

$$\sin \frac{c}{2} \cos \frac{A-B}{2} = \sin \frac{C}{2} \sin \frac{a+b}{2};$$

and the remaining two follow from the polar triangle.

Mr. Crofton remarks, that "Prof. Gudermann, of Cleves, in his work on Spherics, gives a construction like mine, but deduces the formulas in a much more complicated form."

Mr. S. Roberts gave an account of his paper

*On the Order of the Discriminant of a Ternary Form.*

I propose to give a short analytical proof of two results obtained geometrically by Professor Cremona, relative to the influence of multiple points common to the curves of an involution. A few remarks are added on some limiting cases. The formula referred to will be found at pp. 261, 262 of the German translation of Professor Cremona's Introduction to the Geometrical Theory of Plane Curves (1865).

1. Let  $U, V$  be ternary forms of the  $n^{\text{th}}$  degree, so that  $U+kV=0$  represents an involution of curves. In the general case, we obtain the number of possible double points of the involution, or the order relative to  $k$  of the discriminant, by means of the system of determinants

$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \end{vmatrix} = 0.$$

Taking the two equations

$$\frac{du}{dx} \frac{dv}{dz} - \frac{du}{dz} \frac{dv}{dx} = 0 \dots\dots (1), \quad \frac{du}{dy} \frac{dv}{dz} - \frac{du}{dz} \frac{dv}{dy} = 0 \dots\dots (2),$$

we get the gross order  $4(n-1)^2$ , from which must be deducted the extraneous order of

$$\frac{du}{dz} = 0 \dots\dots (3), \quad \frac{dv}{dz} = 0 \dots\dots (4),$$

or  $(n-1)^2$ , leaving  $3(n-1)^2$ .

2. Suppose, however, that all the curves of the involution have a common multiple point at  $x=0, y=0$  (say  $\bar{x}\bar{y}$ ) of the order of multiplicity  $p$ ; and suppose, further, that the tangents at the point are common, and that  $s$  of them coincide with  $y=0$ . We may write

$$\begin{aligned} U &= y^p a z^{n-p} + b z^{n-p-1} + \&c., \\ V &= y^p a z^{n-p} + \beta z^{n-p-1} + \&c. \end{aligned}$$

The highest power of  $z$  in (1), (2) is  $2n-2p-2$ , and its coefficient contains in each case  $y^{p-1}$ . These equations, therefore, represent curves having  $\bar{x}\bar{y}$  for a multiple point of the order  $2p$ , and there are  $s-1$  common tangents; also (3), (4) represent curves having in common

the multiple point  $\bar{xy}$ , with an order of multiplicity  $p$ , and with  $p$  common tangents.

Hence, for the number of double points of the system, not counting the common multiple point, we have

$$4(n-1)^2 - 4p^2 - s + 1 - \{(n-1)^2 - p^2 - p\} = 3(n-1)^2 - p(3p-1) - s + 1.$$

The diminution therefore is  $p(3p-1) + s - 1$ .

3. Let the same supposition be retained, except that the tangents at the multiple point are no longer common.

Then the coefficient of  $z^{2n-2p-1}$  in (1) is of the form

$$(n-p) \left( \frac{da}{dx} A - \frac{dA}{dx} a \right),$$

where  $A, a$  are binary forms of the degree  $p$  in  $x, y$ .

The corresponding coefficient in (2) is of the form

$$(n-p) \left( \frac{da}{dy} A - \frac{dA}{dy} a \right).$$

$$\begin{aligned} \text{But } p \left( \frac{da}{dx} A - \frac{dA}{dx} a \right) &= \frac{da}{dx} \left( x \frac{dA}{dx} + y \frac{dA}{dy} \right) - \frac{dA}{dx} \left( x \frac{da}{dx} + y \frac{da}{dy} \right) \\ &= y \left( \frac{da}{dx} \frac{dA}{dy} - \frac{da}{dy} \frac{dA}{dx} \right). \end{aligned}$$

$$\text{Similarly } p \left( \frac{da}{dy} A - \frac{dA}{dy} a \right) = x \left( \frac{da}{dy} \frac{dA}{dx} - \frac{da}{dx} \frac{dA}{dy} \right).$$

Consequently the curves (1), (2) have  $2p-2$  common tangents at the multiple point, which is of the order  $2p-1$  on each curve. The curves (3), (4) have a common multiple point at  $\bar{xy}$  of the order  $p$ .

The order of the system in question is now

$$\begin{aligned} 4(n-1)^2 - (2p-1)^2 - 2p + 2 - \{(n-1)^2 - p^2\} \\ = 3(n-1)^2 - (p-1)(3p+1). \end{aligned}$$

The diminution on account of the common multiple point is  $(p-1)(3p+1)$ .

4. If there are two common multiple points  $\bar{xy}, \bar{xz}$ , of the orders  $p, q$ , the application of the foregoing result gives for the order generally

$$3(n-1)^2 - (p-1)(3p+1) - (q-1)(3q+1);$$

and if  $p+q = n$ , we have  $6pq - 4(p+q) + 5$ .

This seems to be the order of the discriminant of a ternary form, in which the highest powers of  $z, y$  are  $q$  and  $p$  respectively. An apparent inconsistency here exists, in connection with a formula obtained by Dr. Henrici (Proc. Math. Soc., Nov. 1868, p. 109). His result is  $6pq - 4(p+q) + 4$ , relative to a non-homogeneous form in two variables. Reverting to  $U, V$ , and supposing that these func-

tions represent curves having two common multiple points  $\bar{xy}$ ,  $\bar{xz}$  of the orders  $p$ ,  $q$ , where  $p+q = n$ , we may write them in the forms

$$\begin{aligned} \text{U} &= az^2 + b\bar{xz}z^{-1} + \&c., \\ \text{V} &= az^2 + \beta\bar{xz}z^{-1} + \&c., \end{aligned}$$

where  $a$ ,  $b$ , &c.,  $\alpha$ ,  $\beta$ , &c. are binary forms of the degree  $p$  in  $x$ ,  $y$ .

The curve (2) now contains the line  $x = 0$ ; and if, as in the instance referred to,  $x$  is replaced by unity, this curve is lowered in degree by unity. The curve (1) is of the degree  $2n-2$ , and has  $\bar{xy}$  for a multiple point of the order  $2p-1$ , and  $\bar{xz}$  for a multiple point of the order  $2q-2$ ; the curve (2) is to be taken as of the degree  $2n-3$ , with a multiple point at  $\bar{xy}$  of the order  $2p-2$ , and another at  $\bar{xz}$  of the order  $2q-2$ . There are also common tangents at the multiple points in number  $2p-2$  and  $2q-2$  respectively. The curves (3), (4) have the common multiple points  $\bar{xy}$ ,  $\bar{xz}$  of the orders  $p$ ,  $q-1$  respectively.

The order which we now obtain is

$$2(n-1)(2n-3) - 2(p-1)(2p-1) - 4(q-1)^2 - 2p - 2q + 4 \\ - \{(n-1)^2 - p^2 - (q-1)^2\},$$

or, what is the same thing, in accordance with Henrici's result,

$$6pq - 4(p+q) + 4.$$

5. It is easy to see geometrically that there is, in this point of view, a reduction by unity. For, taking a right line through the multiple points, we have  $n$  of the base-points linear; the remaining base-points lie, therefore, on a curve of the degree  $n-1$  having multiple points at  $\bar{xy}$ ,  $\bar{xz}$  of the orders  $p-1$ ,  $q-1$  respectively. Hence the compound curve belongs to the involution, and the right line meets it again in one point, which constitutes a double point of the system.

If one of the multiple points is of the order  $n-1$ , every transversal through it meets the curve again in a single point, and the sum of the multiplicities is  $n$ . When, therefore, we discard the composite curves, there is no double point distinct from the given multiple point, as is otherwise evident.

If, again, we have any number  $m$  of multiple points on a transversal, the sum of whose multiplicities is  $n$ ; it follows from the same reasoning that one of the curves of the involution contains the right line. When, therefore, we discard this case, which gives rise to  $m-1$  other double points, the order of the discriminant of the corresponding form is

$$3(n-1)^2 - \Sigma_m (p_1-1)(3p_1+1) - m + 1 = 6\Sigma_m p_1 p_2 - 4\Sigma_m p_1 + 4.$$

The President gave an account of his investigations on the centrosurface of an ellipsoid (locus of the centres of curvature of the ellipsoid). The surface has been studied by Dr. Salmon, and also by Prof. Clebsch, but in particular the theory of the nodal curve on the surface admits of further development. The position of a point on the ellipsoid is de-

terminated by means of the parameters, or elliptic coordinates,  $h, k$ ; viz., if as usual  $a, b, c$  are the semi-axes, and if  $(X, Y, Z)$  are the coordinates of the point in question,

$$\text{then } \frac{X^2}{a^2+h} + \frac{Y^2}{b^2+h} + \frac{Z^2}{c^2+h} = 1,$$

$$\frac{X^2}{a^2+k} + \frac{Y^2}{b^2+k} + \frac{Z^2}{c^2+k} = 1;$$

and hence

$$-\beta\gamma X^2 = a^2(a^2+h)(a^2+k),$$

$$-\gamma\alpha Y^2 = b^2(b^2+h)(b^2+k),$$

$$-\alpha\beta Z^2 = c^2(c^2+h)(c^2+k),$$

if for shortness

$$\alpha = b^2 - c^2,$$

$$\beta = c^2 - a^2,$$

$$\gamma = a^2 - b^2,$$

$$(\alpha + \beta + \gamma = 0).$$

This being so, the coordinates of the point of intersection of the normal at  $(X, Y, Z)$  by the normal at the consecutive point of the curve

$$\text{of curvature } \frac{X^2}{a^2+k} + \frac{Y^2}{b^2+k} + \frac{Z^2}{c^2+k} = 1$$

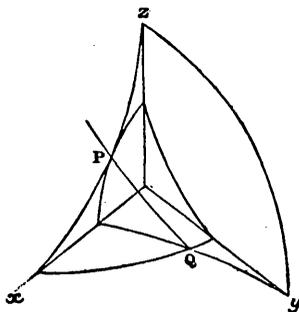
are given by the formulæ

$$-\beta\gamma\alpha^2 x^2 = (a^2+h)^3(a^2+k),$$

$$-\gamma\alpha b^2 y^2 = (b^2+h)^3(b^2+k),$$

$$-\alpha\beta c^2 z^2 = (c^2+h)^3(c^2+k);$$

viz., these equations, considering therein  $(h, k)$  as arbitrary parameters, determine the coordinates  $(x, y, z)$  of a point on the centro-surface. The principal sections (as is known) consist each of them of an ellipse counting three times, and of an evolute of an ellipse; the evolute and ellipse have four contacts (two-fold intersections) and four simple intersections, but the contacts and intersections respectively, are in the different sections real and imaginary; and if (as, without loss of generality, may be assumed)  $a^2 + c^2 > 2b^2$ , then the form of the principal sections is as shown in the figure (which represents only an octant of the surface); viz., there is a real contact at P in the plane of  $zx$ , and a real intersection at Q in the plane of  $xy$ . The surface has thus an exterior and an interior sheet, but (instead of meeting in a conical point, as in the wave surface) these intersect in a nodal curve QP. (The curve has a cusp at Q, and a node at P; viz., the curve extends beyond P, but from that point is acnodal, or without any real sheet of the



surface passing through it.) For the nodal curve there must be two values  $(h, k)$ ,  $(h_1, k_1)$ , giving the same values of  $(x, y, z)$ ; viz., there must exist the relations

$$\begin{aligned}(a^2 + h)^3 (a^2 + k) &= (a^2 + h_1)^3 (a^2 + k_1), \\ (b^2 + h)^3 (b^2 + k) &= (b^2 + h_1)^3 (b^2 + k_1), \\ (c^2 + h)^3 (c^2 + k) &= (c^2 + h_1)^3 (c^2 + k_1); \end{aligned}$$

from which equations eliminating  $h_1$  and  $k_1$ , we should have between  $h, k$  a relation which, combined with the expressions of  $x, y, z$  in terms of  $(h, k)$ , determines the nodal curve. But the better course is to eliminate  $k, k_1$ , thus obtaining a relation between  $h$  and  $h_1$ , in virtue whereof  $h_1$  may be regarded as a known function of  $h$ ;  $k$  and  $k_1$  can then be readily expressed in terms of  $h, h_1$ ; that is, we have  $k$  as a function of  $h, h_1$ , or in effect as a function of  $h$ . The relation between  $h, h_1$  (after a reduction of some complexity) assumes ultimately a form which is very simple and remarkable; viz., writing

$$P = a^2 + b^2 + c^2, \quad Q = b^2c^2 + c^2a^2 + a^2b^2, \quad R = a^2b^2c^2,$$

the relation is

$$\begin{aligned}(6R + 3Qh + Ph^2), \\ + h_1(3Q + 4Ph + 3h^2), \\ + h_1^2(P + 3h) = 0; \end{aligned}$$

this is a (2, 2) correspondence between the two parameters  $h, h_1$ ; the united values  $h_1 = h$ , are given by the equation  $6(R + Qh + Ph^2 + h^3) = 0$ , that is

$$(a^2 + h)(b^2 + h)(c^2 + h) = 0;$$

viz., the two points on the ellipsoid which have their common centre of curvature on the nodal curve are only situate on the same curve of curvature when this curve is a principal section of the ellipsoid.

[Since the date of the foregoing communication, Prof. Cayley has found that the squared coordinates  $x^2, y^2, z^2$  of a point on the nodal curve can be expressed as rational functions of a single variable parameter  $\sigma$ .]

The following present was received:—

“Monatsbericht” (Sept. and Oct. 1869).

January 13th, 1870.

Prof. CAYLEY, President, in the Chair.

Prof. Oppermann, of Copenhagen, was one of the visitors present.

Mr. A. Ramsay was proposed for election.

Mr. J. J. Walker, M.A., gave an account of his paper on the “Equations of Centres and Foci, and conditions of certain Involutions.”

Dr. Henrici, Prof. Hirst, the President, and Mr. Clifford, took part in a discussion upon the paper.

The President then made his first communication to the Society of results he had arrived at in his paper on "Quartic Surfaces."

Mr. S. Roberts exhibited and explained diagrams of the Pedals of Conic Sections which he had constructed by the methods described in his communication of the 14th Jan. 1869 (vol. ii. p. 125).

The following presents were received:—

"A Memoir on Cubic Surfaces, and a Memoir on the Theory of Reciprocal Surfaces," by Prof. Cayley: from the Author.

"Annali di Matematica:" (Dicembre, 1869).

"Crelle's Journal," 71 Band, zweites Heft.

"Note on a Theorem relative to Neutral Series," by A. De Morgan, F.R.A.S., (Cambridge Phil. Society's Transactions, vol. xi. pt. 3): from the Author.

"On Interpolation, Summation, and the Adjustment of Numerical Tables," (Journal of Institute of Actuaries, vol. xi. p. 61); and "On an Improved Theory of Annuities and Assurances," (ditto vol. xv. p. 95,) from the Author, W. S. B. Woolhouse, F.R.A.S.

"On the Nodal Cones of Quadri-nodal Cubics, and the Zomal Conics of Tetrazomal Quartics," by the Rev. R. Townsend, F.R.S.: from the Author.

"Monatsbericht" (Nov. 1869).

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*February 10th, 1870.*

Prof. CAYLEY, President, in the Chair.

Prof. Oppermann, of Copenhagen, was again present.

Mr. A. Ramsay was elected a Member; and Mr. E. A. L. Bradshaw Smith, M.A., Fellow of Christ's College, Cambridge, was proposed for election.

The President having vacated the Chair, communicated the concluding portion of

*A Memoir on Quartic Surfaces.*

The present Memoir is intended as a commencement of the theory of the quartic surfaces which have nodes (conical points). A quartic surface may be without nodes, or it may have any number of nodes up to 16. I show that this is so, and I consider how many of the nodes may be given points. Although it would at first sight appear that the number is 8, it is in fact 7; viz., we can, with 7 given points as nodes