \dfrac{s - s_n}{s_m - s_n} + \sqrt{-\Sigma_n} \dfrac{s_m - s}{s_m - s_n}}{2\sqrt{\{s - s_m \,.\, s - s_n \,.\, s - s_{m+n}\}}},$$

to be verified by differentiation.

7. But it is difficult to combine $I(4v)$ and Φ into a single term; so that, for practical purposes, it is preferable to calculate $I(v)$ and $I(3v)$ separately, and then to combine them into a single term, which we shall denote by θ', with

(1) $$\theta = \theta' - uP(2v).$$

In general, with a parameter

(2) $$v = \frac{2\omega'}{n}, \quad n \text{ odd},$$

(3) $$I(v) = \frac{1}{n} \sin^{-1} \frac{t^{\frac{1}{2}(n-3)} + qt^{\frac{1}{2}(n-5)} + \cdots}{2t^{\frac{1}{2}n}} \sqrt{T}$$

$$= \frac{1}{n} \cos^{-1} \frac{nP(v)t^{\frac{1}{2}(n-1)} + \cdots}{2t^{\frac{1}{2}n}},$$

(4) $$I(3v) = \frac{1}{n} \sin^{-1} \frac{t^{\frac{1}{2}(n-3)} + \cdots}{2(t-y)^{\frac{1}{2}n}} \sqrt{T}$$

$$= \frac{1}{n} \cos^{-1} \frac{nP(3v)t^{\frac{1}{2}(n-1)} + \cdots}{2(t-y)^{\frac{1}{2}n}},$$

so that

(5) $$\theta' = \frac{1}{n} \sin^{-1} \frac{\frac{1}{2}n(Pv + P3v)t^{n-2} + P_1 t^{n-3} + \cdots + P_{n-2}}{2(t^2 - yt)^{\frac{1}{2}n}} \sqrt{T}$$

$$= \frac{1}{n} \cos^{-1} \frac{2t^n + Q_1 t^{n-1} + \cdots + Q_n}{2(t^2 - yt)^{\frac{1}{2}n}}.$$

With a parameter

(6) $$v = \frac{\omega'}{n}, \quad n \text{ odd},$$

(7) $$I(v) = \frac{1}{n} \sin^{-1} \frac{nP(v)t^{\frac{1}{2}(n-3)} + \cdots}{t^{\frac{1}{2}n}} \sqrt{(t - t_1 \,.\, t - t_2)}$$

$$= \frac{1}{n} \cos^{-1} \frac{t^{\frac{1}{2}(n-1)} + \cdots}{t^{\frac{1}{2}n}} \sqrt{(t - t_3)},$$

with a corresponding expression for $I(3v)$, so that θ' assumes the same form as in (5).

With a parameter

(8) $$v = \frac{\omega'}{2n}, \quad n \text{ odd},$$

(9) $$I(v) = \frac{1}{2n} \cos^{-1} \frac{t^{n-1} + \cdots}{t^n} \sqrt{(t - t_1 \,.\, t - t_2)}$$
$$= \frac{1}{2n} \sin^{-1} \frac{2nP(v)t^{n-1} + \cdots}{t^n} \sqrt{(t - t_3)},$$

with a corresponding expression for $I(3v)$; but θ' again assumes the same form as in (5).

8. Writing ϱ for $\frac{r^2}{a^2}$ (again a different use of the letter ϱ to that employed in (1) § 1, which must now be distinguished in the sequel by an accent when used to denote the radius of curvature), and replacing t by $\frac{y}{1-\varrho}$, then (5) § 7 will assume the form

(1) $$\theta' = \frac{1}{n} \sin^{-1} \frac{\varrho^{n-2} + H_1 \varrho^{n-3} + \cdots + H_{n-2}}{Q \varrho^{\frac{1}{2}n}} \sqrt{P}$$

(2) $$= \frac{1}{n} \cos^{-1} \frac{\varrho^n + L_1 \varrho^{n-1} + \cdots + L_n}{Q \varrho^{\frac{1}{2}n}},$$

where

(3) $$P = -\left(\varrho^2 - \frac{2x - y - y^2}{x} \varrho + \frac{x - y + y^2}{x}\right)^2 + 4 \frac{y^4}{x^2} \varrho.$$

From (7) § 1,

(4) $$\theta = \frac{1}{2} \int_{\varrho}^{1} \frac{\varrho^2 - \frac{2x - y - y^2}{x} \varrho + \frac{x - y + y^2}{x}}{\varrho \sqrt{P}} \, d\varrho,$$

so that, from (1) § 7,

(5) $$\theta' = \theta + \frac{P(2v)}{2ay} \int^{a^2} \frac{dr^2}{\sqrt{R}}$$
$$= \theta + \frac{y}{x} P(2v) \int^{1} \frac{d\varrho}{\sqrt{P}}$$
$$= \frac{1}{2} \int \frac{\varrho^2 + \left(2 \frac{y}{x} P 2v - \frac{2x - y - y^2}{x}\right) \varrho + \frac{x - y + y^2}{x}}{\varrho \sqrt{P}} \, d\varrho$$

in which

(6) $$2 \frac{y}{x} P(2v) - \frac{2x - y - y^2}{x} = \frac{y}{x} P(4v).$$

The comparison of the differentiations of θ' in (1) and (2) will lead to sufficient equations, and even superfluous equations to serve as verifications, for the determination of the coefficients H and L, when n, x, y, and $P(2v)$ are assigned.

Omitting the algebraical details, it will be found in this way that the leading coefficients are given by

$$(7) \qquad H_1 = \frac{2x-y-y^2}{x} - n\frac{y}{x}P(2v),$$

$$(8) \qquad L_1 = \qquad\qquad - n\frac{y}{x}P(2v),$$

$$(9) \qquad H_2 = \left[\frac{2x-y-y^2}{x} - n\frac{y}{x}P(2v)\right]\left[\frac{2x-y-y^2}{x} - \frac{1}{2}n\frac{y}{x}P(2v)\right] - \frac{x-y+y^2}{x},$$

$$(10) \qquad L_2 = -\frac{1}{2}n\frac{y}{x}P(2v)\left[\frac{2x-y-y^2}{x} - n\frac{y}{x}P(2v)\right] = \frac{1}{2}L_1H_1,$$

. . . .

$$(11) \qquad L_n = -\frac{x-y+y^2}{x}H_{n-2}.$$

9. It will also be found that, if P is split up into the two factors P_1 and P_2, where

$$(1) \qquad P_1 = -\frac{r^4}{a^4} + \frac{2x-y-y^2}{x}\frac{r^2}{a^2} + 2\frac{y^2}{x}\frac{r}{a} - \frac{x-y+y^2}{x} = \left(1-\frac{r}{a}\right)\left(\frac{r^3}{a^3} + \frac{r^2}{a^2} - \frac{x-y-y^2}{x}\frac{r}{a} - \frac{x-y+y^2}{x}\right),$$

$$(2) \qquad P_2 = \frac{r^4}{a^4} - \frac{2x-y-y^2}{x}\frac{r^2}{a^2} + 2\frac{y^2}{x}\frac{r}{a} + \frac{x-y+y^2}{x} = \left(1+\frac{r}{a}\right)\left(\frac{r^3}{a^3} - \frac{r^2}{a^2} - \frac{x-y-y^2}{x}\frac{r}{a} + \frac{x-y+y^2}{x}\right),$$

we can generally, when n is odd, replace the expressions for θ' in (1) and (2), § 8 by $\left(\text{writing } r \text{ for } \frac{r}{a}\right)$

$$(3) \qquad \theta' = \frac{2}{n}\sin^{-1}\frac{r^{n-2} + h_1r^{n-3} + h_2r^{n-4} + \cdots}{qr^{\frac{1}{2}n}}\sqrt{P_1}$$

$$(4) \qquad = \frac{2}{n}\cos^{-1}\frac{r^{n-2} - h_1r^{n-3} + h_2r^{n-4} - \cdots}{qr^{\frac{1}{2}n}}\sqrt{P_2}$$

leading to the differential relation

$$(5) \qquad \frac{d\theta'}{dr} = -\frac{r^4 + \frac{y}{x}P(4v)r^2 + \frac{x-y+y^2}{x}}{r\sqrt{P}};$$

and now we find

(6) $$h_1 = 0, \quad H_1 = 2h_2, \quad H_2 = h_2^2 + 2h_4, \ldots$$

and the number of coefficients h required is now only half that required in the previous method.

10. The curve becomes an algebraical curve by making

(1) $$P(2v) = 0,$$

and now

(2) $$H_1 = \frac{2x - y - y^2}{x}, \; L_1 = 0, \; H_2 = \left(\frac{2x-y-y^2}{x}\right)^2 - \frac{x-y+y^2}{x},$$
$$L_2 = 0;$$

and further

(3) $$H_3 = \left(\frac{2x-y-y^2}{x}\right)^3 - 2\,\frac{2x-y-y^2}{x} \cdot \frac{x-y+y^2}{x} - \frac{n-6}{3}\,\frac{y^4}{x^2},$$
$$L_3 = -\frac{1}{3}\,n\,\frac{y^4}{x^2}, \ldots.$$

But, in the general case, a secular term $uP(2v)$ is associated with θ, and this is expressed in terms of Legendre's $F\varphi$ by

(4) $$\frac{P(2v)}{\sqrt{(s_1 - s_3)}}\,F\varphi,$$

where

(5) $$\sin^2\varphi = \frac{s_1 - s_3}{s - s_3}, \quad \kappa^2 = \frac{s_2 - s_3}{s_1 - s_3}, \quad \kappa'^2 = \frac{s_1 - s_2}{s_1 - s_3};$$

or by

(6) $$\frac{\frac{1}{2}P(2v)}{\sqrt[4]{(s_1 - s_2 \,.\, s_3 - s_2)}}\,F\varphi,$$

where

(7) $$\cot^2\tfrac{1}{2}\varphi = \frac{s - s_2}{\sqrt{(s_1 - s_2 \,.\, s_3 - s_2)}}, \quad \kappa^2, \kappa'^2 = \frac{1}{2}\left[1 \mp \frac{s_2 - \frac{1}{2}(s_1 + s_3)}{\sqrt{(s_1 - s_2 \,.\, s_3 - s_2)}}\right],$$

according as the discriminant of S in (2) § 2 is positive or negative.

11. As a first application of the preceding analysis, take the parameter

(1) $$v = \frac{2}{3}\,\omega',$$

and the associated integral (L. M. S. XXV, p. 210)

(2) $$I = \frac{1}{2}\int^{\infty} \frac{\frac{1}{3}s + m}{s\sqrt{S}}\,ds$$
$$= \frac{1}{3}\sin^{-1}\frac{s+m}{2s^{\frac{3}{2}}} = \frac{1}{3}\cos^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}},$$

where

(3) $$S = 4s^3 - (s+m)^2.$$

We now substitute

(4) $$\frac{r^2}{a^2} = \frac{1}{s},$$

(5) $$\frac{R}{a^2} = \frac{\frac{1}{4}S}{s^4},$$

so that

(6) $$\frac{p}{a} = \frac{s+m}{2s^2}$$

and

(7) $$Aa^3 = \frac{1}{2}m, \quad Ba = \frac{1}{2}, \quad C = 0.$$

Now

(8) $$s' = \frac{1}{2}\int_0 \frac{dr^2}{\sqrt{R}} = a\int^\infty \frac{ds}{\sqrt{S}} = au;$$

and

(9) $$\theta = \frac{1}{2}\int_0 \frac{\frac{1}{2}mr^2 + \frac{1}{2}}{\sqrt{R}}\, dr^2$$

$$= \frac{1}{2}\int^\infty \frac{s+m}{s\sqrt{S}}\, ds$$

$$= I + \frac{1}{3}u$$

$$= \frac{1}{3}\sin^{-1}\left[\frac{1}{2}\,\frac{r}{a}\left(m\,\frac{r^2}{a^2}+1\right)\right] + \frac{1}{3}\,\frac{s'}{a},$$

or

(10) $$\frac{r}{a}\left(m\,\frac{r^2}{a^2}+1\right) = 2\sin\left(3\vartheta - \frac{s'}{a}\right),$$

where m is an arbitrary number.

The discriminant of S is $-m^3(1+27m)$, which is positive in the region $0 > m > -\frac{1}{27}$.

To obtain a closed curve, the real period ω of the elliptic integral u must be made an aliquot part of π, by an appropriate choice of m.

12. The next case of a parameter

$$v = \frac{1}{2}\omega',$$

is obtained by putting $y = 0$ (L. M. S. XXV, p. 211); but this makes $r^2 = a^2$, so that the curve is circular, and the case is devoid of interest.

13. With a parameter

(1) $$v = \frac{1}{3}\omega',$$

(2) $$\gamma_6 = 0, \quad \text{or} \quad x = y - y^2 \quad \text{(L. M. S. XXV, p. 216);}$$

and now

(3) $$Aa^3 = \frac{1-y}{2y}, \quad Ba = -\frac{1-3y}{2y}, \quad C = 0;$$

(4) $$\varrho = \frac{r^2}{a^2} = 1 - \frac{y}{s+x}.$$

(5) $$t = s + x = y\,\frac{a^2}{a^2 - r^2}$$

(6) $$s - y^2 = y\,\frac{r^2}{a^2 - r^2}$$

(7) $$p = r^2(Ar^2 + B)$$

(8) $$\frac{p}{a} = \frac{1}{2y}\frac{r^2}{a^2}\left[(1-y)\frac{r^2}{a^2} - 1 + 3y\right]$$

(9) $$Ar^2 + B = \frac{2(s+x) - 1 + y}{2a(s+x)}$$

(10) $$\theta = \frac{1}{2}\int^{a^2} \frac{Ar^2 + B}{\sqrt{R}}\,dr^2$$

$$= \frac{1}{2}\int^{\infty} \frac{2y(s+x) - x}{s+x}\,\frac{ds}{\sqrt{S}}$$

$$= \left(y - \frac{1}{2}Pv\right)u + I(v),$$

where

(11) $$P(v) = \frac{2}{3},$$

(12) $$I(v) = \frac{1}{2}\int^{\infty} \frac{\frac{2}{3}(s + y - y^2) - y(1-y)}{s + y - y^2}\,\frac{ds}{\sqrt{S}}$$

$$= \frac{1}{3}\sin^{-1}\frac{\sqrt{\left\{s^2 - \frac{1}{4}(1-y)(1-5y)s + \frac{1}{4}y^2(1-y)^2\right\}}}{(s+y-y^2)^{\frac{3}{2}}}$$

$$= \frac{1}{3}\cos^{-1}\frac{s - \frac{1}{2}(1-y)(1-2y)}{(s+y-y^2)^{\frac{3}{2}}}\sqrt{(s-y^2)}$$

so that

$$(13)\qquad \theta + \left(\frac{1}{3} - y\right)u$$

$$= \frac{1}{3}\sin^{-1}\frac{\sqrt{[(a^2 - r^2)\{4y^2a^4 - (1-y)(1-5y)a^2r^2 + (1-y)^2r^4\}]}}{2ya^3}$$

$$= \frac{1}{3}\cos^{-1}\frac{(1-y)r^3 - (1-3y)a^2r}{2ya^3}$$

$$= \frac{1}{3}\cos^{-1}\frac{p}{r} = \frac{1}{3}\lambda.$$

By putting $y = \frac{1}{3}$, we obtain the algebraical curve

$$(14)\qquad r^3 = a^3\cos 3\vartheta, \quad \text{with} \quad \frac{p}{a} = \left(\frac{r}{a}\right)^4.$$

As y diminishes from $\frac{1}{3}$, points of inflexion come into existence, where

$$(15)\qquad \frac{r^2}{a^2} = \frac{1-3y}{2-2y};$$

also $p = 0$, when

$$(16)\qquad \frac{r^2}{a^2} = \frac{1-3y}{1-y}.$$

The discriminant of S is negative in this region of

$$(17)\qquad 1 > y > \frac{1}{9},$$

and

$$(18)\qquad u = \frac{F\varphi}{2y^{\frac{3}{4}}}, \quad \cot^2\frac{1}{2}\varphi = \frac{s - y^2}{y^{\frac{3}{2}}} = \frac{1}{\sqrt{y}}\frac{r^2}{a^2 - r^2},$$

$$(19)\qquad \varkappa^2 = \frac{\left(1 - y^{\frac{1}{2}}\right)^3\left(1 + 3y^{\frac{1}{2}}\right)}{16y^{\frac{3}{2}}}, \quad \varkappa'^2 = \frac{\left(1 + y^{\frac{1}{2}}\right)^3\left(-1 + 3y^{\frac{1}{2}}\right)}{16y^{\frac{3}{2}}};$$

and the apsidal angle

$$(20)\qquad \Theta = \frac{\pi}{6} - \frac{\frac{1}{3} - y}{y^{\frac{3}{4}}}K.$$

In Fig. 1 the apsidal angle $\Theta = -\frac{1}{4}\pi$, corresponding, by trial, to $y = 0.156$, nearly.

We can write (13) in the form

$$(21)\qquad \theta' = \frac{2}{3}\sin^{-1}\frac{\sqrt{[(a-r)\{2ya^2 - (1-y)ar - (1-y)r^2\}]}}{2\sqrt{(ya^3)}}$$

$$= \frac{2}{3}\cos^{-1}\frac{\sqrt{[(a+r)\{2ya^2 + (1-y)ar - (1-y)r^2\}]}}{2\sqrt{(ya^3)}}.$$

In the region where the discriminant of S is positive we can put (L. M. S. XXV, p. 217)

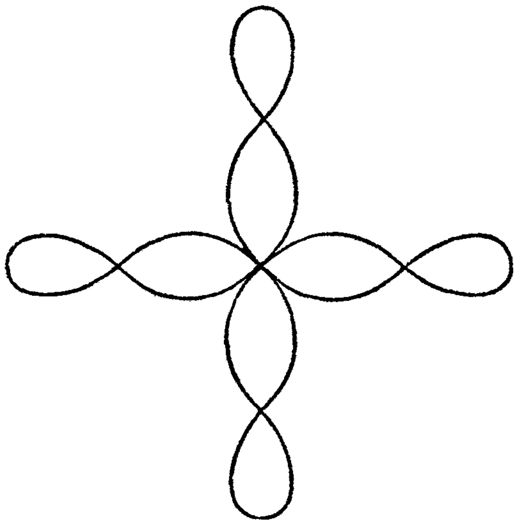

Fig. 1.

(22) $$y = \frac{-c}{2-5c+2c^2},$$

and now write (21) in the form

(23) $$\theta' = \frac{2}{3}\sin^{-1}\sqrt{\left[\frac{1}{2}\left(1-\frac{r}{a}\right)\left(1-\frac{1-c}{c}\frac{r}{a}\right)\left\{1+(1-c)\frac{r}{a}\right\}\right]}$$
$$= \frac{2}{3}\cos^{-1}\sqrt{\left[\frac{1}{2}\left(1+\frac{r}{a}\right)\left(1+\frac{1-c}{c}\frac{r}{a}\right)\left\{1-(1-c)\frac{r}{a}\right\}\right]}.$$

Arranged in descending order

(24) $$s_1 = \left(\frac{c-c^2}{2-5c+2c^2}\right)^2,\quad s_2 = \left(\frac{1-c}{2-5c+2c^2}\right)^2,\quad s_3 = \left(\frac{c}{2-5c+2c^2}\right)^2,$$

with c negative, $c < -1$.

In the outer branch $\left(1 > \frac{r}{a} > \frac{-c}{1-c}\right)$

(25) $$\theta = \theta' - \frac{2}{3}\cdot\frac{1-c+c^2}{\sqrt{(-2c^3+c^4)}}F\varphi,$$

with

(26) $$\sin^2\varphi = \frac{c^2}{1-2c}\frac{a^2-r^2}{r^2},\quad \varkappa^2 = \frac{1-2c}{-2c^3+c^4};$$

and the apsidal angle

(27) $$\Theta = \frac{\pi}{6} - \frac{2}{3}\cdot\frac{1-c+c^2}{\sqrt{(-2c^3+c^4)}}K.$$

Inflexions come into existence when $-1 > c > -\sqrt{2}-1$, and then

(28) $$\frac{r^2}{a^2} = \frac{1-c+c^2}{2(1-c)^2};$$

while $p = 0$, when

$$\frac{r^2}{a^2} = \frac{1-c+c^2}{(1-c)^2}. \tag{29}$$

In the inner branch $\left(\frac{1}{1-c} > \frac{r}{a} > 0\right)$.

$$\theta = \theta' + \frac{2}{3}\,\frac{1-c+c^2}{\sqrt{(-2c^3+c^4)}}\,F\varphi, \tag{30}$$

with

$$\sin^2\varphi = (c^2-2c)\,\frac{r^2}{a^2-r^2}, \tag{31}$$

and an apsidal angle

$$\Theta = \frac{\pi}{6} + \frac{2}{3}\,\frac{1-c+c^2}{\sqrt{(-2c^3+c^4)}}\,K. \tag{32}$$

In Fig. 2 the apsidal angles are made -60^0 and 120^0, by taking $c = -1.5$, about.

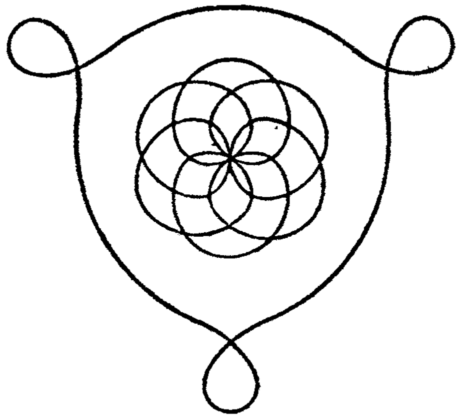

Fig. 2.

We can also make use of a quadric substitution

$$z = \frac{s^2 - \frac{1}{4}(1-y)(1-5y)s + \frac{1}{4}y^2(1-y)^2}{s-y^2} \tag{33}$$

$$= s - \frac{1}{4}(1-6y+y^2) + \frac{y^3}{s-y^2},$$

for the reduction of the secular term u; this will lead to the same modulus as before for a negative discriminant of S, changing to its reciprocal for a positive discriminant.

In the separating case, when $y = \frac{1}{9}$,

$$\frac{p}{a} = \frac{r^2}{a^2}\left(4\frac{r^2}{a^2} - 3\right), \tag{34}$$

$$\frac{R}{a^2} = \frac{r^2}{a^2}\left(1-\frac{r^2}{a^2}\right)\left(1-4\frac{r^2}{a^2}\right)^2; \tag{35}$$

and the curve has an asymptotic circle, of radius $\frac{1}{2}a$; the outer and inner branches being given by

$$\theta = \sin^{-1}\sqrt{\left(1-\frac{r^2}{a^2}\right)} - \frac{2}{3}\sqrt{3}\,\mathrm{sh}^{-1}\sqrt{\frac{a^2-r^2}{4r^2-a^2}}, \tag{36}$$

$$= \cos^{-1}\frac{r}{a} \qquad - \frac{2}{3}\sqrt{3}\,\mathrm{ch}^{-1}\sqrt{\frac{3r^2}{4r^2-a^2}},$$

and

$$\theta = \sin^{-1}\frac{r}{a} \qquad + \frac{2}{3}\sqrt{3}\,\mathrm{sh}^{-1}\sqrt{\frac{3r^2}{a^2-4r^2}}. \tag{37}$$

14. With a parameter

(1) $$v = \frac{2}{5}\omega',$$

(2) $$\gamma_5 = 0, \quad \text{when} \quad x = y \quad \text{(L. M. S. XXV, p. 213)};$$

(3) $$Aa^3 = \frac{1}{2x}, \quad Ba = -\frac{1-x}{2x}, \quad \frac{C}{a} = \frac{1}{2};$$

and

(4) $$\theta' = \frac{1}{5}\sin^{-1}\frac{\varrho^3 + H_1\varrho^2 + H_2\varrho + H_3}{Q\varrho^{\frac{5}{2}}}\sqrt{P}$$

$$= \frac{1}{5}\cos^{-1}\frac{\varrho^5 + L_1\varrho^4 + \ldots + L_5}{Q\varrho^{\frac{5}{2}}}$$

where

(5) $$P = -\{\varrho^2 - (1-x)\varrho + x\}^2 + 4x^2\varrho;$$

also

(6) $$P(2v) = \frac{1-3x}{5}, \quad P(4v) = -\frac{3+x}{5}:$$

so that

(7) $$\frac{d\theta'}{d\varrho} = -\frac{1}{2}\frac{\varrho^2 - \frac{3+x}{5}\varrho + x}{\varrho\sqrt{P}}$$

and we find, as in equations (7) ... (11), § 8,

(8) $$H_1 = 2x, \quad H_2 = x^2, \quad H_3 = -x^2,$$

$$L_1 = -1 + 3x, \quad L_2 = -x + 3x^2, \quad L_3 = -2x^2 + 3x^3, \quad L_4 = -x^2 - 2x^3,$$
$$L_5 = x^3; \quad \text{also} \quad Q = 2x.$$

Written in the form of (3), (4) § 9,

(9) $$\theta' = \frac{2}{5}\sin^{-1}\frac{\frac{r^3}{a^3} + x\frac{r}{a} + x}{2\sqrt{x}\left(\frac{r}{a}\right)^{\frac{5}{2}}}\sqrt{P_1},$$

(10) $$= \frac{2}{5}\cos^{-1}\frac{\frac{r^3}{a^3} + x\frac{r}{a} - x}{2\sqrt{x}\left(\frac{r}{a}\right)^{\frac{5}{2}}}\sqrt{P_2},$$

and

(11) $$P_1 = -\frac{r^4}{a^4} + (1-x)\frac{r^2}{a^2} + 2x\frac{r}{a} - x$$

$$= \left(1 - \frac{r}{a}\right)\left(\frac{r^3}{a^3} + \frac{r^2}{a^2} + x\frac{r}{a} - x\right),$$

(12) $$P_2 = \left(1 + \frac{r}{a}\right)\left(\frac{r^3}{a^3} - \frac{r^2}{a^2} + x\frac{r}{a} + x\right).$$

The algebraical case is obtained by putting

(13) $$x = \frac{1}{3},$$

and $\frac{r}{a}$ now oscillates between 1 and $\frac{1}{3}(\sqrt{10}-1)$, the real root of

(14) $$\frac{r^3}{a^3} + \frac{r^2}{a^2} + \frac{1}{3}\frac{r}{a} - \frac{1}{3} = 0.$$

Now in this algebraical curve,

(15) $$\frac{p}{a} = \frac{3}{2}\frac{r^4}{a^4} - \frac{r^2}{a^2} + \frac{1}{2},$$

so that p never vanishes; and at the inflexions

(16) $$\frac{r}{a} = \frac{1}{3}\sqrt{3}, \qquad \frac{p}{a} = \frac{1}{3}.$$

The curve is shown in the annexed figure 3.

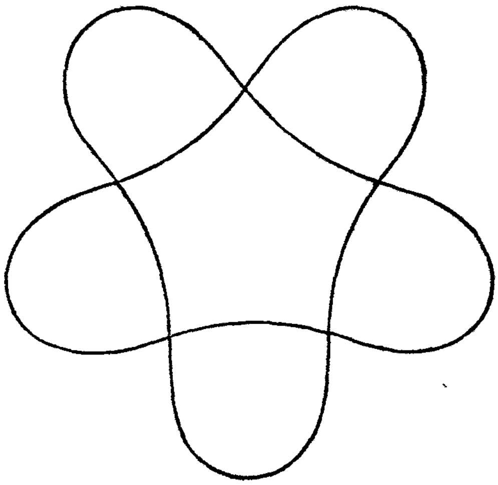

Fig. 3.

15. With a parameter

(1) $$v = \frac{2}{7}\omega',$$

(2) $\gamma_7 = 0$, when $x = z(1-z)^2$, $y = z(1-z)$ (L. M. S. XXV, p. 222),

(3) $$Aa^3 = \frac{1}{2z}, \quad Ba = -\frac{1-3z+z^2}{2z(1-z)}, \quad \frac{C}{a} = -\frac{z}{2(1-z)},$$

(4) $$P(2v) = \frac{3-9z+5z^2}{7}, \quad P(4v) = \frac{-1+3z+3z^2}{7}.$$

Writing r for $\frac{r}{a}$, the differential relations

(5) $$\frac{d\theta}{dr} = -\frac{r^4 - \frac{1-3z+z^2}{1-z}r^2 - \frac{z^2}{1-z}}{r\sqrt{P}},$$

(6) $$\frac{d\theta'}{dr} = -\frac{r^4 - \frac{1-3z+3z^2}{7(1-z)}r^2 - \frac{z^2}{1-z}}{r\sqrt{P}},$$

are satisfied by

(7) $$\theta = \theta' - \frac{3-9z+5z^2}{7(1-z)}\int_\varrho^1 \frac{d\varrho}{\sqrt{P}}$$

$$= \theta' - \frac{3-9z+5z^2}{7}\int^\infty \frac{ds}{\sqrt{S}},$$

where

(8) $$\theta' = \frac{2}{7}\sin^{-1}\frac{r^5 + h_1 r^4 + h_2 r^3 + \cdots + h_5}{qr^{\frac{7}{2}}}\sqrt{P_1},$$

(9) $$= \frac{2}{7}\cos^{-1}\frac{r^5 - h_1 r^4 + h_2 r^3 - \cdots - h_5}{qr^{\frac{7}{2}}}\sqrt{P_2},$$

(10) $$P_1 = -r^4 + \frac{1-3z+z^2}{1-z}r^2 + 2zr + \frac{z^2}{1-z}$$

$$= (1-r)\left(r^3 + r^2 + \frac{2z-z^2}{1-z}r + \frac{z^2}{1-z}\right).$$

and now we find

(11) $h_1 = 0$, $h_2 = -1+2z$, $h_3 = z$, $h_4 = z^2$, $h_5 = -\frac{z^3}{1-z}$;

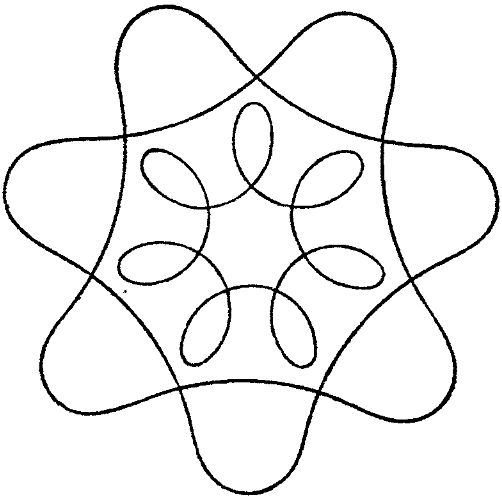

Fig. 4.

$$q^2 = \frac{4z^3}{(1-z)^2}.$$

Putting

(12) $P(2v) = 0$, $z = \frac{9 \pm \sqrt{21}}{10}$,

we obtain two algebraical cases represented in figure 4.

16. With a parameter

(1) $$v = \frac{2}{9}\omega',$$

(2) $$\gamma_9 = 0$$

when

$$x = c^2(1-c)(1-c+c^2), \quad y = c^2(1-c) \quad \text{(L. M. S. XXV, p. 232)},$$

(3) $Aa^3 = \frac{1-c+c^2}{2c^2(1-c)}$, $Ba = -\frac{1-2c+c^2+c^3}{2c^2(1-c)}$, $\frac{C}{a} = -\frac{1-c}{2c}$;

(4) $P(2v) = \frac{1-3c^2+7c^3}{9}$, $P(4v) = \frac{-7+18c-15c^2+5c^3}{9}$;

and, writing r for $\frac{r}{a}$,

(5) $$\theta' = \frac{2}{9}\sin^{-1}\frac{r^7+h_1r^6+h_2r^5+\cdots+h_7}{qr^{\frac{9}{2}}}\sqrt{P_1},$$

(6) $$= \frac{2}{9}\cos^{-1}\frac{r^7-h_1r^6+h_2r^5-\cdots-h_7}{qr^{\frac{9}{2}}}\sqrt{P_2},$$

(7) $$P_1 = -r^4 + \frac{1-2c+c^2+c^3}{1-c+c^2}r^2 + 2\frac{c^2-c^3}{1-c+c^2}r + \frac{c(1-c)^2}{1-c+c^2}$$

$$= (1-r)\left[r^3 + r^2 + \frac{c-c^3}{1-c+c^2}r + \frac{c(1-c)^2}{1-c+c^2}\right],$$

(8) $$h_1 = 0, \quad h_2 = -\frac{c(1-2c+3c^2)}{1-c+c^2}, \quad h_3 = \frac{c^2(1-c)}{1-c+c^2},$$

$$h_4 = c^2\frac{1-3c+3c^2}{1-c+c^2}, \quad h_5 = \frac{c^2(1-c)(1-2c)}{1-c+c^2}, \quad h_6 = -\frac{c^3(1-c)^2}{1-c+c^2},$$

$$h_7 = \frac{c^2(1-c)^3}{1-c+c^2}.$$

In the algebraical case

(9) $P(2v)=0,\ 1-3c^2+7c^3=0,\ c=-\frac{1}{f^{\frac{4}{3}}+f^{-\frac{4}{3}}},\ f=\frac{1}{2}(\sqrt{5}+1)$

and the curve is shown in fig. 5.

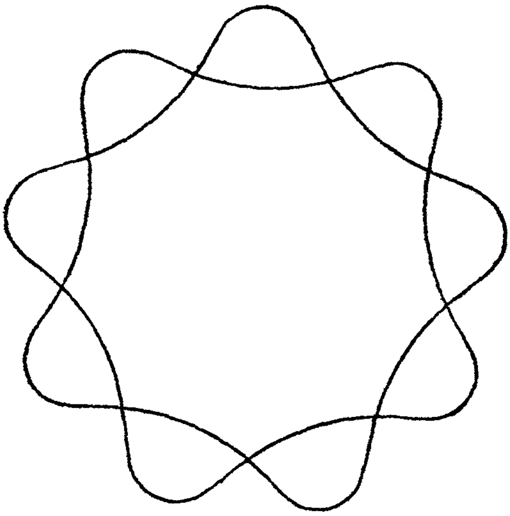

Fig. 5.

17. With a parameter

(1) $$v=\frac{1}{5}\,\omega',$$

we must employ the Transformation of the Tenth Order, and put (L. M. S. XXV, p. 235)

(2) $$x=-\frac{(c+1)(c-1)^3}{c(c^2-4c-1)^2},\quad y=-\frac{(c+1)(c-1)}{c(c^2-4c-1)},$$

(3) $$Aa^3=-\frac{c(c-1)}{c+1},\quad Ba=\frac{c^3+c^2+3c-1}{c^2-1},\quad \frac{C}{a}=-\frac{c+1}{c-1},$$

(4) $$P(v)=\frac{3c-1}{5c},\quad P(2v)=\frac{c^3-c^2+7c-3}{5a(c^2-4c-1)},$$
$$P(4v)=-\frac{3c^3+7c^2+c+1}{5c(c^2-4c-1)}$$

and now we find

(5) $$\theta'=\frac{2}{5}\sin^{-1}\frac{r^3+\frac{r}{c}-\frac{c+1}{c(c-1)}}{qr^{\frac{5}{2}}}\sqrt{P_1},$$

(6) $$=\frac{2}{5}\cos^{-1}\frac{r^3+\frac{r}{a}+\frac{c+1}{c(c-1)}}{qr^{\frac{5}{2}}}\sqrt{P_2},$$

where

(7) $$P_1=(1-r)\left(r+\frac{c+1}{c-1}\right)\left[\left(r-\frac{1}{c-1}\right)^2-\frac{c^2+c-1}{c(c-1)^2}\right],$$

(8) $$P_2=(1+r)\left(r-\frac{c+1}{c-1}\right)\left[\left(r+\frac{1}{c-1}\right)^2-\frac{c^2+c-1}{c(c-1)^2}\right],$$

(9) $$q^2=-4\frac{(c+1)^3}{c(c-1)^3}.$$

In the algebraical case

(10) $P(2v)=0,\ c^3-c^2+7c-3=0,\ c=\frac{1}{3}(\sqrt[3]{10}-1)^2=0.3526,$

so that q^2 is positive, and $\frac{r}{a}$ oscillates between 1 and $\frac{c+1}{c-1}=3.836,$ giving a curve of the same character as fig. 3.

18. To construct the curve for a parameter

(1) $$v=\frac{2\omega'}{11},$$

in the form

(2) $$\theta'=\frac{2}{11}\sin^{-1}\frac{r^9+h_2r^7+\cdots+h_9}{qr^{\frac{9}{2}}}\sqrt{P_1}$$

(3) $$=\frac{2}{11}\cos^{-1}\frac{r^9+h_2r^7-\cdots-h_9}{qr^{\frac{9}{2}}}\sqrt{P_2},$$

we must take (L. M. S. XXV, p 242)

(4) $$x=-\tfrac{1}{2}c(1+c)(1+2c+\sqrt{C}),\quad y=-c\frac{1+4c+2c^2+\sqrt{C}}{2(1+c)},$$

where

(5) $$C=1+4c+8c^2+4c^3,$$

(6) $$P(2v)=\frac{6+27c+44c^2+18c^3+(8+13c)\sqrt{C}}{22(1+c)},$$

(7) $$P(4v)=\frac{12+43c+44c^2+14c^3-(6+9c)\sqrt{C}}{22(1+c)}.$$

With $\gamma_{13}=0$, (L. M. S. XXV, § 50)

(8) $$v=\frac{2\omega'}{13},$$

(9) $$x=\frac{c^2}{2(1+c)}\{1+2c-2c^2+2c^3+6c^4+c^5+0+2c^7+c^8$$
$$+(1-2c-c^2+c^3-c^4-c^5)\sqrt{C}\},$$

(10) $$y=\frac{c^2}{2(1+c)^2}\{1+3c+0+0+2c^4+c^5+(1-c-c^2)\sqrt{C}\},$$

(11) $$C=1+4c+6c^2+2c^3+c^4+2c^5+c^6,$$

(12) $$13P(2v)$$
$$=\frac{6+12c-9c^2-33c^3+4c^4+8c^5-18c^6-11c^7+(4+0-15c^2+7c^3+11c^4)\sqrt{C}}{2(1+c)^2},$$

(13) $$13P(4v)$$
$$=\frac{-14-54c-83c^2-79c^3-70c^4-62c^5-36c^6-9c^7+(8-26c-69c^2-25c^3+9c^4\sqrt{C}}{2(1+c)^2},$$

$$(14)\qquad \theta' = \frac{2}{13} \sin^{-1} \frac{r^{11} + h_2 r^9 + h_3 r^8 + \cdots + h_{11}}{q r^{\frac{13}{2}}} \sqrt{P_1},$$

$$(15)\qquad = \frac{2}{13} \cos^{-1} \frac{r^{11} + h_2 r^9 - h_3 r^8 + \cdots - h_{11}}{q r^{\frac{13}{2}}} \sqrt{P_2}.$$

With $\gamma_{15} = 0$, (L. M. S. XXV, § 56)

$$(16)\qquad v = \frac{2\omega'}{15},$$

$$(17)\qquad x = \frac{c(c+1)}{2(c^2+3c+3)} \{(c^9 + 2c^8 - 2c^7 - 8c^6 - 3c^5 + 6c^4 + 6c^3 + 2c^2 + 0 - 1)(c^2 + 3c + 3) + (c^9 + 4c^8 + 4c^7 - 6c^6 - 15c^5 - 8c^4 + 4c^3 + 6c^2 + 4c + 1)\sqrt{C}\},$$

$$(18)\qquad y = -\frac{c(c+1)}{2(c^2+3c+3)} \{c^4 + 0 - 2c^2 - c + 1)(c^2 + 3c + 3) + (c^4 + 2c^3 + 0 - 3c - 1)\sqrt{C}\},$$

$$(19)\qquad C = (c^2 - c - 1)(c^2 + 3c + 3),$$

$$(20)\qquad 15P(2v)$$

$$= \frac{(13c^6 + 9c^5 - 30c^4 - 35c^3 + 12c^2 + 9c + 6)(c^2 + 3c + 3) + (13c^5 + 35c^4 + 14c^3 - 51c^2 - 48c - 9)c\sqrt{C}}{2(c^2 + 3c + 3)},$$

$$(21)\qquad 15P(4v)$$

$$= \frac{(11c^6 + 33c^5 + 30c^4 - 25c^3 - 66c^2 - 57c - 18)(c^2 + 3c + 3) + (11c^5 + 55c^4 + 118c^3 + 123c^2 + 54c - 3)c\sqrt{C}}{2(c^2 + 3c + 3)},$$

$$(22)\qquad \theta' = \frac{2}{15} \sin^{-1} \frac{r^{13} + h_2 r^{11} + h_3 r^{10} + \cdots + h_{13}}{q r^{\frac{15}{2}}} \sqrt{P_1},$$

$$(23)\qquad = \frac{2}{15} \cos^{-1} \frac{r^{13} + h_2 r^{11} - h_3 r^{10} + \cdots - h_{13}}{q r^{\frac{15}{2}}} \sqrt{P_2}.$$

This is as far as we can go at present with these Elliptic Functions of the First Stage, and their Division-Values (Theilwerthe); the next cases of $\gamma_{17} = 0$, and $\gamma_{19} = 0$ have proved intractable by this method.

19. With a parameter

$$(1)\qquad v = \frac{1}{6}\omega',$$

$$(2)\qquad \theta' = \frac{1}{3} \sin^{-1} \frac{\varrho + H_1}{Q\varrho^{\frac{3}{2}}} \sqrt{P},$$

$$(3)\qquad = \frac{1}{3} \cos^{-1} \frac{\varrho^3 + L_1\varrho^2 + L_2\varrho + L_3}{Q\varrho^{\frac{3}{2}}}$$

and, turning to the Transformation of the Twelfth Order (L. M. S. XXV, p. 248)

$$(4)\qquad x = \frac{-c(1+c)(1+c+c^2)(1+c^2)}{(1-c)^2}, \quad y = \frac{-c(1+c)(1+c+c^2)}{1-c};$$

$$(5)\qquad Aa^3 = -\frac{1+c^2}{2c(1+c)(1+c+c^2)},$$

$$(6)\qquad Ba = \frac{1+2c+4c^2+2c^3+c^4}{2c(1+c)(1+c+c^2)},$$

$$(7)\qquad \frac{C}{a} = \frac{c(1+c)}{2(1+c+c^2)}$$

and we shall find

$$(8)\qquad \begin{aligned} H_1 &= -(1+c)^2, \\ L_1 &= -\frac{2(1+c+c^2)^2}{1+c^2}, \\ L_2 &= \frac{(1+c)^2(1+c+c^2)^2}{1+c^2}, \\ L_3 &= -\frac{c^2(1+c)^4}{1+c^2}, \\ Q &= \frac{2c^2(1+c)}{1+c^2}; \end{aligned}$$

$$(9)\qquad P = \left(1-\frac{r^2}{a^2}\right)\left(\frac{r^2}{a^2}-c^2\right)\left[\left\{\frac{r^2}{a^2}-\frac{c(1+c)^2}{1+c^2}\right\}^2-(1+c)^2\frac{r^2}{a^2}\right],$$

leading on differentiation to

$$(10)\qquad \frac{d\theta'}{d\varrho} = -\frac{1}{2}\,\frac{\varrho^2+\dfrac{1+2c+2c^3+c^4}{3(1+c^2)}\varrho-\dfrac{c^2(1+c)^2}{1+c^2}}{\varrho\sqrt{P}},$$

$$(11)\qquad \frac{d\theta}{d\varrho} = -\frac{1}{2}\,\frac{\varrho^2-\dfrac{1+2c+4c^2+2c^3+c^4}{1+c^2}\varrho-\dfrac{c^2(1+c)^2}{1+c^2}}{\varrho\sqrt{P}};$$

so that

$$(12)\qquad \theta'-\theta = \frac{2}{3}\,\frac{(1+c+c^2)^2}{1+c^2}\int_{\varrho}^{1}\frac{d\varrho}{\sqrt{P}};$$

and the secular term cannot be cancelled, so as to obtain an algebraical curve.

Equations (2) and (3) can also be written in the form

$$(13)\qquad \theta' = \frac{2}{3}\sin^{-1}\frac{\left(\frac{r}{a}+1+c\right)\sqrt{\left[\left(1-\frac{r}{a}\right)\left(\frac{r}{a}-c\right)\left\{\frac{r^2}{a^2}-(1+c)\frac{r}{a}-\frac{c(1+c)^2}{1+c^2}\right\}\right]}}{2c\sqrt{\frac{1+c}{1+c^2}}\left(\frac{r}{a}\right)^{\frac{3}{2}}},$$

$$(14)\qquad = \frac{2}{3}\cos^{-1}\frac{\left(\frac{r}{a}-1-c\right)\sqrt{\left[\left(1+\frac{r}{a}\right)\left(\frac{r}{a}+c\right)\left\{\frac{r^2}{a^2}+(1+c)\frac{r}{a}-\frac{c(1+c)^2}{1+c^2}\right\}\right]}}{2c\sqrt{\frac{1+c}{1+c^2}}\left(\frac{r}{a}\right)^{\frac{3}{2}}}.$$

20. With the parameter

(1) $$v = \frac{\omega'}{4n},$$

where n is an integer, we must proceed to transformations of the order $8n$; and it is convenient to employ the Elliptic Functions of the Second Stage, explained in the *Transformation and Division of Elliptic Functions* Proc. L. M. S. XXVII, p. 449, where we put

(2) $$x = -\frac{m^3\alpha}{(\alpha-m)^2}, \quad y = -\frac{(1-2m)\alpha}{\alpha-m};$$

so that S has the factor

(3) $$s - s_\gamma = \frac{m^2\alpha^2}{(\alpha-m)^2}$$

$$= \frac{m^2\alpha}{\alpha-m}\,\frac{\dfrac{r^2}{a^2} - \left(\dfrac{1-m}{m}\right)^2}{1-\dfrac{r^2}{a^2}},$$

on putting, as in (10) § 3,

(4) $$s + x = \frac{y}{1-\dfrac{r^2}{a^2}},$$

(5) $$s = \frac{m^3\alpha}{(\alpha-m)^2}\,\frac{\dfrac{r^2}{a^2} + \dfrac{(1-2m)\alpha - m(1-m)^2}{m^3}}{\dfrac{r^2}{a^2}-1}.$$

The other factors of S being denoted by $s-s_\alpha$ and $s-s_\beta$, (L. M. S. XXVII, p. 452) we find, after reduction,

(6) $$4(s-s_\alpha)(s-s_\beta) = 4s^2 + \frac{4(1-2m)\alpha - 1}{(\alpha-m)^2}m^2 s + \frac{m^4(1-2m)^2\alpha^2}{(\alpha-m)^4}$$

$$= \frac{\alpha}{(\alpha-m)^3}\,\frac{m^4\dfrac{r^4}{a^4} + \left\{4(1-2m)\alpha - 1 + 2m - 2m^2\right\}m^2\dfrac{r^2}{a^2} + \left\{2(1-2m)\alpha - m + m^2\right\}^2}{\left(1-\dfrac{r^2}{a^2}\right)^2}.$$

It is convenient in the sequel to put

(7) $$(1-2m)\alpha = (m-m^2)\beta = \frac{\beta}{4\gamma+4},$$

(8) $$(2m-1)^2 = \frac{\gamma}{\gamma+1};$$

and to change the scale of the figure by putting

(9) $$\frac{r^2}{b^2} = \varrho, \quad \text{where} \quad \frac{b^2}{a^2} = \frac{1-m}{m};$$

(10) $$s - s_\gamma = \frac{m^2\alpha}{\alpha-m}\,\frac{\varrho - \dfrac{1-m}{m}}{\dfrac{m}{1-m} - \varrho}.$$

(11) $A\alpha^3 = -\dfrac{m^3}{2(1-2m)^2\alpha} = -\dfrac{m^2}{2(1-2m)(1-m)\beta},$

(12) $Ba = -m\dfrac{2(1-2m)\alpha-1+2m-2m^2}{2(1-2m)^2\alpha} = -m\dfrac{\beta-2\gamma-1}{(1-2m)\beta},$

(13) $\dfrac{C}{a} = (1-m)\dfrac{2(1-2m)\alpha-m+m^2}{2(1-2m)^2\alpha} = (1-m\,\dfrac{2\beta-1}{2(1-2m)\beta};$

(14) $$\frac{p}{a} = -\frac{m^2}{2(1-2m)(1-m)\beta}\left[\frac{r^4}{a^4}+2(\beta-2\gamma+1)\frac{1-m}{m}\frac{r^2}{a^2} - (2\beta-1)\left(\frac{1-m}{m}\right)^2\right]$$

$$= -\frac{1-m}{2(1-2m)\beta}\left\{\frac{r^4}{b^4}+2(\beta-2\gamma-1)\frac{r^2}{b^2}-2\beta+1\right\}$$

or

(15) $$\frac{p}{b} = \frac{\sqrt{(m-m^2)}}{2(2m-1)\beta}\{\varrho^2+2(\beta-2\gamma-1)\varrho-2\beta+1\}$$
$$= \frac{\varrho^2+2(\beta-2\gamma-1)\varrho-2\beta+1}{4\beta\sqrt{\gamma}}.$$

When $m-m^2$ in negative, we put

(16) $$\frac{r^2}{b^2} = -\varrho, \quad \text{where} \quad \frac{b^2}{a^2} = \frac{m-1}{m};$$

and now γ is negative, and

(17) $$\frac{p}{b} = \frac{\varrho^2+2(\beta-2\gamma-1)\varrho-2\beta+1}{4\beta\sqrt{(-\gamma)}}.$$

Also

(18) $$4(s-s_\alpha)(s-s_\beta) = \frac{m^4\alpha}{(\alpha-m)^3}\,\frac{\varrho^2+2(2\beta-2\gamma-1)\varrho+(2\beta-1)^2}{\left(\frac{m}{1-m}-\varrho\right)^2},$$

(19) $$s+x = -\frac{m(1-2m)\alpha}{(1-m)(\alpha-m)}\,\frac{1}{\frac{m}{1-m}-\varrho}.$$

(20) $$\frac{R}{b^2} = \frac{m}{1-m}\,\frac{\frac{1}{4}S}{(s+x)^4}$$
$$= \frac{P_1P_2}{16\beta^2\gamma\left(\frac{m}{1-m}-\varrho\right)^4}$$

where

(21) $$P_1 = \left(\frac{m}{1-m}-\varrho\right)\left(\varrho-\frac{1-m}{m}\right)$$
$$= -\varrho^2+2(2\gamma+1)\varrho-1$$

(22) $$P_2 = \varrho^2+2(2\beta-2\gamma-1)\varrho+(2\beta-1)^2,$$

(23) $$P = P_1P_2$$
$$= -\{\varrho^2+2(\beta-2\gamma-1)\varrho-2\beta+1\}^2+16\beta^2\gamma\varrho;$$

and

(24) $$\theta = \frac{1}{2}\int_{\varrho}^{\frac{m}{1-m}} \frac{\varrho^2 + 2(\beta - 2\gamma - 1)\varrho - 2\beta + 1}{\varrho\sqrt{P}}\,d\varrho,$$

(25) $$\theta' = \frac{1}{2}\int \frac{\varrho^2 + \frac{y}{x}P(4v)\frac{m}{1-m}\varrho - 2\beta + 1}{\varrho\sqrt{P}}\,d\varrho,$$

(26) $$\theta' - \theta = \frac{y}{x}P(2v)\frac{m}{1-m}\int_{\varrho}^{\frac{m}{1-m}} \frac{d\varrho}{\sqrt{P}}.$$

21. As the effective parameter in the expression of the curve is $4v$, and now

(1) $$4v = \frac{\omega'}{n},$$

we find that θ' can be reduced to either of the equivalent forms

(2) $$\theta' = \frac{1}{n}\sin^{-1}\frac{\varrho^{n-1} + H_1\varrho^{n-2} + \cdots + H_{n-1}}{Q\varrho^{\frac{1}{2}n}}\sqrt{P_1},$$

(3) $$= \frac{1}{n}\cos^{-1}\frac{\varrho^{n-1} + L_1\varrho^{n-2} + \cdots + L_{n-1}}{Q\varrho^{\frac{1}{2}n}}\sqrt{P_2}.$$

This implies the identical relation

(4) $$Q^2\varrho^n = (\varrho^{n-1} + H_1\varrho^{n-2} + \cdots + H_{n-1})^2\{-\varrho^2 + 2(2\gamma+1)\varrho - 1)\}$$
$$+ (\varrho^{n-1} + L_1\varrho^{n-2} + \cdots + L_{n-1})^2\{\varrho^2 + 2(2\beta - 2\gamma - 1)\varrho$$
$$+ (2\beta - 1)^2\}$$

leading to the relations

(5) $$H_1 - L_1 = 2\beta,$$
$$\cdots\cdots\cdots$$
$$H_{n-1}^2 = (2\beta - 1)^2 L_{n-1}^2,$$

useful for the determination or verification of the coefficients H and L.

The comparison of the differentiations of (2) and (3) will serve better for the determination of these coefficients; and in this manner we find

(6) $$\begin{aligned} H_1 &= 2\gamma + 1 - n\frac{y}{x}P(2v)\frac{m}{1-m}, \\ L_1 &= -2\beta + 2\gamma + 1 - n\frac{y}{x}P(2v)\frac{m}{1-m}, \\ H_2 &= \\ L_2 &= \\ &\cdots\cdots\cdots \\ H_{n-1} &= (2\beta - 1)L_{n-1}. \end{aligned}$$

The curve is algebraical when $P(2v) = 0$, and then

(7) $H_1 = 2\gamma + 1, \qquad L_1 = -2\beta + 2\gamma + 1,$

(8) $H_2 = \frac{3}{2}(2\gamma+1)^2 - \frac{1}{2}, \quad L_2 = 2\beta^2 - 6\beta\gamma + 3\gamma^2 - 4\beta + 6\gamma + 1;$

. . . .

the roots of $P_2 = 0$ are now imaginary, and ϱ oscillates between $\frac{m}{1-m}$ and $\frac{1-m}{m}$, or $[\sqrt{(\gamma+1)} \pm \sqrt{\gamma}]^2$.

22. In the simplest of these cases, when

$$n = 1, \quad 4v = \omega', \quad v = \frac{1}{4}\omega',$$

we shall find that $\beta = 1$, in consequence of the relation

(1) $$s_\gamma - s(4v) = 0,$$

in (329) L. M. S. XXVII, p. 450; and now

(2) $$\theta' = \sin^{-1}\frac{\sqrt{\{-\varrho^2 + 2(1+2\gamma)\varrho - 1\}}}{2\sqrt{\varrho}},$$

(3) $$= \cos^{-1}\frac{\sqrt{\{\varrho^2 + 2(1-2\gamma)\varrho + 1\}}}{2\sqrt{\varrho}},$$

(4) $$= \frac{1}{2}\sin^{-1}\frac{\sqrt{P}}{2\varrho} = \frac{1}{2}\cos^{-1}\frac{\varrho^2 - 4\gamma\varrho + 1}{2\varrho}.$$

Also

$$P(4v) = 0,$$

and

(5) $$\frac{d\theta'}{d\varrho} = -\frac{1}{2}\frac{\varrho^2 - 1}{\varrho\sqrt{P}},$$

(6) $$\frac{d\theta}{d\varrho} = -\frac{1}{2}\frac{\varrho^2 - 4\gamma\varrho - 1}{\varrho\sqrt{P}},$$

(7) $$\theta' - \theta = 2\gamma\int_\varrho^{\frac{m}{1-m}}\frac{d\varrho}{\sqrt{P}}.$$

Making use of the quadric substitution

(8) $$t = \frac{P_1}{P_2} = \frac{-\varrho^2 + 2(1+2\gamma)\varrho - 1}{\varrho^2 + 2(1-2\gamma)\varrho + 1},$$

then $\varrho = \pm 1$ give the turring points of t, where t assumes the values

(9) $$\frac{\gamma}{1-\gamma} \quad \text{and} \quad -\frac{1+\gamma}{\gamma}.$$

We must distinguish between the two cases of $\gamma < 1$, and $\gamma > 1$.

I. $0 < \gamma < 1$; the roots of $P_2 = 0$ are imaginary and ϱ oscillates between

(10) $$\varrho_0, \varrho_3 = [\sqrt{(1+\gamma)} \pm \sqrt{\gamma}]^2,$$

the roots of $P_1 = 0$; also

(11) $$\frac{\gamma}{1-\gamma} - t = \frac{(\varrho-1)^2}{(1-\gamma)P_2},$$

(12) $$t + \frac{1+\gamma}{\gamma} = \frac{(\varrho+1)^2}{\gamma P_2},$$

(13) $$2\gamma \int_{\varrho}^{\varrho_0} \frac{d\varrho}{\sqrt{P}} = \sqrt{\frac{\gamma}{1-\gamma}} \int^{\infty} \frac{dt}{\sqrt{\left(4t\cdot t + \frac{1+\gamma}{\gamma}\cdot\frac{\gamma}{1-\gamma} - t\right)}}$$

an elliptic integral of the first kind, with modulus $\varkappa = \gamma$; and putting

(14) $$\cos\varphi = \sqrt{\left(\frac{1+\varkappa}{\varkappa}\right)}\,\frac{\varrho-1}{\varrho+1},$$

(15) $$2\gamma \int \frac{d\varrho}{\sqrt{P}} = \varkappa \int_0 \frac{d\varphi}{\Delta\varphi} = \varkappa F\varphi$$

so that

(16) $$\theta = \theta' - \varkappa F\varphi,$$

θ, θ', and φ starting from zero, where

(17) $$\varrho = \varrho_0 = [\sqrt{(1+x)} + \sqrt{\varkappa}]^2.$$

When $\varrho = 1$, $\varphi = \frac{1}{2}\pi$, $F\varphi = K$, and the arc of a loop is bisected; and

(18) $$\theta' = \sin^{-1}\sqrt{\varkappa} = \cos^{-1}\sqrt{(1-\varkappa)}.$$

When

$$\varrho = \varrho_3 = [\sqrt{(1+\varkappa)} - \sqrt{\varkappa}]^2,\quad \varphi = \pi,\quad F\varphi = 2K,\quad \theta' = 0;$$

and the apsidal angle

(19) $$\Theta = -2\varkappa K.$$

At the inflexions, $\varrho = 2\varkappa$, which makes

(20) $P_1 = 4\varkappa^2 + 4\varkappa - 1 = (2\varkappa+1)^2 - 2$

which is positive if

$$\varkappa > \frac{1}{2}(\sqrt{2}-1) > \sin 10^0 57';$$

and $p = 0$, when $\varrho^2 - 4\varkappa\varrho - 1 = 0$, which makes $P_1 = 2(\varrho - 1)$, which is positive, so that the curve is looped.

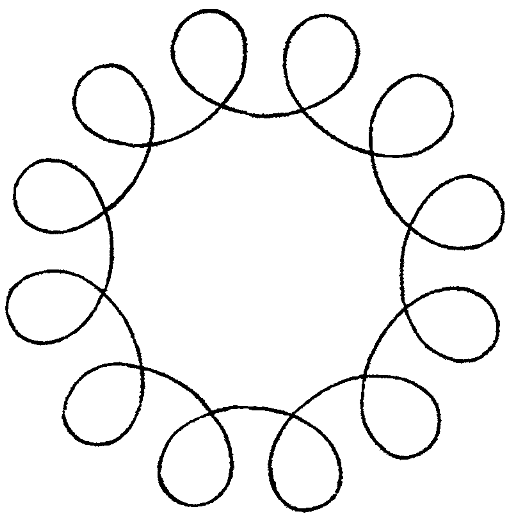
Fig. 6.

Fig. 6 has been drawn for a small modular angle, about $4^0 45'$, so as to show the running pattern, in the penultimate form of the case discussed by Halphen (F. E. II, p. 219).

II. $1 < \gamma < \infty$, the roots of $P_2 = 0$ are real, and denoting them by ϱ_1 and ϱ_2,

(21) $$\varrho_1,\ \varrho_2 = [\sqrt{\gamma} \pm \sqrt{(\gamma-1)}]^2$$
and

(22) $$2\gamma \int_{\varrho,\ \varrho_3}^{\varrho_0,\ \varrho} \frac{d\varrho}{\sqrt{P}} = \sqrt{\left(\frac{\gamma}{\gamma-1}\right)} \int^{\infty} \frac{dt}{\sqrt{\left(4t\cdot t + \frac{\gamma+1}{\gamma}\cdot t + \frac{\gamma}{\gamma-1}\right)}} = E\Phi,$$

with a modulus $\varkappa = \frac{1}{\gamma}$, and

(23) $$\sin^2\varphi = \frac{P_1}{\varkappa(\varrho+1)^2},\quad \cos^2\varphi = \frac{(1+\varkappa)P_2}{\varkappa(\varrho+1)^2},\quad \Delta^2\varphi = (1+\varkappa)\left(\frac{\varrho-1}{\varrho+1}\right)^2.$$

Thus at the bisection of the arc of a loop,

(24) $$\Delta^2\varphi = \sqrt{(1-\varkappa^2)},\quad \text{and}\quad \varrho = \frac{\sqrt[4]{(1+\varkappa)} + \sqrt[4]{(1-\varkappa)}}{\sqrt[4]{(1+\varkappa)} - \sqrt[4]{(1-\varkappa)}}.$$

At the inflexions, $\varrho = \frac{2}{\varkappa}$, which makes $P_2 = 2 - \left(\frac{2}{\varkappa} - 1\right)^2$, so that $\varkappa > 2(\sqrt{2}-1) > \sin 56^0$.

The outer branch $(\varrho_0 > \varrho > \varrho_1)$ has the apsidal angle

(25) $$\Theta = \frac{1}{2}\pi - K,$$

and is of the same character as the curves in Case I for a negative discriminant; and the curve is now completed by the inner branch

$$(\varrho_2 > \varrho > \varrho_3),$$

with the apsidal angle

(26) $$\Theta = \frac{1}{2}\pi + K,$$

forming a running pattern, without points where $p = 0$, and without inflexions.

An example is given in fig. 7 for apsidal angles of -45^0 and 225^0, when the modular angle is about $66^0\,21'$; this may be compared with figures given by Halphen.

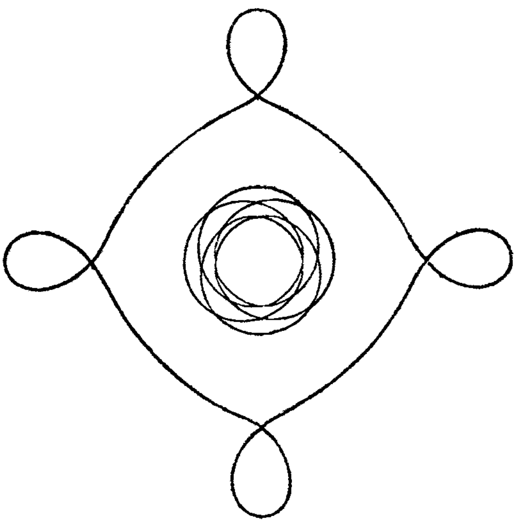

Fig. 7.

In the separating case between I and II, $\varkappa = 1$ and an asymptotic circle, $\varrho = 1$, makes its appearance; we find

(27) $$\theta = \sin^{-1}\frac{\sqrt{(-\varrho^2+6\varrho-1)}}{2\sqrt{\varrho}} - \operatorname{sh}^{-1}\frac{\sqrt{(-\varrho^2+6\varrho-1)}}{\sqrt{2(\varrho-1)}}$$
$$= \cos^{-1}\frac{\varrho-1}{2\sqrt{\varrho}} - \operatorname{ch}^{-1}\frac{\varrho+1}{\sqrt{2(\varrho-1)}},$$

in the outer branch; and

$$(28) \qquad \theta = \sin^{-1}\frac{\sqrt{(-1+6\varrho-\varrho^2)}}{2\sqrt{\varrho}} + \operatorname{sh}^{-1}\frac{\sqrt{(-1+6\varrho-\varrho^2)}}{\sqrt{2(1-\varrho)}}$$
$$= \cos^{-1}\frac{1-\varrho}{2\sqrt{\varrho}} + \operatorname{ch}^{-1}\frac{1+\varrho}{\sqrt{2(1-\varrho)}},$$

in the inner branch.

23. Proceeding to the next case of $n=2$ and a parameter $4v = \frac{1}{2}\omega'$, and considering only the algebraical case with $P(2v)=0$, then, from (2), (3), (7) § 21, the equations can be written

$$(1) \qquad \theta = \frac{1}{2}\sin^{-1}\frac{(\varrho+2\gamma+1)\sqrt{\{-\varrho^2+2(2\gamma+1)\varrho-1\}}}{Q\varrho},$$

$$(2) \qquad = \frac{1}{2}\cos^{-1}\frac{(\varrho-2\beta+2\gamma+1)\sqrt{\{\varrho^2+2(2\beta-2\gamma-1)\varrho+(2\beta-1)^2\}}}{Q\varrho};$$

and we shall find from the differentiations that

$$(3) \qquad \beta = -\frac{\sqrt{3}+1}{2}, \quad \gamma = -\frac{\sqrt{3}+1}{\sqrt{3}}, \quad Q^2 = 2\sqrt{3}(\sqrt{3}+1)^2;$$

so that we must put $\varrho = -\frac{r^2}{b^2}$, and now

$$(4) \qquad \frac{p}{b} = \frac{\left(\frac{r^2}{b^2}-\frac{\sqrt{3}+1}{2\sqrt{3}}\right)^2+\frac{5}{12}(\sqrt{3}+1)^2}{\frac{2}{\sqrt[4]{3}}(\sqrt{3}+1)^{\frac{3}{2}}},$$

so that p never vanishes; but there are points of inflexion where

$$(5) \qquad \frac{r^2}{b^2} = \frac{\sqrt{3}+1}{2\sqrt{3}},$$

$$\frac{p}{b} = \frac{5\sqrt[4]{3}}{24}\sqrt{(\sqrt{3}+1)};$$

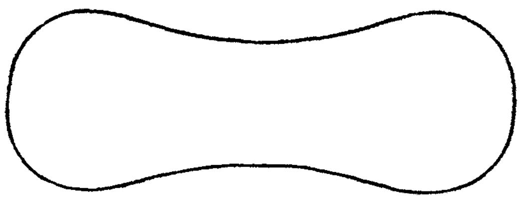

Fig. 8.

as shown in fig. 8.

24. It will not be difficult to work out the corresponding algebraical cases of the parameters

$$(1) \qquad 4v = \frac{1}{3}\omega', \quad \frac{1}{4}\omega', \quad \frac{1}{5}\omega', \ldots$$

when the equation of the curve takes one of the forms

$$(2) \qquad \theta = \frac{1}{3}\sin^{-1}\frac{\varrho^2+(2\gamma+1)\varrho+H_2}{Q\varrho^{\frac{3}{2}}}\sqrt{P_1},$$

$$(3) \qquad = \frac{1}{3}\cos^{-1}\frac{\varrho^2-(2\beta-2\gamma-1)\varrho+L_2}{Q\varrho^{\frac{3}{2}}}\sqrt{P_2};$$

or

$$(4)\qquad \theta = \frac{1}{4}\sin^{-1}\frac{\varrho^3+(2\gamma+1)\varrho^2+H_2\varrho+H_3}{Q\varrho^2}\sqrt{P_1},$$

$$(5)\qquad = \frac{1}{4}\cos^{-1}\frac{\varrho^3-(2\beta-2\gamma-1)\varrho^2+L_2\varrho+L_3}{Q\varrho^2}\sqrt{P_2};$$

. . . .

Writing δ for $2\gamma+1$, and ε for $2\beta-1$, we shall find in this way that the equations (2), (3) lead to

$$(6)\qquad \varepsilon = -2\,\frac{2\delta^2-\delta}{(\delta-1)^2}$$

and

$$(7)\qquad 5\delta^6+3\delta^4+8\delta^3-9\delta^2+1=0,$$

$$(8)\qquad \varepsilon^6+6\varepsilon^5+21\varepsilon^4+34\varepsilon^3+21\varepsilon^2+6\varepsilon+1=0,$$

a reciproced sextic, which can be written

$$(9)\qquad (\varepsilon+1)^6+2\varepsilon^3+6\varepsilon^2(\varepsilon+1)^2=0,$$

as that

$$(10)\qquad (\varepsilon+1)^2+(\sqrt[3]{4}-\sqrt[3]{2})\,\varepsilon=0,$$

$$(11)\qquad \varepsilon = -1-\tfrac{1}{2}\sqrt[3]{4}+\tfrac{1}{2}\sqrt[3]{2}-\left(1-\tfrac{1}{2}\sqrt[3]{2}\right)\sqrt{(\sqrt[3]{4}+1)}$$

$$= -1\,.\,758994.$$

Similarly the equation for δ may be written

$$(12)\qquad \delta^2+1+\sqrt[3]{4}(\delta^2+\delta)+2\sqrt[3]{2}\,\delta=0,$$

giving

$$(13)\qquad \delta = -\frac{1}{\sqrt[3]{2}}-\frac{1}{\sqrt[3]{2}\sqrt{(\sqrt[3]{4}+1)}} = -1.287167,$$

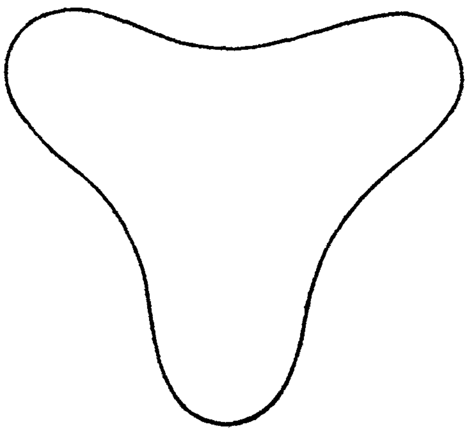

Fig. 9.

and fig. 9 has been drawn in accordance with these numerical results.

25. Having rectified the curve, by means of equation (5) § 1, and found its perimeter L, we may suppose the tube, of which the curve is the cross section, to spring out to a circular form of the same perimeter L and radius $c = L/2\pi$ suppose, on releasing the pressure p given by $p = 8EIA$ (Halphen, F. E. II, p. 194); while according to the formula (F. E. II, p. 235) the external pressure P at which the circular form becomes unstable, and tends to buckle into n waves is given by $P=(n^2-1)\frac{EI}{c^3}$, where $I=\frac{1}{12}(\text{thickness})^3$, and E is the modulus of elasticity of the material.

The length l of half a wave is given by (5) § 1

(1) $$l = \frac{1}{2}\int_{r_3}^{r_2} \frac{dr^2}{\sqrt{R}},$$

so that, with

(2) $$\varrho = -\frac{r^2}{b^2}, \quad \frac{b^2}{a^2} = \frac{m-1}{m},$$

(3) $$\frac{l}{b} = \frac{1}{2Ab^2}\int_{\varrho_3}^{\varrho_0} \frac{d\varrho}{\sqrt{P}} = 2\beta\sqrt{(-\gamma)}\,\frac{2K}{\sqrt[4]{(P_0 P_3)}}$$

where

(4) $$P_0 = \varrho_0{}^2 + 2(2\beta - 2\gamma - 1)\varrho_0 + (2\beta - 1)^2$$
$$= 4\beta(\varrho_0 + \beta - 1),$$

(5) $$P_3 = 4\beta(\varrho_3 + \beta - 1),$$

(6) $$P_0 P_3 = 16\beta^2(\beta^2 + 4\beta\gamma - 4\gamma),$$

and

(7) $$\varkappa^2, \varkappa'^2 = \frac{1}{2} \mp \frac{\beta^2 + 2\beta\gamma - 2\gamma(\gamma+1)}{2\beta\sqrt{(\beta^2 + 4\beta\gamma - 4\gamma)}},$$

(8) $$\varkappa^2\varkappa'^2 = \frac{\gamma^2(\gamma+1)(2\beta - 2\gamma - 1)}{\beta^2(\beta^2 + 4\beta\gamma - 4\gamma)}.$$

The arc is bisected where

(9) $$\varrho = \sqrt{(P_0 P_3)} - \beta + 1,$$

(10) $$\frac{r^2}{b^2} = \beta - 1 + 4\beta\sqrt{(\beta^2 + 4\beta\gamma - 4\gamma)}.$$

Now on releasing the pressure from a tube collapsed into n waves, the tube springs out to a circular form of radius c, such that the half wave becomes an arc of a circle subtending an angle π/n; and thus

(11) $$\frac{\pi c}{n} = l,$$

(12) $$\frac{c}{b} = \frac{n\sqrt{(\beta\gamma)}}{\sqrt[4]{(\beta^2 + 4\beta\gamma - 4\gamma)}}\,\frac{K}{\frac{1}{2}\pi}.$$

Then

(13) $$\frac{p}{P} = \frac{8Ac^3}{n^2 - 1}$$
$$= \frac{2n^3}{n^2 - 1}\,\frac{(-\beta\gamma^2)^{\frac{1}{2}}}{(\beta^2 + 4\beta\gamma - 4\gamma)^{\frac{3}{4}}}\left(\frac{K}{\frac{1}{2}\pi}\right)^3.$$

Thus, in fig. 8, in which $n = 2$, we find that $\varkappa = \frac{1}{3}$, and

(14) $$\frac{p}{P} = \frac{32}{27}\left(\frac{K}{\frac{1}{2}\pi}\right)^3 = 1.285,$$

so that an increase in pressure of $28\frac{1}{2}$ % will collapse the tube from the circular form to that shown in fig. 8.

In fig. 9, where $n = 3$, we shall find that

$$\frac{\varkappa}{\varkappa'} = 2 - \sqrt[3]{4}, \tag{15}$$

$$\frac{p}{P} = 1.319 \tag{16}$$

an increase of nearly 32 % over the pressure at which the tube begins to collapse into three waves; to preserve the stability of this form the tube may be hold at the points of inflexion.

It will be noticed that the moduli of the elliptic functions which occur are the same as those required in the motion of a top having four or three cusps: this is consequent on the relation $P(2v) = 0$; and we may utilise this analogy in proceeding to the next case of equations (4) (5) § 24.

26. For the next case, in which the cross section of the collapsed tube is the algebraical curve, of four waves,

$$\theta = \frac{1}{4} \sin^{-1} \frac{\varrho^3 + H_1\varrho^2 + H_2\varrho + H_3}{Q\varrho^2} \sqrt{P_1}, \tag{1}$$

$$\theta = \frac{1}{4} \cos^{-1} \frac{\varrho^3 + L_1\varrho^2 + L_2\varrho + L_3}{Q\varrho^2} \sqrt{P_2} \tag{2}$$

with

$$H_1 = \delta, \quad L_1 = \delta - \varepsilon - 1, \quad H_2 = \frac{3}{2}\delta^2 - \frac{1}{2}, \tag{3}$$

$$L_2 = \frac{3}{2}(\delta - \varepsilon - 1)^2 - \frac{1}{2}\varepsilon^2, \quad H_3 = \varepsilon L_3,$$

$$L_3 = \frac{3\delta^2 - 1 - 6\varepsilon(\delta - \varepsilon - 1) + 2\varepsilon^3}{2(3\delta\varepsilon + 4\delta - 2\varepsilon - 2)},$$

we are led to a reciprocal equation for ε, of the 18th degree, which, on putting $\varepsilon + \frac{1}{\varepsilon} = x$, reduces to

$$(x+2)(x^2-2)(3x^6 + 8x^5 - 98x^4 - 688x^3 - 1900x^2 - 2624x - 1528) = 0. \tag{4}$$

Putting $x = \frac{2}{y} - 2$, the sextic equation becomes

$$y^6 + 20y^5 + 22y^4 + 64y^3 - 4y^2 + 112y - 24 = 0, \tag{5}$$

an equation of the same structure as (288) L. M. S. XXVII, p. 597, having two real roots, 0.21 and — 19.02176; of which the second gives a value $\varepsilon = -1.3810617$, and thence $\delta = -1.135$, which gives Fig. 10.

27. When the cross section has five waves, the form is given by

$$(1)\qquad \theta = \frac{1}{5}\sin^{-1}\frac{\varrho^4 + H_1\varrho^3 + H_2\varrho^2 + H_3\varrho + H_4}{Q\varrho^{\frac{5}{2}}}\sqrt{P_1},$$

$$(2)\qquad \theta = \frac{1}{5}\cos^{-1}\frac{\varrho^4 + L_1\varrho^3 + L_2\varrho^2 + L_3\varrho + L_4}{Q\varrho^{\frac{5}{2}}}\sqrt{P_2},$$

with the same expressions for H_1, L_1, H_2, L_2 as above; and an investigation is in progress on the elimination of H_3, L_3, H_4 for the formation of the equations for ε and δ; the equation for ε will be

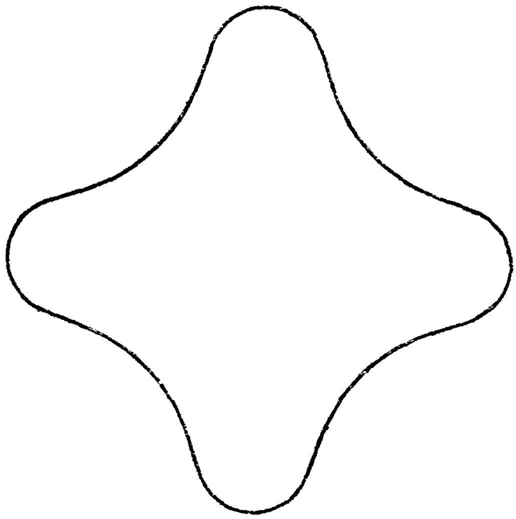
Fig. 10.

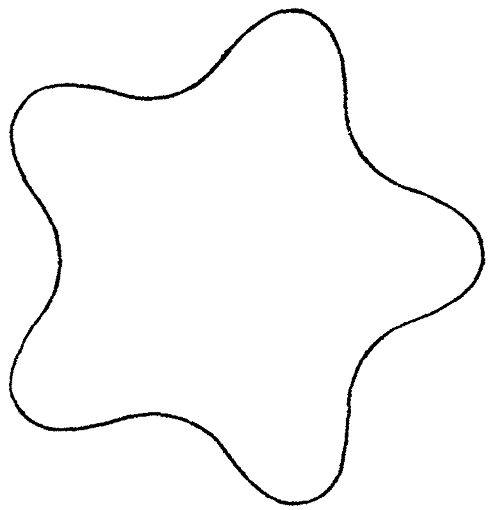
Fig. 11.

reciprocal, by analogy with the preceding cases, and will have the same structure as equation (314) (L. M. S. XXVII, p. 604) which is required for the determination of the motion of a top, complete in five cusps.

It is now found, on examination of equations (326) — (329), L. M. S. XXVII, that we can determine the algebraical curve with n waves by putting

$$(3)\qquad \varepsilon = -\frac{\operatorname{cn}\frac{K'}{2n}}{\operatorname{cn}\frac{3K'}{2n}},$$

$$(4)\qquad \frac{m-1}{m} = \tan\frac{1}{2}\varphi_1 \tan\frac{1}{2}\varphi_2,$$

$$(5)\qquad F\varphi_1 = (2n-3)\frac{K'}{2n},\quad F\varphi_2 = (2n-1)\frac{K'}{2n};$$

the modulus being determined from the condition that

$$(6)\qquad Z\frac{K'}{2n} = \frac{\operatorname{sn}\frac{K'}{2n}\operatorname{dn}\frac{K'}{2n}}{\operatorname{cn}\frac{K'}{2n}},\quad \text{or}\quad Z\frac{K'}{n} = \frac{\operatorname{sn}\frac{K'}{n}\operatorname{dn}\frac{K'}{n}}{1+\operatorname{cn}\frac{K'}{n}}.$$

These formulas have been tested on the preceding algebraical curves, with two, three, and four waves; and they enable us to draw fig. 11, an algebraical curve with five waves, by putting $n = 5$, employing the co-modular angle $66^0\ 9'$ corresponding to the root $c = 7.4$ in equation (314), L. M. S. XXVII, p. 604, for cusps at intervals $\frac{4}{5}\pi$ in azimuth, in the corresponding case of algebraical motion of the Top.

28. Looking back over these and similar calculations required in mechanical problems which introduce the Elliptic Integral of the Third Kind, it will be remarked that a *desideratum* is the formation of the series of functions, analogous to the *sn*, *cn* and *dn* functions of Abel and Jacobi, as defined in Halphen's F. E. I., Chap. VII, p. 222.

By means of these functions it is possible to express

$$(1) \qquad \Phi(u, v) = \frac{\mathfrak{S}(u-v)}{\mathfrak{S}u\,\mathfrak{S}v}\,e^{\nu u},$$

when

$$(2) \qquad v = \frac{2\varpi}{n}, \quad \nu = \frac{2\tilde{\eta}}{n},$$

by the n^{th} root of an integral function of $\wp u$ and $\wp' u$; the logarithm of this function being an Elliptic Integral of the Third Kind; and Abel's pseudo-elliptic integral is reduced to

$$(3) \qquad \frac{1}{2}\log\frac{\Phi(u, v)}{\Phi(-u, v)} = \frac{1}{2}\log\frac{\mathfrak{S}(u-v)}{\mathfrak{S}(u+v)}\,e^{2\nu u}$$

being the integral $iI(v)$ of equation (1) § 2, its differential coefficient with respect to u being

$$(4) \qquad \frac{1}{2}\,i\,\frac{P(v)(s-\sigma) - \sqrt{-\Sigma}}{s-\sigma},$$

with

$$(5) \qquad s - \sigma = \wp u - \wp v, \quad \sqrt{S} = -\wp' u, \quad \sqrt{-\Sigma} = i\wp' v.$$

The requisite materials of the analysis will be found in L. M. S. XXV, by means of which it is possible to make

$$(6) \qquad \left\{\frac{\mathfrak{S}(u-v)}{\mathfrak{S}u\,\mathfrak{S}v}\,e^{\nu u}\right\}^n = A + iB\sqrt{S},$$

$$(7) \qquad \left\{\frac{\mathfrak{S}(u+v)}{\mathfrak{S}u\,\mathfrak{S}v}\,e^{-\nu u}\right\}^n = A - iB\sqrt{S},$$

$$(8) \qquad (s-\sigma)^n = A^2 + B^2 S,$$

where A and B are rational integral functions of s, of the order $\frac{1}{2}(n-1)$ and $\frac{1}{2}(n-3)$ if n is odd, and of the order $\frac{1}{2}n$ and

$\frac{1}{2}(n-4)$ if n is even; and, from L. M. S. XXV, XXVII, the values of A and B can be written down for the numbers

(9) $n = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 22.$

When n is even, the relation can be written in the form

$$(10)\qquad \left\{\frac{\mathfrak{S}(u-v)}{\mathfrak{S}u\,\mathfrak{S}v}\,e^{\nu u}\right\}^{\frac{1}{2}n} = P\sqrt{(s-s_\alpha)} + iQ\sqrt{(s-s_\beta \,.\, s-s_\gamma)}.$$

Also, when n is odd, it will be noticed that S, on putting $s-\sigma = t^2$, can be resolved into two factors T_1 and T_2, cubics in t, and that we can write

$$(11)\qquad \left\{\frac{\mathfrak{S}(u-v)}{\mathfrak{S}u\,\mathfrak{S}v}\,e^{\nu u}\right\}^{\frac{1}{2}n} = L\sqrt{T_1} + iM\sqrt{T_2},$$

with

$$(12)\qquad t^n = L^2 T_1 + M^2 T_2,$$

in which L and M are integral functions of t, of the order $\frac{1}{2}(n-3)$.

Thus for instance, for $n = 11$, we are able to infer from § 18 of the present article, and from L. M. S. XXV, p. 241, that

$$(13)\qquad \left\{\frac{\mathfrak{S}\left(u-\frac{4r\omega'}{11}\right)}{\mathfrak{S}u\,\mathfrak{S}\frac{4r\omega'}{11}}\,e^{\left(iP\frac{4r\omega'}{11}+2\zeta\frac{4r\omega'}{11}\right)u}\right\}^{\frac{11}{2}}$$

$$= \frac{1}{2}(t^4 - a_1 t^3 + a_2 t^2 - a_3 t + a_4)\sqrt{T_2}$$

$$+ \frac{1}{2}i(t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4)\sqrt{T_1},$$

in which

$$(14)\qquad t^2 = s - \sigma = \wp u - \wp\frac{4r\omega'}{11},$$

$$(15)\qquad T_1 = 2t^3 + (1+y)t^2 + 2xt + xy,$$

$$(16)\qquad T_2 = 2t^3 - (1+y)t^2 + 2xt - xy,$$

$$(17)\qquad x = -\frac{1}{2}c(1+c)(1+2c+\sqrt{C}),$$

$$(18)\qquad y = -c\,\frac{1+4c+2c^2+\sqrt{C}}{2(1+c)},$$

$$(19)\qquad C = 1 + 4c + 8c^2 + 4c^3,$$

$$(20)\qquad P\frac{4r\omega'}{11} = \frac{6+27c+44c^2+18c^3+(8+13c)\sqrt{C}}{22(1+c)},$$

$$(21)\quad a_1 = -\frac{1}{2(1+c)}\{2+7c+10c^2+4c^3+(2+3c)\sqrt{C}\},$$

$$(22)\quad a_2 = -\frac{1}{2(1+c)^2}\{1+10c+40c^2+82c^3+86c^4+40c^5 \\ +6c^6+(1+2c)^2(1+4c+2c^2)\sqrt{C}\},$$

$$(23)\quad a_3 = \frac{1}{2}\{1+8c+28c^2+52c^3+50c^4+20c^5+2c^6 \\ +(1+2c)(1+4c+6c^2+2c^3)\sqrt{C}\},$$

$$(24)\quad a_4 = \frac{c}{2(1+c)}\{1+11c+54c^2+151c^3+255c^4+254c^5 \\ +135c^6+32c^7+2c^8 \\ +(1+9c+34c^2+67c^3+69c^4+32c^5+5c^6)\sqrt{C}\}.$$

The „multiplicative elliptic functions“ α, β, γ, δ introduced by Professor Klein into the Theory of the Top are of the same nature as this function $\Phi(u, v)$, qualified by factors which are exponential functions of the time [*Princeton Lectures*, p. 31; Klein-Sommerfeld, *Theorie des Kreisels*, p. 420] and the time is expressible by Legendre's elliptic integral $F\varphi$, the only transcendental function now required to be tabulated.

A sufficient number of these functions will enable us to explore the analytical field of a mechanical problem by a series of the parameter v which are the simplest aliquot parts of a period, between two of which any actual numerical case may be taken to lie.

We are thereby relieved of the necessity of tables of the Θ functions, and even these would be useless when the parameter v was a fraction of the imaginary period, as is generally the case, as seen above.

We can however change v immediately to a fraction of the real period, merely by changing the sign of s and S in our expressions, and thereby tabulate an algebraical series of the „*Theilwerthe*“ of the Θ and Z functions of Jacobi.

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