# Rank-Four Lorentzian Readout from Relational Dynamics

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**Abstract**

We continue the construction of effective dynamics from a premetric relational substrate by isolating the precise algebraic gates that must be passed before a reduced quotient dynamics may be interpreted as carrying an effective Lorentzian spacetime readout. The previous layer supplies a physical quotient, a relational clock covector, and an effective Hessian obtained by stationary hidden-sector elimination. The present paper proves that a Lorentzian readout is licensed only when four independent conditions hold simultaneously: a stable rank-four readout projection exists; the clock covector descends to that sector and defines a one-plus-complement splitting; the clock-neutral block of the descended response is positive; and the clock Schur scalar is negative. Under these hypotheses the response form has inertia $(1, 0, 3)$, admits Lorentz frames, determines a null cone, and is invariant under admissible frame changes. We sharpen the construction by proving lift-independence of the clock Schur scalar, a principal-minor diagnostic for the spatial positivity gate, a quantitative stability theorem with explicit margins, and a Schur-parent audit showing that the Lorentzian sector must be accounted for inside the inertia of the parent response. The result is deliberately conditional. It does not derive matter, gauge fields, cosmology, or the Einstein equations, and it does not assert that every relational substrate becomes spacetime. It identifies the exact finite-dimensional mathematical interface between relational time and effective Lorentzian geometry, together with the failure modes that falsify that interpretation.

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# 1 Introduction

A premetric theory that has already constructed a quotient and a relational clock still has not constructed spacetime. A clock orders physical states, and an effective Hessian describes quadratic response on observable variations, but neither object by itself supplies a Lorentzian metric, a light cone, or a distinction between one temporal and three spatial directions. This paper addresses precisely this intermediate problem. We ask when the quotient-clock-response structure developed at the previous layer admits a four-dimensional Lorentzian readout, and we state the answer as a set of mathematical gates whose failure modes are as important as their successes.

The need for this discipline is not cosmetic. It is easy to smuggle spacetime into a premetric construction by naming one coordinate "time", by declaring three remaining variables to be "space", or by imposing a metric signature at the point where one should have derived it. The present paper avoids that route. The primitive object at this layer is only a reduced quadratic response form on an effective variation space over the physical quotient. The relational clock is represented by a covector, not by a background coordinate. The clock direction is not assumed orthogonal to anything until the response form itself determines the relevant Schur splitting. The spatial complement is not a Euclidean space until positivity of the induced response on that complement is proved or imposed as a gate condition.

The logic is therefore conditional and exact. If the effective variation space has physical rank four, if the clock covector is nonzero and admits an admissible complement, if the reduced response form is nondegenerate on that rank-four readout sector, and if the Schur scalar associated with the clock direction is negative while the complement is positive definite, then the readout form has inertia $(1, 0, 3)$. By Sylvester's law of inertia, this statement is invariant under changes of readout frame. It is therefore meaningful to call the resulting object a Lorentzian readout. What has been obtained is not a coordinate convention but a congruence class of nondegenerate symmetric bilinear forms with one negative and three positive directions.

The paper is deliberately positioned before the introduction of matter, gauge fields, or cosmological history. The only physical content at stake is the emergence of a metric-like readout from relational dynamics. This is already a nontrivial step. The Ehlers-Pirani-Schild program shows how aspects of spacetime geometry can be reconstructed from light rays and freely falling particles, but it assumes operational probes already living in a spacetime setting [3]. Causal-set approaches start from causal order and volume [6, 7]. Lorentzian geometry in general relativity begins with a differentiable manifold and a metric of signature $(-, +, +, +)$ [4, 5]. Here the question is narrower and earlier: how can a response operator, obtained only after quotienting and relational clock selection, pass a rank-and-signature test that licenses an effective Lorentzian interpretation?

The contribution of the paper is fourfold. First, we formalize the rank-four readout gate as a statement about physical variation spaces and projection maps, not as a declaration that spacetime has four dimensions. Second, we prove a clock Schur signature theorem which gives an elementary and useful criterion for one negative clock direction and positive spatial complement. Third, we show that the Lorentzian readout is invariant under admissible frame changes and stable under small perturbations inside an open chamber of the space of symmetric forms. Fourth, we isolate the diagnostics that prevent hidden metric or hidden time assumptions. These diagnostics are essential because a failed gate is not a defect in the mathematics; it is a falsifier for the proposed readout.

The refinement made in the present version is to treat each gate not only as a sufficient condition but also as an audit condition. A rank-four sector that is not stable under redescription is not a physical readout. A clock covector that does not descend to the quotient is a hidden time parameter. A spatial block that is positive only in a preferred presentation is a hidden metric. A negative clock component that disappears after the hidden sector is eliminated is not a dynamical clock sign. The proofs below are organized so that these possibilities are excluded explicitly rather than by terminology. This is why the paper repeatedly distinguishes the substrate, the quotient, the effective response, and the final readout: conflating any two of these objects would make the Lorentzian conclusion look stronger than it is.

## 2 Inputs inherited from the quotient-clock layer

The present construction starts from the output of the preceding quotient-clock analysis. We summarize only the structures needed here, because the proofs below must not depend on any unstated metric or spacetime primitive. Let $\mathcal{Q}$ denote the physical quotient of a relational presentation space. A point $q \in \mathcal{Q}$ is an equivalence class of presentations, not an event of a manifold. The quotient may be finite, discrete, stratified, local, or only partially represented by

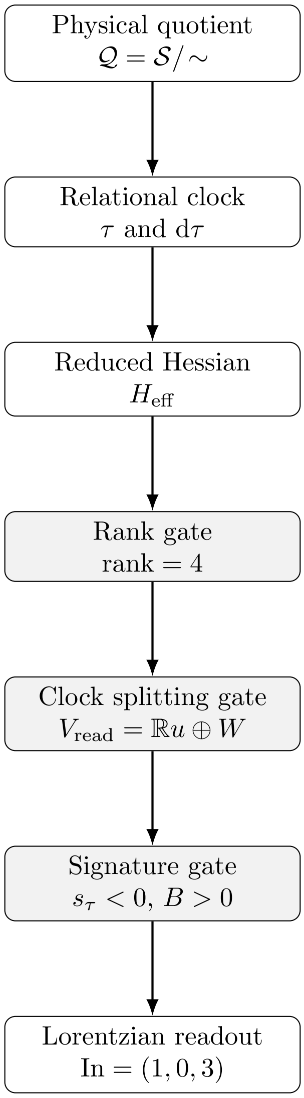


Figure 1: The dependency order used in this paper. The Lorentzian readout is not assumed at the beginning. It is licensed only after the quotient, relational clock, reduced Hessian, rank gate, clock-splitting gate, and signature gate have all been passed.

charts. The present paper works pointwise on finite-dimensional variation spaces over admissible regions of $\mathcal{Q}$, so no global smooth manifold assumption is required.

For each admissible quotient state $q$ we assume a finite-dimensional real vector space $V_q$ of physical variations. These variations represent first-order changes of quotient data that survive presentation equivalence. The assignment $q \mapsto V_q$ may later become a vector bundle when a smooth readout exists, but at this stage it is only a family of vector spaces compatible with admissible transitions. A relational clock is represented locally by a nonzero covector

$$\theta_q := \mathrm{d}\tau_q \in V_q^*, \tag{1}$$

where $\tau$ is a monotone functional on an admissible transition domain in $\mathcal{Q}$. The covector notation is important: the clock is not an external coordinate. It is a quotient-descended functional whose differential measures first-order clock variation.

The third inherited object is a symmetric bilinear form

$$H_q : V_q \times V_q \to \mathbb{R}, \tag{2}$$

obtained as the reduced Hessian of a parent quadratic response by hidden-sector stationary elimination. In matrix form, if a parent variation space has visible-hidden splitting $V_{\text{vis}} \oplus V_{\text{hid}}$ and if the hidden block $D$ is invertible, the reduced visible form is represented by the Schur complement

$$H_{\text{eff}} = A - BD^{-1}C. \tag{3}$$

In this paper we rename the reduced visible form $H_q$ when no ambiguity is possible. The word "Hessian" is used in the algebraic sense: a symmetric quadratic response form on variations. No coordinate metric is being introduced by this terminology.

**Definition 2.1** (Premetric readout datum)**.** *A premetric readout datum over an admissible quotient domain $U \subseteq \mathcal{Q}$ consists of the following pointwise data: finite-dimensional real variation spaces $V_q$, nonzero clock covectors $\theta_q \in V_q^*$, and symmetric bilinear response forms $H_q \in \mathrm{Sym}^2(V_q^*)$ for each $q \in U$. No metric, topology, or differentiable manifold is included in the datum unless it is explicitly constructed later from these objects.*

**Definition 2.2** (Admissible readout projection)**.** *Given a premetric readout datum at $q$, an admissible readout projection is a surjective linear map*

$$P_q : V_q \to R_q \tag{4}$$

*onto a finite-dimensional real vector space $R_q$ such that presentation-equivalent variations have the same image and such that the clock covector descends through $P_q$ whenever a clock readout is claimed. The descended clock covector is denoted $\bar{\theta}_q \in R_q^*$ and satisfies $\theta_q = \bar{\theta}_q \circ P_q$ on the projected sector.*

The projection $P_q$ expresses what the effective observer, apparatus, or mathematical readout is allowed to resolve. It is not assumed to be four-dimensional. Four-dimensionality is a gate condition tested in the next section. The response form on $R_q$ is well-defined when $H_q$ descends to the quotient by the kernel of $P_q$ or when a representative section has been fixed and the final statement is independent of that section. To avoid hiding this issue, we use the following explicit descent assumption.

**Assumption 2.3** (Response descent)**.** *For each admissible $P_q : V_q \to R_q$ there is a symmetric bilinear form $h_q \in \mathrm{Sym}^2(R_q^*)$ such that*

$$H_q(v, w) = h_q(P_q v, P_q w) \tag{5}$$

*for all variations $v, w$ in the readout-relevant sector. Equivalently, $H_q$ is constant along physically invisible directions in the sense required by the chosen readout.*

This assumption is not automatic. If it fails, the apparent readout depends on hidden or presentation-level variations and cannot be treated as physical at this layer. Throughout the paper $h_q$ denotes the descended effective response form on the readout sector. When the state $q$ is fixed, we suppress the subscript.

## 3 The rank-four readout gate

The first gate is rank. A Lorentzian spacetime readout, in the usual local sense, requires a four-dimensional tangent-like sector. This does not mean that the underlying relational variation space has dimension four. It may be larger, because hidden, gauge, internal, or unresolved relational directions may exist. It also does not mean that four dimensions are declared fundamental. The rank-four gate only says that the effective sector on which a metric-like readout is to be defined has dimension four.

**Definition 3.1** (Readout rank). *Let $P : V \to R$ be an admissible readout projection. The readout rank is*

$$r(P) := \dim R = \operatorname{rank} P. \tag{6}$$

*The rank-four gate is passed at $q$ if there exists an admissible readout projection $P_q : V_q \to R_q$ with*

$$r(P_q) = 4 \tag{7}$$

*and with a descended response form $h_q \in \mathrm{Sym}^2(R_q^*)$.*

**Definition 3.2** (Nondegenerate rank-four response). *A rank-four readout sector $(R, h)$ is nondegenerate if the linear map*

$$R \to R^*, \qquad x \mapsto h(x, \cdot) \tag{8}$$

*is an isomorphism. In matrix language, for any basis of $R$, the representing matrix of $h$ has determinant different from zero.*

**Proposition 3.3** (Rank is presentation-invariant). *Let $V$ and $V'$ be two presentation-related representatives of the same physical variation sector, and suppose the representative change is an invertible linear map $A : V \to V'$ compatible with the physical quotient. If $P' : V' \to R$ is defined by $P' = P \circ A^{-1}$, then* $\operatorname{rank} P' = \operatorname{rank} P$.

*Proof.* Since $A$ is an isomorphism, $\operatorname{im}(P \circ A^{-1}) = \operatorname{im} P$. Therefore the two maps have the same image dimension. This proves $\operatorname{rank} P' = \operatorname{rank} P$. The statement is elementary, but it is a useful diagnostic: if a proposed four-dimensional sector changes rank under a presentation transformation, then the sector was not a quotient-level object. □

**Theorem 3.4** (Rank-four readout theorem). *Assume a premetric readout datum at $q$ and an admissible projection $P : V \to R$ satisfying response descent. If $r(P) = 4$ and the descended response form $h$ is nondegenerate, then the pair $(R, h)$ is a four-dimensional nondegenerate quadratic readout sector. In particular, every basis of $R$ represents $h$ by an invertible $4 \times 4$ symmetric matrix, and all such matrices are related by congruence transformations.*

*Proof.* The condition $r(P) = 4$ gives $\dim R = 4$. By response descent, $h$ is a well-defined symmetric bilinear form on $R$. Nondegeneracy says that the associated map $R \to R^*$ is an isomorphism. Choosing a basis $e = (e_0, e_1, e_2, e_3)$ of $R$ identifies $h$ with a symmetric matrix $G$ whose entries are $G_{ab} = h(e_a, e_b)$. The linear map $R \to R^*$ has matrix $G$ in this basis, so nondegeneracy is equivalent to $\det G \neq 0$. If $e' = eA$ is another basis, then the matrix becomes $G' = A^T G A$, which is congruent to $G$. Thus the content of the readout sector is the congruence class of a nondegenerate symmetric bilinear form on a four-dimensional real vector space, not any particular coordinate matrix. □

The theorem does not yet identify the signature. A nondegenerate four-dimensional form can have inertia $(0, 0, 4)$, $(1, 0, 3)$, $(2, 0, 2)$, $(3, 0, 1)$, or $(4, 0, 0)$ depending on sign convention and eigenvalue counts. The Lorentzian gate is the special case with exactly one sign differing from the other three. We next show how the relational clock selects which sign should be interpreted as temporal, without assuming an external metric.

# 4 Clock covectors and one-plus-complement splitting

A relational clock on the quotient supplies a covector $\theta \in R^*$ on the rank-four readout sector. The covector measures infinitesimal change of the clock functional along readout variations. It is not yet a vector, because no metric has been introduced to identify $R$ and $R^*$. To obtain a one-plus-complement splitting, one must choose a vector $u \in R$ normalized by the clock covector,

$$\theta(u) = 1, \tag{9}$$

and define the clock-neutral complement

$$W := \ker \theta. \tag{10}$$

Since $\theta \neq 0$ and $\dim R = 4$, the subspace $W$ has dimension three. The vector $u$ is not unique: replacing $u$ by $u + w$ with $w \in W$ preserves $\theta(u) = 1$. The signature gate must therefore be stated in a way that does not depend on an arbitrary choice of $u$. The Schur scalar gives the required invariant.

**Lemma 4.1** (Clock complement decomposition)**.** *Let $R$ be a four-dimensional real vector space and let $\theta \in R^*$ be nonzero. If $u \in R$ satisfies $\theta(u) = 1$, then*

$$R = \mathbb{R}u \oplus W, \qquad W = \ker \theta, \tag{11}$$

*with* $\dim W = 3$.

*Proof.* For any $x \in R$, define $w = x - \theta(x)u$. Then $\theta(w) = \theta(x) - \theta(x)\theta(u) = 0$, so $w \in W$, and $x = \theta(x)u + w$. This proves that $R = \mathbb{R}u + W$. If $\lambda u \in W$, then $0 = \theta(\lambda u) = \lambda$, hence the intersection is trivial. Since $\theta$ is nonzero, the rank-nullity theorem gives $\dim W = \dim R - 1 = 3$. $\square$

Let $h$ be the descended symmetric response form on $R$. In the decomposition $R = \mathbb{R}u \oplus W$, write

$$h \sim G = \begin{pmatrix} a & c^T \\ c & B \end{pmatrix}, \tag{12}$$

where $a = h(u, u)$, $c_i = h(w_i, u)$ in a basis $(w_1, w_2, w_3)$ of $W$, and $B_{ij} = h(w_i, w_j)$. The block $B$ is the restriction of $h$ to the clock-neutral complement. The clock Schur scalar is

$$s_\tau := a - c^T B^{-1} c, \tag{13}$$

whenever $B$ is invertible.

**Definition 4.2** (Clock-spatial admissibility)**.** *A rank-four readout sector $(R, h, \theta)$ is clock-spatially admissible if $\theta \neq 0$, if $W = \ker \theta$ is three-dimensional, and if the restriction $B = h|_W$ is nondegenerate. It is positive-spatially admissible if, in addition, $B$ is positive definite.*

This definition uses the word “spatial” only as a readout label. It does not assume a pre-existing Euclidean spatial metric. Positivity of $B$ is a statement about the response form restricted to clock-neutral variations. If this restriction is not positive definite, the Lorentzian gate in the sign convention used here fails.

**Proposition 4.3** (Independence of the Schur clock sign)**.** *Assume $B$ is positive definite. The sign of the Schur scalar $s_\tau = a - c^T B^{-1} c$ is independent of the basis chosen in $W$. It is also invariant under replacing the basis of $W$ by any element of* $\mathrm{GL}(W)$.

*Proof.* Under a basis change $w' = wM$ with $M \in \mathrm{GL}(W)$, the matrix blocks transform as $B' = M^T B M$ and $c' = M^T c$. Therefore

$$(c')^T (B')^{-1} c' = c^T M (M^T B M)^{-1} M^T c = c^T M M^{-1} B^{-1} (M^T)^{-1} M^T c = c^T B^{-1} c. \tag{14}$$

The scalar $a = h(u, u)$ is unaffected by changing only the basis of $W$. Hence $s_\tau$ is basis-independent on $W$. □

The next point is important enough to record as a separate proposition. The clock lift $u$ is auxiliary: it is a way to split $R$ into a clock line and the clock-neutral complement, but it is not itself a primitive time vector. The scalar that decides the clock sign must therefore be independent of the chosen normalized lift.

**Proposition 4.4** (Lift-independence of the clock Schur scalar)**.** *Let $R = \mathbb{R}u \oplus W$ with $W = \ker\theta$ and $\theta(u) = 1$. Let the response matrix be*

$$G = \begin{pmatrix} a & c^T \\ c & B \end{pmatrix}, \qquad B \textit{ invertible.} \tag{15}$$

*If $u' = u + r$ for some $r \in W$, and if $a'$, $c'$, and $B$ are the corresponding blocks in the decomposition $R = \mathbb{R}u' \oplus W$, then*

$$a' - c'^T B^{-1} c' = a - c^T B^{-1} c. \tag{16}$$

*Consequently the clock Schur scalar $s_\tau$ is an invariant of $(h, \theta)$ and not of the arbitrary clock lift.*

*Proof.* Write $r$ in the chosen basis of $W$ as a column vector. Since $u' = u + r$, the new clock-clock block is

$$a' = h(u + r, u + r) = a + 2c^T r + r^T B r, \tag{17}$$

while the new clock-spatial coupling is

$$c' = c + Br. \tag{18}$$

Therefore

$$\begin{aligned} a' - c'^T B^{-1} c' &= a + 2c^T r + r^T B r - (c + Br)^T B^{-1} (c + Br) && (19) \\ &= a + 2c^T r + r^T B r - c^T B^{-1} c - 2r^T c - r^T B r && (20) \\ &= a - c^T B^{-1} c. && (21) \end{aligned}$$

The terms depending on the lift displacement $r$ cancel identically. □

**Corollary 4.5** (Clock rescaling)**.** *If the clock covector is rescaled by a nonzero scalar, $\theta' = \alpha\theta$, then the sign of the clock Schur scalar is unchanged. More precisely, with the compatible normalized lift $u' = \alpha^{-1} u$ one has $s_{\tau'} = \alpha^{-2} s_\tau$.*

*Proof.* The complement $W = \ker\theta$ is unchanged by nonzero rescaling. The blocks scale as $a' = \alpha^{-2}a$ and $c' = \alpha^{-1}c$, while $B$ is unchanged. Hence $s_{\tau'} = a' - c'^T B^{-1} c' = \alpha^{-2}(a - c^T B^{-1} c)$. $\square$

Thus the sign condition used in the signature theorem below is not a coordinate artifact, not a choice of clock lift, and not a normalization artifact. It is the response-theoretic sign of the quotient clock direction after its coupling to clock-neutral variations has been eliminated.

## 5 The Lorentzian signature gate

The signature gate is the point at which a metric-like interpretation first becomes legitimate. We use the inertia convention

$$\mathrm{In}(G) = (n_-, n_0, n_+), \tag{22}$$

where $n_-$, $n_0$, and $n_+$ count negative, zero, and positive eigenvalues of any symmetric matrix representing the bilinear form. Sylvester's law of inertia ensures that this triple is invariant under congruence transformations [9, 13].

**Definition 5.1** (Lorentzian signature gate)**.** *A clock-spatially admissible rank-four readout sector $(R, h, \theta)$ passes the Lorentzian signature gate, in the sign convention of this paper, if for some normalized clock lift $u$ and $W = \ker\theta$ the block representation* (12) *satisfies*

$$B > 0, \qquad s_\tau = a - c^T B^{-1} c < 0. \tag{23}$$

*defining one negative clock-Schur direction and three positive clock-neutral directions.*

**Theorem 5.2** (Clock Schur signature theorem)**.** *Let $R$ be a four-dimensional real vector space with nonzero clock covector $\theta$, normalized lift $u$, complement $W = \ker\theta$, and symmetric bilinear form $h$. Suppose that in the decomposition $R = \mathbb{R}u \oplus W$ the matrix of $h$ is*

$$G = \begin{pmatrix} a & c^T \\ c & B \end{pmatrix} \tag{24}$$

*defining a block $B$ that is positive definite. If*

$$s_\tau = a - c^T B^{-1} c < 0, \tag{25}$$

*then $h$ has inertia*

$$\mathrm{In}(h) = (1, 0, 3). \tag{26}$$

*Conversely, if $B > 0$ and $\mathrm{In}(h) = (1, 0, 3)$, then $s_\tau < 0$.*

*Proof.* Because $B$ is invertible, define the congruence matrix

$$T = \begin{pmatrix} 1 & 0 \\ -B^{-1}c & I_3 \end{pmatrix}. \tag{27}$$

A direct multiplication gives

$$T^T G T = \begin{pmatrix} a - c^T B^{-1} c & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} s_\tau & 0 \\ 0 & B \end{pmatrix}. \tag{28}$$

Congruence preserves inertia by Sylvester's law. Since $B$ is positive definite, it contributes three positive eigenvalues and no negative or zero eigenvalues. If $s_\tau < 0$, the scalar block

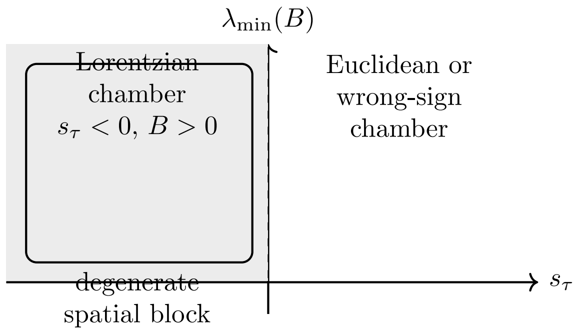


Figure 2: Schematic signature chamber. The Lorentzian readout is available only in the open region where the spatial block is positive and the clock Schur scalar is negative. The boundary corresponds to degenerate clock or spatial response and is not a regular Lorentzian readout.

contributes exactly one negative eigenvalue. Hence $\mathrm{In}(h) = (1, 0, 3)$. Conversely, if $B > 0$ and $\mathrm{In}(h) = (1, 0, 3)$, the block $B$ already accounts for three positive eigenvalues. The remaining Schur scalar must account for the unique negative eigenvalue and cannot be zero or positive. Therefore $s_\tau < 0$. □

**Proposition 5.3** (Principal-minor diagnostic for the spatial gate)**.** *Let $B$ be the $3 \times 3$ matrix representing the restriction of $h$ to the clock-neutral complement $W$ in an ordered basis $(w_1, w_2, w_3)$. Then the spatial positivity gate $B > 0$ is equivalent to positivity of the three leading principal minors in some basis obtained by a nonsingular change of spatial coordinates after symmetric pivoting. In a basis where no pivoting is required, this reads*

$$B_{11} > 0, \qquad \det \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} > 0, \qquad \det B > 0. \tag{29}$$

*Equivalently, the gate may be checked basis-independently by requiring all eigenvalues of $B$ to be positive.*

*Proof.* This is Sylvester's criterion applied to the symmetric matrix representing $h|_W$. In a basis where the leading principal minors are the pivots of the symmetric decomposition, positivity of those pivots is equivalent to positive definiteness. If a different spatial basis is chosen, the matrix changes by congruence $B \mapsto M^T B M$, and positive definiteness is preserved. The eigenvalue formulation is basis-independent because a real symmetric matrix is positive definite if and only if its quadratic form is positive on every nonzero vector, equivalently if and only if all eigenvalues are positive. □

The theorem is a local algebraic version of the step that is often hidden by notation. It shows exactly when one negative clock direction and three positive clock-neutral directions are present. The negative direction is not necessarily the initially chosen vector $u$; after congruence diagonalization it is represented by the Schur-adjusted clock direction

$$u_\perp = u - \sum_{i,j} (B^{-1})^{ij} c_j w_i, \tag{30}$$

which is $h$-orthogonal to $W$ and satisfies $h(u_\perp, u_\perp) = s_\tau$. This vector is obtained from the response form itself, not from a prior metric.

## 6 Lorentzian readout and frame invariance

Once the signature gate is passed, the readout sector carries a Lorentzian bilinear form. It is important to phrase this statement without pretending that a smooth spacetime manifold has already appeared. The immediate output is a four-dimensional vector space with a nondegenerate form of inertia $(1, 0, 3)$. A manifold interpretation requires an additional charting or localization step. The present paper defines the pointwise metric readout and proves its frame invariance.

**Definition 6.1** (Lorentzian readout sector)**.** *A Lorentzian readout sector at $q$ is a triple $(R_q, h_q, \theta_q)$ such that $R_q$ is four-dimensional, $h_q$ is a nondegenerate symmetric bilinear form with inertia $(1, 0, 3)$, and $\theta_q \in R_q^*$ is a nonzero clock covector whose kernel is positive with respect to the clock-Schur splitting of theorem 5.2.*

**Definition 6.2** (Readout frame and metric components)**.** *Let $(R, h)$ be a Lorentzian readout sector. A readout frame is an ordered basis $e = (e_0, e_1, e_2, e_3)$ of $R$. The metric components of the readout in this frame are*

$$g_{ab} = h(e_a, e_b). \tag{31}$$

*A Lorentz frame is a readout frame in which*

$$g_{ab} = \eta_{ab} := \mathrm{diag}(-1, 1, 1, 1). \tag{32}$$

*defining the sign convention used throughout this paper.*

**Theorem 6.3** (Existence of Lorentz frames)**.** *If $(R, h)$ is a four-dimensional real vector space with* $\mathrm{In}(h) = (1, 0, 3)$*, then there exists a basis of $R$ in which the matrix of $h$ is $\eta = \mathrm{diag}(-1, 1, 1, 1)$.*

*Proof.* By the real symmetric bilinear form classification, or equivalently by repeated completion of squares, there exists a basis in which $h$ is diagonal with one negative and three positive diagonal entries. Let this diagonal matrix be $\mathrm{diag}(-\alpha, \beta_1, \beta_2, \beta_3)$ with all $\alpha, \beta_i > 0$. Rescaling the first basis vector by $\alpha^{-1/2}$ and the remaining three by $\beta_i^{-1/2}$ transforms the diagonal entries to $-1, 1, 1, 1$. This proves the existence of a Lorentz frame. □

**Theorem 6.4** (Frame invariance of the Lorentzian readout)**.** *Let $G$ and $G'$ be the matrices of $h$ in two readout frames related by $A \in \mathrm{GL}(4, \mathbb{R})$. Then*

$$G' = A^T G A. \tag{33}$$

*Consequently $G$ and $G'$ have the same inertia. If both frames are Lorentz frames, then $A \in \mathrm{O}(1, 3)$, i.e.*

$$A^T \eta A = \eta. \tag{34}$$

*Proof.* The congruence formula is the standard transformation law for a bilinear form under a basis change. Sylvester's law gives equality of inertia. If both frames are Lorentz frames, then $G = G' = \eta$, so the same congruence formula gives $\eta = A^T \eta A$. This is precisely the defining condition for the Lorentz group in the chosen sign convention. □

This theorem supplies the first legitimate sense in which local Lorentz symmetry appears in the construction. It is not postulated as a primitive spacetime symmetry. It is the residual frame freedom of any four-dimensional readout sector whose effective response form has Lorentzian inertia. The symmetry is therefore a consequence of the signature gate and frame equivalence.

$$\begin{tikzcd}
V_q \arrow[r, "P_q"] \arrow[d, "\text{presentation change}"'] & R_q \arrow[r, "h_q"] \arrow[d, "A\in \mathrm{GL}(4)"] & R_q^* \arrow[d, "(A^{-1})^T"] \\
V_q' \arrow[r, "P_q'"] & R_q' \arrow[r, "h_q'"] & (R_q')^*
\end{tikzcd}$$

Figure 3: Compatibility of presentation change, readout projection, and bilinear response. The physical content is the congruence class of the descended form, not a preferred matrix.

# 7 From pointwise readout to effective line element

The word "metric" becomes meaningful only after one specifies the level at which the Lorentzian form is being read. At the purely algebraic level the metric is the bilinear form $h$ on $R$. If an admissible region of the quotient admits local coordinates or readout labels $x^\mu$, one may express $h$ in component form by a frame field. This is a secondary readout step, not a primitive assumption.

Let $U \subseteq \mathcal{Q}$ be an admissible region for which the rank-four Lorentzian sector varies coherently with $q \in U$. Suppose there is a local readout chart assigning four labels $x^\mu$ and a frame map

$$e : TU_{\text{read}} \to R, \tag{35}$$

where $TU_{\text{read}}$ denotes the tangent-like module of chart variations. In components, write $e^a{}_\mu$ for the frame coefficients. The effective metric components in the readout chart are then

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \tag{36}$$

when $e$ is a Lorentz frame. In a nonorthonormal frame one replaces $\eta_{ab}$ by the matrix $G_{ab} = h(e_a, e_b)$.

**Definition 7.1** (Effective metric readout). *An effective metric readout on an admissible region $U$ is a smoothly or coherently varying family of Lorentzian readout sectors $(R_q, h_q)$ together with local readout labels and frame maps that express $h_q$ in component form $g_{\mu\nu}(q)$. The metric is effective because it is read from $h_q$ after quotient, clock, rank, and signature gates, not placed on the substrate as primitive data.*

**Theorem 7.2** (Metric readout theorem). *Assume an admissible region $U$ carries coherent Lorentzian readout sectors $(R_q, h_q)$ and local frame maps $e_q$ into readout labels. Then the components defined by* (36) *transform as a symmetric covariant rank-two tensor under readout-coordinate changes, provided the frame maps transform by the corresponding change of basis. The signature of $g_{\mu\nu}$ is Lorentzian at each point where the frame is nondegenerate.*

*Proof.* Symmetry follows immediately from symmetry of $\eta_{ab}$ or, in a general frame, from symmetry of $G_{ab}$. If $x^\mu$ and $y^\alpha$ are two readout label systems and $J^\mu{}_\alpha = \partial x^\mu / \partial y^\alpha$ is the change-of-label matrix, then the frame coefficients transform as $e^a{}_\alpha = e^a{}_\mu J^\mu{}_\alpha$. Substitution gives

$$g'_{\alpha\beta} = \eta_{ab} e^a{}_\alpha e^b{}_\beta = \eta_{ab} e^a{}_\mu e^b{}_\nu J^\mu{}_\alpha J^\nu{}_\beta = g_{\mu\nu} J^\mu{}_\alpha J^\nu{}_\beta, \tag{37}$$

which is the covariant tensor transformation law. Nondegeneracy of the frame implies congruence between $g$ and $\eta$, so Sylvester's law gives the same inertia $(1, 0, 3)$. □

The theorem marks the precise boundary between premetric algebra and effective geometry.

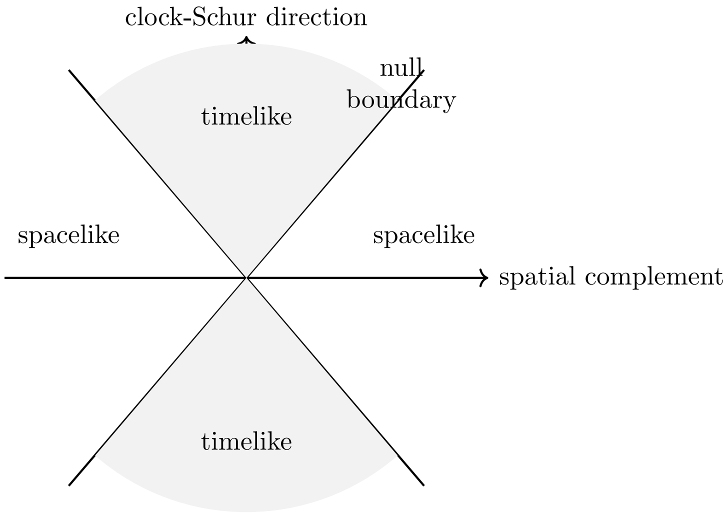


Figure 4: Two-dimensional schematic slice of the null cone after the Lorentzian signature gate. The cone is not primitive causal structure; it is the zero locus of the effective readout form.

Before the frame maps exist, one has Lorentzian readout sectors but not yet a coordinate spacetime. After coherent frame maps exist, one can speak of metric components, null directions, timelike and spacelike variations, and local Lorentz-frame transformations. The construction remains conditional: it does not force every quotient region to admit such a coherent frame.

# 8 Null cone and causal readout

Once a Lorentzian form exists, a null cone can be defined algebraically. This is again not primitive causal order. It is a consequence of the signature gate.

**Definition 8.1** (Readout causal classes)**.** *Let $(R, h)$ be a Lorentzian readout sector with sign convention $(-,+,+,+)$. A nonzero vector $x \in R$ is timelike if $h(x,x) < 0$, null if $h(x,x) = 0$, and spacelike if $h(x,x) > 0$. The null cone is*

$$\mathcal{N}_h = \{x \in R \setminus \{0\} : h(x,x) = 0\}. \tag{38}$$

**Proposition 8.2** (Cone existence after the signature gate)**.** *If a readout sector passes the Lorentzian signature gate, then its null cone is nonempty and separates timelike from spacelike readout variations in every Lorentz frame.*

*Proof.* By theorem 6.3, choose a Lorentz frame in which $h(x,x) = -x_0^2 + x_1^2 + x_2^2 + x_3^2$. Vectors such as $(1,1,0,0)$ are nonzero and null, so the cone is nonempty. The equation $x_0^2 = x_1^2 + x_2^2 + x_3^2$ is the standard double cone in this frame. The sign of $h(x,x)$ distinguishes the interior and exterior of the cone. Since other Lorentz frames are related by transformations preserving $h$, the classification is frame-invariant. □

# 9 Stability of the Lorentzian chamber

A physically useful gate should not be destroyed by infinitesimal perturbations. The rank condition is stable as long as the relevant determinant minors remain nonzero, and the signature condition is stable because eigenvalues of symmetric matrices vary continuously. We record this as a theorem because it clarifies the meaning of a chamber: Lorentzian readout is not a single fine-tuned point but an open region in the space of effective response forms.

**Theorem 9.1** (Openness of the Lorentzian chamber)**.** *Let $R$ be a fixed four-dimensional real vector space and let $\mathcal{S}_R$ be the vector space of symmetric bilinear forms on $R$. The subset*

$$\mathcal{L}_R = \{h \in \mathcal{S}_R : \mathrm{In}(h) = (1, 0, 3)\} \tag{39}$$

*is open in $\mathcal{S}_R$. If a clock splitting $R = \mathbb{R}u \oplus W$ is fixed, the subset defined by $B > 0$ and $s_\tau < 0$ is also open.*

*Proof.* Choose any basis of $R$ and identify $\mathcal{S}_R$ with the space of real symmetric $4 \times 4$ matrices. The eigenvalues of a symmetric matrix depend continuously on its entries. If $G$ has one negative and three positive eigenvalues, none of its eigenvalues is zero. There is therefore a positive distance from the spectrum of $G$ to zero. Small enough perturbations preserve the sign of each eigenvalue, hence preserve inertia. This proves openness of $\mathcal{L}_R$. For the clock-Schur formulation, positive definiteness of $B$ is open because the smallest eigenvalue of $B$ remains positive under small perturbations. The scalar $s_\tau$ depends continuously on $a, c, B$ whenever $B$ is invertible, so the condition $s_\tau < 0$ is open as well. □

**Corollary 9.2** (No fine-tuning at the signature level)**.** *If an effective response form passes the Lorentzian signature gate, then all sufficiently small perturbations of the response form, preserving the same rank-four readout sector and clock splitting, also pass the gate.*

**Proposition 9.3** (Quantitative stability margins)**.** *Fix a clock splitting and suppose the Lorentzian gate is passed with*

$$\beta = \lambda_{\min}(B) > 0, \qquad \sigma = -s_\tau > 0. \tag{40}$$

*Let the spatial block and Schur scalar be perturbed to $B + \Delta B$ and $s_\tau + \Delta s$. If*

$$\|\Delta B\|_2 < \beta, \qquad |\Delta s| < \sigma, \tag{41}$$

*then the perturbed readout still passes the Lorentzian signature gate. In particular, the quantities $\beta$ and $\sigma$ are explicit gate margins.*

*Proof.* By Weyl's eigenvalue inequality for symmetric matrices, every eigenvalue of $B + \Delta B$ differs from the corresponding eigenvalue of $B$ by at most $\|\Delta B\|_2$. Hence the smallest eigenvalue of $B + \Delta B$ is larger than $\beta - \|\Delta B\|_2$, which is positive by hypothesis. Thus the spatial block remains positive definite. The clock Schur scalar remains negative because $s_\tau + \Delta s < -\sigma + \sigma = 0$. The two defining inequalities of the Lorentzian gate are therefore preserved. □

The margin formulation is useful for later computational work. It distinguishes a robust Lorentzian chamber, where $\beta$ and $\sigma$ are comfortably separated from zero, from a boundary solution whose apparent signature can be destroyed by arbitrarily small perturbations. A candidate derivation that produces $\beta \approx 0$ or $\sigma \approx 0$ should be treated as a near-degenerate readout, not as a stable spacetime sector.

The corollary is only local. It does not say that the underlying substrate dynamically prefers the Lorentzian chamber. It says that once a substrate has entered such a chamber, the signature condition is structurally stable. A later dynamical theory may still be required to explain why a particular chamber is selected rather than another.

## 10 Compatibility with Schur reduction

The previous paper introduced the effective Hessian by Schur reduction from a parent visible-hidden decomposition. The present signature analysis must be compatible with that reduction; otherwise the Lorentzian readout might be an artifact of eliminating hidden variables incorrectly. We therefore record the exact relation between parent inertia and readout inertia.

Let the parent quadratic response be represented by

$$H_{\mathrm{par}} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \tag{42}$$

where $D$ is invertible. The Schur complement on the visible sector is

$$H/D = A - BD^{-1}B^T. \tag{43}$$

The Haynsworth inertia additivity formula states

$$\mathrm{In}(H_{\mathrm{par}}) = \mathrm{In}(D) + \mathrm{In}(H/D), \tag{44}$$

with inertia added componentwise [11, 12].

**Theorem 10.1** (Schur-compatible Lorentzian readout)**.** *Assume the effective readout form $h$ is obtained by descending a Schur complement $H/D$ to a rank-four readout sector $R$. If $h$ passes the Lorentzian signature gate and if the descent kernel is nondegenerately separated from $R$, then the Lorentzian inertia of $h$ is the visible contribution to the parent inertia after accounting for the eliminated hidden block and the unobserved visible kernel.*

*Proof.* By Haynsworth additivity, the parent inertia splits into the inertia of $D$ and the inertia of the Schur complement on the visible sector. By the nondegenerate separation assumption, the visible sector decomposes as a direct sum of the readout sector $R$ and an unobserved kernel sector $K$ on which the descended form is block-separated. Therefore

$$\mathrm{In}(H/D) = \mathrm{In}(h) + \mathrm{In}((H/D)|_K). \tag{45}$$

Since $h$ passes the Lorentzian signature gate, theorem 5.2 gives $\mathrm{In}(h) = (1, 0, 3)$. Substitution into (44) yields the claimed accounting of parent inertia. No eigenvalue has disappeared; it has been assigned either to the hidden block, the unobserved visible kernel, or the Lorentzian readout sector. □

This theorem is a bookkeeping safeguard. It prevents the interpretation that Schur reduction simply throws away signs until the desired signature appears. A valid reduction must preserve the inertia accounting of the parent response. The Lorentzian readout is a sectoral statement inside that accounting.

**Proposition 10.2** (Parent inertia audit)**.** *Let the parent response be nondegenerate and let hidden elimination be performed by an invertible block $D$. If a descended rank-four sector has Lorentzian inertia $(1, 0, 3)$, then the parent form must contain at least one negative direction unless the negative direction is exactly cancelled by an opposite-sign contribution in an excluded degenerate accounting. In the nondegenerate Schur setting, no Lorentzian negative direction can be created from a positive-definite parent by a legitimate Schur reduction.*

*Proof.* Haynsworth additivity gives $\mathrm{In}(H_{\mathrm{par}}) = \mathrm{In}(D) + \mathrm{In}(H/D)$. If the readout sector is a nondegenerate direct summand of $H/D$, then $\mathrm{In}(H/D)$ contains the summand $(1, 0, 3)$ plus the inertia of unobserved visible directions. Componentwise additivity implies that the parent negative count is the negative count of $D$ plus the negative count of $H/D$. Therefore the readout negative direction is counted inside the parent inertia. If the parent form is positive definite, all inertia contributions in the additivity formula must have zero negative count, contradicting the presence of the Lorentzian readout summand. Thus a legitimate nondegenerate Schur reduction cannot manufacture a negative clock direction from an everywhere positive parent response. □

This audit is one of the most important safeguards in the paper. It means that a future dynamical derivation of the Lorentzian gate must explain where the negative clock-Schur sign resides in the parent operator. It cannot merely project away inconvenient directions or change sign conventions after reduction.

## 11 Examples and diagnostics

The examples in this section are intentionally small. Their purpose is not empirical modeling but diagnostic clarity. They show how the gates pass, fail, and change under perturbation.

**Example 11.1** (A minimal passing sector)**.** *Let* $R = \mathbb{R}^4$ *with clock covector* $\theta(x) = x_0$ *and choose* $u = (1, 0, 0, 0)$*, so* $W = \{x_0 = 0\}$*. Let*

$$G = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \tag{46}$$

*Then*

$$B = \mathrm{diag}(2, 3, 4) > 0, \qquad c = (1, 0, 0)^T, \qquad a = -2. \tag{47}$$

*The clock Schur scalar is*

$$s_\tau = -2 - (1, 0, 0)\,\mathrm{diag}(1/2, 1/3, 1/4)(1, 0, 0)^T = -\frac{5}{2} < 0. \tag{48}$$

*Therefore the sector passes the Lorentzian gate and has inertia* $(1, 0, 3)$*.*

**Example 11.2** (Rank failure)**.** *Let the effective projection have image dimension three. Even if the resulting* $3 \times 3$ *form has one negative and two positive directions, it is not a four-dimensional Lorentzian readout. It may describe an effective lower-dimensional sector, but it fails the rank-four gate. Conversely, an image dimension five with inertia* $(1, 0, 4)$ *is not a four-dimensional spacetime readout without an additional projection and descent proof. Thus rank is not a decorative condition; it determines the dimensionality of the effective readout itself.*

**Example 11.3** (Wrong-sign chamber)**.** *Let* $B > 0$ *but* $s_\tau > 0$*. Then the congruent diagonal form has four positive entries, so the readout is positive definite rather than Lorentzian. It may support a Euclidean response interpretation, but it does not supply a temporal cone or a Lorentzian causal classification. If* $B$ *has mixed signature, then the sector may have two negative directions or another indefinite pattern. Such cases are not near-misses; they are different signature chambers.*

**Example 11.4** (Null-clock boundary)**.** *If* $B > 0$ *but* $s_\tau = 0$*, then the full form is degenerate. The Schur-adjusted clock direction has zero norm, and the bilinear form has a nontrivial radical.*

*This is a boundary case, not a regular Lorentzian readout. It may be physically interesting as a transition surface between chambers, but it cannot be used as a nondegenerate effective metric without additional data.*

## 12 No-hidden-time and no-hidden-metric diagnostics

The construction can fail by importing the desired structure before the gates are checked. We therefore include explicit diagnostics. They are not additional assumptions; they are tests for whether a proposed derivation is honest.

**Definition 12.1** (Hidden-time import)**.** *A construction contains hidden time import if the clock covector $\theta$ is defined by reference to an external parameter, a preferred background coordinate, or a causal order not derived from quotient transitions, rather than by a quotient-descended relational clock functional.*

**Definition 12.2** (Hidden-metric import)**.** *A construction contains hidden metric import if the identification of clock and spatial directions uses an inner product, orthogonality relation, norm, cone, or signature that is not derived from the effective response form or from explicitly declared non-metric data.*

**Proposition 12.3** (No-hidden-time diagnostic)**.** *If the clock covector $\theta$ fails to descend through the physical quotient or changes under presentation-equivalent descriptions, then the corresponding Lorentzian readout is not physical in the sense of this paper.*

*Proof.* A physical readout must be a function of quotient data. If $\theta$ changes under presentation-equivalent descriptions, then the splitting $R = \mathbb{R}u \oplus \ker\theta$, the Schur scalar $s_\tau$, and the resulting signature classification may also change under pure redescription. The readout would then depend on presentation rather than physical state. This violates admissibility of the readout projection and invalidates the Lorentzian interpretation. □

**Proposition 12.4** (No-hidden-metric diagnostic)**.** *The only orthogonality used in the clock-Schur signature theorem is the orthogonality induced by the effective response form $h$ after the rank-four readout sector has been selected. Any prior use of a metric orthogonality relation to choose $W$ or to assign sign to the clock direction is an additional assumption and must be declared separately.*

*Proof.* The complement $W$ is defined as $\ker\theta$, using only the clock covector. It is not defined as an orthogonal complement. The $h$-orthogonal clock direction is produced only after the block form of $h$ is available and after $B$ is invertible. Its expression is the Schur-adjusted vector $u_\perp = u - B^{-1}c$ in block notation. Therefore any orthogonality used before $h$ is introduced cannot be part of the theorem proved here. □

These diagnostics make the claim of the paper narrower but stronger. The construction does not say that every effective four-dimensional sector should be interpreted geometrically. It says that a sector passing quotient descent, rank four, clock splitting, signature, and no-hidden-structure tests admits a Lorentzian readout with controlled transformation laws.

## 13 Time orientation and clock compatibility

A Lorentzian bilinear form determines two timelike cones at each point, but it does not by itself choose which cone is future-directed. In standard spacetime geometry this choice is called time orientation. In the present construction the natural candidate for time orientation is the relational clock. This is a further reason why the clock covector must be quotient-descended before the Lorentzian readout is interpreted physically. Without it, the signature theorem would produce a Lorentzian form but no physically preferred direction of increase.

Let $(R, h, \theta)$ be a Lorentzian readout sector. A vector $x \in R$ is compatible with increasing relational time if $\theta(x) > 0$. This condition uses the clock covector, not the metric. The Lorentzian form then classifies whether such a vector is timelike, null, or spacelike. The future readout cone is therefore not simply one of the two algebraic timelike components; it is the component selected by positive clock evaluation, provided this selection is consistent on timelike vectors.

**Definition 13.1** (Clock-compatible time orientation). *A Lorentzian readout sector $(R, h, \theta)$ is clock-time-oriented if there exists a timelike vector $T \in R$ such that $\theta(T) > 0$ and if the set of timelike vectors satisfying $\theta(x) > 0$ is contained in a single connected component of the timelike cone. That component is called the future cone of the readout sector.*

**Proposition 13.2** (Schur clock direction selects a time orientation). *Assume a sector passes the Lorentzian signature gate with $B > 0$ and $s_\tau < 0$. Let $u_\perp = u - B^{-1}c$ be the Schur-adjusted clock direction. If $\theta(u_\perp) > 0$, then $u_\perp$ is timelike and determines a clock-compatible time orientation.*

*Proof.* The completion-of-squares calculation gives $h(u_\perp, u_\perp) = s_\tau < 0$, so $u_\perp$ is timelike. Since $\theta$ is continuous as a linear functional and the timelike cone has two connected components, the component containing $u_\perp$ is selected by the sign convention $\theta > 0$ provided no vector in that component crosses the kernel of $\theta$. If such a crossing occurred inside the timelike cone, then a timelike vector would have zero clock rate, contradicting the assumed compatibility of the clock readout. Under the stated condition the component containing $u_\perp$ is therefore the future cone. □

The proposition separates Lorentzian signature from temporal orientation. A sector can be Lorentzian but fail to be time-oriented by the relational clock if the clock covector is badly aligned with the timelike cone. Such a failure is not a contradiction; it is a diagnostic. It means that the clock used to obtain ordering on the quotient is not compatible with the candidate geometric readout in that region. In a physical theory, this would indicate either a wrong readout projection, a transition through a singular chamber boundary, or an incomplete account of the clock variable.

## 14 Coherence of frames over an admissible region

The preceding statements were pointwise. A spacetime-like readout over a region requires more than a Lorentzian form at isolated states. It requires coherent identification of nearby readout sectors. This coherence may later be represented by a connection, by transition functions, or by local trivializations of a rank-four bundle. At this stage it is enough to state the minimal condition needed for a regional effective metric.

**Definition 14.1** (Coherent Lorentzian readout region). *Let $U \subseteq \mathcal{Q}$ be an admissible quotient region. A coherent Lorentzian readout over $U$ is a family of rank-four sectors $R_q$, forms $h_q$ of inertia $(1, 0, 3)$, and clock covectors $\theta_q$, together with local identifications on overlaps such that the transition maps are invertible and preserve the congruence class of $h_q$ and the positive clock orientation. In a local Lorentz frame this means that overlap maps take values in the proper time-oriented Lorentz group whenever orientation and time orientation have been fixed.*

**Theorem 14.2** (Local coherence criterion)**.** *Suppose $U$ is covered by readout patches $U_i$. On each $U_i$ assume there is a Lorentz frame field for $h_q$ and a clock-compatible future cone. If on every overlap $U_i \cap U_j$ the frame change matrix $A_{ij}$ satisfies*

$$A_{ij}^T \eta A_{ij} = \eta \tag{49}$$

*and preserves the selected future cone, then the local readout sectors glue to a coherent Lorentzian readout over $U$.*

*Proof.* The equation $A_{ij}^T \eta A_{ij} = \eta$ says that overlap maps preserve the bilinear form. Therefore the metric components computed in one local frame agree by Lorentz-frame transformation with those computed in another. Preservation of the future cone ensures that the clock orientation does not flip on overlaps. The usual cocycle condition for triple overlaps is inherited from the consistency of changing between local frames. Hence the local sectors define a coherent Lorentzian readout on $U$. □

This criterion is intentionally weaker than the full smooth-bundle machinery used in differential geometry. The reason is conceptual. Smoothness should enter only once the quotient region admits a smooth readout atlas. The algebraic gate already knows what must be preserved: the bilinear form and the clock orientation. Smooth differentiable structure is not part of the primitive substrate; it is a later regularity condition on how readout sectors vary.

## 15 A worked parent-to-readout Schur example

We now give a complete finite-dimensional example showing how a parent response can reduce to a Lorentzian readout sector without assigning Lorentzian signature to the parent by fiat. Consider a visible sector of dimension four coupled to a two-dimensional hidden sector. Let

$$H_{\text{par}} = \begin{pmatrix} A & E \\ E^T & D \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \tag{50}$$

where the visible block and coupling block are

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{51}$$

The Schur-reduced visible response is

$$H/D = A - ED^{-1}E^T = \begin{pmatrix} -\frac{3}{2} & 1 & 0 & 0 \\ 1 & \frac{8}{3} & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \tag{52}$$

Use the clock covector $\theta(x) = x_0$ and the normalized lift $u = (1, 0, 0, 0)$. The clock-neutral block is $B = \operatorname{diag}(8/3, 3, 4)$, which is positive definite, and $c = (1, 0, 0)^T$. The Schur clock scalar is

$$s_\tau = -\frac{3}{2} - c^T B^{-1} c = -\frac{3}{2} - \frac{3}{8} = -\frac{15}{8} < 0. \tag{53}$$

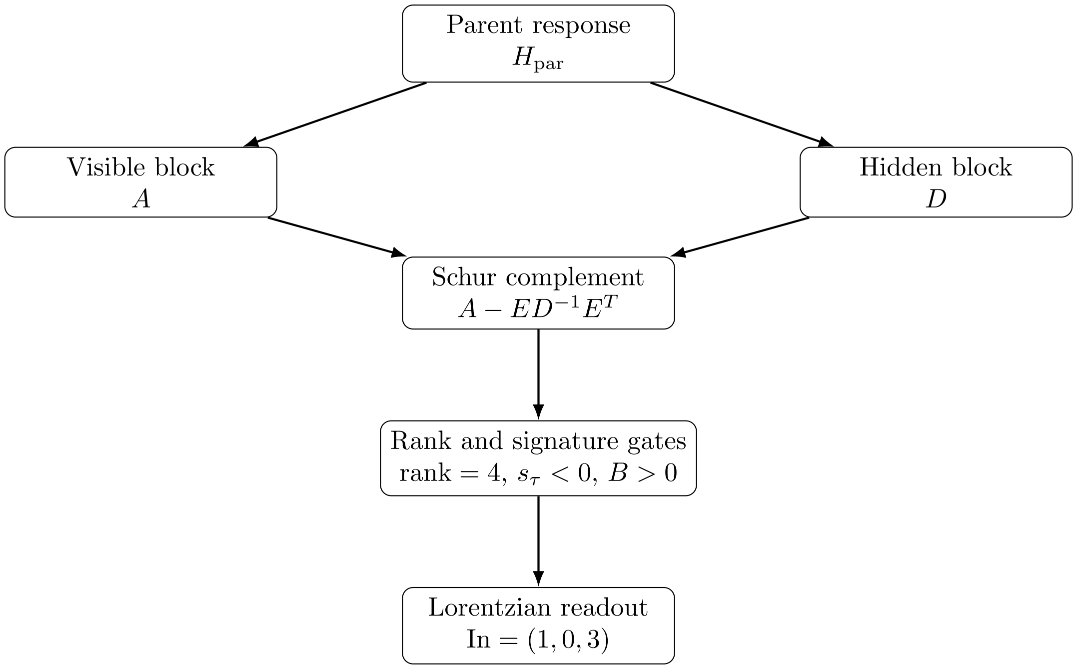


Figure 5: A finite parent-to-readout reduction. Hidden-sector elimination produces the effective visible response; only after descent, rank, and signature tests does the visible response acquire a Lorentzian interpretation.

Thus the readout sector is Lorentzian. Meanwhile the hidden block $D$ contributes two positive directions to the parent inertia. Haynsworth additivity gives

$$\mathrm{In}(H_{\mathrm{par}}) = \mathrm{In}(D) + \mathrm{In}(H/D) = (0, 0, 2) + (1, 0, 3) = (1, 0, 5). \tag{54}$$

The parent response has one negative and five positive directions, but the four-dimensional readout sector alone carries the Lorentzian interpretation.

The example also illustrates why the visible block alone is not sufficient. If one inspected $A$ before eliminating hidden variables, the clock Schur scalar would be different. The effective geometry must be read from the reduced response, not from an unreduced parent block. This is the algebraic analogue of integrating out or stationarizing unobserved variables before assigning effective physical meaning to the remaining sector.

## 16 Gate-check algorithm

The construction can be summarized as a finite gate-check procedure. This is useful both for later mathematical work and for preventing ambiguous claims in prose. Given a candidate parent operator and quotient-clock data, one should not ask immediately whether a Lorentzian metric has been found. One should ask whether the following sequence of tests has been passed.

| Gate | Required check |
|---|---|
| Quotient descent | The variables, variations, observables, clock covector, and response form must be invariant under physical equivalence. |
| Clock gate | A nonzero quotient-descended clock covector $\theta$ must exist on the candidate readout sector. |

| | |
|---|---|
| Reduction gate | Hidden or unresolved variables must be eliminated by a controlled operation, such as Schur reduction with invertible hidden block. |
| Rank gate | The physical readout projection must have four-dimensional image and stable rank under admissible redescription. |
| Nondegeneracy gate | The descended response form on the readout sector must be nondegenerate. |
| Spatial positivity gate | The restriction of the response form to $\ker\theta$ must be positive definite in the chosen sign convention. |
| Clock sign gate | The clock Schur scalar $s_\tau$ must be negative. |
| Inertia accounting | Eliminated and unobserved sectors must be accounted for by an inertia additivity theorem. |
| Frame gate | The resulting form must transform by congruence, with Lorentz frames related by $\mathrm{O}(1,3)$. |

Passing all gates licenses the phrase "effective Lorentzian readout". Failing any gate does not necessarily invalidate the parent theory; it invalidates this particular geometric interpretation at this layer. This distinction matters because a premetric substrate may contain regions, branches, or phases that are non-Lorentzian, lower-dimensional, degenerate, or not clock-compatible. Such possibilities should not be erased by terminology.

### 16.1 Falsification ledger

The gate algorithm can also be read as a falsification ledger. This is editorially useful because it prevents later sections of the broader program from converting conditional statements into narrative claims. Each row below records a precise way in which the Lorentzian readout interpretation can fail.

| Failure mode | Mathematical signal | Consequence |
|---|---|---|
| No quotient descent | Observables, clock, or response depend on representatives rather than equivalence classes | The construction is presentation-level and cannot be assigned physical readout status. |
| Clock collapse | $\theta = 0$ on the candidate sector | No clock-neutral complement of codimension one is defined. |
| Rank mismatch | $\dim R \neq 4$ or rank is unstable under redescription | The sector is not a four-dimensional readout. |
| Degenerate response | $\det h = 0$ | The sector has null response directions and no regular Lorentzian metric readout. |
| Spatial failure | $B$ is not positive definite | The clock-neutral complement is not a three-dimensional spatial sector in the chosen sign convention. |
| Clock sign failure | $s_\tau \geq 0$ | The response does not contain one negative Schur clock direction. |
| Schur audit failure | Parent inertia does not account for the readout inertia | The claimed Lorentzian form is an artifact of reduction or projection. |

| | | |
|---|---|---|
| Frame failure | Transformations do not act by congruence on $h$ | The apparent signature is frame-dependent and therefore not physical. |

This ledger is intentionally severe. A construction that fails one row may still be mathematically interesting, but it is not the Lorentzian readout theorem proved in this paper. The terminology should follow the gates rather than precede them.

## 17 Relation to geometric and physical reconstructions

The present result should be compared carefully with established reconstruction programs. Ehlers, Pirani, and Schild reconstruct conformal and projective structures from light propagation and free fall [3]. Their operational ingredients are physically compelling, but they are already spacetime-facing: rays, particles, and projective behavior are not primitive relational presentations. Malament-type results show how causal structure can determine conformal geometry under suitable hypotheses [8]. Causal-set theory begins from a locally finite partial order interpreted causally [6, 7]. These approaches illuminate how metric geometry can be constrained or reconstructed from physically meaningful structures, but they do not perform the exact rank-and-signature gate studied here.

The present paper is closer in mathematical spirit to linear-algebraic reduction and inertia theory. Schur complements, Haynsworth additivity, and Sylvester's law provide the rigorous skeleton [10, 11, 12, 13]. The physical novelty lies in their placement inside a quotient-clock pipeline. The clock covector determines the candidate one-dimensional temporal factor, the response form determines the actual sign of that factor after coupling to the clock-neutral complement, and inertia theory proves that the result is coordinate-independent. This makes the Lorentzian interpretation conditional, testable, and stable.

The paper also differs from approaches that begin with a tetrad or frame field. In tetrad formulations of general relativity, one starts with a Lorentzian internal metric and a soldering form relating internal and tangent indices. Here the Lorentzian internal form appears only after the effective response has passed the signature gate. A frame field can then be introduced as a readout map, but it is not a primitive of the substrate. This inversion of order is conceptually important: the frame expresses an already selected Lorentzian sector; it does not select the sector by decree.

## 18 Mathematical contract and limitations

For clarity, we state the exact contract of the paper. The results prove that, given a physical quotient, a quotient-descended clock covector, a descended effective Hessian, a rank-four readout projection, a nondegenerate spatial restriction, and a negative clock Schur scalar, the effective readout sector is Lorentzian and frame-invariant. They also prove that this condition is open under perturbation and compatible with Schur inertia accounting. These are mathematical theorems about the indicated structures.

The results do not prove that the universe must have four dimensions. They do not prove that a generic relational substrate selects a Lorentzian chamber. They do not derive the Einstein equations, matter fields, gauge interactions, cosmic history, or particle spectra. They do not replace the need for a dynamical selection principle. The paper provides the gate through which a premetric relational dynamics must pass before it can legitimately be interpreted as supporting effective spacetime geometry.

This limitation is a strength rather than a weakness. It separates the mathematical existence of a Lorentzian readout from the physical problem of why such a readout is selected. A later theory may attempt to derive the rank-four condition and the sign chamber from a parent operator. That later derivation must satisfy the diagnostics developed here. If it fails rank, descent, clock non-nullity, spatial positivity, Schur negativity, or inertia accounting, then it has not produced the Lorentzian readout claimed.

## 19 Conclusion

We have constructed the third layer of the premetric affine program in a form that can be stated independently of the later physical sectors. Starting from a quotient-clock-effective-Hessian structure, we defined the rank-four readout gate, the clock-spatial splitting gate, and the Lorentzian signature gate. We proved that a positive clock-neutral block together with a negative clock Schur scalar gives inertia $(1, 0, 3)$, and we showed that the resulting Lorentzian readout is invariant under frame changes, admits Lorentz frames, produces a null cone only after the signature gate, and remains stable under small perturbations inside an open chamber.

The central message is not that spacetime has been derived from nothing. The central message is that a precise finite-dimensional test exists between relational dynamics and effective Lorentzian geometry. The test is algebraic, quotient-aware, clock-aware, and falsifiable. It identifies exactly what must be true before one may speak of an effective spacetime readout without having assumed a spacetime metric at the start. In that sense the paper supplies the missing bridge between relational time and Lorentzian geometry, while leaving the dynamical selection of the bridge as the next problem.

## A Completion of squares for the clock block

We give the elementary calculation behind theorem 5.2 in coordinate-free language. Let $R = \mathbb{R}u \oplus W$, let $B = h|_W$ be nondegenerate, and define the linear functional $\ell \in W^*$ by $\ell(w) = h(u, w)$. Since $B$ is nondegenerate, there is a unique $z \in W$ such that

$$B(z, w) = \ell(w) \tag{55}$$

for all $w \in W$. In coordinates, $z = B^{-1}c$. Define

$$u_\perp = u - z. \tag{56}$$

Then for every $w \in W$,

$$h(u_\perp, w) = h(u, w) - h(z, w) = \ell(w) - B(z, w) = 0. \tag{57}$$

Thus $u_\perp$ is $h$-orthogonal to $W$. Moreover,

$$\begin{aligned} h(u_\perp, u_\perp) &= h(u - z, u - z) && (58)\\ &= h(u,u) - 2h(u,z) + h(z,z) && (59)\\ &= a - 2B(z,z) + B(z,z) && (60)\\ &= a - B(z,z). && (61) \end{aligned}$$

In coordinates $B(z,z) = c^T B^{-1} c$, so $h(u_\perp, u_\perp) = s_\tau$. Therefore

$$R = \mathbb{R}u_\perp \oplus W \tag{62}$$

is an $h$-orthogonal decomposition, and the inertia of $h$ is the sum of the inertia of the scalar $s_\tau$ and the inertia of $B$.

# B Rank-four minors and local persistence

Let $P(\lambda) : V \to R_0$ be a family of matrices depending continuously on parameters $\lambda$. If $P(\lambda_0)$ has rank four, then some $4 \times 4$ minor has nonzero determinant at $\lambda_0$. By continuity, the same determinant remains nonzero in a neighborhood of $\lambda_0$. Thus the condition $\operatorname{rank} P \geq 4$ is locally stable. If the codomain is four-dimensional, this is exactly the stability of surjectivity. If the codomain dimension is larger, maintaining rank exactly four additionally requires that all five-by-five minors vanish, which is not generally an open condition. For this reason the rank-four gate should be understood as a selected readout projection of fixed four-dimensional codomain, not as a generic rank condition in an arbitrary larger codomain.

This observation is useful when reading a candidate construction. If the construction says that rank four emerges from a larger visible space, it must either provide a stable projection mechanism or explain why higher-rank readout directions are physically invisible. Otherwise rank four is not yet a robust output.

# C Lorentz group as residual frame freedom

Let $e$ and $e'$ be Lorentz frames of the same Lorentzian readout sector. If $A$ is the change-of-frame matrix such that $e' = eA$, then the metric matrices satisfy

$$\eta = A^T \eta A. \tag{63}$$

Thus $A \in \mathrm{O}(1,3)$. Conversely, if $A \in \mathrm{O}(1,3)$ and $e$ is a Lorentz frame, then $eA$ is also a Lorentz frame. This proves that the Lorentz group is exactly the residual freedom among frames that preserve the effective readout form. No independent postulate of Lorentz symmetry is required at this algebraic level once the signature gate has been passed.

# D Quantitative gate margins

For later use we collect the numerical form of the rank and signature checks. Let $P$ be represented by a matrix with four-dimensional codomain. Surjectivity is equivalent to the existence of at least one nonzero $4 \times 4$ minor. A practical stability margin is

$$\rho_P = \max_I |\det P_I|, \tag{64}$$

where $P_I$ ranges over all $4 \times 4$ submatrices obtained by choosing four input columns. If $\rho_P > 0$, the rank-four projection is locally stable under sufficiently small perturbations of $P$. The margin does not by itself explain why the codomain is four-dimensional; it only checks robustness once the codomain has been selected.

For the signature gate, the two useful margins are

$$\beta = \lambda_{\min}(B), \qquad \sigma = -s_\tau. \tag{65}$$

The readout is safely inside the Lorentzian chamber when both are positive and separated from zero. If either margin approaches zero, the sector approaches a boundary: $\beta \to 0$ gives spatial degeneracy, while $\sigma \to 0$ gives a null clock-Schur direction. Thus the same algebra that proves existence also provides a diagnostic scale for numerical or symbolic candidates.

## E Gate checklist for later use

A later derivation that claims to obtain an effective spacetime sector from a parent relational operator should provide the following objects and checks. First, it must identify the physical quotient and show that the relevant variations descend to it. Second, it must provide a quotient-descended clock covector. Third, it must define a reduced response form by a controlled elimination or projection procedure, with inertia accounting for eliminated sectors. Fourth, it must identify a four-dimensional readout projection whose rank is stable under presentation changes. Fifth, it must show that the clock-neutral restriction of the response form is positive definite and that the clock Schur scalar is negative. Sixth, it must verify that the resulting Lorentzian form is not an artifact of a hidden metric or hidden time parameter. Only after these checks is it legitimate, in the terminology of this paper, to speak of an effective Lorentzian spacetime readout.

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