Spectral Matter Architecture: Riesz Projectors, Feshbach Kernels, and Vorton Carrier Gates

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Abstract

This paper develops the spectral matter layer of the coherent affine-substrate program without invoking particle names, Standard Model assignments, fitted mass data, or gauge couplings. The objective is to specify when the effective physical readout operator admits isolated finite-rank spectral sectors that can support later matter interpretation. Starting from the quotient dynamics, Lorentzian readout, and metric-affine effective dynamics of the preceding papers, we introduce Riesz projectors, physical descent, Feshbach kernels, rank gates, compactness diagnostics, and vorton carriers. The term vorton is used here in a precise spectral sense: it denotes a compact finite-rank carrier equipped with a stable internal circulation structure, not a particle label or a phenomenological identification. The main results prove that isolated spectral islands define invariant projector ranges by contour calculus; that these projectors are covariant under representation changes and stable under gap-preserving perturbations; that the Feshbach map gives an exact isospectral kernel whenever the eliminated block is invertible; that stable matter-like carriers require reality, residue, locality, and compactness gates; and that vorton candidates require an additional circulation-index gate before any later physical interpretation is allowed. The paper is therefore a gate theorem for spectral matter architecture. It does not derive the electron, neutrino, quark, color, weak isospin, or generation spectrum. It supplies the operator-theoretic conditions under which such words may become meaningful in later work.

Keywords: spectral projectors; Riesz projection; Feshbach map; Schur complement; matter kernel; finite-rank spectral island; vorton carrier; circulation index; perturbation stability; affine substrate.

Contents

1 Introduction

The previous stages of the construction established a disciplined path from a premetric relational substrate to an effective readout. The first stage separated physical content from presentation by constructing a quotient of relational configurations. The second stage introduced a relational clock functional and an effective Hessian on the quotient. The third stage formulated the rank-four and Lorentzian signature gates under which the effective Hessian admits a physical spacetime readout. The fourth stage decomposed metric-affine dynamics and isolated the nonmetric trace mode as a geometrical degree of freedom subject to descent, kinetic, and source gates. None of these results yet justifies the use of particle names. They provide an arena in which a physical operator can be studied, but they do not by themselves explain why matter sectors should exist, why their ranks should be finite, or why any finite-dimensional block deserves a physical interpretation.

The present paper is devoted to that missing layer. Its purpose is to build the spectral architecture that must exist before a theory can speak responsibly about matter. In particular, it does not begin with electrons, neutrinos, quarks, generations, weak doublets, color triplets, Yukawa matrices, or mass formulas. Those words would be premature at this stage. A mathematically serious construction must first explain what a matter sector is in operator-theoretic terms. The natural answer is that a matter sector is an isolated, physically descending, finite-rank spectral island of the effective readout operator, equipped with a stable reduced kernel obtained by eliminating complementary degrees of freedom through an exact Feshbach or Schur reduction.

This approach changes the logical order of the theory. Instead of assigning a finite-dimensional vector space to each type of particle and then searching for an operator acting on it, we first take the effective physical operator and ask which finite-rank subspaces are forced, or at least admissible, by its spectral decomposition. Only after such subspaces are obtained as Riesz ranges can later papers ask whether one of them carries a rank-three color-like normal form, whether another carries a rank-two weak-like normal form, and whether still later reductions produce numerical couplings and masses. The rank statements

$$\operatorname{rank}(P_C) = 3, \qquad \operatorname{rank}(P_W) = 2 \tag{1.1}$$

therefore appear here only as spectral gates. They are not yet group-theoretic or phenomenological identifications. The notation is intentionally austere: it records the rank and separation of sectors, not their final physical labels.

The main mathematical tools are classical but used here in a specific order. Riesz projectors isolate spectral islands by contour integration of the resolvent. The Feshbach map, equivalently the Schur complement of an operator pencil, removes complementary modes without losing the relevant spectral zeros. Perturbation estimates control when a projector keeps its rank under deformation. Descent criteria guarantee that a spectral sector is not an artifact of a redundant presentation but survives on the physical quotient. Finally, kernel diagnostics prevent one from mistaking a convenient block decomposition for an invariant matter architecture.

The results are deliberately conditional. If the parent physical operator has no isolated finite-rank spectral islands, then the spectral matter architecture fails. If the contour enclosing a proposed sector crosses the spectrum under admissible perturbations, its rank is not stable. If the Feshbach eliminated block is singular at the relevant parameter, the reduced kernel is not defined. If a projector does not descend to the quotient, the corresponding sector is representational rather than physical. These failure modes are part of the theory rather than defects of exposition: they are precisely what makes the construction testable at the mathematical level.

2 Scope and mathematical contract

The paper uses the effective readout setting produced by the previous stages, but it is self-contained at the level needed for spectral analysis. We assume a complex Hilbert space $\mathcal{H}$ or a finite-dimensional complex vector space when explicitly stated, together with a closed densely defined operator K that represents the linearized physical readout operator after quotient, clock, and signature gates have already been passed. In many applications K will be self-adjoint or symmetric with respect to a positive physical inner product, but the contour calculus used below only requires closedness and a nonempty resolvent set in the relevant region. When self-adjointness is used, it is stated explicitly.

The contract of the paper is the following. We prove statements about isolated spectral sectors of K, their finite ranks, their stability, their descent, and their exact reduced kernels. We do not prove that any particular spectral sector is an electron, a neutrino, a quark, a generation, a color representation, or a weak representation. We do not compute physical masses or gauge couplings. We also do not assume a Standard Model representation and then reverse-engineer a projector. The logical direction is from operator to projector, from projector to reduced kernel, from reduced kernel to rank and stability gates, and only later from those gates to possible physical interpretation.

This distinction is essential because a finite-dimensional block can appear for trivial reasons. It may be a basis artifact, a gauge-fixing remnant, a duplicated presentation mode, a non-descending subspace, or a perturbatively unstable cluster. Such a block is not matter in any scientific sense. A matter sector must pass stronger tests: spectral isolation, Riesz construction, quotient descent, rank stability, Feshbach admissibility, and representation independence. The paper formalizes these tests.

Definition 2.1 (Admissible spectral problem)**.** An admissible spectral problem consists of a complex Hilbert space $\mathcal{H}$, a closed densely defined operator $K : D(K) \subset \mathcal{H} \to \mathcal{H}$, and an open region $\Omega \subset \mathbb{C}$ such that the resolvent set $\rho(K)$ meets Ω. A compact subset $\Sigma \subset \Omega \cap \operatorname{spec}(K)$ is called isolated in Ω if there exists a positively oriented finite union of smooth closed contours $\Gamma \subset \Omega \cap \rho(K)$ such that Σ lies in the bounded interior of Γ and $\operatorname{spec}(K) \setminus \Sigma$ lies outside it.

The region Ω is included because the physical operator may have parts of its spectrum that are irrelevant to the readout under consideration. One may work in a finite-energy window, a low-lying sector, or a domain selected by earlier quotient gates. What matters is not that the full spectrum is globally discrete, but that the proposed matter island is separated by a contour on which the resolvent exists.

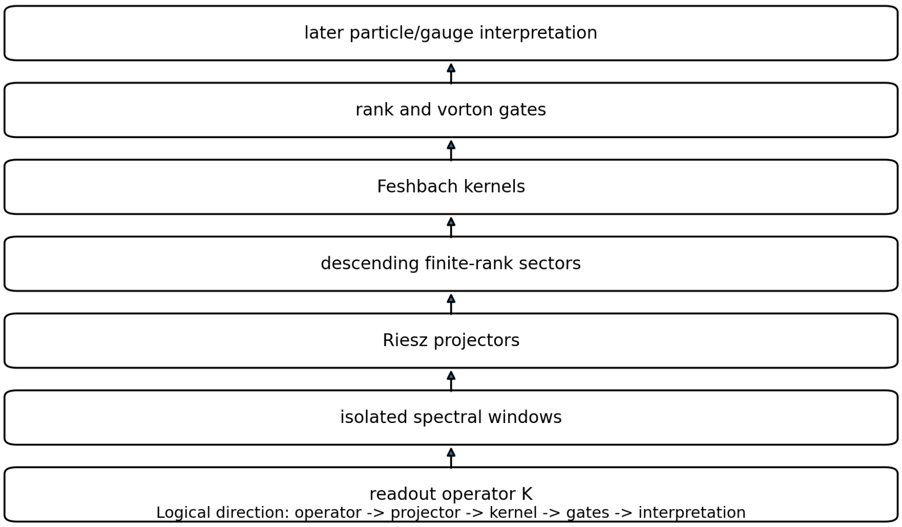

Figure 1: Logical stack of the spectral matter layer. The direction is deliberately one-way: an operator produces isolated spectral projectors, projectors produce exact reduced kernels, and only kernels passing rank, stability, compactness, and carrier gates may be interpreted later. No particle name is used as input.

This stack is also the audit logic used throughout the paper. A calculation that begins with a named particle multiplet has already skipped the spectral layer. A calculation that begins with an arbitrary finite block has not yet shown that the block is invariant, descending, or stable. A calculation that uses only the retained block rather than the Feshbach kernel has not yet computed the parent poles. These distinctions are not matters of style; they decide whether the construction is derivational or merely descriptive.

3 The parent readout operator and physical descent

Let $\mathcal{H}_{\text{pres}}$ denote a presentation Hilbert space and $\mathcal{H}_{\text{phys}}$ the physical Hilbert space obtained after quotienting redundant descriptions. The passage from presentation to physical state space may be represented by a surjective linear map

$$J : \mathcal{H}_{\text{pres}} \longrightarrow \mathcal{H}_{\text{phys}}, \tag{3.1}$$

with $\operatorname{Ker} J$ equal to the closure of null presentation modes. A presentation operator K_{pres} descends to a physical operator K only if it preserves the null space in the required domain sense:

$$K_{\mathrm{pres}}(D(K_{\mathrm{pres}}) \cap \operatorname{Ker} J) \subseteq \operatorname{Ker} J. \tag{3.2}$$

When this holds, there exists a unique operator K on $\mathcal{H}_{\mathrm{phys}}$ satisfying

$$KJx = JK_{\mathrm{pres}}x \tag{3.3}$$

for all admissible presentation vectors x. In finite dimensions this is simply the condition that the operator be well defined on equivalence classes; in unbounded settings the same condition is interpreted on a common invariant core.

This descent requirement is repeated at the level of spectral projectors. A projector constructed before quotienting is not physical unless it maps equivalent presentations to equivalent projected presentations. The criterion is straightforward but important.

Proposition 3.1 (Projector descent criterion)**.** *Let $J : \mathcal{H}_{\mathrm{pres}} \to \mathcal{H}_{\mathrm{phys}}$ be a quotient map and let P be a bounded projection on $\mathcal{H}_{\mathrm{pres}}$. Then P descends to a bounded projection $\bar{P}$ on $\mathcal{H}_{\mathrm{phys}}$ satisfying $\bar{P}J = JP$ if and only if*

$$P(\operatorname{Ker} J) \subseteq \operatorname{Ker} J. \tag{3.4}$$

If the condition holds, $\bar{P}[Jx] = [Px]$ is well defined and unique.

Proof. Assume first that $P(\operatorname{Ker} J) \subseteq \operatorname{Ker} J$. If $Jx = Jy$, then $x - y \in \operatorname{Ker} J$, hence $P(x-y) \in \operatorname{Ker} J$, so $JPx = JPy$. Therefore the formula $\bar{P}(Jx) = JPx$ is independent of the representative. Linearity follows from linearity of P and J. Since $P^2 = P$, one has $\bar{P}^2 J = \bar{P}JP = JP^2 = JP = \bar{P}J$, and because J is surjective, $\bar{P}^2 = \bar{P}$. Conversely, if a projection $\bar{P}$ satisfies $\bar{P}J = JP$, then for $x \in \operatorname{Ker} J$ one has $JPx = \bar{P}Jx = 0$, so $Px \in \operatorname{Ker} J$. Uniqueness follows because every physical vector has the form Jx. $\square$

This proposition prevents an important error. It is possible for a presentation operator to have a beautiful finite-dimensional spectral block that disappears when one passes to the physical quotient, or worse, fails to be well defined there. Such a block cannot be part of matter architecture. Physical spectral sectors must be built on the descended operator or must be shown to descend by the criterion above.

4 Riesz projectors and finite-rank spectral sectors

The Riesz projector associated with an isolated spectral island is the basic mathematical object of this paper. It is independent of the choice of basis, depends only on the spectral contour, and is stable under gap-preserving deformations. It therefore provides the correct language for matter sectors before group representation theory enters.

Let $\Gamma \subset \rho(K)$ be a positively oriented contour enclosing an isolated part Σ of the spectrum and no other spectral points. The resolvent is

$$R(z; K) = (zI - K)^{-1}. \tag{4.1}$$

The associated Riesz projector is

$$P_\Gamma(K) = \frac{1}{2\pi i} \oint_\Gamma R(z;K)\, dz. \tag{4.2}$$

The sign convention is chosen so that for a diagonal finite-dimensional operator with eigenvalues inside Γ, the projector selects precisely the corresponding eigenspaces.

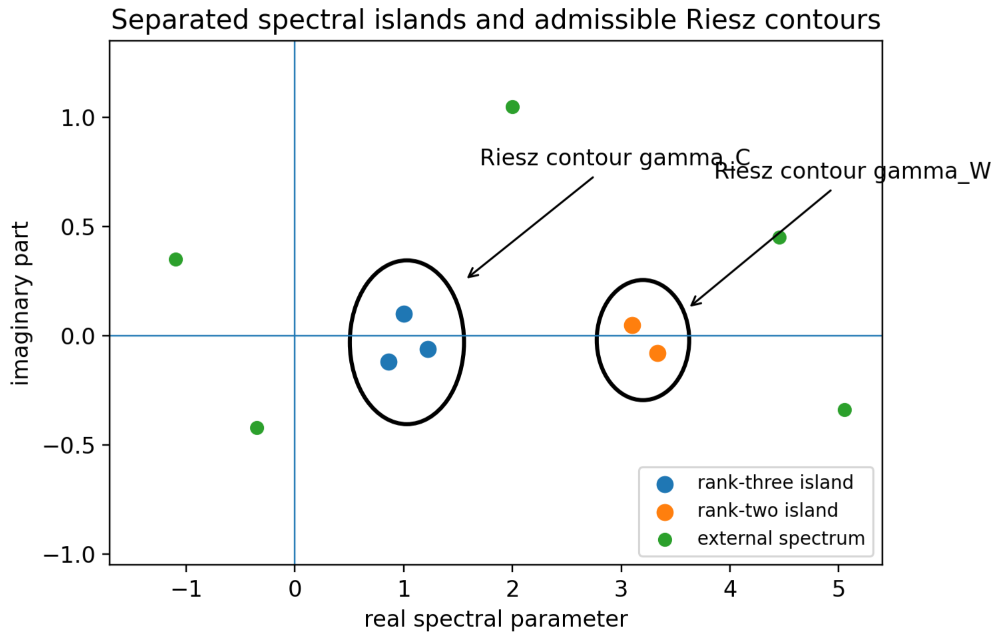

Figure 2: Separated spectral islands determine matter candidates only through contours lying in the resolvent set. The labels “rank-three” and “rank-two” describe spectral ranks at this stage; they do not yet assert color, weak isospin, or any Standard Model interpretation.

Theorem 4.1 (Riesz projector theorem)**.** *Let K be a closed densely defined operator and let $\Gamma \subset \rho(K)$ be a positively oriented contour enclosing an isolated spectral subset Σ. Then $P_\Gamma(K)$ is a bounded projection. Moreover $P_\Gamma(K)$ commutes with every resolvent $R(w;K)$ for $w \in \rho(K)$, its range is invariant under K, and in finite dimension its rank is the algebraic multiplicity of the eigenvalues enclosed by Γ.*

Proof. Boundedness follows because $z \mapsto R(z;K)$ is norm-continuous on the compact contour Γ, hence the contour integral converges in operator norm. To prove that $P_\Gamma^2 = P_\Gamma$, choose two homologous contours Γ_1 and Γ_2 enclosing the same spectral island, with Γ_1 inside Γ_2. Using the resolvent identity

$$R(z;K)R(w;K) = \frac{R(z;K) - R(w;K)}{w - z}, \tag{4.3}$$

one obtains

$$P_\Gamma^2 = \frac{1}{(2\pi i)^2} \oint_{\Gamma_2} \oint_{\Gamma_1} R(z;K)R(w;K)\, dz\, dw \tag{4.4}$$

$$= \frac{1}{(2\pi i)^2} \oint_{\Gamma_2} \oint_{\Gamma_1} \frac{R(z;K) - R(w;K)}{w - z}\, dz\, dw. \tag{4.5}$$

For the term containing $R(w; K)$, the inner integral in z vanishes because w lies outside Γ_1. For the term containing $R(z; K)$, the outer integral in w gives $2\pi i$ because z lies inside Γ_2. Hence $P_\Gamma^2 = P_\Gamma$. Commutation with the resolvent follows from the same identity and the fact that resolvents of a fixed operator commute. Invariance of the range is obtained by writing

$$KR(z; K) = zR(z; K) - I \tag{4.6}$$

on $D(K)$ and integrating along Γ, which gives $KP_\Gamma = P_\Gamma K$ on the relevant domain. In finite dimension the resolvent has a partial fraction expansion around the enclosed eigenvalues. The contour integral extracts the residues of the principal parts, which are precisely the spectral projection onto the generalized eigenspaces. The rank is therefore the algebraic multiplicity. □

Definition 4.2 (Spectral matter candidate)**.** A spectral matter candidate is a Riesz range

$$\mathcal{H}_\Gamma = \operatorname{Ran} P_\Gamma(K) \tag{4.7}$$

associated with an isolated spectral subset Σ, subject to three preliminary requirements: $P_\Gamma(K)$ must descend to the physical quotient, $\operatorname{rank} P_\Gamma(K) < \infty$, and the rank must be stable under the admissible perturbations of the parent operator.

This definition intentionally avoids particle terminology. A finite-rank Riesz range is a matter candidate because it is a stable, invariant, physically defined spectral sector. It is not yet matter in the phenomenological sense. The next sections introduce the additional kernel and rank gates needed before interpretation can proceed.

5 Vorton carriers as compact spectral matter candidates

The word “vorton” is used in this paper only after the spectral machinery has been installed. It does not denote a particle, a Standard Model field, a cosmic string solution, or a fitted phenomenological object. It denotes a possible compact spectral carrier. This distinction is essential. A vorton cannot be introduced as a named degree of freedom and then used to explain why matter sectors exist. It must be obtained as a finite-rank Riesz sector of the parent readout operator, and it must then pass additional compactness and circulation tests.

Definition 5.1 (Vorton carrier candidate)**.** Let K be the physical readout operator and let $P_V = P_{\Gamma_V}(K)$ be a Riesz projector associated with an isolated spectral island Σ_V. A vorton carrier candidate is a tuple

$$\mathfrak{V} = (P_V,\, \mathcal{H}_V,\, \mathcal{K}_V(z),\, J_V,\, \nu_V) \tag{5.1}$$

where $\mathcal{H}_V = \operatorname{Ran} P_V$ is finite-dimensional, $\mathcal{K}_V(z)$ is the exact Feshbach kernel on $\mathcal{H}_V$, J_V is an anti-self-adjoint circulation generator on a nontrivial subspace of $\mathcal{H}_V$ with respect to the physical inner product, and ν_V is a homotopy-invariant integer or parity index associated with the internal circulation whenever such an index is defined.

The definition intentionally separates existence from interpretation. The Riesz projector supplies the sector, the Feshbach kernel supplies the exact reduced dynamics, the compactness gate decides whether the sector can act as a local or controlled nonlocal carrier, and the circulation gate decides whether the word vorton is justified. If P_V is absent, the vorton is not derived. If P_V exists but J_V is absent, the sector may still be a matter candidate, but it is not a vorton carrier in the technical sense used here. If J_V exists but is unstable under admissible deformations, the circulation is not physical.

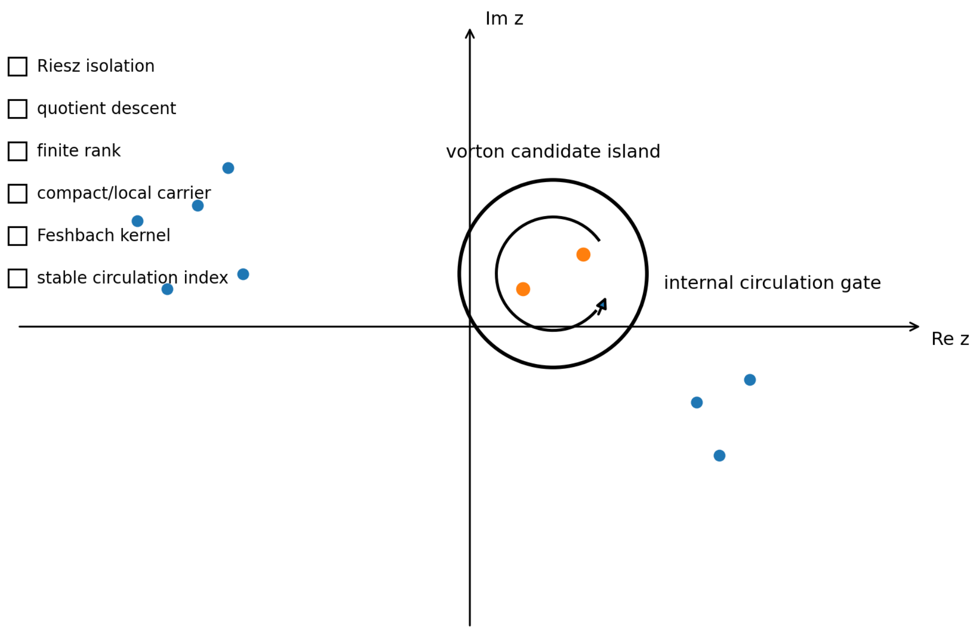

Figure 3: A vorton carrier is not an assumed object. It is a finite-rank Riesz island that descends to the physical quotient, admits an exact Feshbach kernel, satisfies a compactness or locality diagnostic, and carries a stable internal circulation structure. The closed contour represents spectral isolation; the internal arrow represents the additional circulation gate.

Theorem 5.2 (Vorton gate theorem). *Let $P_V = P_{\Gamma_V}(K)$ be a finite-rank Riesz projector of the physical readout operator. Suppose that P_V descends to the physical quotient, that the complementary Feshbach block is invertible on a domain Ω, and that the exact reduced kernel $\mathcal{K}_V(z)$ is well-defined on Ω. Then P_V may be called a vorton carrier only if there exists a circulation generator J_V on $\mathcal{H}_V$ whose defining algebraic relations and associated index are invariant under admissible representation changes and stable under gap-preserving perturbations. In particular, spectral isolation and finite rank are necessary but not sufficient.*

Proof. The first part follows from the definition of a physical sector in this paper. Since

P_V is a Riesz projector, $\mathcal{H}_V = \operatorname{Ran} P_V$ is invariant under the parent operator restricted to the isolated spectral island. Since P_V descends to the quotient, the sector is not a presentation artifact. Since the complementary Feshbach block is invertible, the exact reduced kernel exists and has the same retained poles as the corresponding factor of the parent determinant. These facts are enough to define a finite-rank spectral matter candidate, but none of them supplies circulation. The additional datum J_V is meaningful only if it is intrinsic to the reduced sector, because otherwise it can be created or removed by changing basis. Therefore it must transform covariantly under similarity transformations preserving the physical sector and must retain its index under perturbations that keep the Riesz contour inside the resolvent set. If this stability fails, the would-be circulation is not part of the physical spectral architecture. Thus the vorton terminology requires all previous matter-sector gates plus the additional circulation gate. □

A simple algebraic form of the circulation gate can be stated when the reduced sector carries a positive physical inner product. In the minimal two-plane case, one asks for an anti-self-adjoint endomorphism J_V satisfying

$$J_V^* = -J_V, \qquad J_V^2 = -\omega_V^2 \Pi_V, \qquad \omega_V > 0, \tag{5.2}$$

where Π_V is the identity on the circulating plane. The scalar ω_V is not a mass or a coupling. It is the spectral circulation scale of the carrier. In a semisimple complex representation this condition is equivalent to a pair of opposite imaginary eigenvalues for J_V or, after multiplication by i, to a two-level Hermitian splitting. In a real representation it is the standard finite-dimensional normal form of a stable rotation block.

Proposition 5.3 (Stability of the circulation scale)**.** *Let $J_V(t)$ be a norm-continuous family of anti-self-adjoint circulation generators on a fixed finite-rank Riesz sector, and assume that*

$$J_V(t)^2 = -\omega_V(t)^2 \Pi_V(t) \tag{5.3}$$

on a two-dimensional circulating subspace. If $\omega_V(0) > 0$ and $\|J_V(t) - J_V(0)\| < \omega_V(0)$, then the circulation cannot collapse to zero along the deformation.

Proof. For anti-self-adjoint J_V, the positive operator $-J_V^2$ has eigenvalue ω_V^2 on the circulating plane. The spectrum of a finite-dimensional operator moves upper-semicontinuously under norm perturbations and, for normal operators, obeys the elementary Hausdorff bound by the norm of the perturbation. If the perturbation norm is smaller than the initial gap between zero and the nonzero circulation eigenvalues, then zero cannot enter the circulating spectral pair. Hence $\omega_V(t)$ remains positive. This proves stability of the nonzero circulation scale under the stated perturbation bound. □

This result is deliberately elementary, but it is precisely the kind of estimate that a later matter calculation must display. A figure, a symbol, or a name is not a derivation. A vorton carrier requires an isolated projector, an exact kernel, a compactness gate, and a stable circulation estimate. Without these ingredients the term would only rename an assumed degree of freedom.

6 Representation independence and covariance of projectors

A physical sector cannot depend on the coordinates or basis used to describe it. If two operators are related by an invertible change of representation, their spectral projectors must be carried into each other. This is automatic for Riesz projectors, and it is one of the main reasons they are preferable to ad hoc block selections.

Proposition 6.1 (Similarity covariance)**.** *Let $U : \mathcal{H} \to \mathcal{H}'$ be a bounded invertible map and let $K' = UKU^{-1}$. If $\Gamma \subset \rho(K)$, then $\Gamma \subset \rho(K')$ and*

$$P_\Gamma(K') = UP_\Gamma(K)U^{-1}. \tag{6.1}$$

Consequently $\operatorname{rank} P_\Gamma(K') = \operatorname{rank} P_\Gamma(K)$ *whenever the rank is finite.*

Proof. For every $z \in \rho(K)$,

$$zI - K' = zI - UKU^{-1} = U(zI - K)U^{-1}, \tag{6.2}$$

so

$$R(z; K') = (zI - K')^{-1} = UR(z; K)U^{-1}. \tag{6.3}$$

Integrating around Γ gives

$$P_\Gamma(K') = \frac{1}{2\pi i} \oint_\Gamma UR(z; K)U^{-1}\, dz = UP_\Gamma(K)U^{-1}. \tag{6.4}$$

The rank statement follows because invertible maps preserve dimension of finite-dimensional ranges. □

Diagnostic 6.2 (No-basis-sector diagnostic)**.** A proposed sector fails the no-basis-sector diagnostic if it is defined by choosing a convenient coordinate block rather than by a contour in the resolvent set or by a projector proved to be equivalent to such a contour projector. A block that disappears under a similarity transformation is not a matter sector.

This diagnostic is severe but necessary. In a long derivational chain, it is tempting to recognize a familiar pattern and assign it a particle label. The present paper forbids that move. First one proves that the sector is spectral and representation independent. Only then can a later paper ask whether the sector carries gauge normal forms.

7 Feshbach reduction and exact matter kernels

Once a finite-rank spectral sector has been isolated, the next question is how it interacts with the complement. The complement cannot simply be discarded. It may shift poles, split degeneracies, or generate nonlocal effective terms. The exact way to eliminate it is the Feshbach map, which is the operator-theoretic version of the Schur complement.

Let P be a bounded projection and $Q = I - P$. For a spectral parameter z, write the block operator

$$K - zI = \begin{pmatrix} A(z) & B \\ C & D(z) \end{pmatrix} \tag{7.1}$$

relative to $\mathcal{H} = P\mathcal{H} \oplus Q\mathcal{H}$, where

$$A(z) = P(K - zI)P, \quad B = PKQ, \quad C = QKP, \tag{7.2}$$

and

$$D(z) = Q(K - zI)Q. \tag{7.3}$$

Whenever $D(z)$ is invertible on $Q\mathcal{H}$, the Feshbach map is

$$F_P(z) = A(z) - BD(z)^{-1}C. \tag{7.4}$$

In the spectral matter context, $F_P(z)$ is the exact matter kernel associated with the retained sector $P\mathcal{H}$.

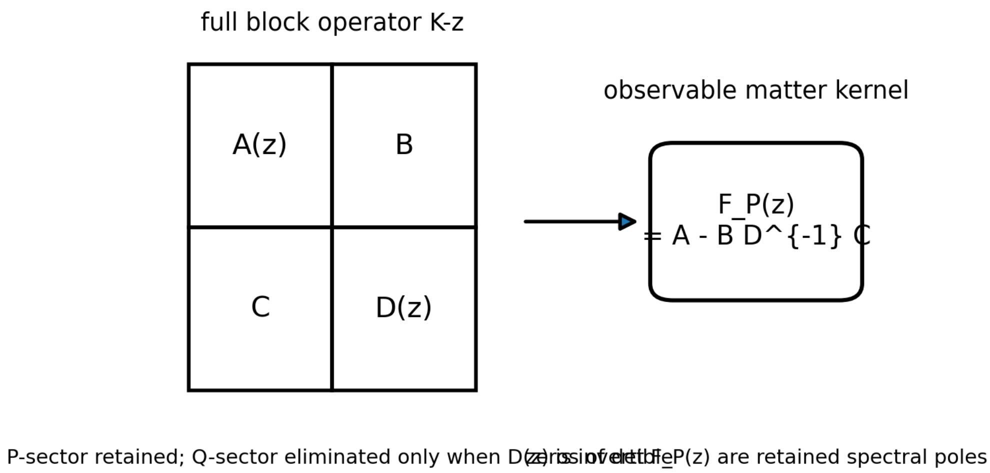

Figure 4: Feshbach reduction is not truncation. The complementary sector contributes the exact term $BD^{-1}C$, which shifts the retained spectral equation and records virtual excursions into the eliminated subspace.

Theorem 7.1 (Feshbach isospectral theorem)**.** *Assume that $D(z)$ is invertible. Then the block operator $K - zI$ is invertible if and only if $F_P(z)$ is invertible. In finite* *dimension,*

$$\det(K - zI) = \det D(z)\, \det F_P(z). \tag{7.5}$$

Moreover, whenever $F_P(z)u = 0$, the vector

$$\Psi = u - D(z)^{-1}Cu \tag{7.6}$$

understood as $u \oplus [-D(z)^{-1}Cu]$ lies in $\mathrm{Ker}(K - zI)$. Conversely, every vector in $\mathrm{Ker}(K - zI)$ is obtained this way from a vector in $\mathrm{Ker}\, F_P(z)$.

Proof. The block factorization

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \tag{7.7}$$

is obtained by direct multiplication. The left and right triangular factors are invertible whenever D is invertible. Therefore the full block operator is invertible if and only if both D and $F_P = A - BD^{-1}C$ are invertible. Since D is assumed invertible, invertibility of the full operator is equivalent to invertibility of F_P. In finite dimension the determinant of each triangular factor is one, hence the determinant identity follows.

If $F_P(z)u = 0$, define $v = -D(z)^{-1}Cu$. Then the lower block equation gives

$$Cu + Dv = Cu - DD^{-1}Cu = 0, \tag{7.8}$$

and the upper block equation gives

$$Au + Bv = Au - BD^{-1}Cu = F_P(z)u = 0. \tag{7.9}$$

Thus $u \oplus v \in \mathrm{Ker}(K - zI)$. Conversely, if $u \oplus v \in \mathrm{Ker}(K - zI)$, the lower block equation implies $v = -D^{-1}Cu$, and substituting into the upper block equation gives $F_P(z)u = 0$. This proves the reconstruction statement. □

The theorem shows that the matter kernel is not an approximation. It is an exact spectral reduction in the region where the eliminated block is invertible. Approximation enters only later, if one expands $D(z)^{-1}$, truncates a series, or replaces the kernel by a local normal form.

8 The matter kernel and pole equations

The operator-valued function $F_P(z)$ is the central object of the spectral matter layer. It contains the retained block $A(z)$, the coupling to the complement B and C, and the exact return through the eliminated sector $D(z)^{-1}$. Its zeros determine the spectral poles of the retained sector after the complement has been accounted for. In finite dimension this statement is just the determinant factorization above; in infinite dimension it must be formulated through Fredholm determinants or analytic Fredholm theory when the relevant compactness hypotheses hold.

Definition 8.1 (Matter kernel)**.** Let P be an admissible finite-rank Riesz projector for K. The matter kernel associated with P is the finite-dimensional operator-valued function

$$\mathcal{K}_P(z) = P(K - zI)P - PKQ\,[Q(K - zI)Q]^{-1}\,QKP, \tag{8.1}$$

defined on the subset of the spectral parameter plane where $Q(K - zI)Q$ is invertible. The matter pole equation is

$$\det \mathcal{K}_P(z) = 0. \tag{8.2}$$

Because P is finite rank, $\mathcal{K}_P(z)$ is a finite matrix once a basis of $P\mathcal{H}$ is chosen. The determinant is basis covariant up to multiplication by the determinant of the basis transformation and its inverse, hence its zero set is basis independent.

Proposition 8.2 (Kernel covariance)**.** *Let U be a bounded invertible transformation, $K' = UKU^{-1}$, and $P' = UPU^{-1}$. Then the corresponding Feshbach kernels satisfy*

$$\mathcal{K}_{P'}^{K'}(z) = U_P\,\mathcal{K}_P^{K}(z)\,U_P^{-1}, \tag{8.3}$$

where $U_P : P\mathcal{H} \to P'\mathcal{H}'$ is the restriction of U. Consequently the pole equation is representation independent.

Proof. Under the decomposition $\mathcal{H} = P\mathcal{H} \oplus Q\mathcal{H}$, the transformed decomposition is $\mathcal{H}' = P'\mathcal{H}' \oplus Q'\mathcal{H}'$, with $Q' = UQU^{-1}$. Each block of $K' - zI$ is the corresponding block of $K - zI$ conjugated by the appropriate restriction of U. In particular,

$$D'(z)^{-1} = U_Q D(z)^{-1} U_Q^{-1}. \tag{8.4}$$

Substitution into the definition of the Feshbach map gives the stated conjugacy. A conjugacy transformation preserves invertibility and therefore preserves the zero set of the determinant in finite dimension. □

9 Rank gates and sector architecture

The spectral layer can now state rank gates without overinterpreting them. A rank gate is a statement about the dimension of a Riesz range. It is not a statement about a gauge representation, a particle family, or a phenomenological multiplicity until later structures have been derived.

Definition 9.1 (Rank gate)**.** Let Γ_a be an admissible spectral contour for K. The rank gate r_a is passed if

$$\operatorname{rank} P_{\Gamma_a}(K) = r_a \tag{9.1}$$

and the rank remains invariant under all perturbations admitted by the physical stability class of the theory.

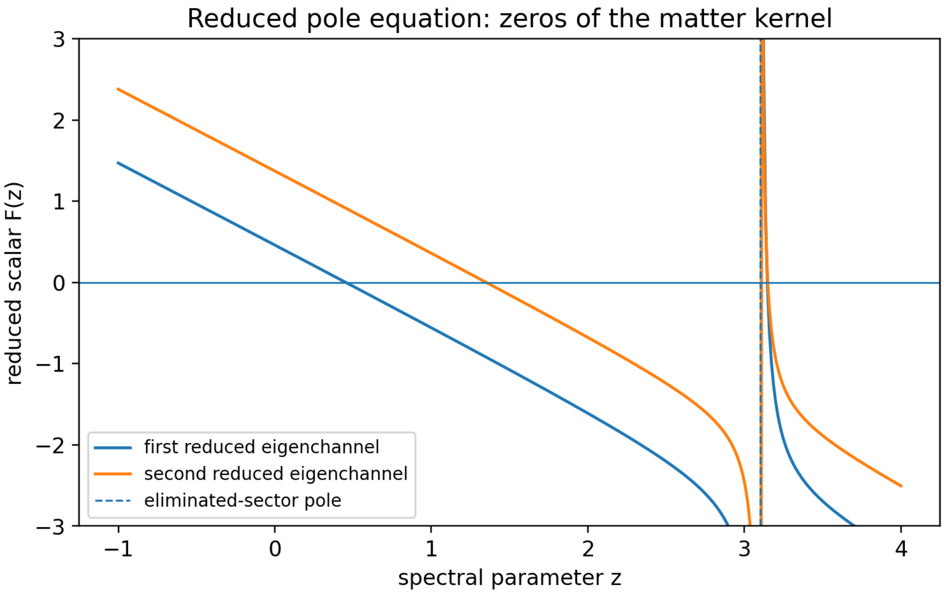

Figure 5: A reduced pole equation can acquire shifts and avoided crossings through the exact return term $BD^{-1}C$. The singular point at the eliminated-sector pole is not part of the admissible Feshbach domain.

The two rank gates expected to be important in later papers are

$$\operatorname{rank}(P_C) = 3, \qquad \operatorname{rank}(P_W) = 2. \tag{9.2}$$

At this stage P_C and P_W mean only "the rank-three candidate projector" and "the rank-two candidate projector." The notation is chosen because later gauge normal forms may attach conventional names, but no such attachment is used here.

Theorem 9.2 (Rank stability under contour-preserving perturbations)**.** *Let K be a bounded operator and let $\Gamma \subset \rho(K)$ be a contour enclosing a finite-rank spectral island. Let*

$$g = \operatorname{dist}(\Gamma, \operatorname{spec}(K)) > 0. \tag{9.3}$$

If E is a bounded perturbation with $\|E\| < g$, then $\Gamma \subset \rho(K + E)$, the Riesz projector $P_\Gamma(K + E)$ is defined, and

$$\|P_\Gamma(K + E) - P_\Gamma(K)\| \le \frac{L(\Gamma)}{2\pi} \frac{\|E\|}{g(g - \|E\|)}, \tag{9.4}$$

where $L(\Gamma)$ is the contour length. In particular, the rank is constant along every norm-continuous path $K(t)$ for which Γ stays in the resolvent set.

Proof. For $z \in \Gamma$, $\|(zI - K)^{-1}\| \le g^{-1}$. Since $\|E\| < g$, the operator

$$I - E(zI - K)^{-1} \tag{9.5}$$

is invertible by the Neumann series, and

$$(zI - K - E)^{-1} = (zI - K)^{-1}[I - E(zI - K)^{-1}]^{-1}. \tag{9.6}$$

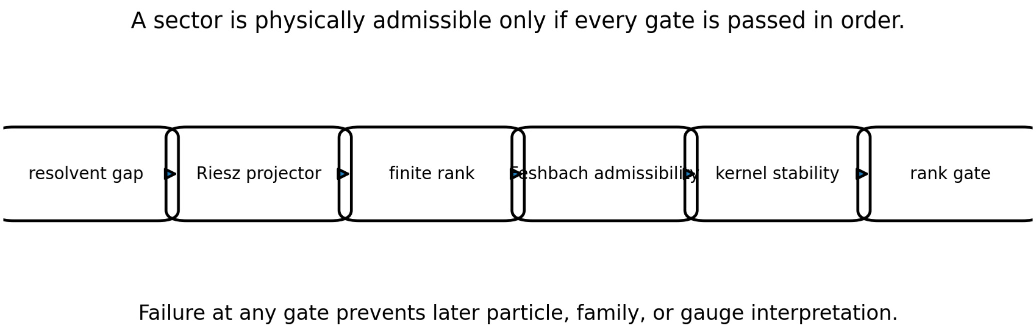

Figure 6: Rank gates are late-stage consequences of earlier spectral tests. A rank number is not meaningful until the contour exists, the Riesz projector is physical, the rank is finite and stable, and the Feshbach kernel is admissible.

Therefore $z \in \rho(K+E)$. The resolvent identity gives

$$R(z;K+E) - R(z;K) = R(z;K+E)ER(z;K). \tag{9.7}$$

Moreover $\|R(z;K+E)\| \le (g-\|E\|)^{-1}$, again by the Neumann estimate. Hence

$$\|R(z;K+E) - R(z;K)\| \le \frac{\|E\|}{g(g-\|E\|)}. \tag{9.8}$$

Integrating around Γ gives the projector estimate. Rank constancy follows because a norm-continuous path of projections cannot change finite rank without leaving the projection manifold; equivalently, if $\|P(t) - P(t_0)\| < 1$, then $\operatorname{Ran} P(t)$ and $\operatorname{Ran} P(t_0)$ are isomorphic. □

The theorem gives a quantitative audit. A rank-three gate is not merely the observation that three eigenvalues happen to sit near each other in one calculation. It requires a contour, a gap, and a perturbation class small enough that the Riesz rank cannot change.

10 Commuting sector projectors and refinement

Matter architecture is rarely a single projector. A realistic spectral structure is a family of projectors that may refine one another. The cleanest situation occurs when the projectors commute. Then their products are again projections, and the ranges form a lattice of sectors.

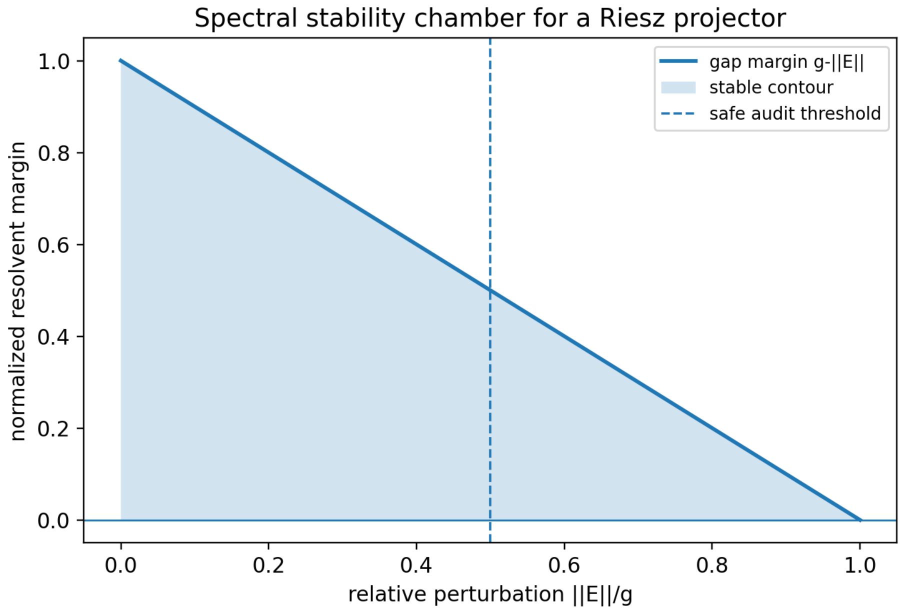

Figure 7: The contour stability margin is controlled by the gap between the contour and the spectrum. When the perturbation reaches the gap, the contour may cross the spectrum and the rank gate can fail.

Proposition 10.1 (Commuting Riesz refinement)**.** *Let $P_1, \ldots, P_n$ be mutually commuting Riesz projectors of K. Then every product*

$$P_I = \prod_{j \in I} P_j \prod_{k \notin I} (I - P_k) \tag{10.1}$$

is a projection commuting with K. The nonzero ranges of the P_I give a direct decomposition of the finite-dimensional space generated by the original projectors.

Proof. Because the projectors commute and satisfy $P_j^2 = P_j$, every factor in the product is a projection and all factors commute. Hence the product is a projection. It commutes with K because each Riesz projector commutes with the resolvent and hence with K on the invariant domain. Orthogonality in the algebraic sense follows from the fact that two products with different index choices contain a factor $P_j(I - P_j) = 0$. Summing all products over all choices gives the identity on the subspace generated by the projectors. □

If projectors fail to commute, sector refinement becomes more delicate. Noncommuting projectors may signal either a genuine mixing structure or an ill-posed attempt to impose incompatible sector labels. In that case the theory must work with the algebra generated by the projectors rather than with a simultaneous decomposition.

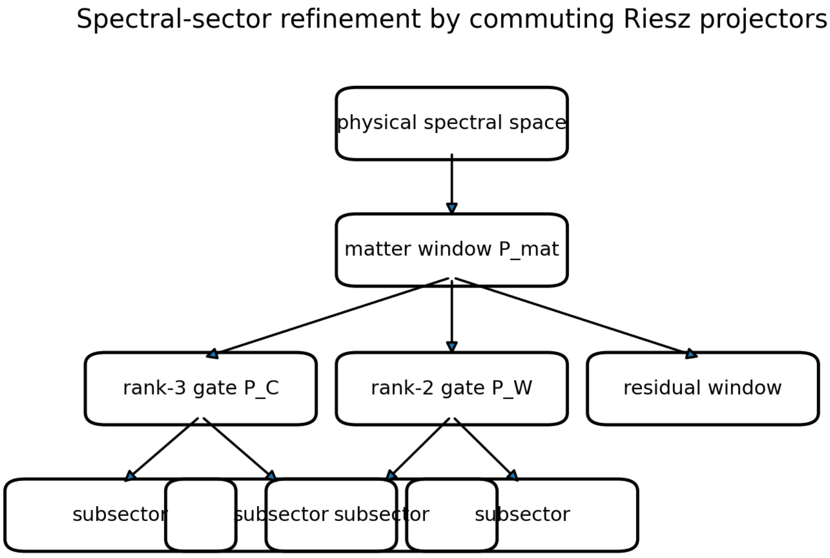

Figure 8: A sector architecture is a lattice of physically descending spectral projectors, not a list of names. Later representation theory can act only on such stable projector ranges.

11 Worked calculation: a two-dimensional retained island

A simple finite-dimensional calculation illustrates the difference between truncation and Feshbach reduction. Consider the operator

$$K_\epsilon = \begin{pmatrix} \lambda_1 & 0 & \epsilon a \\ 0 & \lambda_2 & \epsilon b \\ \epsilon c & \epsilon d & \mu \end{pmatrix}, \tag{11.1}$$

where λ_1, λ_2 form the retained island and μ lies outside the contour. Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q = I - P. \tag{11.2}$$

Then

$$A(z) = \begin{pmatrix} \lambda_1 - z & 0 \\ 0 & \lambda_2 - z \end{pmatrix}, \quad B = \epsilon \begin{pmatrix} a \\ b \end{pmatrix}, \quad C = \epsilon \begin{pmatrix} c & d \end{pmatrix}, \quad D(z) = \mu - z. \tag{11.3}$$

For $z \neq \mu$, the exact matter kernel is

$$\mathcal{K}_P(z) = \begin{pmatrix} \lambda_1 - z & 0 \\ 0 & \lambda_2 - z \end{pmatrix} - \frac{\epsilon^2}{\mu - z} \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}. \tag{11.4}$$

The pole equation is therefore

$$0 = \det \mathcal{K}_P(z) \tag{11.5}$$

$$= (\lambda_1 - z)(\lambda_2 - z) - \frac{\epsilon^2}{\mu - z}\left[bd(\lambda_1 - z) + ac(\lambda_2 - z)\right]. \tag{11.6}$$

Multiplying by $\mu - z$, one obtains

$$(\mu - z)(\lambda_1 - z)(\lambda_2 - z) - \epsilon^2\left[bd(\lambda_1 - z) + ac(\lambda_2 - z)\right] = 0. \tag{11.7}$$

This is exactly the characteristic equation of the full matrix away from $z = \mu$. The eliminated sector has not been thrown away; it produces the second-order return term that shifts the retained poles.

This calculation also shows why premature particle naming is dangerous. If one simply truncates to $A(z)$, the poles are λ_1 and λ_2. The true poles are shifted by the complement unless a, b, c, d vanish or unless the return term is negligible in a controlled expansion. Any later mass calculation must therefore use the Feshbach kernel or prove that the return term is absent by symmetry.

12 Minimal vorton-Feshbach calculation

The abstract vorton gate becomes more transparent in the smallest nontrivial retained sector. Consider a two-dimensional retained Riesz space $\mathcal{H}_V$ with a Hermitian circulation splitting represented by the Pauli matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{12.1}$$

and let the retained block of the operator pencil be

$$A(z) = (m - z)I_2 + \omega\sigma_2, \qquad \omega > 0. \tag{12.2}$$

Let the eliminated sector be one-dimensional with

$$D(z) = M - z, \tag{12.3}$$

and let the coupling from the retained sector to the eliminated sector be $B = be_1$ and $C = B^*$ in a basis where $e_1 = (1, 0)^T$. The exact Feshbach kernel is then

$$F_V(z) = A(z) - BD(z)^{-1}C = (m - z)I_2 + \omega\sigma_2 - \frac{|b|^2}{M - z}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{12.4}$$

This formula is the simplest example of why truncation is not acceptable. The retained block $A(z)$ would give poles at $z = m \pm \omega$. The exact retained-sector poles are instead zeros of

$$\det F_V(z) = 0. \tag{12.5}$$

A direct determinant calculation gives

$$\det F_V(z) = \det \begin{pmatrix} m - z - \frac{|b|^2}{M-z} & -i\omega \\ i\omega & m - z \end{pmatrix} \tag{12.6}$$

$$= \left(m - z - \frac{|b|^2}{M - z} \right)(m - z) - \omega^2. \tag{12.7}$$

Multiplying by $M - z$ removes the denominator and yields the cubic pole equation

$$(M - z)\Big((m - z)^2 - \omega^2\Big) - |b|^2(m - z) = 0. \tag{12.8}$$

This is not an illustrative guess; it is the exact determinant identity for the parent block decomposition under the stated assumptions.

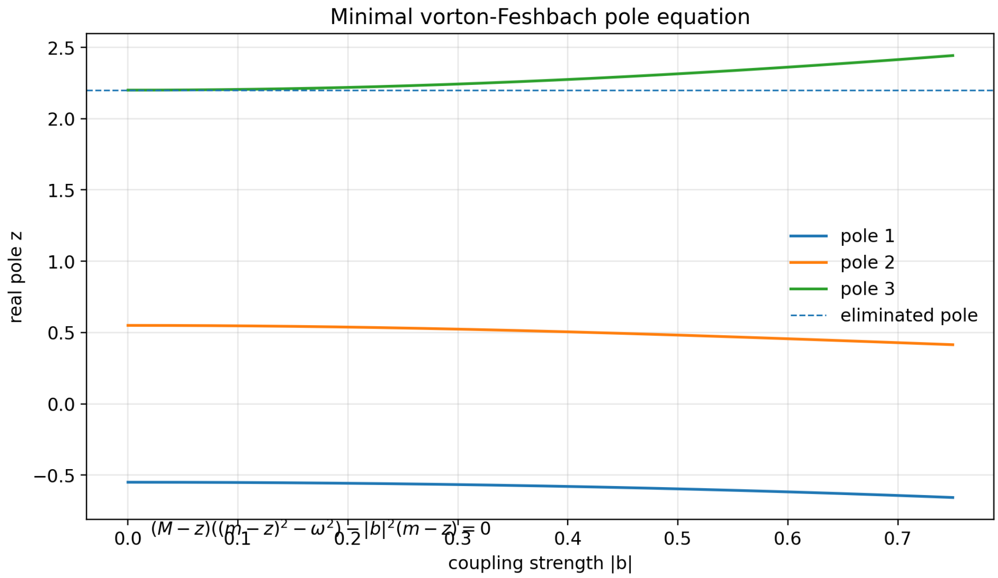

Figure 9: Exact pole motion in the minimal vorton-Feshbach model. The two retained circulation poles are not obtained by truncating the retained block once the eliminated sector couples back. They are zeros of the cubic Feshbach pole equation.

Proposition 12.1 (Small-coupling pole shift in the minimal vorton model)**.** *Assume $M \neq m \pm \omega$. For sufficiently small $|b|$, the two retained poles near $z_\pm^{(0)} = m \pm \omega$ have the expansion*

$$z_\pm = m \pm \omega - \frac{|b|^2}{2(M - m \mp \omega)} + O(|b|^4). \tag{12.9}$$

Proof. Let $x = z - m$ and $M_0 = M - m$. The exact pole equation becomes

$$(M_0 - x)(x^2 - \omega^2) + |b|^2 x = 0. \tag{12.10}$$

For the pole near $x_+ = \omega$, write $x = \omega + \delta$ with $\delta = O(|b|^2)$. Since

$$x^2 - \omega^2 = (\omega + \delta)^2 - \omega^2 = 2\omega\delta + O(\delta^2), \tag{12.11}$$

substitution gives

$$(M_0 - \omega)2\omega\delta + |b|^2\omega + O(|b|^2\delta) + O(\delta^2) = 0. \tag{12.12}$$

Thus

$$\delta = -\frac{|b|^2}{2(M_0 - \omega)} + O(|b|^4). \tag{12.13}$$

Since $z = m + x$, this gives the displayed negative shift for the pole near $m + \omega$. For the pole near $x = -\omega$, write $x = -\omega + \delta$; then $x^2 - \omega^2 = -2\omega\delta + O(\delta^2)$ and the same substitution gives $\delta = -|b|^2/[2(M_0 + \omega)] + O(|b|^4)$. This is the displayed formula for the pole near $m - \omega$. □

The sign in this elementary calculation is an audit point. If one writes the pencil as $zI - K$ instead of $K - zI$, the displayed scalar equation changes by an overall sign but the pole locations do not. A later numerical matter calculation must state which convention is used before quoting shifts. The purpose of this section is not to identify the two retained poles with any particle. It is to show what a genuine vorton-carrier calculation looks like at the first nontrivial level: the carrier is an isolated two-dimensional Riesz island, the circulation is an internal splitting, and the pole positions are roots of the exact Feshbach determinant.

13 Rank-three and rank-two toy gates

A toy example can illustrate the rank gates without assigning physical names. Let

$$K_0 = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1, \ldots, \gamma_m) \tag{13.1}$$

with α_i contained in a contour Γ_C, β_j contained in a disjoint contour Γ_W, and all γ_k outside both contours. Then

$$\mathrm{rank}\, P_{\Gamma_C}(K_0) = 3, \qquad \mathrm{rank}\, P_{\Gamma_W}(K_0) = 2. \tag{13.2}$$

If E is a perturbation satisfying

$$\|E\| < \min\{g_C, g_W\}, \tag{13.3}$$

where g_C and g_W are the contour gaps, then the two ranks remain three and two. If in addition the projectors commute, their direct-sum sector is stable. If they do not commute after perturbation, the ranks remain meaningful but the simultaneous sector decomposition must be replaced by the algebra generated by the projectors.

This example is intentionally elementary. Its purpose is not to model the observed matter spectrum. Its purpose is to state the exact mathematical meaning of a later claim that a rank-three or rank-two sector has been derived. Such a claim must identify contours, prove gaps, compute ranks, control perturbations, and show descent.

14 Contour deformation and rank invariance audit

The rank of a Riesz sector is not attached to a chosen drawing of the contour; it is attached to the homotopy class of that contour inside the resolvent set. This point matters for figures and for calculations. A visually attractive spectral diagram is not evidence of a stable sector unless the allowed deformation domain is specified and the contour is prevented from crossing the spectrum.

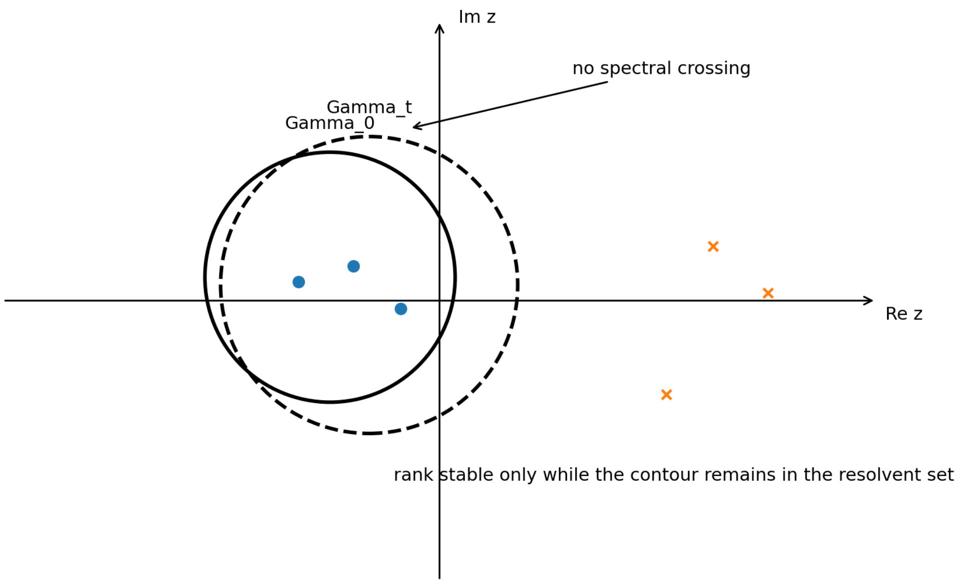

Figure 10: A Riesz contour may be deformed without changing the projector rank only while it remains in the resolvent set and crosses no spectral point. The figure is schematic, but the rule is exact: spectral crossing is the wall at which the sector rank can change.

Proposition 14.1 (Homotopy invariance of the Riesz rank)**.** *Let Γ_s, $0 \le s \le 1$, be a continuous family of closed positively oriented contours contained in $\rho(K)$, and suppose no spectrum of K crosses the swept annulus. Then* rank $P_{\Gamma_s}(K)$ *is independent of s.*

Proof. The Riesz projector is given by the contour integral of the resolvent. Since the resolvent is analytic on the swept region and the contours are homologous there, Cauchy's theorem gives

$$\frac{1}{2\pi i}\int_{\Gamma_0}(zI-K)^{-1}\,dz = \frac{1}{2\pi i}\int_{\Gamma_1}(zI-K)^{-1}\,dz. \tag{14.1}$$

Thus the projector is unchanged. In particular its rank is unchanged. If the contour crosses the spectrum, the resolvent ceases to be analytic on the deformation region and the argument fails. This is exactly the spectral wall at which the rank gate may change. □

This proposition also clarifies the role of graphics. A figure may illustrate spectral separation, but the proof must state the resolvent domain. The vorton carrier figure above is therefore not a claim of existence. It is a diagram of the gates that must be passed for existence to become a mathematical statement.

15 Failure modes and falsification ledger

A spectral matter architecture can fail in several precise ways. The first failure is absence of a spectral gap. Without a contour in the resolvent set, the Riesz projector is undefined. The second failure is infinite rank. A contour may enclose a spectral island, but if the associated range is infinite-dimensional, it cannot serve as a finite matter sector without further reduction. The third failure is non-descent: the projector may exist on a presentation space but fail to preserve the quotient kernel. The fourth failure is Feshbach singularity: if the eliminated block $D(z)$ is not invertible at the relevant parameter, the proposed matter kernel is invalid. The fifth failure is perturbative instability: if admissible deformations move spectrum through the contour, the rank is not robust. The sixth failure is representation dependence: if the sector can be destroyed by a change of basis, it is not physical.

Table 1: Spectral matter gate ledger.

Gate	Mathematical requirement	Failure implication
Isolation	Existence of $\Gamma \subset \rho(K)$ enclosing the proposed island	No Riesz projector exists
Finite rank	$\operatorname{rank} P_\Gamma < \infty$	Sector is not a finite matter candidate
Descent	$P_\Gamma(\operatorname{Ker} J) \subset \operatorname{Ker} J$	Sector is representational
Feshbach admissibility	$Q(K - z)Q$ invertible on the working domain	Reduced kernel is undefined
Rank stability	Gap-preserving perturbation bound	Rank is accidental
Representation independence	Similarity covariance of the projector and kernel	Sector is basis artifact
Kernel closure	Pole equation derived from $\det \mathcal{K}_P(z) = 0$	Pole positions are truncated approximations

The ledger should be read as a set of obligations. Passing one gate does not imply the next. In particular, the existence of a rank-three Riesz range does not by itself imply a color symmetry, just as a rank-two range does not by itself imply a weak doublet. Gauge normal forms require additional algebraic structure and therefore belong to the next paper, not this one.

The audit matrix is the central correction enforced in this version of the paper. The

Gate	Object tested	Failure mode
G1	Riesz contour	contour crosses spectrum
G2	descent map	sector is presentation artifact
G3	finite rank	continuum or dense spectral block
G4	Feshbach block	eliminated inverse is singular
G5	kernel reality	unstable/non-Hermitian carrier
G6	vorton circulation	no stable internal phase index
G7	locality/compactness	uncontrolled return term

The audit prevents named sectors from entering before spectral and kernel gates pass.

Figure 11: Gate audit for spectral matter architecture. The left column names the gate, the middle column identifies the mathematical object being tested, and the right column records the failure mode. This table is included as a figure because it is the practical checklist for preventing premature particle or vorton interpretation.

existence of a finite-dimensional block is never enough. The existence of an attractive pole diagram is never enough. The existence of an internally rotating two-plane is never enough. A matter carrier must survive all gates in the matrix, and a vorton carrier must also pass the circulation gate. This is the difference between a naming convention and a derivation.

16 Relation to gauge normal forms

The corrected order of the series is now clear. This paper constructs finite-rank spectral sectors and exact matter kernels. The next layer may ask whether the algebra acting on these sectors admits normal forms resembling gauge representations, whether kinetic forms are positive, and whether effective couplings can be normalized. That question presupposes the present one. A gauge representation without a stable spectral sector would be a representation in search of a carrier. Conversely, a stable spectral sector without a gauge normal form is still meaningful as spectral architecture but not yet a Standard Model sector.

The separation is not cosmetic. It prevents the most common source of overclaiming in speculative unification programs: importing known particle multiplets first and then treating the resulting pattern as a derivation. The present paper reverses the order. Spectral projectors come first. Representation theory comes second. Numerical couplings come later.

17 Disjoint contours and additive spectral calculus

The previous sections treated a single spectral island. A matter architecture, however, requires several sectors to coexist without being imposed by hand. The correct mathematical statement is not that the operator has many convenient blocks, but that its resolvent admits several disjoint contours, each of which defines a Riesz projector, and that the algebraic relations among those projectors follow from contour calculus. This section records the exact identities needed later when rank-three and rank-two windows are studied together.

Theorem 17.1 (Additivity for disjoint spectral islands)**.** *Let Γ_1 and Γ_2 be disjoint positively oriented contours in $\rho(K)$, enclosing disjoint isolated spectral subsets Σ_1 and Σ_2. Let Γ be a contour homologous in $\rho(K)$ to the union $\Gamma_1 \cup \Gamma_2$. Then*

$$P_\Gamma(K) = P_{\Gamma_1}(K) + P_{\Gamma_2}(K), \tag{17.1}$$

and

$$P_{\Gamma_1}(K)P_{\Gamma_2}(K) = P_{\Gamma_2}(K)P_{\Gamma_1}(K) = 0. \tag{17.2}$$

If the two ranks are finite, then

$$\operatorname{rank} P_\Gamma = \operatorname{rank} P_{\Gamma_1} + \operatorname{rank} P_{\Gamma_2}. \tag{17.3}$$

Proof. The additivity statement follows immediately from Cauchy's theorem applied to the resolvent, because the contour integral over a contour homologous to the disjoint union equals the sum of the two contour integrals. Orthogonality is slightly more informative. Choose nested representatives so that the integration variable for one contour lies outside the other. Using the resolvent identity,

$$R(z;K)R(w;K) = \frac{R(z;K) - R(w;K)}{w - z}, \tag{17.4}$$

we compute the product of the two projectors as a double contour integral. In the term containing $R(w;K)$, the integral over the other contour has no pole; in the term containing $R(z;K)$, the corresponding integral also has no pole because the contours enclose disjoint spectral sets. Both contributions vanish. Therefore the projectors multiply to zero in either order. The rank identity follows from the directness of the sum of ranges. □

The theorem is a useful safeguard. If two proposed matter windows overlap spectrally, the sum of their ranks is not a physical decomposition. If the contours are disjoint but the corresponding projected sectors are later made to mix by a noncommuting additional structure, the mixing must be introduced as new algebraic information rather than hidden inside the spectral definition.

18 Self-adjoint readout and reality of the reduced kernel

Many physically relevant spectral problems are self-adjoint after the correct inner product has been selected by the quotient and readout construction. The present paper does not assume self-adjointness globally, because intermediate affine operators may be non-normal. Nevertheless, when self-adjointness is available, it imposes strong reality and symmetry constraints on the Riesz sectors and on the Feshbach kernel.

Proposition 18.1 (Self-adjoint Riesz projectors)**.** *Assume that $K = K^*$ on $\mathcal{H}$ and that Γ encloses a compact isolated subset of the real spectrum while avoiding the rest of the spectrum. Then $P_\Gamma(K)$ is the orthogonal spectral projection onto the corresponding spectral subspace. In particular,*

$$P_\Gamma(K)^* = P_\Gamma(K). \tag{18.1}$$

Proof. For self-adjoint K, the spectral theorem gives $K = \int_{\mathbb{R}} \lambda \, dE(\lambda)$. For $z \notin \mathbb{R}$, the resolvent is $R(z; K) = \int (z - \lambda)^{-1} dE(\lambda)$, and the same formula extends to contours avoiding the real spectrum except through the enclosed interval. Integrating the scalar kernel $(z - \lambda)^{-1}$ around Γ gives one if λ lies in the enclosed spectral island and zero otherwise. Therefore the contour integral equals $E(\Sigma)$, which is orthogonal and self-adjoint. □

Proposition 18.2 (Reality of the self-adjoint Feshbach kernel)**.** *Let $K = K^*$, let $P = P^*$, and let $Q = I - P$. For real z such that $D(z) = Q(K - zI)Q$ is invertible, the Feshbach kernel*

$$\mathcal{K}_P(z) = A(z) - BD(z)^{-1}C \tag{18.2}$$

is self-adjoint on $P\mathcal{H}$. Consequently its finite-dimensional eigenvalues are real and its pole equation has real zeros or real avoided crossings within the admissible real domain.

Proof. Since $K = K^*$ and $P = P^*$, the blocks satisfy $A(z)^* = A(z)$, $D(z)^* = D(z)$, and $C = B^*$ for real z. If $D(z)$ is invertible and self-adjoint, then $D(z)^{-1}$ is self-adjoint. Hence

$$(BD(z)^{-1}C)^* = C^* D(z)^{-1} B^* = BD(z)^{-1}C. \tag{18.3}$$

Therefore $\mathcal{K}_P(z)^* = \mathcal{K}_P(z)$. □

This result is physically important because a finite spectral sector is not enough. If the readout is intended to support stable matter-like excitations, the reduced kernel should have a reality or symmetry property compatible with the physical inner product. Otherwise the sector may encode damping, growth, leakage, or a non-Hermitian effective channel rather than a stable matter carrier.

19 Analytic Fredholm gate

In infinite-dimensional settings the determinant $\det \mathcal{K}_P(z)$ is automatically meaningful only when $P\mathcal{H}$ is finite-dimensional. For complementary sectors and for later continuum corrections, one needs an analytic Fredholm gate. The full machinery is standard, but the implication for the present program is simple: poles and finite-dimensional spectral data are controlled only when the operator-valued functions involved are analytic Fredholm families in the relevant domain.

Definition 19.1 (Analytic Fredholm admissibility)**.** A Feshbach kernel $\mathcal{K}_P(z)$ is analytically admissible in a domain $U \subset \mathbb{C}$ if it is analytic as an operator-valued function on U except for isolated poles inherited from $D(z)^{-1}$, and if any infinite-dimensional correction entering the retained sector is compact or trace-class in the sense required to define a Fredholm determinant or an equivalent zero-pole counting function.

Proposition 19.2 (Finite-rank automatic admissibility)**.** *If P has finite rank and $D(z)^{-1}$ exists and is analytic on U, then $\mathcal{K}_P(z)$ is a finite-dimensional analytic matrix function on U. Its zeros are isolated unless* $\det \mathcal{K}_P$ *vanishes identically.*

Proof. All entries of $A(z)$ are affine in z, and the entries of $BD(z)^{-1}C$ are analytic because they are obtained by composing bounded maps with an analytic inverse. Therefore $\mathcal{K}_P(z)$ is an analytic matrix function. The determinant is an analytic scalar function. A nonzero analytic scalar function has isolated zeros, while if it vanishes identically the gate fails because the pole equation does not define isolated spectral points. □

The final alternative in the proposition is not merely technical. If $\det \mathcal{K}_P(z) \equiv 0$, the proposed sector contains an identically flat or constrained direction. Such a direction may be gauge, redundant, or protected by an additional symmetry, but it cannot be counted as an ordinary isolated matter pole without further analysis.

20 Explicit rank-three and rank-two block calculation

We now give a fully explicit finite-dimensional calculation that realizes the two rank gates without assigning particle names. Let

$$K_0 = \operatorname{diag}(a_1, a_2, a_3, b_1, b_2, c_1, c_2, c_3, c_4) \tag{20.1}$$

with

$$a_i \in I_C, \qquad b_j \in I_W, \qquad c_k \notin I_C \cup I_W, \tag{20.2}$$

where I_C and I_W are disjoint compact intervals. Choose contours Γ_C and Γ_W enclosing I_C and I_W, respectively, and no other eigenvalues. Then contour integration gives

$$P_C = \operatorname{diag}(1,1,1,0,0,0,0,0,0), \qquad P_W = \operatorname{diag}(0,0,0,1,1,0,0,0,0). \tag{20.3}$$

Thus $\operatorname{rank} P_C = 3$, $\operatorname{rank} P_W = 2$, and $P_C P_W = 0$. Let E be any perturbation with $||E|| < g$, where g is the minimum distance from the two contours to the remaining spectrum. The stability theorem implies that both Riesz projectors persist with the same rank.

The nontrivial part is the effective kernel. Suppose the retained rank-three block couples to the residual block through matrices $B_C \in \mathbb{C}^{3\times 4}$ and $C_C \in \mathbb{C}^{4\times 3}$, with residual diagonal block $D_C(z) = \operatorname{diag}(c_1 - z, \ldots, c_4 - z)$. Then the exact rank-three kernel is

$$\mathcal{K}_C(z) = \operatorname{diag}(a_1 - z, a_2 - z, a_3 - z) - B_C D_C(z)^{-1} C_C. \tag{20.4}$$

Similarly, if the rank-two block couples to its residual complement by B_W and C_W, then

$$\mathcal{K}_W(z) = \operatorname{diag}(b_1 - z, b_2 - z) - B_W D_W(z)^{-1} C_W. \tag{20.5}$$

These two formulas are the only legitimate starting point for later spectral pole calculations. Replacing them by $\operatorname{diag}(a_i - z)$ or $\operatorname{diag}(b_j - z)$ is an additional approximation and must be justified by a decoupling theorem or by a controlled small parameter.

This calculation is deliberately abstract, but it contains the essential physical lesson. The numbers three and two are not inserted as particle labels. They are dimensions of Riesz ranges. The pole structure of those ranges is not their unperturbed diagonal spectrum but the zero set of their Feshbach kernels. If a future derivation claims three color-like channels or two weak-like channels, it must identify which part of the above structure has become a representation-theoretic normal form.

21 Residues, spectral weights, and physical normalization

A pole equation tells us where a retained spectral excitation can occur, but it does not yet specify its normalization or spectral weight. In a finite-dimensional analytic kernel, a simple pole or zero has a residue that determines the strength of the associated mode in the projected resolvent. This is the precursor to later normalization constants, not a substitute for them.

Suppose $F(z)$ is a scalar reduced kernel with a simple zero at z_0. Near z_0,

$$F(z) = F'(z_0)(z - z_0) + O((z - z_0)^2). \tag{21.1}$$

The reduced resolvent has leading behavior

$$F(z)^{-1} = \frac{1}{F'(z_0)} \frac{1}{z - z_0} + O(1). \tag{21.2}$$

Thus the residue is $1/F'(z_0)$. In a matrix kernel, if $F(z_0)u = 0$ and $v^* F(z_0) = 0$ with $v^* F'(z_0) u \neq 0$, the rank-one pole contribution is

$$F(z)^{-1} = \frac{u v^*}{(z - z_0) v^* F'(z_0) u} + O(1). \tag{21.3}$$

This expression is basis covariant and supplies the normalization gate for a simple isolated mode.

The formula also warns against a common mistake. An eigenvalue location alone is not a physical normalization. The derivative of the kernel matters because the eliminated sector contributes energy-dependent return terms. A later mass or coupling calculation that ignores $F'(z_0)$ may get the pole location but not the correct residue or normalization.

22 No-particle-naming theorem

The construction so far allows us to state a negative theorem that will guide the rest of the series. It is not a deep theorem mathematically, but it is important scientifically because it prevents a premature reading of the spectral gates.

Theorem 22.1 (No particle naming before spectral closure)**.** *Let P be a finite-dimensional subspace selected by hand inside $\mathcal{H}$. If P is not proved to be a physically descending Riesz projector, or equivalent to one by representation-independent data, then no particle interpretation assigned to P is derivationally valid within the present framework. If P is a Riesz projector but fails Feshbach admissibility, rank stability, or kernel closure, then its particle interpretation remains at most heuristic.*

Proof. A particle interpretation in this framework requires a physical sector. A physical sector must be independent of presentation, hence it must descend to the quotient. It must be invariant under the parent readout operator, hence it must be spectral or proved equivalent to a spectral range. It must have finite rank if it is to support a finite multiplet interpretation. It must be stable under admissible perturbations, otherwise the dimension of the supposed multiplet is accidental. Finally, it must possess an exact reduced kernel, otherwise the spectral poles attributed to it are not those of the parent operator. If any of these conditions is absent, the interpretation uses information not derived by the framework. Therefore it may be a useful guess, but it is not a derivation. □

This theorem is the reason the present paper contains no electron, neutrino, or quark spectrum. The absence is not a weakness of the paper; it is the discipline that makes a later spectrum calculation meaningful.

23 Perturbative pole shift from the exact kernel

A useful check on any spectral matter calculation is whether the leading pole shifts obtained from the exact Feshbach kernel agree with ordinary perturbation theory when a small coupling parameter exists. The agreement is not assumed; it follows from expanding the Schur return term. This provides a concrete computational bridge between the abstract kernel formalism and the numerical calculations that later papers will need.

Consider a one-dimensional retained sector coupled to an eliminated sector. Let

$$K_\epsilon = \begin{pmatrix} \lambda & \epsilon b^* \\ \epsilon b & D_0 \end{pmatrix}, \tag{23.1}$$

where D_0 is self-adjoint and $\lambda \notin \operatorname{spec}(D_0)$. The Feshbach kernel is the scalar function

$$F_\epsilon(z) = \lambda - z - \epsilon^2 b^*(D_0 - z)^{-1} b. \tag{23.2}$$

A pole z_ϵ near λ satisfies $F_\epsilon(z_\epsilon) = 0$. Writing

$$z_\epsilon = \lambda + \epsilon^2 z_2 + O(\epsilon^4), \tag{23.3}$$

and expanding $(D_0 - z_\epsilon)^{-1}$ around λ, one obtains

$$z_2 = -b^*(D_0 - \lambda)^{-1} b. \tag{23.4}$$

Hence

$$z_\epsilon = \lambda - \epsilon^2 b^*(D_0 - \lambda)^{-1} b + O(\epsilon^4). \tag{23.5}$$

This is the standard second-order shift, but it has been obtained here from the exact reduced kernel rather than by truncating the parent operator. The sign and magnitude are controlled by the placement of the eliminated spectrum relative to the retained pole.

Proposition 23.1 (Second-order Feshbach shift)**.** *Let K_ϵ be the block operator above and assume λ is separated from* $\operatorname{spec}(D_0)$ *by a gap $g > 0$. For sufficiently small ϵ, there exists a unique pole z_ϵ near λ and it has the expansion*

$$z_\epsilon = \lambda - \epsilon^2 b^*(D_0 - \lambda)^{-1} b + O(\epsilon^4), \tag{23.6}$$

where the remainder is bounded by a constant depending on g and $||b||$.

Proof. The reduced scalar kernel is analytic for $|z - \lambda| < g/2$. At $\epsilon = 0$, $F_0(z) = \lambda - z$ has a simple zero at $z = \lambda$. By the analytic implicit function theorem, a unique zero z_ϵ persists for small ϵ. Substitute the ansatz $z_\epsilon = \lambda + \epsilon^2 z_2 + O(\epsilon^4)$ into $F_\epsilon(z_\epsilon) = 0$. The term $\lambda - z_\epsilon$ contributes $-\epsilon^2 z_2 + O(\epsilon^4)$. The return term contributes $-\epsilon^2 b^*(D_0 - \lambda)^{-1} b + O(\epsilon^4)$, since the derivative of the resolvent produces an additional factor of $z_\epsilon - \lambda = O(\epsilon^2)$. Equating the coefficient of ϵ^2 gives the formula for z_2. □

This proposition is a template for later true calculations. Whenever a proposed mass or spectral pole is quoted, one must specify whether it is an uncorrected retained-block eigenvalue, an exact zero of the Feshbach kernel, or an expansion of that zero. These are different statements, and confusing them would invalidate the derivation.

24 Schur complement versus physical truncation

A recurring source of error in finite-dimensional model building is the replacement of a Schur complement by a truncation. A truncation keeps $A(z)$ and discards $BD^{-1}C$. A Schur reduction keeps $A(z) - BD^{-1}C$. The two coincide only under strict conditions. The following theorem makes that condition explicit.

Theorem 24.1 (Exact decoupling criterion). *Let $K - zI$ have block form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with D invertible. The Feshbach kernel equals the truncated retained block, $F_P(z) = A(z)$, if and only if*

$$BD(z)^{-1}C = 0. \tag{24.1}$$

In particular, if $D(z)^{-1}$ is injective and B has trivial right annihilator on $\operatorname{Ran} D^{-1}C$, *then exact truncation requires $C = 0$. If $C = B^*$ in a self-adjoint problem, exact truncation follows from $B = 0$.*

Proof. The first statement is the definition of F_P. If D^{-1} is injective and B has no nonzero vector in its null space on the range of $D^{-1}C$, the condition $BD^{-1}C = 0$ forces $D^{-1}C = 0$, hence $C = 0$. In the self-adjoint case $C = B^*$. If $B = 0$, the return term vanishes. Conversely, if exact truncation is required for all nearby real z and $D(z)^{-1}$ is positive or negative definite on an interval, then $BD^{-1}B^* = 0$ implies $B = 0$ by positivity of the quadratic form. □

The theorem is included because it is a practical audit rule. If a later calculation uses a small retained block as though it were closed, the paper must either prove a symmetry that kills the return term or display a small parameter controlling it. Otherwise the calculation is not a derivation of the parent spectrum.

25 Angles between spectral sectors

Finite-rank projectors may remain stable while their ranges rotate. If a later physical interpretation depends not only on the rank but also on the alignment of two sectors, one must control the angle between their ranges. Let P and P' be finite-rank projections of the same rank. When $||P - P'|| < 1$, their ranges are isomorphic and the gap metric

$$\delta(P, P') = ||P - P'|| \tag{25.1}$$

controls the maximal angle between the subspaces. In the self-adjoint case this is the usual operator-norm gap between orthogonal projections.

Proposition 25.1 (Subspace angle control). *Let P and P' be orthogonal projections with $||P - P'|| = \delta < 1$. Then the restriction $P'|_{\operatorname{Ran} P} : \operatorname{Ran} P \to \operatorname{Ran} P'$ is injective, and the principal angles θ_j between the two ranges satisfy*

$$\sin \theta_{\max} \leq \delta. \tag{25.2}$$

Proof. If $x \in \operatorname{Ran} P$ and $P'x = 0$, then $P - P')x = x$, hence $||x|| = ||(P - P')x|| \leq \delta ||x||$. Since $\delta < 1$, this forces $x = 0$, so the restriction is injective. For orthogonal projections, the operator norm $||P - P'||$ equals the sine of the largest principal angle between the ranges. This standard identity follows by reducing the pair of projections to Halmos two-subspace normal form; the stated inequality is the part needed here. □

The relevance is that a stable rank does not automatically mean a stable physical orientation. If the later gauge or Yukawa layer depends on overlaps among spectral ranges, the gap metric of projectors becomes a calculable input rather than a free mixing angle.

26 Locality and compactness gates for matter kernels

The Feshbach kernel can be exact and still physically problematic. Eliminating a complement generally produces nonlocal or energy-dependent terms. In a field-theoretic setting, later interpretation requires a locality or controlled nonlocality gate. At the abstract level this can be stated without committing to a particular spacetime model.

A finite-dimensional kernel $\mathcal{K}_P(z)$ is acceptable as a matter kernel if its dependence on z and on the readout variables admits either an exact local normal form or a convergent expansion whose discarded terms are bounded in the domain of interest. If the return term $BD^{-1}C$ contains uncontrolled branch cuts, dense poles, or noncompact continuum feedback, then the finite-rank sector may exist but may not define a local matter carrier. This is not a failure of spectral theory; it is a failure of the stronger physical gate.

Diagnostic 26.1 (Kernel locality diagnostic)**.** A finite-rank spectral sector passes the locality diagnostic in a working readout domain if the exact Feshbach return term can be represented there by a finite local normal form, or by a convergent nonlocal expansion with explicit remainder bounds. It fails the diagnostic if its spectral dependence is uncontrolled in the domain in which the sector is to be interpreted physically.

This diagnostic is placed at the spectral layer because gauge normal forms should not be built on sectors whose kernels are not physically controlled. The logical order is strict: first spectral isolation, then kernel closure, then locality or controlled nonlocality, and only after that representation theory.

27 What would count as a completed spectral matter derivation

The paper has intentionally avoided claiming that a full matter spectrum has been derived. It is nevertheless useful to state what would count as completion of this layer. A completed spectral matter derivation, including any vorton sector, would provide a parent physical operator K obtained from the previous readout papers; a collection of contours $\Gamma_a \subset \rho(K)$; proof that the corresponding Riesz projectors descend to the physical quotient; finite-rank calculations for each projector; perturbation margins showing rank stability; exact Feshbach kernels for each sector; pole equations with residue normalization; and locality or controlled nonlocality diagnostics. Only after this list is complete may one attach physical labels; only after the additional circulation and compactness gates are complete may one attach the technical word vorton.

In particular, a later claim that rank $P_C = 3$ must contain an actual contour or an equivalent spectral construction. A later claim that rank $P_W = 2$ must do the same. A later claim about three generations must identify whether the number three is a Riesz rank, a refinement multiplicity, an index of a family of contours, or a degeneracy of a kernel pole. These are different mathematical statements. The present paper supplies the language required not to confuse them.

28 Conclusion

We have constructed the spectral matter layer required by a premetric affine-substrate program before any particle labels are introduced. The central mathematical object is the finite-rank Riesz projector associated with an isolated spectral island of the physical readout operator. Such a projector defines a matter candidate only if it descends to the physical quotient, remains stable under admissible perturbations, and admits an exact Feshbach kernel after complementary modes are eliminated. The reduced kernel is not an approximation: it is an isospectral Schur complement on its domain of definition, and its determinant supplies the pole equation for the retained sector.

The paper also clarified the status of rank-three and rank-two statements. They are legitimate spectral gates, not yet gauge or particle identifications. A later paper may attempt to show that such sectors carry gauge normal forms and effective couplings, but that attempt must rest on the spectral architecture developed here. In this sense the present work is deliberately prior to phenomenology. It is the mathematical checkpoint that decides whether matter-like finite sectors exist at all.

A Resolvent identities used in the Riesz calculus

For completeness we record the basic resolvent identities. If $z, w \in \rho(K)$, then

$$R(z;K) - R(w;K) = (w - z)R(z;K)R(w;K). \tag{A.1}$$

This follows by subtracting the identities $(zI - K)R(z;K) = I$ and $(wI - K)R(w;K) = I$, or directly by multiplying

$$R(z;K)[(wI - K) - (zI - K)]R(w;K). \tag{A.2}$$

The perturbative identity used in the stability theorem is

$$R(z;K+E) - R(z;K) = R(z;K+E)ER(z;K), \tag{A.3}$$

which follows from the same calculation applied to $K + E$ and K. If $\|ER(z; K)\| < 1$, then

$$R(z; K + E) = R(z; K)[I - ER(z; K)]^{-1} \tag{A.4}$$

with the inverse given by the Neumann series.

B Finite-dimensional determinant factorization

Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{B.1}$$

with D invertible. The Schur complement is $F = A - BD^{-1}C$. The factorization

$$M = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \tag{B.2}$$

implies

$$\det M = \det F \, \det D. \tag{B.3}$$

This identity is the finite-dimensional form of the Feshbach isospectral theorem. It also makes clear why the eliminated block cannot be ignored. Its determinant carries the eliminated-sector poles, while the Schur complement carries the retained-sector poles after coupling to the eliminated sector.

C Quantitative projector stability

The bound in Theorem 8.2 can be sharpened in several directions, but the simple form used in the paper is sufficient for gate auditing. Let Γ be a circle of radius r around a finite spectral island and let the distance from Γ to the rest of the spectrum be g. If $\|E\| \leq \eta g$ with $0 < \eta < 1$, then

$$\|P_\Gamma(K + E) - P_\Gamma(K)\| \leq \frac{L(\Gamma)}{2\pi g} \frac{\eta}{1 - \eta}. \tag{C.1}$$

Thus a conservative audit may require $\eta \leq 1/2$, giving

$$\|P_\Gamma(K + E) - P_\Gamma(K)\| \leq \frac{L(\Gamma)}{2\pi g}. \tag{C.2}$$

This is not a universal smallness guarantee unless the geometric factor is also controlled, but it gives an explicit threshold for checking whether the chosen contour is safely separated.

D Semisimplicity and generalized sectors

A Riesz projector isolates generalized eigenspaces, not necessarily diagonal eigenspaces. If the restriction of K to $\operatorname{Ran} P_\Gamma$ contains nilpotent Jordan parts, then the sector is finite-dimensional but not semisimple. In physical applications this distinction matters. A semisimple sector can be diagonalized within the Riesz range, while a nonsemisimple sector carries polynomial time or clock dependence in the associated evolution. Therefore a later particle interpretation must add a semisimplicity gate:

$$K|_{\operatorname{Ran} P_\Gamma} = K_{\text{diag}} \quad \text{or equivalently} \quad N_\Gamma = 0 \tag{D.1}$$

where N_Γ is the nilpotent part in the Jordan-Chevalley decomposition of the finite-dimensional restriction. The present paper does not assume semisimplicity; it records it as an additional gate.

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