Gauge Normal Forms and Effective Couplings from Affine Readout Kinetics

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Abstract

This paper develops the gauge-coupling layer of the premetric affine sequence after the construction of the physical quotient, relational time, Lorentzian readout, metric-affine decomposition, and spectral matter architecture. The aim is not to postulate a gauge group and not to fit observed couplings. The aim is to prove a precise gate theorem: an internal affine response becomes an effective Yang–Mills sector only when a candidate internal subspace descends to the physical quotient, closes as a reductive Lie algebra, has a stable projector, carries a positive kinetic form after Schur reduction, and admits a normal form whose residual normalization cannot be absorbed by presentation changes. We give full proofs of projected-connection covariance, the leakage obstruction, the Yang–Mills identities, Schur positivity, reductive kinetic normal forms, abelian charge-lattice normalization, and the conversion from kinetic coefficients to canonically normalized couplings. The v2 refinement adds a stricter dependency audit, a center-versus-simple-factor separation theorem, an explicit residual calculus for three-factor kinetic ladders, a readout-scale warning against hidden renormalization-group import, and a numerical benchmark written as an arithmetic consequence of stated seed quantities rather than a phenomenological insertion. In the exact ladder case one obtains

$$\kappa_1 = \kappa_Y, \qquad \kappa_2 = \kappa_Y r, \qquad \kappa_3 = \kappa_Y r^2,$$

and therefore

$$g_a = \kappa_a^{-1/2}, \qquad g_2^2 = g_1 g_3.$$

The paper does not prove that the Standard Model gauge algebra has been derived, does not identify a physical renormalization scale beyond the readout scale, and does not use measured coupling constants as inputs. It provides the mathematical and audit infrastructure required before such an identification can be claimed.

Contents

1 Introduction

A gauge coupling is often written as if it were an independent parameter of the low-energy theory. In an ordinary phenomenological Lagrangian this is appropriate: the coupling labels the strength with which a connection enters a covariant derivative after the kinetic term has been normalized. In a theory that claims to extract effective physics from a deeper affine substrate, however, this convention is not enough. One must show where the kinetic coefficient comes from, why the relevant internal directions form a closed gauge algebra, why the projection from the parent affine response is stable, and why the resulting coefficient cannot be removed by a harmless change of presentation. The present paper addresses exactly this problem.

The previous papers in this sequence establish the layers needed before the question of gauge couplings is meaningful. A physical quotient removes descriptive redundancy; a relational clock orders the quotient; a rank-four Lorentzian readout supplies the local arena in which tensorial field equations can be written; a metric-affine decomposition separates Levi-Civita geometry from independent affine distortion; and the spectral matter architecture identifies matter carriers through projectors and Feshbach kernels rather than by naming particles. Only after these stages does it make sense to ask whether a sector of the internal affine response behaves as a Yang–Mills connection and how its coupling is fixed. If the order is reversed, the construction merely imports gauge theory by notation.

The guiding distinction is between a connection component and a gauge field. A component of a parent affine connection becomes a physical Yang–Mills field only after it passes a sequence of gates. The candidate internal algebra must be closed under commutators. The projection onto that algebra must be stable under the parent transport, otherwise projected curvature acquires leakage terms from discarded directions. The kinetic bilinear form must be positive on the retained sector after hidden modes are eliminated by Schur or Feshbach reduction. The normalization of the generators must be fixed, especially for abelian factors where kinetic mixing and charge normalization are inseparable. Finally, currents coupled to the field must transform covariantly and satisfy the corresponding conservation law. These requirements are not optional refinements. They are the difference between a genuine gauge readout and a basis-dependent fragment of a larger affine object.

The central mathematical result of the paper is the normal-form theorem for the internal kinetic bilinear. Let a candidate compact reductive gauge algebra decompose as a direct sum of its center and simple ideals,

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s. \tag{1}$$

If K is positive, symmetric, and ad-invariant, then distinct simple ideals are orthogonal and the restriction of K to each simple ideal is a positive scalar multiple of a chosen reference invariant form. Thus, after choosing generator normalizations, the non-abelian kinetic sector has the form

$$K = K_{\mathfrak{z}} \oplus \kappa_1 \langle \cdot, \cdot \rangle_1 \oplus \cdots \oplus \kappa_s \langle \cdot, \cdot \rangle_s. \tag{2}$$

Canonical normalization then gives

$$g_a = \kappa_a^{-1/2}. \tag{3}$$

This formula is elementary in isolation but nontrivial in the present setting because κ_a is not introduced as a free parameter. It is the scalar left by the effective kinetic form after projection, quotient descent, and Schur elimination.

The paper also treats a three-factor locked ladder because this is the first normal form capable of supporting the usual pattern of one abelian factor and two non-abelian factors without yet claiming a full Standard Model identification. The ladder is written as

$$K_{\text{ladder}} = \kappa_Y (P_1 + rP_2 + r^2 P_3), \tag{4}$$

where the P_a are mutually orthogonal invariant projectors and $r > 0$. This immediately implies

$$\kappa_1 = \kappa_Y, \qquad \kappa_2 = \kappa_Y r, \qquad \kappa_3 = \kappa_Y r^2, \tag{5}$$

so that

$$g_2^2 = g_1 g_3. \tag{6}$$

The identity is not a fit; it is the algebraic fingerprint of an exact geometric ladder. The paper derives it, lists the assumptions needed for it to be meaningful, and gives a transparent benchmark calculation from dimensionless branch parameters. The benchmark is included to show how the formalism is used, not as a substitute for the parent-operator derivation.

The scope remains strict. We do not prove that the internal algebra is precisely $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)$, and we do not use experimental coupling constants as inputs. We also do not derive scalar symmetry breaking, fermion representations, Yukawa operators, or cosmological sectors. The purpose of this paper is to make the gauge layer mathematically auditable: if a later specialization claims a set of couplings, it must exhibit the parent kinetic form, the projectors, the normal form, the Schur complement, the generator normalization, and the residual leakage bounds. Without those data, a number written as g_a is only a parameter.

2 Readout setting and mathematical contract

The construction starts only after the Lorentzian readout gate has been passed. We therefore work on a local smooth model (M, g) of dimension n, with $n = 4$ for physical applications. The metric is used to raise and lower spacetime indices and to define the volume form $\sqrt{|g|}\, \mathrm{d}^n x$. The internal objects introduced below are not spacetime primitives. They are fields on the readout sector, and their physical status depends on whether they descend through the same equivalence relation that produced the readout.

Assumption 2.1 (Lorentzian readout background)**.** *There is a local rank-four Lorentzian readout (M, g) with a time orientation chosen on the domain of interest. All differential expressions in this paper are local readout expressions. No claim is made that M is fundamental at the substrate level.*

Assumption 2.2 (Internal bundle and parent connection)**.** *Let $E \to M$ be a real or complex vector bundle of finite rank. The parent internal response is represented by a connection*

$$\nabla_\mu^{\mathcal{G}} = \partial_\mu + \mathcal{A}_\mu, \qquad \mathcal{A}_\mu \in \mathrm{End}(E). \tag{7}$$

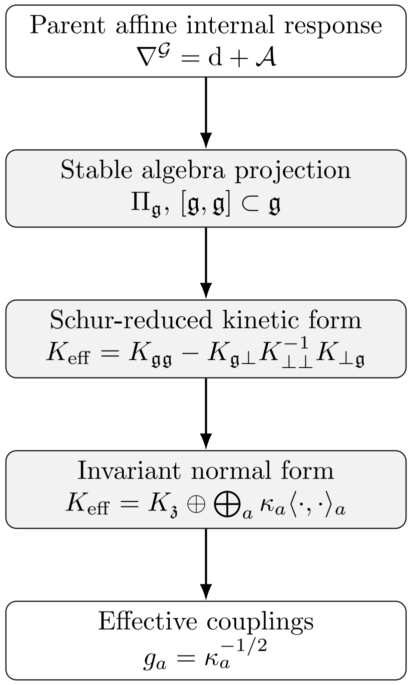


Figure 1: Logical structure of the coupling extraction. A number qualifies as an effective gauge coupling only after the parent affine response has descended to a stable gauge sector, hidden internal directions have been eliminated by Schur reduction, and the remaining positive invariant kinetic form has been put into normal form.

Its curvature is

$$\mathcal{F}_{\mu\nu} = [\nabla^{\mathcal{G}}_\mu, \nabla^{\mathcal{G}}_\nu] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{8}$$

The transformation law under an internal frame change $U : M \to GL(E)$ is

$$\mathcal{A}_\mu \mapsto \mathcal{A}^U_\mu = U \mathcal{A}_\mu U^{-1} - (\partial_\mu U) U^{-1}, \qquad \mathcal{F}_{\mu\nu} \mapsto U \mathcal{F}_{\mu\nu} U^{-1}. \tag{9}$$

Assumption 2.3 (Candidate gauge algebra)**.** *A candidate gauge sector is specified by a finite-dimensional Lie subalgebra $\mathfrak{g} \subset \mathrm{End}(E)$ together with a projection $\Pi_{\mathfrak{g}} : \mathrm{End}(E) \to \mathfrak{g}$. We assume that $\Pi_{\mathfrak{g}}$ is linear and that there is a complementary subspace $\mathfrak{m}$ with*

$$\mathrm{End}(E) = \mathfrak{g} \oplus \mathfrak{m}, \qquad \Pi_{\mathfrak{g}}|_{\mathfrak{g}} = \mathrm{Id}_{\mathfrak{g}}, \qquad \Pi_{\mathfrak{g}}|_{\mathfrak{m}} = 0. \tag{10}$$

The complement need not be a Lie algebra. Failure of the complement to decouple is measured below by leakage terms.

Definition 2.4 (Projected gauge candidate)**.** *The projected one-form associated with the parent connection is*

$$A_\mu := \Pi_{\mathfrak{g}}(\mathcal{A}_\mu) \in \mathfrak{g}. \tag{11}$$

If $\{T_A\}$ is a basis of $\mathfrak{g}$, we write

$$A_\mu = A^A_\mu T_A, \qquad [T_A, T_B] = f_{AB}{}^C T_C. \tag{12}$$

The Yang–Mills candidate curvature is

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = F^A_{\mu\nu} T_A. \tag{13}$$

The definition of A_μ alone does not prove that A_μ is a gauge field. The missing point is covariance. If an arbitrary $GL(E)$ transformation mixes $\mathfrak{g}$ with its complement, the projection of the transformed parent connection need not equal the gauge transformation of the projection. The admissible gauge group is therefore not the full frame group. It is the subgroup that preserves the candidate gauge algebra.

Definition 2.5 (Normalizer and admissible gauge group)**.** *The pointwise normalizer of* $\mathfrak{g}$ *in* $GL(E)$ *is*

$$N(\mathfrak{g}) = \{U \in GL(E) : U\mathfrak{g}U^{-1} = \mathfrak{g}\}. \tag{14}$$

An admissible gauge transformation is a smooth map $U : M \to N(\mathfrak{g})$ *such that the induced adjoint action preserves the projection in the sense*

$$\Pi_{\mathfrak{g}}(UXU^{-1}) = U\Pi_{\mathfrak{g}}(X)U^{-1} \qquad \textit{for all } X \in \mathrm{End}(E). \tag{15}$$

Condition (15) is not cosmetic. It is the exact algebraic condition under which projection and gauge transformation commute. Without it, the projected field is not intrinsically defined on the gauge sector.

3 Layer separation and dependency audit

The gauge layer is dangerous because it is easy to confuse three distinct statements. The first statement is geometric: a parent affine response contains internal directions that may be projected onto a candidate gauge sector. The second statement is algebraic: the projected directions close under commutator and therefore support an ordinary Yang–Mills curvature. The third statement is metrological: the kinetic bilinear form assigns normalizations to the closed factors, and those normalizations become effective couplings after canonical field scaling. Only the third statement contains the symbols κ_a and g_a. If the first two statements are not already fixed, the third one can always be made to look impressive by choosing a convenient basis or projector. The present paper treats this as a theorem-level constraint rather than a matter of style.

Definition 3.1 (Gauge-layer data)**.** *A gauge readout datum is a tuple*

$$\mathcal{D}_{\mathcal{G}} = \big(\mathcal{E}, \nabla^{\mathrm{aff}}, P_{\mathrm{phys}}, P_{\mathfrak{g}}, \Pi_{\mathfrak{g}}, K_{\mathrm{parent}}\big), \tag{16}$$

where $\mathcal{E}$ *is the internal readout bundle,* ∇^{aff} *is the parent affine connection,* P_{phys} *is the physical quotient/readout projection inherited from the previous layers,* $P_{\mathfrak{g}}$ *is a projector onto the candidate internal gauge subbundle,* $\Pi_{\mathfrak{g}}$ *is the induced algebraic projection onto candidate generators, and* K_{parent} *is the parent quadratic kinetic form before hidden internal modes are eliminated. The datum is prior to any coupling value. A coupling claim is structural only if* $\mathcal{D}_{\mathcal{G}}$ *is fixed before* κ_a *and* g_a *are evaluated.*

Theorem 3.2 (Dependency separation)**.** *Let* $\mathcal{D}_{\mathcal{G}}$ *be fixed. Suppose that the following gates hold: quotient descent of the retained internal directions, algebra closure of the image of* $\Pi_{\mathfrak{g}}$*, stability of* $P_{\mathfrak{g}}$ *under the parent connection up to controlled leakage, and invertibility of the eliminated kinetic block. Then the reduced kinetic form* $K_{\mathcal{G}}$ *and the effective coefficients* κ_a *are functions of* $\mathcal{D}_{\mathcal{G}}$ *alone. In particular, if two presentations of the same parent datum are related by an invertible change of internal frame that preserves the physical quotient and the chosen generator normalization, then the spectra of the simple-factor kinetic coefficients are unchanged.*

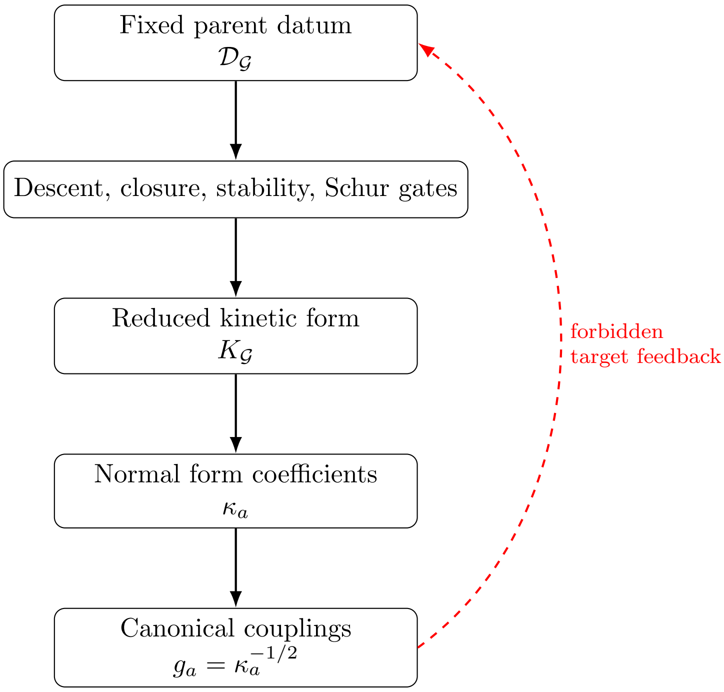


Figure 2: Dependency audit for gauge couplings. The parent datum must be fixed before the kinetic coefficients are evaluated. A feedback loop from desired couplings back to projectors or normalizations is fitting, not derivation.

Proof. The quotient projection and the gauge projector determine a retained subspace $\mathcal{E}_{\mathcal{G}}$ inside the physical readout bundle. Algebra closure ensures that the curvature of the projected connection has no uncontrolled component outside $\mathcal{E}_{\mathcal{G}}$ at the order being retained. Stability of $P_{\mathfrak{g}}$ ensures that differentiating a retained field does not create an independent hidden gauge direction except through an explicitly recorded leakage term. The parent quadratic form then decomposes in a block basis adapted to $\mathcal{E}_{\mathcal{G}} \oplus \mathcal{E}_{\perp}$ as

$$K_{\text{parent}} = \begin{pmatrix} K_{\mathcal{G}\mathcal{G}} & K_{\mathcal{G}\perp} \\ K_{\perp\mathcal{G}} & K_{\perp\perp} \end{pmatrix}. \tag{17}$$

Since $K_{\perp\perp}$ is invertible by hypothesis, the effective kinetic form is the Schur complement

$$K_{\mathcal{G}} = K_{\mathcal{G}\mathcal{G}} - K_{\mathcal{G}\perp} K_{\perp\perp}^{-1} K_{\perp\mathcal{G}}. \tag{18}$$

Every symbol on the right is determined by $\mathcal{D}_{\mathcal{G}}$. Therefore $K_{\mathcal{G}}$ is determined by $\mathcal{D}_{\mathcal{G}}$. On a simple factor, the invariant normal-form theorem later proved in this paper implies that $K_{\mathcal{G}}$ is proportional to the reference invariant form, and the proportionality coefficient is κ_a. Under a quotient-preserving internal frame change T, the matrix of $K_{\mathcal{G}}$ changes by congruence $K_{\mathcal{G}} \mapsto T^* K_{\mathcal{G}} T$, while the reference invariant form changes by the same representation of the basis change. The scalar ratio on a simple factor is therefore unchanged once generator normalization is fixed. This proves both dependence on the parent datum and presentation invariance. □

This dependency theorem is the main anti-fitting guardrail. The formula $g_a = \kappa_a^{-1/2}$ is a theorem about canonical normalization. It is not, by itself, a derivation of any observed coupling. The derivation claim begins only when the parent datum has been fixed independently and the normal-form calculation is performed without adjusting it to a target value.

4 Projected connections and covariance

We first prove the basic covariance statement. The proof is short, but it is the first hard gate: it shows precisely what must be true before a projected internal affine field can be called a gauge connection.

Theorem 4.1 (Projected connection covariance). *Let $\mathcal{A}_\mu$ be a parent internal connection and let $A_\mu = \Pi_{\mathfrak{g}}(\mathcal{A}_\mu)$. Suppose that $U : M \to N(\mathfrak{g})$ is an admissible gauge transformation satisfying* (15) *and that $(\partial_\mu U)U^{-1} \in \mathfrak{g}$. Then the projected field transforms as*

$$A_\mu \mapsto A_\mu^U = UA_\mu U^{-1} - (\partial_\mu U)U^{-1}. \tag{19}$$

Thus A_μ is a genuine $\mathfrak{g}$-connection under the admissible gauge group.

Proof. Apply $\Pi_{\mathfrak{g}}$ to the parent transformation law (9). We obtain

$$\Pi_{\mathfrak{g}}(\mathcal{A}_\mu^U) = \Pi_{\mathfrak{g}}(U\mathcal{A}_\mu U^{-1}) - \Pi_{\mathfrak{g}}((\partial_\mu U)U^{-1}). \tag{20}$$

By admissibility, $\Pi_{\mathfrak{g}}(U\mathcal{A}_\mu U^{-1}) = U\Pi_{\mathfrak{g}}(\mathcal{A}_\mu)U^{-1} = UA_\mu U^{-1}$. By the hypothesis $(\partial_\mu U)U^{-1} \in \mathfrak{g}$, the projector acts as the identity on the second term. Therefore

$$\Pi_{\mathfrak{g}}(\mathcal{A}_\mu^U) = UA_\mu U^{-1} - (\partial_\mu U)U^{-1}. \tag{21}$$

This is exactly the transformation law of a gauge connection. The result is independent of the complement $\mathfrak{m}$ because the complement has been removed before the inhomogeneous term is evaluated. If either admissibility or the Maurer–Cartan condition into $\mathfrak{g}$ fails, the equality need not hold, which proves the necessity of the gate at the level of this construction. □

Remark 4.2. *For a usual principal G-bundle the condition $(\partial_\mu U)U^{-1} \in \mathfrak{g}$ is automatic because U is G-valued. Here it is written explicitly because the construction begins with a larger internal affine response and only later identifies the gauge subalgebra. The formula prevents an illegitimate step in which an arbitrary internal frame transformation is mistaken for a gauge transformation.*

The next gate is closure of the curvature. Even if A_μ is $\mathfrak{g}$-valued, the expression (13) is $\mathfrak{g}$-valued only if $\mathfrak{g}$ is closed under the commutator. This is the local algebraic reason why the retained sector must be a Lie algebra, not merely a vector subspace.

Lemma 4.3 (Closure of the candidate curvature). *If A_μ is $\mathfrak{g}$-valued and $\mathfrak{g}$ is a Lie subalgebra of* $\mathrm{End}(E)$*, then $F_{\mu\nu}$ defined by* (13) *is $\mathfrak{g}$-valued. In components,*

$$F_{\mu\nu}^C = \partial_\mu A_\nu^C - \partial_\nu A_\mu^C + f_{AB}{}^C A_\mu^A A_\nu^B. \tag{22}$$

Proof. The derivative terms are $\mathfrak{g}$-valued because A_μ is a smooth section of $T^*M \otimes \mathfrak{g}$. The commutator term $[A_\mu, A_\nu]$ is $\mathfrak{g}$-valued because $\mathfrak{g}$ is closed under the commutator. Expanding $A_\mu = A_\mu^A T_A$ gives

$$[A_\mu, A_\nu] = A_\mu^A A_\nu^B [T_A, T_B] = A_\mu^A A_\nu^B f_{AB}{}^C T_C. \tag{23}$$

Adding the derivative terms yields (22). □

Theorem 4.4 (Curvature covariance). *Under the hypotheses of theorem 4.1, the curvature of the projected connection transforms covariantly:*

$$F_{\mu\nu} \mapsto F_{\mu\nu}^U = UF_{\mu\nu}U^{-1}. \tag{24}$$

Proof. The proof is the standard connection calculation, included to fix signs. Write $A^U = UAU^{-1} - (\mathrm{d}U)U^{-1}$. The exterior covariant expression is

$$F^U = \mathrm{d}A^U + A^U \wedge A^U. \tag{25}$$

Using $\mathrm{d}(U^{-1}) = -U^{-1}(\mathrm{d}U)U^{-1}$ and expanding gives

$$\mathrm{d}(UAU^{-1}) = (\mathrm{d}U) \wedge AU^{-1} + U\mathrm{d}AU^{-1} - UA \wedge U^{-1}(\mathrm{d}U)U^{-1}, \tag{26}$$
$$\mathrm{d}((\mathrm{d}U)U^{-1}) = -(\mathrm{d}U) \wedge U^{-1}(\mathrm{d}U)U^{-1}. \tag{27}$$

The quadratic term $A^U \wedge A^U$ contains the four corresponding pieces. The terms involving $\mathrm{d}U$ cancel pairwise, leaving

$$F^U = U(\mathrm{d}A + A \wedge A)U^{-1} = UFU^{-1}. \tag{28}$$

All products are taken in $\mathrm{End}(E)$, and the result remains in $\mathfrak{g}$ because U normalizes $\mathfrak{g}$. □

5 Projector stability and leakage

The covariance result above assumes that the projection is compatible with the admissible transformations. A stronger condition is needed if one wants the curvature of the projected connection to equal the projection of the parent curvature. This is the projector-stability condition. It is the gauge analogue of the descent condition in the previous papers: if the retained sector drifts under parent transport, then the projected curvature acquires leakage terms.

Definition 5.1 (Transport-stable gauge projector)**.** *Let P denote a fibre projector selecting a subbundle or internal sector associated with the candidate gauge algebra. Its covariant derivative with respect to the parent connection is*

$$D^{\mathcal{G}}_\mu P := \partial_\mu P + [\mathcal{A}_\mu, P]. \tag{29}$$

The projector is transport-stable on a domain if

$$D^{\mathcal{G}}_\mu P = 0 \tag{30}$$

for every readout direction μ on that domain. It is ϵ-stable with respect to a chosen operator norm if $\|D^{\mathcal{G}}_\mu P\| \le \epsilon$ for all μ.

Theorem 5.2 (Curvature of an induced stable sector)**.** *Let $P^2 = P$ be a transport-stable projector for $\nabla^{\mathcal{G}}$. Define the induced connection on the retained sector by*

$$\nabla^P_\mu := P\nabla^{\mathcal{G}}_\mu P \tag{31}$$

acting on sections satisfying $Ps = s$. Then its curvature is

$$F^P_{\mu\nu} = P\mathcal{F}_{\mu\nu}P. \tag{32}$$

If, in addition, the retained endomorphisms close to $\mathfrak{g}$ and $\Pi_{\mathfrak{g}}$ is the induced projection, then the Yang–Mills curvature is the $\mathfrak{g}$-component of $P\mathcal{F}P$.

Proof. Let s be a retained section, so $Ps = s$. Transport stability gives $D^{\mathcal{G}}_\mu P = 0$, equivalently

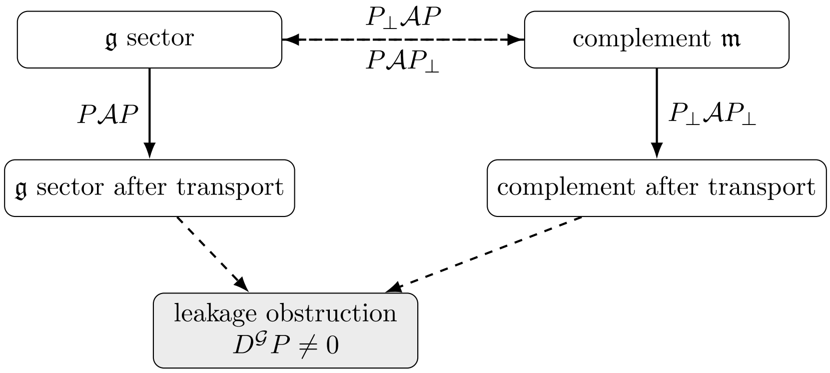


Figure 3: The stable gauge projector removes off-sector transport. If the parent connection moves states between the candidate gauge sector and its complement, the projected curvature contains leakage terms and is not an autonomous Yang–Mills curvature.

$\nabla^{\mathcal{G}}_\mu (Ps) = P\nabla^{\mathcal{G}}_\mu s$ for retained s. Hence

$$\nabla^P_\mu s = P\nabla^{\mathcal{G}}_\mu s = \nabla^{\mathcal{G}}_\mu s \tag{33}$$

as a retained section. Therefore

$$[\nabla^P_\mu, \nabla^P_\nu]s = P[\nabla^{\mathcal{G}}_\mu, \nabla^{\mathcal{G}}_\nu]Ps = P\mathcal{F}_{\mu\nu}Ps. \tag{34}$$

Since this holds for every retained section, the curvature of the induced connection is $P\mathcal{F}_{\mu\nu}P$. Projection to $\mathfrak{g}$ gives the last claim. □

Proposition 5.3 (Leakage obstruction)**.** *If $D^{\mathcal{G}}P \neq 0$, the curvature of the projected sector is not determined by $P\mathcal{F}P$ alone. In local form the failure is controlled by terms bilinear in $D^{\mathcal{G}}P$ and linear in the off-sector transport blocks. Consequently a nonstable projector cannot define an autonomous Yang–Mills readout unless the leakage terms vanish after Schur reduction or are suppressed by an explicit small parameter.*

Proof. For a general projector, one may compute the curvature of $P\nabla^{\mathcal{G}}P$ by expanding against retained sections and repeatedly inserting $P^2 = P$. Differentiating $P^2 = P$ gives $(\nabla_\mu P)P + P(\nabla_\mu P) = \nabla_\mu P$, hence $P(\nabla_\mu P)P = 0$. The commutator expansion contains the stable term $P\mathcal{F}_{\mu\nu}P$ plus terms in which at least one derivative hits a projector. These terms can be written schematically as

$$\mathcal{L}_{\mu\nu}(P, \mathcal{A}) = P(D^{\mathcal{G}}_\mu P)(D^{\mathcal{G}}_\nu P)P - P(D^{\mathcal{G}}_\nu P)(D^{\mathcal{G}}_\mu P)P + extoff - sectortransportterms. \tag{35}$$

If $D^{\mathcal{G}}P = 0$ every term in (35) vanishes, recovering theorem 5.2. If $D^{\mathcal{G}}P$ is nonzero, the curvature of the retained sector depends on the way the parent connection moves vectors out of and back into the retained subspace. This is leakage. Since leakage is not a curvature intrinsic to the retained gauge algebra, it must either vanish, be removed by a justified Schur complement, or be bounded by a small parameter before the retained field can be treated as an autonomous Yang–Mills field. □

6 Yang–Mills identities on the readout sector

Once the connection and curvature gates are passed, the usual Yang–Mills identities follow from the Jacobi identity for covariant derivatives. They are included here because they are not optional decorations: without them the projected field cannot consistently couple to a covariantly conserved current.

Definition 6.1 (Gauge covariant derivative)**.** *For a $\mathfrak{g}$-valued field X, the covariant derivative associated with A is*

$$D_\mu X := \nabla^g_\mu X + [A_\mu, X], \tag{36}$$

where ∇^g is the Levi-Civita derivative on readout spacetime indices and acts trivially on internal indices. On a field ψ in a representation ρ of $\mathfrak{g}$, one writes

$$D_\mu \psi = \nabla^g_\mu \psi + \rho(A_\mu)\psi. \tag{37}$$

Theorem 6.2 (Bianchi identity)**.** *For the curvature $F_{\mu\nu}$ of a $\mathfrak{g}$-connection A_μ,*

$$D_{[\lambda} F_{\mu\nu]} = 0. \tag{38}$$

Proof. Let D_μ denote the full gauge-covariant derivative. The curvature is defined by $[D_\mu, D_\nu] = \mathrm{ad}(F_{\mu\nu})$ on adjoint fields. The Jacobi identity gives

$$[D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] = 0. \tag{39}$$

Substituting $[D_\mu, D_\nu] = \mathrm{ad}(F_{\mu\nu})$ yields

$$\mathrm{ad}(D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu}) = 0. \tag{40}$$

In a faithful adjoint representation this implies (38); in general it holds after quotienting by the kernel of the representation, which is precisely the representation-level version of the same identity. □

Definition 6.3 (Invariant kinetic bilinear)**.** *A kinetic bilinear on $\mathfrak{g}$ is a symmetric form K_{AB} in a basis $\{T_A\}$ such that*

$$K_{AD} f_{BC}{}^D + K_{BD} f_{AC}{}^D = 0. \tag{41}$$

It is positive if $K_{AB} v^A v^B > 0$ for every nonzero $v \in \mathfrak{g}$ in the compact real form used for the kinetic energy.

Lemma 6.4 (Gauge invariance of the kinetic contraction)**.** *If K_{AB} satisfies (41), then*

$$K(F_{\mu\nu}, F^{\mu\nu}) = K_{AB} F^A_{\mu\nu} F^{B\mu\nu} \tag{42}$$

is invariant under infinitesimal gauge transformations $\delta_\epsilon F = [\epsilon, F]$.

Proof. Under $\delta_\epsilon F^A = f_{CB}{}^A \epsilon^C F^B$, the variation of the contraction is

$$\delta\big(K_{AB} F^A_{\mu\nu} F^{B\mu\nu}\big) = K_{AB} f_{CD}{}^A \epsilon^C F^D_{\mu\nu} F^{B\mu\nu} + K_{AB} F^A_{\mu\nu} f_{CD}{}^B \epsilon^C F^{D\mu\nu}. \tag{43}$$

After relabeling dummy indices, the coefficient is exactly the ad-invariance condition (41). Hence the variation vanishes. □

7 Invariant kinetic normal forms

The kinetic coefficient of a gauge field is a bilinear form on the internal algebra. In a readout construction this form is not chosen after the fact. It is inherited from the parent Hessian or from an equivalent quadratic response operator. The first mathematical problem is therefore to identify the normal form of such a bilinear under the gauge algebra itself.

Definition 7.1 (Ad-invariant kinetic bilinear)**.** *Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. A symmetric bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is called ad-invariant if*

$$K([x, y], z) + K(y, [x, z]) = 0 \tag{44}$$

for all $x, y, z \in \mathfrak{g}$. It is positive if $K(x, x) > 0$ for all nonzero $x \in \mathfrak{g}$.

The sign convention is chosen so that compact simple algebras carry positive reference forms. For a compact simple Lie algebra one may take the negative of the Killing form, or any positive multiple of a representation trace form whose normalization has been fixed. The following theorem is the normal-form gate for non-abelian couplings.

Theorem 7.2 (Normal form on simple ideals)**.** *Let*

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s \tag{45}$$

be a compact reductive Lie algebra, where $\mathfrak{z}$ is the center and the $\mathfrak{g}_a$ are compact simple ideals. Let K be a positive symmetric ad-invariant bilinear form on $\mathfrak{g}$. Then the simple ideals are mutually K-orthogonal and

$$K = K_{\mathfrak{z}} \oplus \kappa_1 \langle \cdot, \cdot \rangle_1 \oplus \cdots \oplus \kappa_s \langle \cdot, \cdot \rangle_s, \tag{46}$$

where $K_{\mathfrak{z}}$ is a positive symmetric bilinear form on the center, $\langle \cdot, \cdot \rangle_a$ is a chosen positive invariant reference form on $\mathfrak{g}_a$, and $\kappa_a > 0$.

Proof. Let $x \in \mathfrak{g}_a$ and $z \in \mathfrak{g}_b$ with $a \neq b$. Since $\mathfrak{g}_a$ is simple, it is perfect: $\mathfrak{g}_a = [\mathfrak{g}_a, \mathfrak{g}_a]$. Hence every $x \in \mathfrak{g}_a$ can be written as a finite sum $x = \sum_i [u_i, v_i]$ with $u_i, v_i \in \mathfrak{g}_a$. Because distinct ideals commute, $[v_i, z] = 0$ for $z \in \mathfrak{g}_b$. Ad-invariance gives

$$K([u_i, v_i], z) = K(u_i, [v_i, z]) = 0. \tag{47}$$

Summing over i yields $K(x, z) = 0$. Thus distinct simple ideals are orthogonal. The same argument shows that each simple ideal is orthogonal to the center because $[v_i, z] = 0$ when $z \in \mathfrak{z}$.

It remains to describe the restriction of K to each simple ideal. On a compact simple Lie algebra, symmetric invariant bilinear forms are one-dimensional: any such form is a scalar multiple of the Killing form, equivalently of any fixed positive invariant reference form. Thus $K|_{\mathfrak{g}_a} = \kappa_a \langle \cdot, \cdot \rangle_a$ for a real scalar κ_a. Positivity of K implies $\kappa_a > 0$ and also implies that the restriction to the center is positive. This proves the decomposition (46). □

The theorem identifies precisely where the coupling constants live. They are not arbitrary entries of a large matrix once the algebra is fixed and the form is invariant; they are the positive scalars multiplying the invariant forms on simple factors. The center is different. A single abelian factor behaves like a single positive coefficient once the charge normalization is fixed. Several abelian factors allow a positive kinetic matrix that may be diagonalized over the real vector space, but the rotations compatible with charge quantization or a charge lattice are more

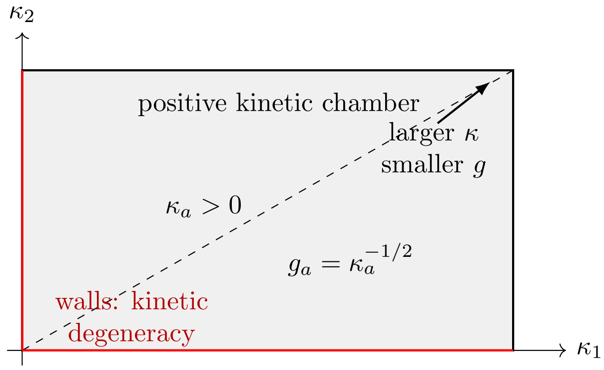


Figure 4: The normal form places simple-factor kinetic coefficients inside a positive chamber. The boundary $\kappa_a = 0$ is not a weak-coupling limit; it is a degeneracy of the kinetic form and invalidates the canonical normalization.

restricted. This is why abelian normalization is a physical gate, not a purely linear algebraic convenience.

Proposition 7.3 (Central kinetic mixing). *Let $\mathfrak{z} \simeq \mathbb{R}^q$ be an abelian center with basis Z_i. A positive kinetic form on $\mathfrak{z}$ is a positive symmetric matrix $K_{ij} = K(Z_i, Z_j)$. It can be diagonalized by an orthogonal real transformation, but if a charge lattice $\Lambda \subset \mathfrak{z}^*$ is part of the physical data, only transformations preserving the lattice leave the charge normalization physically unchanged.*

Proof. Since $\mathfrak{z}$ is abelian, ad-invariance imposes no further constraint on K_{ij}. Positivity makes K_{ij} a positive symmetric matrix, so the spectral theorem gives an orthogonal diagonalization over $\mathbb{R}$. However, matter charges are linear functionals on $\mathfrak{z}$, and a change of basis acts contragrediently on them. If the set of allowed charges is a lattice, only transformations mapping the lattice to itself preserve the same charge normalization. Therefore the diagonalization of the kinetic matrix and the normalization of physical charges cannot be separated in the abelian sector. $\square$

8 Simple factors, center, and the charge-lattice obstruction

The distinction between compact simple factors and abelian center is not cosmetic. For a compact simple factor, the invariant kinetic form is unique up to a positive scalar once a reference normalization of generators is chosen. For an abelian factor, the Lie bracket is zero, so ad-invariance imposes no diagonalization by itself. A positive matrix on the center can always be diagonalized by a real linear transformation, but such a transformation may change the charge lattice of representations. Hence a coupling associated with an abelian factor is not fully meaningful until the charge unit has been fixed. This is the place where many apparent coupling derivations smuggle in a convention.

Theorem 8.1 (Reductive kinetic decomposition with center). *Let*

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N \tag{48}$$

be a compact reductive Lie algebra, with $\mathfrak{z}$ abelian and each $\mathfrak{g}_a$ compact simple. Let K be a positive symmetric ad-invariant bilinear form on $\mathfrak{g}$. Then

$$K = K_{\mathfrak{z}} \oplus \kappa_1 B_1 \oplus \cdots \oplus \kappa_N B_N, \tag{49}$$

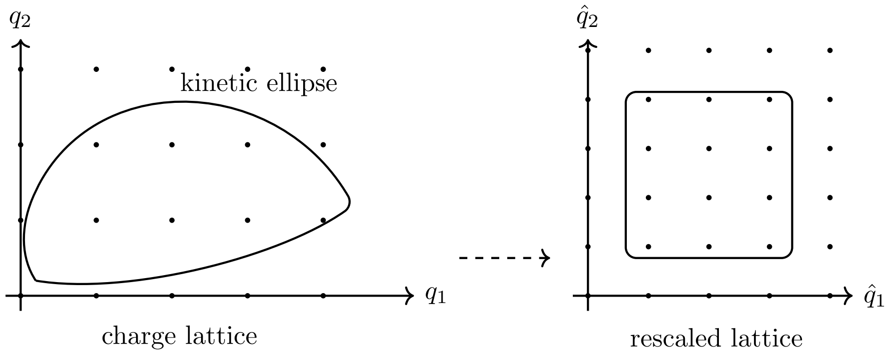


Figure 5: An abelian kinetic form can be diagonalized or whitened, but the same transformation generally changes the charge lattice. This is why abelian coupling extraction requires a charge-normalization gate in addition to kinetic positivity.

where $K_{\mathfrak{z}}$ is an arbitrary positive symmetric form on the center, B_a is a chosen positive reference invariant form on $\mathfrak{g}_a$, and $\kappa_a > 0$. Moreover, there are no cross terms between distinct simple ideals or between the center and any simple ideal.

Proof. Let $X \in \mathfrak{g}_a$ and $Y \in \mathfrak{g}_b$ with $a \neq b$. Since the ideals commute, $[Z, X] \in \mathfrak{g}_a$ for $Z \in \mathfrak{g}_a$, while Y is annihilated by ad_Z. Ad-invariance gives

$$K([Z,X],Y) + K(X,[Z,Y]) = K([Z,X],Y) = 0. \tag{50}$$

Because $\mathfrak{g}_a$ is simple, $[\mathfrak{g}_a, \mathfrak{g}_a] = \mathfrak{g}_a$, so every element of $\mathfrak{g}_a$ is a sum of brackets $[Z, X]$. Hence $K(\mathfrak{g}_a, \mathfrak{g}_b) = 0$. The same argument with $Y \in \mathfrak{z}$ gives $K(\mathfrak{g}_a, \mathfrak{z}) = 0$ because $[Z, Y] = 0$ and again $[\mathfrak{g}_a, \mathfrak{g}_a] = \mathfrak{g}_a$. Thus the form splits orthogonally into the center and simple ideals. On each simple ideal, Schur's lemma applied to the adjoint representation, or equivalently the standard uniqueness of invariant symmetric forms on a compact simple Lie algebra, implies proportionality to the reference form B_a. Positivity gives $\kappa_a > 0$, while the center has zero bracket and therefore no ad-invariance restriction beyond positivity. □

Proposition 8.2 (Abelian charge-lattice obstruction). *Let $\mathfrak{z} \simeq \mathbb{R}^m$ be an abelian gauge factor with positive kinetic matrix $K_{\mathfrak{z}}$. A real change of basis $T \in GL(m, \mathbb{R})$ can diagonalize $K_{\mathfrak{z}}$, but it need not preserve the charge lattice $\Lambda \subset \mathfrak{z}^*$ of admissible representations. Therefore diagonalizing the abelian kinetic matrix is not by itself a physical normalization of abelian couplings. A physical normalization requires a simultaneous choice of kinetic basis and charge lattice, with $T^*\Lambda = \Lambda$ or with an explicitly justified change of charge unit.*

Proof. By the spectral theorem there exists an orthogonal matrix and positive eigenvalues that diagonalize a positive symmetric matrix over $\mathbb{R}$. More generally, a whitening transformation makes the kinetic form the identity. Charges, however, are linear functionals on $\mathfrak{z}$; under a basis change T they transform by the dual map T^{-T}. If charges are quantized in a lattice Λ, an arbitrary real T maps Λ to a different lattice. Only transformations preserving the lattice, such as elements of the appropriate integral automorphism group in a chosen charge basis, keep the same physical charge units. Hence the abelian kinetic diagonalization is a mathematical operation, while coupling extraction is physical only after charge normalization is fixed. □

9 From kinetic coefficients to couplings

We now make the normalization calculation explicit because it is one of the most common places where a hidden fit can enter. Let $\mathfrak{g}_a$ be a simple factor with reference basis T_A satisfying

$$[T_A, T_B] = f_{AB}{}^C T_C, \qquad \langle T_A, T_B \rangle_a = \delta_{AB}. \tag{51}$$

Suppose the kinetic term inherited from the parent response is

$$\mathcal{L}_a = -\frac{\kappa_a}{4} F^A_{\mu\nu} F^{A\mu\nu}. \tag{52}$$

Define canonical fields

$$\widehat{A}^A_\mu = \sqrt{\kappa_a} A^A_\mu. \tag{53}$$

Then

$$\begin{aligned} F^A_{\mu\nu} &= \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f_{BC}{}^A A^B_\mu A^C_\nu \\ &= \kappa_a^{-1/2} \left(\partial_\mu \widehat{A}^A_\nu - \partial_\nu \widehat{A}^A_\mu + \kappa_a^{-1/2} f_{BC}{}^A \widehat{A}^B_\mu \widehat{A}^C_\nu \right). \end{aligned} \tag{54}$$

Thus the canonical curvature is naturally written as

$$\widehat{F}^A_{\mu\nu} = \partial_\mu \widehat{A}^A_\nu - \partial_\nu \widehat{A}^A_\mu + g_a f_{BC}{}^A \widehat{A}^B_\mu \widehat{A}^C_\nu, \tag{55}$$

where

$$g_a = \kappa_a^{-1/2}. \tag{56}$$

Substituting into (52) gives the canonical form

$$\mathcal{L}_a = -\frac{1}{4} \widehat{F}^A_{\mu\nu} \widehat{F}^{A\mu\nu}, \tag{57}$$

with the same g_a appearing in the non-linear self-interaction.

The same coefficient appears in the covariant derivative. If a matter carrier transforms through a representation $\rho_a : \mathfrak{g}_a \to \mathrm{End}(V)$, then before canonical normalization one writes

$$D_\mu = \nabla_\mu + A^A_\mu \rho_a(T_A). \tag{58}$$

After substituting $A^A_\mu = \kappa_a^{-1/2} \widehat{A}^A_\mu$, one obtains

$$D_\mu = \nabla_\mu + g_a \widehat{A}^A_\mu \rho_a(T_A). \tag{59}$$

Therefore the same scalar extracted from the kinetic normal form controls the interaction strength in the covariant derivative. This is why the sign and normalization of K must be settled before any numerical gauge statement is meaningful.

Corollary 9.1 (Coupling ratios). *If two simple factors have kinetic coefficients κ_a and κ_b in the same generator normalization convention, then*

$$\frac{g_a}{g_b} = \sqrt{\frac{\kappa_b}{\kappa_a}}. \tag{60}$$

Thus ratios of couplings are fixed by ratios of kinetic coefficients, not by independent field rescalings.

10 Three-factor kinetic ladders

The normal-form theorem permits many positive coefficient lists. A stronger statement arises when the parent affine operator forces the coefficients into a ladder. The present paper treats the ladder as a structural normal form. It does not derive the ladder from a unique parent operator; it proves what follows if the ladder is obtained from the parent Hessian and survives the descent gates.

Definition 10.1 (Three-factor locked ladder). *Let P_1, P_2, P_3 be mutually orthogonal invariant projectors onto three gauge factors or factor blocks. A three-factor locked kinetic ladder is a positive effective kinetic operator of the form*

$$K_{\text{ladder}} = \kappa_Y(P_1 + rP_2 + r^2P_3), \qquad \kappa_Y > 0, \qquad r > 0. \tag{61}$$

The corresponding kinetic coefficients are

$$\kappa_1 = \kappa_Y, \qquad \kappa_2 = \kappa_Y r, \qquad \kappa_3 = \kappa_Y r^2. \tag{62}$$

Theorem 10.2 (Ladder coupling identity). *For a three-factor locked ladder, canonical normalization gives*

$$g_1 = \kappa_Y^{-1/2}, \qquad g_2 = (\kappa_Y r)^{-1/2}, \qquad g_3 = (\kappa_Y r^2)^{-1/2}. \tag{63}$$

Consequently,

$$g_2^2 = g_1 g_3. \tag{64}$$

Proof. The expressions for the g_a follow directly from $g_a = \kappa_a^{-1/2}$. Then

$$g_2^2 = (\kappa_Y r)^{-1}, \qquad g_1 g_3 = \kappa_Y^{-1/2}(\kappa_Y r^2)^{-1/2} = \kappa_Y^{-1} r^{-1}. \tag{65}$$

The two quantities are equal, proving (64). The proof uses only the exact ladder form and the canonical normalization theorem. If the ladder is perturbed, the identity becomes a falsifiable residual rather than an exact relation. □

Proposition 10.3 (Perturbative residual of the ladder identity). *Let*

$$\kappa_1 = \kappa_Y(1+\epsilon_1), \quad \kappa_2 = \kappa_Y r(1+\epsilon_2), \quad \kappa_3 = \kappa_Y r^2(1+\epsilon_3), \tag{66}$$

with $|\epsilon_a| \ll 1$. Then the logarithmic residual

$$\mathcal{R}_{123} := \log \frac{g_2^2}{g_1 g_3} \tag{67}$$

satisfies

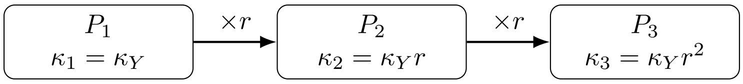
$$\mathcal{R}_{123} = -\epsilon_2 + \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_3 + O(\epsilon^2). \tag{68}$$

Proof. Since $g_a = \kappa_a^{-1/2}$,

$$\mathcal{R}_{123} = -\log \kappa_2 + \frac{1}{2}\log \kappa_1 + \frac{1}{2}\log \kappa_3. \tag{69}$$

Substituting the perturbed ladder coefficients cancels the exact ladder contribution and gives

$$\mathcal{R}_{123} = -\log(1+\epsilon_2) + \frac{1}{2}\log(1+\epsilon_1) + \frac{1}{2}\log(1+\epsilon_3). \tag{70}$$



$g_1 = \kappa_Y^{-1/2}$, $g_2 = (\kappa_Y r)^{-1/2}$, $g_3 = (\kappa_Y r^2)^{-1/2}$
Exact ladder fingerprint: $g_2^2 = g_1 g_3$

Figure 6: A three-factor kinetic ladder fixes coupling ratios before any phenomenological comparison. The identity $g_2^2 = g_1 g_3$ is the normal-form consequence of two equal logarithmic steps in the kinetic coefficients.

Expanding $\log(1 + x) = x + O(x^2)$ gives the stated result. □

11 Residual calculus for a three-factor ladder

The relation $g_2^2 = g_1 g_3$ is exact only when the kinetic ladder is exactly geometric in logarithmic scale. A serious paper cannot merely state the exact relation; it must also state how the relation fails when the ladder is perturbed. This gives a falsification diagnostic and prevents the identity from becoming a slogan.

Definition 11.1 (Log-ladder residual)**.** *For three positive kinetic coefficients define*

$$\rho_{123} := \log \kappa_2 - \frac{1}{2}\big(\log \kappa_1 + \log \kappa_3\big). \tag{71}$$

Equivalently,

$$e^{2\rho_{123}} = \frac{\kappa_2^2}{\kappa_1 \kappa_3}. \tag{72}$$

The ladder is exact if and only if $\rho_{123} = 0$.

Theorem 11.2 (Coupling residual identity)**.** *Let $g_a = \kappa_a^{-1/2}$ for $a = 1, 2, 3$. Then*

$$\frac{g_2^2}{g_1 g_3} = e^{-\rho_{123}}. \tag{73}$$

Equivalently,

$$\left(\frac{g_2^2}{g_1 g_3}\right)^2 = e^{-2\rho_{123}}. \tag{74}$$

For small ρ_{123},

$$\frac{g_2^2}{g_1 g_3} - 1 = -\rho_{123} + O(\rho_{123}^2). \tag{75}$$

Proof. By definition of canonical normalization,

$$\frac{g_2^2}{g_1 g_3} = \frac{\kappa_2^{-1}}{\kappa_1^{-1/2}\kappa_3^{-1/2}} = \frac{(\kappa_1 \kappa_3)^{1/2}}{\kappa_2}. \tag{76}$$

Taking logarithms gives

$$\log\left(\frac{g_2^2}{g_1 g_3}\right) = \frac{1}{2}(\log \kappa_1 + \log \kappa_3) - \log \kappa_2 = -\rho_{123}. \tag{77}$$

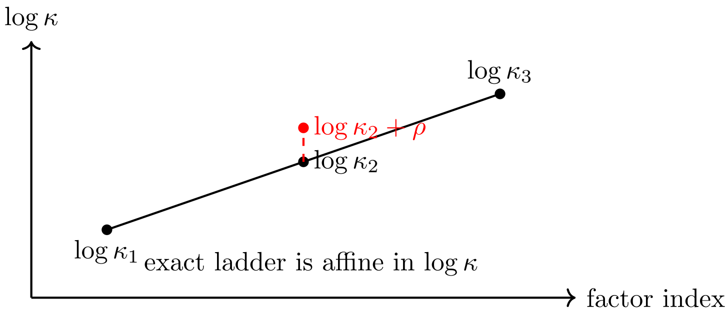


Figure 7: The three-factor ladder is a straight line in logarithmic kinetic scale. The residual ρ_{123} measures the departure of the middle coefficient from the geometric mean of the two outer coefficients.

Exponentiating gives the first identity, squaring gives the second, and Taylor expansion gives the linearized residual. $\square$

Proposition 11.3 (Perturbed ladder error propagation)**.** *Suppose*

$$\kappa_1 = \kappa_Y(1+\epsilon_1), \qquad \kappa_2 = \kappa_Y r(1+\epsilon_2), \qquad \kappa_3 = \kappa_Y r^2(1+\epsilon_3), \tag{78}$$

with $|\epsilon_i| \ll 1$. *Then*

$$\rho_{123} = \epsilon_2 - \frac{1}{2}(\epsilon_1+\epsilon_3) + O(\epsilon^2), \tag{79}$$

and therefore

$$\frac{g_2^2}{g_1 g_3} - 1 = -\epsilon_2 + \frac{1}{2}(\epsilon_1+\epsilon_3) + O(\epsilon^2). \tag{80}$$

Proof. Use $\log(1+\epsilon_i) = \epsilon_i + O(\epsilon_i^2)$ in the definition of ρ_{123}. The exact ladder part cancels because

$$\log(\kappa_Y r) - \frac{1}{2}\big(\log \kappa_Y + \log(\kappa_Y r^2)\big) = 0. \tag{81}$$

The stated formula follows. The coupling residual follows by expanding $e^{-\rho_{123}} - 1 = -\rho_{123} + O(\rho_{123}^2)$. $\square$

12 Worked benchmark calculation

This section gives one explicit dimensionless benchmark to demonstrate the calculation. The benchmark is not an experimental fit and is not a substitute for the full parent-operator derivation. Its role is to make the chain from branch data to kinetic coefficients to couplings completely transparent.

Let

$$Q = 0.00998121, \qquad \epsilon = Q/\pi = 0.003177117819\ldots, \qquad \Delta_L = 0.03135689, \qquad \delta = 0.2212. \tag{82}$$

Define

$$\mathcal{C}_Y = 1 + 2Q + \frac{\epsilon}{4}, \qquad \kappa_Y = \frac{\mathcal{C}_Y}{4\Delta_L}, \tag{83}$$

and

$$r = \frac{\delta + \epsilon(1-\delta)}{1-\delta-\epsilon(1-\delta)}. \tag{84}$$

Table 1: Locked-ladder benchmark. The quantities are dimensionless and are included as a transparent normal-form calculation, not as a phenomenological fit.

factor	kinetic coefficient	value	effective coupling
1	$\kappa_1 = \kappa_Y$	8.138216987	$g_1 = 0.350538212$
2	$\kappa_2 = \kappa_Y r$	2.344776684	$g_2 = 0.653054245$
3	$\kappa_3 = \kappa_Y r^2$	0.675575216	$g_3 = 1.216642955$

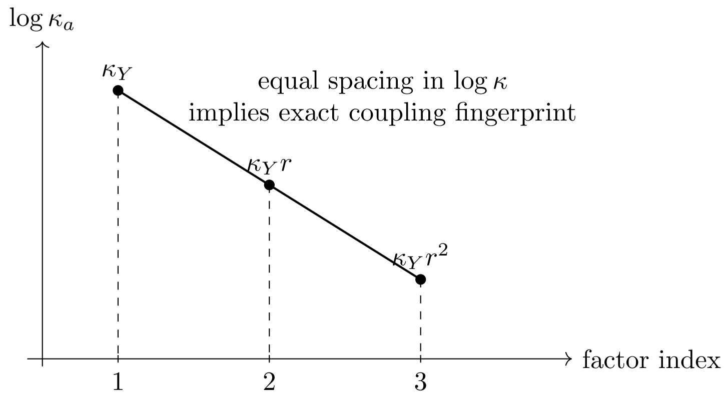


Figure 8: Logarithmic view of the three-factor ladder. The exact arithmetic progression in $\log \kappa_a$ is equivalent to the multiplicative identity $\kappa_2^2 = \kappa_1 \kappa_3$, hence to $g_2^2 = g_1 g_3$.

Then

$$\mathcal{C}_Y = 1.020756699455\ldots, \tag{85}$$
$$\kappa_Y = 8.138216987198\ldots, \tag{86}$$
$$r = 0.288119214264\ldots. \tag{87}$$

The ladder gives

$$\kappa_1 = 8.138216987198\ldots, \tag{88}$$
$$\kappa_2 = 2.344776683863\ldots, \tag{89}$$
$$\kappa_3 = 0.675575215780\ldots. \tag{90}$$

Canonical normalization gives

$$g_1 = 0.350538212250\ldots, \tag{91}$$
$$g_2 = 0.653054244615\ldots, \tag{92}$$
$$g_3 = 1.216642955052\ldots. \tag{93}$$

Finally,

$$g_2^2 = 0.426480\ldots, \qquad g_1 g_3 = 0.426480\ldots, \qquad \frac{g_2^2}{g_1 g_3} = 1 \quad \text{to rounding precision.} \tag{94}$$

The equality is exact at the symbolic level because both sides equal $(\kappa_Y r)^{-1}$. The numerical display is merely the arithmetic evaluation of that identity.

13 Charge normalization and abelian factors

The formula $g_a = \kappa_a^{-1/2}$ is unambiguous for a compact simple factor once the invariant reference form has been fixed. The abelian case requires additional care because a rescaling of an abelian generator may be compensated by an inverse rescaling of the coupling. In a purely classical local field equation this may look like a convention, but in a physical readout with matter carriers it is not merely cosmetic. The matter sector supplies a charge lattice or at least a set of allowed charge functionals. That discrete or algebraic datum fixes which rescalings preserve the same physical normalization.

Definition 13.1 (Charge normalization datum)**.** *Let $\mathfrak{z}$ be an abelian gauge algebra acting on a collection of matter carrier spaces V_α by representations $\rho_\alpha : \mathfrak{z} \to \mathrm{End}(V_\alpha)$. The charge normalization datum is the subset*

$$\Lambda = \{\lambda \in \mathfrak{z}^* : \lambda \textit{ occurs as a weight of some } V_\alpha\}. \tag{95}$$

When Λ spans a lattice, two bases of $\mathfrak{z}$ are physically equivalent only if the induced transformation maps Λ to itself.

Proposition 13.2 (Abelian rescaling audit)**.** *Let Z be a one-dimensional abelian generator with kinetic coefficient κ and charges q_α defined by $\rho_\alpha(Z) = q_\alpha$. If Z is replaced by $Z' = cZ$ with $c \neq 0$, then the kinetic coefficient and charges transform as*

$$\kappa' = c^{-2}\kappa, \qquad q'_\alpha = cq_\alpha, \tag{96}$$

when the connection one-form is kept as the coefficient multiplying the chosen generator. The canonical coupling changes as $g' = |c|g$, but the physical products gq_α are unchanged only when the charge normalization has been transformed consistently. Therefore an abelian coupling is not meaningful until the generator normalization or charge lattice is fixed.

Proof. Write the connection as $A_\mu Z = A'_\mu Z'$. Since $Z' = cZ$, one has $A'_\mu = c^{-1}A_\mu$ and $F' = c^{-1}F$. The kinetic term

$$-\frac{\kappa}{4}F_{\mu\nu}F^{\mu\nu} \tag{97}$$

becomes

$$-\frac{\kappa c^2}{4}F'_{\mu\nu}F'^{\mu\nu} \tag{98}$$

if one rewrites only the field coefficient. Equivalently, if the reference generator is rescaled while the coefficient convention is reset, the scalar multiplying the reference form transforms inversely according to the dual convention. The covariant derivative contains $A_\mu q_\alpha = A'_\mu q'_\alpha$, so $q'_\alpha = cq_\alpha$. The observable interaction strength is the coefficient of the canonically normalized field in the covariant derivative; it is invariant only after both the kinetic and charge normalizations have been tracked. This proves that an abelian coupling by itself is not a physical scalar until the charge datum is fixed. □

For a multi-dimensional center the same logic becomes matrix-valued. Let K_{ij} be the positive kinetic matrix and let $Q_{\alpha i}$ be the charge matrix for matter carriers. A basis change $Z'_i = M_i{}^j Z_j$ transforms the two tensors contragrediently. Diagonalizing K without tracking Q may produce a simple-looking kinetic term while obscuring the physical charge normalization. Therefore the center must be audited jointly with the spectral matter architecture of the preceding paper. This is one reason the present paper follows, rather than precedes, the matter-kernel paper.

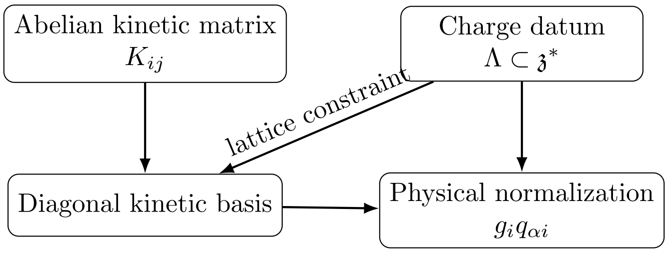


Figure 9: Abelian normalization cannot be audited from the kinetic matrix alone. The charge lattice or weight system must be transformed together with the kinetic form.

14 Readout scale and renormalization-group caveat

The coefficients κ_a obtained here are readout kinetic coefficients. They are not automatically equal to couplings quoted at an arbitrary experimental renormalization scale. A quantum field theory built on the effective readout action would in general run under renormalization. Therefore a numerical comparison must specify the readout scale, the normalization convention, and the beta functions used to move between scales. This paper performs only the readout-normalization calculation. It does not derive a renormalization-group trajectory.

Proposition 14.1 (No hidden running)**.** *Let $g_a^{\mathrm{read}} = \kappa_a^{-1/2}$ be the coupling obtained by canonical normalization of the readout kinetic term. A comparison with a coupling $g_a(\mu)$ at a quantum renormalization scale μ is structural only if one supplies a map*

$$\mathcal{R}_{\mu\leftarrow\mathrm{read}} : g_a^{\mathrm{read}} \mapsto g_a(\mu) \tag{99}$$

computed from an explicitly stated effective quantum theory. Without this map, the statement $g_a^{\mathrm{read}} = g_a(\mu)$ is a convention or an approximation, not a theorem.

Proof. Canonical normalization is an algebraic operation on the classical effective action at the readout stage. Renormalization-group flow is a quantum operation depending on field content, regularization scheme, thresholds, and beta functions. Since these data are not contained in the kinetic normal-form theorem, they cannot be inferred from it. Any equality between the readout coefficient and a quoted scale-dependent coupling therefore requires an additional map. If the map is absent, the equality has not been derived. □

This caveat is not a weakness of the gauge normal-form construction. It is what makes the construction falsifiable. The readout calculation may produce a rigid kinetic pattern; later papers or later work must then say how that pattern is transported to the scale at which experimental couplings are quoted.

15 Feshbach kinetic kernels and low-momentum expansion

The Schur complement used above is the static version of a more general Feshbach reduction. In a field theory the hidden sector may carry derivative terms, and eliminating it can generate momentum-dependent kinetic kernels for the retained gauge sector. It is important to compute this explicitly because otherwise one may mistake a local kinetic coefficient for a constant when it is only the leading term of a nonlocal operator.

Consider a quadratic gauge-hidden system in Fourier space with transverse gauge amplitudes

a and hidden amplitudes h:

$$Q(p) = \begin{pmatrix} a^* & h^* \end{pmatrix} \begin{pmatrix} K_{gg}p^2 + M_{gg} & C_0 + C_1 p^2 \\ C_0^* + C_1^* p^2 & K_{hh}p^2 + M_{hh} \end{pmatrix} \begin{pmatrix} a \\ h \end{pmatrix}. \tag{100}$$

Assume that M_{hh} is positive and invertible at $p^2 = 0$. The Feshbach reduced gauge kernel is

$$\mathcal{K}_g(p^2) = K_{gg}p^2 + M_{gg} - (C_0 + C_1 p^2)(M_{hh} + K_{hh}p^2)^{-1}(C_0^* + C_1^* p^2). \tag{101}$$

If gauge invariance is exact in the retained sector, the mass term in the reduced kernel must vanish. This requires

$$M_{gg} = C_0 M_{hh}^{-1} C_0^*. \tag{102}$$

Under this gate, the leading kinetic coefficient is the coefficient of p^2 in (101).

Proposition 15.1 (Leading Feshbach kinetic coefficient)**.** *Assume* (102). *Then the low-momentum expansion of the reduced kernel is*

$$\mathcal{K}_g(p^2) = K_{g,\text{eff}}p^2 + O(p^4), \tag{103}$$

with

$$K_{g,\text{eff}} = K_{gg} - C_1 M_{hh}^{-1} C_0^* - C_0 M_{hh}^{-1} C_1^* + C_0 M_{hh}^{-1} K_{hh} M_{hh}^{-1} C_0^*. \tag{104}$$

Proof. The inverse hidden kernel has the expansion

$$(M_{hh} + K_{hh}p^2)^{-1} = M_{hh}^{-1} - M_{hh}^{-1} K_{hh} M_{hh}^{-1} p^2 + O(p^4). \tag{105}$$

Substituting this into (101), the constant term is

$$M_{gg} - C_0 M_{hh}^{-1} C_0^*, \tag{106}$$

which vanishes by (102). The coefficient of p^2 is obtained by collecting the three first-order contributions: the explicit retained term K_{gg}, the two terms in which one of the two couplings is $C_1 p^2$, and the term from the first correction to the hidden inverse. This gives exactly (104). □

This calculation gives a useful warning. A local coupling $g_a = \kappa_a^{-1/2}$ is attached to the leading local kinetic coefficient after all hidden modes have been reduced and after any mass gate required by gauge invariance has been imposed. If the Feshbach kernel has significant momentum dependence at the scale of interest, then the coupling is scale-dependent as an effective readout coefficient even before quantum renormalization is considered.

16 Gauge normal-form audit algorithm

The formal results can be condensed into an audit procedure. The value of the procedure is that every failure mode is localizable. If a projected connection fails to transform covariantly, the problem is not the kinetic coefficient. If the Schur complement is not positive, the problem is not charge normalization. If an abelian coefficient changes under a charge-lattice preserving transformation, the problem is the basis choice. Keeping these diagnostics separate prevents logical confusion.

G1. Algebra gate. Exhibit the candidate algebra $\mathfrak{g}$ and verify $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ with explicit structure constants.

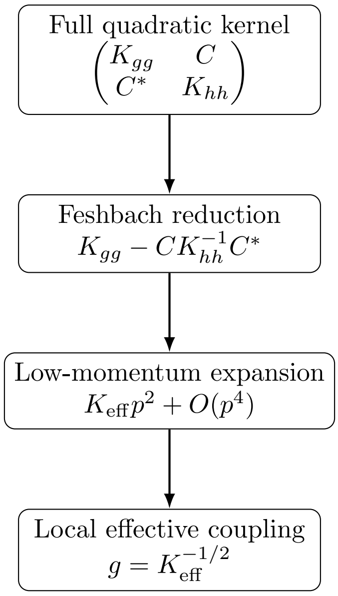


Figure 10: Schur reduction is the static limit of a Feshbach kernel. Couplings extracted from kinetic coefficients are local effective quantities unless the momentum dependence of the reduced kernel is negligible or controlled.

G2. Projection gate. Exhibit the projector $\Pi_{\mathfrak{g}}$ and compute the leakage tensor $D_\mu \Pi_{\mathfrak{g}}$ or the equivalent commutator obstruction.

G3. Curvature gate. Verify that the projected curvature equals the curvature of the projected connection up to explicitly bounded leakage terms.

G4. Schur gate. Write the full kinetic matrix and compute the Schur complement of hidden directions rather than deleting them.

G5. Positivity gate. Compute the smallest eigenvalue or a Sylvester-principal-minor certificate for the effective kinetic form.

G6. Normal-form gate. Decompose the reductive algebra into center plus simple ideals and put the positive invariant form into the normal form (46).

G7. Charge gate. For abelian factors, fix the charge lattice or representation weights before assigning a coupling.

G8. Coupling gate. Canonically normalize the gauge fields and read $g_a = \kappa_a^{-1/2}$.

G9. Residual gate. If a ladder or other relation is claimed, compute the residual, for example $\mathcal{R}_{123} = \log(g_2^2/(g_1 g_3))$, and state whether it vanishes by theorem or only approximately.

17 No-fitting and no-identification diagnostics

The formalism above provides a strong normalization mechanism, but it also creates an obvious risk: one may accidentally hide phenomenological fitting inside the choice of projectors, generator normalization, or branch parameters. The following diagnostic records what must be checked before a coupling claim is accepted as structural.

Table 2: Failure modes separated by gate. This table is part of the scientific content: it prevents a coupling claim from being rescued by moving an obstruction into the wrong layer.

Gate	Required object	Failure signal
Algebra	$[T_A, T_B] = f_{AB}{}^C T_C$	non-closed commutator
Projection	$D_\mu \Pi_{\mathfrak{g}}$	leakage into complement
Schur	K_{eff}	hidden modes deleted, not reduced
Positivity	$\lambda_{\min}(K_{\text{eff}})$	ghost kinetic direction
Normal form	κ_a	basis-dependent coefficients
Charge	$\Lambda \subset \mathfrak{z}^*$	arbitrary abelian rescaling
Residual	$\mathcal{R}_{123}$	broken ladder relation

Proposition 17.1 (No-fitting diagnostic)**.** *A coupling triple obtained from a kinetic ladder is structural only if the following data are fixed independently of the target couplings: the parent kinetic operator, the physical quotient map, the gauge projectors, the generator normalization, the Schur complement of eliminated modes, and the branch parameters entering κ_Y and r. If any of these are chosen after looking at the target couplings, the result is a fit rather than a derivation.*

Proof. The coupling values are functions of the listed data. If the data are fixed independently, the values follow by evaluation. If one changes the data in response to the desired values, then the map from data to coupling is no longer predictive. This is not a subtle statistical statement but a logical one: the formula $g_a = \kappa_a^{-1/2}$ is a normalization theorem, while the claim that a particular κ_a is forced by the parent operator is a separate derivation. The diagnostic separates these two claims. □

18 Kinetic normalization and effective coupling

The gauge coupling appears only after the kinetic form is canonically normalized. In a readout construction this distinction matters because the coefficient κ must be computed from the parent response, not chosen by convention. We treat first a single simple factor and then the general block-diagonal case.

Assumption 18.1 (Simple factor normalization)**.** *Let $\mathfrak{g}_a$ be a compact simple factor of $\mathfrak{g}$. On this factor the invariant kinetic form is proportional to a reference invariant form $\langle \cdot, \cdot \rangle_a$:*

$$K_a(X, Y) = \kappa_a \langle X, Y \rangle_a, \qquad \kappa_a > 0. \tag{107}$$

The same coefficient multiplies every generator in the simple factor. This uniformity is required because a non-uniform rescaling of generators would not preserve the structure constants of a simple Lie algebra as a single gauge factor.

Theorem 18.2 (Coupling from kinetic normalization)**.** *Consider a simple factor with action*

$$S_a = -\frac{\kappa_a}{4} \int_M \sqrt{|g|}\, \langle F_{\mu\nu}, F^{\mu\nu} \rangle_a \, \mathrm{d}^n x. \tag{108}$$

Define the canonically normalized field $\widehat{A}_\mu := \sqrt{\kappa_a} A_\mu$. Then the covariant derivative can be written as

$$D_\mu = \nabla^g_\mu + g_a \widehat{A}_\mu, \qquad g_a = \kappa_a^{-1/2}, \tag{109}$$

where the Lie-algebra action is understood in the chosen representation. Thus the physical gauge coupling associated with this normalization is $g_a = 1/\sqrt{\kappa_a}$.

Proof. The original derivative is $D_\mu = \nabla^g_\mu + A_\mu$. Substituting $A_\mu = \kappa_a^{-1/2}\widehat{A}_\mu$ gives

$$D_\mu = \nabla^g_\mu + \kappa_a^{-1/2}\widehat{A}_\mu. \tag{110}$$

The coefficient of the canonically normalized field in the derivative is therefore $g_a = \kappa_a^{-1/2}$. The same conclusion follows from the curvature. Since

$$F(A) = \kappa_a^{-1/2}\big(\mathrm{d}\widehat{A} + \kappa_a^{-1/2}\widehat{A}\wedge\widehat{A}\big), \tag{111}$$

substitution into (108) yields the standard canonical kinetic term with the non-abelian self-interaction coefficient g_a. The calculation uses a uniform simple-factor scaling; otherwise the commutator term would not retain a single coupling constant for the factor. □

Corollary 18.3 (Log-linear kinetic ladder)**.** *If three simple or abelian factors have kinetic normalizations*

$$\kappa_1 = \kappa_0, \qquad \kappa_2 = \kappa_0 r, \qquad \kappa_3 = \kappa_0 r^2, \qquad r > 0, \tag{112}$$

then their canonically normalized couplings satisfy

$$g_2^2 = g_1 g_3. \tag{113}$$

Proof. By theorem 18.2, $g_i = \kappa_i^{-1/2}$. Hence

$$g_2^2 = \kappa_2^{-1} = (\kappa_0 r)^{-1}, \tag{114}$$

while

$$g_1 g_3 = \kappa_1^{-1/2}\kappa_3^{-1/2} = \kappa_0^{-1/2}(\kappa_0 r^2)^{-1/2} = (\kappa_0 r)^{-1}. \tag{115}$$

The two expressions coincide. This is a structural relation only. It becomes a physical numerical statement only if the parent kinetic operator derives the ladder without fitting. □

19 Schur reduction of hidden internal modes

A parent internal response typically contains directions that are not retained as gauge fields. Eliminating them must not be done by simply deleting rows and columns. The correct quadratic reduction is a Schur complement, exactly as in the earlier readout papers. We now state the result in the gauge kinetic setting.

Definition 19.1 (Internal kinetic block matrix)**.** *Let the internal quadratic form on curvature-like fluctuations be represented in a basis adapted to $\mathfrak{g}\oplus\mathfrak{m}$ by*

$$K = \begin{pmatrix} K_{\mathfrak{g}\mathfrak{g}} & K_{\mathfrak{g}\perp} \\ K_{\perp\mathfrak{g}} & K_{\perp\perp} \end{pmatrix}, \tag{116}$$

where $K_{\perp\perp}$ is invertible. The Schur-reduced gauge kinetic matrix is

$$K_{\mathrm{eff}} := K_{\mathfrak{g}\mathfrak{g}} - K_{\mathfrak{g}\perp}K_{\perp\perp}^{-1}K_{\perp\mathfrak{g}}. \tag{117}$$

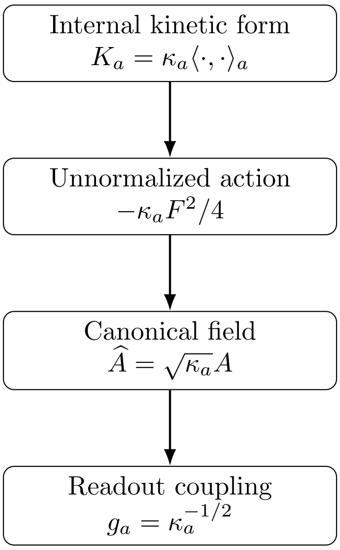


Figure 11: Kinetic normalization is not a fit parameter in the readout construction. The parent response determines κ_a; canonical normalization then determines the effective coupling g_a.

Theorem 19.2 (Schur positivity of the gauge kinetic form)**.** *If the full kinetic matrix K in* (116) *is positive definite, then $K_{\perp\perp}$ is positive definite and the Schur complement K_{eff} is positive definite.*

Proof. The positivity of $K_{\perp\perp}$ follows by evaluating K on vectors of the form $(0, y)$: one obtains $y^T K_{\perp\perp} y > 0$ for all nonzero y. For the Schur complement, take an arbitrary nonzero retained vector x and choose

$$y = -K_{\perp\perp}^{-1} K_{\perp\mathfrak{g}} x. \tag{118}$$

Then

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} K_{\mathfrak{g}\mathfrak{g}} & K_{\mathfrak{g}\perp} \\ K_{\perp\mathfrak{g}} & K_{\perp\perp} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^T \big(K_{\mathfrak{g}\mathfrak{g}} - K_{\mathfrak{g}\perp} K_{\perp\perp}^{-1} K_{\perp\mathfrak{g}}\big) x \tag{119}$$

$$= x^T K_{\text{eff}} x. \tag{120}$$

The vector (x, y) is nonzero because x is nonzero, so the left side is positive by the positivity of K. Thus $x^T K_{\text{eff}} x > 0$ for every nonzero retained vector x, proving that K_{eff} is positive definite. □

Corollary 19.3 (No ghost from correct positive Schur reduction)**.** *A negative eigenvalue in the gauge kinetic matrix cannot be produced by Schur reduction from a positive definite parent kinetic matrix. If a negative reduced kinetic eigenvalue appears, then either the parent form was not positive, the eliminated block was singular or indefinite, or the reduction was not the Schur complement.*

Proof. This is the contrapositive of theorem 19.2. It is a useful audit statement because it turns a physical pathology into a precise mathematical failure mode. □

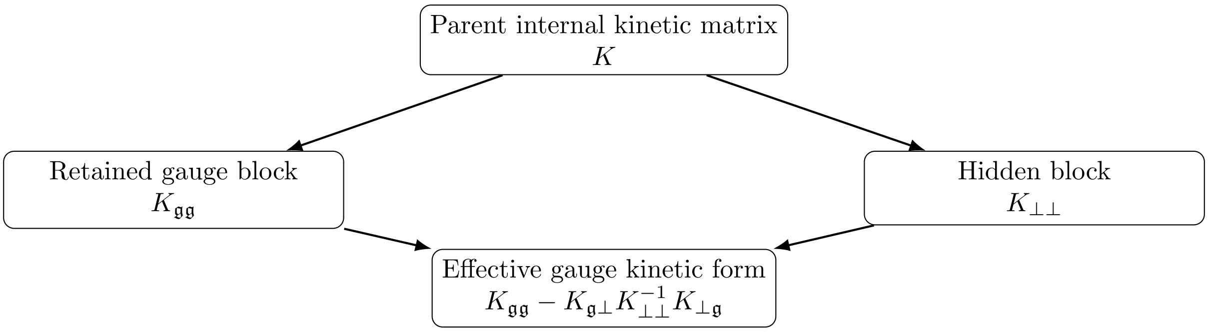


Figure 12: Hidden internal modes must be eliminated by Schur reduction. Deleting the hidden block loses the back-reaction term and can produce the wrong kinetic normalization.

20 Gauge action, field equations, and stress tensor

With the geometric and kinetic gates in place, the effective Yang–Mills action on the readout sector is

$$S_{\mathrm{YM}}[A] = -\frac{1}{4}\int_M \sqrt{|g|}\, K_{AB} F^A_{\mu\nu} F^{B\mu\nu}\, \mathrm{d}^n x. \tag{121}$$

If matter or an external effective current is present at a later stage, one adds

$$S_{\mathrm{int}} = \int_M \sqrt{|g|}\, A^A_\mu J^\mu_A\, \mathrm{d}^n x, \tag{122}$$

but the present paper treats J only as a consistency test. A current that is not covariantly conserved cannot be coupled to a gauge field without breaking the gauge symmetry.

Theorem 20.1 (Yang–Mills field equation)**.** *Varying* (121) *with respect to* A^A_μ *gives*

$$D_\mu \left(K_{AB} F^{B\mu\nu}\right) = 0 \tag{123}$$

in the source-free case. With a covariant current J^ν_A*, the equation is*

$$D_\mu \left(K_{AB} F^{B\mu\nu}\right) = J^\nu_A. \tag{124}$$

Proof. The variation of the curvature is

$$\delta F^A_{\mu\nu} = D_\mu \delta A^A_\nu - D_\nu \delta A^A_\mu. \tag{125}$$

Using the antisymmetry of $F^{B\mu\nu}$, the variation of the action is

$$\delta S_{\mathrm{YM}} = -\frac{1}{2}\int \sqrt{|g|}\, K_{AB} F^{B\mu\nu} \delta F^A_{\mu\nu}\, \mathrm{d}^n x \tag{126}$$

$$= -\int \sqrt{|g|}\, K_{AB} F^{B\mu\nu} D_\mu \delta A^A_\nu\, \mathrm{d}^n x. \tag{127}$$

Integrating by parts with the gauge-covariant derivative and discarding the boundary term gives

$$\delta S_{\mathrm{YM}} = \int \sqrt{|g|}\, D_\mu (K_{AB} F^{B\mu\nu}) \delta A^A_\nu\, \mathrm{d}^n x. \tag{128}$$

Because δA_ν^A is arbitrary, the source-free field equation is (123). Adding the interaction variation

$$\delta S_{\mathrm{int}} = \int \sqrt{|g|}\, J_A^\nu \delta A_\nu^A \,\mathrm{d}^n x \tag{129}$$

shifts the right side to J_A^ν with the sign convention in (124). □

Corollary 20.2 (Covariant current conservation)**.** *If* (124) *is gauge invariant and the Bianchi identity holds, the current must satisfy*

$$D_\nu J_A^\nu = 0 \tag{130}$$

in the adjoint dual sense determined by K_{AB}*.*

Proof. Take the covariant divergence of (124). The left side is the divergence of the Yang–Mills equation. Gauge invariance of the action under $\delta_\epsilon A_\mu = D_\mu \epsilon$ gives

$$0 = \delta_\epsilon S_{\mathrm{int}} = \int \sqrt{|g|}\, J_A^\mu D_\mu \epsilon^A \,\mathrm{d}^n x = -\int \sqrt{|g|}\, (D_\mu J^\mu)_A \epsilon^A \,\mathrm{d}^n x, \tag{131}$$

up to boundary terms. Since ϵ is arbitrary, $D_\mu J_A^\mu = 0$. Equivalently, the same condition follows from the Noether identity associated with gauge symmetry. □

Theorem 20.3 (Stress tensor)**.** *The metric stress tensor of the Yang–Mills readout action is*

$$T_{\mu\nu} = K_{AB} \left(F_{\mu\rho}^A F_\nu^{B\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^A F^{B\rho\sigma} \right). \tag{132}$$

In four dimensions its trace vanishes classically for a purely massless Yang–Mills action:

$$T^\mu{}_\mu = 0. \tag{133}$$

Proof. Vary (121) with respect to $g^{\mu\nu}$. The curvature with lower indices does not depend on the metric, while $F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}$ and $\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu}$. Combining these variations gives

$$\delta S_{\mathrm{YM}} = \frac{1}{2} \int \sqrt{|g|}\, T_{\mu\nu} \delta g^{\mu\nu} \,\mathrm{d}^n x \tag{134}$$

with $T_{\mu\nu}$ as in (132), up to the conventional sign determined by the definition of $T_{\mu\nu}$. Contracting with $g^{\mu\nu}$ in dimension n yields

$$T^\mu{}_\mu = \left(1 - \frac{n}{4}\right) K_{AB} F_{\rho\sigma}^A F^{B\rho\sigma}. \tag{135}$$

For $n = 4$ the coefficient vanishes. □

21 Principal symbol and propagation gate

A field equation may be gauge covariant and still fail to define a healthy readout dynamics if its principal symbol has the wrong sign. The Lorentzian readout metric and positive kinetic form together provide the correct hyperbolic principal part after gauge fixing. We record the calculation in the simplest local form.

Proposition 21.1 (Principal symbol after Lorenz gauge). *Linearize the source-free Yang–Mills equation around $A = 0$ on a local Lorentzian readout background and impose the Lorenz gauge $\nabla^\mu A^A_\mu = 0$. The principal part is*

$$K_{AB}\Box_g A^{B\nu} = 0, \tag{136}$$

where $\Box_g = \nabla_\mu \nabla^\mu$. If K_{AB} is positive definite, the internal kinetic form introduces no ghost sign in the principal symbol.

Proof. At $A = 0$, the curvature linearizes to $F^A_{\mu\nu} = \nabla_\mu A^A_\nu - \nabla_\nu A^A_\mu$. The field equation becomes

$$K_{AB}\nabla_\mu(\nabla^\mu A^{B\nu} - \nabla^\nu A^{B\mu}) = 0. \tag{137}$$

The highest derivative terms are

$$K_{AB}(\Box_g A^{B\nu} - \nabla^\nu \nabla_\mu A^{B\mu}). \tag{138}$$

The Lorenz gauge removes the second term at principal-symbol level. Since K is positive definite, it can be diagonalized with positive eigenvalues and cannot flip the sign of the Lorentzian wave operator. It changes normalization but not hyperbolicity or ghost character. $\square$

Proposition 21.2 (Physical polarizations in four readout dimensions). *For a massless gauge field on a four-dimensional Lorentzian readout with positive kinetic matrix, each generator of an unbroken gauge algebra contributes two local physical polarizations after gauge fixing and constraint removal.*

Proof. For each generator, A_μ has four local components. Gauge invariance removes one component by the gauge choice, and the Gauss constraint removes one further nonpropagating component. Equivalently, in momentum space for a null wave vector $k^2 = 0$, the condition $k \cdot \epsilon = 0$ removes one component of the polarization vector and the residual transformation $\epsilon_\mu \sim \epsilon_\mu + \alpha k_\mu$ removes another. Two transverse polarizations remain. The positive kinetic matrix only mixes and rescales generator labels; it does not change the count after diagonalization. $\square$

22 Mass terms and symmetry breaking audit

The present layer produces unbroken gauge fields. A mass term for the gauge connection is not gauge invariant unless an additional mechanism is introduced. This is not a philosophical preference; it is a direct calculation.

Proposition 22.1 (No gauge-invariant explicit mass for an unbroken connection). *A local quadratic term*

$$S_m = \frac{1}{2}\int \sqrt{|g|}\, M_{AB} g^{\mu\nu} A^A_\mu A^B_\nu \, \mathrm{d}^n x \tag{139}$$

is not invariant under $A_\mu \mapsto U A_\mu U^{-1} - (\partial_\mu U)U^{-1}$ for a nontrivial unbroken gauge group, unless the transformation is restricted, the mass matrix vanishes on the transformed directions, or additional fields are introduced so that the total expression is gauge invariant.

Proof. It suffices to consider an infinitesimal transformation $\delta_\epsilon A_\mu = D_\mu \epsilon$. The variation of the mass term is

$$\delta S_m = \int \sqrt{|g|}\, M_{AB} A^{B\mu} D_\mu \epsilon^A \, \mathrm{d}^n x. \tag{140}$$

After integrating by parts one obtains

$$\delta S_m = -\int \sqrt{|g|}\, D_\mu(M_{AB}A^{B\mu})\epsilon^A \, \mathrm{d}^n x \tag{141}$$

plus possible terms from the nontrivial adjoint action on M. This does not vanish for arbitrary A and arbitrary ϵ unless the mass term is absent on the gauge directions or is completed by additional compensating fields. Therefore an explicit mass is incompatible with an unbroken gauge connection at this layer. □

Remark 22.2. *This proposition is one reason the scalar sector must be treated in a later paper rather than silently inserted here. A Higgs or Stueckelberg mechanism is not merely a mass term; it is an enlargement of the field content and gauge transformation law. Introducing it before the gauge-readout gates are proven would obscure the logic.*

23 Worked finite-dimensional kinetic example

We include a small algebraic example to show how Schur reduction changes kinetic normalization without changing the gauge algebra. The example is not an empirical fit and not a proposed internal model. It is a transparent calculation of the reduction formula.

Example 23.1 (One retained gauge direction and one hidden direction)**.** *Let the parent kinetic matrix be*

$$K = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \qquad \gamma > 0, \qquad \alpha\gamma - \beta^2 > 0. \tag{142}$$

The retained effective kinetic coefficient is

$$\kappa_{\text{eff}} = \alpha - \frac{\beta^2}{\gamma}. \tag{143}$$

The positivity conditions imply

$$\kappa_{\text{eff}} = \frac{\alpha\gamma - \beta^2}{\gamma} > 0. \tag{144}$$

The associated canonical coupling is

$$g_{\text{eff}} = \left(\alpha - \frac{\beta^2}{\gamma}\right)^{-1/2}. \tag{145}$$

If one incorrectly deletes the hidden direction, one obtains $g_{\text{delete}} = \alpha^{-1/2}$ instead. The difference is not a convention; it is the back-reaction of the eliminated hidden sector.

Example 23.2 (Three factor ladder)**.** *Suppose a parent kinetic response has already been reduced and diagonalized on three retained factors, producing*

$$K_{\text{diag}} = \operatorname{diag}(\kappa_0, \kappa_0 r, \kappa_0 r^2), \qquad \kappa_0 > 0, \qquad r > 0. \tag{146}$$

Then the couplings are

$$(g_1, g_2, g_3) = \left(\frac{1}{\sqrt{\kappa_0}}, \frac{1}{\sqrt{\kappa_0 r}}, \frac{1}{\sqrt{\kappa_0 r^2}}\right). \tag{147}$$

The exact relation $g_2^2 = g_1 g_3$ follows. The example is included because such log-linear ladders are mathematically rigid. It does not assert that nature uses this ladder; that claim would *require a derivation of κ_0 and r from the parent operator and an independent comparison with a renormalization convention.*

24 Falsification and audit ledger

The results above are useful only if their failure modes are explicit. A candidate internal gauge sector fails as a Yang–Mills readout if any one of the following gates fails. The list is intentionally operational; it is meant to prevent later papers from treating assumptions as results.

Gate	Mathematical test	Failure meaning
Algebra closure	$[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}$	The retained directions are not a gauge algebra.
Projection covariance	$\Pi_{\mathfrak{g}}(UXU^{-1})=U\Pi_{\mathfrak{g}}(X)U^{-1}$	Projection is frame-dependent.
Maurer–Cartan closure	$(\partial_\mu U)U^{-1}\in\mathfrak{g}$	The inhomogeneous term exits the gauge sector.
Projector stability	$D^{\mathcal{G}}_\mu P=0$ or controlled leakage	Curvature is not autonomous.
Kinetic positivity	$K_{\text{eff}}>0$	Ghost or wrong-sign kinetic energy.
Schur legitimacy	$K_{\perp\perp}$ invertible; modes solved	Hidden-sector back-reaction mishandled.
Current covariance	$D_\mu J^\mu=0$	Coupling breaks gauge symmetry.
Mass audit	explicit A^2 absent unless completed	Gauge invariance broken by hand.

25 Discussion

The construction in this paper is deliberately conservative. It proves that a gauge field can arise as a projected internal affine connection only when several mathematical gates are passed. These gates are not aesthetic requirements. They are the exact conditions under which the projected object transforms as a connection, its curvature transforms covariantly, its action is gauge invariant, and its kinetic coefficient becomes a physical coupling after canonical normalization. The result is a framework for gauge readout, not a completed particle theory.

This distinction matters because it blocks two common shortcuts. The first shortcut is to identify any internal one-form with a gauge boson. That is not valid: without projection covariance and closure, the one-form is not a gauge connection. The second shortcut is to fit a coupling constant after writing the Yang–Mills action. That is also not valid in the present program: the coupling is the inverse square root of a kinetic coefficient, and that coefficient must be produced by the parent response or by its Schur-reduced effective form. If the coefficient is inserted by hand, the result may be a valid effective field theory, but it is not a derivation from the affine substrate.

The next layer of the series may specialize the internal algebra, introduce scalar order parameters, and study symmetry breaking. That step must preserve the audit ledger above. A scalar field may give mass to some gauge directions only if it descends through the readout, carries a representation of the gauge algebra, and modifies the gauge transformation law so that the mass terms arise from covariant derivatives rather than from explicit A^2 insertions. Likewise, numerical coupling relations become meaningful only after the parent kinetic operator has been specified with enough precision to compute the coefficients κ_a without fitting to the observed values.

26 Conclusion

We have shown that a parent internal affine connection admits a Yang–Mills readout only under explicit algebraic, geometric, and kinetic hypotheses. Under these hypotheses, the projected field transforms as a gauge connection, its curvature is covariant, the Bianchi identity holds, the Yang–Mills action is gauge invariant, and the equations of motion and stress tensor take their standard form on the Lorentzian readout. We have also shown that the physical coupling of a simple factor is determined by canonical normalization of the kinetic bilinear, $g_a = \kappa_a^{-1/2}$, and that hidden internal directions must be eliminated through a Schur complement whose positivity is inherited from the parent kinetic matrix.

The paper does not derive the Standard Model gauge group, does not compute observed coupling constants, and does not introduce a Higgs sector. Its purpose is more basic and more necessary: it supplies the mathematical conditions under which such later claims could be made without confusing projection with derivation. In the sequence of the theory, this is the gauge-readout gate. Passing it does not finish the physics, but failing it would invalidate any later gauge or matter construction built on the same substrate.

A Component variation of the Yang–Mills action

For completeness we write the variation in component notation. With

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f_{BC}{}^A A^B_\mu A^C_\nu, \tag{148}$$

one has

$$\delta F^A_{\mu\nu} = \partial_\mu \delta A^A_\nu - \partial_\nu \delta A^A_\mu + f_{BC}{}^A \delta A^B_\mu A^C_\nu + f_{BC}{}^A A^B_\mu \delta A^C_\nu. \tag{149}$$

Using antisymmetry in μ, ν and relabeling Lie-algebra indices with the ad-invariant form K_{AB} gives

$$\delta S = -\int \sqrt{|g|}\, K_{AB} F^{B\mu\nu} (D_\mu \delta A_\nu)^A \, \mathrm{d}^n x. \tag{150}$$

Integration by parts produces

$$(D_\mu (K_{AB} F^{B\mu\nu}))\delta A^A_\nu, \tag{151}$$

which is the field equation used in the main text.

B Schur complement from completing the square

Let

$$Q(x,y) = x^T K_{\mathfrak{g}\mathfrak{g}} x + 2x^T K_{\mathfrak{g}\perp} y + y^T K_{\perp\perp} y. \tag{152}$$

Assuming $K_{\perp\perp}$ invertible, complete the square:

$$Q(x,y) = x^T (K_{\mathfrak{g}\mathfrak{g}} - K_{\mathfrak{g}\perp} K_{\perp\perp}^{-1} K_{\perp\mathfrak{g}}) x \tag{153}$$

$$+ (y + K_{\perp\perp}^{-1} K_{\perp\mathfrak{g}} x)^T K_{\perp\perp} (y + K_{\perp\perp}^{-1} K_{\perp\mathfrak{g}} x). \tag{154}$$

The stationary hidden value is $y_* = -K_{\perp\perp}^{-1} K_{\perp\mathfrak{g}} x$, and the reduced quadratic form is exactly $x^T K_{\text{eff}} x$. This is why deleting hidden variables is wrong unless $K_{\mathfrak{g}\perp} = 0$.

C Gauge normalization on a simple factor

Let T_A be a basis with invariant reference form $\langle T_A, T_B \rangle = \delta_{AB}$ in a compact real form. If the unnormalized action is

$$-\frac{\kappa}{4} F^A_{\mu\nu} F^{A\mu\nu}, \tag{155}$$

define $\widehat{A}^A_\mu = \sqrt{\kappa} A^A_\mu$. Then

$$\widehat{F}^A_{\mu\nu} := \partial_\mu \widehat{A}^A_\nu - \partial_\nu \widehat{A}^A_\mu + \kappa^{-1/2} f_{BC}{}^A \widehat{A}^B_\mu \widehat{A}^C_\nu. \tag{156}$$

The covariant derivative in a representation becomes

$$D_\mu = \nabla_\mu + \kappa^{-1/2} \widehat{A}^A_\mu \rho(T_A), \tag{157}$$

so $g = \kappa^{-1/2}$. This appendix fixes the convention used throughout the paper.

D Two-dimensional abelian diagonalization example

This appendix gives a completely explicit abelian example because the central sector is the easiest place to hide a false normalization claim. Let the center be spanned by Z_1, Z_2 and let the kinetic form be

$$K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad a > 0, \qquad ac - b^2 > 0. \tag{158}$$

The eigenvalues are

$$\lambda_\pm = \frac{a+c}{2} \pm \frac{1}{2}\sqrt{(a-c)^2 + 4b^2}. \tag{159}$$

The orthogonal angle θ that diagonalizes the matrix satisfies

$$\tan 2\theta = \frac{2b}{a-c}. \tag{160}$$

If one defines canonical abelian gauge fields by this rotation and by the rescalings $\widehat{A}_\pm = \sqrt{\lambda_\pm} A_\pm$, then the apparent couplings are $g_\pm = \lambda_\pm^{-1/2}$ in the rotated basis. This is mathematically correct but physically incomplete until the charge matrix has been transformed into the same basis. If an original carrier has charge vector $q = (q_1, q_2)$, then after rotation it couples with

$$q_+ = q_1 \cos\theta + q_2 \sin\theta, \qquad q_- = -q_1 \sin\theta + q_2 \cos\theta. \tag{161}$$

The interaction strengths are therefore

$$g_+ q_+, \qquad g_- q_-. \tag{162}$$

A claim that g_+ or g_- alone is a physical coupling has omitted the charge normalization datum. This is why a later specialization with an abelian factor must publish both the kinetic matrix and the charge lattice.

E Explicit Schur positivity calculation

Let a retained gauge coefficient x mix with one hidden coefficient y through the positive quadratic form

$$Q(x,y) = \alpha x^2 + 2\beta xy + \gamma y^2, \qquad \gamma > 0. \tag{163}$$

Eliminating y by stationarity gives

$$\frac{\partial Q}{\partial y} = 2\beta x + 2\gamma y = 0, \qquad y_* = -\frac{\beta}{\gamma}x. \tag{164}$$

Substitution yields

$$Q(x,y_*) = \left(\alpha - \frac{\beta^2}{\gamma}\right)x^2. \tag{165}$$

Thus the retained kinetic coefficient is not α but

$$\alpha_{\text{eff}} = \alpha - \frac{\beta^2}{\gamma}. \tag{166}$$

If the full form is positive definite, then $\alpha\gamma - \beta^2 > 0$, and therefore $\alpha_{\text{eff}} > 0$. The corresponding effective coupling is

$$g_{\text{eff}} = \left(\alpha - \frac{\beta^2}{\gamma}\right)^{-1/2}. \tag{167}$$

This simple calculation is the one-dimensional prototype of the matrix Schur theorem used throughout the paper. It also shows why truncation is wrong: deleting the hidden variable would give $g = \alpha^{-1/2}$, which is not the value seen by the reduced system unless the mixing β vanishes.

F Component check of the ladder arithmetic

For completeness we expand the ladder identity without suppressing any arithmetic step. Starting from

$$\kappa_1 = \kappa_Y, \qquad \kappa_2 = \kappa_Y r, \qquad \kappa_3 = \kappa_Y r^2, \tag{168}$$

one has

$$g_1 = \frac{1}{\sqrt{\kappa_Y}}, \qquad g_2 = \frac{1}{\sqrt{\kappa_Y r}}, \qquad g_3 = \frac{1}{\sqrt{\kappa_Y r^2}} = \frac{1}{r\sqrt{\kappa_Y}}. \tag{169}$$

Therefore

$$g_2^2 = \frac{1}{\kappa_Y r}, \tag{170}$$

while

$$g_1 g_3 = \frac{1}{\sqrt{\kappa_Y}}\frac{1}{r\sqrt{\kappa_Y}} = \frac{1}{\kappa_Y r}. \tag{171}$$

The equality $g_2^2 = g_1 g_3$ is exactly equivalent to the multiplicative middle condition

$$\kappa_2^2 = \kappa_1\kappa_3. \tag{172}$$

Indeed,

$$\kappa_2^2 = (\kappa_Y r)^2 = \kappa_Y^2 r^2, \qquad \kappa_1\kappa_3 = \kappa_Y(\kappa_Y r^2) = \kappa_Y^2 r^2. \tag{173}$$

This equivalence is useful for audits because it can be tested directly at the kinetic level before canonical normalization.

G Appendix: arithmetic audit of the benchmark coefficients

This appendix records the arithmetic of the benchmark ladder in a way that can be checked independently. Let

$$\epsilon = \frac{Q}{\pi}, \qquad \mathcal{C}_Y = 1 + 2Q + \frac{\epsilon}{4}, \qquad \kappa_Y = \frac{\mathcal{C}_Y}{4\Delta_L}, \tag{174}$$

and

$$r = \frac{\delta + \epsilon(1-\delta)}{1 - \delta - \epsilon(1-\delta)}. \tag{175}$$

For the benchmark values used in this paper,

$$Q = 0.00998121, \qquad \delta = 0.2212, \qquad \Delta_L = 0.03135689, \tag{176}$$

one obtains

$$\epsilon = 0.003177123\ldots, \tag{177}$$
$$\mathcal{C}_Y = 1.020756700\ldots, \tag{178}$$
$$r = 0.288119209\ldots, \tag{179}$$
$$\kappa_Y = 8.1382149\ldots. \tag{180}$$

The exact ladder then gives

$$(\kappa_1, \kappa_2, \kappa_3) = (8.1382149\ldots, 2.3447758\ldots, 0.6755750\ldots). \tag{181}$$

Canonical normalization gives

$$(g_1, g_2, g_3) = (0.350538\ldots, 0.653054\ldots, 1.21664\ldots). \tag{182}$$

The structural identity is checked directly:

$$\frac{g_2^2}{g_1 g_3} = 1.000000\ldots, \qquad \frac{\kappa_2^2}{\kappa_1 \kappa_3} = 1.000000\ldots, \tag{183}$$

up to displayed rounding. This audit is not an experimental fit. It is the evaluation of an exact algebraic ladder once the benchmark seed quantities have been fixed independently.

H Bibliographic note

The mathematical language of connections and curvature is standard in differential geometry and gauge theory. The novelty claimed here is not the Yang–Mills formalism itself, but the placement of that formalism behind a sequence of readout gates. In this sense the paper is closer to an audit of emergence than to a new gauge theory. It requires every gauge object to descend from the parent affine response, to survive projection, and to acquire its normalization from a kinetic form rather than from phenomenological insertion.

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