ON THE RELATION BETWEEN THE HILBERT SPACE AND THE CALCUL FONCTIONNEL OF FRÉCHET.

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Mr. FRÉCHET [1]) has called attention to the fact that the subset of the HILBERT space Ω [2]) which is homeomorphic to the class of all continuous functions is not closed, and suggests that in order to pass with security from the theory of functions of an infinity of variables to the Calcul Fonctionnel it is desirable to show that the class F of all continuous functions is the image of a perfect subset of the space Ω.

It is my purpose to show that this end can be attained by means of a suitable definition of distance (écart) [3]) for the space Ω. The definition of distance to be given implies that limit in the space Ω is equivalent to uniform convergence, except at a set of points of measure zero, in the space Ω_1, of all measureable functions of a single real variable whose squares are summable.

We consider the class Ω_1, of all measureable functions whose squares are summable on the interval $I = (0 \leqq x \leqq 2\pi)$. Functions which differ only at the points of a subset of I of measure zero, a null set, are considered as identical in the following discussion.

Let $f(x)$ be a function of Ω_1. We define:

$$(1) \qquad u_0(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = a_0 \qquad (m = 1, 2, 3, \ldots)$$

$$u_m(x) = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos m(\alpha - x)\,dx = a_m \cos mx + b_m \sin mx.$$

[1]) M. FRÉCHET, *Les ensembles abstraits et le Calcul fonctionnel* [Rendiconti del Circolo Matematico di Palermo, t. XXX (2° semestre 1910), pp. 1-26], p. 15, § 27.

[2]) D. HILBERT, *Wesen und Ziele einer Analysis der unendlichvielen unabhängigen Variabeln* [Rendiconti del Circolo Matematico di Palermo, t. XXVII (1° semestre 1909), pp. 59-74].

[3]) Cf. M. FRÉCHET, *Relations entre les notions de limite et de distance* [Transactions of the American Mathematical Society, v. XIX (1918), pp. 53-65], p. 54.

The series

$$(1') \qquad u_0(x) + u_1(x) + \cdots + u_m(x) + \cdots$$

is the FOURIER series for $f(x)$. The functions:

$$(2) \qquad \begin{cases} \varphi_0(x) = u_0(x), \qquad \varphi_1(x) = u_0(x) + u_1(x), \\ \varphi_2(x) = u_0(x) + u_2(x) + u_3(x), \ \dots \end{cases}$$

are the FOURIER convergents of $f(x)$. They converge to $f(x)$ in mean, that is

$$\lim_{m \to \infty} \int_0^{2\pi} [f(x) - \varphi_m(x)]^2 \, dx = 0 \quad [4]).$$

The CESÀRO-HÖLDER means of the first order of the functions $\varphi_m(x)$ have been studied by FEJÉR [5]). We set:

$$(3) \qquad S_m(x) = \frac{\varphi_0(x) + \cdots + \varphi_{m-1}(x)}{m}.$$

The following formulas are easily established:

$$(4) \quad \int_0^{2\pi} S_m(x) \cos u x \, dx = \frac{m-n}{m} \pi a_n, \quad \int_0^{2\pi} S_m(x) \sin n x \, dx = \frac{m-n}{m} \pi b_n \qquad (m > n)$$

$$(5) \qquad S_m(x) = \frac{1}{m\pi} \int_{-\frac{x}{2}}^{\pi - \frac{x}{2}} f(x + 2\beta) \left(\frac{\sin m\beta}{\sin \beta} \right)^2 d\beta \quad [5]).$$

We shall have need of the following proposition:

If a sequence $\varphi_0(x), \varphi_1(x), \dots, \varphi_m(x), \dots$ converges in mean to a function $f(x)$ the CESÀRO-HÖLDER *means of the first order of the sequence converge to $f(x)$ in mean likewise.*

If $S_m(x) = \dfrac{\varphi_0(x) + \cdots + \varphi_m(x)}{m}$, we have to prove that

$$\lim_{m \to \infty} \int_0^{2\pi} [f(x) - S_m(x)]^2 \, dx = 0.$$

Consider the following inequalities:

$$\int_0^{2\pi} [f(x) - S_m(x)]^2 \, dx = \int_0^{2\pi} \left(\frac{\sum\limits_{k=1}^{k=m} f(x) - \varphi_k(x)}{m} \right)^2 dx$$

$$\leq \frac{3}{m^2} \sum_{k=1}^{k=m} \int_0^{2\pi} [f(x) - \varphi_k(x)]^2 \, dx.$$

4) Cf. T. LALESCO, *Introduction à la théorie des equations intégrales.* (Paris. A. Hermann & Fils, 1912), pp. 91-96.

5) L. FEJÉR, *Untersuchungen über* FOURIER*sche Reihen* [Mathematische Annalen, Bd. LVIII (1904), pp. 51-69].

Denote by m_e the greatest value of m for which

$$\int_0^{2\pi} [f(x) - \varphi_m(x)]^2 dx > e.$$

Then we have for all $m \geqq m_e$

$$\int_0^{2\pi} [f(x) - S_m(x)]^2 dx \leqq \frac{3}{m^2}\left\{\sum_{k=1}^{k=m_e} \int_0^{2\pi} [f(x) - \varphi_k(x)]^2 dx + (m - m_e)e\right\}$$

and therefore for all m sufficiently large,

$$\int_0^{2\pi} [f(x) - S_m(x)]^2 \leqq e,$$

which was to be proved.

There corresponds, by means of the equations (1), to every function $f(x)$ of the class Ω_1, a unique point A of the HILBERT space Ω, in fact the point whose coordinates are

(6) $$A = (a_0, a_1, b_1, a_2, b_2, \ldots, a_m, b_m, \ldots),$$

since, as is well known, the series:

(7) $$a_0^2 + a_1^2 + b_1^2 + \cdots + a_m^2 + b_m^2 + \cdots$$

converges. Conversely, to every point X of the space Ω

(8) $$X = (x_1, x_2, x_3, \ldots, x_k, \ldots)$$

there corresponds a sequence:

(9) $$u_0(x) = x_1, \ldots, u_m(x) = x_{2m} \cos mx + x_{2m+1} \sin mx, \ldots$$

of continuous functions which determines a unique function of Ω_1.

The functions $S_m(x)$ of equations (3) are defined for every point of Ω and function of Ω_1. Let A, B be any two points of Ω; $g(x)$, $h(x)$ the corresponding functions of Ω_1, and $S'_m(x)$, $S''_m(x)$, the corresponding FEJÉR convergents. We define the distances between the points A, B and the distance (g, h) between the functions $g(x)$, $h(x)$ simultaneously as follows:

(10) $$(A, B) = (g, h) = \lim_{m \to \infty} \max \frac{|S'_m(x) - S''_m(x)|}{1 + |S'_m(x) - S''_m(x)|}.$$

As defined above distance in Ω has the following properties: $(A, B) = 0$ is equivalent to $A = B$; $0 \leqq (A, B) \leqq 1$; $(A, B) = (B, A)$. Furthermore, if A, B, C are any three points of Ω, then

(11) $$(A, B) \leqq (A, C) + (C, B).$$

The above inequality is an evident consequence of the inequality:

$$|S'_m(x) - S''_m(x)| \leqq |S'_m(x) - S'''_m(x)| + |S'''_m(x) - S''_m(x)|$$

where $S'_m(x)$, $S''_m(x)$, $S'''_m(x)$, are the respective convergents of A, B, C, defined by equations (8), (9), (2), and (3).

It is evident that if $(A, B) = 0$, then $S'_m(x) - S''_m(x)$ converges to zero uniformly in x. If the coordinates of A and B are respectively:

$$A = (a_1, a_2, \ldots, a_k, \ldots),$$
$$B = (b_1, b_2, \ldots, b_k, \ldots),$$

we obtain from formulas (4), for every p and $m > p$;

$$\frac{m-n}{m}\pi(a_{2p} - b_{2p}) = \int_0^{2\pi}[S'_m(x) - S''_m(x)]\cos p x\, d x,$$

$$\frac{m-n}{m}\pi(a_{2p+1} - b_{2p+1}) = \int_0^{2\pi}[S'_m(x) - S''_m(x)]\sin p x\, d x,$$

and it follows that $a_{2p} = b_{2p}$, $a_{2p+1} = b_{2p+1}$, $(p = 1, 2, 3, \ldots)$. That is, $A = B$.

Evidently the distance (g, h) has like properties. We note especially the following inequality:

(11′) $$(g, h) \leqq (g, f) + (f, h).$$

If $f(x)$ is continuous, $f(x) = \lim_{m \to \infty} S_m(x)$ [5]. Hence, if $g(x)$, $h(x)$, are continuous functions, we have

$$(g, h) = \lim_{m\to\infty} \max \frac{|S'_m(x) - S''_m(x)|}{1 + |S'_m(x) - S''_m(x)|} = \max. \frac{|g(x) - h(x)|}{1 + |g(x) - h(x)|}.$$

It follows that if $h(x)$ approaches $g(x)$ uniformly, (g, h) approaches zero, and conversely. That is, limit as defined in terms of distance in Ω or Ω_1 is equivalent to uniform convergence in the class of continuous functions.

We will now show that the image φ in Ω of the class F of all continuous functions on the interval I is closed relative to the present definition of distance. Suppose that a point B of Ω is the limit of a sequence of points, $C_1, C_2, \ldots, C_n, \ldots$ of φ; that is, $\lim_{m\to\infty}(B, C_m) = 0$. Because of inequality (11), there exists for every small positive number e an integer n_e such that if n' and n'' are any two integers greater than n_e then $(C'_n, C''_n) \leqq e$. Let $h_{n'}(x)$, $h_{n''}(x)$ be the continuous functions which correspond respectively to $C_{n'}$, $C_{n''}$. We have at once, $(h_{n'}, h_{n''}) \leqq e$, and therefore

$$|h_{n'}(x) - h_{n''}(x)| \leqq e$$

for all values of x in the interval I.

It follows that the sequence $h_1(x), h_2(x), \ldots, h_n(x), \ldots$ converges uniformly to a continuous limit function $g(x)$. Let B' denote the point of φ which corresponds to $g(x)$. Since

$$(g, h_n) = (B', C_n)$$

and we have

$$(B, B') \leqq (B', C_n) + (C_n, B)$$

it follows that $(B, B') = 0$ and therefore $B = B'$. That is, B belongs to φ, which was to be proved.

It remains to determine the type of convergence in Ω_1, which corresponds to limit in Ω, and thus completely justify the application of the calcul fonctionnel of FRECHÉT to the classes Ω_1, and Ω interchangeably. We have the theorem:

THEOREM. — *A necessary and sufficient condition that* $\lim_{n\to\infty}(g, h_n) = 0$, *where* $g(x)$, $h_n(x)$ *are functions of* $\Omega_1 = (n = 1, 2, 3, \ldots)$, *is that the sequence of functions* $h_n(x)$ *converge to* $g(x)$ *uniformly except at a set of points of measure zero.*

Let S_m, S_{mn} denote the FEJÉR convergents of rank m of $g(x)$, $h_n(x)$ respectively. If the sequence $h_n(x)$ converges to $g(x)$ uniformly, except at a set of points of measure zero, we have for every n sufficiently large:

$$|S_m(x) - S_{mn}(x)| = \frac{1}{m\pi}\left|\int_{-\frac{x}{2}}^{\pi-\frac{x}{2}} [g(x+2\beta) - h_n(x+2\beta)]\left(\frac{\sin m\beta}{\sin\beta}\right)^2 d\beta\right|$$

$$\leqq e.\int_{-\frac{x}{2}}^{\pi-\frac{x}{2}} \left(\frac{\sin m\beta}{\sin\beta}\right)^2 d\beta \leqq e.1 \leqq e \quad ^{6)}.$$

Therefore $(g, h_n) \leqq e$. That is, $\lim_{n\to\infty}(g, h_n) = 0$, which proves the condition sufficient.

Conversely, if for sufficiently large values of n,

$$(g, h_n) \leqq e/2,$$

we will have for all m sufficiently large

$$\frac{|S_m(x) - S_{mn}(x)|}{1 + |S_m(x) - S_m(x)|} \leqq e,$$

and therefore

$$|S_m(x) - S_{mn}(x)| \leqq e.$$

From the functions $S_m(x) - S_{mn}(x)$ we may select a sequence which converges to $g(x) - h_n(x)$ uniformly in general [4]). Therefore, except at the points of a subset

[6]) FEJÉR, loc. cit. [5]), p. 55.

of I of measure ε, small at pleasure, we must have

$$|g(x) - h_n(x)| \leqq e.$$

But the inequality above holds for all values of $\varepsilon > 0$, and therefore holds for all values of x, excepting a null set.

It follows since the sum of an enumerable infinity of null sets is a null set, that

$$g(x) = \lim_{n \to \infty} h_n(x)$$

and that the convergence is uniform except at the points of a null set. This completes the proof of the theorem.

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