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LII. Additions to the theory of eclipses, and the methods of calculating their results

Professor Bessel

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The whole of the foregoing remarks are submitted, with much deference, to the consideration of geologists; as supplementary to Dr. Macculloch's memoirs on the concretionary structure in rocks, and as hints, for future investigation, by those who have it in their power to institute extensive researches in Geology.

Hazelwood School, near Birmingham,
Oct. 10th, 1830.

LI. *On the Discharge of a Jet of Water under Water.* *By* R. W. Fox, *Esq.*

To the Editors of the Philosophical Magazine and Annals.

I AM not aware that it has been before noticed, that a jet of water discharges the same quantity, in water, as in air, in a given time, without reference to the depth or the motion of the water; at least within certain limits.

Thus when the experiment was tried with a head of water six feet high, the same orifice discharged equal quantities in equal times in air, in still water, and in a rapid stream, moving at the rate of about six feet in a second; the jet having in one case been turned with the current, and in another against it: and when by lengthening the tube, the aperture was submerged to the depth of fifteen feet, the effect was the same as at the surface, under the pressure of an equal column above it.

These results have been obtained by my brother Alfred Fox, and myself;—and you may, perhaps, think them deserving a place in your Magazine, if they should appear to you to be new.

We sometimes coloured the water, when the jet appeared to pass unbroken to a considerable distance under the water.

Falmouth, 10th month, 9th, 1830. R. W. Fox.

LII. *Additions to the Theory of Eclipses, and the Methods of calculating their Results.* *By Professor* Bessel.

[Continued from page 275.]

[7.] THE calculation of an occultation of a star can now be performed, after the preparatory operations explained in the preceding section, in two different ways. The first supposes that the same value of T is to be applied to all observations which are to be calculated, in which case p and q correspond to the value a in the arrangement above given. On this

this supposition the values of P and Q corresponding to the time T + T′, are obtained by this formula:

$$(7) \ldots a + T'.b + \frac{T'^2}{2}.c + \frac{T'.T'^2-1}{2.3}.d + \frac{T'^2.T'^2-1}{2.3.4}e + \&c.$$

consequently, p' and q' by this formula:

$$(8) \ldots b + \frac{T'}{2}c + \frac{T'^2-1}{2.3}d + \frac{T'.T'^2-1}{2.3.4} + \&c.$$

by which N and log n are found. If, however, several observations are to be calculated at the same time, it is more convenient at once to calculate the values of N and log n for different values of T′; they are obtained by putting for n sin N and n cos N these expressions:

For $T = -2$ .... $b - c + \frac{1}{2}d - \frac{1}{4}e$
$= -1$ .... $b - \frac{1}{2}c$
$= \ \ 0$ .... $b - \frac{1}{6}d$
$= +1$ .... $b + \frac{1}{2}c$
$= +2$ .... $b + c + \frac{1}{2}d + \frac{1}{4}e.$

For every single observation M and log m are then to be calculated by the formulæ

$$m.\sin M = p - u, \qquad m \cos M = q - v$$

in which p and q are the known values of P and Q for the time T; at last the sixth formula is to be calculated. A second approximation is not necessary if the value of $T' = t - d - T$, or the assumed difference of meridians employed in the first is correct within some minutes of time, which may always be supposed. Such an approximation would, however, cause little trouble, as the calculation of formula (6), only, would be affected by the alterations. A second way of performing the calculation would be to calculate by interpolation (7) from the values of P and Q corresponding to T, T ∓ 1, &c. those values of these quantities which belong to the time of observation, reduced to the first meridian by an assumed difference of meridians, and to use these values instead of p and q and their differential quotients, instead of p' and q'. The latter are found by the formula

$$(9) \ldots b + T'.c + \frac{3T'^2-1}{2.3}d + \frac{2T'^3-T'}{3.4}e + \&c.$$

by the application of which, consequently, N and log n will be obtained. If it should be preferred at once to find N and log n, and to interpolate between the values thus calculated, they might be obtained by the expression for n sin N and n cos N,

For

$$\begin{aligned}\text{For } T' &= -2 \ldots\ldots b - 2c + \tfrac{11}{6} d - \tfrac{7}{6} e \\ &= -1 \ldots\ldots b - \ \ c + \tfrac{1}{3} d - \tfrac{1}{12} e \\ &= \ \ \ 0 \ldots\ldots b - \tfrac{1}{6} d \\ &= +1 \ldots\ldots b + \ \ c + \tfrac{1}{3} d + \tfrac{1}{12} e \\ &= +2 \ldots\ldots b + 2c + \tfrac{11}{6} d + \tfrac{7}{6} e\end{aligned}$$

The remainder of the calculation is as above.

This second manner of conducting the calculation, supposes, therefore, the determination of p and q by interpolation, while the former one constantly proceeds from the same values of these quantities. But it has the advantage of a more easy calculation of the term (6) $\frac{m}{n} \frac{\cos(M - N - \psi)}{\cos\psi}$ which is the correction of the assumed difference of meridians, and which is, of course, commonly very small; its convergency to the truth is likewise the greatest possible, and the error of the first approximation arises only from the moon's motion during the interval between the supposed and the real difference of meridians being taken as it would be at the beginning, or at the end of this interval of time, while it ought to have been taken for the middle of it. For the observatories of Europe, whose meridians are very nearly known, the square of $n(T' + i)$ might even be neglected, and the equation to be resolved might thus be reduced to one of the first degree. But all these advantages of the second method of calculating appear to me to be insignificant in comparison to the trouble of the interpolation for finding p and q. I therefore prefer the first. In the application of these formulæ, however, the interpolation will never require to be carried beyond the second differences, and consequently three values of P and Q will be sufficient.

[8.] The method here explained leaves it to the choice of the calculator, whether the quantities α, δ, shall refer to the equator or to the ecliptic. In the result of the calculation there is nothing referring to either of these great circles, and they serve only to denote the relative situation of the various points of the celestial sphere referred to in the problem. The former is, however, always more easy, if the places of the moon in relation to the equator are contained in an ephemeris; and even if this should not be the case, whenever several observations are to be calculated at the same time. But when there are few observations, and when the places of the moon are to be derived from the tables themselves, or are to be be taken from an ephemeris containing the moon's longitude and latitude, the preparatory calculations required in finding the quantities

quantities referring to the equator become more troublesome than the calculation of the longitude and latitude of the zenith: in that case the ecliptic deserves the preference. The tables give the longitude, latitude, and parallax of the moon for the time T, and likewise their variations for the preceding and following hours. These three longitudes and latitudes are to be converted into right ascension and declination, if the equator is to be used. This trouble is saved if we calculate with longitude and latitude; but, on the contrary, in using right ascension and declination we dispense with the calculation of the longitude and latitude of the star and the zenith. If, therefore, in the case that the places of the moon in relation to the equator are unknown, only two or three observations are to be calculated, the ecliptic appears to deserve the preference. If a single observation is to be compared with the ephemerides or the tables, it is more advantageous to calculate p' and q' from the hourly variations of α, δ, π, than to derive them from the three values of P and Q. In this case it will be most convenient to assume for T, the time of the observation reduced to the first meridian, by applying an approximate value of the difference of meridians, and to calculate for this moment P and Q, as also their differential quotients p' and q', by the formulæ

$$(10)\begin{cases} p' = \left(\dfrac{\cos\delta\cos(\alpha - A)}{\omega\cos\pi} \cdot \dfrac{d\alpha}{dt} - \dfrac{\sin\delta\sin(\alpha - A)}{\omega\sin\pi} \cdot \dfrac{d\delta}{dt} - \dfrac{p}{\omega\,\mathrm{tang}\,\pi} \cdot \dfrac{d\pi}{dt}\right. \\ q' = \left(\dfrac{\cos\delta\sin D\sin(\alpha - A)}{\omega\sin\pi} \cdot \dfrac{d\alpha}{dt} + \dfrac{\cos\delta\cos D + \sin\delta\sin D\cos(\alpha - A)}{\omega\sin\pi} \cdot \dfrac{d\delta}{dt}\right. \\ \qquad\qquad \left. - \dfrac{q}{\omega\,\mathrm{tang}\,\pi} \cdot \dfrac{d\pi}{dt}\right) \end{cases}$$

in which $\frac{d\alpha}{dt}$, $\frac{d\delta}{dt}$, $\frac{d\pi}{dt}$ signify the hourly motions, and ω the radius of the circle expressed in seconds.

[9.] It now remains further to develope that part of formula (6) which is dependent on the corrections of the elements of the calculation. The result obtained by the method here explained, is not so much the difference of the meridian of the place of observation, as the relation between it and the elements used in the calculation, and by the combination of several observations one or more of the elements of the calculation are eliminated, and the result is thus made partly or entirely independent of the tables.

In section [5] the quantities i and i' have been so assumed as to give these equations:

$$p'i - q'.i' = a\Delta\alpha + b\Delta\delta + c\Delta\pi + d\Delta e^2$$
$$q'i + p'i' = a'\Delta\alpha + b'\Delta\delta + c'\Delta\pi + d'\Delta e^2$$

$\Delta\alpha$, $\Delta\delta$, &c. are here assumed as expressed in parts of the radius; they must, therefore, be divided by $\omega = 206265$ if they are meant to be given in seconds. The coefficients $a, b \ldots a' b'$ are the differential quotients of $P - u$, and $Q - v$ in relation to α, δ, π and e^2; neglecting in their values the small quantities of the order of $\alpha - A$ and $\delta - D$, which, on account of the smallness of $\Delta\alpha$, $\Delta\delta$, &c. will produce no error of any consequence, the expressions for them will become very simple: $a = \frac{\cos\delta}{\sin\pi}$; $a' = 0$; $b = 0$; $b' = \frac{1}{\sin\pi}$; $c = -\frac{P}{\text{tang}\,\pi}$; $c' = -\frac{Q}{\text{tang}\,\pi}$; $d = -\frac{du}{d.e^2}$; $d' = -\frac{dv}{d.e^2}$. Adopting these expressions, and substituting $n \sin N$ and $n \cos N$ for p' and q', and h for $\frac{s}{\omega . n . \sin\pi}$, where s represents the number of seconds equal to the hour to which p' and q' belong [6], we obtain

$$i = h \sin N . \cos\delta\,\Delta\alpha + h\cos N\,\Delta\delta - h\cos\pi . \Delta\pi\,[P\sin N + Q\cos N] - h\,\omega\sin\pi . \Delta e^2\left[\frac{du}{d.e^2}\sin N + \frac{dv}{d.e^2}\cos N\right]$$

$$i' = -h\cos N . \cos\delta . \Delta\alpha + h\sin N . \Delta\delta + h\cos\pi . \Delta\pi\,[P\cos N - Q\sin N] + h.\omega\sin\pi . \Delta . e^2\left[\frac{du}{d.e^2}\cos N - \frac{dv}{d.e^2}\sin N\right].$$

Hence,

$$\left(i + \frac{i'}{\text{tang}\,\psi}\right)\frac{\sin\psi}{h} = -\cos(N+\psi).\cos\delta.\Delta\alpha + \sin(N+\psi)\Delta\delta + \cos\pi . \Delta\pi\,[P\cos(N+\psi) - Q\sin(N+\psi)] + \omega\sin\pi\,\Delta e^2 \times \left[\frac{du}{d.e^2}\cos(N+\psi) - \frac{dv}{d.e^2}\sin(N+\psi)\right].$$

The part dependent on $\Delta\pi$ may be reduced to a more convenient form, and the one dependent on $\Delta . e^2$ may be further developed. Substituting in the former $p + n \sin N . T'$ and $q + n\cos N . T'$ for P and Q, it becomes $\cos\pi . \Delta\pi\,[p\cos(N+\psi) - q\sin(N+\psi) - n . T'\sin\psi] = \cos\pi . \Delta\pi\,[(p\cos N - q\sin N)\cos\psi - (p\sin N + q\cos N + n\,T')\sin\psi]$; and further, making $x = q\sin N - p\cos N$, $n\tau = n\,T - p\sin N - q\cos N$, and putting for T' its expression $\frac{t-d-T}{s}$, it assumes this form

$$-\cos\pi . \Delta\pi\left[x\cos\psi + \frac{n}{s}(t - d - \tau)\sin\psi\right]$$

It will be easily perceived that $x\,\omega\sin\pi$ is the smallest distance of the true path of the moon from the star, positive if the moon passes to the north, negative if she passes to the south

south of it, and τ the time of the nearest conjunction counted from the first meridian.

The development of the influence of $\Delta . e^2$ depends on the differential quotients in relation to e^2 of the quantities $r \cos \phi'$ and $r \sin \phi'$ which occur in the expression for v and u. But these quantities are differently expressed by e^2 and the observed latitude ϕ, according as ϕ' denotes the declination or the latitude of the zenith. The formulæ to be employed in the two cases must, therefore, be separately developed, while all the preceding ones equally apply to both cases. I begin with the case of ϕ' denoting the declination.

We have in this case, $r \cos \phi' = \frac{\cos \phi}{\sqrt{(1-e^2 \sin \phi^2)}}$; $r \sin \phi' = \frac{(1-e^2) \sin \phi}{\sqrt{(1-e^2 \sin \phi^2)}}$ whence $\frac{d . r \cos \phi'}{d . e^2} = r \cos \phi' . \frac{r^2 \sin \phi'^2}{2(1-e^2)^2}$; $\frac{d r \sin \phi'}{d . e^2} = r \sin \phi' . \frac{r^2 \sin \phi'^2}{2(1-e^2)^2} - \frac{r \sin \phi'}{1-e^2}$ or putting $\beta = \frac{r \sin \phi'}{1-e^2}$,
$\frac{d . r \cos \phi'}{d . e^2} = \frac{1}{2} \beta^2 r \cos \phi'$; $\frac{d . r \sin \phi'}{d . e^2} = \frac{1}{2} \beta^2 . r \sin \phi' - \beta$.

The expression for u and v being these:

$$u = r \cos \phi' . \sin (\mu - A) ; \; v = r \sin \phi' \cos D - r \cos \phi' \sin D \cos (\mu - A),$$

we obtain $\frac{d u}{d . e^2} = \frac{1}{2} \beta^2 . u$ and $\frac{d v}{d . e^2} = \frac{1}{2} \beta^2 . v - \beta \cos D$;
and, next, the part dependent on $\Delta e^2 = \omega \sin \pi . \Delta e^2 \times \left[\frac{1}{2} \beta^2 \Big(u \cos (N+\psi) - v \sin (N+\psi)\Big) + \beta \cos D \sin (N+\psi)\right]$

That part of this expression which is multiplied into $\frac{1}{2} \beta^2$ may be written: $P \cos (N+\psi) - Q \sin (N+\psi) - (P-u) \times \cos (N+\psi) + (Q-v) \sin (N+\psi)$; and we may substitute for $P-u$ and $Q-v$ their equivalents $m \sin M + n \sin N . T'$, and $m \cos M + n \cos N . T'$ by which we have $(P-u) \cos (N+\psi) - (Q-v) \times \sin (N+\psi) = m . \sin (M - N - \psi) - n T' \sin \psi =$ (substituting for T' its value [5] —

$$\frac{m . (\cos M - N - \psi)}{n \cos \psi}\Big) - \frac{. m \sin (M-N)}{\cos \psi} = k.$$

If we here apply the above transformation of $P \cos (N+\psi) - Q \sin (N+\psi)$ the expression dependent on $\Delta . e e$ will become: =

$$-\omega \sin \pi . \Delta e^2 \Big[\tfrac{1}{2} \beta^2 [x \cos \psi + \tfrac{n}{s} (t - d - \tau) \sin \psi - k] - \beta \cos D \sin (N+\psi)\Big].$$

[To be continued.]

 LIII. *Notes*