CAS Lagrangian Atlas:
Variational Layers, Effective Actions, and Descent Maps

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Abstract

This atlas collects and organizes the action-level content of the CAS series. Its purpose is not to introduce a new sector, but to answer a structural question that a physicist will naturally ask after reading the first seven papers: what is the action? The answer is necessarily layered. Before a Lorentzian readout is selected, the theory has premetric variational functionals and quadratic Hessian forms rather than ordinary spacetime lagrangian densities. After rank, signature, and localization gates pass, some of these objects descend to local effective actions: metric-affine actions, trace-mode actions, spectral matter kernels, and gauge kinetic lagrangians. This document separates these layers, states their variables and symmetries, derives the main Schur and Feshbach reductions explicitly, records which coefficients are derived or conditional, and identifies the remaining gates required before a fully renormalized quantum field theoretic lagrangian can be claimed.

Contents

1 Purpose and scientific contract

The previous papers in the series deliberately avoided presenting a single final lagrangian at the beginning of the construction. That avoidance was not cosmetic. A usual local lagrangian density on a smooth Lorentzian manifold already presupposes several structures that the early CAS papers do not assume: a differentiable spacetime, a metric volume form, a signature, a causal split, and field variables living over a readout base. In the premetric part of the construction these structures are not available. The correct object at that level is therefore not a local density of the form $\mathcal{L}(\phi, \partial\phi, g)\sqrt{|g|}\,\mathrm{d}^4x$, but a variational functional on relational presentations, together with a quotient rule that decides which variations are physically meaningful.

At the same time, a theory of physics cannot remain forever at the level of abstract operators and gates. If it is to be read by external physicists, it must say what the action is at each level, how one action descends to the next, which variables are dynamical, which variables are eliminated, and which coefficients are derived rather than fitted. This document is an atlas for that purpose. It is intentionally written as a cross-paper technical guide rather than as another foundational paper. It identifies the lagrangian or action object appropriate to each layer of the theory and records its descent conditions.

The word "lagrangian" will be used in three related but distinct senses. First, a *premetric variational functional* is an action-like object defined on relational data without an a priori spacetime density. Second, an *effective quadratic action* is the second variation of such a functional after quotienting and after eliminating non-observed directions by Schur reduction. Third, a *local readout lagrangian* is an ordinary field-theoretic density that exists only after rank, signature, frame, and locality gates have passed. Confusing these three meanings is one of the easiest ways to overstate the theory. The atlas therefore keeps them separate throughout.

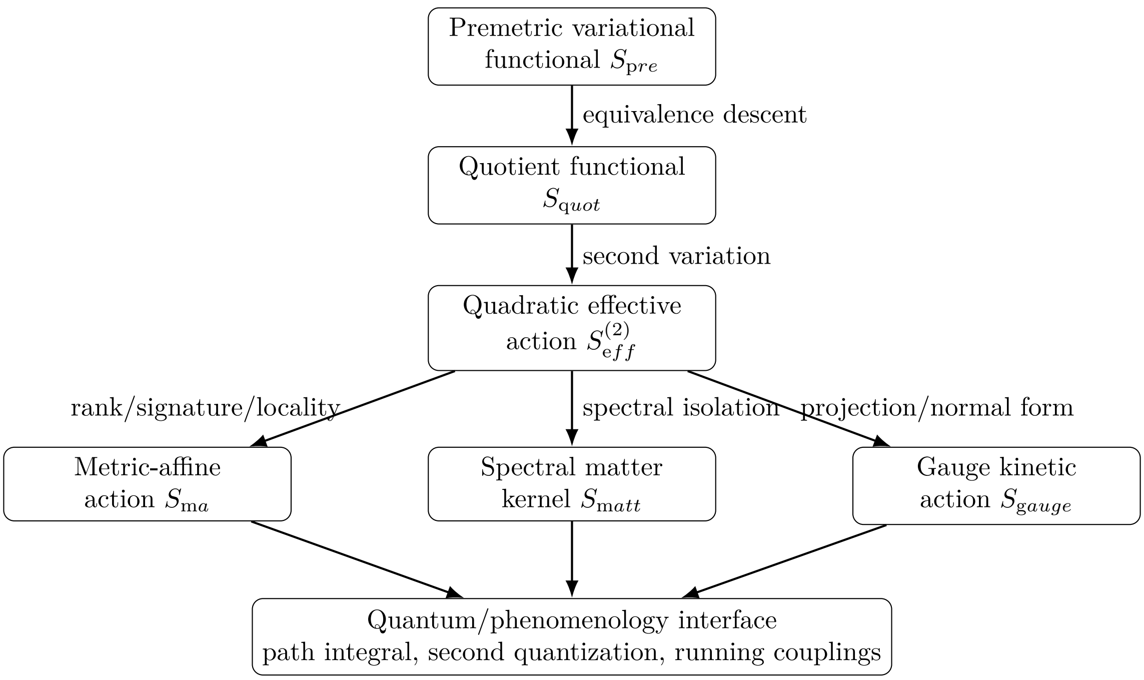

Figure 1: The lagrangian atlas is layered. A premetric functional is not yet a local spacetime density. A local action appears only after quotient, second variation, reduction, rank, signature, and locality gates have passed.

The scientific contract is the following. This atlas does not claim that a unique final CAS lagrangian has already been derived in full. It claims that the actions used across the series can be organized into a coherent descent chain, that several reduction steps are mathematically controlled, and that the remaining open points can be stated as precise gates rather than hidden assumptions.

2 Global map of variational layers

The seven-paper series contains several action-level objects. The safest terminology is to call them *layers*. Each layer has its own variables, locality status, and admissible variations. A local field lagrangian is only one possible endpoint of the chain.

Table 1: Lagrangian layers and their status.

Layer	Action object	Variables	Locality status	Main gate
L_0	Premetric functional S_{pre}	relational presentations, affine transport data	not a spacetime density	quotient invariance
L_1	Quotient functional S_{quot}	physical equivalence classes and physical variations	nonlocal or finite-dimensional	descent to $\mathcal{S}/\sim$
L_2	Quadratic action $S^{(2)}_{eff}$	retained and eliminated physical variations	operator or finite-dimensional quadratic form	Schur invertibility
L_3	Lorentzian readout action	rank-four frame, clock split, spatial block	local after readout	signature gate
L_4	Metric-affine action S_{ma}	$g_{\mu\nu}$, ${\Gamma^\lambda}_{\mu\nu}$, distortion components	local after metric-affine readout	tensorial decomposition
L_5	Spectral matter kernel S_{matt}	operators, Riesz projectors, Feshbach sectors	operatorial before derivative expansion	spectral gap/locality
L_6	Gauge kinetic action S_{gauge}	projected internal connection, curvature, kinetic form	local after gauge readout	algebraic closure/positivity
L_Q	Quantum interface	fields, measures, operator kernels	incomplete QFT until measure and renormalization are supplied	quantization gate

This table is not a list of independent postulates. The layers are ordered by descent. Invariance under physical equivalence allows S_{pre} to descend to S_{quot}. A stationary branch and a well-defined physical tangent allow the second variation to define $S^{(2)}_{\mathrm{eff}}$. Non-observed or constrained directions may then be eliminated by Schur or Feshbach reduction. Only after the rank-four Lorentzian readout and locality gates pass can ordinary field-theoretic densities be written without importing spacetime by hand.

3 The premetric variational functional

Let $\mathcal{S}$ be the space of relational presentations and let $\sim$ denote physical equivalence. At the earliest layer the action is a functional

$$S_{pre} : \mathcal{S} \longrightarrow \mathbb{R}, \tag{3.1}$$

or, more generally, a functional on relational histories or transition domains. It is not assumed to be an integral over spacetime. It may depend on incidence data, affine transport data, and relational compatibility data. We write this schematically as

$$S_{\text{pre}}[X] = \Phi(\mathcal{R}[X], \mathcal{A}[X], \mathcal{C}[X]), \tag{3.2}$$

where $\mathcal{R}$ denotes relational incidence, $\mathcal{A}$ denotes affine transport, and $\mathcal{C}$ denotes coherence or compatibility constraints. The notation is deliberately schematic because the early papers do not require a unique microscopic parent functional. What they require is that the functional has a well-defined first and second variation along admissible physical directions.

Definition 3.1 (Physical descent of a variational functional)**.** *A functional S_{pre} descends to the quotient $\mathcal{S}/\sim$ if*

$$x \sim y \quad \Longrightarrow \quad S_{\text{pre}}[x] = S_{\text{pre}}[y]. \tag{3.3}$$

When this condition holds, there exists a unique functional S_{quot} on $\mathcal{S}/\sim$ satisfying

$$S_{\text{quot}}([x]) = S_{\text{pre}}[x]. \tag{3.4}$$

Theorem 3.2 (Quotient action descent)**.** *If S_{pre} is constant on physical equivalence classes, then the quotient functional S_{quot} exists and is unique. Conversely, if a functional S_{quot} on the quotient satisfies $S_{\text{pre}} = S_{\text{quot}} \circ \pi$, where $\pi : \mathcal{S} \to \mathcal{S}/\sim$ is the quotient map, then S_{pre} is constant on equivalence classes.*

Proof. Assume S_{pre} is constant on equivalence classes. Define $S_{\text{quot}}([x]) = S_{\text{pre}}[x]$. If $[x] = [y]$, then $x \sim y$ and therefore $S_{\text{pre}}[x] = S_{\text{pre}}[y]$, so the definition is independent of representative. Uniqueness follows because every point of the quotient has the form $[x]$ and hence any descending functional must take the value $S_{\text{pre}}[x]$ there. Conversely, if $S_{\text{pre}} = S_{\text{quot}} \circ \pi$ and $x \sim y$, then $\pi(x) = \pi(y)$, so $S_{\text{pre}}[x] = S_{\text{quot}}(\pi(x)) = S_{\text{quot}}(\pi(y)) = S_{\text{pre}}[y]$. □

This theorem is elementary, but it is essential. It says that any proposed parent action is unphysical unless it is invariant under the same equivalences that define physical state. It also shows why adding arbitrary terms to the parent functional is dangerous: a term may be mathematically allowed on presentation space while failing to descend to physical state space.

4 From quotient action to Hessian action

Assume now that S_{quot} is defined on a differentiable chart of the physical quotient, or on a finite-dimensional transition domain admitting variations. Let q denote local coordinates on this physical domain, not coordinates of spacetime. Suppose q_* is a stationary point or stationary branch:

$$\left.\frac{\partial S_{\text{quot}}}{\partial q^i}\right|_{q_*} = 0. \tag{4.1}$$

Then the second variation is

$$\delta^2 S_{\text{quot}}[q_*] = \frac{1}{2}\,\delta q^i H_{ij} \delta q^j, \qquad H_{ij} = \left.\frac{\partial^2 S_{\text{quot}}}{\partial q^i \partial q^j}\right|_{q_*}. \tag{4.2}$$

In coordinate-free notation this is

$$S^{(2)}_{eff}[\xi] = \frac{1}{2}\langle \xi, H\xi \rangle, \qquad \xi \in T_{[q_*]}(\mathcal{S}/\sim). \tag{4.3}$$

This is the first point at which the series obtains an object resembling an ordinary quadratic lagrangian. It is still not necessarily local in spacetime. It is a quadratic form on physical variations.

Diagnostic 4.1 (No hidden spacetime test)**.** *A quadratic action $\frac{1}{2}\langle \xi, H\xi \rangle$ is not yet a local field action unless the physical variation space has been identified with sections of a bundle over a readout base and the operator H admits a local or controlled pseudolocal representation on that base.*

5 Schur reduction as action reduction

Let the variation split into retained and eliminated components,

$$\xi = (u, v), \tag{5.1}$$

with quadratic action

$$S^{(2)}[u, v] = \frac{1}{2} \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.2}$$

Assume D is invertible. The eliminated equation is

$$\frac{\partial S^{(2)}}{\partial v} = B^T u + Dv = 0, \qquad v_* = -D^{-1} B^T u. \tag{5.3}$$

Substitution into (5.2) gives

$$S^{(2)}_{\text{red}}[u] = \frac{1}{2} u^T A u + u^T B v_* + \frac{1}{2} v_*^T D v_* \tag{5.4}$$

$$= \frac{1}{2} u^T A u - u^T B D^{-1} B^T u + \frac{1}{2} u^T B D^{-1} B^T u \tag{5.5}$$

$$= \frac{1}{2} u^T \left(A - B D^{-1} B^T \right) u. \tag{5.6}$$

Thus

$$H_{eff} = A - B D^{-1} B^T. \tag{5.7}$$

This is an action-level reduction, not merely an algebraic trick. It says that if the eliminated variables are solved at the quadratic stationary level, the retained dynamics is governed by the Schur complement.

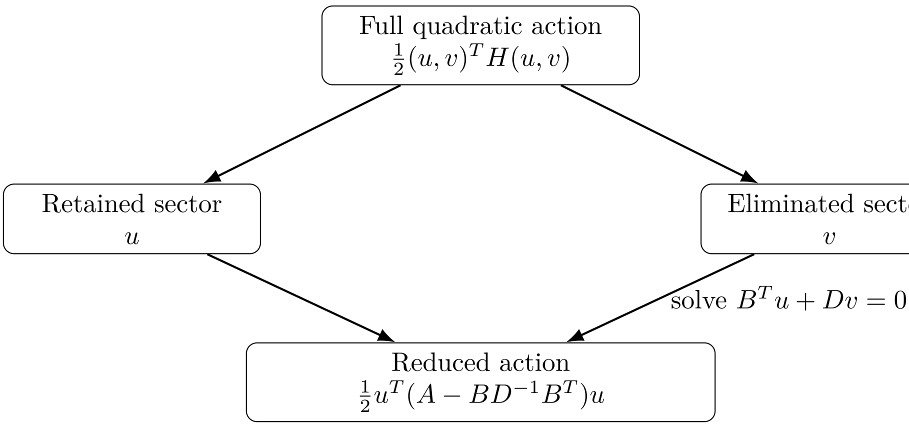

Figure 2: Schur reduction is an action reduction. The eliminated variables are not discarded; they are solved and substituted back into the quadratic action.

Theorem 5.1 (Schur action theorem)**.** *Let D be invertible in* (5.2)*. Then integrating out or classically eliminating v at the quadratic stationary level yields the reduced Hessian $H_{eff} = A - BD^{-1}B^T$. The reduced action is unique among quadratic actions in u that reproduce the stationary value of the full action on the eliminated branch.*

Proof. The stationary eliminated branch is uniquely $v_* = -D^{-1} B^T u$ because D is invertible. Direct substitution gives the Schur complement as shown above. If another quadratic action $\frac{1}{2} u^T G u$ reproduced the stationary value for all retained variations u, then $u^T G u = u^T (A - BD^{-1} B^T) u$ for all u. Since both matrices are symmetric, equality of the quadratic forms for all u implies equality of the matrices. □

The distinction between Schur reduction and truncation is important. Truncation would set $v = 0$ and produce A. Schur reduction produces $A - BD^{-1}B^T$, including the backreaction of eliminated directions. A claim that a retained sector has a given lagrangian must specify which operation was used.

6 Lorentzian readout and local kinetic form

Once the reduced Hessian is obtained, a Lorentzian readout is possible only under rank and signature conditions. Let the retained physical sector have rank four and let a clock covector select a split

$$U = \mathbb{R}\tau \oplus S, \tag{6.1}$$

where S is a three-dimensional spatial complement. In this split the reduced quadratic form takes the block shape

$$K = \begin{pmatrix} a & c^T \\ c & B \end{pmatrix}. \tag{6.2}$$

If B is positive definite and the Schur scalar

$$s_\tau = a - c^T B^{-1} c \tag{6.3}$$

is negative, then K has Lorentzian inertia $(1, 3)$.

The associated local field action is not obtained simply by naming this matrix a metric. A readout frame $e^A{}_\mu$ must identify the rank-four physical sector with tangent directions on an effective base. Then

$$g^{eff}_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu, \qquad \eta = \mathrm{diag}(-1, 1, 1, 1), \tag{6.4}$$

and local lagrangian densities can be written using g^{eff} only after this frame and signature gate have passed.

Proposition 6.1 (Signature-to-lagrangian condition)**.** *A reduced quadratic form H_{eff} supplies a Lorentzian kinetic seed for a local lagrangian only if the following conditions hold: its physical rank is four, its clock Schur scalar is negative, its spatial block is positive, and there exists a nondegenerate readout frame identifying the rank-four sector with local tangent directions.*

Proof. Rank four supplies the dimension of the readout tangent sector. Positivity of B and negativity of (6.3) imply Lorentzian inertia by the standard Schur congruence

$$\begin{pmatrix} 1 & 0 \\ -B^{-1}c & I \end{pmatrix} \begin{pmatrix} a & c^T \\ c & B \end{pmatrix} \begin{pmatrix} 1 & -c^T B^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} s_\tau & 0 \\ 0 & B \end{pmatrix}. \tag{6.5}$$

A nondegenerate frame is then required to pull the internal rank-four form to a local base. Without it one has a Lorentzian algebraic form, not a spacetime lagrangian. □

7 Metric-affine lagrangian layer

After Lorentzian readout, a standard metric-affine action may be written. The independent variables are a metric $g_{\mu\nu}$ and an affine connection $\Gamma^\lambda{}_{\mu\nu}$. The action has the schematic form

$$S_{\mathrm{ma}}[g, \Gamma] = \int_M \mathrm{d}^4x \sqrt{|g|}\, \mathcal{L}_{\mathrm{ma}}(g, \Gamma, R(\Gamma), T(\Gamma), Q(\Gamma)), \tag{7.1}$$

where torsion and non-metricity are

$$T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\mu\nu]}, \qquad Q_{\lambda\mu\nu} := -\nabla^\Gamma_\lambda g_{\mu\nu}. \tag{7.2}$$

The previous paper focused on the pure trace non-metric component. With the convention

$$Q_{\lambda\mu\nu} = 2W_\lambda g_{\mu\nu}, \tag{7.3}$$

the torsion-free Weyl-type connection is

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\}_g + \delta^\lambda_\mu W_\nu + \delta^\lambda_\nu W_\mu - g_{\mu\nu} W^\lambda. \tag{7.4}$$

Indeed, inserting (7.4) gives $\nabla^\Gamma_\lambda g_{\mu\nu} = -2W_\lambda g_{\mu\nu}$, hence (7.3).

For this connection in dimension n, the scalar curvature is

$$R(\Gamma) = R(g) - 2(n-1)\nabla_\mu W^\mu - (n-1)(n-2)W_\mu W^\mu. \tag{7.5}$$

In four dimensions,

$$R(\Gamma) = R(g) - 6\nabla_\mu W^\mu - 6W_\mu W^\mu. \tag{7.6}$$

The divergence term is a boundary term in the Einstein-Hilbert truncation. Therefore an action

$$S_{\mathrm{EH}}[g, W] = \frac{1}{2\kappa}\int \sqrt{|g|}\, R(\Gamma) \tag{7.7}$$

contains, up to boundary terms,

$$S_{\mathrm{EH}}[g, W] = \frac{1}{2\kappa}\int \sqrt{|g|}\, R(g) - \frac{3}{\kappa}\int \sqrt{|g|}\, W_\mu W^\mu. \tag{7.8}$$

Thus W_μ is algebraic in the Einstein-Hilbert truncation. It is not automatically a propagating field.

Theorem 7.1 (Trace-mode kinetic gate)**.** *The trace non-metric mode W_μ is dynamically propagating only if the effective action contains a nondegenerate kinetic operator for W_μ, for example a Maxwell-type term*

$$-\frac{1}{4} Z_W F_{\mu\nu}(W) F^{\mu\nu}(W), \qquad F_{\mu\nu}(W) = \nabla_\mu W_\nu - \nabla_\nu W_\mu, \tag{7.9}$$

with $Z_W > 0$, or an equivalent positive principal symbol. Otherwise W_μ remains auxiliary at this truncation level.

Proof. The Euler-Lagrange equation for W_μ receives derivative terms only from the kinetic operator. The Einstein-Hilbert contribution derived above is algebraic in W_μ after the boundary divergence is removed. Hence, without a derivative term with positive principal symbol, the equation for W_μ is a constraint rather than a wave equation. With the Maxwell-type term and a mass-like algebraic term, the principal part is $Z_W(\nabla^2 \delta^\mu{}_\nu - \nabla^\mu \nabla_\nu)$ up to gauge or Proca constraints, so the field has propagating degrees of freedom when the remaining constraint structure is nondegenerate. □

The conditional local action for this layer is therefore

$$S_W[g, W] = \int \mathrm{d}^4x \sqrt{|g|}\left[-\frac{1}{4} Z_W F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m_W^2 W_\mu W^\mu + J^\mu_W W_\mu + \cdots\right]. \tag{7.10}$$

Here m_W is a trace-mode mass parameter and must not be confused with the electroweak W boson mass. The notation W_μ is geometric.

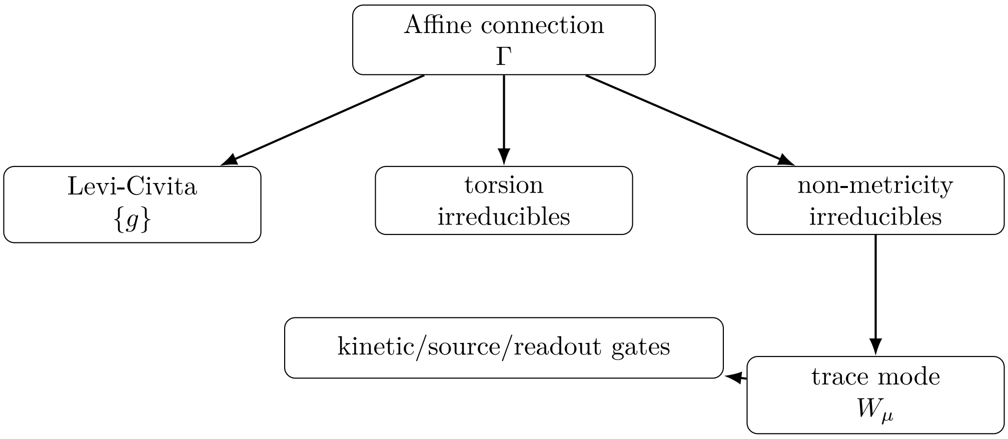

Figure 3: The metric-affine layer splits the affine connection into Levi-Civita, torsion, and non-metricity pieces. The trace mode W_μ is a candidate geometric mode only after its own kinetic and source gates pass.

8 Spectral matter kernels as operator lagrangians

The matter architecture paper uses operator kernels before naming particles. The action-level object is not initially a local spinor or scalar lagrangian, but a quadratic operator functional

$$S_{\mathrm{matt}}[\Psi; z] = \frac{1}{2}\langle \Psi, (K - z)\Psi \rangle_{\mathcal{H}}, \tag{8.1}$$

where K is a self-adjoint or suitably closed operator on a Hilbert space $\mathcal{H}$, and z is a spectral parameter. If an isolated part of the spectrum is enclosed by a contour Γ, the Riesz projector is

$$P_\Gamma(K) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\zeta - K)^{-1}\, \mathrm{d}\zeta. \tag{8.2}$$

The corresponding spectral sector is $\mathcal{H}_\Gamma = \operatorname{Ran} P_\Gamma(K)$.

Theorem 8.1 (Spectral action descent)**.** *If Γ encloses an isolated finite-rank spectral island of K, then the quadratic form (8.1) splits into projected blocks relative to $P_\Gamma \oplus (1 - P_\Gamma)$. If the external block is resolvent-invertible at z, the dynamics on $\mathcal{H}_\Gamma$ is governed by the Feshbach kernel*

$$F_\Gamma(z) = K_{\Gamma\Gamma} - z - K_{\Gamma E}(K_{EE} - z)^{-1} K_{E\Gamma}. \tag{8.3}$$

Proof. Write $\Psi = \psi + \chi$ with $\psi = P_\Gamma \Psi$ and $\chi = (1 - P_\Gamma)\Psi$. The stationary equation in the external sector is

$$(K_{EE} - z)\chi + K_{E\Gamma}\psi = 0. \tag{8.4}$$

If $K_{EE} - z$ is invertible, then $\chi_* = -(K_{EE} - z)^{-1} K_{E\Gamma}\psi$. Substitution into (8.1) gives the projected kernel (8.3), exactly as in Schur reduction, with the inverse replaced by a resolvent. The finite-rank and gap assumptions ensure that the projected sector is stable under small contour-preserving perturbations. □

A compact vorton carrier is represented at this level by a distinguished finite-rank spectral island,

$$P_V = P_{\Gamma_V}(K), \qquad \mathcal{H}_V = \operatorname{Ran} P_V, \qquad K_V(z) = K_{VV} - z - K_{VE}(K_{EE} - z)^{-1} K_{EV}. \tag{8.5}$$

This definition is deliberately spectral. It does not yet identify the carrier with an electron, neutrino, quark, or any observed particle. A particle interpretation requires additional residue, localization, spin/statistics, charge, and scattering gates.

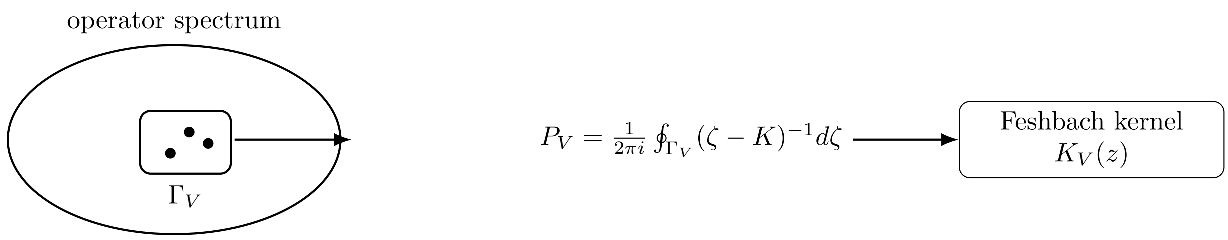

Figure 4: A vorton carrier is treated as a finite-rank spectral island with a Riesz projector and a Feshbach kernel. At this stage it is a spectral carrier, not yet a named particle.

8.1 Explicit two-mode carrier calculation

A minimal carrier calculation clarifies the role of the kernel. Consider a two-mode compact sector coupled to one heavy external mode:

$$K_{VV} = mI_2 + \omega\sigma_2, \qquad K_{EE} = M, \qquad K_{VE} = \begin{pmatrix} b \\ 0 \end{pmatrix}. \tag{8.6}$$

Then

$$F_V(z) = (m - z)I_2 + \omega\sigma_2 - \frac{|b|^2}{M - z}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{8.7}$$

The pole condition is

$$\det F_V(z) = 0. \tag{8.8}$$

Multiplying by $M - z$ gives the exact cubic equation

$$(M - z)\big((m - z)^2 - \omega^2\big) - |b|^2(m - z) = 0. \tag{8.9}$$

For small $|b|$, the two carrier poles are

$$z_\pm = m \pm \omega - \frac{|b|^2}{2(M - m \mp \omega)} + O(|b|^4). \tag{8.10}$$

This calculation is not a fit. It shows how external eliminated modes shift compact spectral carriers through a resolvent denominator.

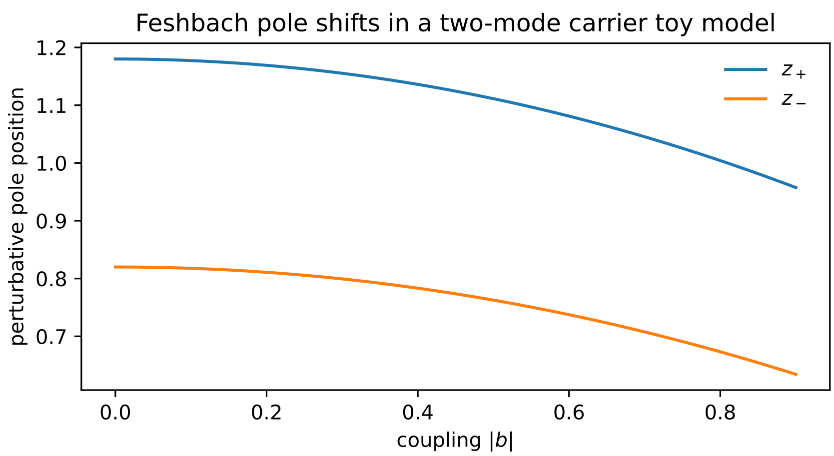

Figure 5: Perturbative pole shifts from (8.10) for a minimal two-mode carrier with $m = 1$, $\omega = 0.18$, and $M = 3$. The plot is illustrative, but the formula is derived explicitly from the Feshbach determinant.

9 From spectral kernels to local matter lagrangians

A local matter lagrangian can be written only after an additional derivative expansion and localization gate. Suppose a Feshbach kernel has an isolated simple pole near $z = z_0$ and admits, after readout, an expansion in local momenta,

$$K_F(p) = Z(p^2 - z_0) + \sum_{r\geq 2} c_r \mathcal{O}_r(p), \tag{9.1}$$

where the operators $\mathcal{O}_r$ are local or controlled pseudolocal readout operators. Then a local effective action may take the schematic form

$$S_{\rm local}^{matter} = \int \mathrm{d}^4x\sqrt{|g|}\left[\bar{\Psi}(Z_\Psi i\gamma^\mu D_\mu - M_\Psi)\Psi + \sum_{r\geq 5}\frac{c_r}{\Lambda^{r-4}}\mathcal{O}_r\right], \tag{9.2}$$

for spinorial carriers, or the analogous scalar/vector form for other representations. Equation (9.2) is conditional. It cannot be asserted merely from the existence of a spectral island.

Diagnostic 9.1 (No particle naming theorem)**.** *A finite-rank spectral island is not, by itself, an identified particle. Particle naming requires a local field representation, a positive residue or norm, a spin/statistics assignment, a charge representation, a scattering channel, and an observable calibration or prediction protocol.*

This diagnostic answers a central external criticism. The vorton can and should appear in the atlas, but it must appear as a carrier gate. If it is named as a particle too early, the theory loses its strongest methodological feature.

10 Gauge kinetic lagrangians and coupling normalization

The gauge layer begins with an internal affine connection Γ_μ^{int} and a projector P_g onto a candidate gauge subspace. The projected gauge potential is

$$A_\mu = P_g \Gamma_\mu^{\rm int} P_g, \tag{10.1}$$

provided the projector is stable and the projected generators close:

$$[T_A, T_B] = f_{AB}{}^C T_C. \tag{10.2}$$

The curvature is then

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{10.3}$$

and the kinetic action is

$$S_{gauge} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{|g|}\, K_{AB} F^A_{\mu\nu} F^{B\mu\nu}. \tag{10.4}$$

If the internal kinetic form block-diagonalizes over simple or abelian factors, one obtains

$$S_{gauge} = -\frac{1}{4}\sum_a \kappa_a \int \mathrm{d}^4x\sqrt{|g|}\, F^a_{\mu\nu} F_a^{\mu\nu}. \tag{10.5}$$

Canonical normalization rescales fields so that

$$g_a = \kappa_a^{-1/2}. \tag{10.6}$$

This formula is simple, but its interpretation is not. The κ_a are readout-scale kinetic normalizations. They are not automatically the same as renormalized couplings at an experimental scale. A comparison with QFT requires a running map, a scheme convention, thresholds, and uncertainty propagation.

Theorem 10.1 (Gauge kinetic positivity gate). *The gauge kinetic action* (10.4) *is ghost-free at the quadratic level only if the kinetic form K_{AB} is positive definite on the physical gauge subspace after quotient and Schur/Feshbach elimination.*

Proof. The principal quadratic energy of gauge fluctuations is determined by K_{AB} contracted with the spacetime kinetic form. If K has a negative eigenvalue on the physical gauge subspace, then a corresponding linear combination of gauge fields has wrong-sign kinetic energy. If K has a null direction, the kinetic operator is degenerate and the corresponding mode is either constrained, gauge-redundant, or ill-defined unless removed by an additional quotient. Positive definiteness on the physical subspace is therefore the necessary ghost-free condition. □

11 Putting the effective action together

After the relevant gates pass, the local effective action has the schematic form

$$S^{eff}_{\text{local}} = S_{\text{grav}} + S_W + S_{\text{gauge}} + S_{\text{matter}} + S_{\text{int}} + S_{\text{ct}} + \cdots . \tag{11.1}$$

Each term in (11.1) has a different provenance. The gravitational term descends from the Lorentzian metric-affine readout. The trace-mode term descends from the non-metricity trace and requires a kinetic gate. The gauge term descends from internal affine projection and kinetic normalization. The matter term descends from spectral islands and Feshbach kernels. Interaction terms require compatibility of spectral, gauge, and geometric projections. Counterterms or higher-derivative terms enter only after a quantum or effective-field-theory interface is specified.

A safer expanded template is

$$S^{eff}_{\text{local}} = \int \mathrm{d}^4x\sqrt{|g|}\Big[\frac{1}{2\kappa_g}R(g) - \frac{1}{4}Z_W F_{\mu\nu}(W)F^{\mu\nu}(W) - \frac{1}{2}m_W^2 W_\mu W^\mu \tag{11.2}$$

$$- \frac{1}{4}\sum_a \kappa_a F^a_{\mu\nu}F_a^{\mu\nu} + \mathcal{L}^{\text{local}}_{\text{matter}} + \mathcal{L}^{\text{allowed}}_{\text{int}} + \sum_{r>4}\frac{c_r}{\Lambda^{r-4}}\mathcal{O}_r\Big]. \tag{11.3}$$

This template is not a final claim. It is an atlas entry: each coefficient and term must be assigned a derivation status.

Table 2: Coefficient status in the action template.

Coefficient	Meaning	Status	Required check
κ_g	effective gravitational normalization	not fixed in the atlas	Newtonian/readout calibration
Z_W	kinetic coefficient of trace non-metric mode	conditional	positive principal symbol
m_W^2	trace-mode mass parameter	conditional	no tachyon/auxiliary audit
κ_a	gauge kinetic normalizations	derived once internal kinetic form passes gates	positivity and charge-lattice audit
g_a	canonical gauge couplings	$g_a = \kappa_a^{-1/2}$ at readout scale	running/scheme bridge
M_Ψ	local matter carrier mass	not derived from spectral island alone	pole, residue, and localization gates

Coefficient	Meaning	Status	Required check
c_r	higher-operator coefficients	open or conditional	EFT/renormalization interface

12 Quantum status of the lagrangian

A local classical or semiclassical action is not yet a complete quantum field theory. The quantum interface requires at least three additional structures: a field configuration space, a measure or canonical commutation structure, and a renormalization prescription. Formally one would like to write

$$Z[J] = \int \mathcal{D}\Phi \, \exp\left(\mathrm{i} S^{eff}_{local}[\Phi] + \mathrm{i} \int J\Phi \right), \tag{12.1}$$

but this expression is only meaningful after the measure, gauge fixing, ghost terms, anomaly checks, and renormalization scheme are supplied.

The spectral language of Paper 5 is close to quantum mechanics, but it is not by itself a second-quantized QFT. Operator spectra, Riesz projectors, and Feshbach kernels can provide one-particle or finite-sector data. A field theory requires a Fock or algebraic lift, statistics, local commutators or causal propagators, and a prescription for loops. Therefore the atlas records the current quantum status as follows:

$$\text{CAS operator spectrum} \quad \not\equiv \quad \text{complete QFT spectrum.} \tag{12.2}$$

The arrow between the two is an open interface, not a solved theorem.

Diagnostic 12.1 (Quantum-interface gate)**.** *A CAS effective action may be promoted to a QFT candidate only after passing: locality or controlled pseudolocality, positivity of residues, spin/statistics assignment, gauge fixing and ghost consistency, anomaly cancellation or control, renormalization closure, and a scale/scheme map to observables.*

13 Phenomenological interface

The action atlas also clarifies what would count as a physical prediction. A coefficient in an effective action becomes predictive only if it is determined upstream and pushed forward to an observable without inverse fitting. Schematically,

$$\text{parent data} \longrightarrow \text{projectors} \longrightarrow \text{effective action} \longrightarrow \text{renormalized observable} \longrightarrow \text{measurement.} \tag{13.1}$$

At each arrow there is a possible calibration. A calibration is acceptable if it fixes units or a previously declared scale convention. It is not acceptable if it is used to choose the coefficient that is later claimed as predicted.

Theorem 13.1 (No inverse-fitting criterion)**.** *Let O be an observable computed from an effective action coefficient c. If c is selected by minimizing the discrepancy between the computed value of O and the measured value of O, then O is not a prediction of the theory. If c is fixed upstream by quotient, spectral, Schur, or normal-form gates independent of O, and the map from c to O is specified before comparison, then O is a prediction conditional on those upstream gates.*

Proof. In the first case the measured value of O is an input to the determination of c, so the subsequent agreement is tautological with respect to O. In the second case c is fixed without using O, and the map to O is specified before comparison. The measured value therefore tests the chain rather than defining it. This is precisely the distinction between fitting and prediction. □

14 Failure modes and required next calculations

The atlas identifies several failure modes that are more useful than vague caveats. A parent functional can fail to descend to the quotient. A Hessian can fail to be nondegenerate on the physical sector. A Schur block can be singular. A Lorentzian readout can fail by rank defect, wrong signature, or absence of a nondegenerate frame. A trace non-metric mode can remain auxiliary. A spectral island can lose its gap. A Feshbach kernel can be nonlocal beyond control. A gauge kinetic form can have wrong sign or arbitrary abelian normalization. A proposed QFT can fail at the measure, anomaly, or renormalization gate.

The next concrete calculations should therefore not add terminology. They should compute: the parent-to-Hessian map for a selected branch, the derivative expansion of a spectral carrier kernel, the running map from readout-scale g_a to a standard renormalization scale, the residue positivity of candidate matter carriers, and the anomaly/charge-lattice consistency of the gauge sector.

15 Euler-Lagrange checks for the local layers

The atlas would be incomplete if it only listed actions without showing what their variations produce. At the local readout level, the standard test is whether the proposed density yields the expected equations of motion and whether the principal symbol has the correct sign. This section records the elementary variations for the three local-looking sectors that appear in the series: the trace non-metric mode, the gauge kinetic sector, and the local matter expansion after a spectral kernel has passed the localization gate.

For the trace non-metric mode, take the conditional action

$$S_W[W] = \int d^4x\sqrt{|g|}\left[-\frac{1}{4}Z_W F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m_W^2 W_\mu W^\mu + J_W^\mu W_\mu\right]. \tag{15.1}$$

Using $\delta F_{\mu\nu} = \nabla_\mu \delta W_\nu - \nabla_\nu \delta W_\mu$ and integrating by parts, the first variation is

$$\delta S_W = \int d^4x\sqrt{|g|}\left[-\frac{1}{2}Z_W F^{\mu\nu}(\nabla_\mu\delta W_\nu - \nabla_\nu\delta W_\mu) - m_W^2 W^\mu\delta W_\mu + J_W^\mu\delta W_\mu\right] \tag{15.2}$$

$$= \int d^4x\sqrt{|g|}\left[Z_W\nabla_\nu F^{\nu\mu} - m_W^2 W^\mu + J_W^\mu\right]\delta W_\mu + \text{boundary}. \tag{15.3}$$

The Euler-Lagrange equation is therefore

$$Z_W\nabla_\nu F^{\nu\mu} - m_W^2 W^\mu + J_W^\mu = 0. \tag{15.4}$$

This equation is meaningful as a propagating field equation only when $Z_W > 0$ and the remaining constraint structure is nondegenerate. If $Z_W = 0$, it collapses to an algebraic relation between W^μ and its source.

For the gauge sector, start from

$$S_g[A] = -\frac{1}{4}\int d^4x\sqrt{|g|}\,K_{AB}F^A_{\mu\nu}F^{B\mu\nu} + \int d^4x\sqrt{|g|}\,J^\mu_A A^A_\mu. \tag{15.5}$$

The field strength variation is $\delta F^A_{\mu\nu} = D_\mu\delta A^A_\nu - D_\nu\delta A^A_\mu$. Assuming that K_{AB} is covariantly constant on the selected gauge subspace, integration by parts gives

$$D_\nu\left(K_{AB}F^{B\nu\mu}\right) = J^\mu_A. \tag{15.6}$$

If K_{AB} is block diagonal with factor values κ_a, then canonical field normalization turns (15.6) into the usual Yang–Mills form with coupling $g_a = \kappa_a^{-1/2}$. This is the precise location at which gauge couplings enter the local action.

For a localized matter carrier, the symbolic action

$$S_\Psi = \int \mathrm{d}^4x\sqrt{|g|}\,\bar{\Psi}\,(Z_\Psi i\gamma^\mu D_\mu - M_\Psi)\,\Psi \tag{15.7}$$

yields

$$(Z_\Psi i\gamma^\mu D_\mu - M_\Psi)\,\Psi = 0. \tag{15.8}$$

However, in CAS this expression is downstream of a spectral calculation. The coefficients Z_Ψ and M_Ψ cannot be declared fundamental unless they are obtained from a pole, residue, and derivative expansion of a Feshbach kernel. Thus the local Dirac-like formula is an endpoint template, not an upstream assumption.

Proposition 15.1 (Variation consistency of local templates)**.** *The local templates* (15.1)*,* (15.5)*, and* (15.7) *are acceptable CAS readout lagrangians only when their coefficients are fixed by upstream gates and their Euler-Lagrange equations agree with the principal symbols and constraints inherited from those gates.*

Proof. The variations above show that each local template produces a definite differential equation. If the upstream reduced operator has a different principal symbol, a different nullspace, or a different constraint algebra, then the local template is not a faithful readout of the CAS kernel. Conversely, if the derivative expansion of the upstream operator has the same principal part, residue, and symmetry structure, the local template is the leading local representative of that reduced operator. Therefore variation consistency is not optional; it is the test that the local lagrangian is actually descended rather than guessed. □

16 No single-final-action theorem

It is tempting to compress the entire theory into one displayed formula. That would be misleading at the present stage. The early layers are not local spacetime densities, the spectral matter layer is operatorial before localization, and the quantum layer is incomplete until the measure and renormalization gates are supplied.

Theorem 16.1 (No single-final-action theorem at the current stage)**.** *At the current level of the series, there is no mathematically justified single local action $S_{\mathrm{CAS}}^{\mathrm{f}inal}$ whose terms simultaneously represent the premetric substrate, quotient dynamics, Lorentzian readout, metric-affine trace sector, spectral matter carriers, gauge kinetic normal forms, and quantum-field-theoretic completion without additional descent data.*

Proof. A single local field action requires a common base manifold, common field variables, a common integration measure, and a common variational class. The premetric layer has none of these as primitives. The quotient and Hessian layers live on physical variation spaces rather than on fields over spacetime. The spectral matter layer is formulated by operator kernels and Riesz/Feshbach reductions, which become local only under a derivative expansion gate. The gauge and metric-affine layers are local only after readout. The quantum layer requires measure and renormalization data not contained in the classical local templates. Therefore combining all layers into one local expression would either omit necessary descent maps or import downstream structures upstream. A final action may be a future endpoint, but it is not yet a justified present object. □

This theorem is not a weakness. It prevents the theory from overstating itself. It also tells the reader exactly what must be built next: a complete descent from premetric functional to local effective action, followed by a quantization and renormalization interface.

17 Combined Schur-Feshbach example

The Schur and Feshbach reductions used in different papers are not unrelated. Schur reduction is the finite-dimensional or algebraic version; Feshbach reduction is the spectral-parameter-dependent operator version. A combined example helps fix the logic. Let the physical variation sector split as

$$\mathcal{H} = \mathcal{H}_R \oplus \mathcal{H}_V \oplus \mathcal{H}_E, \tag{17.1}$$

where $\mathcal{H}_R$ is a retained geometric readout sector, $\mathcal{H}_V$ is a compact carrier sector, and $\mathcal{H}_E$ is an external eliminated sector. Let the quadratic operator be

$$K - z = \begin{pmatrix} A_R - z & C_{RV} & C_{RE} \\ C_{VR} & K_{VV} - z & K_{VE} \\ C_{ER} & K_{EV} & K_{EE} - z \end{pmatrix}. \tag{17.2}$$

If $K_{EE} - z$ is invertible, eliminating $\mathcal{H}_E$ gives the reduced two-block kernel

$$K_{RV}^{\mathrm{red}}(z) = \begin{pmatrix} A_R - z & C_{RV} \\ C_{VR} & K_{VV} - z \end{pmatrix} - \begin{pmatrix} C_{RE} \\ K_{VE} \end{pmatrix} (K_{EE} - z)^{-1} \begin{pmatrix} C_{ER} & K_{EV} \end{pmatrix}. \tag{17.3}$$

If the readout sector is then treated as slowly varying or constrained, a second Schur elimination may produce a carrier-only kernel

$$K_V^{eff}(z) = K_{VV} - z - K_{VE}(K_{EE} - z)^{-1} K_{EV} - \Delta_R(z), \tag{17.4}$$

where

$$\Delta_R(z) = \left(C_{VR} - K_{VE}(K_{EE} - z)^{-1} C_{ER}\right) A_R^{eff}(z)^{-1} \left(C_{RV} - C_{RE}(K_{EE} - z)^{-1} K_{EV}\right). \tag{17.5}$$

This formula shows why a carrier mass or pole cannot be read from K_{VV} alone. The external sector and the geometric readout sector both shift the effective pole positions. A phenomenological prediction must therefore specify the entire reduction order or prove reduction-order invariance under the relevant hypotheses.

Diagnostic 17.1 (Reduction-order audit)**.** *Whenever a quantity is extracted from a reduced kernel, the calculation must state which blocks have been eliminated, in which order, and under which invertibility or gap assumptions. If different admissible orders produce different values, the value is not yet a well-defined prediction.*

18 Dictionary to standard physics language

Part of the external readability problem is semantic. The CAS terms are useful internally, but a physicist needs a translation layer. The following dictionary is intentionally conservative.

Table 3: Minimal dictionary from CAS action language to standard physics language.

CAS phrase	Closest standard language	Important caveat
Premetric functional	action-like variational principle before spacetime	not a local lagrangian density
Quotient action	gauge-invariant or redundancy-reduced action	equivalence is broader than ordinary gauge until specified
Clock functional	internal time/order parameter	not external Newtonian time

CAS phrase	Closest standard language	Important caveat
Schur-reduced Hessian	effective quadratic operator after integrating out constrained modes	not a truncation
Lorentzian readout	emergence of a metric signature sector	conditional on rank/signature gates
Trace non-metric mode W_μ	Weyl/nonmetricity trace vector	not the electroweak W boson
Vorton carrier	compact finite-rank spectral island	not yet a particle
Feshbach kernel	energy-dependent effective Hamiltonian/operator	locality requires derivative expansion
Gauge normal form	canonical normalization of gauge kinetic terms	requires charge-lattice and renormalization-scale checks

This dictionary should be included, at least in compressed form, whenever the series is sent to an external reader. It reduces the risk that proprietary vocabulary is mistaken for unsupported physics.

19 What would make the atlas stronger

The atlas makes the current action structure explicit, but several calculations would make it substantially stronger. First, a representative parent functional should be carried through to its Hessian without skipping the second-variation calculation. Second, at least one spectral carrier should be taken from a Feshbach kernel to a local derivative expansion with residue normalization. Third, the readout-scale gauge normalizations should be propagated through a standard renormalization-group map with clearly stated scheme and thresholds. Fourth, the trace-mode sector should be tested against a concrete source and boundary-value problem to distinguish auxiliary behavior from propagation. Fifth, the quantum interface should specify whether the intended completion is path-integral, canonical, algebraic, or effective-field-theoretic.

These are not cosmetic additions. They are the exact points at which a mathematical architecture begins to become a predictive physical theory.

20 Conclusion

The CAS series should not be described as having one already-final lagrangian. It has a controlled hierarchy of variational objects. At the premetric level it has a relational functional whose physical admissibility depends on quotient descent. At the quadratic level it has Hessian actions and Schur-reduced effective actions. At the geometric readout level it has metric-affine and trace-mode lagrangians, with the non-metric trace mode propagating only if a kinetic gate passes. At the spectral matter level it has operator kernels, Riesz projectors, and Feshbach reductions, with local particle lagrangians conditional on additional localization and quantum gates. At the gauge level it has projected Yang–Mills kinetic forms and effective couplings derived from kinetic normalization, with QFT comparison requiring a scale and scheme bridge.

This layered answer is less rhetorically simple than writing down a single final expression, but it is scientifically safer. It tells an external reader exactly where ordinary lagrangian physics enters, which terms are derived, which are conditional, and what remains open before the theory can claim full quantum-field-theoretic and phenomenological completion.

A Schur complement derivation in operator form

Let $\mathcal{H} = \mathcal{H}_R \oplus \mathcal{H}_E$ and let

$$H = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \tag{A.1}$$

be self-adjoint on a common dense domain, with D invertible on the eliminated sector. For variations $\xi = (u, v)$, the quadratic action is

$$S[\xi] = \frac{1}{2}\langle u, Au\rangle + \mathrm{Re}\langle u, Bv\rangle + \frac{1}{2}\langle v, Dv\rangle. \tag{A.2}$$

The eliminated equation is $B^*u + Dv = 0$, so $v_* = -D^{-1}B^*u$. Substitution gives

$$S_{\mathrm{red}}[u] = \frac{1}{2}\langle u, (A - BD^{-1}B^*)u\rangle. \tag{A.3}$$

The same calculation holds for bounded operators directly and for unbounded operators under the usual domain and closed-form hypotheses. The finite-dimensional formula used in the main text is the cleanest presentation of this argument.

B Feshbach reduction and pole equations

For an operator K split by $P \oplus Q$, write

$$K - z = \begin{pmatrix} K_{PP} - z & K_{PQ} \\ K_{QP} & K_{QQ} - z \end{pmatrix}. \tag{B.1}$$

If $K_{QQ} - z$ is invertible, then the Q equation gives

$$\chi = -(K_{QQ} - z)^{-1}K_{QP}\psi. \tag{B.2}$$

The P equation becomes

$$\left(K_{PP} - z - K_{PQ}(K_{QQ} - z)^{-1}K_{QP}\right)\psi = 0. \tag{B.3}$$

The zeros of the determinant of this finite-rank kernel are the effective poles. For the two-mode carrier in the main text, direct determinant evaluation yields (8.9).

C Derivation of the Weyl scalar curvature formula

Let $C^\lambda{}_{\mu\nu} = \delta^\lambda_\mu W_\nu + \delta^\lambda_\nu W_\mu - g_{\mu\nu}W^\lambda$. The curvature difference between $\Gamma = \{g\} + C$ and the Levi-Civita connection is

$$R^\lambda{}_{\rho\mu\nu}(\Gamma) = R^\lambda{}_{\rho\mu\nu}(g) + 2\nabla_{[\mu}C^\lambda{}_{\nu]\rho} + 2C^\lambda{}_{[\mu|\sigma|}C^\sigma{}_{\nu]\rho}. \tag{C.1}$$

Contraction gives

$$R(\Gamma) = R(g) - 2(n-1)\nabla_\mu W^\mu - (n-1)(n-2)W_\mu W^\mu. \tag{C.2}$$

The divergence contributes a boundary term to the Einstein-Hilbert action under standard boundary assumptions. The remaining term is algebraic in W_μ.

D Canonical normalization of gauge kinetic terms

For a single gauge factor with action

$$S = -\frac{1}{4}\kappa \int F_{\mu\nu}F^{\mu\nu}, \tag{D.1}$$

rescale the gauge potential by $A_c = \sqrt{\kappa}A$. Then the kinetic term becomes canonical. If matter couples through $D_\mu = \partial_\mu + A_\mu$ in the original normalization, then in canonical variables

$$D_\mu = \partial_\mu + \kappa^{-1/2}A_{c\mu}. \tag{D.2}$$

Thus the effective coupling is $g = \kappa^{-1/2}$. This argument assumes that the charge generator normalization has already been fixed. In abelian sectors this is a separate charge-lattice gate, not an automatic consequence of kinetic positivity.

E Public-facing claim guide

A safe public-facing statement is: CAS provides a layered variational architecture in which local field lagrangians arise only after quotient, readout, spectral, and kinetic gates pass. It is not yet a single completed QFT lagrangian. A stronger claim, such as derivation of observed particle masses or fully renormalized Standard Model couplings, requires additional calculations not supplied by this atlas alone.

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