Metric-Affine Decomposition and the Nonmetric Trace Mode

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Abstract

We continue the premetric-to-readout construction by studying the first genuinely metric-affine object that becomes meaningful after a rank-four Lorentzian readout has been licensed. The previous layer establishes a conditional four-dimensional Lorentzian response sector. The present paper assumes only that output and then performs the affine calculation that follows. An independent affine connection decomposes into the Levi-Civita connection of the readout metric plus a tensorial distortion; the distortion decomposes into torsion and nonmetricity; and the pure trace part of nonmetricity defines a covector mode W_μ only if it descends through the physical readout. We give explicit sign conventions, prove tensoriality and extraction statements, derive the pure trace connection, compute length and volume transport, evaluate the trace curvature, compute both the Ricci-level and scalar curvature formulas, and prove that the Einstein-Hilbert scalar of a pure nonmetric trace connection does not by itself generate a Maxwell kinetic term. In dimension four it contributes only a boundary term and an algebraic W^2 term. A propagating trace mode therefore requires a trace-curvature-square term, an equivalent reduced Hessian principal symbol, or an explicitly specified source sector. Under such a kinetic gate we derive the Maxwell-Proca equations, their constraint structure, stress tensor, and flat-readout degrees of freedom. The paper is deliberately conditional: it does not identify W_μ with a known particle, interaction, or cosmological component. It supplies the exact metric-affine algebra, variational bookkeeping, descent conditions, stability diagnostics, and falsification gates required before any later physical interpretation of a nonmetric vector mode can be credible.

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1 Introduction

The first three papers in this series are designed to avoid a common failure mode in emergent-spacetime programs. A relational substrate can be described, a physical quotient can be formed, and a clock functional can be constructed without yet having a spacetime metric. Even after a Lorentzian rank-four readout has been licensed, one has not yet obtained matter, gauge fields, or the Einstein equations. What one has is more modest and more precise: a four-dimensional effective readout sector equipped with a Lorentzian symmetric form. The natural next question is therefore not the particle spectrum, and not cosmology, but the status of the affine connection that acts on the readout sector.

This paper addresses that question. Once a Lorentzian readout metric g exists on a four-dimensional sector, an independent affine connection Γ can be compared with the Levi-Civita connection of g. Their difference is a tensor. That tensor is the first mathematically controlled place where non-Riemannian information may live. It contains torsion, nonmetricity, and trace components that are invisible in pure pseudo-Riemannian geometry. Among these components, the trace of nonmetricity is especially important because it is represented by a covector and because its curvature is an exterior derivative. In a readout language this covector is the most economical candidate for a genuine nonmetric vector mode.

The paper has a narrow purpose. It does not claim that the nonmetric trace is automatically present, automatically propagating, or automatically identified with a known interaction. It proves the metric-affine bookkeeping required to make those later questions meaningful. The calculations are standard enough to be checkable, but they are written here with explicit conventions because sign choices in metric-affine geometry easily create spurious conclusions. We use the convention

$$Q_{\lambda\mu\nu} := -\nabla^{\Gamma}_{\lambda} g_{\mu\nu}, \tag{1}$$

so a pure trace nonmetricity field satisfies $Q_{\lambda\mu\nu} = 2W_{\lambda} g_{\mu\nu}$. With this convention the corresponding torsion-free connection is

$$\Gamma^{\lambda}{}_{\mu\nu} = \{^{\lambda}{}_{\mu\nu}\} + \delta^{\lambda}_{\mu} W_{\nu} + \delta^{\lambda}_{\nu} W_{\mu} - g_{\mu\nu} W^{\lambda}. \tag{2}$$

This formula is not a definition smuggled in for convenience; it is derived below from the nonmetricity equation. It is the central local expression around which the rest of the paper is organized.

The most important physical calculation is the curvature calculation. For the pure trace nonmetric connection in dimension n one obtains

$$R(\Gamma) = R(g) - 2(n-1)\nabla^{\{\}}_{\mu} W^{\mu} - (n-1)(n-2) W_{\mu} W^{\mu}. \tag{3}$$

In dimension four this becomes

$$R(\Gamma) = R(g) - 6\nabla^{\{\}}_{\mu} W^{\mu} - 6 W_{\mu} W^{\mu}. \tag{4}$$

Thus the Einstein-Hilbert scalar, by itself, does not produce the kinetic term $F_{\mu\nu}F^{\mu\nu}$ for $F = \mathrm{d}W$. Up to a boundary term it gives an algebraic contribution proportional to W^2. A propagating nonmetric trace mode therefore requires an additional curvature-square term, an effective Hessian contribution, or a source sector. This point is crucial. It prevents the theory from pretending that a vector field has been dynamically derived merely because a covector appears in the connection.

The second important calculation concerns the trace curvature. For the same pure trace connection one finds

$$R^{\lambda}{}_{\lambda\mu\nu}(\Gamma) = n(\partial_{\mu} W_{\nu} - \partial_{\nu} W_{\mu}) = nF_{\mu\nu}. \tag{5}$$

Consequently a curvature-square term built from the trace curvature produces a genuine Maxwell-type kinetic term. If an additional quadratic response supplies an algebraic term $m_W^2 W^2$, then the resulting field equation is of Maxwell-Proca type. This paper derives that equation explicitly

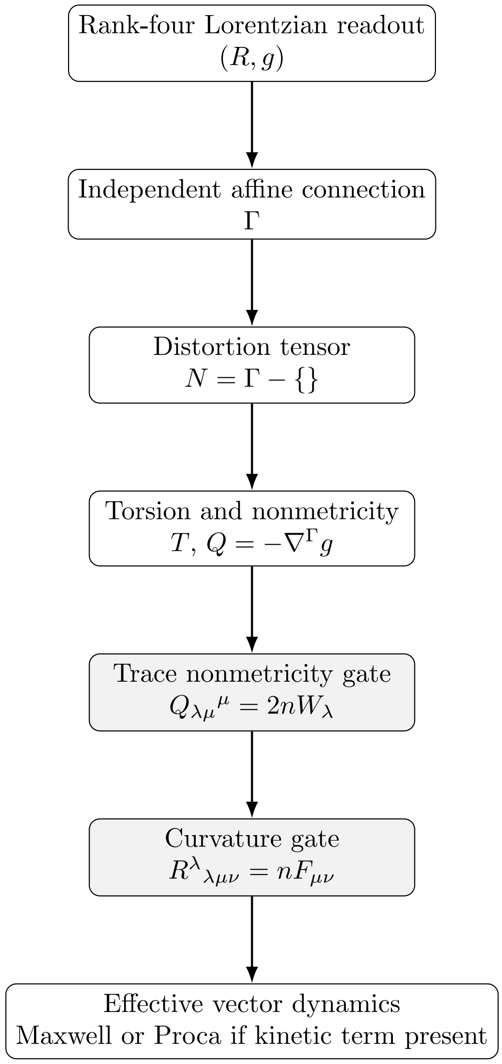

Figure 1: Logical position of the nonmetric trace mode. The mode W_μ is not inserted before the Lorentzian readout exists. It is extracted from the trace of nonmetricity after an independent affine connection is compared with the Levi-Civita connection of the readout metric.

and records the associated constraint and flat-readout dispersion law. These are elementary calculations, but they are indispensable because they separate three distinct claims: the existence of a trace mode, the existence of its curvature, and the existence of its propagation.

The literature background is broad. Weyl geometry introduced the idea that length may change under parallel transport [1]. Metric-affine gravity treats metric and connection as independent fields and decomposes the connection into torsion and nonmetricity [3]. Modern reviews of generalized geometric formulations emphasize how curvature, torsion, and nonmetricity encode distinct gravitational data [4]. The present paper does not try to replace those frameworks. It uses their local mathematics as a controlled continuation of the relational readout program. The distinctive requirement here is that every affine object must be justified as a descendant of the previously licensed readout, not as an independent spacetime primitive.

2 Mathematical contract and inherited assumptions

We start from a four-dimensional Lorentzian readout sector produced by the previous layer. Locally this means that there is a four-dimensional real vector space, or a smooth readout chart after localization, equipped with a nondegenerate symmetric bilinear form of inertia $(1, 0, 3)$. In this paper we write the local readout as a Lorentzian manifold (M, g) only after this gate has been passed. The notation does not mean that a smooth spacetime manifold was assumed at the substrate level. It means that the rank-four Lorentzian readout has been promoted to a local differentiable model for the purpose of affine calculation.

Assumption 2.1 (Readout localization)**.** *On the domain considered in this paper the rank-four Lorentzian readout admits a smooth local model (M, g) of dimension n, with $n = 4$ in the physical case. The metric g has Lorentzian inertia $(1, 0, n - 1)$, and all tensor operations below are performed on this local readout. No claim is made that this local model is globally complete or fundamental at the substrate level.*

Assumption 2.2 (Independent affine response)**.** *The readout admits an affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ not assumed to equal the Levi-Civita connection of g. The connection is required to descend through the same physical equivalence relation that licensed the readout. If it fails to descend, the affine mode is presentation-dependent and is not physical in the sense of this paper.*

Definition 2.3 (Levi-Civita connection and distortion)**.** *Let $\{^{\lambda}{}_{\mu\nu}\}$ denote the unique torsion-free metric-compatible connection of g. The distortion tensor of an affine connection Γ is*

$$N^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \{^{\lambda}{}_{\mu\nu}\}. \tag{6}$$

Since the difference of two affine connections is tensorial, N is a $(1, 2)$ tensor on the readout domain.

Definition 2.4 (Torsion and nonmetricity)**.** *The torsion of Γ is*

$$T^{\lambda}{}_{\mu\nu} := 2\Gamma^{\lambda}{}_{[\mu\nu]} = 2N^{\lambda}{}_{[\mu\nu]}, \tag{7}$$

and the nonmetricity tensor is defined in this paper by

$$Q_{\lambda\mu\nu} := -\nabla^{\Gamma}_{\lambda} g_{\mu\nu}. \tag{8}$$

With this sign convention, positive pure trace nonmetricity is written $Q_{\lambda\mu\nu} = 2W_{\lambda} g_{\mu\nu}$.

The following theorem records the basic decomposition used throughout the paper. It is included explicitly because different sign conventions in the literature change the displayed formula. The convention here is fixed by $Q = -\nabla^{\Gamma} g$.

Theorem 2.5 (Distortion decomposition)**.** *Let (M, g) be a readout metric and let Γ be an affine connection with distortion $N = \Gamma - \{\}$. With torsion $T^{\lambda}{}_{\mu\nu} = 2N^{\lambda}{}_{[\mu\nu]}$ and nonmetricity $Q_{\lambda\mu\nu} = -\nabla^{\Gamma}_{\lambda} g_{\mu\nu}$, the distortion decomposes as*

$$N^{\lambda}{}_{\mu\nu} = K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}, \tag{9}$$

where

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu}\right) \tag{10}$$

is the contortion contribution and

$$L^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(Q_{\mu\nu\rho} + Q_{\nu\mu\rho} - Q_{\rho\mu\nu} \right) \tag{11}$$

is the disformation contribution in the sign convention $Q = -\nabla^{\Gamma} g$.

Proof. Lower the first index of the distortion by $N_{\alpha\mu\nu} := g_{\alpha\lambda} N^{\lambda}{}_{\mu\nu}$. Since the Levi-Civita connection is metric compatible, the nonmetricity convention gives

$$Q_{\lambda\mu\nu} = N_{\nu\lambda\mu} + N_{\mu\lambda\nu}. \tag{12}$$

The torsion equation gives

$$T_{\alpha\mu\nu} = N_{\alpha\mu\nu} - N_{\alpha\nu\mu}. \tag{13}$$

The metric-compatible part of N is determined by the antisymmetric equations in the last two slots and yields the displayed expression for K; substituting K into the torsion equation verifies $2K^{\lambda}{}_{[\mu\nu]} = T^{\lambda}{}_{\mu\nu}$, while substituting it into $K_{\nu\lambda\mu} + K_{\mu\lambda\nu}$ gives zero. The remaining symmetric-in-metricity part must therefore solve the three linear equations obtained by cyclic permutations of $Q_{\lambda\mu\nu} = L_{\nu\lambda\mu} + L_{\mu\lambda\nu}$. Adding the first two cyclic equations and subtracting the third gives

$$2L_{\alpha\mu\nu} = Q_{\mu\nu\alpha} + Q_{\nu\mu\alpha} - Q_{\alpha\mu\nu}, \tag{14}$$

which raises to the displayed formula for $L^{\lambda}{}_{\mu\nu}$. Since both K and L are determined by linear equations whose sum reproduces the torsion and nonmetricity of N, the decomposition is unique. □

3 Conventions, sign checks, and tensoriality of distortion

The rest of the paper depends on a small number of sign and index conventions. Because the nonmetric trace mode is a one-form extracted from a connection, a sign error can turn an auxiliary mode into an apparently propagating one, or can reverse the interpretation of length transport. We therefore record the convention audit before any physical claim is made. Throughout, the readout metric has Lorentzian signature fixed by the previous paper. The connection $\nabla^{\{\}}$ denotes the Levi-Civita connection of g, while ∇^{Γ} denotes an independent affine connection acting on the same readout bundle. The distortion tensor is defined by

$$N^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \{^{\lambda}{}_{\mu\nu}\}. \tag{15}$$

The sign convention for nonmetricity is

$$Q_{\lambda\mu\nu} := -\nabla^{\Gamma}_{\lambda} g_{\mu\nu}. \tag{16}$$

The pure trace branch studied in this paper is the branch in which

$$Q_{\lambda\mu\nu} = 2W_{\lambda} g_{\mu\nu}. \tag{17}$$

This choice fixes all factors of two. In particular, the contraction convention is

$$Q_{\lambda} := Q_{\lambda\mu}{}^{\mu} = 2nW_{\lambda}, \tag{18}$$

so that $W_\lambda = Q_\lambda/(2n)$ in dimension n. No independent normalization of W is introduced later. If a later physical paper rescales W, it must also rescale the kinetic coefficient and the source coupling. Otherwise it is changing the theory rather than changing notation.

Lemma 3.1 (Distortion is a tensor)**.** *Let Γ and $\{\}$ be two affine connections on the same readout tangent bundle. Their difference $N^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \{^\lambda{}_{\mu\nu}\}$ transforms tensorially under readout coordinate changes.*

Proof. Under a coordinate change, each connection coefficient transforms with a homogeneous tensorial term plus the same inhomogeneous second-derivative term generated by the change of coordinates. Subtracting the two transformation laws cancels the inhomogeneous part. The remaining transformation law is exactly the $(1,2)$ tensor law. This is the reason the comparison between an independent affine connection and the Levi-Civita connection is meaningful only after a readout metric has been licensed. Before the readout metric exists there is no Levi-Civita reference connection and therefore no tensorial distortion of this kind. □

Lemma 3.2 (Pure trace sign check)**.** *Suppose N is torsion-free and has the form*

$$N^\lambda{}_{\mu\nu} = \delta^\lambda_\mu W_\nu + \delta^\lambda_\nu W_\mu - g_{\mu\nu} W^\lambda. \tag{19}$$

Then $-\nabla^\Gamma_\lambda g_{\mu\nu} = 2W_\lambda g_{\mu\nu}$.

Proof. Because $\nabla^{\{\}} g = 0$, the only contribution to $\nabla^\Gamma_\lambda g_{\mu\nu}$ comes from the distortion acting on the two covariant indices:

$$\nabla^\Gamma_\lambda g_{\mu\nu} = -N^\rho{}_{\lambda\mu} g_{\rho\nu} - N^\rho{}_{\lambda\nu} g_{\mu\rho}. \tag{20}$$

Using (19), the first term is

$$N^\rho{}_{\lambda\mu} g_{\rho\nu} = (\delta^\rho_\lambda W_\mu + \delta^\rho_\mu W_\lambda - g_{\lambda\mu} W^\rho) g_{\rho\nu} \tag{21}$$

$$= g_{\lambda\nu} W_\mu + g_{\mu\nu} W_\lambda - g_{\lambda\mu} W_\nu, \tag{22}$$

while the second term is

$$N^\rho{}_{\lambda\nu} g_{\mu\rho} = g_{\lambda\mu} W_\nu + g_{\nu\mu} W_\lambda - g_{\lambda\nu} W_\mu. \tag{23}$$

Adding the two expressions cancels the mixed terms and leaves $2W_\lambda g_{\mu\nu}$. Therefore $\nabla^\Gamma_\lambda g_{\mu\nu} = -2W_\lambda g_{\mu\nu}$, which is precisely $Q_{\lambda\mu\nu} = 2W_\lambda g_{\mu\nu}$ with our sign convention. □

Proposition 3.3 (No metric-affine shortcut)**.** *The one-form W is not present in the premetric substrate as a spacetime vector. It is a readout-level object. It becomes meaningful only after three prior structures have been licensed: a rank-four readout sector, a Lorentzian symmetric form on that sector, and an independent affine connection that can be compared with the Levi-Civita connection of that form.*

Proof. The definition of W uses g to raise and contract indices and uses the Levi-Civita connection of g to define the tensorial distortion. Without g there is no trace $Q_{\lambda\mu}{}^\mu$ in the metric sense and no decomposition of Γ into Levi-Civita plus distortion. Thus any attempt to introduce W before the readout layer either assumes a metric structure in advance or defines a different object. The proposition is therefore a bookkeeping statement, but it is a physically important one: it prevents the nonmetric trace mode from being counted as a primitive field of the substrate. □

This section is deliberately elementary. Its purpose is to close the most common loophole in metric-affine interpretations. The trace mode is not a symbol that can be moved freely between layers of the construction. It is a tensorial remnant of a comparison that exists only once a metric readout has already been produced.

4 The nonmetric trace mode

The nonmetricity tensor has several irreducible components. The present paper isolates the trace that is directly represented by a covector. This is the component that can behave as a vector mode after a kinetic term is present. The construction is purely algebraic and therefore does not require equations of motion.

Definition 4.1 (Weyl trace covector)**.** *In dimension n, define the nonmetric trace covector*

$$W_\lambda := \frac{1}{2n} Q_{\lambda\mu}{}^{\mu} = \frac{1}{2n} g^{\mu\nu} Q_{\lambda\mu\nu}. \tag{24}$$

The trace-free remainder with respect to this contraction is

$$\widehat{Q}_{\lambda\mu\nu} := Q_{\lambda\mu\nu} - 2W_\lambda g_{\mu\nu}, \tag{25}$$

so that $g^{\mu\nu}\widehat{Q}_{\lambda\mu\nu} = 0$.

Theorem 4.2 (Pure trace nonmetric connection)**.** *Assume torsion vanishes and nonmetricity is pure trace,*

$$T^\lambda{}_{\mu\nu} = 0, \qquad Q_{\lambda\mu\nu} = 2W_\lambda g_{\mu\nu}. \tag{26}$$

Then the affine connection is

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\} + \delta^\lambda_\mu W_\nu + \delta^\lambda_\nu W_\mu - g_{\mu\nu} W^\lambda. \tag{27}$$

Conversely, any connection of the form (27) *is torsion-free and satisfies $Q_{\lambda\mu\nu} = 2W_\lambda g_{\mu\nu}$.*

Proof. With $T = 0$ the contortion term in theorem 2.5 vanishes. Substituting $Q_{\lambda\mu\nu} = 2W_\lambda g_{\mu\nu}$ into the disformation formula gives

$$L^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(2W_\mu g_{\nu\rho} + 2W_\nu g_{\mu\rho} - 2W_\rho g_{\mu\nu}\right) \tag{28}$$

$$= \delta^\lambda_\nu W_\mu + \delta^\lambda_\mu W_\nu - g_{\mu\nu} W^\lambda. \tag{29}$$

This proves the displayed connection. Conversely, the additional term is symmetric in μ, ν, so torsion vanishes. Using $\nabla^{\{\}} g = 0$ and the definition $Q = -\nabla^\Gamma g$, the distortion contribution gives

$$Q_{\lambda\mu\nu} = N_{\nu\lambda\mu} + N_{\mu\lambda\nu} = 2W_\lambda g_{\mu\nu}, \tag{30}$$

which proves the converse. □

Corollary 4.3 (Trace extraction is exact)**.** *For pure trace nonmetricity one has*

$$Q_{\lambda\mu}{}^{\mu} = 2nW_\lambda. \tag{31}$$

Thus the normalization in the definition of W is not conventional once the pure trace ansatz is fixed.

The status of W_μ in this paper is therefore precise. It is not any arbitrary vector field. It is the exact trace of nonmetricity on the readout connection, normalized so that the pure trace component of Q is $2W_\lambda g_{\mu\nu}$. If the full nonmetricity has trace-free pieces, the paper does not discard them by fiat. It studies the trace mode as a separately auditable sector and records the conditions under which the remaining components are either absent, hidden, or dynamically decoupled.

5 Length transport and integrability

The operational meaning of the pure trace nonmetricity mode is most transparent in parallel transport. If a vector is parallel transported by Γ, its length with respect to g need not be preserved. The failure of length preservation is governed exactly by W.

Theorem 5.1 (Length transport law)**.** *Let $\gamma(s)$ be a curve with tangent $u^\mu = \dot{\gamma}^\mu$, and let $X^\mu(s)$ be parallel transported along γ by the pure trace connection* (27)*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\, g(X,X) = -2W_\mu u^\mu\, g(X,X). \tag{32}$$

Consequently, when $g(X,X)$ is nonzero,

$$g(X,X)(s) = g(X,X)(s_0)\exp\left[-2\int_{s_0}^{s} W_\mu \dot{\gamma}^\mu\, \mathrm{d}s'\right]. \tag{33}$$

Proof. By parallel transport, $\nabla^\Gamma_u X = 0$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}s} g(X,X) = (\nabla^\Gamma_u g)(X,X) + 2g(\nabla^\Gamma_u X, X). \tag{34}$$

The second term vanishes. Since $Q = -\nabla^\Gamma g = 2W \otimes g$, one has $\nabla^\Gamma_u g = -2W(u)g$. Substitution gives the differential equation, and integration gives the exponential formula. □

Corollary 5.2 (Closed-loop length holonomy)**.** *If γ is a closed curve, the length-squared holonomy of a non-null parallel transported vector is*

$$\frac{g(X,X)_{\text{final}}}{g(X,X)_{\text{initial}}} = \exp\left[-2\oint_\gamma W\right]. \tag{35}$$

If $W = \mathrm{d}\phi$ locally, then the holonomy around contractible closed curves vanishes. If $F = \mathrm{d}W$ is nonzero on a spanning surface, length holonomy is controlled by the flux of F through that surface.

This calculation is one of the sharpest diagnostics in the paper. If W is claimed as a physical mode, it must have an operational readout such as length holonomy, trace curvature, or coupling to an appropriate source. If it is pure gauge on the relevant domain, it may be removable by a Weyl rescaling and should not be counted as an independent propagating degree of freedom.

6 Autoparallels, null curves, and volume transport

The connection (27) also determines autoparallels. This calculation is physically useful because it separates light-cone information from scale-transport information. A pure nonmetric trace changes lengths, but it does not change the unparametrized null curves of the readout metric.

Thus the mode is not simply a deformation of the light cone. It is a scale connection on top of the Lorentzian cone already licensed by the previous paper.

Proposition 6.1 (Autoparallel equation)**.** *Let $u^\mu = \dot{x}^\mu$ be the tangent to an affine autoparallel of the pure trace connection. Then*

$$u^\nu \nabla^{\{\}}_\nu u^\lambda + 2(W_\mu u^\mu)u^\lambda - (g_{\mu\nu}u^\mu u^\nu)W^\lambda = 0. \tag{36}$$

Proof. The affine autoparallel equation is $u^\nu \nabla^\Gamma_\nu u^\lambda = 0$. Substituting $\Gamma = \{\} + N$ and $N^\lambda{}_{\mu\nu} = \delta^\lambda_\mu W_\nu + \delta^\lambda_\nu W_\mu - g_{\mu\nu}W^\lambda$ gives

$$\begin{aligned} 0 &= u^\nu \nabla^{\{\}}_\nu u^\lambda + N^\lambda{}_{\mu\nu}u^\mu u^\nu & (37)\\ &= u^\nu \nabla^{\{\}}_\nu u^\lambda + 2(W_\mu u^\mu)u^\lambda - (g_{\mu\nu}u^\mu u^\nu)W^\lambda, & (38) \end{aligned}$$

which is the displayed formula. □

Theorem 6.2 (Null autoparallels are projectively Levi-Civita)**.** *If u is null, $g(u,u) = 0$, then every pure-trace affine autoparallel satisfies*

$$u^\nu \nabla^{\{\}}_\nu u^\lambda = -2(W_\mu u^\mu)u^\lambda. \tag{39}$$

Therefore its unparametrized curve is a Levi-Civita null geodesic. The pure nonmetric trace changes the parametrization but not the null trajectory.

Proof. Set $g(u,u) = 0$ in (36). The remaining Levi-Civita acceleration is proportional to u^λ. A curve whose Levi-Civita acceleration is proportional to its tangent is a geodesic with non-affine parametrization. Reparametrizing the curve removes the proportionality term locally, so the unparametrized trajectory is a Levi-Civita null geodesic. □

Corollary 6.3 (Light-cone stability of the pure trace)**.** *The pure nonmetric trace mode does not open or close the Lorentzian null cone of the readout metric. Its direct geometric effect is scale transport, trace curvature, and non-affine parametrization of null autoparallels, not a change of the cone itself.*

This result is an important guardrail for physical interpretation. If a later model predicts a change in the null cone, the source of that change cannot be the pure Weyl trace alone. It must come from a change in the readout metric, from trace-free nonmetricity entering the effective principal symbol, or from another sector not included in the present pure trace calculation.

Proposition 6.4 (Volume transport)**.** *Let ϵ_g be the metric volume form. For the pure trace connection,*

$$\nabla^\Gamma_\lambda \epsilon_g = -nW_\lambda \epsilon_g. \tag{40}$$

Consequently a volume element transported along a curve with tangent u changes by the factor

$$\exp\left[-n\int W_\mu u^\mu \,\mathrm{d}s\right]. \tag{41}$$

Proof. Under a connection with $\nabla^\Gamma_\lambda g_{\mu\nu} = -2W_\lambda g_{\mu\nu}$, the determinant scales by the trace of the metric variation. Since $g^{\mu\nu}\nabla^\Gamma_\lambda g_{\mu\nu} = -2nW_\lambda$, the square root of the absolute determinant scales with one half of this trace, giving $\nabla^\Gamma_\lambda\sqrt{|g|} = -nW_\lambda\sqrt{|g|}$. The same relation holds for the volume form. Integration along a transported volume element gives the exponential factor. □

7 Three calibration examples

The following examples are not phenomenological claims. They are calibration tests for the formulas above. Each is included because it makes one possible branch of the nonmetric trace mode concrete and because each branch has a different physical status.

Example 7.1 (Exact trace mode). *Let $W = \mathrm{d}\phi$ on a simply connected readout patch. Then $F = \mathrm{d}W = 0$, and the trace curvature vanishes. By theorem 8.1, the conformal rescaling $g'_{\mu\nu} = e^{2\phi} g_{\mu\nu}$ with $W' = 0$ leaves the same affine connection. In this branch the local trace mode is removable by a Weyl-pair transformation and has no local curvature. It may still affect boundary data or global holonomy on non-simply connected domains, but it should not be counted as a local propagating vector.*

Example 7.2 (Constant trace mode on flat readout). *On flat readout space with $g = \eta$ and constant W_μ, one has $F = 0$ but W^2 may be nonzero. The scalar curvature formula gives $R(\Gamma) = -6W^2$ in four dimensions. The mode is curvature-scalar active but trace-curvature inactive. If no kinetic term or source is present, the Einstein-Hilbert contribution treats this configuration algebraically rather than radiatively.*

Example 7.3 (Plane-wave trace curvature). *Let $W_\mu = \epsilon_\mu e^{ik\cdot x}$ on a flat readout background. Then $F_{\mu\nu} = i(k_\mu \epsilon_\nu - k_\nu \epsilon_\mu)e^{ik\cdot x}$. The massless kinetic equation gives $k^2 = 0$ after gauge reduction, while the massive equation gives $k^2 = -M_W^2/Z_W$ and $k \cdot \epsilon = 0$. This is the explicit local test that separates a Maxwell branch from a Proca branch.*

8 Conformal covariance and trace curvature

The pure trace connection has a familiar Weyl covariance. The pair (g, W) may be changed without changing the affine connection, provided the metric is conformally rescaled and W is shifted appropriately.

Proposition 8.1 (Weyl-pair invariance). *Let*

$$g'_{\mu\nu} = e^{2\sigma} g_{\mu\nu}, \qquad W'_\mu = W_\mu - \partial_\mu \sigma. \tag{42}$$

Then the pure trace connection built from (g', W') is equal to the pure trace connection built from (g, W).

Proof. The Levi-Civita connection of $g' = e^{2\sigma} g$ is

$$\{^\lambda{}_{\mu\nu}\}' = \{^\lambda{}_{\mu\nu}\} + \delta^\lambda_\mu \partial_\nu \sigma + \delta^\lambda_\nu \partial_\mu \sigma - g_{\mu\nu} \partial^\lambda \sigma. \tag{43}$$

The pure trace distortion built from $W' = W - \mathrm{d}\sigma$ contributes

$$\delta^\lambda_\mu (W_\nu - \partial_\nu \sigma) + \delta^\lambda_\nu (W_\mu - \partial_\mu \sigma) - g_{\mu\nu}(W^\lambda - \partial^\lambda \sigma), \tag{44}$$

where the conformal factors cancel in the last term because $g'_{\mu\nu} W'^\lambda_{g'} = g_{\mu\nu}(W^\lambda - \partial^\lambda \sigma)$. Adding the two displayed expressions cancels every derivative of σ and leaves the original connection. $\square$

Definition 8.2 (Trace curvature). *The trace curvature of an affine connection is the two-form*

$$\mathcal{R}_{\mu\nu} := R^\lambda{}_{\lambda\mu\nu}(\Gamma). \tag{45}$$

It is sometimes called the homothetic curvature in metric-affine geometry.

Theorem 8.3 (Trace curvature of the pure nonmetric trace). *For the pure trace connection* (27) *in dimension n,*

$$\mathcal{R}_{\mu\nu} = nF_{\mu\nu}, \qquad F_{\mu\nu} := \partial_\mu W_\nu - \partial_\nu W_\mu. \tag{46}$$

Proof. Taking the connection trace gives

$$\Gamma^\lambda{}_{\lambda\mu} = \{^\lambda{}_{\lambda\mu}\} + nW_\mu. \tag{47}$$

The trace of the commutator term in the curvature cancels because it is a trace of a matrix commutator. Therefore

$$R^\lambda{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\lambda{}_{\lambda\nu} - \partial_\nu \Gamma^\lambda{}_{\lambda\mu}. \tag{48}$$

The Levi-Civita trace contributes an exact derivative whose curl vanishes locally, since $\{^\lambda{}_{\lambda\mu}\} = \partial_\mu \log\sqrt{|g|}$. The remaining contribution is $n(\partial_\mu W_\nu - \partial_\nu W_\mu)$. □

This theorem supplies the cleanest kinetic candidate for W. A term quadratic in $\mathcal{R}_{\mu\nu}$ is, up to the constant n^2, a Maxwell kinetic term for $F = \mathrm{d}W$. It is important that this statement follows from the curvature trace, not from the scalar curvature. The next section shows why.

9 Scalar curvature calculation

The scalar curvature of the pure trace connection is an essential test. If the scalar curvature already contained F^2, then a propagating vector mode could arise from an Einstein-Hilbert action alone. It does not. The calculation below is included in full because it is one of the places where an overstrong claim would be physically misleading.

Theorem 9.1 (Scalar curvature of the pure trace connection). *Let Γ be the torsion-free pure trace nonmetric connection* (27) *on an n-dimensional readout metric* (M, g). *Then*

$$R(\Gamma) = R(g) - 2(n-1)\nabla^{\{\}}_\mu W^\mu - (n-1)(n-2)W_\mu W^\mu. \tag{49}$$

In dimension four,

$$R(\Gamma) = R(g) - 6\nabla^{\{\}}_\mu W^\mu - 6W_\mu W^\mu. \tag{50}$$

Proof. Let $N^\lambda{}_{\mu\nu} = \delta^\lambda_\mu W_\nu + \delta^\lambda_\nu W_\mu - g_{\mu\nu}W^\lambda$. The curvature difference between $\Gamma = \{\} + N$ and the Levi-Civita connection is

$$R^\rho{}_{\sigma\mu\nu}(\Gamma) = R^\rho{}_{\sigma\mu\nu}(g) + 2\nabla^{\{\}}_{[\mu} N^\rho{}_{\nu]\sigma} + 2N^\rho{}_{[\mu|\lambda|}N^\lambda{}_{\nu]\sigma}. \tag{51}$$

Contracting ρ with μ gives the Ricci tensor. The derivative part contributes to the scalar

$$-2(n-1)\nabla^{\{\}}_\mu W^\mu. \tag{52}$$

The quadratic part is obtained from

$$N^\rho{}_{\rho\lambda} = nW_\lambda \tag{53}$$

and from the direct contraction of $N^\rho{}_{\nu\lambda}N^\lambda{}_{\rho\sigma}$. The scalar quadratic contribution is

$$-(n-1)(n-2)W_\mu W^\mu. \tag{54}$$

Combining the Levi-Civita scalar, the derivative contribution, and the quadratic contribution yields (49); setting $n = 4$ yields (50). A component-level expansion of the quadratic contraction is given in section A. $\square$

Corollary 9.2 (Einstein-Hilbert term does not propagate W)**.** *In dimension four,*

$$\int_M \sqrt{|g|}\, R(\Gamma) = \int_M \sqrt{|g|}\, R(g) - 6 \int_M \sqrt{|g|}\, W_\mu W^\mu - 6 \int_M \partial_\mu(\sqrt{|g|} W^\mu), \tag{55}$$

so the Einstein-Hilbert scalar contains no $F_{\mu\nu}F^{\mu\nu}$ kinetic term for the nonmetric trace mode.

Proof. Use (50) and the identity $\sqrt{|g|}\nabla^{\{\}}_\mu W^\mu = \partial_\mu(\sqrt{|g|}W^\mu)$. $\square$

The corollary is a constraint on the theory. If the nonmetric trace is to be a propagating vector field, its kinetic term must enter through another part of the effective action, most naturally through $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ or through a Hessian sector that reduces to the same quadratic form. Without such a term, and in the absence of a source, W is algebraic.

10 Ricci-level bookkeeping and the auxiliary-field theorem

The scalar curvature formula is enough to show that the Einstein-Hilbert term does not generate a Maxwell kinetic term. Nevertheless, the Ricci-level formula is useful because it identifies where the antisymmetric curvature lives and separates it from the scalar contraction. This is the point at which many erroneous arguments arise: the trace curvature contains $F = \mathrm{d}W$, but the scalar curvature contraction does not contain $F_{\mu\nu}F^{\mu\nu}$. The antisymmetric two-form is real; it is simply not activated as a propagating mode by the linear curvature scalar alone.

Theorem 10.1 (Ricci tensor of the pure trace connection)**.** *Let Γ be the torsion-free pure trace connection*

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\} + \delta^\lambda_\mu W_\nu + \delta^\lambda_\nu W_\mu - g_{\mu\nu}W^\lambda. \tag{56}$$

With $F_{\mu\nu} = \nabla^{\{\}}_\mu W_\nu - \nabla^{\{\}}_\nu W_\mu$, the Ricci tensor defined by $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ is

$$R_{\mu\nu}(\Gamma) = R_{\mu\nu}(g) + F_{\mu\nu} - (n-2)\nabla^{\{\}}_\nu W_\mu - g_{\mu\nu}\nabla^{\{\}}_\rho W^\rho \tag{57}$$

$$+ (n-2)\big(W_\mu W_\nu - g_{\mu\nu}W_\rho W^\rho\big). \tag{58}$$

Consequently,

$$R_{\mu\nu}(\Gamma) - R_{\nu\mu}(\Gamma) = nF_{\mu\nu}. \tag{59}$$

and the scalar contraction is

$$g^{\mu\nu}R_{\mu\nu}(\Gamma) = R(g) - 2(n-1)\nabla^{\{\}}_\mu W^\mu - (n-1)(n-2)W_\mu W^\mu. \tag{60}$$

Proof. The curvature difference between two connections is

$$R^\rho{}_{\sigma\mu\nu}(\Gamma) - R^\rho{}_{\sigma\mu\nu}(g) = \nabla^{\{\}}_\mu N^\rho{}_{\nu\sigma} - \nabla^{\{\}}_\nu N^\rho{}_{\mu\sigma} \tag{61}$$

$$+ N^\rho{}_{\mu\lambda}N^\lambda{}_{\nu\sigma} - N^\rho{}_{\nu\lambda}N^\lambda{}_{\mu\sigma}. \tag{62}$$

Contracting ρ with μ gives the Ricci difference. The derivative contribution is

$$\nabla^{\{\}}_\rho N^\rho{}_{\nu\mu} - \nabla^{\{\}}_\nu N^\rho{}_{\rho\mu} = \nabla^{\{\}}_\mu W_\nu - (n-1)\nabla^{\{\}}_\nu W_\mu - g_{\mu\nu}\nabla^{\{\}}_\rho W^\rho \tag{63}$$

$$= F_{\mu\nu} - (n-2)\nabla^{\{\}}_\nu W_\mu - g_{\mu\nu}\nabla^{\{\}}_\rho W^\rho. \tag{64}$$

The quadratic contribution is obtained from $N^\rho{}_{\rho\lambda} = nW_\lambda$ and from the direct contraction

$$N^\rho{}_{\nu\lambda} N^\lambda{}_{\rho\mu} = (n+2)W_\mu W_\nu - 2g_{\mu\nu}W^2, \tag{65}$$

whereas

$$N^\rho{}_{\rho\lambda} N^\lambda{}_{\nu\mu} = n(2W_\mu W_\nu - g_{\mu\nu}W^2). \tag{66}$$

Their difference is $(n-2)(W_\mu W_\nu - g_{\mu\nu}W^2)$. Adding derivative and quadratic parts gives (57). Antisymmetrizing gives $nF_{\mu\nu}$, and contraction with $g^{\mu\nu}$ cancels the antisymmetric term and gives the scalar formula. □

Corollary 10.2 (Einstein-Hilbert pure trace variation)**.** *In dimension $n \neq 1, 2$, the pure Einstein-Hilbert functional for the pure trace connection, after discarding the boundary divergence, gives an algebraic equation for W rather than a propagation equation. In vacuum it sets $W_\mu = 0$ if no additional source or kinetic term is present.*

Proof. Using the scalar formula, the pure trace part of the Einstein-Hilbert action differs from the Levi-Civita Einstein-Hilbert action by

$$S_{\text{trace}} = -\frac{(n-1)(n-2)}{2\kappa} \int_M \sqrt{|g|}\, W_\mu W^\mu \, \mathrm{d}^n x \tag{67}$$

after the divergence term is placed on the boundary. Varying with respect to W_ν yields

$$\delta S_{\text{trace}} = -\frac{(n-1)(n-2)}{\kappa} \int_M \sqrt{|g|}\, W^\nu \delta W_\nu \, \mathrm{d}^n x. \tag{68}$$

For arbitrary compactly supported δW_ν, the Euler-Lagrange equation is $W^\nu = 0$. No second derivatives of W appear. Thus the field is auxiliary in this branch. □

Remark 10.3 (Why this matters)**.** *The corollary is not a negative result for the whole program. It is a protection against overclaiming. A nonmetric vector can be geometrically present but dynamically auxiliary. A later physical identification must therefore exhibit a kinetic gate. The admissible sources of that gate are limited: a curvature-square term such as $R^\lambda{}_{\lambda\mu\nu}R^\rho{}_\rho{}^{\mu\nu}$, a reduced Hessian principal symbol inherited from the parent relational dynamics, or an explicitly stated source sector whose variation supplies derivatives of W. Without one of these mechanisms, a propagating vector mode has not been derived.*

11 Quadratic dynamics of the trace mode

The minimal effective dynamics for a descended nonmetric trace mode is obtained by combining a Maxwell-type kinetic term with an optional algebraic term. We write the calculation in four dimensions and with Lorentzian signature $(-,+,+,+)$. The constants are left symbolic because this paper is not a parameter-fitting paper. The point is to determine which structures are mathematically licensed and what equations follow from them.

Definition 11.1 (Quadratic trace-mode action)**.** *Let $F_{\mu\nu} = 2\partial_{[\mu}W_{\nu]}$. The quadratic action for a nonmetric trace mode on a fixed readout background is*

$$S_W[g, W] = \int_M \sqrt{|g|} \left(-\frac{Z_W}{4} F_{\mu\nu}F^{\mu\nu} - \frac{M_W^2}{2} W_\mu W^\mu + J_\mu W^\mu \right) \mathrm{d}^4 x, \tag{69}$$

where $Z_W > 0$ is the kinetic normalization, $M_W^2 \geq 0$ is an algebraic response coefficient, and J_μ is an external or effective source current.

Theorem 11.2 (Euler-Lagrange equation). *Variation of (69) with respect to W_μ gives*

$$Z_W \nabla_\nu^{\{\}} F^{\nu\mu} - M_W^2 W^\mu + J^\mu = 0. \tag{70}$$

If $M_W = 0$, the equation is invariant under $W \mapsto W + \mathrm{d}\chi$ provided the source is conserved. If $M_W \neq 0$ and $J = 0$, the equation implies the Proca constraint

$$\nabla_\mu^{\{\}} W^\mu = 0. \tag{71}$$

Proof. The variation of the field strength is $\delta F_{\mu\nu} = 2\nabla_{[\mu}^{\{\}} \delta W_{\nu]}$. Since F is antisymmetric, the kinetic variation is

$$\delta\left(-\frac{Z_W}{4}\sqrt{|g|} F_{\mu\nu} F^{\mu\nu}\right) = -\frac{Z_W}{2}\sqrt{|g|} F^{\mu\nu} \delta F_{\mu\nu} \tag{72}$$

$$= -Z_W \sqrt{|g|} F^{\mu\nu} \nabla_\mu^{\{\}} \delta W_\nu. \tag{73}$$

Integrating by parts and discarding the boundary term gives $Z_W \sqrt{|g|}(\nabla_\mu^{\{\}} F^{\mu\nu})\delta W_\nu$. The variation of the algebraic term is $-M_W^2 \sqrt{|g|} W^\nu \delta W_\nu$, and the source variation is $\sqrt{|g|} J^\nu \delta W_\nu$. Setting the coefficient of δW_ν to zero gives (70). For $M_W = 0$, the action changes under $W \mapsto W + \mathrm{d}\chi$ only by $\int \sqrt{|g|} J^\mu \partial_\mu \chi$, which is a boundary term when $\nabla_\mu^{\{\}} J^\mu = 0$. For $M_W \neq 0$ and $J = 0$, taking the divergence of (70) gives $-M_W^2 \nabla_\mu^{\{\}} W^\mu = 0$ because the double divergence of an antisymmetric two-form vanishes. □

Corollary 11.3 (Flat-readout dispersion). *On a flat readout background with $g = \eta$ and $J = 0$, a plane wave $W_\mu = \epsilon_\mu e^{ik\cdot x}$ satisfies*

$$k^2 = 0 \tag{74}$$

for the massless gauge case after gauge reduction, and

$$k^2 = -M_W^2 / Z_W, \qquad k_\mu \epsilon^\mu = 0 \tag{75}$$

for the massive Proca case.

Proof. In Lorenz gauge for the massless case, or using the Proca constraint for the massive case, (70) reduces to $(Z_W \Box - M_W^2) W^\mu = 0$. Substituting $W^\mu = \epsilon^\mu e^{ik\cdot x}$ and using $\Box e^{ik\cdot x} = -k^2 e^{ik\cdot x}$ gives the stated dispersion relations, with the massless case retaining the usual gauge redundancy. □

12 How a kinetic term can arise

The previous section deliberately wrote an effective quadratic action instead of pretending that the Einstein-Hilbert scalar generates the kinetic term. There are two mathematically clean routes by which such a term can appear. The first is to include the square of the trace curvature. The second is for the parent relational Hessian to reduce, after quotient and readout, to a quadratic form whose principal symbol is that of $\mathrm{d}W$.

Proposition 12.1 (Trace-curvature square). *For the pure trace connection in dimension n,*

$$\mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} = n^2 F_{\mu\nu} F^{\mu\nu}. \tag{76}$$

Consequently an action term

$$-\frac{\beta}{4}\int \sqrt{|g|}\,\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \tag{77}$$

contributes a Maxwell kinetic normalization $Z_W = \beta n^2$.

Proof. This is an immediate consequence of theorem 8.3. Raising indices with the readout metric gives $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} = n^2 F_{\mu\nu}F^{\mu\nu}$. □

Definition 12.2 (Principal-symbol kinetic gate)**.** *A reduced Hessian supplies a propagating trace-mode kinetic term if, after readout localization and gauge fixing of non-propagating redundancies, the second-derivative principal symbol acting on* W *is proportional to*

$$\Pi^{\mu\nu}(k) = Z_W(k^2 g^{\mu\nu} - k^\mu k^\nu) \tag{78}$$

with $Z_W > 0$ *on the physical subspace.*

This definition is intentionally phrased as a gate. It is not enough for a one-form W to appear in the connection. Its quadratic response must contain the correct principal symbol. In the absence of this symbol, W is at most an auxiliary field or a gauge artifact at this level.

13 Worked parent calculation: scalar curvature plus trace curvature

We now combine the previous results in a compact but explicit calculation. Consider the local effective action

$$S[g, W] = \int \sqrt{|g|}\left[\frac{1}{2\kappa}R(\Gamma) - \frac{\beta}{4}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\right]\mathrm{d}^4x \tag{79}$$

for the pure trace connection. Using theorems 8.3 and 9.1 in dimension four gives, up to a boundary term,

$$S[g, W] = \int \sqrt{|g|}\left[\frac{1}{2\kappa}R(g) - \frac{3}{\kappa}W_\mu W^\mu - 4\beta F_{\mu\nu}F^{\mu\nu}\right]\mathrm{d}^4x. \tag{80}$$

Matching to (69) yields

$$Z_W = 16\beta, \qquad M_W^2 = \frac{6}{\kappa}, \tag{81}$$

with the sign conventions of (69). If $\beta > 0$ and $\kappa > 0$, the kinetic normalization is positive and the algebraic coefficient is positive in the Proca convention used here. The resulting equation is

$$16\beta\nabla_\nu^{\{\}}F^{\nu\mu} - \frac{6}{\kappa}W^\mu = 0. \tag{82}$$

Thus the trace curvature square makes W propagate, while the scalar curvature supplies the algebraic scale.

This example is not presented as the final CAS action. It is a calibration calculation. It shows exactly which local terms do what and prevents an ambiguity that would otherwise contaminate later physical claims. If a later theory supplies a different normalization or a different sign, it must explain which parent Hessian block or curvature invariant changed the coefficients.

14 Readout descent and falsification diagnostics

A covector field extracted from nonmetricity is not automatically physical. It must pass descent, invariance, and dynamics gates. These gates are not optional because each prevents a distinct error.

Definition 14.1 (Physical nonmetric trace mode)**.** *A nonmetric trace covector W is physical on a readout domain if the following conditions hold. First, W descends from the affine response through the physical quotient, so presentation-equivalent affine data give the same W. Second, its curvature $F = \mathrm{d}W$ is invariant under admissible readout frame changes and gauge transformations. Third, the reduced Hessian or action contains either a positive kinetic gate for F or a specified algebraic constraint explaining why W is auxiliary. Fourth, any coupling to sources is expressed in quotient-descended observables rather than in presentation variables.*

Theorem 14.2 (No-hidden-vector diagnostic)**.** *If two presentation-equivalent affine connections induce different trace covectors W and W' on the same readout sector, and if the difference is not an admissible gauge transformation leaving F and all couplings invariant, then the trace covector is not a physical mode of the quotient theory.*

Proof. A physical mode is, by definition, a function of quotient data. If equivalent presentations induce inequivalent values of W that change curvature or couplings, then an observer using only quotient data could not assign a unique value to the mode. The ambiguity is not a coordinate artifact unless it is removed by an admissible gauge transformation. Therefore the mode fails descent. □

Table 1: Gate audit for the nonmetric trace mode.

Gate	Mathematical test	Failure interpretation
Readout gate	Rank-four Lorentzian sector already licensed	No metric-affine calculation is meaningful yet
Affine descent	Γ descends through quotient equivalence	Connection is presentation-level data
Trace extraction	$W_\lambda = (2n)^{-1} Q_{\lambda\mu}{}^{\mu}$ well-defined	No unique vector trace exists
Curvature gate	$F = \mathrm{d}W$ descends and is invariant	Mode is gauge or presentation artifact
Kinetic gate	Principal symbol $Z_W(k^2 g^{\mu\nu} - k^\mu k^\nu)$ with $Z_W > 0$	No propagating vector mode
Algebraic gate	If no kinetic term, field equation is stated as constraint	Auxiliary field mistaken for radiation
Source gate	Couplings use quotient-descended currents	Hidden matter input

15 Descent, stability, and gate audit

The preceding sections are local metric-affine calculations. A local calculation is not yet a physical mode of the relational theory. For W_μ to be physically usable, it must descend through the readout map, have a nondegenerate kinetic principal symbol if it is to propagate, and couple only to sources that respect the branch in which it is placed. This section states the audit explicitly. It is intentionally stricter than what is often written in metric-affine model building, because the goal here is not to show that a vector can be written down. The goal is to prevent the later theory from confusing notation, gauge, auxiliary variables, and physical degrees of freedom.

Let $\pi : \mathcal{U} \to R$ denote a local readout projection from a representative region of the premetric or quotient construction to the rank-four Lorentzian readout sector R. A one-form W_R on R is the physical nonmetric trace mode only if the trace extracted upstairs agrees with the pullback of W_R on horizontal variations and is invariant under vertical re-descriptions.

Theorem 15.1 (Readout descent criterion)**.** *Let $W_{\mathcal{U}}$ be the one-form obtained by trace extraction from an affine representative on $\mathcal{U}$. A readout one-form W_R exists with $W_{\mathcal{U}} = \pi^* W_R$ if and only if the following two conditions hold: first, $W_{\mathcal{U}}(V) = 0$ for every vertical vector $V \in \ker \mathrm{d}\pi$; second, $\mathcal{L}_V W_{\mathcal{U}} = 0$ for every infinitesimal vertical re-description field V that preserves the physical class. When these conditions hold, W_R is unique on the image of π.*

Proof. This is the standard basic-form criterion specialized to a one-form. If $W_{\mathcal{U}} = \pi^* W_R$, then $W_{\mathcal{U}}$ annihilates vertical vectors because $\mathrm{d}\pi(V) = 0$, and it is invariant along vertical re-descriptions because pullbacks from the base are constant on fibers. Conversely, if the one-form is horizontal and invariant, its value on a tangent vector X_R at a readout point can be defined by choosing any lift $X_{\mathcal{U}}$ with $\mathrm{d}\pi(X_{\mathcal{U}}) = X_R$ and setting $W_R(X_R) = W_{\mathcal{U}}(X_{\mathcal{U}})$. Horizontality makes the definition independent of adding vertical vectors to the lift, and invariance makes it independent of the representative in the same physical fiber. Uniqueness follows because π is surjective onto the readout patch under consideration. □

Corollary 15.2 (Failure of descent is physical failure)**.** *If either condition in theorem 15.1 fails, then the extracted trace is not a physical readout one-form. It may still be a representative artifact, a gauge variable, or an auxiliary bookkeeping device, but it cannot be used as a physical field in a later sector without adding further structure.*

Proposition 15.3 (Kinetic stability gate)**.** *Consider the quadratic trace-mode action on a fixed Lorentzian readout background,*

$$S_W = \int_M \sqrt{|g|} \left[-\frac{Z_W}{4} F_{\mu\nu} F^{\mu\nu} - \frac{M_W^2}{2} W_\mu W^\mu + J_\mu W^\mu \right] \mathrm{d}^4 x, \tag{83}$$

using the mostly-plus convention for the local inertial energy analysis. The transverse propagating sector has the Maxwell sign only when $Z_W > 0$. If $Z_W < 0$, the transverse sector is a ghost. If $Z_W = 0$, the mode is not a propagating Maxwell-type vector at quadratic order.

Proof. In a local inertial frame, the kinetic part is $Z_W(E^2 - B^2)/2$ at the Lagrangian level and gives a Hamiltonian density proportional to $Z_W(E^2 + B^2)/2$ for the transverse Maxwell sector after imposing the Gauss constraint in the massless case or the Proca constraint in the massive case. Thus $Z_W > 0$ is required for positive transverse energy. If $Z_W < 0$, the sign of the transverse Hamiltonian is reversed. If $Z_W = 0$, there is no time-derivative quadratic form for the transverse modes and no Maxwell-type propagation. □

Proposition 15.4 (Massless source gate)**.** *On the massless branch $M_W = 0$, gauge compatibility of the source coupling requires $\nabla_\mu^{\{\}} J^\mu = 0$. On the massive branch $M_W \neq 0$, the divergence of the equation of motion gives the constraint*

$$M_W^2 \nabla_\mu^{\{\}} W^\mu = \nabla_\mu^{\{\}} J^\mu \tag{84}$$

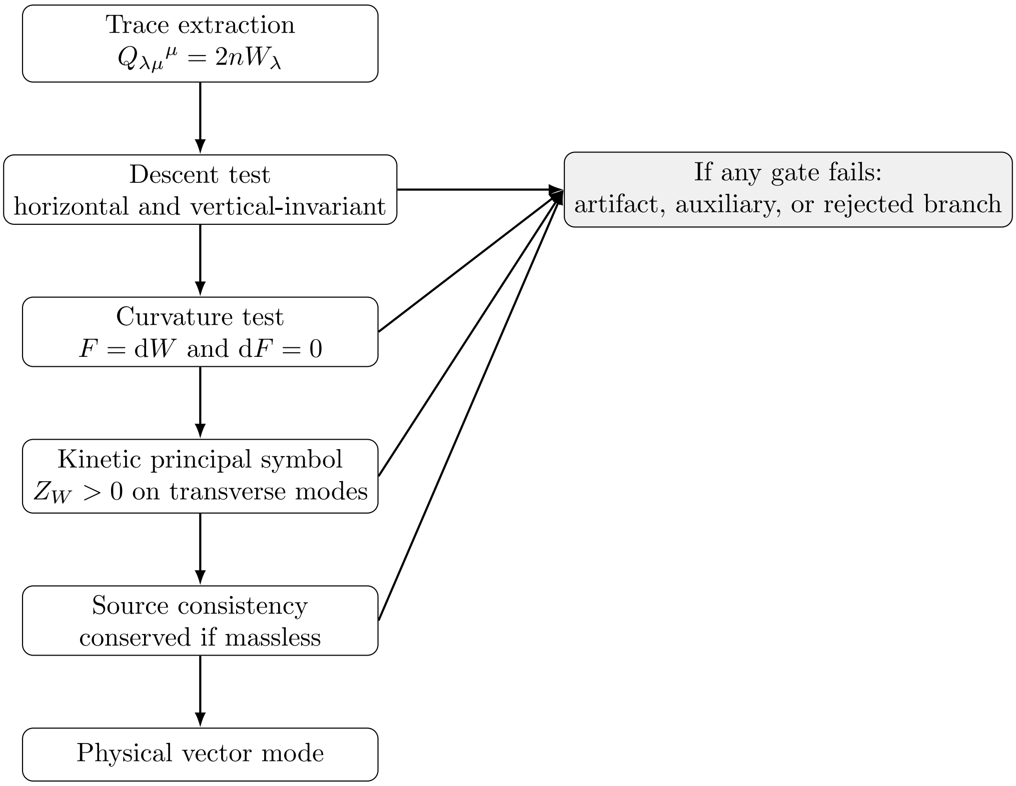

Figure 2: Gate audit for the nonmetric trace. The figure deliberately separates trace extraction, readout descent, curvature, kinetic propagation, and source consistency. A later paper may use W_μ as a physical mode only after all relevant gates have been passed.

up to curvature terms already contained in the covariant divergence of F.

Proof. For $M_W = 0$, the transformation $W \mapsto W + \mathrm{d}\chi$ changes the source action by $\int \sqrt{|g|} J^\mu \nabla^{\{\}}_\mu \chi = -\int \sqrt{|g|}\chi \nabla^{\{\}}_\mu J^\mu$ after integration by parts. Gauge compatibility for arbitrary χ therefore requires conservation. For $M_W \neq 0$, take the divergence of $Z_W \nabla^{\{\}}_\nu F^{\nu\mu} + M_W^2 W^\mu = J^\mu$. The double divergence of the antisymmetric field strength vanishes in the usual contracted sense, leaving the stated constraint. The precise sign follows the source convention in the action. □

Table 2: Minimal audit ledger for later use of the nonmetric trace mode.

Gate	Required calculation	Failure interpretation
Trace extraction	$Q_{\lambda\mu}{}^{\mu} = 2nW_\lambda$ with fixed sign convention	No well-defined trace one-form
Readout descent	Horizontality and vertical invariance of W	Representative artifact rather than physical field
Curvature	$R^\lambda{}_{\lambda\mu\nu} = nF_{\mu\nu}$ and $\mathrm{d}F = 0$	No closed field strength for the trace branch
Scalar curvature	$R(\Gamma) = R(g) - 2(n-1)\nabla \cdot W - (n-1)(n-2)W^2$	Einstein-Hilbert term alone gives no Maxwell propagation
Kinetic gate	Positive transverse principal symbol, $Z_W > 0$	Ghost or auxiliary branch

Gate	Required calculation	Failure interpretation
Mass/source gate	Conserved source if massless; Proca constraint if massive	Inconsistent coupling
Projective audit	Separate Weyl trace from projective distortion	Gauge artifact or mixed distortion misidentified as trace mode

The audit table is part of the mathematical content of the paper. It tells a later calculation what must be shown before a numerical or phenomenological claim can be attached to W_μ. In particular, passing the trace extraction gate and failing the kinetic gate means that the mode exists geometrically but is not a propagating particle-like field. Passing the kinetic gate and failing the descent gate means that a vector exists in a representative but not in the physical quotient. These distinctions are necessary if the construction is to remain a theory rather than an exercise in notation.

16 Relation to metric-affine gravity and Weyl geometry

The construction is close to Weyl geometry in its pure trace sector but differs in its logical role. In standard Weyl geometry, the pair (g, W) is introduced as a geometric structure and the length transport law is then derived. Here the Lorentzian metric is already an effective readout licensed by a prior rank-and-signature gate, and W is extracted from the nonmetricity of an independent affine connection on that readout. This changes the burden of proof. The theory may not assume the Weyl one-form as primitive; it must show that the affine response contains a quotient-descended trace.

The construction is also close to metric-affine gravity, where torsion and nonmetricity arise naturally when the connection is independent of the metric. The present paper uses the same tensor calculus, but it does not assert the full metric-affine gauge framework. It isolates one sector, the nonmetric trace, because that is the minimal vector-like affine mode. The remaining torsion and trace-free nonmetricity components are not denied; they are simply not promoted in this paper unless their own descent and kinetic gates are supplied.

Finally, the construction must not be confused with the Einstein-Maxwell identification sometimes associated historically with Weyl's original idea. The trace curvature is Maxwell-like as a two-form, but identifying it with electromagnetism requires matter charges, gauge coupling normalization, phenomenological constraints, and experimental comparison. None of that is done here. The result of this paper is weaker and stronger at the same time: weaker because it does not identify W with a known field, stronger because it proves exactly which calculations license W as a possible propagating nonmetric vector.

17 Projective shifts and why the Weyl trace is not a projective artifact

A common ambiguity in affine geometry is projective freedom. Since projective transformations preserve unparametrized autoparallels, it is tempting to identify every one-form appearing in a connection with a projective gauge artifact. That temptation is dangerous here. The nonmetric trace covector studied in this paper is not the same object as an arbitrary projective shift. The difference can be checked directly.

Definition 17.1 (Projective shift)**.** *A projective shift of an affine connection is*

$$\Gamma^\lambda{}_{\mu\nu} \longmapsto \Gamma^\lambda{}_{\mu\nu} + \delta^\lambda_\mu \xi_\nu, \tag{85}$$

where ξ_ν is a one-form. This transformation leaves the unparametrized autoparallel curves invariant but changes the affine parametrization.

Proposition 17.2 (Projective shift does not equal pure Weyl nonmetricity)**.** *Under the projective shift $\delta\Gamma^\lambda{}_{\mu\nu} = \delta^\lambda_\mu \xi_\nu$, torsion and nonmetricity change by*

$$\delta T^\lambda{}_{\mu\nu} = 2\delta^\lambda{}_{[\mu}\xi_{\nu]}, \tag{86}$$

and, with the convention $Q = -\nabla^\Gamma g$,

$$\delta Q_{\lambda\mu\nu} = \xi_\mu g_{\lambda\nu} + \xi_\nu g_{\lambda\mu}. \tag{87}$$

This variation is not of the pure trace Weyl form $2\omega_\lambda g_{\mu\nu}$ unless it vanishes in dimension greater than one. Therefore the nonmetric trace mode $W_\lambda = (2n)^{-1} Q_{\lambda\mu}{}^\mu$ is not, by itself, an arbitrary projective gauge parameter.

Proof. The torsion variation follows immediately from antisymmetrizing the two lower indices. For nonmetricity, use

$$Q_{\lambda\mu\nu} = -\partial_\lambda g_{\mu\nu} + \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho{}_{\lambda\nu} g_{\mu\rho}. \tag{88}$$

Substituting $\delta\Gamma^\rho{}_{\lambda\mu} = \delta^\rho_\lambda \xi_\mu$ and $\delta\Gamma^\rho{}_{\lambda\nu} = \delta^\rho_\lambda \xi_\nu$ gives the displayed variation of Q. If this were equal to $2\omega_\lambda g_{\mu\nu}$ for all μ, ν, contraction with $g^{\mu\nu}$ would give $2\xi_\lambda = 2n\omega_\lambda$, so $\omega_\lambda = \xi_\lambda / n$. Substituting back would require $\xi_\mu g_{\lambda\nu} + \xi_\nu g_{\lambda\mu} = 2n^{-1}\xi_\lambda g_{\mu\nu}$ for all indices, which forces $\xi = 0$ in a nondegenerate metric of dimension greater than one. □

This result does not deny projective freedom. It states only that the pure nonmetric trace sector and the projective sector are different affine directions. A complete parent theory must specify whether projective symmetry is a gauge redundancy, is broken by the action, or is fixed by matter couplings. The audit here prevents a later paper from removing W by invoking projective invariance without doing the required tensor calculation.

18 Stress tensor and energy sign of the propagated trace mode

A vector mode with a Maxwell-type kinetic term must also pass an energy-sign test. The sign $Z_W > 0$ in the quadratic action was not decorative. With Lorentzian signature $(-, +, +, +)$, it is the condition that the Maxwell principal energy density has the standard sign on a physical time orientation.

Proposition 18.1 (Stress tensor of the quadratic trace mode)**.** *For the source-free Lagrangian*

$$\mathcal{L}_W = -\frac{Z_W}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{M_W^2}{2} W_\alpha W^\alpha, \tag{89}$$

variation with respect to the readout metric gives

$$T^{(W)}_{\mu\nu} = Z_W \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) + M_W^2 \left(W_\mu W_\nu - \frac{1}{2} g_{\mu\nu} W_\alpha W^\alpha \right). \tag{90}$$

For $Z_W > 0$ the kinetic contribution has positive energy density with respect to a standard local inertial observer.

Proof. The metric variation of $F_{\alpha\beta}$ itself vanishes because $F = \mathrm{d}W$ is an exterior derivative of the one-form W. The metric enters through the inverse metrics raising indices and through $\sqrt{|g|}$. The standard variation identities $\delta\sqrt{|g|} = -(1/2)\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu}$ and $\delta(F_{\alpha\beta}F^{\alpha\beta}) = 2F_{\mu\rho}F_{\nu}{}^{\rho}\delta g^{\mu\nu}$ give the first term of (90). The algebraic term is varied similarly and gives the second term. In a local inertial frame with observer $u^\mu = (1,0,0,0)$, the Maxwell part yields $T^{(F)}_{00} = (Z_W/2)(|\mathbf{E}|^2 + |\mathbf{B}|^2)$, so its sign is positive precisely when $Z_W > 0$. □

Corollary 18.2 (Ghost diagnostic)**.** *If the reduced Hessian produces $Z_W < 0$ for the trace curvature square, the propagated nonmetric trace mode is a ghost with respect to the readout metric. Such a sector cannot be promoted to a stable physical vector mode without an additional mechanism changing the sign of the physical kinetic form.*

The stress calculation is another place where the paper refuses a purely formal success. A two-form curvature may exist even when the kinetic sign is wrong. In that case one has a mathematically defined affine curvature but not a healthy propagating degree of freedom.

19 Degrees of freedom and constraints

The equations derived above also determine the number of local field degrees of freedom. This count is not a substitute for a Hamiltonian analysis of a complete interacting theory, but it is the correct linearized count for the quadratic trace-mode sector and should be recorded before the mode is used in later physical arguments.

Proposition 19.1 (Linear degree-of-freedom count)**.** *In four readout dimensions, the source-free massless trace mode with $Z_W > 0$ carries two transverse local polarizations after gauge reduction. The source-free massive trace mode with $Z_W > 0$ and $M_W^2 > 0$ carries three local polarizations. If $Z_W = 0$ and $M_W^2 \neq 0$, the field is auxiliary and carries no propagating polarization.*

Proof. For $M_W = 0$, the action is invariant under $W_\mu \mapsto W_\mu + \partial_\mu\chi$. The four components of W_μ are reduced by one gauge function and one gauge condition, leaving two transverse polarizations on shell. For $M_W^2 > 0$, gauge invariance is absent, but the Proca equation implies the constraint $\partial_\mu W^\mu = 0$ in flat readout, leaving three independent components on shell. If $Z_W = 0$, the equation of motion is algebraic, $M_W^2 W^\mu = J^\mu$ in the presence of a source and $W = 0$ without one; an algebraic equation carries no propagating initial data. □

This count will matter later. If a later sector counts the nonmetric trace as a gauge field, it must be in the massless branch or in a branch where gauge symmetry is restored. If it counts it as a massive vector, it must include the third polarization and the Proca constraint. If it eliminates it, it must not also use its curvature as an independent radiative observable.

20 Bianchi identity and current consistency

Because the trace curvature of the pure trace mode is an exterior derivative, it satisfies the ordinary differential Bianchi identity. This is another exact calculation that constrains admissible sources.

Proposition 20.1 (Trace Bianchi identity)**.** *Let $F = \mathrm{d}W$. Then*

$$\nabla^{\{\}}_{[\rho} F_{\mu\nu]} = 0. \tag{91}$$

If the massless field equation $Z_W \nabla^{\{\}}_{\nu} F^{\nu\mu} + J^{\mu} = 0$ holds, then the source must satisfy

$$\nabla^{\{\}}_{\mu} J^{\mu} = 0. \tag{92}$$

Proof. The identity $\mathrm{d}^2 = 0$ gives the first equation in exterior form. Taking the divergence of the massless equation gives $\nabla^{\{\}}_{\mu} J^{\mu} = -Z_W \nabla^{\{\}}_{\mu} \nabla^{\{\}}_{\nu} F^{\nu\mu}$. The double divergence of an antisymmetric two-form vanishes after the curvature commutator contraction, so the current is conserved. □

A nonconserved source is therefore not a small imperfection. It means that the assumed massless gauge branch is inconsistent. One must either supply a massive branch, add compensating fields, or reject the coupling.

21 Conclusion

This paper has supplied the metric-affine continuation of the relational readout series at the first point where affine geometry becomes physically meaningful. Starting from a rank-four Lorentzian readout, we introduced an independent affine connection and decomposed its distortion into torsion and nonmetricity with explicit conventions. We then isolated the nonmetric trace covector W_{μ}, derived the pure trace connection, proved its length transport law, computed its trace curvature, computed the scalar curvature of the pure trace connection, and derived the quadratic equations of motion when a kinetic term is present.

The central physical lesson is that the existence of a nonmetric trace covector is not the same as the existence of a propagating vector field. The Einstein-Hilbert scalar of the pure trace connection gives no F^2 term. It yields a boundary term and an algebraic W^2 contribution. Propagation requires a trace-curvature-square term or an equivalent reduced Hessian principal symbol. This distinction is essential for keeping the construction honest. It prevents the later theory from counting a field as physical merely because it appears in the connection.

The paper therefore ends with a clean interface for the next layer. A later matter or gauge-sector paper may use W_{μ} only if it passes the descent, curvature, kinetic, and source gates stated here. If these gates fail, the nonmetric trace remains auxiliary, gauge, or presentation-dependent. If they pass, then W_{μ} is a mathematically controlled vector mode of the effective metric-affine readout.

A Component check of the scalar curvature

For completeness we record the scalar contractions used in theorem 9.1. Let

$$N^{\lambda}{}_{\mu\nu} = \delta^{\lambda}_{\mu} W_{\nu} + \delta^{\lambda}_{\nu} W_{\mu} - g_{\mu\nu} W^{\lambda}. \tag{93}$$

Then

$$N^{\rho}{}_{\rho\lambda} = n W_{\lambda}. \tag{94}$$

The derivative part of the Ricci contraction is

$$g^{\sigma\nu}\left(\nabla^{\{\}}_{\rho}N^{\rho}{}_{\nu\sigma}-\nabla^{\{\}}_{\nu}N^{\rho}{}_{\rho\sigma}\right)=-2(n-1)\nabla^{\{\}}_{\mu}W^{\mu}. \tag{95}$$

The quadratic part is

$$g^{\sigma\nu}\left(N^{\rho}{}_{\rho\lambda}N^{\lambda}{}_{\nu\sigma}-N^{\rho}{}_{\nu\lambda}N^{\lambda}{}_{\rho\sigma}\right)=-(n-1)(n-2)W_{\mu}W^{\mu}. \tag{96}$$

Adding these two contributions to $R(g)$ gives (49).

B Variation of the Maxwell-Proca functional

The variational calculation for (69) is insensitive to whether F arises from ordinary exterior differentiation or from the trace curvature, because the trace curvature is nF. The only change is the normalization Z_W. Boundary terms are

$$-Z_W\int_{\partial M}\sqrt{|\gamma|}\,n_{\mu}F^{\mu\nu}\delta W_{\nu}, \tag{97}$$

so the variational principle is well posed if either δW vanishes on the boundary or the boundary flux is fixed. The source term is gauge-compatible in the massless case only if

$$\nabla^{\{\}}_{\mu}J^{\mu}=0, \tag{98}$$

otherwise the transformation $W\mapsto W+\mathrm{d}\chi$ changes the action by a non-boundary term.

C Checklist for later physical use

A later paper that uses the nonmetric trace mode must state explicitly which of the following situations holds. If $Z_W>0$ and $M_W=0$, the mode is a massless gauge one-form with gauge redundancy and conserved source. If $Z_W>0$ and $M_W\neq 0$, the mode is Proca-like and carries the corresponding transversality constraint in the source-free case. If $Z_W=0$ and $M_W\neq 0$, the mode is auxiliary and must be eliminated algebraically. If both coefficients vanish, the trace is either gauge, undetermined, or fixed only by sources and boundary data. These alternatives are physically inequivalent and must not be conflated.

D Projective-shift component calculation

The projective shift calculation in theorem 17.2 can be checked at the level of traces. Under $\delta\Gamma^{\lambda}{}_{\mu\nu}=\delta^{\lambda}_{\mu}\xi_{\nu}$ one has

$$\delta Q_{\lambda\mu}{}^{\mu}=2\xi_{\lambda}. \tag{99}$$

The Weyl trace extracted from this variation would therefore be $\delta W_{\lambda}=\xi_{\lambda}/n$. However, substituting this value into the pure Weyl form gives $2\delta W_{\lambda}g_{\mu\nu}=2n^{-1}\xi_{\lambda}g_{\mu\nu}$, which is not equal to $\xi_{\mu}g_{\lambda\nu}+\xi_{\nu}g_{\lambda\mu}$ except in the trivial case. Thus a projective shift changes the Weyl trace contraction but also creates non-Weyl nonmetricity and torsion. This is why trace extraction alone is not a gauge-fixing argument.

E Hamiltonian-level caution

The degree-of-freedom count in theorem 19.1 is the correct linearized count for the isolated quadratic trace-mode sector. A full Hamiltonian analysis of an interacting metric-affine theory may introduce additional primary or secondary constraints, especially when projective symmetry, torsion, or trace-free nonmetricity are included. The present paper therefore uses the count only as a local diagnostic. It is strong enough to prevent double counting of the trace mode, but it is not advertised as a complete Hamiltonian proof for any later full theory.

F Ricci tensor component derivation

This appendix records the Ricci-level calculation in a form that can be checked line by line. For

$$N^{\lambda}{}_{\mu\nu} = \delta^{\lambda}_{\mu} W_{\nu} + \delta^{\lambda}_{\nu} W_{\mu} - g_{\mu\nu} W^{\lambda}, \tag{100}$$

one has

$$N^{\rho}{}_{\rho\mu} = n W_{\mu}, \tag{101}$$

and therefore

$$\nabla^{\{\}}_{\rho} N^{\rho}{}_{\nu\mu} = \nabla^{\{\}}_{\nu} W_{\mu} + \nabla^{\{\}}_{\mu} W_{\nu} - g_{\mu\nu} \nabla^{\{\}}_{\rho} W^{\rho}, \tag{102}$$

$$\nabla^{\{\}}_{\nu} N^{\rho}{}_{\rho\mu} = n \nabla^{\{\}}_{\nu} W_{\mu}. \tag{103}$$

The derivative contribution to $R_{\mu\nu}(\Gamma) - R_{\mu\nu}(g)$ is consequently

$$\nabla^{\{\}}_{\mu} W_{\nu} - (n-1) \nabla^{\{\}}_{\nu} W_{\mu} - g_{\mu\nu} \nabla^{\{\}}_{\rho} W^{\rho}. \tag{104}$$

The quadratic contractions are

$$N^{\rho}{}_{\rho\lambda} N^{\lambda}{}_{\nu\mu} = n(2W_{\mu}W_{\nu} - g_{\mu\nu}W^2), \tag{105}$$

$$N^{\rho}{}_{\nu\lambda} N^{\lambda}{}_{\rho\mu} = (n+2) W_{\mu}W_{\nu} - 2 g_{\mu\nu} W^2. \tag{106}$$

Subtracting gives $(n-2)(W_{\mu}W_{\nu} - g_{\mu\nu}W^2)$. Combining derivative and quadratic parts gives theorem 10.1. This appendix is included because the scalar result alone hides the antisymmetric Ricci component; the trace curvature is visible before contraction and disappears from the curvature scalar because $g^{\mu\nu}F_{\mu\nu} = 0$.

G Pure Einstein-Hilbert trace variation

The pure trace Einstein-Hilbert calculation is sometimes misread as a kinetic derivation. The precise variational statement is as follows. In dimension n,

$$S_{EH}[g, W] = \frac{1}{2\kappa} \int_M \sqrt{|g|} \left(R(g) - 2(n-1) \nabla^{\{\}}_{\mu} W^{\mu} - (n-1)(n-2) W^2 \right) \mathrm{d}^n x. \tag{107}$$

If the boundary value of W is fixed or the divergence term is canceled by an appropriate boundary term, the only bulk W-dependence is algebraic. The variation is

$$\delta_W S_{EH} = -\frac{(n-1)(n-2)}{\kappa} \int_M \sqrt{|g|}\, W^{\mu} \delta W_{\mu} \, \mathrm{d}^n x. \tag{108}$$

Therefore the bulk equation is $W^\mu = 0$ for $n \neq 1, 2$. In $n = 2$, the W^2 coefficient vanishes and the trace contribution is a boundary term. Neither case produces Maxwell propagation. This is the shortest possible diagnostic for whether a later paper has smuggled in dynamics: if only the linear curvature scalar is used, no F^2 equation follows.

H Principal symbol and polarization count

In the flat-readout limit, the source-free Maxwell-Proca equation takes the form

$$Z_W \partial_\nu F^{\nu\mu} + M_W^2 W^\mu = 0. \tag{109}$$

For a plane wave $W^\mu = \varepsilon^\mu e^{ik\cdot x}$ the principal algebraic system is

$$\left[-Z_W(k^2\delta^\mu{}_\nu - k^\mu k_\nu) + M_W^2 \delta^\mu{}_\nu\right]\varepsilon^\nu = 0, \tag{110}$$

with the sign of k^2 determined by the chosen Lorentzian convention. If $M_W \neq 0$, contraction with k_μ gives $M_W^2 k \cdot \varepsilon = 0$, so the polarization is transverse and three massive polarizations remain on shell. If $M_W = 0$, the system is gauge-invariant under $\varepsilon_\mu \mapsto \varepsilon_\mu + ik_\mu\chi$ and two transverse polarizations remain after gauge fixing. This count is local and quadratic. It is not advertised as a full nonlinear Hamiltonian analysis of an arbitrary metric-affine theory.

References

[1] H. Weyl, "Gravitation und Elektrizitaet," Sitzungsberichte der Koeniglich Preussischen Akademie der Wissenschaften, 1918, pp. 465–480.

[2] J. Ehlers, F. A. E. Pirani, and A. Schild, "The geometry of free fall and light propagation," in *General Relativity: Papers in Honour of J. L. Synge*, edited by L. O'Raifeartaigh, Clarendon Press, Oxford, 1972.

[3] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, "Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance," *Physics Reports* **258**, 1–171, 1995.

[4] J. Beltran Jimenez, L. Heisenberg, and T. Koivisto, "The geometrical trinity of gravity," *Universe* **5**, 173, 2019.

[5] R. M. Wald, *General Relativity*, University of Chicago Press, 1984.

[6] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Wiley, 1963.

[7] M. Nakahara, *Geometry, Topology and Physics*, 2nd ed., Institute of Physics Publishing, 2003.

[8] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, 1973.

[9] J. A. Schouten, *Ricci-Calculus: An Introduction to Tensor Analysis and Its Geometrical Applications*, 2nd ed., Springer, 1954.

[10] E. V. Haynsworth, "Determination of the inertia of a partitioned Hermitian matrix," *Linear Algebra and its Applications* **1**, 73–81, 1968.