Relational Time, Quotient Dynamics and Effective Readout from a Premetric Affine Substrate

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Abstract

We develop a mathematical construction of relational time and effective readout starting from a premetric affine substrate. The primitive data are not spacetime points, coordinates, a metric tensor, or an external temporal parameter, but relational presentations, a physical equivalence relation, invariant observables, and admissible transitions between quotient states. The first part of the paper proves the universal quotient property and the descent theorem for observables, making explicit the condition under which a proposed physical quantity is independent of presentation. The second part formulates relational clock existence as a rankability problem. Finite acyclic transition domains admit strict clocks by the longest-chain construction, cyclic domains obstruct strict clocks, and interval-compatible clocks are unique up to positive affine reparametrization. The third part introduces finite-dimensional variation spaces and derives an effective Hessian on observable variations by stationary hidden-sector elimination. The Schur complement is then paired with an explicit inertia theorem, so that spectral information removed from the visible sector remains accounted for by the hidden block. The result is not a derivation of Lorentzian spacetime, matter, or cosmology. It is a controlled pre-spacetime layer: a quotient, a clock, and a reduced response operator obtained without hidden use of metric or temporal primitives.

Contents

1 Introduction

The concept of time plays two incompatible roles in fundamental physics. It is indispensable for formulating dynamics, yet its status varies sharply across theories. In Newtonian mechanics time is an external universal parameter [2]. In general relativity it is encoded in Lorentzian geometry and measured along worldlines, so it is no longer external but it is still available through the differentiable spacetime structure [4]. In ordinary quantum theory the temporal parameter typically enters at the kinematical level, either explicitly or through the background spacetime on which fields are defined. These frameworks disagree about what time is, but they agree on one operational fact: a sufficiently robust temporal ordering exists before the main dynamical laws are written.

The purpose of this paper is to isolate the mathematical step that must precede any claim that time is emergent. Before one can derive a metric, a light cone, a Hamiltonian, or a cosmological history, one must first explain how an ordering parameter can be obtained from data that do not already contain time in disguise. This is a delicate problem. A coordinate label can mimic time, a gauge choice can impose an apparent direction, and a causal order can silently import the structure one meant to derive. We therefore begin from a deliberately austere object: a premetric relational substrate. It consists of presentations, physical equivalence between presentations, invariant observables, and admissible transitions between physical quotient states. It carries no primitive manifold, topology, metric tensor, causal cone, or external evolution parameter.

The guiding idea is that dynamics should not be assigned to presentations. A presentation is only a way of encoding relational content. Different presentations may describe the same physical state. The first step is therefore to form the quotient by physical equivalence. Only objects that descend to this quotient are admitted as physical in the sense used here. This quotient discipline is not a formal luxury; it is the mechanism that prevents a clock from being a property of notation, parametrization, or redundant representation. If a candidate time variable does not descend to the quotient, it may be useful as a gauge coordinate, but it is not a relational clock.

Once the quotient is formed, the problem becomes order-theoretic. A transition relation on quotient states represents admissible succession, replacement, refinement, or comparison. A strict relational clock is a real-valued function that increases along every admissible transition. Such a clock need not exist. Directed cycles are immediate obstructions, and infinite domains require rankability or an equivalent order-representation hypothesis. The paper therefore treats clock existence as a theorem with hypotheses, not as a slogan. In finite acyclic domains we prove existence by the longest-chain construction; in cyclic domains we prove obstruction; in domains where interval comparison is physically meaningful we prove that clock freedom reduces to positive affine reparametrization.

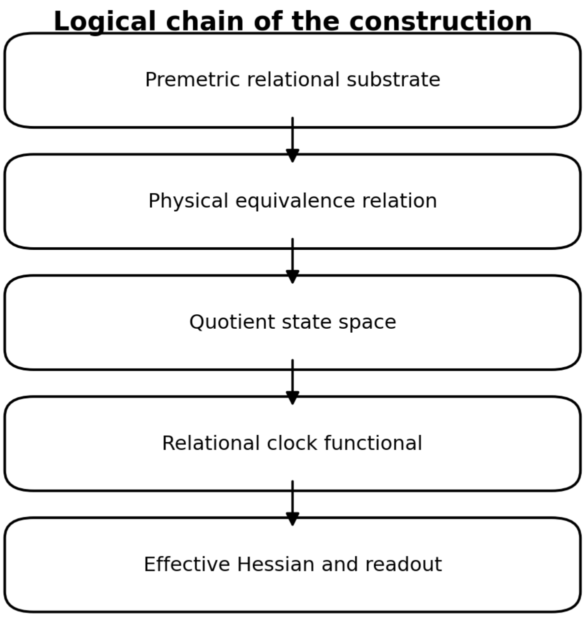

Figure 1: Dependency structure of the paper. The construction first removes presentation redundancy by quotienting, then asks whether the quotient transition structure is clock-admissible, and only afterwards introduces a quadratic response operator. This order is the main safeguard against hidden spacetime assumptions.

The next problem is readout. A clock orders states, but an effective dynamics requires a response operator on variations. Since no manifold is assumed, variations are introduced algebraically as finite-dimensional vector spaces assigned to physical states. A parent quadratic response form is split into observable and hidden sectors. Under invertibility of the hidden block, stationary elimination gives the Schur complement. The reduced Hessian is therefore not guessed; it is obtained by minimizing or stationarizing the hidden sector while holding the observable variation fixed. The corresponding inertia theorem shows that eliminated directions are not discarded: the inertia of the parent form is the componentwise sum of the inertia of the hidden block and the inertia of the effective Hessian.

This paper is the second layer of a larger foundational program, but it is written to stand on its own. It uses standard quotient reasoning, elementary order theory, affine comparison, Schur complements, and inertia additivity. The conceptual novelty is not a new algebraic theorem in isolation. It is the dependency order: quotient first, clock second, response third. This order is what makes it possible to discuss effective dynamics without assuming spacetime primitives. The work is related to relational motivations associated with Leibniz, Mach, and modern background-independent approaches, to operational reconstructions such as Ehlers-Pirani-Schild, to causal set theory, and to standard linear-algebraic reduction methods. It differs from all of these at the point of departure: causal order, metric geometry, and worldlines are not assumed.

1.1 Main results

The paper proves six groups of results. First, theorem 3.1 gives the universal property of the physical quotient and theorem 3.2 characterizes exactly which quantities descend to physical observables. Second, theorem 4.3 gives the descent criterion for transition relations, and theorem 4.4 formulates the corresponding no-hidden-time diagnostic. Third, theorems 4.7 and 5.1 prove the cycle obstruction and finite acyclic clock existence theorem. Fourth, theorem 5.7 proves positive affine uniqueness when interval scale is part of the physical structure. Fifth, theorem 7.1 derives the effective Hessian by stationary hidden-sector elimination. Sixth, theorem 7.5 proves inertia additivity by congruence, ensuring that the spectral content of the parent response is not lost under reduction. The remaining sections supply examples, functoriality under redescription, strongly connected condensation, a finite clock algorithm, singular-block warnings, a formal definition of premetric effective dynamics, and the boundary between order, duration, and metric time.

1.2 Reading guide and mathematical contract

The word “time” is used in this paper in a restricted mathematical sense. A relational clock is an invariant rank or order-representation function on physical quotient states. It is not yet metric proper time, thermodynamic time, quantum measurement time, or cosmological time. The word “dynamics” is also used in a restricted sense. An effective readout consists of a clock, an observable variation sector, and a reduced quadratic response operator. It is not yet a nonlinear field equation. This contract is important because the later layers of a physical theory may require Lorentzian signature, smooth patching, hyperbolicity, matter coupling, and observational calibration. None of those later layers is imported into the proofs below.

1.3 What is not claimed

We do not prove that every relational substrate has a clock. Cyclic domains and non-rankable partial orders provide immediate counterexamples. We do not prove that the clock constructed here is the time measured by macroscopic clocks in our universe. That identification requires additional readout, stability, and metric gates. We do not prove Lorentzian signature, spatial dimensionality, matter content, gauge structure, or cosmological dynamics. The present paper supplies the mathematical preconditions for posing those later problems without circularity.

2 Premetric relational substrates

2.1 Presentations and physical equivalence

Let $\mathcal{S}$ be a nonempty set. Elements of $\mathcal{S}$ are called *relational presentations.* At this stage an element $x \in \mathcal{S}$ is not a point of space, not an event in spacetime, and not a coordinate tuple. It is a complete formal presentation of the relational data available at the level under consideration. The word presentation is deliberately used because the same physical content may have different formal encodings.

Definition 2.1 (Physical equivalence)**.** *A physical equivalence relation on $\mathcal{S}$ is an equivalence relation $\sim$ on $\mathcal{S}$. If $x \sim y$, the presentations x and y are physically indistinguishable at the level of structure represented by $\sim$. The equivalence class of x is denoted*

$$[x] = \{y \in \mathcal{S} : y \sim x\}.$$

The quotient set of physical states is

$$\mathcal{Q} = \mathcal{S}/\!\sim = \{[x] : x \in \mathcal{S}\}.$$

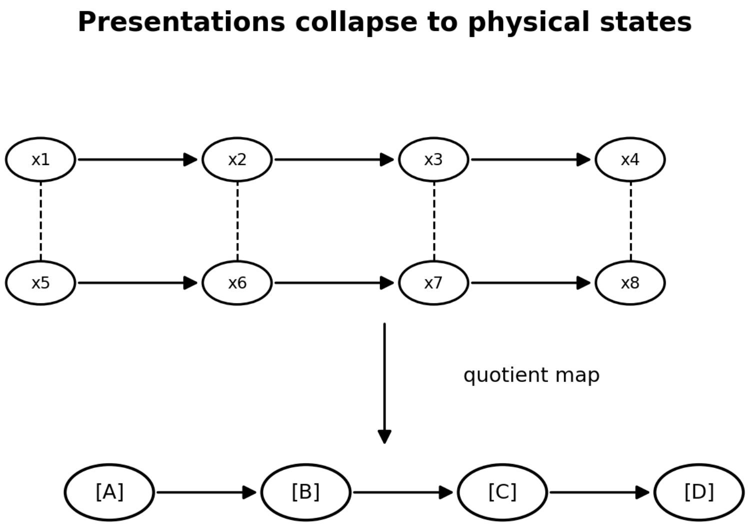

Figure 2: Presentation-level data may contain redundant copies of the same physical state. The quotient is the layer on which observables, transitions, and clocks must be defined if they are to be invariant under redescription.

The quotient map is denoted $\pi : \mathcal{S} \to \mathcal{Q}$, $\pi(x) = [x]$.

The definition does not specify how $\sim$ is chosen. In concrete applications it may be induced by gauge transformations, redescription symmetries, observational indistinguishability, or equality of a chosen family of primitive relational observables. The present paper treats $\sim$ as part of the substrate data, because theorems about quotient descent do not depend on its origin.

The separation between presentation and physical state is essential. If one works directly on $\mathcal{S}$, any construction may depend on descriptive redundancy. If one works on $\mathcal{Q}$, every construction is invariant under $\sim$ by definition. This is the first anti-circularity step: time cannot be read from a presentation-dependent coordinate if the clock is required to live on $\mathcal{Q}$.

2.2 Observables and invariant content

Let Y be a set of possible values. A function $O : \mathcal{S} \to Y$ is not automatically physical. It is physical only if it does not distinguish presentations declared equivalent by $\sim$. This is expressed by the following definition.

Definition 2.2 (Equivalence-invariant observable)**.** *A function* $O : \mathcal{S} \to Y$ *is* $\sim$*-invariant if*

$$x \sim y \implies O(x) = O(y).$$

When $Y = \mathbb{R}$*, such functions will be called real relational observables.*

A family $\mathcal{F}$ of invariant real observables can be used to define or test the equivalence relation. One common case is

$$x \sim_{\mathcal{F}} y \iff f(x) = f(y) \quad \text{for all } f \in \mathcal{F}.$$

If $\sim = \sim_{\mathcal{F}}$, the family $\mathcal{F}$ is separating on the quotient by construction. In a more general setting, $\mathcal{F}$ may be smaller than the full invariant algebra and may fail to separate all quotient states.

This distinction matters for clock existence, because a clock is a particular invariant observable with a monotonicity property.

Definition 2.3 (Separating family)**.** *A family $\mathcal{F}$ of $\sim$-invariant real observables separates $\mathcal{Q}$ if for any two distinct classes $[x] \neq [y]$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*

Separation is weaker than coordinatization. A separating family may be infinite, may not define a manifold chart, and may not carry a metric. It merely ensures that distinct physical states can be distinguished by some invariant readout. This is the minimal form of distinguishability required in the present paper.

2.3 Affine transport as comparison data

The previous definitions concern states and observables. Dynamics also requires comparison between states. At the premetric level comparison is not distance measurement and not parallel transport on a manifold. We model comparison as transport of readable affine data along admissible transitions.

Let A be an affine space modeled on a vector space E. A readable affine datum is an element of A. An affine comparison from one presentation to another is an affine map $T : \mathsf{A} \to \mathsf{A}$. If $T(a) = L(a) + b$ after choosing an origin, the linear part $L : E \to E$ and translation part $b \in E$ encode how relational data are re-expressed between presentations. No metric is needed to define affine maps.

Definition 2.4 (Affine transport substrate)**.** *An affine transport substrate consists of $(\mathcal{S}, \sim, \mathsf{A}, \mathcal{T})$, where $\mathcal{S}$ is a set of presentations, $\sim$ is a physical equivalence relation, A is an affine space, and $\mathcal{T}$ assigns to each admissible comparison $x \to y$ an affine map $T_{yx} : \mathsf{A} \to \mathsf{A}$. Whenever a composite comparison $x \to y \to z$ is admissible, the associated defect is*

$$\Omega(z, y, x) = T_{xz}^{-1} T_{yz} T_{xy},$$

when the inverse exists.

The defect Ω measures path dependence of comparison. In a smooth geometry it would resemble holonomy. Here it is only an affine comparison defect. This distinction is important because holonomy language can suggest an already existing connection on a manifold. The present structure is weaker and earlier.

3 The physical quotient

3.1 Universal property

The quotient $\mathcal{Q}$ is not only a set of classes. It is characterized by a universal property: any map from presentations to another set that is constant on equivalence classes factors uniquely through the quotient. This is the exact formal content behind the statement that physical constructions must descend to $\mathcal{Q}$.

Theorem 3.1 (Quotient universal property)**.** *Let $\pi : \mathcal{S} \to \mathcal{Q} = \mathcal{S}/\sim$ be the quotient map. For every set Y and every function $F : \mathcal{S} \to Y$ satisfying*

$$x \sim y \quad \Longrightarrow \quad F(x) = F(y),$$

there exists a unique function $\bar{F} : \mathcal{Q} \to Y$ such that

$$F = \bar{F} \circ \pi.$$

Equivalently, the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, "F"] \arrow[d, "\pi"'] & Y \\ \mathcal{Q} \arrow[ur, dashed, "\bar{F}"'] & \end{tikzcd}$$

commutes.

Proof. Define $\bar{F}([x]) = F(x)$. This is well-defined because if $[x] = [y]$, then $x \sim y$, and by hypothesis $F(x) = F(y)$. The equality $F = \bar{F} \circ \pi$ follows immediately from $(\bar{F} \circ \pi)(x) = \bar{F}([x]) = F(x)$. If $G : \mathcal{Q} \to Y$ also satisfies $F = G \circ \pi$, then for every class $[x]$ one has $G([x]) = G(\pi(x)) = F(x) = \bar{F}([x])$. Thus $G = \bar{F}$, proving uniqueness. □

Corollary 3.2 (Observable descent). *A function $O : \mathcal{S} \to Y$ defines a physical observable on $\mathcal{Q}$ if and only if it is $\sim$-invariant. In that case the induced observable $\bar{O} : \mathcal{Q} \to Y$ is unique.*

Proof. If O is invariant, the result is theorem 3.1. Conversely, if $O = \bar{O} \circ \pi$, then $x \sim y$ implies $\pi(x) = \pi(y)$, hence $O(x) = \bar{O}(\pi(x)) = \bar{O}(\pi(y)) = O(y)$. □

This theorem is elementary, but it is the reason that later constructions are made on $\mathcal{Q}$. If a proposed clock is a function on $\mathcal{S}$ that does not descend to $\mathcal{Q}$, then it measures presentation rather than physical state. Such a function may be useful as a gauge-fixing coordinate, but it cannot be the relational clock of the physical quotient.

3.2 Equivalence generated by observables

The converse construction is also important. Given a family of observables on $\mathcal{S}$, one can define physical equivalence as equality of all those observables. This produces a quotient whose points are exactly the empirically indistinguishable classes with respect to the chosen family.

Proposition 3.3 (Observable-generated quotient). *Let $\mathcal{F}$ be a family of functions $f : \mathcal{S} \to \mathbb{R}$. Define*

$$x \sim_{\mathcal{F}} y \quad \Longleftrightarrow \quad f(x) = f(y) \text{ for all } f \in \mathcal{F}.$$

Then $\sim_{\mathcal{F}}$ is an equivalence relation. Each $f \in \mathcal{F}$ descends to a function $\bar{f} : \mathcal{S}/{\sim_{\mathcal{F}}} \to \mathbb{R}$, and the descended family separates the quotient.

Proof. Reflexivity, symmetry, and transitivity follow from equality in $\mathbb{R}$. Each $f \in \mathcal{F}$ is invariant by definition of $\sim_{\mathcal{F}}$, so it descends by theorem 3.2. If $[x] \neq [y]$, then $x \not\sim_{\mathcal{F}} y$, hence there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Thus $\bar{f}([x]) \neq \bar{f}([y])$, so the family separates. □

The proposition gives a clean way to pass from raw distinguishability to a physical state space. It also clarifies a limitation. If $\mathcal{F}$ is chosen too small, the quotient may identify states that a richer theory would distinguish. If $\mathcal{F}$ is chosen too large, it may fail to respect physical redundancy. The quotient is only as meaningful as the equivalence relation that defines it.

4 Transition relations and quotient dynamics

4.1 Physical transition structures

A quotient state space by itself is static. To discuss relational time one needs a structure that says when one physical state can succeed, replace, refine, or be compared after another. We encode this by a relation on $\mathcal{Q}$, not on $\mathcal{S}$.

Definition 4.1 (Transition relation). *A physical transition relation on $\mathcal{Q}$ is a binary relation*

$$\rightsquigarrow \subseteq \mathcal{Q} \times \mathcal{Q}.$$

We write $q \rightsquigarrow q'$ when q' is an admissible successor of q. A finite chain is a sequence $q_0, q_1, \ldots, q_n$ such that $q_i \rightsquigarrow q_{i+1}$ for every i.

The relation $\rightsquigarrow$ is not assumed to be total, antisymmetric, transitive, causal, or generated by a differential equation. In applications it may arise from admissible transformations of relational data, allowed updates of a substrate, or coarse-grained transition classes. The only requirement at this stage is that it be defined on physical states rather than presentations.

Definition 4.2 (Presentation-level compatibility). *Suppose $\rightsquigarrow_S$ is a relation on $\mathcal{S}$. It is compatible with $\sim$ if whenever $x \sim x'$, $y \sim y'$, and $x \rightsquigarrow_S y$, one has $[x'] \rightsquigarrow [y']$ whenever the quotient relation is defined by $[x] \rightsquigarrow [y]$. Equivalently, $\rightsquigarrow_S$ descends to a well-defined relation on $\mathcal{Q}$.*

Proposition 4.3 (Descent of transitions). *A presentation-level relation $\rightsquigarrow_S$ descends to a well-defined relation $\rightsquigarrow$ on $\mathcal{Q}$ by*

$$[x] \rightsquigarrow [y] \quad \Longleftrightarrow \quad x \rightsquigarrow_S y$$

if and only if the truth value of $x \rightsquigarrow_S y$ is invariant under replacing x and y by equivalent presentations.

Proof. If the quotient relation is well-defined, then $[x] = [x']$ and $[y] = [y']$ imply that $[x] \rightsquigarrow [y]$ and $[x'] \rightsquigarrow [y']$ have the same truth value. Hence $x \rightsquigarrow_S y$ is invariant under equivalent replacements. Conversely, if this invariance holds, the proposed definition of $[x] \rightsquigarrow [y]$ does not depend on the representatives chosen, so it defines a relation on the quotient. $\square$

This proposition is a second anti-circularity condition. It prevents one from introducing a time-ordering relation that is secretly a relation between coordinate descriptions. If the transition relation fails to descend, it is not yet physical.

Proposition 4.4 (No-hidden-time diagnostic). *Let $\theta : \mathcal{S} \to \mathbb{R}$ be a presentation-level ordering label and suppose that a transition relation is defined by the rule*

$$x \rightsquigarrow_S y \quad \Longrightarrow \quad \theta(y) > \theta(x).$$

If θ is not constant on equivalence classes, then θ does not define a physical relational clock on $\mathcal{Q}$. Moreover, any transition rule whose truth value changes under replacement of representatives cannot define a physical transition relation on the quotient.

Proof. If θ is not constant on equivalence classes, there exist $x \sim x'$ with $\theta(x) \neq \theta(x')$. Hence there is no function $\bar{\theta} : \mathcal{Q} \to \mathbb{R}$ satisfying $\theta = \bar{\theta} \circ \pi$, by the observable descent theorem. Thus

θ is a presentation label rather than a quotient observable. If the transition rule depends on such a label in a way that changes when representatives are replaced, then the truth value of $x \rightsquigarrow_S y$ is not invariant under equivalence. By theorem 4.3, the relation fails to descend to $\mathcal{Q}$. Therefore the rule cannot define a physical transition relation at the level required for a relational clock. $\square$

This proposition is the formal version of the main methodological warning of the paper. One can always write a parameter on a presentation space. The relevant question is whether the parameter survives quotienting. Only after this test has been passed can it be considered a candidate relational clock.

4.2 Acyclicity and the obstruction to strict clocks

A strict clock cannot exist on a directed cycle. This simple fact is the main reason clock existence must be formulated as a theorem with hypotheses rather than as a slogan.

Definition 4.5 (Directed cycle)**.** *A directed cycle in $(\mathcal{Q}, \rightsquigarrow)$ is a finite sequence $q_0, q_1, \ldots, q_{n-1}$ with $n \geq 1$, all q_i distinct, such that*

$$q_i \rightsquigarrow q_{i+1} \quad (0 \leq i < n-1), \qquad q_{n-1} \rightsquigarrow q_0.$$

The relation is acyclic if it has no directed cycles.

Definition 4.6 (Strict relational clock)**.** *A strict relational clock on $(\mathcal{Q}, \rightsquigarrow)$ is a function $\tau : \mathcal{Q} \to \mathbb{R}$ such that*

$$q \rightsquigarrow q' \quad \Longrightarrow \quad \tau(q') > \tau(q).$$

Theorem 4.7 (Cycle obstruction)**.** *If $(\mathcal{Q}, \rightsquigarrow)$ contains a directed cycle, then no strict relational clock exists on $\mathcal{Q}$.*

Proof. Assume there is a directed cycle $q_0 \rightsquigarrow q_1 \rightsquigarrow \cdots \rightsquigarrow q_{n-1} \rightsquigarrow q_0$ and a strict clock τ. Then

$$\tau(q_1) > \tau(q_0), \quad \tau(q_2) > \tau(q_1), \quad \ldots, \quad \tau(q_0) > \tau(q_{n-1}).$$

Combining the inequalities gives $\tau(q_0) > \tau(q_0)$, a contradiction. $\square$

Directed cycles do not necessarily make a theory inconsistent. They mean that the cycle cannot be strictly ordered by a real clock. One may collapse strongly connected components, allow periodic internal phases, or work with a non-strict time variable. Those are additional choices. The strict clock theorem applies only after the relevant cyclic structure has been removed or interpreted.

5 Clock theorems

5.1 Finite acyclic domains

The cleanest existence theorem is finite. It is also physically useful because many premetric substrates are naturally discrete or coarse-grained at the level where the first clock is sought.

Theorem 5.1 (Finite relational clock theorem)**.** *Let $(\mathcal{Q}, \rightsquigarrow)$ be a finite directed graph. If $\rightsquigarrow$ is acyclic, then there exists a strict relational clock $\tau : \mathcal{Q} \to \mathbb{N} \subset \mathbb{R}$. Moreover τ may be chosen as*

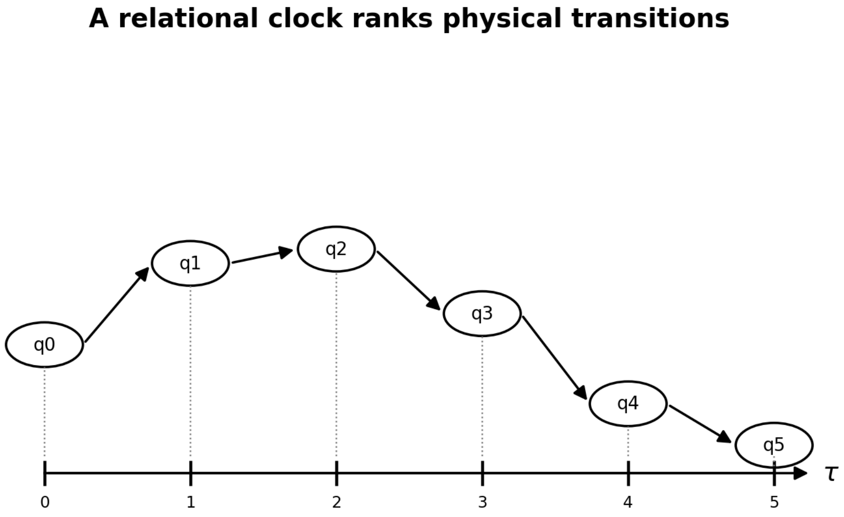

Figure 3: A strict relational clock is an order-representation of admissible transitions on quotient states. Equal values may occur only between incomparable states; every directed physical transition must increase the clock value.

the length of the longest directed chain ending at each vertex:

$$\tau(q) = \max\{n \in \mathbb{N} : \exists q_0 \rightsquigarrow q_1 \rightsquigarrow \cdots \rightsquigarrow q_n = q\}.$$

Proof. Because the graph is finite and acyclic, every directed chain has bounded length. Therefore the maximum in the definition of $\tau(q)$ exists. If $q \rightsquigarrow q'$, then any longest chain ending at q can be extended by the edge $q \rightsquigarrow q'$, giving a chain ending at q' whose length is $\tau(q) + 1$. Hence $\tau(q') \geq \tau(q) + 1 > \tau(q)$. Thus τ is a strict relational clock. □

The construction is canonical once the transition graph is given, but it is not unique. Many other strictly increasing functions exist. The longest-chain rank is useful because it is intrinsic to the directed graph and does not require choosing coordinates.

Corollary 5.2 (Topological clock)**.** *Every finite acyclic transition graph admits an injective refinement of the longest-chain clock. In particular there exists $\tau : \mathcal{Q} \to \mathbb{R}$ such that $q \rightsquigarrow q'$ implies $\tau(q') > \tau(q)$, and $\tau(q) \neq \tau(q')$ for all distinct states if desired.*

Proof. The longest-chain clock may assign the same value to incomparable states. Since $\mathcal{Q}$ is finite, choose any total ordering that refines the partial order generated by $\rightsquigarrow$. Assign consecutive real values in that total order. The resulting function is injective and strictly increases along every edge. □

5.2 Rankable domains beyond the finite case

The finite theorem is not the most general possible statement. For infinite domains, one must impose a representation condition. The correct abstraction is rankability.

Definition 5.3 (Rankable transition domain)**.** *A transition domain $(\mathcal{Q}, \rightsquigarrow)$ is rankable if there exists a function $\rho : \mathcal{Q} \to \mathbb{R}$ such that $q \rightsquigarrow q'$ implies $\rho(q') > \rho(q)$.*

This definition may look tautological because it is exactly the existence of a strict clock. Its value is methodological: it prevents hidden assumptions. For infinite domains, acyclicity alone need not guarantee a real-valued rank function without additional set-theoretic or order-theoretic hypotheses. Rankability is therefore the explicit gate required for a real clock.

Proposition 5.4 (Order embedding sufficient condition)**.** *Let $\preceq$ be the reflexive transitive closure of $\rightsquigarrow$. If there exists an order-preserving embedding $\iota : (\mathcal{Q}, \preceq) \to (\mathbb{R}, \leq)$ such that $q \rightsquigarrow q'$ implies $\iota(q) < \iota(q')$, then $\tau = \iota$ is a strict relational clock.*

Proof. This is immediate from the defining property of ι. For every transition $q \rightsquigarrow q'$, one has $\tau(q) = \iota(q) < \iota(q') = \tau(q')$. □

The proposition clarifies the relation between relational clocks and order theory. A clock is an order representation into the real line. The physical content lies in the transition relation; the real line is used only as a convenient ordered codomain.

5.3 Affine uniqueness of clocks

If a clock is required only to preserve order, then any strictly increasing reparametrization is equally valid. Physical clocks normally do more: they compare intervals. The moment interval composition becomes part of the structure, the freedom reduces from arbitrary monotone transformations to positive affine transformations.

Definition 5.5 (Interval-compatible clock)**.** *Let $(\mathcal{Q}, \rightsquigarrow)$ admit a class $\mathcal{C}$ of composable chains. A strict clock $\tau : \mathcal{Q} \to \mathbb{R}$ is interval-compatible if for any composable chains $\gamma_1 : q_0 \to q_1$ and $\gamma_2 : q_1 \to q_2$, the interval assigned to the composite satisfies*

$$\Delta_\tau(\gamma_2 \circ \gamma_1) = \Delta_\tau(\gamma_1) + \Delta_\tau(\gamma_2),$$

where $\Delta_\tau(q_i \to q_j) = \tau(q_j) - \tau(q_i)$.

For any real-valued function the displayed additivity holds algebraically for endpoints. The nontrivial requirement is that the physical interval scale be the same for two clocks. We formulate this as preservation of interval ratios.

Definition 5.6 (Same affine interval scale)**.** *Two strict clocks $\tau, \tau' : \mathcal{Q} \to \mathbb{R}$ represent the same affine interval scale if for all chains with nonzero τ-intervals, the ratio*

$$\frac{\tau'(q') - \tau'(q)}{\tau(q') - \tau(q)}$$

is independent of the chain endpoints (q, q') within each connected clock domain.

Theorem 5.7 (Affine uniqueness)**.** *Let $D \subseteq \mathcal{Q}$ be a connected clock domain. Suppose $\tau, \tau' : D \to \mathbb{R}$ are strict clocks representing the same affine interval scale. Then there exist constants $a > 0$ and $b \in \mathbb{R}$ such that*

$$\tau' = a\tau + b$$

on D.

Proof. By assumption the ratio of clock intervals is constant on the connected domain. Let that constant be a. Since both clocks are strict in the same transition orientation, $a > 0$. Fix a base state $q_0 \in D$ and define $b = \tau'(q_0) - a\tau(q_0)$. For any $q \in D$ connected to q_0 by a chain of comparable intervals, the same-scale condition gives

$$\tau'(q) - \tau'(q_0) = a(\tau(q) - \tau(q_0)).$$

Rearranging yields $\tau'(q) = a\tau(q) + b$. Connectedness extends the equality to all of D. $\square$

Remark 5.8. *If only order matters, affine uniqueness is false: $\tau' = f \circ \tau$ is a strict clock for every strictly increasing function f. The affine theorem therefore expresses an additional physical decision: intervals, not merely order, are being compared.*

6 Variations and tangent-like structures without manifolds

A dynamical readout requires variations. In a smooth theory variations are tangent vectors. In the present setting there is no manifold, so tangent vectors cannot be assumed. We instead introduce linearized variation spaces as additional local data at a quotient state.

Definition 6.1 (Variation space)**.** *A variation assignment on $\mathcal{Q}$ associates to each $q \in \mathcal{Q}$ a real vector space V_q. Elements $\delta q \in V_q$ are formal first-order physical variations at q. A family of observables $\bar{f} : \mathcal{Q} \to \mathbb{R}$ is differentiable relative to this assignment if each $\bar{f}$ has a linear first variation $\mathrm{d}\bar{f}_q : V_q \to \mathbb{R}$.*

This definition is intentionally weak. It does not require that $\mathcal{Q}$ be a differentiable manifold. It only requires that the observables under consideration have first variations on a chosen linear model. Such a model may arise from a chart, a finite-dimensional embedding, a discrete linearization, or a formal perturbation calculus.

Definition 6.2 (Separating first variations)**.** *A family $\mathcal{F}$ of differentiable observables separates variations at q if*

$$\mathrm{d}\bar{f}_q(v) = 0 \quad \text{for all } \bar{f} \in \mathcal{F}$$

implies $v = 0$ in V_q.

If first variations fail to separate, there are infinitesimal directions invisible to the chosen observables. Such directions may be gauge, hidden, or genuinely missing from the observational family. The effective Hessian constructed below must be interpreted relative to this choice.

Proposition 6.3 (Observable embedding of variations)**.** *Let $\mathcal{F} = \{\bar{f}_1, \dots, \bar{f}_m\}$ be a finite family separating variations at q. Then the linear map*

$$D_q : V_q \to \mathbb{R}^m, \qquad D_q(v) = (\mathrm{d}\bar{f}_1(v), \dots, \mathrm{d}\bar{f}_m(v))$$

is injective.

Proof. The kernel of D_q consists of vectors annihilated by all $\mathrm{d}\bar{f}_i$. Since the family separates variations, the only such vector is 0. Hence D_q is injective. $\square$

This proposition provides a minimal substitute for local coordinates. If the observable differentials separate variations, then variations can be represented by their observable responses. No metric is used.

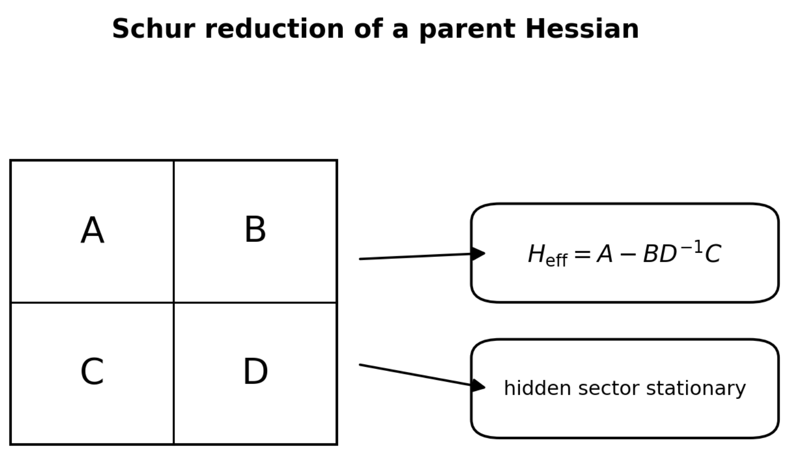

Figure 4: Stationary hidden-sector elimination. The reduced operator is not obtained by deleting hidden variables, but by solving the hidden stationary equation and substituting the solution into the parent quadratic response.

7 Quadratic response and effective Hessians

7.1 Parent quadratic forms

Let V be a finite-dimensional real vector space of variations at a fixed physical state. A parent quadratic response is a symmetric bilinear form

$$H : V \times V \to \mathbb{R}.$$

Equivalently, after choosing a basis, H is represented by a real symmetric matrix. The interpretation is that $H(v, v)$ is the second-order response of some local functional to the variation v. The functional itself need not be a spacetime action at this stage.

Suppose V is decomposed as

$$V = V_o \oplus V_h,$$

where V_o is an observable sector and V_h is a hidden, constrained, or eliminated sector. In block form,

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $C = B^T$ in the symmetric case. We assume D is invertible when performing stationary elimination.

Theorem 7.1 (Schur reduction theorem)**.** *Let $V = V_o \oplus V_h$ be finite-dimensional and let*

$$Q(x, y) = \frac{1}{2}x^T Ax + x^T By + \frac{1}{2}y^T Dy$$

be a quadratic form with $D = D^T$ invertible. For fixed x, the stationary hidden variation is

$$y_*(x) = -D^{-1}B^T x.$$

Substitution gives the reduced quadratic form

$$Q_{\text{eff}}(x) = \frac{1}{2}x^T(A - BD^{-1}B^T)x.$$

Thus the effective Hessian on V_o is

$$H_{\text{eff}} = A - BD^{-1}B^T.$$

Proof. For fixed x, differentiate $Q(x, y)$ with respect to y. The first variation in a hidden direction $\eta \in V_h$ is

$$\delta_y Q(x, y)[\eta] = x^T B\eta + y^T D\eta = (B^T x + Dy)^T \eta.$$

Stationarity for all η is therefore equivalent to $B^T x + Dy = 0$. Since D is invertible, the unique stationary point is $y_*(x) = -D^{-1}B^T x$. Substituting into Q gives

$$Q(x, y_*) = \frac{1}{2}x^T Ax - x^T BD^{-1}B^T x + \frac{1}{2}x^T BD^{-1}DD^{-1}B^T x.$$

The last two terms combine to $-\frac{1}{2}x^T BD^{-1}B^T x$, yielding

$$Q_{\text{eff}}(x) = \frac{1}{2}x^T(A - BD^{-1}B^T)x.$$

□

Remark 7.2. *The theorem is not a statement that hidden modes do not exist. It is a statement about stationary elimination under an invertible hidden block. If D has zero modes, if the hidden sector is dynamical, or if the stationary point is not physically admissible, the Schur formula is not the correct reduction.*

7.2 Completion of squares and congruence

The Schur complement also appears through a completion of squares. This form is useful because it leads directly to inertia additivity.

Lemma 7.3 (Block congruence factorization)**.** *Let*

$$H = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

with D invertible. Then

$$\begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}B^T & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix} = H.$$

Hence H is congruent to $H_{\text{eff}} \oplus D$.

Proof. Multiply the last two factors first:

$$\begin{pmatrix} H_{\text{eff}} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix} = \begin{pmatrix} H_{\text{eff}} & 0 \\ B^T & D \end{pmatrix}.$$

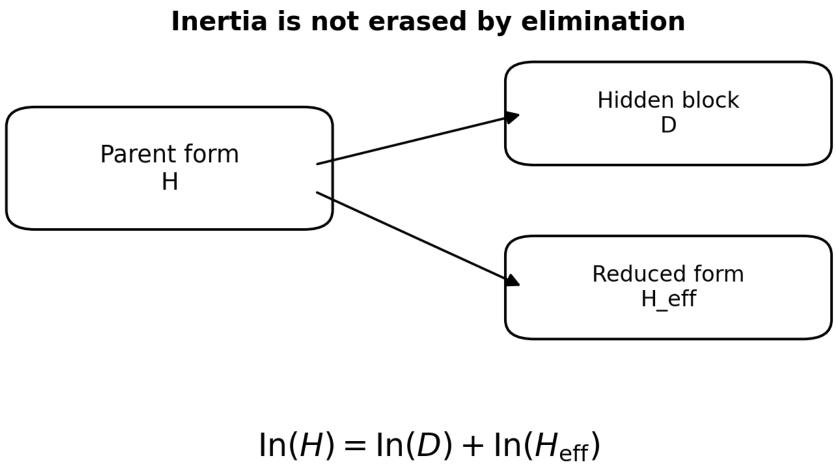

Figure 5: Inertia bookkeeping under Schur reduction. The effective Hessian carries the visible spectral contribution, while the hidden block retains the eliminated contribution; the parent inertia is the componentwise sum.

Multiplying on the left gives

$$\begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} H_{\text{eff}} & 0 \\ B^T & D \end{pmatrix} = \begin{pmatrix} H_{\text{eff}} + BD^{-1}B^T & B \\ B^T & D \end{pmatrix}.$$

Since $H_{\text{eff}} = A - BD^{-1}B^T$, the upper-left block is A, and the product is H. $\square$

Definition 7.4 (Inertia)**.** *For a real symmetric matrix M, the inertia is*

$$\mathrm{In}(M) = (n_+(M), n_-(M), n_0(M)),$$

where n_+, n_-, and n_0 are the numbers of positive, negative, and zero eigenvalues counted with algebraic multiplicity.

Theorem 7.5 (Haynsworth inertia additivity)**.** *Let $H = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$ be real symmetric with D invertible. Then*

$$\mathrm{In}(H) = \mathrm{In}(D) + \mathrm{In}(A - BD^{-1}B^T),$$

where the addition is componentwise.

Proof. By theorem 7.3, H is congruent to $H_{\text{eff}} \oplus D$. Sylvester's law of inertia states that congruent real symmetric matrices have the same inertia. The inertia of a direct sum is the componentwise sum of the inertias. Hence $\mathrm{In}(H) = \mathrm{In}(H_{\text{eff}}) + \mathrm{In}(D)$. $\square$

The inertia theorem is the main safeguard against a common mistake. Eliminating a hidden sector changes the visible quadratic form, but the full spectral accounting is not lost. If a negative direction resides entirely in the hidden block, it remains recorded in $\mathrm{In}(D)$. If it is transferred into the visible sector through coupling, it appears in $\mathrm{In}(H_{\mathrm{eff}})$. Either way, the parent system controls the reduced signature.

8 Effective readout without external time

8.1 Clock-parameterized response

Once a relational clock τ exists, one may use it as an ordering parameter on chains of quotient states. This does not mean that τ is an external time. It is an invariant function on physical states. A response equation written with respect to τ is therefore a readout along the quotient, not a background evolution law.

Let $q(\lambda)$ be a parametrized representative of a chain in $\mathcal{Q}$, where λ is only a bookkeeping parameter. If $\tau(q(\lambda))$ is strictly monotone, then locally one may reparametrize the chain by τ. This gives derivatives such as $\mathrm{d}/\mathrm{d}\tau$ along the chain, but these derivatives are defined only after the relational clock exists.

Definition 8.1 (Effective readout triple)**.** *An effective readout triple at a physical state is* $(\tau, V_o, H_{\mathrm{eff}})$*, where* τ *is a relational clock on a clock domain,* V_o *is an observable variation space, and* H_{eff} *is a Schur-reduced Hessian on* V_o*.*

The triple is not yet a spacetime theory. It is the minimal data needed to speak of ordered response: which physical states are later along the relational chain, which variations are observed, and how the quadratic response acts on those variations.

Proposition 8.2 (No metric requirement)**.** *The construction of an effective readout triple* $(\tau, V_o, H_{\mathrm{eff}})$ *requires no metric on* $\mathcal{Q}$ *and no metric on* V_o*. It requires only a quotient state space, a rankable transition relation, a vector space of variations, and a symmetric quadratic response operator with an invertible eliminated block.*

Proof. The quotient state space is defined set-theoretically. The clock is a real-valued rank function on transitions. The variation space is a vector space by assignment, and H_{eff} is obtained by linear algebra from the parent quadratic form. None of these operations requires a distance function, inner product, metric tensor, topology, or differentiable structure on $\mathcal{Q}$. □

8.2 Status of dynamics

The word dynamics is used carefully. A relational clock orders states; an effective Hessian gives linearized response; together they provide an effective readout of change. They do not by themselves give a full nonlinear law, a Hamiltonian system, or a spacetime field equation. Those structures require additional assumptions: a class of admissible paths, a variational principle, a symplectic or Poisson structure, or a smooth manifold approximation.

For this reason, the principal output of the present paper is not a final equation of motion. It is the following controlled implication:

$$\begin{aligned}&\text{physical quotient} + \text{rankable transitions}\\ &\qquad + \ \text{quadratic response} \Longrightarrow \text{relational clock} + \text{effective Hessian}.\end{aligned}$$

This implication is the mathematical bridge needed before any later theory can claim to obtain spacetime dynamics rather than assume it.

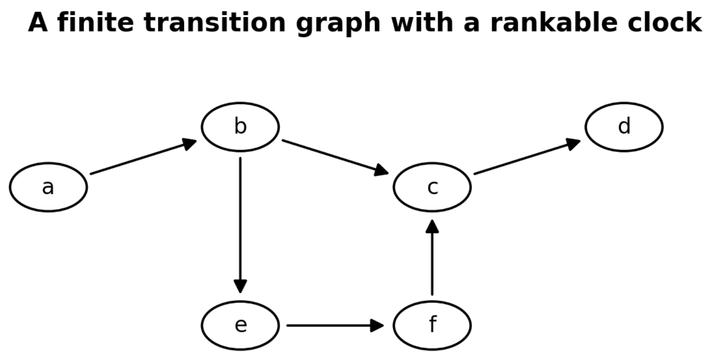

Figure 6: Finite acyclic transition graph used in the examples. The longest-chain rank is computed intrinsically from the directed transition structure and therefore supplies a strict relational clock without an external temporal parameter.

9 Examples

9.1 Finite acyclic transition graph

Consider the finite graph in figure 6. Its vertices are quotient states and its directed edges are transitions. Since the graph is acyclic, theorem 5.1 assigns to each vertex the length of the longest directed chain ending at that vertex. This produces a strict relational clock. Different incomparable branches can have the same clock value if they have the same rank; an injective refinement can separate them if desired, but the physical ordering along edges is already fixed.

Let the vertices be a, b, c, d, e, f, with edges

$$a \rightsquigarrow b, \quad b \rightsquigarrow c, \quad c \rightsquigarrow d, \quad b \rightsquigarrow e, \quad e \rightsquigarrow f, \quad f \rightsquigarrow c.$$

The longest-chain clock gives

$$\tau(a) = 0, \quad \tau(b) = 1, \quad \tau(e) = 2, \quad \tau(f) = 3, \quad \tau(c) = 4, \quad \tau(d) = 5.$$

The values are not assigned from an external time coordinate. They are computed from the transition relation itself.

9.2 Cycle and strongly connected reduction

Let $q_0 \rightsquigarrow q_1 \rightsquigarrow q_2 \rightsquigarrow q_0$. By theorem 4.7, no strict clock exists on these three states. A possible resolution is to collapse the strongly connected component $\{q_0, q_1, q_2\}$ to a single macro-state and seek a clock on the resulting condensation graph. This operation is not automatic physics; it is a modeling decision. It may correspond to treating the cycle as an internal phase, an unresolved gauge orbit, or a recurrent subsystem.

The example shows why cyclicity cannot be ignored. If a purported derivation of time assigns increasing clock values around a physical cycle, it is inconsistent. If it assigns equal values, then the clock is not strict on the cycle. The substrate must decide how cycles are interpreted before strict relational time can be claimed.

9.3 A numerical Schur example

Let

$$H = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 5 \end{pmatrix}$$

and split the first two coordinates as observable and the third as hidden. Then

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad D = (5).$$

The Schur complement is

$$H_{\text{eff}} = A - BD^{-1}B^T = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11/5 & 3/5 \\ 3/5 & 19/5 \end{pmatrix}.$$

Both D and H_{eff} are positive definite, so H is positive definite by inertia additivity. This example is simple, but it illustrates the principle: the visible response is shifted by hidden-sector coupling, and the full spectral status is recovered by adding the inertia of the hidden block.

10 Functoriality and redescription invariance

The quotient construction must be stable under change of presentation. Otherwise a clock could be an artifact of a particular encoding of the substrate. We therefore record a functorial version of the preceding results. The purpose is not to introduce category theory for its own sake, but to make precise the statement that physical content is invariant under redescription.

Definition 10.1 (Morphism of relational substrates)**.** *Let $(\mathcal{S}_1, \sim_1)$ and $(\mathcal{S}_2, \sim_2)$ be relational presentation spaces with physical equivalence relations. A map $F : \mathcal{S}_1 \to \mathcal{S}_2$ is equivalence-respecting if*

$$x \sim_1 y \quad \Longrightarrow \quad F(x) \sim_2 F(y).$$

Such a map is called a morphism of relational substrates at the quotient level.

Proposition 10.2 (Induced map on physical quotients)**.** *If $F : \mathcal{S}_1 \to \mathcal{S}_2$ is equivalence-respecting, then there exists a unique map*

$$\bar{F} : \mathcal{S}_1/\!\sim_1 \to \mathcal{S}_2/\!\sim_2$$

such that $\bar{F}([x]) = [F(x)]$ and $\pi_2 \circ F = \bar{F} \circ \pi_1$.

Proof. Define $\bar{F}([x]) = [F(x)]$. If $[x] = [y]$, then $x \sim_1 y$. Since F is equivalence-respecting, $F(x) \sim_2 F(y)$, hence $[F(x)] = [F(y)]$. Thus the definition is independent of representatives. The commuting relation follows directly, and uniqueness follows because every class in $\mathcal{S}_1/\!\sim_1$ is of the form $[x]$. □

Corollary 10.3 (Redescription invariance of quotient clocks)**.** *Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be equivalence-respecting and let $\bar{F}$ be the induced quotient map. If $\tau_2 : \mathcal{S}_2/\!\sim_2 \to \mathbb{R}$ is a relational clock on the* *image of $\bar{F}$, then $\tau_1 = \tau_2 \circ \bar{F}$ is an invariant clock candidate on $\mathcal{S}_1/\!\sim_1$ wherever the pulled-back transition relation is defined and rankable.*

Proof. The function τ_1 is defined on equivalence classes of the first substrate, hence is invariant under redescription in $\mathcal{S}_1$. If $[x] \rightsquigarrow_1 [y]$ implies $\bar{F}([x]) \rightsquigarrow_2 \bar{F}([y])$, then strict monotonicity of τ_2 gives

$$\tau_1([y]) = \tau_2(\bar{F}([y])) > \tau_2(\bar{F}([x])) = \tau_1([x]).$$

□

This result shows how a clock can be transported across equivalent descriptions without becoming presentation-dependent. It also identifies a failure mode: if the map between presentations does not respect physical equivalence, then it is not a physical redescription map. If it respects equivalence but fails to preserve the transition relation, then it may carry states but not dynamics.

11 Condensation of recurrent components

Strict clocks are obstructed by directed cycles, but many physical systems contain recurrent internal structure. The correct operation is not to pretend that cycles are ordered; it is to separate internal recurrence from external progression. For finite graphs this is achieved by the condensation graph of strongly connected components.

Definition 11.1 (Strongly connected component)**.** *Let $(\mathcal{Q}, \rightsquigarrow)$ be a finite transition graph. Two states $q, q' \in \mathcal{Q}$ are mutually reachable if there is a directed path from q to q' and a directed path from q' to q. The equivalence classes of mutual reachability are called strongly connected components. The condensation graph has these components as vertices and an edge $C \to C'$ whenever there exist $q \in C$, $q' \in C'$, $C \neq C'$, with $q \rightsquigarrow q'$.*

Theorem 11.2 (Condensation clock theorem)**.** *The condensation graph of any finite directed graph is acyclic. Consequently every finite transition graph admits a strict relational clock on its strongly connected components.*

Proof. Assume the condensation graph had a directed cycle $C_0 \to C_1 \to \cdots \to C_{n-1} \to C_0$ with distinct strongly connected components. By definition of the edges, there are paths from a state in C_i to a state in C_{i+1} for each i, with indices modulo n. Since states within each C_i are mutually reachable, concatenating these paths shows that every component in the cycle is mutually reachable with every other. Hence they should belong to a single strongly connected component, contradicting distinctness. Therefore the condensation graph is acyclic. The existence of a strict clock on the components follows from theorem 5.1. □

This theorem provides a mathematically precise way to treat systems with internal cyclicity. A strict clock may fail on microscopic states but exist on recurrent components. The physical question then becomes whether a strongly connected component should be interpreted as an internal phase, gauge orbit, coarse cell, or genuine recurrence. The theorem does not decide that interpretation; it gives the exact quotient on which strict ordering becomes possible.

12 Clock construction as an algorithm

For finite substrates, the clock theorem is constructive. This is useful because it turns the existence proof into an audit procedure. Given a finite transition graph on physical states, one can test acyclicity, form strongly connected components if needed, and compute a longest-chain clock. The output is either a strict clock or a certificate explaining why no strict clock exists at the chosen resolution.

Step	Operation
1	Form the physical quotient $\mathcal{Q} = \mathcal{S}/\sim$ and verify that the transition relation descends from presentations to quotient states.
2	Build the directed graph $G = (\mathcal{Q}, \rightsquigarrow)$ on physical states or on a finite sampled domain.
3	Detect directed cycles. If none exist, proceed directly to longest-chain ranking.
4	If cycles exist, compute strongly connected components and build the condensation graph. Interpret the components before treating them as clock states.
5	On the acyclic graph, compute $\tau(q)$ as the length of the longest directed chain ending at q.
6	Verify strict monotonicity along every edge. Store failures as transition or quotient defects.

Table 1: Finite-domain clock construction. The procedure distinguishes failure of quotient descent, failure of strict rankability, and successful construction of a relational clock.

The algorithm is deliberately conservative. It does not transform every graph into a physical clock by fiat. If cycles are present, it produces a clock only after replacing recurrent components by component-states. That replacement is a coarse-graining step and must be justified in the physical application.

13 Descent of quadratic forms to the quotient

The Schur formalism assumes that a quadratic response form has already been placed on a physical variation space. In many constructions one begins instead with variations of presentations. We therefore need a condition under which a quadratic form descends from presentation variations to quotient variations.

Let W_x be a vector space of presentation-level variations at x, and let $R_x \subset W_x$ be the vertical subspace tangent to the equivalence class $[x]$, whenever such a tangent model is available. The quotient variation space is modeled as $V_{[x]} = W_x/R_x$. A bilinear form H_x on W_x descends to $V_{[x]}$ precisely when it does not see vertical directions.

Proposition 13.1 (Quadratic descent criterion)**.** *Let W be a real vector space, $R \subset W$ a subspace, and $H : W \times W \to \mathbb{R}$ a symmetric bilinear form. There exists a unique symmetric bilinear form $\bar{H} : (W/R) \times (W/R) \to \mathbb{R}$ satisfying*

$$\bar{H}([u], [v]) = H(u, v)$$

if and only if $R \subseteq \mathrm{Rad}(H)$*, where*

$$\mathrm{Rad}(H) = \{r \in W : H(r, w) = 0 \text{ for all } w \in W\}.$$

Proof. Suppose first that $R \subseteq \mathrm{Rad}(H)$. If $[u] = [u']$ and $[v] = [v']$, then $u' = u + r$ and $v' = v + s$ for some $r, s \in R$. Hence

$$H(u', v') = H(u + r, v + s) = H(u, v) + H(r, v) + H(u, s) + H(r, s) = H(u, v),$$

because $r, s \in \mathrm{Rad}(H)$. Thus $\bar{H}([u], [v]) = H(u, v)$ is well-defined. Bilinearity and symmetry are inherited. Uniqueness follows because quotient classes exhaust W/R. Conversely, if $\bar{H}$ is well-defined and $r \in R$, then $[r] = [0]$, so for any $w \in W$,

$$H(r, w) = \bar{H}([r], [w]) = \bar{H}([0], [w]) = 0.$$

Thus $r \in \mathrm{Rad}(H)$, proving $R \subseteq \mathrm{Rad}(H)$. □

If the vertical directions are not in the radical, the quadratic form does not descend. One must then either fix a gauge, choose a complement, modify the form, or treat the vertical directions as physical. This is an important diagnostic: a presentation-level Hessian is not automatically a physical Hessian.

14 Gauge complements and dependence on splitting

When a quadratic form does not descend directly, one often chooses a complement $W = V \oplus R$. This produces a representative physical sector V, but the resulting restricted form may depend on the complement. Such dependence is acceptable only if later constructions are shown to be invariant under admissible changes of complement.

Definition 14.1 (Admissible complement)**.** *Let* W *be a presentation variation space and* $R \subset W$ *a vertical subspace. A complement* V *is admissible for a bilinear form* H *if* $W = V \oplus R$ *and the restriction* $H|_V$ *is nondegenerate on the observable directions being retained.*

Proposition 14.2 (Complement-change warning)**.** *Let* H *be a bilinear form on* W *and let* V, V' *be two complements of* R*. If* $R \not\subseteq \mathrm{Rad}(H)$*, then the restricted forms* $H|_V$ *and* $H|_{V'}$ *need not be congruent.*

Proof. It suffices to give a two-dimensional example. Let $W = \mathbb{R}^2$, $R = \mathrm{span}(e_2)$, and let H be represented by the identity matrix. Take $V = \mathrm{span}(e_1)$ and $V' = \mathrm{span}(e_1 + \alpha e_2)$. The restricted quadratic value on the basis vector of V is 1, while on the basis vector of V' it is $1 + \alpha^2$. As one-dimensional forms they are congruent over $\mathbb{R}$ if both are positive, but their normalization differs. With indefinite forms, even the sign can change for suitable choices. Since $e_2 \notin \mathrm{Rad}(H)$, vertical displacement can alter restricted values. Hence complement independence is not automatic. □

The proposition explains why the present paper distinguishes descent from gauge choice. Direct descent is invariant. Complement restriction is a gauge-dependent construction until an invariance theorem is supplied. Later geometric readout must therefore specify whether it uses a descended form, a fixed gauge, or a Schur-complemented effective form whose dependence has been controlled.

15 A formal definition of effective dynamics

We now collect the ingredients into a single definition. This definition is intentionally minimal: it does not require a Hamiltonian structure, a Lagrangian path integral, or a spacetime field equation.

Definition 15.1 (Premetric effective dynamics)**.** *A premetric effective dynamics on a domain $D \subseteq \mathcal{Q}$ consists of:*

(i) a strict relational clock $\tau : D \to \mathbb{R}$;

(ii) a variation assignment $q \mapsto V_q$ for $q \in D$;

(iii) a decomposition $V_q = V_{o,q} \oplus V_{h,q}$ into observable and hidden sectors;

(iv) a symmetric parent Hessian H_q whose hidden block D_q is invertible;

(v) the effective Hessian $H_{\mathrm{eff}}(q) = A_q - B_q D_q^{-1} B_q^T$ on $V_{o,q}$.

The dynamics is called regular on D if these objects vary consistently along chains in D and if no hidden block becomes singular.

This definition gives a precise meaning to effective dynamics before spacetime. The clock supplies order, the variation spaces supply linear response directions, and the effective Hessian supplies the local quadratic response in observable variables. The definition is purposely silent about smoothness until a smooth approximation is introduced.

Proposition 15.2 (Failure modes of premetric effective dynamics)**.** *A premetric effective dynamics fails at a state q if at least one of the following occurs: the clock is not defined at q, the transition relation is not representative-invariant, the variation space is not specified, the observable/hidden split is ambiguous, the hidden block is singular, or the reduced Hessian depends on an uncontrolled complement choice.*

Proof. Each listed item is one of the data or hypotheses in the definition. If any item fails, the corresponding object in the effective dynamics is undefined or non-unique. Therefore the full structure is not well-defined at q. □

16 Status ledger for the present paper

The results of the paper have different logical statuses. Recording them explicitly prevents overinterpretation.

Claim	Status	Meaning
Quotient existence	theorem	Follows from any equivalence relation on presentations.
Observable descent	theorem	Exactly invariant observables descend to physical states.
Finite clock existence	theorem	Holds for finite acyclic transition graphs.
Cycle obstruction	theorem	Directed cycles forbid strict real-valued clocks on the cycle.

Condensation clock	theorem	Finite graphs admit strict clocks on strongly connected components.
Affine clock uniqueness	conditional theorem	Requires same affine interval scale, not merely order.
Schur effective Hessian	theorem	Requires invertible hidden block and stationary elimination.
Inertia preservation	theorem	Follows from congruence and Sylvester's law.
Identification with physical time	not claimed	Requires later metric, stability, and empirical readout.
Lorentzian signature	not claimed	Requires an additional inertia/signature gate.

Table 2: Logical status of the main claims.

This ledger is part of the scientific content. It makes clear that the construction is not a derivation of the whole physical world. It is a precise mathematical bridge from physical equivalence to relational clock and effective response.

17 Dependency structure of the theorems

The construction has a strict dependency order. The quotient theorem does not depend on clocks, variations, or Hessians. The clock theorem depends on the quotient and on a transition relation that has already descended to it. The Schur reduction theorem depends on a variation space and a block decomposition, but not on a clock. The effective readout triple depends on both: the clock supplies ordered comparison, while the reduced Hessian supplies linearized response. This dependency order is summarized algebraically by the chain

$$(\mathcal{S}, \sim) \longrightarrow \mathcal{Q} \longrightarrow (\mathcal{Q}, \rightsquigarrow) \longrightarrow (D, \tau) \longrightarrow (V_o, V_h, H) \longrightarrow H_{\mathrm{eff}}.$$

Every arrow in this chain has an associated failure mode. If $\sim$ is not an equivalence relation, the quotient is not defined. If observables are not invariant, they do not descend. If transitions depend on representatives, the transition relation is not physical. If the transition domain is cyclic and not condensed, strict clocks fail. If the hidden block is singular, ordinary Schur elimination fails. This ordered list of failure modes is useful because it prevents a later successful construction from being read backward as a proof of its own prerequisites.

Proposition 17.1 (No backward promotion)**.** *Suppose an effective readout triple $(\tau, V_o, H_{\mathrm{eff}})$ is constructed on a domain $D \subseteq \mathcal{Q}$. The existence of this triple does not by itself prove that the original presentation space had a unique physical equivalence relation, that the transition relation was forced, or that the observable/hidden split was unique.*

Proof. The readout triple is constructed after choosing or deriving the equivalence relation, transition relation, variation assignment, and sector split. The data of the triple contain only τ, V_o, and H_{eff}. Different upstream choices may produce isomorphic or even identical downstream triples. Therefore the downstream object cannot logically determine uniqueness of the upstream data unless an additional inverse theorem is supplied. □

This proposition is methodological but also mathematical. It says that the implication established in this paper is directed. Later empirical or geometric success of a readout cannot be used as a proof that all earlier choices were inevitable. Each inverse claim requires its own theorem.

18 Boundary between order, duration, and metric time

A relational clock can mean three different things. First, it can mean only an order parameter: $q \rightsquigarrow q'$ implies $\tau(q') > \tau(q)$. Second, it can mean an affine interval scale: differences $\tau(q') - \tau(q)$ are physically comparable up to a positive affine transformation. Third, it can mean metric time: intervals are measured by a spacetime metric or by a physical clock model embedded in a metric theory. The present paper proves results at the first level and gives a conditional theorem for the second. It does not reach the third.

The distinction is not semantic. An arbitrary strictly increasing map $f \circ \tau$ preserves order but changes intervals. If one uses only the order, τ and $f \circ \tau$ are equivalent. If one uses interval ratios, most such reparametrizations are no longer equivalent. If one uses a metric clock, even affine freedom may be fixed by units and calibration. Thus a claim about emergence of time must specify which level of temporal structure has emerged.

Definition 18.1 (Three clock levels)**.** *Let $D \subseteq \mathcal{Q}$ be a transition domain. An order clock is a strict monotone map $D \to \mathbb{R}$. An affine clock is an order clock together with an interval-comparison structure defined up to positive affine transformations. A metric clock is an affine clock calibrated by an additional metric or physical clock model.*

Proposition 18.2 (Clock-level separation)**.** *Existence of an order clock does not imply existence of a metric clock. Existence of an affine clock does not imply existence of a Lorentzian metric. Conversely, a metric clock restricts to an affine clock locally only after a choice of unit and clock model.*

Proof. An order clock is only a monotone real-valued function on a transition relation. It contains no bilinear form on tangent directions and no rule for measuring lengths of curves. Hence it cannot determine a metric. An affine clock adds interval comparison but still provides only one-dimensional duration data, not a spacetime inner product. A metric clock requires additional geometric structure, and once that structure is given its proper-time parameter can be read as an affine clock after calibration. None of these implications reverses without extra assumptions. $\square$

This separation is the reason the paper stops at effective readout. The next problem is not to repeat the clock theorem, but to determine when the effective Hessian and later geometric data select a Lorentzian signature. That question belongs to a subsequent layer.

19 Relation to existing frameworks

The relational motivation of the paper is historically close to Leibnizian and Machian ideas, but the formal construction is intentionally narrower. We do not attempt to derive all inertial structure from matter distribution, and we do not assume a spatial configuration space whose trajectories are already parametrized. The relational content used here is weaker: presentations, equivalence, observables, transitions, and affine comparison. This makes the construction less physically complete, but it also makes the anti-circularity test sharper.

The work is also adjacent to canonical and covariant approaches to quantum gravity in which time is treated as a subtle or emergent quantity. In those settings the problem is often posed after a geometric or Hamiltonian structure has already been introduced. Here the question is earlier. We ask what can be built before a metric, before a symplectic structure, and before an external evolution parameter. The output is therefore not a solution to the problem of time in quantum gravity; it is a pre-geometric layer that any such solution would have to respect if it claims not to assume time.

Causal set theory begins with a partial order that is interpreted causally. The present construction does not. A transition relation on the quotient may later acquire causal interpretation, but it is initially only an admissibility relation between physical states. This distinction matters because causal order already contains a strong temporal orientation. Our clock theorem shows when an order-representation exists; it does not assume that the order is causal at the primitive level.

The Ehlers-Pirani-Schild reconstruction program begins from operational structures associated with light propagation and freely falling particles. That program is much closer to spacetime physics than the present one: null cones and projective structures already have geometric meaning. Here there are no light rays, no test particles, and no differentiable manifold. The paper therefore sits logically before such reconstructions. Its role is to say how a quotient and a clock can be obtained before the objects used in EPS-type arguments are available.

Finally, the Schur complement and inertia results belong to classical linear algebra. Their function here is not to introduce a new matrix theorem, but to enforce spectral honesty. A hidden sector cannot be discarded merely because one wishes to write an effective theory. If it is eliminated by a stationary equation, the Schur complement gives the visible response and Haynsworth inertia additivity records what has been moved into the hidden block. This is the algebraic control required before later signature or stability claims can be made.

20 Discussion

The central result of the paper is a controlled path from relational substrate to effective readout. The construction starts with presentations and a physical equivalence relation, forms the quotient, descends observables, imposes a transition relation on quotient states, proves clock existence under acyclicity or rankability, and derives an effective Hessian by Schur reduction. Each step has an explicit failure mode. Observables that are not invariant do not descend. Transition relations that depend on representatives are not physical. Directed cycles obstruct strict clocks. Infinite transition domains require rankability or an equivalent order-representation hypothesis. Hidden blocks with zero modes cannot be eliminated by the simple Schur formula without additional constraint analysis. Spectral information must be tracked through inertia, not guessed from the visible block alone.

This failure structure is not a weakness of the framework. It is the reason the construction is useful. A foundational theory should not merely state that time, spacetime, or dynamics emerge; it should specify the mathematical gates at which the claim may fail. In the present paper the first gate is quotient descent, the second is transition descent, the third is clock admissibility, and the fourth is controlled response reduction. The result is a map of what has to be true before one can speak of premetric effective dynamics without circularity.

The next mathematical layer is signature selection. Once an effective Hessian exists, one may ask whether the physical reduced sector contains one distinguished negative direction and a positive transverse sector, and whether such an inertia pattern can be stabilized and patched across domains. That question is not answered here. It is deliberately separated from clock existence. A clock orders transitions; Lorentzian signature concerns the inertia and geometric interpretation of a response operator. Conflating these would be exactly the kind of hidden assumption the quotient-clock-response order is designed to avoid.

A second later layer is smooth readout. Nothing in the present construction requires $\mathcal{Q}$ to be a manifold. If a smooth spacetime approximation is to appear, it must come from additional hypotheses: local charts of quotient states, compatibility of transition patches, nondegenerate frame maps, and stability of the effective Hessian. The present paper provides the algebraic and order-theoretic prerequisites for such a readout, but it does not claim to complete it.

21 Conclusion

We have proved that a class of premetric relational substrates can support a mathematically controlled notion of relational time and effective readout without assuming spacetime primitives. The construction proceeds through four disciplined steps. First, one quotients presentations by physical equivalence and admits only quantities that descend to the quotient. Second, one defines transitions on quotient states and tests whether they admit a strict real-valued clock. Third, one introduces algebraic variation spaces and a parent quadratic response form. Fourth, one eliminates hidden stationary variables by Schur reduction and uses inertia additivity to keep exact spectral account of the parent system.

The conclusion is intentionally conditional. The paper does not assert that time exists for every relational substrate, nor that the clock constructed here is already metric time. It proves that if quotient descent, transition descent, clock admissibility, and hidden-sector invertibility hold, then one obtains a relational clock and an effective Hessian without using an external temporal parameter, coordinate time, metric tensor, or smooth manifold. That is the precise mathematical content of the result.

This establishes the layer needed before a serious derivation of Lorentzian readout can be attempted. The natural next problem is not to add matter or cosmology, but to ask when the reduced Hessian carries a stable signature pattern suitable for spacetime interpretation. The present work supplies the quotient, the order parameter, and the response operator on which that question must be posed.

A Proof details for quotient constructions

This appendix records several elementary but useful facts about quotients. Let $\pi : \mathcal{S} \to \mathcal{Q}$ be the quotient map. A subset $U \subseteq \mathcal{S}$ is saturated if $x \in U$ and $x \sim y$ imply $y \in U$. Saturated subsets are precisely inverse images of subsets of $\mathcal{Q}$. Indeed, if $A \subseteq \mathcal{Q}$, then $\pi^{-1}(A)$ is saturated. Conversely, if U is saturated, then $U = \pi^{-1}(\pi(U))$. This observation is useful when one later adds topology: quotient-open sets are exactly those whose inverse images are open and saturated.

If $\mathcal{F}$ generates $\sim$, the quotient map can be represented by the evaluation map

$$E_{\mathcal{F}} : \mathcal{S} \to \mathbb{R}^{\mathcal{F}}, \qquad E_{\mathcal{F}}(x) = (f(x))_{f \in \mathcal{F}}.$$

Then $x \sim_{\mathcal{F}} y$ if and only if $E_{\mathcal{F}}(x) = E_{\mathcal{F}}(y)$. The quotient is therefore in bijection with the image $E_{\mathcal{F}}(\mathcal{S})$. This gives a concrete representation of the quotient whenever the observable family is fixed.

B Schur complements and singular hidden blocks

The Schur reduction theorem assumed D invertible. If D is singular, several outcomes are possible. If $B^T x$ lies outside the image of D, the hidden stationary equation $Dy = -B^T x$ has no solution. If it lies in the image, solutions exist but are not unique up to $\ker D$. In that case one may use a Moore-Penrose inverse after choosing an inner product, or one may reduce further by quotienting the hidden zero modes. Both operations introduce additional structure and cannot be treated as the same theorem.

A clean singular-block statement is obtained by replacing invertibility with a constraint: stationary elimination is valid on the set of observable variations satisfying $B^T x \in \operatorname{im} D$, and the reduced form is defined only modulo hidden null directions. This is often the correct structure in gauge systems, but it is not the setting of the main theorem because it requires a separate constraint analysis.

C Finite clock algorithm in pseudocode

For completeness we write the finite construction in pseudocode. The input is a finite transition graph on physical quotient states. The output is either a strict clock or a certificate of the recurrent components that must be interpreted before strict ordering is possible.

Input:	$G = (V, E)$
Step 1:	compute strongly connected components $C_1, \ldots, C_k$;
Step 2:	if every C_i is a singleton and no self-loop exists, set $G_c = G$;
Step 3:	otherwise form the condensation graph G_c;
Step 4:	topologically order G_c;
Step 5:	$\tau(C) = \max\{n : \text{there exists a directed chain of length } n \text{ ending at } C\}$;
Output:	τ on G_c and the component map $V \to V(G_c)$.

The clock produced by this algorithm is strict on the condensation graph. It is strict on the original graph if and only if all strongly connected components are singletons and there are no self-loops. This gives an immediate audit of whether a proposed finite relational clock is genuine or only coarse-grained.

D Additional Schur identities

The Schur complement can be written in two equivalent ways depending on whether the hidden or observable block is eliminated. If A is invertible, one may form

$$D - B^T A^{-1} B.$$

The two complements encode different conditional responses. Eliminating hidden variables gives $A - BD^{-1}B^T$, while eliminating observable variables gives $D - B^T A^{-1} B$. These are not interchangeable. In physical applications the choice of eliminated block must correspond to a specified stationary or constrained sector.

One also has the determinant identity

$$\det H = \det D \, \det(A - BD^{-1}B^T)$$

whenever D is invertible. This identity follows immediately from the block factorization in theorem 7.3, because the congruence factors have determinant one. It provides another diagnostic: zeros of the reduced determinant correspond to zeros of the parent determinant not already accounted for by the hidden block.

E Glossary of symbols

Symbol	Meaning
$\mathcal{S}$	set of relational presentations
$\sim$	physical equivalence relation on presentations
$\mathcal{Q}$	quotient state space $\mathcal{S}/\sim$
π	quotient projection $\mathcal{S} \to \mathcal{Q}$
$\mathcal{F}$	family of relational observables
$\rightsquigarrow$	transition relation on physical states
τ	relational clock functional
V_q	variation space at a quotient state
V_o, V_h	observable and hidden variation sectors
H	parent symmetric quadratic response operator
D	hidden block of the parent Hessian
H_{eff}	Schur-reduced effective Hessian
$\text{In}(M)$	inertia of a real symmetric matrix

F Bibliographic notes

The quotient and universal-property language used in the first part of the paper is standard in algebra and category theory; Mac Lane remains a useful reference for the categorical viewpoint. The relational motivations are historically connected with Leibniz and Mach, while modern discussions of time and background independence appear in Rovelli's work and in Barbour's relational program. Operational reconstruction of spacetime geometry from light propagation and free fall is classically associated with Ehlers, Pirani, and Schild. Causal set theory provides a contrasting framework in which order is primitive. The Schur complement and inertia results used in the reduction part of the paper are standard tools in matrix analysis; the inertia identity used here is usually associated with Haynsworth and ultimately rests on Sylvester's law of inertia.

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