LETTER TO EXAMINING BODIES FROM THE MATHEMATICAL ASSOCIATION.

GENTLEMEN,—I am instructed to lay before you a copy of the following resolution passed at a meeting of the Mathematical Association Committee on September 28th :

"That, in view of the doubt felt by many teachers as to the practical use and educational value of the ordinary rules for contracted multiplication and division, examining bodies be requested not to insist on the use of these methods; and that such bodies as agree be requested to publish their decision."

My committee will be glad to hear your decision.

I enclose a few copies of this circular, hoping that you will lay the above resolution before your examiners.—I am, yours faithfully,

A. W. SIDDONS,
Hon. Sec., Mathematical Association Committee.

HARROW-ON-THE-HILL,
October 21st, 1907.

THE INTRODUCTION OF THE MATHEMATICAL IDEA OF INFINITY

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(*Address delivered before the British Association, at Leicester*, 1907.)

I.

IT is a commonplace that ideas which seem abstruse or even incomprehensible to the learned of one generation are easy and familiar to the children of the next. This is, and must be so, with the mathematical idea of infinity. While there are still scientific people who regard the attempt to conceive anything beyond a finite number of things as visionary, I know of at least one child of 10 years old who dallies lovingly with the idea of infinity and the symbol ∞. To him, as to me, the idea of the number of reflexions of two looking-glasses, placed facing one another, is as definite and as unavoidable as that of the number of ants in an ant-hill.

The concept of a number is distinct from the manipulation or arithmetic of numbers. The idea of a number involves that of one number being comparable with other numbers, that is to say, it involves, more or less directly, the ideas of greater, equal and less, but not of the processes by which we answer the question—Is this number greater than that? Thus to recognise that there is a certain definite number—the number of ants in an ant-hill, and another definite number—the number of reflexions of two looking-glasses, placed facing one another, although neither of these numbers can be determined by counting, is one thing, but it is a step farther to realise that the second of these numbers cannot be reached or surpassed, by addition and multiplication, while the number of ants in an ant-hill may be calculated more or less accurately by observation and arithmetic. It is indeed clear that the size of an ant is more than $\frac{1}{10,000}$ cubic inch, and that of an ant-hill less than 100,000 cubic inches, so that the number in question is less than $100,000 \div \frac{1}{10,000}$, that is, less than a thousand million.* But, barring the imperfections of the mirrors and of our eyesight, there are a thousand million reflexions and more. Every integer is too small for the number of reflexions, and so we say that *integers are finite numbers,* while the number of reflexions is *an infinite number.*

Each reflexion corresponds to a definite integer; there is the first, the second, the thousand-millionth, and so on, and so we call the number of reflexions *the countably infinite,* to distinguish it from other infinite numbers, of which more anon.

These infinite numbers, and particularly the countably infinite, played an unrecognised part in the mental life of thousands before Georg Cantor put them and their properties lucidly and systematically before the world. The mathematical idea involved is clear and precise, and my present object is to give some indication how it may gradually be elucidated in the ordinary course of a modern education.

II.

There are three methods of dealing with mathematics:

1. The Logical Method,
2. The Formal Method,
3. The Practical Method.

The second of these has held the field for a long time in English mathematical training, and to it is due the undoubted facility many of our schoolboys and undergraduates possess in the manipulation of numbers and symbols. But it has absorbed an undue amount of attention. I myself stand for both the logical and the practical methods.

The practical view of mathematics and mathematical training is one that is now rapidly gaining ground in this country, so that it is the less necessary to plead for it here. For this reason it is with the first point of view, the logical, that I propose chiefly to deal in the present remarks.

I remember a story of J. J. Thomson's of a pupil of a practical turn of mind who took no interest in his mathematical work until Thomson took two marbles and made them collide. By that simple trick he won the interest of his pupil, who would otherwise have neglected his mathematics, and, Thomson thinks,

* Actually very much less, I believe.

probably have failed in the Tripos; as it was, he ultimately came out among the Senior Optimes.

So I believe some of the best minds have been lost to English mathematics by neglect of the logical method of treatment. I can parallel J. J. Thomson's story by one of my own of a man who told me he had given up mathematics after a year at Cambridge because of $\sqrt{-1}$. He said its use involved a contradiction of all previous axioms.

You must follow the *spirit* of J. J. Thomson's lead if you mean to catch the logically minded boy for mathematics. You must not wear him out solving quadratic equations, or working out by multiplication the number of bricks in a brick wall, or determining the cubic contents of a box by the method of weighing. His mind craves for more than these things, reasons, ideas; his imagination hungers for the unknown.

The moral effect of the logical method must not be overlooked. The second and third methods are alike in this—that they avoid difficulties; the first method alone looks facts in the face.

III.

The different methods are well exemplified in Geometry, and this will bring us to the subject of my paper.

The practical method involves drawings and diagrams, perspective, actual construction of solid models and of sections of solids, processes much neglected in England, but now coming in.

The formal method is the one still to some extent in possession of the field. In it results are regarded as interesting solely for their own sake, and theories only so far as they are productive of new results.

The logical method is that in vogue in Italy and, to a greater or less extent, all over the continent of Europe. On the one hand it concerns itself with the framing of a consistent code of axioms, and on the other it classifies the results obtained on a methodical and intelligent system.

Euclid, though of great interest to the antiquarian, represents a point of view unsatisfactory to all three schools of thought: not enough variety of results for the formalist, not enough logic for the modern mathematician, while the upholders of the third method complain of the thoroughly unpractical point of view exemplified by the proof that a straight line cannot meet a circle in more than two points, without making use of obvious considerations (of symmetry, for instance) that occur to the practical mind.

Infinity is never mentioned in Euclid. The idea is indeed of much later date. It was, I believe, in France that the word *infinity* intruded itself into geometry; Desargues, Descartes and Pascal seem to be responsible for it. It arose, not from philosophical speculations as to the ultimate, but as a help to grasp the anomaly presented by parallel lines in the theory of straight lines and their intersections.

The way I like to put it is this: The introduction of the terms "point at infinity" and "line at infinity" is merely a convenient symbolism. When we say, "These two straight lines have in common a point at infinity" we use the expression merely as standing for "These two straight lines are parallel."

It is easy to explain, even to a child, what we gain by this alternative mode of expression. That the whole subject of projective geometry, with its carefully selected code of axioms, takes its rise, in a certain sense, from this convention, may perhaps appeal to few who have been educated on this side of the Channel; but the fact that it is possible to replace half a dozen enunciations and as many different proofs by a single theorem * cannot fail to startle the student and to engage his interest. In particular, the engineer will soon perceive that he is enabled to remember a number of facts of practical value to him, which, when presented in an unsystematic manner, were too multitudinous for him to manipulate.

There is no hypothecation here of the *existence* of points, or lines, at infinity. There is, in fact, no new idea introduced at this stage at all. In the language of the theory of sets of points, our actual conception of the geometrical universe is that of an entirely open region without a boundary, or what is sometimes called *a domain*. Our senses limit the extent to which we can probe that domain, but our thought wanders unbounded throughout its depths. The imagination may, perhaps, like to carry over the symbolic language introduced for convenience, and to picture to itself the universe as a closed set of points with a plane at infinity. If, however, we have grasped the fact that the language is merely a conventional one, the student, when he comes to Theory of Functions of a

* Instead of saying:

"Two straight lines in a plane have a common point, unless they are parallel"	we say	"Two straight lines in a plane always have a single common point."

Or instead of the following:

"Three planes have a single common point, unless they have none at all, or a straight line common. If they have no common point, there are three cases: (1) they are all parallel; (2) two are parallel; (3) no two are parallel, but the three straight lines of intersection of the planes in pairs are parallel."	we say	"Three planes always have one common point, and, if they have two, they have in common the whole straight line joining those two points."

Examples might be multiplied, but it is not only that we have here a more handy form of statement, we are enabled to discuss simultaneously a number of special cases, and so to avoid the tedious repetition of arguments in which no new idea is introduced, and which therefore render the subject unnecessarily tedious. A good example of this is the celebrated theorem that, if the corresponding sides of two triangles ABC and $A'B'C'$ meet on a straight line, then the rays joining corresponding points, AA', BB', CC', meet at a point, and *vice versa*.

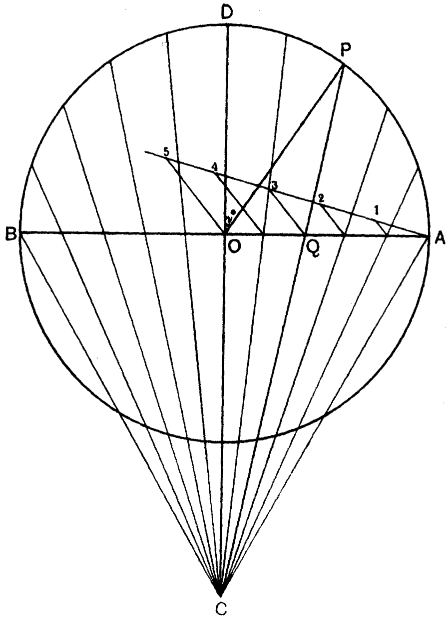

FIG. I.

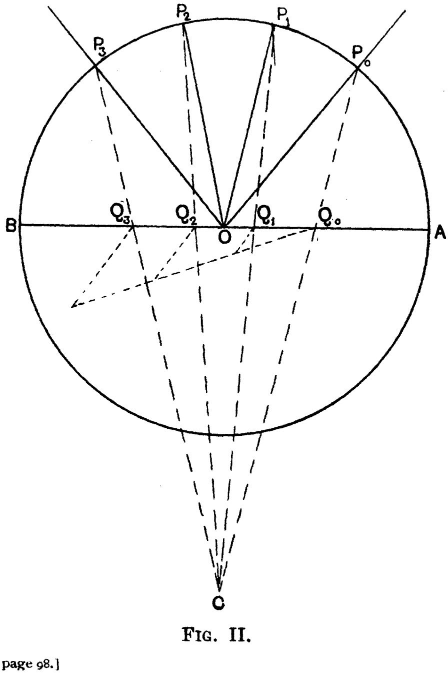

FIG. II.

To be bound up opposite page 98.]

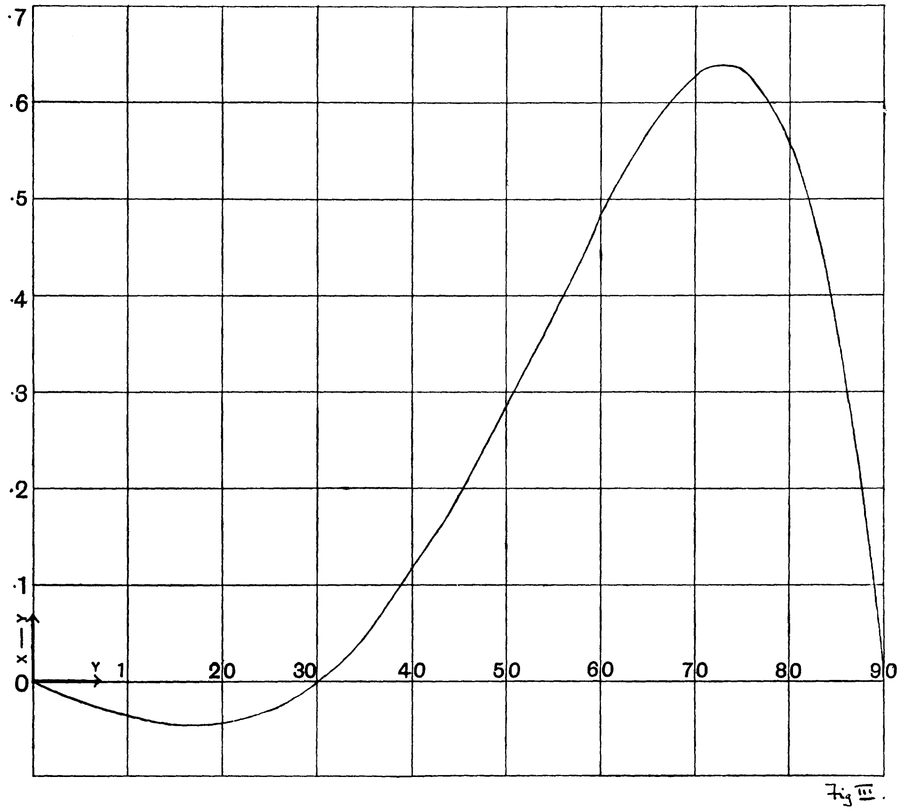

FIG. III.

Complex Variable, will have no difficulty in adopting a different symbolic language, and, if he gives play to his imagination, may prefer to think of his unlimited domain as a three-dimensional sphere, closed by a single point at infinity.

IV.

I have placed geometry first, because of the vivid appeal which it makes both to the practical mind and to the imagination for the introduction of the infinite. But it is not through Geometry that the mathematical idea of the infinite is, in the natural course of a modern education, unavoidably brought before the mind even of the child.

A circulating decimal, with all the difficulties involved in conceiving an infinite number of *processes*, is forced upon the pupil from the formal or the practical standpoint, when he attempts, as he is bound to do, whatever the system * under which he is brought up, to express $\frac{1}{3}$ as a decimal fraction. Later on the theory and practice of logarithms brings the same logical difficulty.

It is desirable that the boy, or girl, should be given a rigid proof of, for example, the theorem that there is no number, integral or fractional, whose square is 2. The old Greek proof will serve.

If there were such a number, we could denote it by $\frac{m}{n}$, where m and n are integers without a common factor. Then m is either odd or even. But the square of an odd number is odd, while

$$m^2 = 2n^2,$$

and is therefore even; thus m must be even.

Put then

$$m = 2p,$$

where p is an integer. Then $4p^2 = 2n^2$,

that is,

$$2p^2 = n^2,$$

so that n^2 and therefore n is also even, contrary to the fact that m and n, having no common factor, are certainly not both divisible by 2.

Therefore there is no such number $\frac{m}{n}$. Q.E.D.

In the same way a proof should be given that there is no number, integral or fractional, which is the logarithm of 2 to the base 10. For, if $\frac{m}{n}$ were such a number, it would mean that

$$2 = 10^{\frac{m}{n}},$$

so that

$$2^n = 10^m.$$

* I am glad to say that circulating decimals have no longer the prominence in the ordinary curriculum that they enjoyed when I was a boy. We must not, however, entirely banish them, or fail to assign to them their proper place, when devising our logical scheme of education.

But 2^n ends in either 2, 4, 8 or 6, and cannot therefore end in a 0. Thus this is impossible, and there is no such number $\frac{m}{n}$. Q.E.D.

At this stage I believe that the right point of view is not the existence, but the non-existence of the irrational* number, and that the use of the term is, as in Geometry, merely a convenient convention. When I say, "There is no number, integral or fractional, whose square is 2," and when I say, "There is a number, an irrational number, whose square is 2," I mean the same thing. If the latter statement is more suggestive, it is because, by our convention, it is equivalent to the positive assertion that, *to any required degree of accuracy, any rational number within a certain interval will serve our purpose.*

The importance of this convention in relation to the concept of length should be clearly brought into view. It does not follow that the lengths of two objects, taken at random, are to one another as two numbers, unless and until this convention has been made. It is invariably true, of course, that two lengths, as measured, however accurately, are in the ratio of two integers, but these integers will in general vary according to the delicacy with which the measurement was made. The quantitative concept of length indeed itself involves the idea of an infinite number of processes; still more is this true of the ideas of area, volume and density.

V.

The familiar measuring rod or rule connects with points on the straight line the idea of definite numbers, their distances from the zero point of the rod. Certain of these numbers are indicated on the straight edge by marks; the integral points 1, 2, 3,... most certainly, while probably the rod is subdivided into tenths, perhaps into hundredths, and a further subdivision is usually effected by the eye. The points so identified by continued subdivision will be the end-points of intervals; any other point must be thought of as the limit of an interval, which is at first a marked interval and gets smaller and smaller till it shrinks up to a point.

Thus, for instance, the point $\frac{1}{3}$ is never obtained by the decimal subdivision. It lies first between the points ·3 and ·4, that is in the interval (·3, ·4). The next subdivision may or may not be feasible, but we only have to imagine that we look through a magnifying glass which enlarges 10 times, and it will be clear to us in what interval our point will lie. For

*A rational number is the general term for any integer or fraction. An irrational number is then, in the first instance, the negation of the rational, and is subsequently used for any of the new numbers conventionally introduced.

that which we must add to $\cdot 3$ to get $\frac{1}{3}$ is $\frac{1}{3} \cdot \frac{1}{10}$, that is $\frac{1}{3}$ of the smaller interval ($\cdot 3$, $\cdot 4$), so that when we look through our microscope the point $\frac{1}{3}$ will seem to have the same position in the interval ($\cdot 3$, $\cdot 4$) as it originally had in the interval (0, 1) as seen by the naked eye. Thus it will lie in the third interval when the small interval ($\cdot 3$, $\cdot 4$) is subdivided into hundredths, that is it will lie between $\cdot 33$ and $\cdot 34$, or in the interval ($\cdot 33$, $\cdot 34$).

This is the same at every subsequent stage, so that the next interval in which $\frac{1}{3}$ lies will always be the third subdivision of the preceding interval; in this interval, if we have a sufficiently strong magnifier, our point will always seem to lie.

The intervals we get then, which are at first marked intervals and afterwards only imagined, are, ($\cdot 3$, $\cdot 4$), ($\cdot 33$, $\cdot 34$), ($\cdot 333$, $\cdot 334$), ($\cdot 3333$, $\cdot 3334$),... and so on *ad infinitum*; these intervals shrink up to the point $\frac{1}{3}$. Having once made this clear to our minds, the idea that intervals lying inside one another, and getting indefinitely small, do actually shrink up to a point *wherever the intervals may be located on the straight line* is instinctive: the uniformity of the unmarked straight line demands this.

The point to which the intervals shrink up is called *the limiting point* of the intervals, or of the end-points of the intervals.

The distance of a point P from the zero point O may, as in the case of the point $\frac{1}{3}$, be a number, integral or fractional—a rational number—but this need not be the case. Suppose OP is the long side of a right-angled triangle, whose other sides are each of length unity. Here, in truth,

$$OP^2 = 2,$$

so that, as we saw, the length of OP is not a rational number, but, to any required degree of accuracy any rational number within a certain interval will serve our purpose. We find ourselves constrained to say that the distance OP is *an irrational number*, and to call P *an irrational point*. At this stage we begin to regard irrational numbers as on a par with rational numbers, and to conceive the straight line as made up of rational and irrational points.*

VI.

What are the odds that, if we take a set of intervals located at random and shrinking up to a point, that that point will

* The irrational point P, corresponding to the irrational number $\sqrt{2}$, is constructed without the use of intervals as the intersection of two straight lines. It cannot be proved that this is the case with *any* irrational number, and it is known that in the case of some irrational numbers, *e.g.* the length of the circumference of a circle of diameter unity, the construction (so called "squaring of the circle") cannot be carried out by means of ruler and compasses. It is assumed as an axiom in the theory of sets of points that every irrational number, by whatever mathematical process it is obtained or defined, *determines* a point on the straight line.

be a rational point? Here you have a pretty problem in infinite numbers. There is an infinite number of rational points, but there is also an infinite number of irrational points; yet the question has a definite answer, just as it would if the numbers in question were finite.

The answer is that you are tolerably certain to hit upon an irrational number; for there are, to put it in schoolboy language, heaps more irrational points than rational ones. I mention this to shew how one is led on from one infinity to another, and to illustrate what I said at the beginning about other infinities beside the countably infinite.

The rational points are countably infinite, as you easily see, for you can arrange them in the following way:

$$\tfrac{1}{2}, \quad \tfrac{1}{3}, \tfrac{2}{3}, \quad \tfrac{1}{4}, \tfrac{3}{4}, \quad \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \ldots$$

so as to get them all, and each in its definite place, so that you can say which is the first, or the twentieth or the thousand-millionth, just as we could in the case of the two mirrors.

Now, it is not hard to shew, by the use of decimal fractions, that the irrational points are *more than countably infinite*; there are, of course, countably infinite sets of them, but no countably infinite arrangement is possible that takes in all.

The two infinite numbers we have here learnt to know are commonly called a and c, a being less than c. There is nothing indefinite or hazy about these numbers; they are recognisable, and they constantly crop up; c is, by the way, startling as it may seem, also the number of points on a straight line, or in a plane, or in space.

VII.

The concept, above referred to, of a point P being *a limiting point of a set of points*, is a very valuable one. The sole and only criterion is that every interval, however small, containing P, encloses a point of the set, distinct from P itself.* When, and only when, this is the case, P is said to be a limiting point of the set.

This does not seem to be a complicated notion, and it is simple enough to detect whether, or no, any given point P is, or is not, a limiting point of a given set. On the other hand, given only the set, the identification of its limiting point, or one of its limiting points, involves an infinite number of processes.

It is only by degrees that the idea of an infinite number of processes can be rendered familiar. If this has been done, the notion of *a limit* will no longer present serious difficulties. A limit is merely the equivalent in arithmetic of that which

* P may, or may not, belong to the set, according as the set is defined. Thus any point is a limiting point of the rational points, but it only belongs to the set if it is itself a rational point

we called *a limiting point* in geometry, the identification of numbers with points effects this. Whatever numbers we have to deal with, for instance the values of any number of areas, we can plot down on a straight line, using a suitable system of units;* if there are only a finite number of points, there will be no limiting point; but, if the points to be plotted are infinite in number (in which case, they cannot, usually, be all actually plotted down) they will certainly have at least one limiting point, which may, or may not, be capable of being plotted down, but can certainly be determined within the limits of vision, which is only another way of making the old statement about the limits of accuracy.

Too often, at present, the pupil comes openly face to face with the idea of infinity first in connection with infinite series, which are defined by means of limits,† *e.g.*

$$\underset{n=\infty}{Lt}\left[\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}\right]=1,$$

or

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\dots=\frac{1}{3}.$$

Generally the sum to infinity of geometric series, such as the above, is sprung upon him, after learning the rule for summing a finite number of terms. In other words, he is supposed to have intuitively a notion of what is meant by a "limit," and, in this way, to be able to explain to himself the idea of a infinite number of terms, whereas almost as soon as he is capable of abstracting the idea of number at all, the concept of infinity should begin to play its appropriate part in his mental drama of the universe.

VIII.

The number of familiar notions which are absolutely dependent on the idea of a simple limit, or limiting point, is large. It is no exaggeration to say that few people have more than a vague idea what they mean by length, area, volume, mass, moment of inertia, current.

To take the notion of area as typical. It is easy to define the area of a triangle, and to prove that it is "half the base × altitude." We naturally say that the area of any number of non-overlapping

* For instance the area 2 sq. metres may be plotted down as the point whose distance from the zero point is 2 millimetres, and so on.

† Plotting down the points $\frac{1}{2}$, $(\frac{1}{2}+\frac{1}{4})$ or $\frac{3}{4}$, $(\frac{1}{2}+\frac{1}{4}+\frac{1}{8})$ or $\frac{7}{8}$, and so on, taking each time one more term into the sum, we get on the straight line points, which soon, within the limits of vision, are indistinguishable from the point 1; this point 1 is in fact the limiting point of the set, and the number 1 is the limit of the set of numbers. The successive sums of the second series are plotted down as the left-hand end-points of the intervals of § VI., which, as we saw, shrank up to the point $\frac{1}{3}$, and the whole infinite series is only another way of writing the circulating decimal $\dot{3}$.

triangles is the sum of their areas, and so get an expression for the area of any polygonal figure. But what do we mean by the area of a circle, or of a figure of more irregular form?

It is proved in elementary books that, if we inscribe a polygon in a circle, and circumscribe one about the circle, the difference between the areas of the two polygons may be made as small as we please, and depends only on the smallness of the sides of the polygons. In other words, the areas of the two polygons will agree to more and more decimal places, the shorter the sides are constructed, in which case they become, of course, more and more numerous. We thus get a definite limit defined, which is not itself the area of any one of these polygons, and depends entirely on their being inscribed in and circumscribed about this particular circle. This limit is what we call the area of the circle.

This is a notion precisely of the same kind as that of length, triangles and polygonal regions taking the place of the rational lengths. If we plot down these areas on a straight line, we get, as before, a series of intervals—between the areas of the inscribed and circumscribed polygons—intervals which get smaller and smaller and finally shrink up to a point.

This definition of the area of a circle suggests that the same kind of definition would do for more irregular figures. But are we sure that the difference between the areas of the inscribed and circumscribed polygons would always diminish indefinitely?

Usually this is so, but it is stimulating to realise that it is possible to find a figure with a crinkly boundary where this is not the case. Clifford is certainly not the only Englishman who has revelled in the possibility of the incredible. Incredible as it sounds that a curve, however crinkly, should take up so much room that the areas of the inscribed and circumscribed polygons never agree beyond a certain point, it has been rigidly proved to be true. When, and only when, the boundary of the region considered is not of this nature, we do get, as in the base of the circle, a definite limit, which is then called the area of the region.

The definition of volume is of precisely the same nature as that of area; that of mass is somewhat more complicated; the other familiar notions I mentioned, as well as that of velocity, involve the idea of a simple limit. It is the same thing to say that all these notions involve the conception of a limiting point on a straight line.

Just in the same way as we may describe a portion of a plane cut out by a curve, so as to have no area, so we can isolate in our minds a portion of space which has no volume, and imagine a suitable surface drawn in a body so that the portion inside has no mass. We may even conceive of the possibility of a body moving continuously in such a way that no point of it has any velocity.

IX.

From limiting points on the straight line, it is a natural step to limiting points in the plane and in space. These are still more interesting because they introduce new ideas and new processes of thought, but they are as easy of conception as limiting points on the straight line. We have, instead of intervals shrinking up to a point, small plane domains, or small solid domains in space, lying inside one another and shrinking indefinitely in size. We already had an example of this, in the case of our two mirrors; on the face of one of them the reflexions are plane domains, lying inside one another, and shrinking up to a point.

The practical importance of the ideas so easily obtained cannot be exaggerated; for the idea of a limiting point in a plane is concomitant to that of a double limit,* and the conception of a double limit is a real difficulty in analysis and in applied mathematics. Still more is this the case with a triple limit, which is however concomitant to a limiting point in space. The proper comprehension of these ideas enables us to be lucid as well as precise in dealing with these important subjects.

X.

Let us now utilise the idea of a limiting point in space to clear up our notions about *the centre of gravity.*

We all use the term centre of gravity, but what do we mean by it? The centre of gravity, you tell me, is a definite point in a body through which the weight always acts. How do we know that there is such a point? Is there, in fact, always such a point? Anyhow, how do we find it?

Now I do not mean to enter here into the difficulties introduced by the fact that gravity does not really act in parallel lines, and so forth. Assume that gravity does act in parallel lines, if you like, or, as I prefer to do, let us use the term *centre of mass,* instead of centre of gravity, the questions just asked still present real difficulties.

It is easy to see by symmetry that a cube or a sphere of uniform matter has its centre of mass at its centre of symmetry. It is no hard practical problem to find the centre of gravity of a uniform rod, or a billiard cue. We have an easy rule for finding graphically or numerically the centre of mass of any two bodies together, provided we know their masses, and their centres of mass separately. But how are we even to define the centre of mass of a body of irregular form and variable specific gravity?

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* This is quite distinct from two simple limits in succession.

If anyone says "we define it by triple integration," * it is clear that this is not a definition likely to appeal to the schoolboy, or the man in the street. But the idea involved is not really occult, it is merely that of a limiting point in space.

Imagine space divided up by parallel planes into small equal cubes. Certain of these will be entirely filled by the body under consideration, certain others will be partially filled, and the rest will contain no point of the body. The cubes containing points of the body will, however, not necessarily be uniform, so that we do not know their centres of mass. We imagine however the mass inside each to be distributed uniformly, so that the centre of mass of each is its centre of symmetry, and we find the centre of mass G_1 of all the cubes which are completely filled with the matter of the body under consideration. We also find the centre of mass G_2 of all the cubes which contain any points whatever of the body.

We now subdivide space into cubes of $\frac{1}{8}$ of the size, by interpolating planes between each pair of dividing planes, and repeating the process we get two new points G_3 and G_4.

Proceeding thus *ad infinitum* we obtain a countably infinite set of such centres of mass $G_1, G_2, G_3, G_4, G_5, G_6, \ldots$

Now it is one of the first things we have learnt about limiting points that every infinite set of points has at least one limiting point. Thus these centres of mass have at least one limiting point, and Nature provides that, just as we do not meet casually with bodies which have no area or volume, so these centres of mass are found habitually to have only one limiting point; moreover, this limiting point is found to be the same however the dividing planes have been located. This limiting point G is called the centre of gravity, or more correctly the centre of mass, of the body.

You see at once that you have here the same hypothesis as to the existence of the centre of mass as in the case of area, volume, or any idea involving limits, but the concept is hardly more complicated than any of the others already touched upon.

XI.

Before closing, there is one familiar idea which I must recall to your minds. It is the concept of time.

There is no idea which is more familiar or more unavoidable in modern life. For ordinary purposes a Waterbury watch or a Dutch clock will perhaps suffice to give us a vivid idea of the nature and use of time. But any really fine observational work brings us face to face with the actual problem, "How

* It may be added that such a definition shirks the real difficulty. The triple integrals may indeed not exist.

are we to make our time-keepers accurate?" and with the logical question, "What is time, and how is it measured?"

That the first question depends on the second is evident. That the answer to the second question involves in a number of ways the idea of infinity is the reason why it appears so inscrutable to most ordinary people who have once tried to understand it. The measurement of time involves the mathematical ideas of distance and velocity (or angle and angular velocity), approximation, disturbance, curve, and a number of other notions summed up in the term limit; notions which will not seem hard or abstruse to people who have never avoided, but courted, the mathematical idea of infinity.

I have done. Allow me to sum up in one sentence the gist of what I have said. The mathematical idea of infinity and of an infinite number of processes is unavoidable, nor is it desirable that it should be avoided. It has been rendered precise, and its contemplation tends to clearness of thought. Its introduction from the earliest stages as a recognised object of thought is in every way to be encouraged.

PERSPECTIVE THROUGH THE STEREOSCOPE.

(*Concluded from* p. 122.)

Slide IX. (fig. 7). Here the Ground Plane is rotated about the Intersection carrying P with it along a circular arc PP' whose centre is M. At the same time E and J are rotated in vertical circles whose centres are I and O through angles EIE', JOJ' each equal to PMP'. Since $E'I$ is parallel to MP', they must lie in a plane cutting the Picture Plane in IM. Also since $E'P'$ is in the plane containing $E'I$ and MP', the point in which $E'P'$ cuts the P. Pl. lies in IM and divides it in the ratio of $E'I : MP'$, *i.e.*, of $EI : MP$.

Hence $E'P'$ cuts the P. Pl. in p. Or thus: Since EIE', PMP' are similar figures with corresponding sides parallel, therefore EP, IM, $E'P'$ meet in a point.

The same applies to any other point in the G. Pl. Hence

Theorem (5) *If the Ground Plane and the eye be both rotated in the same sense through equal angles, the first about the Intersection and the Second about the I-line, the picture remains unaltered.*

Hence E may be rotated into the Picture Plane itself into a position above or below the I-line at a distance equal to the Distance of the Picture EI, as we have previously shewn.

From the previous investigation it follows that the I-line is still the Vanishing Line of the inclined plane in the position shewn and *with the eye at* E'.

Hence the Vanishing Line for the same inclined plane with