1.

ANALYTICAL DEVELOPMENT OF FRESNEL'S OPTICAL THEORY OF CRYSTALS.

[*Philosophical Magazine*, XI. (1837), pp. 461—469, 537—541; XII. (1838), pp. 73—83, 341—345.]

THE following is, I believe, the first successful attempt to obtain the full development of Fresnel's Theory of Crystals by direct geometrical methods. Hitherto little has been done beyond finding and investigating the properties of the wave surface, a subject certainly curious and interesting, but not of chief importance for ordinary practical purposes. Mr Kelland, in a most valuable contribution to the *Cambridge Philosophical Transactions**, has incidentally obtained the difference of the squares of the velocities of a plane front in terms of the angles made by it with the optic axes. I have obtained each of the velocities *separately*, and in a form precisely the same for biaxal as for uniaxal crystals.

I have also assigned in my last proposition the place of the lines of vibration in terms of the like quantities, and *that* in a shape remarkably convenient for determining the *plane* of polarization when the ray is given. For at first sight there appears to be some ambiguity in selecting *which* of the *two* lines of vibration is to be chosen when the front is known. If p be the perpendicular from the centre of the surface of elasticity let fall upon the front, ι_1, ι_2 the angles made by the front with the optic planes, ϵ_1, ϵ_2 the angles between its *due* line of vibration and the optic axes, I have shown that

$$\cos\epsilon_1 = \sqrt{\left(\frac{b^2-p^2}{a^2-c^2}\cdot\frac{\sin\iota_1}{\sin\iota_2}\right)},\quad \cos\epsilon_2 = \sqrt{\left(\frac{b^2-p^2}{a^2-c^2}\cdot\frac{\sin\iota_2}{\sin\iota_1}\right)},$$

so that all doubt is completely removed. The equation preparatory to obtaining the wave surface is found in Prop. 6 by common algebra, without any use of the properties of maxima and minima, and various other curious relations are discussed.

Without the most careful attention to preserve pure symmetry, the expressions could never have been reduced to their present simple forms.

* See *Lond. and Edinb. Phil. Mag.* Vol. x. p. 336.

Analytical Reduction of Fresnel's Optical Theory of Crystals.

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In Proposition 1, a plane front within a crystal being given, the two lines of vibration are investigated.

In Proposition 2 it is shown that the product of the cosines of the inclinations of one of the axes of elasticity to the two lines of vibration, is to the same for either other axis of elasticity in a constant ratio for the same crystal; and the two lines of vibration are proved to be perpendicular to each other.

In Proposition 3, a line of vibration being given, the front to which it belongs is determined; and it is proved that there is only one such, and consequently any line of vibration has but one other line conjugate to it.

In Proposition 4, certain relations are instituted between the positions of, and velocities due to, conjugate lines.

In Proposition 5, the angles made by the front with the planes of elasticity are found in terms of the velocities only.

In Proposition 6, the above is reversed.

In Proposition 7, the position of the planes in which the two velocities are equal (viz. the optic planes) is determined.

In Proposition 8, the position of a front in respect to the optic axes is expressed in terms of the velocities.

In Proposition 9, the problem is reversed, and it is shown that if v_1, v_2 be the two normal velocities with which any front can move perpendicular to itself, and ι_1, ι_2 the angles which it makes with the optic planes, then

$$v_1^2 = a^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2,$$

$$v_2^2 = a^2\left(\sin\frac{\iota_1-\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1-\iota_2}{2}\right)^2.$$

In the 10th the angle made by a line of vibration with the axes of elasticity is expressed in terms of the two velocities of the front to which it belongs.

In the 11th Proposition the velocity due to any line of vibration is expressed in terms of the angles which it makes with the optic axes, viz.

$$v^2 - b^2 = (a^2 - c^2)\cos\epsilon_1\cos\epsilon_2.$$

In the 12th Proposition ϵ_1, ϵ_2 are separately expressed in terms of ι_1, ι_2.

In the Appendix I have given the polar or rather radio-angular equation to the wave surface, from which the celebrated proposition of the ray flows as an immediate consequence.

PROPOSITION 1.

If $$lx + my + nz = 0 \qquad (a)$$

be the equation to a given front, to determine the lines of vibration therein.

It is clear that if x, y, z be any point in one of these lines, the force acting on a particle placed there when resolved into the plane must tend to the centre. Consequently the line of force at x, y, z must meet the perpendicular drawn upon the front from the origin. Now the equation to this perpendicular is

$$\frac{X}{l} = \frac{Y}{m} = \frac{Z}{n} \qquad (1)$$

and the forces acting at x, y, z are a^2x, b^2y, c^2z parallel to x, y, z, so that the equation to the line of force is

$$\frac{X-x}{a^2x} = \frac{Y-y}{b^2y} = \frac{Z-z}{c^2z}. \qquad (2)$$

From (2) we obtain

$$b^2yX - a^2xY = (b^2 - a^2)\,xy \qquad (3)$$

$$c^2zY - b^2yZ = (c^2 - b^2)\,yz \qquad (4)$$

$$a^2xZ - c^2zX = (a^2 - c^2)\,zx. \qquad (5)$$

Hence

$$(b^2 - a^2)\,xyn + (c^2 - b^2)\,yzl + (a^2 - c^2)\,zxm$$
$$= b^2y\,(nX - lZ) + c^2z\,(lY - mX) + a^2x\,(mZ - nY);$$

but by equations (1)

$$lZ - nX = 0, \quad mX - lY = 0, \quad nY - mZ = 0$$

therefore

$$(b^2 - a^2)\frac{n}{z} + (c^2 - b^2)\frac{l}{x} + (a^2 - c^2)\frac{m}{y} = 0. \qquad (b)$$

Also we have

$$nz + lx + my = 0 \qquad (a)$$

therefore

$$(b^2 - a^2)\,n^2 + (c^2 - b^2)\,l^2 + nl\left((c^2 - b^2)\frac{z}{x} + (b^2 - a^2)\frac{x}{z}\right) = (a^2 - c^2)\,m^2$$

or

$$(c^2 - b^2)\left(\frac{z}{x}\right)^2 + \frac{1}{nl}\{(c^2 - b^2)\,l^2 + (b^2 - a^2)\,n^2 - (a^2 - c^2)\,m^2\}\frac{z}{x} + (b^2 - a^2) = 0.$$

And in like manner interchanging b, y, m with c, z, n

$$(b^2 - c^2)\left(\frac{y}{x}\right)^2 + \frac{1}{ml}\{(b^2 - c^2)\,l^2 + (c^2 - a^2)\,m^2 - (a^2 - b^2)\,n^2\}\frac{y}{x} + (c^2 - a^2) = 0.$$

Hence if $\left(\frac{y_1}{x_1}, \frac{z_1}{x_1}\right)\left(\frac{y_2}{x_2}, \frac{z_2}{x_2}\right)$ be the two systems of values of $\frac{y}{x}, \frac{z}{x}$, then

$$\left(\frac{Y}{X} = \frac{y_1}{x_1}, \quad \frac{Z}{X} = \frac{z_1}{x_1}\right)\left(\frac{Y}{X} = \frac{y_2}{x_2}, \quad \frac{Z}{X} = \frac{z_2}{x_2}\right)$$

are the two lines of vibration required.

Proposition 2.

By last proposition it appears that

$$\frac{y_1 y_2}{x_1 x_2} = \frac{c^2 - a^2}{b^2 - c^2} \qquad (c)$$

and

$$\frac{z_1 z_2}{x_1 x_2} = \frac{b^2 - a^2}{c^2 - b^2} \qquad (d)$$

therefore

$$\frac{y_1 y_2 + z_1 z_2}{x_1 x_2} = \frac{c^2 - b^2}{b^2 - c^2} = -1$$

therefore

$$x_1 x_2 + y_1 y_2 + z_1 z_2 = 0.$$

And therefore the two lines of vibration are perpendicular to each other.

N.B. Equations (c) and (d) must not be overlooked.

Proposition 3.

A line of vibration is given $\left(\text{that is } \frac{y_1}{x_1}, \frac{z_1}{x_1} \text{ are given}\right)$ *and the position of the front is to be determined.*

Let $lx + my + nz = 0$ be the front required, then $lx_1 + my_1 + nz_1 = 0$, and

$$(b^2 - c^2)\frac{l}{x_1} + (c^2 - a^2)\frac{m}{y_1} + (a^2 - b^2)\frac{n}{z_1} = 0.$$

Eliminating n we get

$$l\left((a^2 - b^2)\frac{x_1}{z_1} - (b^2 - c^2)\frac{z_1}{x_1}\right) + m\left((a^2 - b^2)\frac{y_1}{z_1} - (c^2 - a^2)\frac{z_1}{y_1}\right) = 0$$

therefore

$$\frac{l}{m} = \frac{x_1}{y_1}\,\frac{(a^2 - b^2)\, y_1^2 - (c^2 - a^2)\, z_1^2}{(b^2 - c^2)\, z_1^2 - (a^2 - b^2)\, x_1^2}$$

$$= \frac{x_1}{y_1}\,\frac{a^2 (x_1^2 + y_1^2 + z_1^2) - (a^2 x_1^2 + b^2 y_1^2 + c^2 z_1^2)}{b^2 (x_1^2 + y_1^2 + z_1^2) - (a^2 x_1^2 + b^2 y_1^2 + c^2 z_1^2)}.$$

If now we make $x_1^2 + y_1^2 + z_1^2 = 1$

$$a^2x_1^2 + b^2y_1^2 + c^2z_1^2 = v_1^2$$

and therefore

$$\frac{l}{m} = \frac{x_1}{y_1} \cdot \frac{a^2 - v_1^2}{b^2 - v_1^2}$$

and in like manner

$$\frac{l}{n} = \frac{x_1}{z_1} \cdot \frac{a^2 - v_1^2}{c^2 - v_1^2};$$

therefore

$$(a^2 - v_1^2)\, x_1x + (b^2 - v_1^2)\, y_1y + (c^2 - v_1^2)\, z_1z = 0$$

is the equation required.

PROPOSITION 4.

$\frac{l}{m}, \frac{l}{n}$ having each only one value, shows that only one front corresponds to the given line of vibration. Let x_2, y_2, z_2, v_2 correspond to x_1, y_1, z_1, v_1 for the conjugate line of vibration, then the equation to the front may be expressed likewise by

$$(a^2 - v_2^2)\, x_2x + (b^2 - v_2^2)\, y_2y + (c^2 - v_2^2)\, z_2z = 0,$$

so that

$$\frac{(a^2 - v_1^2)\, x_1}{(a^2 - v_2^2)\, x_2} = \frac{(b^2 - v_1^2)\, y_1}{(b^2 - v_2^2)\, y_2} = \frac{(c^2 - v_1^2)\, z_1}{(c^2 - v_2^2)\, z_2}.$$

PROPOSITION 5.

To find ω, ϕ, ψ, the angles made by the front with the planes of elasticity in terms of v_1, v_2.

By the last proposition

$$(\cos\omega)^2 = \frac{(a^2 - v_1^2)^2\, x_1^2}{(a^2 - v_1^2)^2\, x_1^2 + (b^2 - v_1^2)^2\, y_1^2 + (c^2 - v_1^2)^2\, z_1^2}$$

$$= \frac{(a^2 - v_1^2)(a^2 - v_2^2)\, x_1x_2}{(a^2 - v_1^2)(a^2 - v_2^2)\, x_1x_2 + (b^2 - v_1^2)(b^2 - v_2^2)\, y_1y_2 + (c^2 - v_1^2)(c^2 - v_2^2)\, z_1z_2}.$$

Now, by Proposition 2,

$$\frac{x_1x_2}{c^2 - b^2} = \frac{y_1y_2}{a^2 - c^2} = \frac{z_1z_2}{b^2 - a^2}$$

therefore $(\cos\omega)^2$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)}{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)+(b^2-v_1^2)(b^2-v_2^2)(a^2-c^2)+(c^2-v_1^2)(c^2-v_2^2)(b^2-a^2)}$$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)}{a^4(c^2-b^2)+b^4(a^2-c^2)+c^4(a^2-b^2)}$$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}.$$

Similarly,

$$(\cos\phi)^2=\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)},$$

$$(\cos\psi)^2=\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}.$$

Proposition 6.

To find v_1, v_2 in terms of ω, ϕ, ψ.

By the last proposition

$$\frac{(\cos\omega)^2}{a^2-v_1^2}=\frac{a^2}{(a^2-b^2)(a^2-c^2)}-v_2^2\cdot\frac{1}{(a^2-b^2)(a^2-c^2)}$$

$$\frac{(\cos\phi)^2}{b^2-v_1^2}=\frac{b^2}{(b^2-a^2)(b^2-c^2)}-v_2^2\cdot\frac{1}{(b^2-a^2)(b^2-c^2)}$$

$$\frac{(\cos\psi)^2}{c^2-v_1^2}=\frac{c^2}{(c^2-b^2)(c^2-a^2)}-v_2^2\cdot\frac{1}{(c^2-a^2)(c^2-b^2)}$$

therefore

$$\frac{(\cos\omega)^2}{a^2-v_1^2}+\frac{(\cos\phi)^2}{b^2-v_1^2}+\frac{(\cos\psi)^2}{c^2-v_1^2}=0.$$

Just in the same way

$$\frac{(\cos\omega)^2}{a^2-v_2^2}+\frac{(\cos\phi)^2}{b^2-v_2^2}+\frac{(\cos\psi)^2}{c^2-v_2^2}=0,$$

so that v_1^2, v_2^2 are the two roots of the equation

$$\frac{(\cos\omega)^2}{a^2-v^2}+\frac{(\cos\phi)^2}{b^2-v^2}+\frac{(\cos\psi)^2}{c^2-v^2}=0.$$

Cor. Hence the equation to the wave surface may be obtained by making

$$(\cos\omega)x+(\cos\phi)y+(\cos\psi)z=v,$$

or if we please to apply Prop. 5, we may make

$$\sqrt{\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}}\cdot x+\sqrt{\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)}}\cdot y$$
$$+\sqrt{\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}}\cdot z=v_1,$$

or, if we please*,

$$\sqrt{\frac{(a^2u^2-1)(a^2-v^2)}{(a^2-b^2)(a^2-c^2)}}\cdot x+\sqrt{\frac{(b^2u^2-1)(b^2-v^2)}{(b^2-a^2)(b^2-c^2)}}\cdot y$$
$$+\sqrt{\frac{(c^2u^2-1)(c^2-v^2)}{(c^2-a^2)(c^2-b^2)}}\cdot z=1.$$

PROPOSITION 7.

To find when $v_1=v_2$.

By Prop. 4,

$$\frac{x_1(v_1^2-a^2)}{x_2(v_2^2-a^2)}=\frac{y_1(v_1^2-b^2)}{y_2(v_2^2-b^2)}=\frac{z_1(v_1^2-c^2)}{z_2(v_2^2-c^2)}. \qquad (\theta)$$

Hence when $v_1=v_2$ we have, generally speaking,

$$\frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}.$$

Now $$x_1x_2+y_1y_2+z_1z_2=0;$$

therefore $x_1^2+y_1^2+z_1^2$ would $=0$, which is absurd.

The only case therefore when v_1 can $=v_2$ is when one of those terms of equation (θ) becomes $\frac{0}{0}$: thus suppose $v_1=b$, then we have $\frac{x_1}{x_2}=\frac{z_1}{z_2}=\frac{0}{0}$, and we can no longer infer $\frac{x_1}{x_2}=\frac{y_1}{y_2}$.

Let now $(\omega_1, \phi_1, \psi_1)(\omega_2, \phi_2, \psi_2)$ be the two systems of values which ω, ϕ, ψ assume when $v_1=v_2=b$, then applying the equation of Prop. 5 we have

$$\cos\omega_1=\sqrt{\frac{a^2-b^2}{a^2-c^2}} \qquad \cos\omega_2=\sqrt{\frac{a^2-b^2}{a^2-c^2}}$$
$$\cos\phi_1=0 \qquad \cos\phi_2=0$$
$$\cos\psi_1=\sqrt{\frac{b^2-c^2}{a^2-c^2}} \qquad \cos\psi_2=\sqrt{\frac{b^2-c^2}{a^2-c^2}},$$

so that b must correspond to the mean axis.

[* See below, p. 27. ED.]

Proposition 8.

ι_1, ι_2 being the angles made by the front with the optic planes, to find ι_1, ι_2 in terms of v_1, v_2.

By analytical geometry

$$\cos\iota_1 = \cos\omega \,.\, \cos\omega_1 + \cos\phi \,.\, \cos\phi_1 + \cos\psi \,.\, \cos\psi_1$$

$$= \sqrt{\frac{(v_1^2-a^2)(v_2^2-a^2)}{(a^2-b^2)(a^2-c^2)}} \,.\, \sqrt{\frac{a^2-b^2}{a^2-c^2}}$$

$$+ \sqrt{\frac{(v_1^2-c^2)(v_2^2-c^2)}{(c^2-a^2)(c^2-b^2)}} \,.\, \sqrt{\frac{c^2-b^2}{c^2-a^2}}$$

$$= \frac{\sqrt{\{(v_1^2-a^2)(v_2^2-a^2)\}} + \sqrt{\{(v_1^2-c^2)(v_2^2-c^2)\}}}{a^2-c^2},$$

and similarly

$$\cos\iota_2 = \cos\omega \,.\, \cos\omega_2 + \cos\phi \,.\, \cos\phi_2 + \cos\psi \,.\, \cos\psi_2$$

$$= \frac{\sqrt{\{(v_1^2-a^2)(v_2^2-a^2)\}} - \sqrt{\{(v_1^2-c^2)(v_2^2-c^2)\}}}{a^2-c^2}.$$

Proposition 9.

To find v_1, v_2 in terms of ι_1, ι_2.

By the last proposition

$$\cos\iota_1 \,.\, \cos\iota_2 = \frac{(v_1^2-a^2)(v_2^2-a^2)-(v_1^2-c^2)(v_2^2-c^2)}{(a^2-c^2)^2}$$

$$= \frac{(a^4-c^4)-(a^2-c^2)(v_1^2+v_2^2)}{(a^2-c^2)^2}$$

$$= \frac{(a^2+c^2)-(v_1^2+v_2^2)}{(a^2-c^2)}$$

therefore

$$v_1^2+v_2^2 = a^2+c^2-(a^2-c^2)\cos\iota_1\cos\iota_2.$$

Again, $(\sin\iota_1)^2 . (\sin\iota_2)^2 = 1-(\cos\iota_1)^2-(\cos\iota_2)^2+(\cos\iota_1)^2(\cos\iota_2)^2$

$$= 1 - 2 \,.\, \frac{(v_1^2-a^2)(v_2^2-a^2)+(v_1^2-c^2)(v_2^2-c^2)}{(a^2-c^2)^2}$$

$$+ \frac{(a^2+c^2)^2-2(a^2+c^2)(v_1^2+v_2^2)+(v_1^2+v_2^2)^2}{(a^2-c^2)^2}$$

$$= \frac{v_1^4-2v_1^2v_2^2+v_2^4}{(a^2-c^2)^2}$$

therefore

$$v_1^2 - v_2^2 = (a^2 - c^2) \sin \iota_1 . \sin \iota_2$$

but
$$v_1^2 + v_2^2 = (a^2 + c^2) - (a^2 - c^2) \cos \iota_1 \cos \iota_2$$

therefore

$$v_1^2 = \frac{a^2 + c^2}{2} - \frac{a^2 - c^2}{2} \cos (\iota_1 + \iota_2)$$

$$= a^2 \left(\sin \frac{\iota_1 + \iota_2}{2} \right)^2 + c^2 \left(\cos \frac{\iota_1 + \iota_2}{2} \right)^2$$

$$v_2^2 = \frac{a^2 + c^2}{2} - \frac{a^2 - c^2}{2} \cos (\iota_1 - \iota_2)$$

$$= a^2 \left(\sin \frac{\iota_1 - \iota_2}{2} \right)^2 + c^2 \left(\cos \frac{\iota_1 - \iota_2}{2} \right)^2.$$

Thus for uniaxal crystals where $\iota_1 + \iota_2 = 180°$

$$v_1^2 = a^2$$

$$v_2^2 = a^2 (\cos \iota)^2 + c^2 (\sin \iota)^2.$$

Cor. Hence we may reduce the discovery of the two fronts into which a plane front is refracted on entering a crystal to the following trigonometrical problem.

Let a sphere be described about any point in the line in which the air front intersects the plane of incidence. Let the great circle PI denote the latter plane, IF the former, OA, OC also great circles, the planes of single velocity. Suppose IGH to be one of the refracted fronts intersecting OA, OC in G and H, then

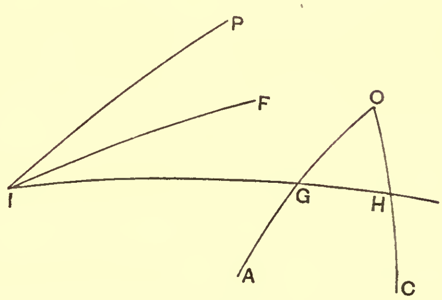


Fig. 1.

$$\frac{(a^2 + c^2) - (a^2 - c^2) \cos (G + H)}{2\,(\text{vel. in air})^2} = \frac{(\sin PIF)^2}{(\sin PIGH)^2}.$$

The double sign will give rise to two positions of the refracted front IGH.

The propositions which follow are perhaps more curious than immediately useful.

PROPOSITION 10.

To determine the portion of a line of vibration in terms of the two velocities of its corresponding front.

We have here to determine the quantities $\frac{y_1}{x_1}$, $\frac{z_1}{x_1}$ (of Prop. 1) in terms of v_1, v_2, or on putting $x_1^2+y_1^2+z_1^2=1$, x_1, y_1, z_1 are to be found in terms of v_1, v_2.

By Prop. 3

$$x_1 : y_1 : z_1 :: \frac{l}{a^2-v_1^2} : \frac{m}{b^2-v_1^2} : \frac{n}{c^2-v_1^2}$$

and by Prop. 5

$$\begin{aligned} l^2 : m^2 : n^2 :: &(b^2-c^2)(a^2-v_1^2)(a^2-v_2^2) \\ :&(c^2-a^2)(b^2-v_1^2)(b^2-v_2^2) \\ :&(a^2-b^2)(c^2-v_1^2)(c^2-v_2^2)\,; \end{aligned}$$

therefore

$$x_1^2 \quad : \quad y_1^2 \quad : \quad z_1^2$$

$$:: (b^2-c^2)\frac{a^2-v_2^2}{a^2-v_1^2} : (c^2-a^2)\frac{b^2-v_2^2}{b^2-v_1^2} : (a^2-b^2)\frac{c^2-v_2^2}{c^2-v_1^2}\,.$$

Let α, β, γ be the angles made by the given line of vibration with the elastic axes, then

$$\begin{aligned} (\cos\alpha)^2 &= \frac{x_1^2}{x_1^2+y_1^2+z_1^2} \\ &= (b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2) \end{aligned}$$

divided by

$$(b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2)+(c^2-a^2)(b^2-v_2^2)(c^2-v_1^2)(a^2-v_1^2) \\ +(a^2-b^2)(c^2-v_2^2)(a^2-v_1^2)(b^2-v_1^2)$$

and therefore

$$= \frac{(b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2)}{(v_1^2-v_2^2)(a^2-b^2)(b^2-c^2)(c^2-a^2)}$$

(where it is to be observed that the reduction of the denominator is simply the effect of a vast heap of terms disappearing under the influence of contact with the magic circuit (a^2-b^2), (b^2-c^2), (c^2-a^2), a simpler instance of which was seen in Proposition 5).

In fact the coefficient of $v^4 \,.\, v^2$

$$\begin{aligned} &= (b^2-c^2)+(c^2-a^2)+(a^2-b^2) \\ &= 0 \end{aligned}$$

that of $v_1^2 \,.\, v_2^2$

$$\begin{aligned} &= (c^2+b^2).(c^2-b^2) \\ &\quad + (a^2+c^2).(a^2-c^2) \\ &\quad + (b^2+a^2).(b^2-a^2) \\ &= (c^4-b^4)+(a^4-c^4)+(b^4-a^4) \\ &= 0. \end{aligned}$$

The term in which neither v_1 nor v_2 enters

$$\begin{aligned} &= a^2b^2c^2\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\} \\ &= 0. \end{aligned}$$

The coefficient of

$$-v_1^2 = a^2.(b^4-c^4)+b^2.(c^4-a^4)+c^2.(a^4-b^4)$$

and that of

$$v_2^2 = b^2c^2.(c^2-b^2)+c^2a^2.(a^2-c^2)+a^2b^2.(b^2-a^2)$$

each of which

$$= (a^2-b^2).(b^2-c^2).(c^2-a^2).$$

Hence

$$(\cos\alpha)^2 = \frac{v_1^2-b^2}{v_1^2-v_2^2}\cdot\frac{(a^2-v_2^2)(c^2-v_1^2)}{(a^2-b^2)(a^2-c^2)},$$

in like manner $(\cos\beta)^2 = \&c.$

and

$$(\cos\gamma)^2 = \frac{v_1^2-b^2}{v_1^2-v_2^2}\cdot\frac{(c^2-v_2^2)(a^2-v_1^2)}{(c^2-b^2)(c^2-a^2)}.$$

PROPOSITION 11.

ϵ_1, ϵ_2 being the angles between any line of vibration and the optic axes, required the velocity due to that line in terms of ϵ_1, ϵ_2.

By analytical geometry,

$$\cos\epsilon_1 = \cos\alpha\,.\,\cos\phi_1 + \cos\gamma\,.\,\cos\psi_1$$

$$\cos\epsilon_2 = \cos\alpha\,.\,\cos\phi_1 - \cos\gamma\,.\,\cos\psi_1$$

therefore

$$\begin{aligned} \cos\epsilon_1\,.\,\cos\epsilon_2 &= (\cos\alpha)^2(\cos\phi_1)^2 - (\cos\gamma)^2(\cos\psi_1)^2 \\ &= \frac{v_1^2-b^2}{v_1^2-v_2^2}\cdot\left\{\frac{(a^2-v_2^2).(c^2-v_1^2)-(c^2-v_2^2).(a^2-v_1^2)}{(a^2-c^2)^2}\right\} \\ &= \frac{v_1^2-b^2}{v_1^2-v_2^2}\cdot\frac{(a^2-c^2)(v_2^2-v_1^2)}{(a^2-c^2)^2} \\ &= \frac{b^2-v_1^2}{a^2-c^2}. \end{aligned}$$

Hence

$$v_1^2 = b^2-(a^2-c^2)\cos\epsilon_1\cos\epsilon_2,$$

and in like manner, for the *conjugate* line of vibration

$$v_2^2 = b^2-(a^2-c^2)\cos\epsilon_1'\cos\epsilon_2'.$$

Proposition 12.

To find ϵ_1, ϵ_2 *in terms of* ι_1, ι_2.

$$(\cos\epsilon_1)^2 + (\cos\epsilon_2)^2 = 2(\cos\alpha)^2 . (\cos\phi_1)^2 + 2(\cos\gamma)^2 . (\cos\psi_1)^2$$

$$= 2\,\frac{v_1^2 - b^2}{v_1^2 - v_2^2}\left\{\frac{(a^2 - v_2^2).(c^2 - v_1^2) + (c^2 - v_2^2).(a^2 - v_1^2)}{(a^2 - c^2)^2}\right\}$$

but by Prop. 9

$$v_1^2 = a^2\left(\sin\frac{\iota_1 + \iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1 + \iota_2}{2}\right)^2$$

$$v_2^2 = a^2\left(\sin\frac{\iota_1 - \iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1 - \iota_2}{2}\right)^2$$

therefore

$$(\cos\epsilon_1)^2 + (\cos\epsilon_2)^2 = \frac{b^2 - v_1^2}{(a^2 - c^2)\sin\iota_1 . \sin\iota_2}$$

multiplied by

$$\frac{2(a^2 - c^2)^2\left[\left(\cos\frac{\iota_1 + \iota_2}{2}\right)^2\left(\sin\frac{\iota_1 - \iota_2}{2}\right)^2 + \left(\cos\frac{\iota_1 - \iota_2}{2}\right)^2\left(\sin\frac{\iota_1 + \iota_2}{2}\right)^2\right]}{(a^2 - c^2)^2}$$

$$= \frac{b^2 - v_1^2}{(a^2 - c^2)\sin\iota_1 . \sin\iota_2}\{(\sin\iota_1)^2 + (\sin\iota_2)^2\}$$

and we have seen that

$$\cos\epsilon_1 \cos\epsilon_2 = \frac{b^2 - v_1^2}{a^2 - c^2}$$

therefore

$$\cos\epsilon_1 + \cos\epsilon_2 = \sqrt{\left(\frac{b^2 - v_1^2}{a^2 - c^2}\right)} . \frac{\sin\iota_1 + \sin\iota_2}{\sqrt{(\sin\iota_1 . \sin\iota_2)}}$$

$$\cos\epsilon_1 - \cos\epsilon_2 = \sqrt{\left(\frac{b^2 - v_1^2}{a^2 - c^2}\right)} . \frac{\sin\iota_1 - \sin\iota_2}{\sqrt{(\sin\iota_1 . \sin\iota_2)}}$$

therefore

$$\cos\epsilon_1 = \sqrt{\left\{\frac{b^2 - v_1^2}{a^2 - c^2} . \frac{\sin\iota_1}{\sin\iota_2}\right\}}$$

$$\cos\epsilon_2 = \sqrt{\left\{\frac{b^2 - v_1^2}{a^2 - c^2} . \frac{\sin\iota_2}{\sin\iota_1}\right\}}$$

and in like manner

$$\cos\epsilon_1' = \sqrt{\left\{\frac{b^2 - v_2^2}{a^2 - c^2} . \frac{\sin\iota_1}{\sin\iota_2}\right\}}$$

$$\cos\epsilon_2' = \sqrt{\left\{\frac{b^2 - v_2^2}{a^2 - c^2} . \frac{\sin\iota_2}{\sin\iota_1}\right\}}$$

where v_1, v_2 for the sake of neatness are left *unexpressed* in terms of ι_1, ι_2.

This is the simplest form by which the position of the lines of vibration can be denoted.

Cor. From the last proposition it appears that

$$\frac{\cos \epsilon_1}{\cos \epsilon_2} = \frac{\sin \iota_1}{\sin \iota_2}.$$

Hence we may construct geometrically for the two planes of polarization.

Let I, K be the projections of the two optic axes on a sphere, E the projection of the normal to the front, P the projection of one line of vibration; then

$$\frac{\cos PK}{\cos PI} = \frac{\sin KE}{\sin IE}.$$

Draw FEG the circle of which P is the pole, meeting PK, PI produced in G and F.

Then $\cos PK = \sin KG$,

and $\cos PI = \sin IF$,

therefore

$$\frac{\sin KG}{\sin FI} = \frac{\sin KE}{\sin IE}$$

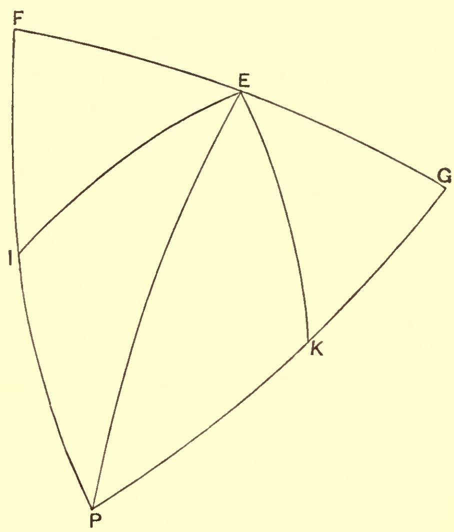


Fig. 2.

therefore

$$\frac{\sin KG}{\sin KE} = \frac{\sin IF}{\sin IE}$$

therefore

$$\sin KEG = \sin IEF$$

therefore $KEG = IEF$ or $180° - IEF$. But $PEF = PEG$, therefore EP bisects either the angle IEK or the supplement to it.

These two positions of EP give the two planes of polarization. The construction is the same as that given in Mr Airy's tracts, and originally proposed, I believe, by Mr MacCullagh.

ADDENDUM.

If in the equation of Prop. 6, viz.

$$\frac{(\cos\omega)^2}{a^2-v^2}+\frac{(\cos\phi)^2}{b^2-v^2}+\frac{(\cos\psi)^2}{c^2-v^2}=0$$

we change a, b, c, v into $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{v}$, and consider v to be the length of a line drawn perpendicular to the plane

$$\cos\omega\,.\,x+\cos\phi\,.\,y+\cos\psi\,.\,z=0,$$

the equation to the extremity thereof must be

$$\frac{a^2r^2(\cos\omega)^2}{a^2-r^2}+\frac{b^2r^2(\cos\phi)^2}{b^2-r^2}+\frac{c^2r^2(\cos\psi)^2}{c^2-r^2}$$

where ω, ϕ, ψ denote the angles between the radius vector r, and the axes of x, y, z, so that the equation may be written

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0,$$

which is that of the wave surface.

But we have seen that

$$v^2=c^2\left\{\cos\left(\frac{\iota_1\pm\iota_2}{2}\right)\right\}^2+a^2\left\{\sin\left(\frac{\iota_1\pm\iota_2}{2}\right)\right\}^2,$$

therefore the equation to the wave surface may be written

$$\frac{1}{r^2}=\frac{\left(\cos\frac{\iota_1\pm\iota_2}{2}\right)^2}{c^2}+\frac{\left(\sin\frac{\iota_1\pm\iota_2}{2}\right)^2}{a^2},$$

where ι_1, ι_2 denote the angles between the radius vector v and the two lines which would be the optic axes if a, b, c were changed into $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ so that if e be the inclination of either to the mean axis of elasticity

$$\cos e=\sqrt{\left(\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^2}-\frac{1}{c^2}}\right)}=\frac{c}{b}\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)}$$

$$\sin e=\sqrt{\left(\frac{\frac{1}{b^2}-\frac{1}{c^2}}{\frac{1}{a^2}-\frac{1}{c^2}}\right)}=\frac{a}{b}\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)}.$$

These lines I shall call by way of distinction the prime radii*.

* Upon the authority of Professor Airy I have appropriated the term optic axes to the lines normal to the fronts of single velocity.

Cor. 1. If r_1, r_2 be the two values of r corresponding to the same values of ι_1, ι_2 we have

$$\frac{1}{r_1^2}-\frac{1}{r_2^2}=\frac{1}{c^2}\left\{\left(\cos\frac{\iota_1-\iota_2}{2}\right)^2-\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2\right\}$$
$$+\frac{1}{a^2}\left\{\left(\sin\frac{\iota_1-\iota_2}{2}\right)^2-\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2\right\}$$
$$=\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\sin\iota_1\,.\,\sin\iota_2,$$

which proves the celebrated problem of *two rays* having a common direction in a crystal.

Cor. 2. The intersection of any concentric sphere with the wave surface is found by making r constant. Hence $\iota_1 \pm \iota_2$ becomes constant, and therefore $r\iota_1 \pm r\iota_2 = \text{constant}$. Hence the curve of intersection is the locus of points, the sum or difference of whose distances from two poles when measured by the arcs of great circles is constant; the poles being the points in which the prime radii pierce the sphere.

In three cases these spherico-ellipses or spherico-hyperbolas become great circles:

(1) When $\iota_1 \pm \iota_2 =$ the angle between the two poles, in which case the curve of intersection is the great circle which comprises the two poles.

(2) When $\iota_1 - \iota_2 = 0$, when the locus is a great circle perpendicular to the former and bisecting the angle between the optic axes.

(3) When $\iota_1 + \iota_2 = 180°$, when the locus is a great circle perpendicular to the two above, and bisecting the supplemental angle between the two axes.

Various other properties may be with the greatest simplicity deduced from the radio-angular equation. The hurry of the press leaves me time only to subjoin the following

Proposition.

To find the inclination of the radius vector to the tangent plane, in terms of the angles which the radius vector makes with the prime radii.

Let O be the centre of the wave surface, OA, OB the two prime radii, OP any radius vector. Let $OP = v$, $POA = \iota_1$, $POB = \iota_2$, and let the inclination of the planes POA, $POB = \mu$;

then
$$\frac{1}{r^2}=\frac{\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2}{a^2}+\frac{\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2}{c^2},$$

(taking only the positive sign for the sake of brevity).

Let OQ, OR be the two adjacent radii vectores, so assumed that

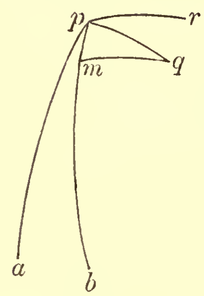


Fig. 3.

$$QOA = POA, \quad QOB = POB + \delta\iota_2,$$
$$ROB = POB, \quad ROA = POA + \delta\iota_1,$$

and let p, q, r, a, b be the projections of P, Q, R, A, B on a sphere of which O is the centre, then it is clear that

$$qpa = 90°, \quad rpb = 90°,$$

draw qm perpendicular to pb, then $pm = \delta\iota_2$, and therefore

$$pq = \frac{pm}{\sin pqm} = \frac{pm}{\sin apb} = \frac{\delta\iota_2}{\sin\mu}.$$

In like manner

$$pr = \frac{\delta\iota_1}{\sin\mu}.$$

Now the angle QPO

$$= \tan^{-1}.\frac{r.POQ}{OQ - OP} = \tan^{-1}.\frac{r.pq}{\dfrac{dr}{d\iota_2}.\delta\iota_2};$$

also

$$\frac{d.\dfrac{1}{r^2}}{d\iota_2} = d\iota_2\left\{\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2\right\}$$

$$= -\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\left(\sin\frac{\iota_1+\iota_2}{2}\right)\left(\cos\frac{\iota_1+\iota_2}{2}\right);$$

therefore

$$\frac{dr}{rd\iota_2} = \tfrac{1}{4}r^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin(\iota_1+\iota_2),$$

therefore

$$\cot QPO = \frac{r^2}{4}\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin(\iota_1+\iota_2)\sin\mu.$$

In like manner

$$\cot RPO = \frac{r^2}{4}\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin(\iota_1+\iota_2)\sin\mu,$$

therefore

$$QPO = RPO.$$

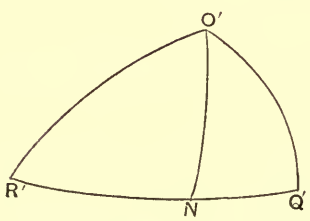


Fig. 4.

Also it is clear that $rpq = apb = \mu$. And to find the inclination of OP to RPQ, we have only to describe a sphere of which P is the centre, and intersecting PQ, PR, PO in Q', R', O'.

Then $R'O'Q' = \mu$, and

$$O'Q' = O'R' = \cot^{-1}\left\{\frac{r^2}{4}\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin(\iota_1+\iota_2)\sin\mu\right\}.$$

Draw $O'N$ perpendicular to $R'Q'$, then $O'N$ measures the inclination of the radius vector to the tangent plane*.

And $$Q'O'N = \frac{\mu}{2},$$

therefore $$\cos\frac{\mu}{2} = \tan O'N \cdot \cot O'Q',$$

therefore $$\cot O'N = \frac{\cot O'Q'}{\cos\frac{\mu}{2}},$$

and therefore
$$\cot O'N = \tfrac{1}{2}r^2 \cdot \left(\frac{1}{a^2} - \frac{1}{c^2}\right)\sin\frac{\mu}{2} \cdot \sin(\iota_1 + \iota_2).$$

Let AOB the angle between the optic axes $= 2e$, then by mere trigonometry
$$\sin\frac{\mu}{2} = \sqrt{\frac{\sin\left(e + \frac{\iota_1 - \iota_2}{2}\right)\sin\left(e - \frac{\iota_1 - \iota_2}{2}\right)}{\sin\iota_1 \cdot \sin\iota_2}},$$

therefore the tangent of the inclination between the radius vector and the normal
$$= \tfrac{1}{2}r^2 \cdot \left(\frac{1}{a^2} - \frac{1}{c^2}\right)\sin(\iota_1 + \iota_2) \cdot \sqrt{\frac{\sin\left(e + \frac{\iota_1 - \iota_2}{2}\right)\sin\left(e - \frac{\iota_1 - \iota_2}{2}\right)}{\sin\iota_1 \cdot \sin\iota_2}}.$$

In like manner the tangent of the inclination between the same radius vector and the normal at the other point of the wave-surface pierced by it
$$= \tfrac{1}{2}(r_1)^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\sin(\iota_1 - \iota_2) \cdot \sqrt{\frac{\sin\left(e + \frac{\iota_1 + \iota_2}{2}\right)\sin\left(e - \frac{\iota_1 + \iota_2}{2}\right)}{\sin\iota_1 \cdot \sin\iota_2}}.$$

We may, in the same way, find the inclination of the tangent plane to either of the prime radii, and to the plane which contains them both, in terms of ι_1 and ι_2; the former by a remarkably elegant construction; but the final expressions do not present themselves under the same simple aspect.

If we call ϕ the angle between the ray and the front, we may still further reduce by substituting for r^2 its values in terms of ι_1, ι_2 and we shall obtain
$$\cot\phi = \frac{2(c^2 - a^2)}{c^2\tan\frac{\iota_1 \mp \iota_2}{2} + a^2\cot\frac{\iota_1 \mp \iota_2}{2}}$$
$$\times\sqrt{\left\{\sin\left(e + \frac{\iota_1 \pm \iota_2}{2}\right)\sin\left(e - \frac{\iota_1 \pm \iota_2}{2}\right) \cdot \operatorname{cosec}\iota_1 \cdot \operatorname{cosec}\iota_2\right\}}.$$

* O' is the projection of the ray and $R'O'$ of the tangent plane. Therefore $O'N$ being perpendicular to $R'Q'$ represents their inclination.

And if π_1, π_2 be the inclinations of the normal to the two prime radii, it may be shown that

$$\cos \pi_1 = \cos \phi \sin \iota_1 \mp \sin \phi \cos \iota_1 \sin \frac{\mu}{2},$$

$$\cos \pi_2 = \cos \phi \sin \iota_2 \pm \sin \phi \cos \iota_2 \sin \frac{\mu}{2}.$$

COR. 1. For uniaxal crystals $\frac{\mu}{2} = 90°$ and $\iota_1 + \iota_2 = 180°$, so that the tangent of the inclination of normal to radius vector

$$= \tfrac{1}{2} r^2 . \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin 2\iota \text{ for one point,}$$

and $= 0$ for the other.

COR. 2. For every point in the circular section which passes through the poles $\sin \frac{\mu}{2} = 0$, and for the other two circular sections $\iota_1 \pm \iota_2 = 0$ or $180°$.

Therefore every point in the three circular sections is an apse.

COR. 3. When a nearly $= c$, $\frac{1}{a^2} - \frac{1}{c^2}$ is very small; and therefore the normal and radius vector very nearly coincide.

COR. 4. Referring to fig. 4 we see that $O'N$ bisects the angle $R'O'Q'$. Now $R'O$, $Q'O$ are respectively perpendicular to the planes passing through O' and the optic axes; and therefore the meridian plane as we may term it, that is, the plane containing both the ray and the normal, always bisects the angle formed by the two planes drawn through the ray and the two optic axes.

COR. 5. When

$$\iota_1 \text{ or } \iota_2 = 0,$$

$$\iota_2 \text{ or } \iota_1 = e.$$

And therefore ϕ assumes the form $\frac{0}{0}$, which indicates that the extremities of the four prime radii are singular points.

In concluding for the present it behoves me to state that one step has been omitted in the foregoing paper*, viz. the actual performance of the eliminations which lead to the rectilinear equation to the wave-surface. But Mr Archibald Smith's elegant and brief Memoir in the *Cambridge Philosophical Transactions*† of last year leaves nothing to be desired further on that head.

[* See below, p. 27. ED.] [† Vol. VI. Also *Phil. Mag.* April, 1838, p. 335. ED.]

That I have not exhibited it in its proper place (Prop. 6) arises only from my respect to the principle of literary propriety. With this important blank supplied the Analytical Theory may be pronounced to be complete.

For all errors and imperfections in what precedes my excuse must be press of time and a total want of the materials to be derived from consulting works of reference.

Since writing the above I have had an opportunity of reading the paper of our living Laplace inserted as part of the Third Supplement to his System of Rays in the *Transactions* of the Royal Irish Academy, in which the principal foregoing results are obtained by aid of a more refined and transcendental analysis.

The nature of the four singular points is there discussed and the existence of four circles of plane contact demonstrated.

The former may be very easily shown thus: when ι_1 is very small $\iota_2 = 2e - \iota_1 \cos\psi$ very nearly, ψ denoting the inclination of the plane in which e is reckoned to the plane in which ι_1 is reckoned.

Hence

$$\left(\frac{1}{r}\right)^2 = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{c^2}\right) - \frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\cos\{2e - \iota_1(\cos\psi \pm 1)\}$$

$$= \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{c^2}\right) - \frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\cos 2e - \frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\sin 2e\,(\cos\psi \pm 1)\,\iota_1$$

$$= \frac{1}{b^2} - \frac{1}{b^2 ac}\sqrt{\{(a^2-b^2)(b^2-c^2)\}}\,(\cos\psi \pm 1)\,\iota_1,$$

therefore

$$r = b\left\{1 + \frac{1}{2}(\cos\psi \pm 1)\left(1-\frac{b^2}{a^2}\right)^{\frac{1}{2}}\left(\frac{b^2}{c^2}-1\right)^{\frac{1}{2}}\iota_1\right\}.$$

Take ψ constant and let the abscissæ and ordinates be reckoned respectively along and perpendicular to the prime ray.

Then $$\iota_1 = \frac{y}{x} \text{ nearly, and } r = \sqrt{(y^2+x^2)} = x,$$

or, if we change the origin to the other extremity of the prime ray,

$$\iota_1 = \frac{y}{b}, \quad r = b - x,$$

so that the equation becomes

$$-\frac{x}{y} = \frac{1}{2}(\cos\psi \pm 1)\sqrt{\left\{\left(1-\frac{b^2}{a^2}\right)\left(\frac{b^2}{c^2}-1\right)\right\}}.$$

Hence at each singular point the surface is touched by a cone, the equation to the generating line of which is given by the above, the extreme angle between it and the prime ray being

$$\cot^{-1}\left[\sqrt{\left\{\left(1-\frac{b^2}{a^2}\right)\left(\frac{b^2}{c^2}-1\right)\right\}}\right].$$

When $b=a$, ψ always $=\frac{\pi}{2}$ and the cone returns into a plane.

Again, let us suppose that the position of any perpendicular from the centre is given, and that of the corresponding radius vector required.

Let OA, OB* denote what we have termed the optic axes, but which it will be more agreeable to analogy to term the prime perpendiculars from centre, and let OP be the given normal. Take OQ, OR contiguous perpendiculars from centre in planes POQ, ROP, perpendicular to POA, POB respectively, then the inclination of the two former will be the same as that of the two latter, and may be termed μ.

Let ι_1, ι_2 now denote the angles POA, POB respectively, then

$$QOA=\iota_1, \qquad QOB=\iota_2+\delta\iota_2,$$
$$ROA=\iota_1+\delta\iota_1, \quad ROB=\iota_2.$$

The ray will be found by joining O with the intersection of three planes drawn at P, Q, R, perpendicular to OP, OQ, OR, respectively.

Now from Prop. 9 it appears that

$$OP=\sqrt{\left\{a^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2+c^2\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2\right\}},$$

using only one sign for the sake of simplicity, which we may do by throwing the ambiguity upon the *way* in which ι_1 or ι_2 is measured, also

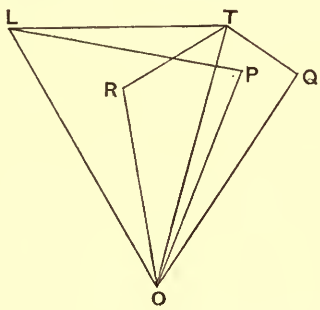


Fig. 5.

$$OQ=OP+\frac{d\,.\,OP}{d\iota_2}\delta\iota_2,$$

$$OR=OP+\frac{d\,.\,OP}{d\iota_1}\delta\iota_1.$$

Let $\delta\iota_1=\delta\iota_2$, then it is clear that $OQ=OR$, and the intersection of the two planes perpendicular to OQ, OR is therefore a line perpendicular to the plane QOR, and to the line which bisects the angle QOR.

In fact if we draw QT, RT perpendicular to OQ, OR respectively in the plane QOR, the intersection in question passes through T and is perpendicular to OT; also

$$OT=OQ\,.\sec\left(\tfrac{1}{2}ROQ\right)=OQ$$

to the first order of smallness.

* OA, OB are not expressed in the figure.

Now it is easy to see (just as on p. 16) that

$$ROP = \frac{\delta\iota_1}{\sin\mu},$$

and also

$$QOP = \frac{\delta\iota_2}{\sin\mu},$$

therefore $ROP = QOP$ and therefore POT is perpendicular to QOR.

Hence the problem is reduced to finding L the intersection of two lines TL, PL drawn in the *same* plane POT.

Now because OTL, OPL are each right angles, a circle may be made to pass through L, T, P, O.

Hence the angle

$$PLO = PTO = \tan^{-1}\frac{OP \times POT}{OT - OP}$$

$$= \tan^{-1}\frac{OP \times POR \,.\, \cos\tfrac{1}{2}\mu}{\dfrac{d\,.\,OP}{d\iota_2}\,\delta\iota_2} = \tan^{-1}\frac{OP \times \dfrac{\delta\iota_2}{\sin\mu}\cos\tfrac{1}{2}\mu}{\dfrac{d\,.\,OP}{d\iota_2}\,\delta\iota_2},$$

and

$$OL = OP \,.\, \sec POL.$$

Also the position of the plane POL is known, and therefore the radius is completely determined in magnitude and position.

It may be worth while also to remark that the above constructions enable us to form a series of equations between the magnitude of the radius and its inclinations to the two prime perpendiculars.

In fact, if we call π_1, π_2 the two inclinations in question

$$\cos\pi_1 = \cos POL \cos\iota_1 \pm \sin POL \sin\iota_1 \,.\, \sin\frac{\mu}{2},$$

$$\cos\pi_2 = \cos POL \cos\iota_2 \mp \sin POL \sin\iota_2 \,.\, \sin\frac{\mu}{2},$$

and of course if we call the angle between the two prime normals $2E$

$$\sin\frac{\mu}{2} = \sqrt{\frac{\sin\left(E + \frac{\iota_1+\iota_2}{2}\right)\sin\left(E - \frac{\iota_1+\iota_2}{2}\right)}{\sin\iota_1 \sin\iota_2}}.$$

Cor. 1. When ι_1 or $\iota_2 = 0$, $\tan POL$ assumes the form $\frac{0}{0}$ which may be interpreted analogously to the method used in the reverse problem, but may be more elegantly illustrated by

Cor. 2. Which is that the meridian plane POT (that is, the plane in which both normal and radius lie) bisects the angle formed by ROP, QOP, and therefore that formed by the planes drawn through the normal and the two prime normals to which these two are perpendicular.

Now we have found (Cor. 4, page 18), that it also bisects the angle formed by the two planes passing through the radius and the two prime radii. Hence when the ray is given, we may find by the easiest geometry the normal and the tangent plane, and *vice versâ*.

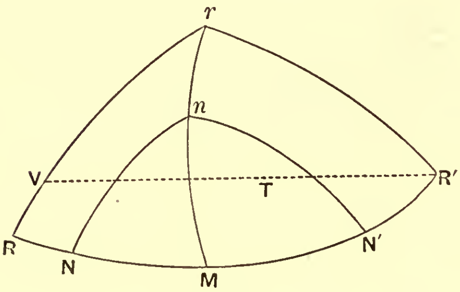


Fig. 6.

Thus suppose (N, N') (R, R') to be the projections of the prime perpendiculars and prime radii on a sphere concentric with the wave surface.

Let n be the projection of any given perpendicular on the same sphere; join nN, nN'; bisect NnN' by nM, which will be the meridian plane.

Draw from R', $R'TV$ perpendicular to nM and make $R'T = TV$. Produce RV to meet Mn in r, then $RrM = R'rM$, and therefore r is the projection of the radius. Just in the same way when r is given we may find n.

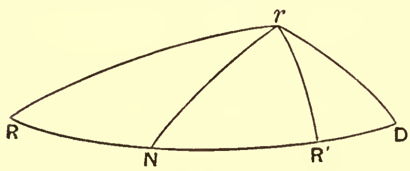


Fig. 7.

Now suppose n to come to N, then the position of the meridian plane nM becomes indeterminate, and r from a point becomes a locus, subject to the condition that $R'rN = RrN$. From r draw rD perpendicular to rN.

Then it is clear that because rN bisects RrR'

$$\frac{\sin RD}{\sin R'D} = \frac{\sin Rr}{\sin R'r} = \frac{\sin RN}{\sin R'N},$$

and therefore D is a fixed point and ND a fixed length, and

$$\cos rND = \tan rN \,.\, \cot ND\,;$$

therefore the projection of the locus of r upon a plane drawn at N perpendicular to the line joining N with the centre O is given by the equation

$$\rho = ON \,.\, \cot ND \,.\, \cos\theta,$$

N being the origin and the projection of ND the prime radius; which is the equation to a circle passing through N, and whose diameter $= ON \cot ND$.

Hence at the extremity of each prime perpendicular the tangent plane meets the surface in a circle passing through that extremity and whose radius $= \frac{1}{2}b \cot\alpha$, α being to be found from the equation

$$\frac{\sin(2E+\alpha)}{\sin\alpha} = \frac{\sin(E+e)}{\sin(E-e)},$$

that is $$\tan(E+\alpha) = (\tan E)^2 \cot e.$$

Just in the same way it may be shown that the trace of the perpendiculars to the tangent planes of the surface at the point where it is pierced by any prime radius upon a plane perpendicular to that radius at its extremity, is also a circle passing through it, and curved in an opposite direction from the circle of plane contact nearest to it.

Hence the enveloping cone at these points may be described as being perpendicular to the circular cone, formed by drawing lines from the centre to the above described circle; that is every generating line of the one will be perpendicular to the generating line which it meets of the other.

More generally it easily appears from fig. 6 that if a series of great circles (representing meridian planes) be taken intersecting the great circle $NRR'N'$ in a fixed point, a plane perpendicular to the radius passing through that point, will intersect the cone of rays as well as the cone of perpendiculars corresponding to those meridian planes, in two *circles*. So that there exist an indefinite number of *circular* cones of rays corresponding to *circular* cones of perpendiculars touching each other in a line lying in the plane containing the extreme axes, and having their circular sections perpendicular to that line.

The cusps are explained by the cone of *rays* degenerating into a right line, and the circles of plane contact by the cone of perpendiculars so degenerating.

Furthermore I observe in conclusion that when a ray is given it follows from the general geometrical construction above that there will be two meridian planes according as we take R with R', or with a point 180° from R', and consequently these two planes will be perpendicular to each other.

And similarly when a *normal* is given there will be two meridian planes perpendicular to each other.

Thus the planes passing through any radius and the two normals at the points where it pierces the wave surface, are *perpendicular* to each other, as are also the two planes passing through any normal and *its* two corresponding radii.

Moreover a glance at fig. 2 will show that the two lines of vibration corresponding to any front lie respectively in the two meridian planes passing through the perpendicular to that front or, in other words, the intersection of a plane drawn through either ray belonging to a front perpendicular thereunto is *always* a line of vibration in that front.

This has been noticed, I think, by Sir William Hamilton for the particular case of the singular points.

As two fronts belong to every ray, so two rays pertain to every front. And from what has been said above it appears that the two lines of vibration in any front are the projections of its two rays upon its own plane.

Note 1.

In the paper above, it is shown that the meridian plane, that is, the plane containing the ray and normal, always passes through a line of vibration in the corresponding point. Now the line of force called into action by a displacement in the line of vibration clearly lies in this very plane; for the resolved part of it lies in the line of vibration itself.

Harmony and analogy concur in suggesting that as two of these four lines are perpendicular to each other, so are also the other two, or in other words, that the ray is always perpendicular to the direction of unresolved force.

The following investigation verifies this conjecture.

Let x, y, z be the coordinates of a point taken at distance unity from the origin and in any line of vibration; then the cosines of the angles made by the line of force with the axes are as $a^2x : b^2y : c^2z$ respectively.

Let a be the inclination between the line of vibration and the line of force, then

$$\cos\omega = \frac{a^2x \,.\, x + b^2y \,.\, y + c^2z \,.\, z}{\sqrt{\{(a^4x^2 + b^4y^2 + c^4z^2)(x^2 + y^2 + z^2)\}}} = \frac{a^2x^2 + b^2y^2 + c^2z^2}{\sqrt{(a^4x^2 + b^4y^2 + c^4z^2)}}.$$

Let $$\sqrt{(a^4x^2 + b^4y^2 + c^4z^2)} = P,$$

then $$P^2 = v^4 (\sec\omega)^2.$$

Now let α, β, γ be the angles of inclination between the coordinate planes and the front in which the line of vibration lies, and λ some quantity to be determined. I have shown in Prop. 3 that if

$$\lambda \cos\alpha = (a^2 - v^2)\, x,$$

then will $$\lambda \cos\beta = (b^2 - v^2)\, y,$$

and $$\lambda \cos\gamma = (c^2 - v^2)\, z\,;$$

therefore $$\lambda^2 = a^4x^2 + b^4y^2 + c^4z^2 - 2v^2(a^2x^2 + b^2y^2 + c^2z^2) + v^4 = P^2 - v^4.$$

Again,

$$\lambda^2 \,.\left(\frac{(\cos\alpha)^2}{(a^2 - v^2)^2} + \frac{(\cos\beta)^2}{(b^2 - v^2)^2} + \frac{(\cos\gamma)^2}{(c^2 - v^2)^2}\right) = x^2 + y^2 + z^2 = 1\,;$$

therefore $$\frac{1}{P^2 - v^4} = \frac{(\cos\alpha)^2}{(a^2 - v^2)^2} + \frac{(\cos\beta)^2}{(b^2 - v^2)^2} + \frac{(\cos\gamma)^2}{(c^2 - v^2)^2}.$$

Now $$\frac{1}{P^2 - v^4} = \frac{1}{v^4(\sec\omega)^2 - v^4} = \frac{1}{v^4}(\cot\omega)^2.$$

And in Mr Smith's investigation of the form of the wave surface (already alluded to*) by great good fortune I find ready to my hand

$$\frac{(\cos\alpha)^2}{(a^2-v^2)^2}+\frac{(\cos\beta)^2}{(b^2-v^2)^2}+\frac{(\cos\gamma)^2}{(c^2-v^2)^2}=\frac{1}{v^2(r^2-v^2)},$$

r being the radius vector to the point whose tangent plane is parallel to the point in question.

Hence

$$(\cot\omega)^2=\frac{v^4}{v^2(r^2-v^2)}=\frac{v^2}{r^2-v^2}=\frac{p^2}{r^2-p^2},$$

p being the length of the perpendicular from the centre upon the tangent plane, for $p=v$.

Hence $(\cot\omega)^2=$ the square of the cotangent of the angle between radius vector and normal.

Or, in other words, the line of force is as much inclined to the line of vibration as the ray is to the normal.

Now the normal is perpendicular to the line of vibration, and all four lines lie in one plane.

Therefore the ray is perpendicular to the line of force. Q. E. D.

I may be allowed to conclude this long paper with a summary of some of the most remarkable consequences which I have extricated from Fresnel's hypothesis.

(1) The two meridian planes corresponding to any given radius are perpendicular to each other†.

(2) So are the two corresponding to any given normal.

(3) Every meridian plane bisects the angle formed by two planes drawn through the radius and the two prime radii.

(4) It also bisects the angle formed by two planes drawn through the normal and the two prime normals.

(5) Each meridian plane contains one line of vibration and the corresponding line of force.

(6) The ray is perpendicular to the line of force.

All these conclusions, except the fourth, are, I believe, original.

* See above, p. 18.

† I have defined the meridian plane to be that which contains radius vector and normal belonging to the same point.

The theory of external and internal conical refraction follows immediately as a particular consequence from the third and fourth combined as already shown; the same propositions also enable us to draw a tangent plane to any point of the wave surface by mere Euclidean geometry. May not some of these conclusions serve to suggest to physical inquirers the question, Has the theory been started from the most natural point of view?

Note 2. *Investigation* of the Wave Surface.*

Since the appearance of the preceding parts, I have succeeded in completing the self-sufficiency of my method by deducing the equation to the wave surface from the expressions given in Prop. 5 for the angles between a front and the principal planes in terms of its two velocities. If these angles be ω, ϕ, ψ, and the two velocities v_1, v_2 we found

$$\cos\omega = \sqrt{\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}},$$

$$\cos\phi = \sqrt{\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)}},$$

$$\cos\psi = \sqrt{\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}}.$$

Let the tangent plane to the wave surface be written

$$\frac{\cos\omega}{v_1}.x + \frac{\cos\phi}{v_1}.y + \frac{\cos\psi}{v_1}.z = 1, \qquad (\alpha)\dagger$$

then

$$\frac{d\dfrac{\cos\omega}{v_1}\ddagger}{d\left(\dfrac{1}{v_1}\right)^2}x + \frac{d\dfrac{\cos\phi}{v_1}}{d\left(\dfrac{1}{v_1}\right)^2}y + \frac{d\dfrac{\cos\psi}{v_1}}{d\left(\dfrac{1}{v_1}\right)^2}z = 0, \qquad (\beta)$$

$$\frac{d\cos\omega}{d(v_2)^2}x + \frac{d\cos\phi}{d(v_2)^2}y + \frac{d\cos\psi}{d(v_2)^2}z = 0. \qquad (\gamma)$$

Let

$$\frac{1}{v_1}\sqrt{\frac{(a^2-v_1^2)}{(a^2-v_2^2)}} = \xi, \qquad \sqrt{\{(a^2-b^2)(a^2-c^2)\}} = \frac{1}{A},$$

$$\frac{1}{v_1}\sqrt{\frac{(b^2-v_1^2)}{(b^2-v_2^2)}} = \eta, \qquad \sqrt{\{(b^2-a^2)(b^2-c^2)\}} = \frac{1}{B},$$

$$\frac{1}{v_1}\sqrt{\frac{(c^2-v_1^2)}{(c^2-v_2^2)}} = \zeta, \qquad \sqrt{\{(c^2-a^2)(c^2-b^2)\}} = \frac{1}{C},$$

* This investigation supplies the step which Mr Tovey was desirous should appear in the *Magazine*. [*Phil. Mag.* March, 1838, p. 261. Ed.]

† In lieu of v_1 we might write v_2 in the denominator without affecting the result.

‡ Observe, that $\dfrac{\cos\omega}{v_1} = \dfrac{\sqrt{\left\{\left(\dfrac{a^2}{v_1^2}-1\right)(a^2-v_2^2)\right\}}}{\sqrt{\{(a^2-b^2)(a^2-c^2)\}}}$, and so on for the rest.

then equation (γ) becomes

$$A\xi x + B\eta y + C\zeta z = 0, \tag{1}$$

and equation (β)

$$\frac{Aa^2}{\xi} x + \frac{Bb^2}{\eta} y + \frac{Cc^2}{\zeta} z = 0, \tag{2}$$

and equation (α) may be written under two forms, viz.

$$(a^2 - v_2^2) A\xi x + (b^2 - v_2^2) B\eta y + (c^2 - v_2^2) C\zeta z = 1, \tag{3}$$

or

$$\left(\frac{a^2}{v_1^2} - 1\right)\frac{A}{\xi} x + \left(\frac{b^2}{v_1^2} - 1\right)\frac{B}{\eta} y + \left(\frac{c^2}{v_1^2} - 1\right)\frac{C}{\zeta} z = 1. \tag{4}$$

From (1)

$$A\xi x + B\eta y = - C\zeta z. \tag{5}$$

From (2)

$$\frac{Aa^2}{\xi} x + \frac{Bb^2}{\eta} y = - \frac{Cc^2}{\zeta} z. \tag{6}$$

From (3) and (1)

$$A\,(a^2 - c^2)\,\xi x + B\,(b^2 - c^2)\,\eta y = 1. \tag{7}$$

From (2) and (4)

$$A\,(a^2 - c^2)\frac{x}{\xi} + B\,(b^2 - c^2)\frac{y}{\eta} = c^2. \tag{8}$$

From (5) and (6)

$$C^2c^2z^2 - B^2b^2y^2 - A^2a^2x^2 = ABxy\left(a^2\frac{\eta}{\xi} + b^2\frac{\xi}{\eta}\right). \tag{9}$$

From (7) and (8)

$$c^2 - B^2(b^2 - c^2)^2 y^2 - A^2(a^2 - c^2)^2 x^2 = ABxy\left(\frac{\eta}{\xi} + \frac{\xi}{\eta}\right) \times (a^2 - c^2)(b^2 - c^2). \tag{10}$$

From (9) and (10)

$$\begin{aligned} AB\,(a^2 - b^2)\,(a^2 - c^2)\,(b^2 - c^2)\,xy\frac{\xi}{\eta} &= a^2c^2 - (a^2 - c^2)\,(b^2 - c^2)\,C^2c^2z^2 \\ &- \{a^2(b^2 - c^2)^2 - b^2\,(a^2 - c^2)\,(b^2 - c^2)\}\,B^2y^2 \\ &- \{a^2\,(a^2 - c^2)^2 - a^2\,(a^2 - c^2)(b^2 - c^2)\}\,A^2x^2 = a^2c^2 - c^2z^2 - c^2y^2 - a^2x^2. \end{aligned} \tag{11}$$

From (11), interchanging (a, x, ξ) with (b, y, η) we have

$$AB\,(b^2 - a^2)\,(b^2 - c^2)\,(a^2 - c^2)\,xy\frac{\eta}{\xi} = b^2c^2 - c^2z^2 - c^2x^2 - b^2y^2. \tag{12}$$

Finally, from (11) and (12) we have

$$\{a^2c^2 - (a^2 - c^2)\,x^2 - c^2\,(x^2 + y^2 + z^2)\}\,\{b^2c^2 - (b^2 - c^2)\,y^2 - c^2\,(x^2 + y^2 + z^2)\}$$
$$= (a^2 - c^2)\,(b^2 - c^2)\,x^2y^2,$$

that is

$$(x^2 + y^2 + z^2)\,(a^2x^2 + b^2y^2 + c^2z^2) - a^2\,(b^2 + c^2)\,x^2$$
$$- b^2\,(a^2 + c^2)\,y^2 - c^2\,(b^2 + a^2)\,z^2 + a^2b^2c^2 = 0$$

the equation required.

2.

ON THE MOTION AND REST OF FLUIDS.

[*Philosophical Magazine*, XIII. (1838), pp. 449—453.]

M. OSTROGRADSKY'S memoir on this subject inserted in the *Scientific Memoirs* seems to have excited much attention, and has been made the occasion of some annotations* by a distinguished writer in the *Philosophical Magazine.* Mr Ivory's recent papers in the same periodical must still more tend to invest with a new interest all such speculations. It seems to me desirable therefore to present the theory of fluids in all the simplicity of which it is susceptible.

I consider a fluid as a collection of particles subject to some law of relative position other than that of rigidity. These particles by their mutual actions maintain the connections of the system. As to the law of force between them we know nothing; but I assume it is a general principle of nature, that for each instant of time the sum of the internal actions (reckoned by the product of each particle into the square of the space due to the internal force acting on it) is a minimum. This in fact is Gauss's principle of least restraint. We may if we please split this principle into two parts; that is to say, assume that the internal system of forces is always such as if acting alone would keep the fluid at rest; and then again assume that any equilibrating system of forces must be subject to the law of virtual velocities. I say *assume,* because it is impossible *à priori* to prove this.

Lagrange's so-called demonstration is unworthy of his name, and (albeit sanctioned by the powerful oral authority of an ex-Cambridge Professor) contrary alike to sense and honesty. It is better therefore at once to proceed upon Gauss's principle. It might easily be shown that this is in effect tantamount in all cases to D'Alembert's and Lagrange's principles combined.

Before entering upon the investigation I may call attention to one point of great analytical interest, and relating to the difficult subject of the algebraical sign, viz. that if the density of a point (x, y) in any circumscribed space be expressed by the quantity $\frac{du}{dx} + \frac{dv}{dy}$ so that the mass is

$$\iint dx\,dy \left(\frac{du}{dx}\right) + \iint dx dy \left(\frac{dv}{dy}\right),$$

[* *Phil. Mag.* May, 1838, p. 385. ED.]

that is not equivalent to

$$\int (udy + vdx),$$

that is if we please

$$\int \left(u \frac{dy}{ds} + v \frac{dx}{ds} \right) ds,$$

(where s is for clearness' sake and to avoid double limits taken an element of the bounding curve) as at first sight it might appear to be, but is in fact equal to

$$\int \left(u \frac{dy}{ds} - v \frac{dx}{ds} \right) ds.$$

I shall demonstrate this point in the next number* of the *Magazine.* It at first caused me some trouble in conducting the annexed inquiry. I shall also take occasion at some other time to revert to a new species (as I believe) of partial differential equations; that is to say, where there are fewer of them than of the principal variables, which may be called therefore Indeterminate Partial Differential Equations. A complete solution of one of these appears in the subjoined

Investigation.

For the sake of simplicity I take an incompressible fluid. The method is nowise different for a fluid of varying density.

Let Δx, Δy, Δz be any displacement undergone by a particle at the point x, y, z parallel to the axes x, y, z respectively; it is easily shown that to satisfy the condition of invariability of mass we must have

$$\frac{d\Delta x}{dx} + \frac{d\Delta y}{dy} + \frac{d\Delta z}{dz} = 0. \qquad (\alpha)$$

One relation between u, v, w the velocities parallel to x, y, z is obtained immediately by putting $u\delta t$, $v\delta t$, $w\delta t$, for Δx, Δy, Δz, which gives

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \qquad (1)$$

as usual.

Again, if X, Y, Z be the impressed forces, and X_1, Y_1, Z_1 the internal forces acting on any particle parallel to the axes, we have

$$X_1 + X = \frac{du}{dt} + \frac{du}{dx} u + \frac{du}{dy} v + \frac{du}{dz} w, \qquad (2)$$

$$Y_1 + Y = \frac{dv}{dt} + \frac{dv}{dx} u + \frac{dv}{dy} v + \frac{dv}{dz} w, \qquad (3)$$

$$Z_1 + Z = \frac{dw}{dt} + \frac{dw}{dx} u + \frac{dw}{dy} v + \frac{dw}{dz} w, \qquad (4)$$

from the mere geometry of the question.

[* p. 36, below. Ed.]

Finally, Gauss's principle teaches us that

$$\iiint dxdydz\,\{X_1\Delta X_1 + Y_1\Delta Y_1 + Z_1\Delta Z_1\} = 0. \qquad (\beta)$$

Now
$$\frac{d(X+X_1)}{dx} + \frac{d(Y+Y_1)}{dy} + \frac{d(Z+Z_1)}{dz}$$
$$= \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2 + 2\left\{\frac{dv}{dz}\frac{dw}{dy} + \frac{dw}{dx}\frac{du}{dz} + \frac{du}{dy}\frac{dv}{dx}\right\},$$

as appears from the equations (1), (2), (3), (4); and hence

$$\frac{d\Delta X_1}{dx} + \frac{d\Delta Y_1}{dy} + \frac{d\Delta Z_1}{dz} = 0,$$

the complete solution of which, free from the sign of integration, is

$$\Delta X_1 = \frac{d\psi}{dy} - \frac{d\phi}{dz},$$

$$\Delta Y_1 = \frac{d\omega}{dz} - \frac{d\psi}{dx},$$

$$\Delta Z_1 = \frac{d\phi}{dx} - \frac{d\omega}{dy},$$

ω, ϕ, ψ being any three independent functions of x, y, z.

On substituting these values in equation (β) we obtain

$$\iiint dxdydz\left\{X_1\frac{d\psi}{dy} - Y_1\frac{d\psi}{dx}\right\} + \iiint dxdydz\left\{Y_1\frac{d\omega}{dz} - Z_1\frac{d\omega}{dy}\right\}$$
$$+ \iiint dxdydz\left\{Z_1\frac{d\phi}{dx} - X_1\frac{d\phi}{dz}\right\} = 0.$$

This may be put under the form

$$\int dz\iint dxdy\left\{\frac{d}{dy}(\psi X_1) - \frac{d}{dx}(\psi Y_1)\right\}$$
$$+ \int dx\iint dydz\left\{\frac{d}{dz}(\omega Y_1) - \frac{d}{dy}(\omega Z_1)\right\}$$
$$+ \int dy\iint dzdx\left\{\frac{d}{dx}(\phi Z_1) - \frac{d}{dz}(\phi X_1)\right\}$$
$$- \iiint dxdydz\,.\,\psi\left(\frac{dX_1}{dy} - \frac{dY_1}{dx}\right)$$
$$- \iiint dxdydz\,.\,\omega\left(\frac{dY_1}{dz} - \frac{dZ_1}{dy}\right)$$
$$- \iiint dxdydz\,.\,\phi\left(\frac{dZ_1}{dx} - \frac{dX_1}{dz}\right) = 0.$$

Here it must be remembered that ω, ϕ, ψ are perfectly independent of each other. Also the values of the three first written quantities depend upon the values of X_1, Y_1, Z_1 at the bounding surface; the values of the three last-written depend upon the general values of X_1, Y_1, Z_1. It is clear therefore that each system of three equations and each member of each system must be separately zero.

The three latter equations give

$$\left.\begin{aligned} \frac{dX_1}{dy} - \frac{dY_1}{dx} &= 0 \\ \frac{dY_1}{dz} - \frac{dZ_1}{dy} &= 0 \\ \frac{dZ_1}{dx} - \frac{dX_1}{dz} &= 0 \end{aligned}\right\}. \qquad (\gamma)$$

The three former require that for each section of the surface parallel to the plane xy

$$\int \psi (X_1 dx + Y_1 dy) = 0,$$

for each section parallel to yz

$$\int \omega (Y_1 dy + Z_1 dz) = 0, \qquad (\delta)^*$$

for each section parallel to zx

$$\int \phi (Z_1 dz + X_1 dx) = 0$$

and these equations are to hold good whatever ψ, ϕ, ω may be. From the equations (γ) we derive

$$X_1 dx + Y_1 dy + Z_1 dz = df, \qquad (5)$$

from equations (δ) we obtain

f = constant for all points in any section of the bounding surface parallel to the plane of xy,

f = constant for all points in any section of the bounding surface parallel to the plane of yz,

f = constant for all points in any section of the bounding surface parallel to the plane of zx.

Now by drawing through all the points in a plane parallel to xy, planes parallel to yz, we may cover the whole surface; hence f is constant all over the surface bounding the fluid.

* See remark at introduction.

Therefore $$X_1 dx + Y_1 dy + Z_1 dz = 0, \qquad (6)$$

for all variations of dx, dy, dz taken upon the surface.

The equations (1, 2, 3, 4, 5, 6) are coincident with those obtained by the usual method; with this difference, that X_1, Y_1, Z_1 here take the place of

$$-\frac{dp}{dx}, \ -\frac{dp}{dy}, \ -\frac{dp}{dz}.$$

Thus then we have obtained all the conditions requisite for determining the motion of fluids from the universal principle of least constraint conjoined with the specific character of the system in question.

General Remarks.

In the case of equilibrium, that is in the case where no particle moves, we have $X_1 + X = 0$, $Y_1 + Y = 0$, $Z_1 + Z = 0$. Hence $Xdx + Ydy + Zdz$ is a complete differential always and zero for the surface.

The above results have been obtained upon the principles of the differential calculus, and the continuity of the forces has been tacitly assumed. If now we were to suppose forces of finite magnitude (as compared with the *whole sum* acting upon the entire system) to be applied to a layer of single particles or to a layer of a thickness of the same order of magnitude as the distances between the particles themselves, (which has been treated as an infinitesimal) it would appear that our results would be no longer applicable, just in the same manner as it would be erroneous to apply the principle of *vis-viva* (for example) without modification, to the case of impulsive forces, because we had deduced it by the calculus in the case of the motion being continuous. Hence the above equations ought not strictly to apply to the motion or rest of a fluid *contained between physical surfaces*; for the pressure afforded by these surfaces, whatever its actual value may be, we know *à priori* is commensurable with the whole amount of force acting on the fluid; but the immediate application of this pressure (*alias* repulsive force) is confined to the bounding layer of fluid particles, or at most extends to a distance bearing a low ratio to the distances between the particles themselves.

Accordingly, to the non-applicability of the equations for free fluids to the case of fluids confined at the boundaries, and to an independent investigation upon the minimum principle for this class of problems, it is, that I look for the true explanation of the phenomena of capillary attraction (vulgarly so called).

3.

ON THE MOTION AND REST OF RIGID BODIES.

[*Philosophical Magazine*, XIV. (1839), pp. 188—190.]

IN the subjoined investigation, which, as far as I know, is my own, I apply the same method to rigid as in the preceding paper I applied to fluid systems.

Let x, y, z be the coordinates of any particle in a rigid body; x', y', z' the coordinates of some other particle, and let

$$x' = x + h, \quad y' = y + k, \quad z' = z + l.$$

Call $\Delta x, \Delta y, \Delta z$ the increments which x, y, z receive after the lapse of a small interval of time; so that terms in which they enter in two or more dimensions may be neglected.

Then
$$\Delta(x') = \Delta x + \frac{d\Delta x}{dx} h + \frac{d\Delta x}{dy} k + \frac{d\Delta x}{dz} l + P,$$

$$\Delta(y') = \Delta y + \frac{d\Delta y}{dx} h + \frac{d\Delta y}{dy} k + \frac{d\Delta y}{dz} l + Q,$$

$$\Delta(z') = \Delta z + \frac{d\Delta z}{dx} h + \frac{d\Delta z}{dy} k + \frac{d\Delta z}{dz} l + R,$$

P, Q, R containing binary and higher combinations of h, k, l, which we shall have no occasion to express.

At the commencement of the interval the squared distance of the two particles was $(x'-x)^2 + (y'-y)^2 + (z'-z)^2$; at the end of the interval the distance squared is

$$(x' - x + \Delta(x') - \Delta x)^2 + (y' - y + \Delta(y') - \Delta y)^2 + (z' - z + \Delta(z') - \Delta z)^2,$$

and these two expressions must be the same by the conditions of rigidity whatever h, k, and l may be; that is

$$\begin{aligned} h^2 + k^2 + l^2 = &\left(h + \frac{d\Delta x}{dx} h + \frac{d\Delta x}{dy} k + \frac{d\Delta x}{dz} l + P\right)^2 \\ &+ \left(k + \frac{d\Delta y}{dx} h + \frac{d\Delta y}{dy} k + \frac{d\Delta y}{dz} l + Q\right)^2 \\ &+ \left(l + \frac{d\Delta z}{dx} h + \frac{d\Delta z}{dy} k + \frac{d\Delta z}{dz} l + R\right)^2, \end{aligned}$$

for all values of h, k, and l.

Hence rejecting infinitesimals of the second order and equating to zero separately the coefficients of h^2, k^2, l^2, and of kl, lh, hk, we have

$$\frac{d\Delta x}{dx} = 0. \qquad (a) \qquad\qquad \frac{d\Delta y}{dz} + \frac{d\Delta z}{dy} = 0. \qquad (d)$$

$$\frac{d\Delta y}{dy} = 0. \qquad (b) \qquad\qquad \frac{d\Delta z}{dx} + \frac{d\Delta x}{dz} = 0. \qquad (e)$$

$$\frac{d\Delta z}{dz} = 0. \qquad (c) \qquad\qquad \frac{d\Delta x}{dy} + \frac{d\Delta y}{dx} = 0. \qquad (f)$$

By differentiating (d), (e), (f) with respect to z, x, y respectively, and substituting from (a), (b), (c), we obtain

$$\frac{d^2\Delta y}{dz^2} = 0, \quad \frac{d^2\Delta z}{dx^2} = 0, \quad \frac{d^2\Delta x}{dy^2} = 0.$$

By differentiating the same with respect to y, z, x respectively, and proceeding as before, we have

$$\frac{d^2\Delta z}{dy^2} = 0, \quad \frac{d^2\Delta x}{dz^2} = 0, \quad \frac{d^2\Delta y}{dx^2} = 0.$$

Thus, then, we have

$$\frac{d\Delta x}{dx} = 0, \quad \frac{d^2\Delta x}{dy^2} = 0, \quad \frac{d^2\Delta x}{dz^2} = 0,$$

$$\frac{d\Delta y}{dy} = 0, \quad \frac{d^2\Delta y}{dz^2} = 0, \quad \frac{d^2\Delta y}{dx^2} = 0,$$

$$\frac{d\Delta z}{dz} = 0, \quad \frac{d^2\Delta z}{dx^2} = 0, \quad \frac{d^2\Delta z}{dy^2} = 0,$$

therefore

$$\Delta x = A + By + Cz, \qquad (o)$$

$$\Delta y = D + Ez + Fx, \qquad (p)$$

$$\Delta z = G + Hx + Ky, \qquad (q)$$

A, B, C, D, E, F, being constant for a *given instant* of time; between which by virtue of the equations (d), (e), (f), we have the relations

$$E + K = 0, \qquad H + C = 0, \qquad B + F = 0.$$

If we call u, v, w the three component velocities of the particles at x, y, z parallel to the three axes, and X_1, Y_1, Z_1, the three internal forces, it is at once seen that u, v, w, as also ΔX_1, ΔY_1, ΔZ_1 must be subject to the same equations as limit Δx, Δy, Δz; so that

$$u = a + \gamma y - \beta z, \qquad (1)$$

$$v = b + \alpha z - \gamma x, \qquad (2)$$

$$w = c + \beta x - \alpha y, \qquad (3)$$

$$\Delta X_1 = a_1 + \gamma_1 y - \beta_1 z, \qquad (h)$$

$$\Delta Y_1 = b_1 + \alpha_1 z - \gamma_1 x, \qquad (j)$$

$$\Delta Z_1 = c_1 + \beta_1 x - \alpha_1 y. \qquad (k)$$

Also if X, Y, Z be the impressed forces, we have

$$X_1 + X = \frac{du}{dt}, \tag{4}$$

$$Y_1 + Y = \frac{dv}{dt}, \tag{5}$$

$$Z_1 + Z = \frac{dw}{dt}. \tag{6}$$

And by Gauss's principle, calling m the mass of the particle at x, y, z,

$$\Delta \Sigma m\,(X_1^2 + Y_1^2 + Z_1^2) = 0.$$

Hence equating separately to zero the coefficients of a_1, b_1, c_1 and of α_1, β_1, γ_1 in the quantity $\Sigma m\,(X_1\Delta X_1 + Y_1\Delta Y_1 + Z_1\Delta Z_1)$ we have

$$\left.\begin{aligned} \Sigma m\, X_1 &= 0 \\ \Sigma m\, Y_1 &= 0 \\ \Sigma m\, Z_1 &= 0 \\ \Sigma m\,(Z_1 y - Y_1 z) &= 0 \\ \Sigma m\,(X_1 z - Z_1 x) &= 0 \\ \Sigma m\,(Y_1 x - X_1 y) &= 0 \end{aligned}\right\}. \tag{7—12}$$

Lastly, we have the equations

$$u = \frac{dx}{dt}, \tag{13}$$

$$v = \frac{dy}{dt}, \tag{14}$$

$$w = \frac{dz}{dt}. \tag{15}$$

From the fifteen equations marked (1) to (15), the motion may be determined by assigning the position of each particle at the end of the time t in terms of its three initial coordinates, its three initial velocities, and the initial values of the nine quantities

$$\begin{array}{lll} \Sigma mx, & \Sigma myz, & \Sigma mx^2, \\ \Sigma my, & \Sigma mzx, & \Sigma my^2, \\ \Sigma mz, & \Sigma mxy, & \Sigma mz^2. \end{array}$$

In the case of rest $X_1 = -X$, $Y_1 = -Y$, $Z_1 = -Z$, and the equations (7) to (12) inclusively taken, express the conditions of equilibrium.

The equations (*o*), (*p*), (*q*), which have been obtained from conditions *purely geometrical*, establish the well-known but interesting and *not obvious* fact, that any *small* motion of a rigid body may be conceived as made up of a motion of translation and a motion about *one* axis.

4.

ON DEFINITE DOUBLE INTEGRATION, SUPPLEMENTARY TO A FORMER PAPER ON THE MOTION AND REST OF FLUIDS.

[*Philosophical Magazine*, XIV. (1839), pp. 298—300.]

IN a paper on Fluids which appeared in the December Number of this *Magazine*, I had occasion to remark, that the mass of an area having at the point (x, y) a density $\frac{du}{dx} + \frac{dv}{dy}$ could be expressed by the simple formula

$$\int_l^0 \left\{u \frac{dy}{ds} - v \frac{dx}{ds}\right\} ds;$$

l being the length, and ds an element of the bounding curve: this may be thought to require some explanation.

(1) Let $APBq$ represent any oval; PpL, QqM any two contiguous ordinates cutting the curve in Pp, Qq respectively, AC, BD the two extreme tangents parallel to Oy, and ρ the density at any point (x, y). The expression $\iint \rho dx dy$ will serve to denote the mass of the oval area $APBq$, and the limits may be twice taken, that is, (i) the two values of y corresponding to any one of x; and (ii) the two values of x corresponding to C and D. This method is in fact tantamount to taking the sum of the columns Pp qQ; but this is not necessary, for $APBq$ may be considered as the *algebraical* sum of the mixtilinear area $APQBDC$, and the mixtilinear area $BDCApq$, or (if any line $O'C'D'$ be drawn parallel to $OCLMD$) of $APQBD'C'$ and $BD'C'Apq$.

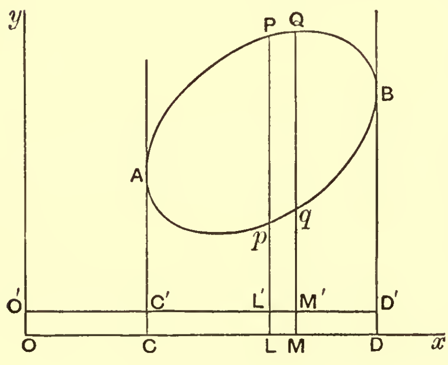


Fig. 1.

Thus then the mass $= \int dx \left(\int \rho dy\right)$, $\int \rho dy$ being left indeterminate, and the extremity of x travelled round from C to D, and back again from D to C.

This will be better expressed by transforming the variable, and summing with respect to some quantity, such as the arc of the curve, which continuously increases, or if we please, with respect to θ, the angle subtending any point taken within the curve.

The mass is then

$$=\pm\int_{2\pi}^{0} d\theta\left\{(\int\rho dy)\frac{dx}{d\theta}\right\};$$

always remembering that *no* constant need be added to $\int\rho dy$, and that the doubtful sign arises from the choice of ways in which θ may be measured round. If the area be not included by one line; but by several, as for example, by a curve and a right line, the above integral, if broken up into as many parts as there are breaches of continuity, will still apply.

(2) Let us suppose that we have two areas exactly coinciding with, and overlapping one another; but the density of the one at (x, y) to be ρ, and of the other ρ'.

Let the mass of the first be treated as the sum of columns parallel to Oy, and that of the second as the sum of columns parallel to Ox.

The one will be represented by

$$\pm\int_{2\pi}^{0} d\theta\,(\int\rho dy)\,\frac{dx}{d\theta},$$

the other will be represented by

$$\pm\int_{2\pi}^{0} d\theta\,(\int\rho' dx)\,\frac{dy}{d\theta},$$

and the sum of the two, or the joint mass, by

$$\pm\int_{2\pi}^{0} d\theta\,\{\int\rho dy\}\,\frac{dx}{d\theta}\pm\int_{2\pi}^{0} d\theta\,\{\int\rho' dx\}\,\frac{dy}{d\theta}.$$

So long as these two operations are performed separately, the doubtful signs may be preserved in each term, because s need not be travelled round in the same direction for the two summations; but if we perform the second integration conjointly for the two masses, their sum

$$=\pm\int_{2\pi}^{0} d\theta\left\{(\int\rho dy)\frac{dx}{d\theta}\pm ?\,(\int\rho' dx)\frac{dy}{d\theta}\right\},$$

the mark of interrogation denoting that *one or the other*, but not *either* of the signs $\pm$ must be used, and the question is, which ?

This will be answered by taking different points in the bounding line which may be continuous or not. Now every line returning into itself, whether continuous or not, will naturally divide with respect of any given system of axes, into at most four parts, or sets of parts; two in which dx and dy both increase or both decrease, and two in which one increases and the other decreases.

Take P_1, P_2, P_3, P_4, any points in the four quadrants respectively, it will be observed that,

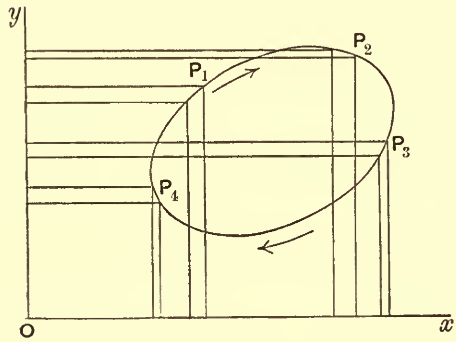


Fig. 2.

At P_1 the ρ column enters additively, and the ρ' column subtractively.

At P_2 both columns are additive.

At P_3 the ρ' column is additive and the ρ column subtractive.

At P_4 both columns enter subtractively.

Again, reckoning round in the direction of the arrows,

At P_1, x and y are both increasing.

At P_2, x is increasing and y decreasing.

At P_3, x and y both decrease.

At P_4, x is decreasing and y increasing.

Thus when $\int\rho dy$ and $\int\rho' dx$ are affected with the same signs, dx and dy are of opposite signs; and when $\int\rho dy$, $\int\rho' dx$ are of opposite signs, dx and dy are of the same sign.

Hence it appears that the mass of the area, whose density at (x, y) is $\rho + \rho'$, is capable of being represented by

$$\pm \int_{2\pi}^{0} d\theta \left\{ (\int\rho dy) \frac{dx}{d\theta} - (\int\rho' dx) \frac{dy}{d\theta} \right\}.$$

5.

ON AN EXTENSION OF SIR JOHN WILSON'S THEOREM TO ALL NUMBERS WHATEVER.

[*Philosophical Magazine,* XIII. (1838), p. 454.]

THE annexed original theorem in numbers will serve as a pendant to the elegant discovery announced by the ever-to-be-lamented and commemorated Horner*, with his dying voice, in your valued pages†.

THEOREM.

If N be any number whatever and

$$p_1, p_2, p_3 \cdots\cdots p_c$$

be all the numbers less than N and prime to it, then either

$$p_1 \cdot p_2 \cdot p_3 \cdots\cdots p_c + 1,$$

or else

$$p_1 \cdot p_2 \cdot p_3 \cdots\cdots p_c - 1,$$

is a multiple of N.

6.

NOTE TO THE FOREGOING.

[*Philosophical Magazine,* XIV. (1839), pp. 47, 48.]

I HAVE to apologize for calling "original" (in the last Number of the *Magazine*) the theorem of numbers which I termed "a pendant to Horner's theorem." This Mr Ivory has done me the honour to inform me may be found in Gauss's *Disquisitiones Arithmeticæ,* p. 76. As Horner's extension of Fermat's theorem suggested this extension of Sir John Wilson's to me, so I concluded that had this extension of Wilson's been known to the world it would naturally have suggested his to Horner. No acknowledgment of this kind having been made, I took it for granted that the theorem I gave was new. Undoubtedly had Mr Horner been aware of Gauss's theorem he would have made mention of it.

I take this opportunity of adding that my acquaintance with Gauss's principle‡ has not been derived from the study of his works, but from a casual statement of it in an English work, *Dynamics*, by Mr Earnshaw, of St John's College, Cambridge.

* Horner's proof is highly valuable as a novel and highly ingenious form of reasoning, but his theorem may be deduced with infinitely more ease and brevity from Fermat's than he seems to have been aware of.

[† *Phil. Mag.* Vol. XI. p. 456. ED.]

[‡ See p. 28 above. ED.]

7.

ON RATIONAL DERIVATION FROM EQUATIONS OF COEXISTENCE, THAT IS TO SAY, A NEW AND EXTENDED THEORY OF ELIMINATION*. PART I.

[*Philosophical Magazine*, XV. (1839), pp. 428—435.]

ANY number of equations existing at the same time and having the same quantities repeated, may be termed equations of coexistence: in the *present* paper we consider only the case of *two* algebraical equations:

$$x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots\ldots + a_m = 0,$$

$$x^n + b_1 x^{n-1} + b_2 x^{n-2} + \ldots\ldots + b_n = 0.$$

The above being "equations of coexistence," x is called "the repeating term."

If we suppose the equation

$$c_0 x^r + c_1 x^{r-1} + c_2 x^{r-2} + \ldots\ldots + c_r = 0$$

to be capable of being deduced from the two above, and, therefore, necessarily implied by them, this will be called "a Particular Derivative" from the equations of coexistence, of the rth degree, (r being supposed less than m and n†, and the coefficients being *rational* functions of the coefficients of the equations of coexistence).

There will be an *indefinite* number in general of such derivatives, and the form involving arbitrary quantities which includes them *all* is called "the general derivative of the rth degree."

Any "Particular Derivative," in which the terms are all integral, numerically as well as literally speaking, is called an "Integral Derivative."

That "Integral Derivative" of any given degree in which the literal parts of the coefficients are of the *lowest possible dimensions*‡, and the numerical parts as low as they can be made, is called the "Prime Derivative"

[* The results of this and some following papers were repeated, with demonstrations, in the paper "On a Theory of the Syzygetic Relations of two rational integral functions comprising an application to the Theory of Sturm's Functions, and that of the greatest Algebraical Common Measure," *Phil. Trans. Royal Soc.* Vol. CXLIII., Part I. pp. 407—548, 1853. See below Section II. Art. (16) of that paper. ED.]

† This restriction upon the value of r is not essentially requisite, and is only introduced to keep the attention fixed upon the particular objects of this first Part.

‡ Of course the dimensions of the coefficients in the equations of coexistence are to be understood as denoted by the indices subscribed.

of that degree. So that there is nothing left ambiguous in the prime derivative save the sign.

The "Derivative by succession" is that particular derivative which is obtained by performing upon the equations of coexistence the process commonly employed for the discovery of the greatest common measure, and equating the *successive remainders* to zero.

To express the product of the sums formed by adding each of one row of quantities to each of another row, we simply write the one row above the other; a notation clearly capable of extension to any number of rows, which would not be the case if we spoke of *differences* instead of *sums**.

Theorem 1.

Let $h_1, h_2, \ldots h_m$, be the roots of one equation of coexistence, $k_1, k_2, \ldots k_n$, the roots of the other. The general derivative of the rth degree is represented by

$$\Sigma\left(SR(h_1, h_2, h_3 \ldots h_r)\{(x-h_1)(x-h_2)\ldots(x-h_r)\} \times \begin{Bmatrix} h_{r+1}, h_{r+2} \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix}\right) = 0,$$

$SR(h_1, h_2, h_3 \ldots h_r)$ denoting any *symmetrical rational* (integral or fractional) function of $h_1, h_2 \ldots h_r$;

$$\begin{Bmatrix} h_{r+1}, h_{r+2} \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix}$$

being to be interpreted as above explained, and Σ of course including as many terms as there are ways of putting n things r and r together†.

A form tantamount to the above, and which may be substituted for it, is its analogue,

$$\Sigma\left(SR(k_1, k_2 \ldots k_r)\{(x-k_1)(x-k_2)\ldots(x-k_r)\} \times \begin{Bmatrix} k_{r+1}, k_{r+2} \ldots k_n \\ -h_1, -h_2 \ldots -h_m \end{Bmatrix}\right) = 0.$$

When $r = 0$ the theorem gives simply

$$\begin{Bmatrix} h_1, h_2 \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix} = 0,$$

and is coincident with that given by Bezout in his Theory of *Elimination.*

* The wider views which I have attained since writing the above, and which will be developed in a future paper, lead me to request that this notation may be considered only as temporary. It would have been more in accordance with these views to have used the two rows to denote products of differences than of sums. But a change now in the text would be very apt to cause errors in printing.

† The general derivative may clearly be expressed also by the sum of any two particular derivatives affected respectively with arbitrary rational coefficients. The equivalency of an arbitrary *function* to two arbitrary *multipliers* is very remarkable, and analogous to what occurs in the solution of certain differential equations.

Subsidiary Theorem (A).

If $h_1, h_2 \dots h_m$ be the roots of the equation

$$x^m + a_1 x^{m-1} + a_2 x^{m-2} + \dots\dots + a_m = 0,$$

and if

$$e^m + a_1 e^{m-1} + a_2 e^{m-2} + \dots\dots + a_m - u = 0,$$

then

$$\Sigma \frac{h_1^r}{(h_1 - h_2)(h_1 - h_3)\dots(h_1 - h_m)} = \frac{1}{r+1} \frac{d}{du} \Sigma (e^{r+1}),$$

u being made zero after differentiation.

Cor. If $R(h_1)$ denote any integral rational function of h_1, then

$$\Sigma \frac{R(h_1)}{(h_1 - h_2)(h_1 - h_3)\dots(h_1 - h_m)}$$

is *always* integral and is *zero* when the dimensions of $R(h_1)$ fall short of $(m-1)$.

Subsidiary Theorem (B).

$$\Sigma \frac{SR(h_1, h_2 \dots h_r)}{\begin{Bmatrix} h_1, h_2 \dots h_r \\ -h_{r+1}, -h_{r+2} \dots -h_m \end{Bmatrix}}$$

can be expressed by the sum of terms, each of which is the product of series of the form

$$\Sigma \frac{R(h_1)}{(h_1 - h_2)(h_1 - h_3)\dots(h_1 - h_m)},$$

it is *always* integral, and when the dimensions of the numerator fall short of $(m-r)r$ it vanishes*.

Subsidiary Theorem (C).

The *only* modes of satisfying the equation

$$\Sigma \{f(h_1, h_2 \dots h_r) \times SR(h_1, h_2 \dots h_r)\} = 0,$$

for all forms of the latter factors short of $(m-r)(n-r)$ dimensions, are to put $f(h_1, h_2 \dots h_r) = 0$, or else

$$f(h_1, h_2 \dots h_r) = \frac{\text{constant}}{\begin{pmatrix} h_1, h_2 \dots h_r \\ -h_{r+1}, -h_{r+2} \dots -h_m \end{pmatrix}}.$$

* It may be remarked also in passing, that any term in the numerator which contains *any* one power not greater than $m-2r$ may be neglected and thrown out of calculation. Moreover, an analogous proposition may be stated of fractions in the denominators of which *any number* of rows are written one under the other; see the first note, page 41.

Theorem 2.

By virtue of the subsidiary theorem (B), the two equations

$$\pm\Sigma\left((x-h_1)(x-h_2)\dots(x-h_r)\times\frac{\begin{Bmatrix}h_{r+1}, h_{r+2}\dots h_m\\ -k_1, -k_2\dots -k_n\end{Bmatrix}}{\begin{Bmatrix}h_{r+1}, h_{r+2}\dots h_m\\ -h_1, -h_2\dots -h_r\end{Bmatrix}}\right)=0,$$

$$\pm\Sigma\left((x-k_1)(x-k_2)\dots(x-k_r)\times\frac{\begin{Bmatrix}k_{r+1}, k_{r+2}\dots k_n\\ -h_1, -h_2\dots -h_m\end{Bmatrix}}{\begin{Bmatrix}k_{r+1}, k_{r+2}\dots k_n\\ -k_1, -k_2\dots -k_r\end{Bmatrix}}\right)=0,$$

are each integer derivatives of the rth degree.

Theorem 3.

And by virtue of the subsidiary theorem (C), the two above equations are the "Prime Integer Derivatives," and are exactly identical with each other.

Cor. 1. The leading coefficient of the "prime derivative" of the rth degree is always of $(m-r)(n-r)$ dimensions.

Cor. 2. If P_r be the prime derivative of the rth degree and if $(X=0, Y=0)$ be the two equations of coexistence, and λ_r, μ_r the two "prime constituents of multiplication" to the said derivative, that is if λ_r and μ_r satisfy the equation $\lambda_r X+\mu_r Y=P_r$, then the coefficient of the leading terms in λ_r and in μ_r is of $(m-r-1)(n-r-1)$ dimensions.

Theorem 4.

The "Prime Derivative" of any given degree is an exact factor of the "derivative by succession," of the same degree. The quotient resulting from striking out this factor is called "the quotient of succession."

Theorem 5.

If L_1, L_2, L_3, &c., be the leading coefficients of the derivatives occurring first, second, third, &c., in order after the equations of coexistence, and if Q_1, Q_2, Q_3, &c., represent the first, second, third, "quotients of succession" reckoned in the same order, then

$$Q_1=1,$$

$$Q_2=\frac{1}{L_1^2},$$

$$Q_3=\frac{L_1^4}{L_2^2},$$

$$Q_4=\frac{L_2^4}{L_1^4L_3^2},$$

and in general

$$Q_{2n} = \frac{L_2^4 L_4^4 \ldots L_{2n-4}^4 L_{2n-2}^4}{L_1^4 L_3^4 \ldots L_{2n-3}^4 L_{2n-1}^2} *,$$

$$Q_{2n+1} = \frac{L_1^4 L_3^4 \ldots L_{2n-3}^4 L_{2n-1}^4}{L_2^4 L_4^4 \ldots L_{2n-2}^4 L_{2n}^2} \dagger.$$

COR. Hence, in place of Sturm's auxiliary functions, we may substitute the functions derived from the equations of coexistence $\left(fx = 0,\ \frac{dfx}{dx} = 0\right)$ according to Theorem 2, due regard being had to the sign.

Scholium. Hitherto it has been supposed that the values of the coefficients in the equations of coexistence are independent of one another, but particular relations may be supposed to exist which shall cause the leading terms given by Theorem 2 to vanish, giving rise to anormal or singular primes, as they may be called, of the degree r of fewer than $(m-r)(n-r)$ dimensions. The theory of this, the failing case (so to say), is highly interesting, and I have already discovered the law of formation for the quotients of succession on the supposition of *any number* of primes vanishing consecutively; but I forbear to vex the patience of my reader further, the more so, as I hope soon to be able to present a complete memoir, with all the steps here indicated filled up, and numerous important additions, (the perfect image of which this is but a rough mould), as homage to the learned and illustrious society which has lately done me the honour of admitting me into its ranks.

Why this has not already been done must be excused, by the fact of the theory having suggested itself abroad in the intervals of sickness‡. Yet thus much will I add in general terms, namely, that as many primes as vanish consecutively, so many units must be added to the index 2 of the accessions

* That the appearance of the index 4 may not startle, let my reader bear in mind that there are what may be termed secondary derivatives of succession for every degree appearing in the process of successive division.

† The prime derivatives must be capable of yielding an *internal* evidence of the truth of Sturm's theorem. In fact, for the case of all the roots being possible, a little consideration will serve to show that the leading term of each prime derivative of the equation $\left\{fx \frac{dfx}{dx}\right\} = 0$ will consist of a series of fractions, each of which fractions is, *numerically* speaking, of the *same sign.*

‡ The reflections which Sturm's memorable theorem had originally excited, were revived by happening to be present at a sitting of the French Institute, where a letter was read from the Minister of Public Instruction, requesting an opinion upon the expediency of forming tables of elimination between two equations as high as the 5th or 6th degree containing one repeating term. The offer was rejected, on the ground of the excessive labour that would be required. I think that this has been very much overrated; and probably many will be of the same opinion who have dwelt upon the fact that no numerical quantity will occur in the result higher than the highest index of the repeating term. Would it not redound to the honour of British science that some painstaking ingenious person should gird himself to the task? and would not this be a proper object to meet with encouragement from the Scientific Association of Great Britain?

received in the numerator and denominator of the subsequent quotient; and in the quotient after that, it is not the square of the leading term of the penultimate prime,—but the product of this term by the leading term of that anormal prime of the same degree which has the lowest dimensions,—that finds its way into the numerator. The rest of the formation remaining undisturbed, *unless* and *until* a new failure have taken place.

Note on Sturm's Theorem.

When one of the equations of coexistence is the differential coefficient with respect to the repeated term of the other, the prime derivatives given in Theorem 2 which coincide in this case with Sturm's auxiliary functions *reduced* to their lowest terms, may be exhibited under an integral *aspect*.

Let SPD intimate that the squared product of the differences is to be taken of the quantities which follow it.

Let S_1 indicate the sum of the quantities to which it is prefixed.

S_2 the sum of the binary products.

S_3 the sum of the ternary products, and so on

Let $h_1, h_2 \dots h_n$ be the roots of any equation.

Then Sturm's last auxiliary function may be *replaced* by

$$SPD\,(h_1, h_2 \dots h_n).$$

The last but one may be replaced by

$$\Sigma SPD\,(h_1, h_2 \dots h_{n-1})\,x + \Sigma S_1\,(h_2, h_3 \dots h_{n-1})\,SPD\,(h_1, h_2 \dots h_{n-1}).$$

The one preceding by

$$\Sigma SPD\,(h_1, h_2 \dots h_{n-2})\,x^2 + \Sigma S_1\,(h_1, h_2 \dots h_{n-2})\,SPD\,(h_1, h_2 \dots h_{n-2})\,x$$
$$+ \Sigma S_2\,(h_1, h_2 \dots h_{n-2})\,SPD\,(h_1, h_2 \dots h_{n-2}),$$

and so on.

Thus then Sturm's rule for determining the absolute number of real roots in an equation is based wholly and solely upon the following

Algebraical Proposition.

If there be n quantities, real and imaginary, the imaginary ones entering in pairs, as many changes of sign as there are in the terms

$$\Sigma SPD\,(h_1, h_2),$$
$$\Sigma SPD\,(h_1, h_2, h_3),$$
$$\dots\dots\dots\dots$$
$$\Sigma SPD\,(h_1, h_2 \dots h_{n-1}),$$
$$\Sigma SPD\,(h_1, h_2 \dots h_n),$$

so many in number are these pairs.

Query (1). Is there no proposition applicable to any n quantities *whatever*?

Query (2). Is there no faintly analogous proposition applicable to higher powers than the squares?

Query (3). Seeing that in forming the coefficients in the equation of the squares of the differences, we pass from n functions of the roots to $n\frac{n-1}{2}$ and not n functions, of their squared differences, does not a natural passage to the former lie through n functions of the squared differences?

In other words, may not the quantities $\Sigma SPD(h_1, h_2 \ldots h_n)$, &c., serve as natural and valuable intermediaries between the coefficients of an equation involving simple quantities and the coefficients of the equation involving the squares of their differences?

P.S. In the next part I trust to be able to present the readers of this *Magazine* with a *direct* and *symmetrical* method of eliminating any number of unknown quantities between any number of equations of any degree, by a newly invented process of symbolical multiplication, and the use of *compound* symbols of notation.

I must not omit to state that the constituents of multiplication λ_r and μ_r explained in Cor. 2 to Theorem 3 are equal to the expression

$$\Sigma (x-k_1)(x-k_2)\ldots(x-k_{n-r-1}) \frac{\begin{pmatrix} k_1, k_2 \ldots k_{n-r-1} \\ -h_1, -h_2 \ldots -h_m \end{pmatrix}}{\begin{pmatrix} k_1, k_2 \ldots k_{n-r-1} \\ -k_{n-r} \ldots -k_n \end{pmatrix}},$$

and its analogue respectively.

8.

ON DERIVATION OF COEXISTENCE. PART II. BEING THE THEORY OF SIMULTANEOUS SIMPLE HOMOGENEOUS EQUATIONS.

[*Philosophical Magazine*, XVI. (1840), pp. 37—43.]

Art. (1). We shall have constant occasion in this paper to denote different quantities by the same letter affected with different subscribed numerical indices.

Such a letter is to be termed a "Base."

Every character consisting of a base and an inferior index, this index is called an argument of the base, namely, the first, second, or nth argument, according as 1, 2, or in general n, be the number subscribed.

Art. (2). I use the symbol PD to denote the product of the differences of the quantities to which it is prefixed (each being to be subtracted *from* each that follows); thus

$PD\,(a, b, c)$ indicates $(b-a)(c-a)(c-b)$.

$PD\,(0, a, b, c)$ indicates $abc\,(b-a)(c-a)(c-b)$.

$PD\,(0, a, b, c \ldots l)$ indicates $abc \ldots l \times PD\,(a, b, c \ldots l)$.

Art. (3). For want of a better symbol I use the Greek letter ζ to denote that the product of factors to which it is prefixed is to be effected after a certain symbolical manner. This I shall distinguish as the zeta-ic product.

The symbol ζ will never be prefixed except to factors, each of which is made up of one or more terms, consisting solely of linear arguments of different bases, that is, characters bearing indices below but none above.

I am thereby enabled to give this short rule for zeta-ic multiplication: "Imagine all the inferior indices to become superior, so that each argument is transformed into a *power* of its base; multiply according to the rules of ordinary algebra; after the multiplication has been *done fully out* depress all the indices into their original position; the result is the zeta-ic product*."

* It is scarcely necessary to add that an analogous interpretation may be extended to any zeta-ic function whatever. Thus

$$\zeta\,(a_1+b_1)^2 = a_2 + 2a_1b_1 + b_2,$$

$$\zeta \cos\,(a_1) = 1 - \frac{a_2}{1\,.\,2} + \frac{a_4}{1\,.\,2\,.\,3\,.\,4}, \text{ \&c.}$$

Thus for example $\zeta(a_r, b_s)$ is the same as simply $a_r b_s$, but $\zeta(a_r, a_s)$ represents not $a_r a_s$ but a_{r+s}.

So in like manner

$$\zeta\{(a_h - b_k)(a_l - b_m)\}$$
$$= a_{h+l} - a_h b_m - b_k a_l + b_{m+k},$$
$$\zeta\{(a_1 - b_1)(a_1 - c_1)(b_1 - c_1)\}$$
$$= \text{the } \textit{depressed product} \text{ of } (a-b)(a-c)(b-c)$$
$$= \text{the } \textit{depressed} \text{ value of } a^2(b-c) + b^2(c-a) + c^2(a-b),$$

that is, $= a_2 b_1 - a_2 c_1 + b_2 c_1 - b_2 a_1 + c_2 a_1 - c_2 b_1$.

Art. (4). We shall have occasion in this part to combine the two symbols ζ, PD: thus we shall use

$$\zeta PD(a_1 b_1) \text{ to denote } \zeta(b_1 - a_1),$$
$$\zeta PD(a_1 b_1 c_1) \text{ to denote } \zeta\{(b_1 - a_1)(c_1 - a_1)(c_1 - b_1)\}.$$

Art. (5). For the sake of elegance of diction I shall in future sometimes omit to insert the inferior index when it is unity; but the reader must always bear in mind that it is to be *understood* though not expressed.

I shall thus be able to speak of the zeta-ic product of such and such bases mentioned by name.

Art. (6). We are not yet come to the limit of the powers of our notation. The zeta-ic product of the sum of arguments will consist of the sum of products of arguments, each argument being (as I have defined) made up of a base and an inferior index. Now we may imagine each index of every term of the zeta-ic product *after it is fully expanded* to be increased or diminished by unity, or each at the same time to be increased or diminished by 2, or each in general to be increased or diminished by r. I shall denote this alteration by affixing an r with the positive or negative sign to the ζ. Thus

$$\zeta(a_1 - b_1)(a_1 - c_1) \text{ being equal to } a_2 - a_1 c_1 + b_1 c_1 - b_1 a_1,$$
$$\zeta_{+1}(a_1 - b_1)(a_1 - c_1) \text{ is equal to } a_3 - a_2 c_2 + b_2 c_2 - b_2 a_2,$$
$$\zeta_{-1}(a_1 - b_1)(a_1 - c_1) \text{ is equal to } a_1 - a_0 c_0 + b_0 c_0 - b_0 a_0.$$

In like manner $\zeta PD(a, b, c)$ indicating

$$b_2 a_1 - b_2 c_1 + c_2 b_1 - c_2 a_1 + a_2 c_1 - a_2 b_1,$$

$\zeta_{\pm r} PD(a, b, c)$ indicates

$$b_{2\pm r} a_{1\pm r} - b_{2\pm r} c_{1\pm r} + c_{2\pm r} b_{1\pm r} - c_{2\pm r} a_{1\pm r} + a_{2\pm r} c_{1\pm r} - a_{2\pm r} b_{1\pm r}.$$

I shall in general denote $\zeta_{+r} PD(a, b, c \ldots l)$ *actually expanded* as the zeta-ic product of $a, b, c, \ldots l$ in its rth *phase*.

Art. (7). *General Properties of Zeta-ic Products of Differences.*

If there be made one interchange in the order of the bases to which ζ is prefixed, the zeta-ic product, in whatever phase it be taken, remains unaltered in magnitude, but changes its sign.

If in any *phase* of a zeta-ic product two of the bases be made to coincide, the expansion vanishes.

Let $\int_1$ be used, agreeably to the ordinary notation, to denote the sum of the quantities to which it is prefixed, $\int_2$ to denote the sum of the binary products, $\int_3$ of the ternary ones, and so on.

Thus let $\int_1 (a_1b_1c_1)$ or $\int_1 (a, b, c)$ indicate $a_1 + b_1 + c_1$,

and $\int_2 (a_1b_1c_1)$ or $\int_2 (a, b, c)$ indicate $a_1b_1 + a_1c_1 + b_1c_1$,

and $\int_3 (a_1b_1c_1)$ or $\int_3 (a, b, c)$ indicate $a_1b_1c_1$,

we shall be able now to state the following remarkable proposition connecting the several phases of certain the same zeta-ic products.

Art. (8). Let $a, b, c, \ldots l$, denote any number of independent bases, say $(n-1)$; but let the arguments of each base be periodic, and the number of terms in each period the same for every base, namely n, so that

$$\begin{array}{ll} a_r = a_{r+n} = a_{r-n}, & a_n = a_0 = a_{-n}, \\ b_r = b_{r+n} = b_{r-n}, & b_n = b_0 = b_{-n}, \\ c_r = c_{r+n} = c_{r-n}, & c_n = c_0 = c_{-n}, \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ l_r = l_{r+n} = l_{r-n}, & l_n = l_0 = l_{-n}, \end{array}$$

r being any number whatever. Then

$$\begin{array}{l} \zeta_{-1}PD\,(0, a, b, c \ldots l) = \zeta\,\{\int_1 (a, b, c \ldots l)\ \ \zeta PD\,(0, a, b, c \ldots l)\}, \\ \zeta_{-2}PD\,(0, a, b, c \ldots l) = \zeta\,\{\int_2 (a, b, c \ldots l)\ \ \zeta PD\,(0, a, b, c \ldots l)\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \zeta_{-r}PD\,(0, a, b, c \ldots l) = \zeta\,\{\int_r (a, b, c \ldots l)\ \ \zeta PD\,(0, a, b, c \ldots l)\}. \end{array}$$

This proposition admits of a great generalization*, but we have now all that is requisite for enabling us to arrive at a proposition exhibiting under one *coup d'œil* every combination and every effect of every combination that can possibly be made with any number of coexisting equations of the first degree, containing any number of *repeated*, or to use the ordinary language of analysts, (variable or) unknown quantities.

* See the Postscript to this paper for *one* specimen.

For the sake of symmetry I make every equation homogeneous; so that to eliminate n repeated terms, no more than n equations will be required.

In like manner the problem of determining n quantities from n equations will be here represented by the case in which we have to determine the *ratios* of $(n+1)$ quantities from n equations.

Art. (9). *Statement of the Equations of Coexistence.*

Let there be any number of bases $(a, b, c \dots l)$, and as many repeated terms $(x, y, z \dots t)$, and let the number of equations be any whatever, say n. The system may be represented by the *type* equation

$$a_r x + b_r y + c_r z + \dots + l_r t = 0,$$

in which r can take up all integer values from $-\infty$ to $+\infty$. The specific number of equations given will be represented by making the arguments of each base *periodic*, so that

$$a_r = a_{\mu n+r}, \ b_r = b_{\mu n+r}, \ c_r = c_{\mu n+r}, \ \dots \ l_r = l_{\mu n+r},$$

μ being any integer whatever.

Art. (10). *Combination of the given Equations.—Leading Theorem.*

Take $f, g, \dots k$ as the *arbitrary* bases of new and absolutely independent but periodic arguments, having the same index of periodicity (n) as $a, b, c \dots l$, and being in number $(n-1)$, that is, one fewer than there are units in that index.

The number of *differing* arbitrary constants thus *manufactured* is $n(n-1)$.

Let $Ax + By + Cz + \dots + Lt = 0$ be the general *prime* derivative from the given equations, then we may make

$$\begin{aligned} A &= \zeta PD(0, a, f, g \dots k), \\ B &= \zeta PD(0, b, f, g \dots k), \\ C &= \zeta PD(0, c, f, g \dots k), \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ L &= \zeta PD(0, l, f, g \dots k). \end{aligned}$$

Art. (11). Cor. 1. *Inferences from the Leading Theorem.*

Let the number of equations, or, which is the same thing, the index of periodicity (n), be the same as the number of repeated terms $(x, y, z \dots t)$, then one relation exists between the coefficients: this is found by making the $(n-1)$ new bases coincide with $(n-1)$ out of the old bases. We get accordingly, as the result of elimination,

$$\zeta PD(0, a, b, c \dots l) = 0.$$

Art. (11). Cor. 2. Let the number of equations be one more than that of the given bases, there will then be two equations of condition. These are represented by preserving one new arbitrary base, as λ. The result of elimination being in this case

$$\zeta PD\,(0, a, b, c \ldots l, \lambda) = 0.$$

Example. The result of eliminating between

$$a_1 x + b_1 y = 0,$$

$$a_2 x + b_2 y = 0,$$

$$a_3 x + b_3 y = 0,$$

is $\zeta PD\,(0, a, b, \lambda) = 0$, that is

$$\lambda_3 b_2 a_1 - \lambda_3 b_1 a_2 + \lambda_1 b_3 a_2 - \lambda_1 b_2 a_3 + \lambda_2 b_1 a_3 - \lambda_2 b_3 a_1 = 0,$$

from which we infer, seeing that λ_3, λ_2, λ_1 are independent,

$$b_2 a_1 - b_1 a_2 = 0,$$

$$b_3 a_2 - b_2 a_3 = 0,$$

$$b_1 a_3 - b_3 a_1 = 0,$$

any *two* of which imply the third.

In like manner, in general, if the number of equations exceed in any manner the number of bases or repeated terms, the rule is to introduce so many *new* and *arbitrary* bases as together with the old bases shall make up the number of equations, and then equate the zeta-ic product of the differences of zero, the old bases and the new bases, to nothing.

Art. (12). Cor. 3. Let the number of equations be *one* fewer than the number (n) of bases or repeated terms; the number of introduced bases in the general theorem is here $(n-2)$. Make these $(n-2)$ bases equal severally to the bases which in the type equation are affixed to $z, u \ldots t$, then

$$C = 0,$$

$$D = 0,$$

$$\cdots\cdots$$

$$L = 0,$$

and we have left simply

$$\zeta PD\,(0, a, c, d \ldots kl)\, x + \zeta PD\,(0, b, c, d \ldots kl)\, y = 0.$$

In like manner we may make to vanish all but A and C, and thus get

$$\zeta PD\,(0, a, b, d \ldots kl)\, x + \zeta PD\,(0, c, b, d \ldots kl)\, z = 0,$$

and similarly

$$\zeta PD(0, a, b \dots k)\, x + \zeta PD(0, b, c \dots l)\, t = 0.$$

Hence

$$\left.\begin{array}{r} x \\ y \\ z \\ . \\ . \\ . \\ t \end{array}\right\} \text{are severally as} \left\{\begin{array}{l} \zeta PD(0, b, c \dots l) \\ \zeta PD(a, 0, c \dots l) \\ \zeta PD(a, b, 0 \dots l) \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ \zeta PD(a, b, c \dots 0). \end{array}\right.$$

This is the symbolical representation as a *formula* of the remarkable *method* discovered by Cramer, perfected by Bezout and demonstrated by Laplace for the solution of simultaneous simple equations.

Art. (13). Cor. 4. In like manner if the number of repeated terms be two greater than the number of equations, we have for the relation between any *three* of them, taken at pleasure, for instance, x, y, z,

$$\zeta PD(0, a, d \dots l)\, x + \zeta PD(0, b, d \dots l)\, y + \zeta PD(0, c, d \dots l)\, z = 0.$$

And in like manner we may proceed, however much in excess the number of repeated terms (unknown quantities) is over the number of equations.

Art. (14). *Subcorollary to Corollary* 3.

If there be any number of bases $(a, b, c \dots l)$, and any other two fewer in number $(f, g \dots k)$

$$\left.\begin{array}{l} \zeta PD(a, f, g \dots k) \times \zeta PD(b, c \dots l) \\ + \zeta PD(b, f, g \dots k) \times \zeta PD(a, c \dots l) \\ + \zeta PD(c, f, g \dots k) \times \zeta PD(b, c \dots l) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + \zeta PD(l, f, g \dots k) \times \zeta PD(a, b, c \dots \ ) = 0 \end{array}\right\}^{*},$$

a formula that from its very nature suggests and *proves* a wide extension of itself.

In conclusion I feel myself bound to state that the principal substance of *Corollaries* (1), (2) and (3) may be found in Garnier's *Analyse Algébrique*, in the chapter headed "Dévéloppement de la Théorie donnée par M. Laplace, &c." But I am not aware of having been anticipated either in the fertile notation which serves to express them nor in the general theorems to which it has given birth.

P.S. I shall content myself for the present with barely enunciating a theorem, one of a class destined it seems to the author to play no secondary part in the development of some of the most curious and interesting points of analysis.

* The cross is used to denote *ordinary* algebraical multiplication.

Let there be $(n-1)$ bases $a, b, c \dots l$, and let the arguments of each be "recurrents of the nth order*," that is to say let

$$a_\iota = \phi\left(\cos\frac{2\pi\iota}{n}\right),\ b_\iota = \psi\left(\cos\frac{2\pi\iota}{n}\right),\ c_\iota = \chi\left(\cos\frac{2\pi\iota}{n}\right),\ \dots\dots l_\iota = \omega\left(\cos\frac{2\pi\iota}{n}\right).$$

Let R_r denote that any symmetrical function of the rth degree is to be taken of the quantities in a parenthesis which come after it, and let ϑ indicate any function whatever. Then the zeta-ic product

$$\zeta\{\zeta R_r(a, b, c \dots l) \times \zeta_\rho \vartheta PD(0, a, b, c \dots l)\}$$

is equal to the product of the *number*

$$R_r\left\{\left(\cos\frac{2\pi}{n} + \sqrt{(-1)}\sin\frac{2\pi}{n}\right), \left(\cos\frac{4\pi}{n} + \sqrt{(-1)}\sin\frac{4\pi}{n}\right), \left(\cos\frac{6\pi}{n} + \sqrt{(-1)}\sin\frac{6\pi}{n}\right),\right.$$

$$\left.\dots\dots, \left(\cos\frac{2(n-1)\pi}{n} + \sqrt{(-1)}\sin\frac{2(n-1)\pi}{n}\right)\right\},$$

multiplied by the zeta-ic phase

$$\zeta_{\rho - r}\vartheta PD(0, a, b, c \dots l)!!$$

* I am indebted for this term to Professor De Morgan, whose pupil I may boast to have been. I have the sanction also of his authority, and that of another profound analyst, my colleague Mr Graves, for the use of the arbitrary terms zeta-ic, zeta-ically. I take this opportunity of retracting the symbol *SPD* used in my last paper, the letter *S* having no meaning except for English readers. I substitute for it *QDP*, where *Q* represents the Latin word Quadratus. On some future occasion I shall enlarge upon a new method of notation, whereby the language of analysis may be rendered much more expressive, depending essentially upon the use of similar figures inserted within one another, and containing numbers or letters, according as quantities or operations are to be denoted. This system to be carried out would require special but very simple printing types to be founded for the purpose.

In the next part of this paper an easy and *symmetrical* mode will be given of representing any polynomial either in its developable or expanded form.

9.

A METHOD OF DETERMINING BY MERE INSPECTION THE DERIVATIVES FROM TWO EQUATIONS OF ANY DEGREE.

[*Philosophical Magazine*, XVI. (1840), pp. 132—135.]

LET there be two equations, one of the nth, the other of the mth degree in x; let the coefficients of the first equation be $a_n, a_{n-1}, a_{n-2} \ldots a_0$, *each* power of x having a coefficient attached to it, a_n belonging to x^n and a_0 to the constant term.

In like manner let $b_m, b_{m-1} \ldots b_0$ be the coefficients of the second equation.

I begin with

A Rule for absolutely eliminating x.

Form out of the (a) progression of coefficients m lines, and in like manner out of the (b) progression of coefficients form n lines in the following manner:

1. (a) Attach $(m-1)$ zeros all to the *right* of the terms in the (a) progression; next attach $(m-2)$ zeros to the right and carry over to the left; next attach $(m-3)$ zeros to the right and carry over 2 to the left. Proceed in like manner until all the $(m-1)$ zeros are carried over to the left and none remain on the right.

The m lines thus formed are to be written under one another.

1. (b) Proceed in like manner to form n lines out of the (b) progression by scattering $(n-1)$ zeros between the right and left.

2. If we write these n lines under the m lines last obtained, we shall have a solid square $(m+n)$ terms *deep* and $(m+n)$ terms *broad.*

3. Denote the lines of this square by arbitrary characters, which write down in vertical order and permute in every possible way, but separate the permutations that can be derived from one another by an even number of interchanges (effected between *contiguous* terms) from the rest; there will thus be half of one kind and half of another.

4. Now arrange the $(m+n)$ lines accordingly, so as to obtain

$$\tfrac{1}{2}\{(m+n)(m+n-1)\ldots 2\,.\,1\}$$

squares of one kind which shall be called positive squares, and an equal number of the opposite kind which shall be called negative.

Draw diagonals in the same direction in all the squares; multiply the coefficients that stand in any diagonal line together: take the sum of the diagonal products of the *positive* squares, and the sum of the diagonal products of the *negative* squares; the difference between these two sums is the prime derivative of the zero degree, that is, is the result of elimination between the two given equations reduced to its ultimate state of simplicity, there will be no irrelevant factors to reject, and no terms which mutually destroy.

Example. To *eliminate* between

$$ax^2+bx+c=0,$$
$$lx^2+mx+n=0,$$

I write down

$$a,\ b,\ c,\ 0, \qquad (1)$$
$$0,\ a,\ b,\ c, \qquad (2)$$
$$l,\ m,\ n,\ 0, \qquad (3)$$
$$0,\ l,\ m,\ n. \qquad (4)$$

I permute the four characters (1), (2), (3), (4), distinguishing them into positive and negative; thus I write together

Positive Permutations.

1	2	3	1	2	3	2	1	3	4	4	4
2	3	1	4	4	4	1	3	2	2	1	3
3	1	2	2	3	1	4	4	4	1	3	2
4	4	4	3	1	2	3	2	1	3	2	1

and again

Negative Permutations.

1	2	3	4	4	4	2	1	3	2	1	3
2	3	1	1	2	3	4	4	4	1	3	2
4	4	4	2	3	1	1	3	2	3	2	1
3	1	2	3	1	2	3	2	1	4	4	4

I reject from the permutations of each species all those where 1 or 3 appear in the fourth place, and also those where 2 or 4 appear in the first place, for these will be presently seen to give rise to diagonal products which are zero.

The permutations remaining are

Positive effectual permutations.

1	3	3	1
2	1	4	3
3	2	1	4
4	4	2	2

Negative effectual permutations.

3	1	1	3
1	4	3	2
4	3	2	1
2	2	4	4

I now accordingly form four positive squares, which are

$$\begin{array}{cccc} a, & b, & c, & 0, \\ 0, & a, & b, & c, \\ l, & m, & n, & 0, \\ 0, & l, & m, & n, \end{array} \qquad \begin{array}{cccc} l, & m, & n, & 0, \\ a, & b, & c, & 0, \\ 0, & a, & b, & c, \\ 0, & l, & m, & n, \end{array} \qquad \begin{array}{cccc} l, & m, & n, & 0, \\ 0, & l, & m, & n, \\ a, & b, & c, & 0, \\ 0, & a, & b, & c, \end{array} \qquad \begin{array}{cccc} a, & b, & c, & 0, \\ l, & m, & n, & 0, \\ 0, & l, & m, & n, \\ 0, & a, & b, & c. \end{array}$$

Drawing diagonal lines from left to right, and taking the sum of the diagonal products, I obtain $a^2n^2 + lb^2n + l^2c^2 + am^2c$. Again, the four negative squares

$$\begin{array}{cccc} l, & m, & n, & 0, \\ a, & b, & c, & 0, \\ 0, & l, & m, & n, \\ 0, & a, & b, & c, \end{array} \qquad \begin{array}{cccc} a, & b, & c, & 0, \\ 0, & l, & m, & n, \\ l, & m, & n, & 0, \\ 0, & a, & b, & c, \end{array} \qquad \begin{array}{cccc} a, & b, & c, & 0, \\ l, & m, & n, & 0, \\ 0, & a, & b, & c, \\ 0, & l, & m, & n, \end{array} \qquad \begin{array}{cccc} l, & m, & n, & 0, \\ 0, & a, & b, & c, \\ a, & b, & c, & 0, \\ 0, & l, & m, & n, \end{array}$$

give as the sum of the diagonal products

$$lbmc + alnc + ambn + lacn,$$

that is,
$$lbmc + ambn + 2acln.$$

Thus the result of eliminating between

$$ax^2 + bx + c = 0,$$

$$lx^2 + mx + n = 0,$$

ought to be, and is

$$a^2n^2 + l^2c^2 - 2acln + lb^2n + am^2c - lbmc - ambn = 0.$$

Rule for finding the prime derivative of the first degree, which is of the form $Ax - B$.

Begin as before, only attach one zero less to each progression; we shall thus obtain *not* a square, but an oblong broader than it is deep, containing $(m+n-2)$ rows, and $(m+n-1)$ terms in each row: in a word, $(m+n-2)$ rows, and $(m+n-1)$ *columns.*

To find A reject the column at the extreme right, we thus recover a square arrangement $(m+n-2)$ terms broad and deep.

Proceed with this new square as with the former one; the difference between the sums of the positive and negative diagonal products will give A.

To find B, do just the same thing, with the exception of striking off not the last column, but the last but *one.*

Rule for finding the prime derivative of any degree, say the rth, namely, $A_r x^r - A_{r-1} x^{r-1} + \ldots\ldots \pm A_0$.

Begin with adding zeros as before, but the number to be added to the (a) progression is $(m-r)$ and to the (b) progression $(n-r)$.

There will thus be formed an oblong containing $(m+n-2r)$ rows, and $(m+n-r)$ terms in each row, and therefore the same number of *columns.*

To find any coefficient as A_s, strike off all the last $(r+1)$ columns except that which is (s) places distant from the extreme right, and proceed with the resulting squares as before.

Through the well-known ingenuity and kindly proferred help of a distinguished friend, I trust to be able to get a machine made for working Sturm's theorem, and indeed all problems of derivation, after the method here expounded; on which subject I have a great deal more yet to say, than can be inferred from this or my preceding papers.

10.

NOTE ON ELIMINATION.

[*Philosophical Magazine*, XVII. (1840), pp. 379, 380.]

THE object of this brief note is to generalise Theorem 2 in my paper on Elimination* which appeared in the last December number of this *Magazine*. The theorem so generalised presents a symmetry which before was wanting. Here, as in so many other instances, the whole occupies in the memory a *less* space than the part.

To avoid the ill-looking and slippery negative symbols, I warn my reader that I now use two rows of quantities written one over the other, to denote the product of the terms resulting from *taking away* each quantity in the under from each in the upper row.

Let $h_1, h_2 \dots h_m$ be the roots of one equation of coexistence,

$k_1, k_2 \dots k_n$ of the other,

and let the prime derivative of the degree r be required. Take *any* two integers p and q, such that $p+q=r$. The derivative in question may be written

$$\Sigma\left\{(x-h_1)\dots(x-h_p)(x-k_1)\dots(x-k_q)\frac{\begin{pmatrix}h_1 h_2 \dots h_p\\ k_1 k_2 \dots k_q\end{pmatrix}\cdot\begin{pmatrix}h_{p+1} h_{p+2} \dots h_m\\ k_{q+1} k_{q+2} \dots k_n\end{pmatrix}}{\begin{pmatrix}h_1 \;\; h_2 \;\; \dots h_p\\ h_{p+1} h_{p+2} \dots h_m\end{pmatrix}\cdot\begin{pmatrix}k_1 \;\; k_2 \;\; \dots k_q\\ k_{q+1} k_{q+2} \dots k_n\end{pmatrix}}\right\}.$$

N.B. Whatever p and q be taken, so long only as $p+q=r$, the above expression changes nothing but its sign; which, therefore, upon transcendental grounds, it is easy to see is of one name or another, according as p is odd or even.

In the original paper, I asserted this theorem only for the case of $p=0$, or $q=0$.

[* p. 43 above. ED.]

11.

ON THE RELATION OF STURM'S AUXILIARY FUNCTIONS TO THE ROOTS OF AN ALGEBRAIC EQUATION.

[*Plymouth British Association Report* 1841, (Pt II.), pp. 23, 24.]

THE author availed himself of the present meeting of the British Association to bring under the more general notice of mathematicians his discovery, made in the year 1839, of the real nature and constitution of the auxiliary functions (so-called) which Sturm makes use of in *locating* the roots of an equation: these are obtained by proceeding with the left-hand side of the equation and its first differential coefficient as if it were our object to obtain their greatest common factor; the successive remainders, with their signs *alternately* changed and preserved, constitute the functions in question. Each of these may be put under the form of a fraction, the denominator of which is a perfect square, or in fact the product of *many*: likewise the numerator contains a huge heap of factors of a similar form.

These therefore, as well as the denominator, since they cannot influence the series of *signs*, may be rejected; and furthermore we may, if we please, again make every other function, beginning from the last but one, change its sign, if we consent to use changes wherever Sturm speaks of continuations of sign, and *vice versâ*.

The functions of Sturm, thus modified and purged of irrelevancy, the author, by way of distinction, and still to attribute honour where it is really most due, proposes to call "Sturm's Determinators"; and he proceeds to lay bare the internal anatomy of these remarkable forms.

He uses the Greek letter "ζ" to indicate that the squared product of the differences of the letters before which it is prefixed is to be taken.

Let the roots of the equation be called respectively $a, b, c, e \ldots l$, the determinators taken in the inverse order are as follows:—

$$\zeta(a, b, c, e \ldots l).$$
$$\Sigma\zeta(b, c, e \ldots l)\, x - \Sigma a\zeta(b, c, e \ldots l).$$
$$\Sigma\zeta(c, e \ldots l)\, x^2 - \Sigma(a+b) \,.\, \zeta(c, e \ldots l)\, x + \Sigma ab \,.\, \zeta(c, e \ldots l).$$

* * * * * * *

$$\Sigma\,\{\zeta(k, l)\,(x-a)\,(x-b)\,(x-c)\,(x-e) \ldots (x-h)\}.$$

It may be here remarked, that the work of assigning the total number of real and of imaginary roots falls exclusively upon the coefficients of the leading terms, which the author proposes to call "Sturm's Superiors": these superiors are only *partial* symmetric functions of the *squared differences*, but *complete* symmetric functions of the *roots themselves*, differing in the former respect from those other (at first sight similar-looking) functions of the squared differences of the roots, in which, from the time of Waring downwards, the conditions of reality have been sought for. It seems to have escaped observation, that the series of terms constituting any one of the coefficients in the equation of the squares of the differences (with the exception of the first and last) each admit of being separated and classified into various subordinate groups in such a way, that instead of being treated as a single symmetric function of the *roots*, they ought to be viewed as aggregates of many. In fact, Sturm's superior No. 1 is identical with Waring's coefficient No. 1; Sturm's superior No. 2 is a *part* of Waring's coefficient No. 3; Sturm's superior No. 3 is a *part* of Waring's coefficient No. 6; and so forth till we come to Sturm's final superior, which is again coextensive and identical with the last coefficient in the equation of the squares of the differences. The theory of symmetric functions of forms which are themselves symmetric functions of simple letters, or even of other forms, the author states his belief is here for the first time shadowed forth, but would be beside his present object to enter further into. He would conclude by calling attention to the importance to the general interests of algebraical and arithmetical science that a searching investigation should be instituted for showing, *à priori*, how, when a set of quantities is known to be made up partly of possible and partly of *pairs* of impossible values, symmetrical functions of these, one less in number than the quantities themselves, may be formed, from the signs of the ratios of which to unity and to one another the respective amounts of possible and impossible quantities may at once be inferred: in short, we ought not to rest satisfied, until, from the very *form* of Sturm's Determinators, without caring to know how they have been obtained, we are able to pronounce upon the uses to which they may be applied.

12.

EXAMPLES OF THE DIALYTIC METHOD OF ELIMINATION AS APPLIED TO TERNARY SYSTEMS OF EQUATIONS.

[*Cambridge Mathematical Journal*, II. (1841), pp. 232—236.]

THIS method is of universal application, and at once enables us to reduce any case of elimination to the form of a problem, where that operation is to be effected between quantities linearly involved in the equations which contain them.

As applied to a binary system, $fx = 0$, $\phi x = 0$, the method furnishes a rule by which we may unfailingly arrive at *the determinant*, free from every species of irrelevancy, whether of a linear, factorial, or numerical kind.

The rule itself is given in the *Philosophical Magazine* (London and Edinburgh, Dec. 1840). The principle of the rule will be found correctly stated by Professor Richelot, of Königsberg, in a late number of *Crelle's Journal*, at the commencement of a memoir in Latin bordering on the same subject ("Nota ad Eliminationem pertinens").

My object at present is to supply a few instances of its application to ternary systems of equations.

Ex. 1. To eliminate x, y, z, between the three homogeneous equations

$$Ay^2 - 2C'xy + Bx^2 = 0, \tag{1}$$

$$Bz^2 - 2A'yz + Cy^2 = 0, \tag{2}$$

$$Cx^2 - 2B'zx + Az^2 = 0. \tag{3}$$

Multiply the equations in order by $-z^2$, x^2, y^2, add together, and divide out by $2xy$; we obtain

$$C'z^2 + Cxy - A'xz - B'yz = 0. \tag{4}$$

By similar processes we obtain

$$A'x^2 + Ayz - B'yx - C'zx = 0, \tag{5}$$

$$B'y^2 + Bzx - C'zy - A'xy = 0. \tag{6}$$

Between these six, treated as simple equations, the six functions of x, y, z, namely, x^2, y^2, z^2, xy, xz, yz, treated as *independent* of each other, may be eliminated; the results may be seen, by mere inspection, to come out

$$ABC(ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C') = 0,$$

or rejecting the special (N.B. not *irrelevant*) factor ABC, we obtain

$$ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C' = 0.$$

I may remark, that the equations (1), (2), (3), or (4), (5), (6), express the condition of

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy,$$

having a factor $\lambda x + \mu y + \nu z$; a general symbolical formula of which I am in possession for determining in general the condition of any polynomial of any degree having a factor, furnishes me at once with either of the two systems indifferently. The aversion I felt to reject *either*, led me to employ both, and thus was the occasion of the Dialytic Principle of Solution manifesting itself.

Ex. 2.
$$Ax^2 + ayz + bzx + cxy = 0, \tag{1}$$
$$My^2 + lyz + mzx + nxy = 0, \tag{2}$$
$$Rz^2 + pyz + qzx + rxy = 0. \tag{3}$$

Multiply equation (1) by $\beta y + \gamma z$, equations (2) and (3) by νz and κy respectively, and add the products together, we obtain terms of which y^2z and yz^2 are the only two into which x does not enter.

Make now the coefficients of each of these zero, and we have

$$a\gamma + l\nu + R\kappa = 0,$$
$$a\beta + M\nu + p\kappa = 0.$$

Let $\nu = a$, $\kappa = a$, then $\gamma = -(l + R)$, $\beta = -(M + p)$.

Hence, multiplying as directed, and then dividing out by x, we obtain

$$(m\nu + b\gamma) z^2 + (r\kappa + c\beta) y^2 + (b\beta + c\gamma + n\nu + q\kappa) yz + A\beta xy + A\gamma xz = 0,$$

or by substitution,

$$\{ra - c(M+p)\} y^2 + \{ma - b(l+R)\} z^2 + \{an + aq - b(M+p) - c(l+R)\} yz$$
$$- A(M+p) xy - A(M+p) xz = 0. \tag{4}$$

Similarly, by preparing the equations so as to admit in turn of y and z as a divisor, we obtain

$$\{ma - l(R+b)\} z^2 + \{mr - n(A+q)\} x^2 + \{mc + mp - n(R+b) - l(A+y)\} xz$$
$$- M(R+b) yz - A(A+q) xy = 0, \tag{5}$$

$$\{rm - q(A+n)\} x^2 + \{ra - p(M+c)\} y^2 + \{rl + rb - p(A+n) - q(M+c)\} xy$$
$$- R(A+n) xz - R(M+c) yz = 0. \tag{6}$$

Between the six equations (1), (2), (3), (4), (5), (6), x^2, y^2, z^2, xy, xz, yz, may be eliminated; the result will be a function of nine letters {three out of each equation (1), (2), (3)} equated to zero. *Perhaps* the determinant may be found to contain a special factor of three letters; and if so, may be replaced by a simpler function of six letters only.

Ex. 3. To eliminate between the three general equations

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 0,$$

$$Lx^2 + My^2 + Nz^2 + 2Pyz + 2Qzx + 2Rxy = 0,$$

$$fx + gy + hz = 0.$$

By virtue of *one* of the two canons which limit the forms in which the letters can appear combined in the determinant of a general system of equations, we know that the determinant in this case (freed of irrelevant factors) ought to be made up in every term of eight letters (powers being counted as repetitions), namely, (A, B, C, D, E, F) must enter in binary combinations, (L, M, N, P, Q, R) the same, whereas f, g, h must enter in *quaternary* combinations.

To obtain the determinant, write

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy = 0, \qquad (1)$$

$$Lx^2 + My^2 + Nz^2 + Pyz + Qzx + Rxy = 0, \qquad (2)$$

$$fx^2 + gyx + hzx = 0, \qquad (3)$$

$$fxy + gy^2 + hzy = 0, \qquad (4)$$

$$fxz + gyz + hz^2 = 0. \qquad (5)$$

We want one equation more of *three* letters between x^2, y^2, z^2, xy, xz, yz. To obtain this, write

$$(Ax + Ez + Fy)\, x_1 + (By + Fx + Dz)\, y_1 + (Cz + Dy + Ex)\, z_1 = 0,$$

$$(Lx + Qz + Ry)\, x_1 + (My + Rx + Pz)\, y_1 + (Nz + Py + Qx)\, z_1 = 0,$$

$$fx_1 + \qquad gy_1 + \qquad hz_1 = 0.$$

Forget that $x_1 = x$, $y_1 = y$, $z_1 = z$, and eliminate x_1, y_1, z_1, we obtain

$$h \begin{Bmatrix} (Ax + Ez + Fy)(My + Rx + Pz) \\ -(By + Fx + Dz)(Lx + Qz + Ry) \end{Bmatrix}$$

$$+ g \begin{Bmatrix} (Cz + Dy + Ex)(Lx + Qz + Ry) \\ -(Nz + Py + Qx)(Ax + Ez + Fy) \end{Bmatrix}$$

$$+ f \begin{Bmatrix} (Nz + Py + Qx)(By + Fx + Dz) \\ -(Cz + Dy + Ex)(My + Rx + Pz) \end{Bmatrix} = 0.$$

This may be put under the form

$$\alpha x^2+\beta y^2+\gamma z^2+\alpha' yz+\beta' zx+\gamma' xy=0, \qquad (6)$$

where the coefficients are of the first order in respect to f, g, h, L, M, N, P, Q, R, A, B, C, D, E, F; in all of the third order.

Between the equations marked from (1) to (6), the process of linear elimination being gone through, we obtain as equated to zero a function of $5+3$, or of eight letters, two belonging to the first equation, two to the second, and four to the third; so that the determinant is clear of all factorial irrelevancy.

Ex. 4. To eliminate x, y, z between the three equations

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy=0,$$
$$Lx^2+My^2+Nz^2+2L'yz+2M'zx+2N'xy=0,$$
$$Px^2+Qy^2+Rz^2+2P'yz+2Q'zx+2R'xy=0.$$

Call these three equations $U=0$, $V=0$, $W=0$, respectively. Write

$$xU=0, \quad (1) \qquad yU=0, \quad (2) \qquad zU=0, \quad (3)$$
$$xV=0, \quad (4) \qquad yV=0, \quad (5) \qquad zV=0, \quad (6)$$
$$xW=0, \quad (7) \qquad yW=0, \quad (8) \qquad zW=0. \quad (9)$$

We have here nine unilateral equations: one more is wanted to enable us to eliminate *linearly* the ten quantities

$$x^3,\ y^3,\ z^3,\ x^2y,\ x^2z,\ xy^2,\ xz^2,\ xyz,\ y^2z,\ yz^2.$$

This tenth may be found by eliminating x, y, z between the three equations

$$x(Ax+B'z+C'y)+y(By+C'x+A'z)+z(Cz+A'y+B'x)=0,$$
$$x(Lx+M'z+N'y)+y(My+N'x+L'z)+z(Nz+L'y+M'x)=0,$$
$$x(Px+Q'z+R'y)+y(Qy+R'x+P'z)+z(Rz+P'y+Q'x)=0;$$

for, by forgetting the relations between the bracketed and unbracketed letters, we obtain

$$(Ax+B'z+C'y)\begin{Bmatrix}(My+N'x+L'z)(Rz+P'y+Q'x)\\-(Qy+R'x+P'z)(Nz+L'y+M'x)\end{Bmatrix}$$
$$+\&c.+\&c.=0,$$

which may be put under the form

$$\alpha x^3+\beta y^3+\gamma z^3+\delta x^2y+\ldots\ldots=0^*. \qquad (10)$$

* We might dispense with a 10th equation, using the nine above given, to determine the ratios of the ten quantities involved to one another; and then by means of any such relations as

$$x^3y\times xy^3=x^2y^2\times x^2y^2, \text{ or } x^3\times y^3=x^2y\times xy^2, \ \&c.$$

obtain a determinant. But it is easy to see that this would be made up of terms, each containing literal combinations of the 18th order.

Again, we might use five out of the nine equations to obtain a new equation free from y^3, y^2z, yz^2, z^3; that is, containing x in every term: which being divided by x, and multiplied

By eliminating linearly between the equations marked from (1) to (10), we obtain as zero a quantity of the twelfth order in all, being of the fourth order in respect to the coefficients of each of the three equations, which is therefore the determinant in its simplest form.

I have purposely, in this brief paper, avoided discussing any theoretical question. I may take some other opportunity of enlarging upon several points which have hitherto been little considered in the theory of elimination, such as the Canons of Form,—the Doctrine of Special Factors,—the Method of Multipliers as extended to a system of any order,—the Connexion between the method of Multipliers and the Dialytic Process,—the Idea of Derivations and of Prime Derivatives extended to ultra-binary Systems. For the present I conclude with the expression of my best wishes for the continued success of this valuable Journal.

by y, or by z, would furnish a 10th equation no longer linearly involved in the 9 already found. The determinant, however, found in this way, would consist of 14-ary combinations of letters.

Finally, we might, instead of a system of ten equations, employ a system of 15, obtained by multiplying each of the given three by any 5 out of the 6 quantities x^2, y^2, z^2, xy, xz, yz; but the determinant, besides being not *totally* symmetrical, would contain combinations of the 15th order.

I may take this opportunity of just adverting to the fact, that the method in the text does in fact contain a solution of the equation

$$\lambda U + \mu V + \nu W = x^r y^s z^t,$$

where $r+s+t=4$, and λ, μ, ν are functions of the second degree in regard to x, y, z to be determined.

13.

INTRODUCTION TO AN ESSAY ON THE AMOUNT AND DISTRIBUTION OF THE MULTIPLICITY OF THE ROOTS OF AN ALGEBRAIC EQUATION.

[*Philosophical Magazine*, XVIII. (1841), pp. 136—139.]

I USE the word *multiplicity* to denote a number, and distinguish between the total and partial multiplicities of the roots of an algebraic equation.

There may be r different roots repeated respectively $h_1, h_2 \ldots h_r$ times.

r is the index of distribution.

$h_1, h_2 \ldots h_r$ are the partial multiplicities, and if $h = h_1 + h_2 + \ldots + h_r$

h is the *total* multiplicity.

The total multiplicity it is clear may be defined as the difference between the index of the equation and the number of its roots distinguishable from one another.

In this Introduction, I propose merely to consider how existing methods may be applied to determine the amount and distribution of multiplicity in a given equation, and conversely, how equations of condition can be formed which shall imply a *given* distribution and amount.

Let the greatest common factor between fx (the argument of the proposed equation) and $\frac{dfx}{dx}$ be called f_1x.

And in like manner, let the greatest common factor of f_1x and $\frac{df_1x}{dx}$ be called f_2x and so on, till in the end we come to f_rx, which has no common factor with $\frac{df_rx}{dx}$.

Let $k_1, k_2 \ldots k_r$ denote the degrees in x of $fx, f_1x \ldots f_rx$ respectively.

It is easy to see that

$k_1 - k_2$, partial multiplicities, are less than 2, that is, are each units.

$k_2 - k_3$, partial multiplicities, will be less than 3, and therefore either 1 or 2 in value respectively, and so on till we come to

$k_{r-1} - k_r$ which will severally be between zero and $r - 1$, and

$k_r - 0$ of values intermediate between zero and r.

Hence there will be

$k_1 - 2k_2 + k_3$ multiplicities each of the value 1,

$k_2 - 2k_3 + k_4$ " " " 2,

....

$k_{r-1} - 2k_r$ ... of the value $r-1$,

and k_r .... of the value r.

In place of fx with $\frac{dfx}{dx}$ we might employ $\frac{dfx}{dx}$ with $\frac{d^2fx}{dx^2}$ and so on for the rest; the values of $k_2, k_3 \dots k_r$ will remain unaffected by this change; but the former method would be more expeditious in practice.

The total multiplicity is, of course, $= k_1$.

Suppose now that we propose to ourselves the converse problem to determine the conditions that an algebraic equation may have a given amount of multiplicity distributed in a given manner.

If $h_1, h_2, h_3 \dots h_r$ be used to denote the given number of partial multiplicities which are respectively of the values $1, 2, 3 \dots r$, it is easy to see that the quantities derived above by $k_1, k_2 \dots k_r$ are respectively equal to

$$h_1 + 2h_2 + \dots\dots + rh_r,$$
$$h_2 + 2h_3 + \dots\dots + rh_{r-1},$$
$$h_3 + 2h_4 + \dots\dots + rh_{r-2},$$
$$\dots\dots\dots\dots\dots\dots$$
$$h_r.$$

Now from $\frac{dfx}{dx}$ having a factor of the degree k_1 common with fx we obtain k_1 conditions, from $\frac{df_1x}{dx}$ having a factor of the degree k_2 common with f_1x we obtain k_2 more, and so on. So that altogether we obtain in this way

$$k_1 + k_2 + \dots\dots + k_r \text{ conditions.}$$

But it may easily be seen that the total multiplicity being k_1, the number of conditions *need* never to exceed k_1 in number, no matter what its distribution may be. Hence, besides the enormous labour of the process, and the extreme complexity of the results, we obtain by this method more equations by far than are necessary, and it requires some caution to know which to reject.

In my forthcoming paper (to appear in *Philosophical Magazine* of next month) I shall show, by a most simple means, how without the use of derived or other subsidiary functions, to obtain the simplest equations of condition which correspond to a given distribution of a given amount of multiplicity.

The total multiplicity, say m, being given in as many ways as that number can be broken into parts, so many different systems of m equations can be formed differing each from the other in the dimensions of the terms.

These systems may be arranged in order so that each in the series shall imply all those that follow it, and be implied in all those that go before, without the converse being satisfied.

The subject of the unreciprocal implication of systems of equations is a very curious one, upon which the limits assigned to me prevent me from enlarging at present. It is closely connected with a part of the theory of elimination, which, as far as I am aware, has either been overlooked, or has not met with the attention which it deserves; I mean the theory of *Special Factors.*

An *example* may make what I mean by these clear.

Let C be a function (if my reader please) void of x, which equivalent to zero implies two given equations in x having a common root.

Let C be rid of all irrelevant factors, that is, let C be the simplest form of the determinant, when the coefficients of the two equations are perfectly independent qualities. Now suppose, as is *quite possible in a variety of ways,* that such relations are instituted between the coefficients alluded to as make C split up into factors, so that $C = L \times M \times N = 0$.

Only one of the factors L, M, N will satisfy the condition of the co-existence of the two given equations: the others are clearly, however, not to be confounded with factors of solution, or irrelevant factors, as they are termed, but are of quite a different nature, and enjoy remarkable properties, which point to an enlarged theory of elimination, and constitute what I call special or singular factors.

I shall feel much obliged to any of the readers of your widely circulated Journal, interested in the subject of this paper, who would do me the honour of communicating with me upon it, and especially if they would (between now and the next coming out of the *Magazine*) inform me whether anything, and if so how much, different from what is here stated has been done in the matter of determining the relations between the coefficients of an equation corresponding to a given amount and distribution of multiplicity in its roots.

I ought to add, that my method enables me not merely to determine the conditions of multiplicity, but also to decompose the equations containing multiple roots into others free of multiplicity, that is, to find, *à priori,* the values of the several quantities

$$\frac{fx\,f_2x}{(f_1x)^2},\ \frac{f_1x\,f_3x}{(f_2x)^2},\ \ldots\ldots,\ \frac{f_{r-1}x}{(f_rx)^2},\ f_rx.$$

Moreover, other decompositions, not necessary to be enlarged upon in this place, may be obtained with equal facility.

14.

A NEW AND MORE GENERAL THEORY OF MULTIPLE ROOTS.

[*Philosophical Magazine*, XVIII. (1841), pp. 249—254.]

I SHALL begin with developing the theory of polynomials containing perfect *square factors*, one or more.

First, let us proceed to determine the relations which must exist between the coefficients of such polynomials, and afterwards show how they may be broken up into others of an inferior degree.

A parallelogram filled with letters standing in *one* row is intended to express the product of the squared difference of the quantities contained. Thus $\left(\overline{\underline{ab}}\right)$ indicates $(a-b)^2$, $\left(\overline{\underline{abc}}\right)$ is used to indicate $(a-b)^2(a-c)^2(b-c)^2$, and so forth.

Suppose now that two of the roots $e_1, e_2 \ldots e_n$ belonging to the equation $fx = 0$ are equal to one another, it is clear that $\left(\overline{\underline{e_1, e_2 \ldots e_n}}\right) = 0$; and moreover is a symmetric function, and can be calculated in terms of the coefficients of fx.

Next let us suppose that we have two couples of equals (as for instance a and b, two of the roots equal, as also c and d two others), it is clear, that on leaving any one of the roots out, the $(n-1)$ that are left will still contain one equality, and therefore we have

$$\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right) = 0, \quad \left(\overline{\underline{e_1, e_3 \ldots e_n}}\right) = 0 \ldots \left(\overline{\underline{e_1, e_2 \ldots e_{n-1}}}\right) = 0.$$

None of the parallelogrammatic functions above taken *singly*, are symmetric functions of the coefficients, but their sum is; so also is the sum of the product of each into the quantity left out.

Now in general, suppose that the polynomial fx contains r perfect square factors, so that we have r couples of equal roots belonging to the equation $fx = 0$, it is clear that $\left(\overline{\underline{e_r, e_{r+1} \ldots e_n}}\right)$ and all the other $\dfrac{n(n-1)\ldots(n-r+2)}{1.2\ldots(r-1)}$ functions of which it is the type are severally zero. Moreover, the sum of these or the sum of the products of each by *any* symmetrical function of the $(r-1)$ letters left out will be a symmetrical function of the coefficients of the powers of x in fx. To express now the *affirmative** conditions corresponding to the case of there being r pairs of equal roots, we *might* employ the r equations,

$$\begin{aligned}
\left(\overline{e_1, e_2 \ldots e_n}\right) &= 0, \\
\Sigma \left(\overline{e_2, e_1 \ldots e_n}\right) &= 0, \\
\Sigma \left(\overline{e_3 \ldots e_n}\right) &= 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
\Sigma \left(\overline{e_r, e_{r+1} \ldots e_n}\right) &= 0.
\end{aligned}$$

But these, except the last, are not the *simplest* that can be employed; that is to say, we can write down r others, the terms of which shall be of lower dimensions in respect to the roots.

Let $\int_\mu$ denote that any rational symmetrical function of the μth degree is to be taken of the quantities which it precedes.

Then the r equations in question are all contained in the general equation

$$\Sigma \left\{\int_\mu (e_1, e_2 \ldots e_{r-1}) \times \left(\overline{e_r, e_{r+1} \ldots e_n}\right)\right\} = 0\,;$$

μ being taken from 0 up to $(r-1)$ we obtain r equations, which in respect to the roots are respectively of all degrees between

$$\frac{n(n-1)\ldots(n-r+2)}{1.2\ldots(r-1)} \text{ and } \frac{n(n-1)\ldots(n-r+2)}{1.2\ldots(r-1)} + (r-1)$$

reckoned inclusively.

Now at this stage it is important to remark that the above r equations, although *necessary*, are not *sufficient*; and indeed, no mere affirmations of equality can be sufficient to ensure there being r pairs of equal roots.

To make this manifest, suppose $r = 2$. Then in order that an equation *may* have two pairs of equal roots, we must have by the above formula

$$\Sigma \left(\overline{e_2, e_3 \ldots e_n}\right) = 0, \quad \Sigma \left\{e_1 \left(\overline{e_2, e_3 \ldots e_n}\right)\right\} = 0.$$

But if instead of there being two perfect square factors there be one perfect *cube* factor in fx, it may be shown by the same reasoning as above, that the very same two equations apply. In fact, it may be shown in general that no such equations as those given above can be *affirmed* in consequence of there being an amount r of multiplicity consisting of unit parts which may not be affirmed with equal truth as necessary consequences of the same

* The importance of the restriction hinted at by the use of the word affirmative will appear hereafter.

amount distributed *in any other manner* whatever. How to obtain affirmative equations sufficient as well as necessary (under certain limitations) will appear at the close of this present paper.

It is worthy of being remarked, that if we make $\int_\mu$ denote the sum of the products of the quantities to which it is prefixed, taken μ and μ together, the equations of affirmation become identical with those obtained by eliminating between fx and $\frac{dfx}{dx}$ *.

It can scarcely be doubted that the illustrious Lagrange, had he chosen to perfect the incomplete theory of equal roots given in the *Résolution Numérique*, by applying to it his own favourite engine of symmetric functions, could scarcely have failed of stumbling by a back passage upon Sturm's memorable theorem.

Let us now proceed to show how a polynomial known to contain one or more perfect square factors may be decomposed.

Let us begin with supposing that it contains but one such factor; so that $fx = \phi x (x-a)^2$.

I shall show how to obtain the equations

$$C(x-a)=0, \quad D\phi x(x-a)=0, \quad E(x-a)^2=0, \quad F(\phi x)=0,$$

each in its lowest terms.

1. To form the equation $Lx + M = 0$, where $x = a$, it is easy to see that if we write down in general the expression $(x-e_1)\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right)$ this will become zero whenever the root e_1 left out is not one of the equal roots (a): so that in fact (calling the two equal roots e_1, e_2 respectively)

$$\Sigma\left\{(x-e_1)\times\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right)\right\} = (x-e_1)\times\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right) + (x-e_2)\times\left(\overline{\underline{e_1, e_3 \ldots e_n}}\right),$$

or simply $$= 2(x-a)\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right).$$

Hence by making

$$x\Sigma\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right) - \Sigma\left\{e_1 \times \left(\overline{\underline{e_2, e_3 \ldots e_n}}\right)\right\} = 0,$$

we have an equation for finding the equal roots e_1, e_2.

Again, it is easily seen upon the same hypothesis, that

$$\Sigma\left\{(x-e_2)(x-e_3)(x-e_4)\ldots(x-e_n)\times\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right)\right\}$$
$$= 2(x-e_2)(x-e_3)\ldots(x-e_n)\times\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right).$$

* See my note on Sturm's Theorem, *Phil. Mag.*, December, 1839 [p. 45 above. Ed.].

Hence, to form the equation having the same roots as $(x-a)\,\phi x$, we have only to make

$$x^{n-1}\Sigma\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right) - x^{n-2}\Sigma\left\{(e_2+e_3+\ldots e_n)\times\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right)\right\}\ldots\ldots$$
$$\pm\,\Sigma\left\{(e_2e_3\ldots e_n)\times\left(\overline{\underline{e_2, e_3 \ldots e_n}}\right)\right\}=0.$$

Suppose now in general that we have r perfect square factors, so that

$$fx = \phi x\,(x-a_1)^2(x-a_2)^2\ldots(x-a_r)^2.$$

To form the equation $C(x-a_1)(x-a_2)\ldots(x-a_r)=0$, we have only to make

$$\Sigma\left\{(x-e_1)(x-e_2)\ldots(x-e_r)\times\left(\overline{\underline{e_{r+1}, e_{r+2}\ldots e_n}}\right)\right\}=0.$$

And to obtain

$$D\phi x\times(x-a_1)(x-a_2)\ldots(x-a_r)=0,$$

we must make

$$\Sigma\left\{(x-e_{r+1})(x-e_{r+2})\ldots(x-e_n)\times\left(\overline{\underline{e_{r+1}, e_{r+2}\ldots e_n}}\right)\right\}=0.$$

The theory of perfect square factors is not yet complete until it has been shown how to obtain constructively ϕx, and, as analogy suggests, the complementary part $D'(x-a_1)^2(x-a_2)^2\ldots(x-a_r)^2$, each in its lowest terms. To effect the latter it might be said that it is only necessary to take the square of $C(x-a_1)(x-a_2)\ldots(x-a_r)$. It is true the polynomial so formed would contain every pair of equal factors, but not in the lowest terms as regards the coefficients (as we shall presently show).

To solve this last part of the problem, let it be agreed that two rows of letters inclosed in a parenthesis shall indicate the product of the squares of the differences got by subtracting each in the row from each in the other, so that

$$\begin{pmatrix}a\\b\end{pmatrix}=(a-b)^2,\quad \begin{pmatrix}a & \\ b & c\end{pmatrix}=(a-b)^2(a-c)^2,\quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}=(a-c)^2(a-d)^2(b-c)^2(b-d)^2.$$

Let us begin with supposing that fx has one pair only of equal roots; to form the simplest quadratic equation containing this pair, write down

$$(x-e_1)(x-e_2)\times\left(\overline{\underline{e_3, e_4\ldots e_n}}\right)\times\begin{pmatrix}e_1, e_2 & \\ e_3, e_4\ldots e_n\end{pmatrix}.$$

Now if e_1 and e_2 are the two equal roots in question neither of the multipliers of $(x-e_1)(x-e_2)$ vanishes.

If e_1 and e_2 are neither of them equal roots $\left(\overline{\underline{e_3, e_4\ldots e_n}}\right)=0$.

If one of the two only belong to the pair of equal roots

$$\begin{pmatrix}e_1, e_2 & \\ e_3, e_4\ldots e_n\end{pmatrix}=0.$$

Hence it is clear that

$$\Sigma \left\{(x-e_1)(x-e_2) \times \left(\overline{e_3, e_4 \ldots e_n}\right) \times \binom{e_1, e_2}{e_3, e_4 \ldots e_n}\right\} = 0$$

is the equation desired.

In like manner if there be r pairs of equal roots the equation of the $(2r)$th degree which contains them all may be written

$$\Sigma \left\{(x-e_1)(x-e_2) \ldots (x-e_{2r}) \times \left(\overline{e_{2r+1} \ldots e_n}\right) \times \binom{e_1, e_2 \ldots e_{2r}}{e_{2r+1} \ldots e_n}\right\} = 0.$$

The coefficient of x^{2r} in this equation is clearly of

$$(n-2r)(n-2r-1) + 4r(n-2r),$$

that is, of $(n+2r-1)(n-2r)$ dimensions. The coefficient of x^r in the equation which contains the r equal roots unyoked together is of $(n-r)(n-r-1)$ dimensions, and consequently the coefficient of x^{2r} in the square of this equation would be of $2(n-r)(n-r-1)$ dimensions, that is, would be $n^2 + 6r^2 - (4r+1)n$ dimensions higher than needful.

Finally, to obtain an equation clear of *simple* as well as double appearances of the equal roots, we have only to write the complementary form

$$\Sigma \left\{(x-e_{2r+1})(x-e_{2r+2}) \ldots (x-e_n) \times \left(\overline{e_{2r+1} + e_n}\right) \times \binom{e_1, e_2 \ldots e_{2r}}{e_{2r+1} \ldots e_n}\right\} = 0.$$

Let us, now that we are more familiarized with the notation essential to this method, revert to the question with which we set out, and endeavour to obtain r such equations as shall imply *unambiguously* the existence of r pairs of equal roots.

The existence of r such pairs enables us to assert the following disjunctive proposition, which cannot be asserted when the *same amount* of multiplicity is distributed in any other way.

To wit, on selecting any r roots out of the entire number, either these r will all be found again in those that are left, *or* those that are left will contain *inter se*, one repetition at least; so that except on the latter supposition any $(r-1)$ may be absolutely sunk out of those that are left, and there will still be *one* root common to the $(n-2r+1)$ remaining, and to the r originally selected to be left out.

Wherefore calling the roots $e_1, e_2 \ldots e_n$, and giving μ any value whatever, we have

$$\Sigma \left\{\int_\mu (e_1, e_2 \ldots e_r) \times \left(\overline{e_{r+1}, e_{r+2} \ldots e_n}\right) \times \Sigma \binom{e_1, e_2 \ldots e_r}{e_{2r}, e_{2r+1} \ldots e_n}\right\} = 0.$$

Hence the simplest distinctive equations indicative of the existence of r pairs of equal roots are to be found by putting μ equal in succession to all values from 0 up to $(r-1)$.

For instance, if we require that an equation of the seventh degree shall have three pairs of equal roots, we need only to call the seven roots respectively a, b, c, d, e, f, g, and then our type equation becomes

$$\Sigma \left\{ \int_\mu (a\,b\,c) \times \left(\overline{\underline{d\,e\,f\,g}}\right) \times \left\{ \begin{array}{l} \begin{pmatrix} d\,e \\ a\,b\,c \end{pmatrix} + \begin{pmatrix} d\,f \\ a\,b\,c \end{pmatrix} + \begin{pmatrix} d\,g \\ a\,b\,c \end{pmatrix} \\ + \begin{pmatrix} e\,f \\ a\,b\,c \end{pmatrix} + \begin{pmatrix} e\,g \\ a\,b\,c \end{pmatrix} + \begin{pmatrix} f\,g \\ a\,b\,c \end{pmatrix} \end{array} \right\} \right\} = 0.$$

From this it appears that the r distinctive equations for r pairs of equal roots are of different dimensions from the r general or overlying ones corresponding to the multiples r, anyhow distributed; the lowest of the latter being of $(n-r+1)(n-r)$, the lowest of the former of

$$(n-r)(n-r-1) + 2r(n-2r+1),$$

that is, of $n(n-1) - 3r(n-1)$ dimensions. In general we shall find that the more unequally distributed the multiplicity may be the lower are the dimensions of the distinctive equations, and are accordingly lowest when the multiplicity is absolutely undistributed*.

* It must not, however, be overlooked, that the equations above given, although decisive as to the existence of r pairs of equal roots *when* the multiplicity is known to be not greater than r, do not enable us to affirm with certainty their existence when this limitation is absent: for should the multiplicity exceed r, then inevitably (no matter how it may be distributed) $\left(\overline{\underline{e_{r+1}, e_{r+2} \ldots e_n}}\right)$ is always zero, and consequently nullifies each term of every one of the equations in question. In fact (repugnant as it may appear to be to the ordinary assumptions of analytical reasoning), it is not possible to express with *absolute* unambiguity the conditions of there being a multiplicity (r) distributed in any assigned manner by means of r *affirmative equations alone.*

15.

ON A LINEAR METHOD OF ELIMINATING BETWEEN DOUBLE, *TREBLE*, AND OTHER SYSTEMS OF ALGEBRAIC EQUATIONS.

[*Philosophical Magazine*, XVIII. (1841), pp. 425—435.]

PART I. BINARY SYSTEMS.

LET U and V be two integer complete homogeneous functions of x and y, one of the mth, the other of the nth degree; and let it be required to express the condition of the coexistence of the two equations $U=0$, $V=0$ by means of the equation $C=0$, where C is free from all appearances of x or y.

This equation, according to the system of notation developed in a preceding paper, and which has been since adopted and sanctioned by the high authority of M. Cauchy, I call the final derivative: the quantity C is designated the final derivee: and it is our present object to show how this may be obtained in a *prime* form, that is to say, divested of irrelevant factors: in this state it must consist of terms, each containing $m+n$ letters, of which n belong to the coefficients of U, and m to those of V.

Of course in applying this rule it is to be understood that every combination of powers in U or V has a single letter prefixed for its coefficient, and that in the final derivee powers are represented by repetitions of the same character.

Every term in U or V being of the form Cx^py^q, x^py^q is called an argument, C its prefix.

Assume two integer positive numbers r and r', and also two others s and s', such that $r+r'=n-1$, $s+s'=m-1$, and form from $U=0$, $V=0$ two new equations,

$$x^r y^{r'} U=0, \quad x^s y^{s'} V=0.$$

Such equations are termed the augmentatives of the two given ones respectively; also $x^r y^{r'} U$ and its fellow are termed the augmentees of U and V.

r and r' are termed the indices of augmentation belonging to U, s and s' the same belonging to V.

Finally, it will be useful hereafter to call the given polynomials U and V themselves the proposees, and the given equations which assert their nullity, the propositive equations, or, briefly, the propositives.

Now as many augmentees of either proposee can be formed as there are ways of stowing away between two lockers (vacancies admissible) a number of things equal to the index of the other*; hence we shall have n augmentees of U, and m of V: thus there will be $m+n$ augmentatives each of the degree $m+n-1$, and the number of arguments is clearly $m+n$ also, so that they can be eliminated linearly, and the final derivee thus found, containing $m+n$ letters (properly aggregated) in each term, will be in its prime form, that is, incapable of further reduction, and void of irrelevant factors.

It is worthy of remark, that the final derivee obtained by arranging in square battalion the prefixes of the augmentees, permuting the rows or columns, and reading off diagonal products, affected each with the proper sign (according to the well known rule of Duality), will not only be free from factorial irrelevancy, but also of linear redundancy, which latter term I use to signify the reappearance of the same combination of prefixes, sometimes with positive and sometimes with negative signs: furthermore, it follows obviously from the nature of the process that no numerical quantity in the final derivee will be greater than the higher of the indices of the two given polynomials.

Part II. Ternary Systems.

Case A. *Indices all equal.*

Method 1.

Let there be now three proposees, U, V, W, integer complete homogeneous functions of x, y, z, each of the degree n: let

$$r+r'+r''=n-1, \quad s+s'+s''=n-1, \quad t+t'+t''=n-1,$$

$$x^r y^{r'} z^{r''} U, \ x^s y^{s'} z^{s''} V, \ x^t y^{t'} z^{t''} W,$$

will, as above, be called the augmentees of U, V, W, and every other part of the notation previously described is to be preserved.

* "Tot Augmenta utriusvis ex æquationibus propositis formari possunt quot modi sint inter duo receptacula (utrivis vel ambobus omnino vacare licet) rerum, quarum numerus indicem alterius æquat, distributionem faciendi."

Suppose now

$$U = 0, \quad V = 0, \quad W = 0,$$

we shall have as many augmentative equations formed from each proposee as there are ways of stowing away n things between *three* lockers (vacancies admissible)*, that is, $n\frac{n+1}{2}$ of each kind; in all, therefore, $3\frac{n(n+1)}{2}$, and every one of these will be of the degree $2n-1$, so that the number of arguments to be eliminated is equal to the number of ways of stowing away $2n-1$ things between three lockers (empty ones counting), that is

$$\frac{2n(2n+1)}{2}.$$

As yet, then, we have not *enough* equations for eliminating these linearly.

Make, however,

$$\alpha + \beta + \gamma = n + 1,$$

and write

$$U = x^\alpha F + y^\beta F' + z^\gamma F'' = 0,$$
$$V = x^\alpha G + y^\beta G' + z^\gamma G'' = 0,$$
$$W = x^\alpha H + y^\beta H' + z^\gamma H'' = 0,$$

it will always be possible to make the multipliers of x^α, y^β, z^γ integer functions: for if we look to any argument in U, V, or W, it is of the form $x^a y^b z^c$, and one of the letters a, b, c must be *not less* than its correspondent α, β, γ, for otherwise $a+b+c$ would be not greater than $\alpha+\beta+\gamma-3$, that is, n would be not greater than $(n+1)-3$, or $n-2$, which is absurd: if now any one, as a, be equal to or greater than α, it may be made to supply an integer part to the multiplier of x^α.

Here it may be asked what is to be done with such terms as $Kx^a y^b z^c$, when two letters a, b are each not less than their correspondents α, β: the answer is, such terms may be made to enter under the multiplier of x^α, or of x^β, or to supply a part to both in any proportion at pleasure†.

From the equations above we get, by linear elimination,

$$FG'H'' + GH'F'' + HF'G'' - GF'H'' - HG'F'' - FH'G'' = 0.$$

This may be denoted thus: $\Pi(\alpha, \beta, \gamma) = 0$, which equation I call a *secondary* derivative, and the left side of it a *secondary* derivee; α, β, γ may likewise be termed the indices of derivation (as r, s, t, &c. are of augmentation).

Now since $\alpha + \beta + \gamma = n + 1$, it is clear that the index of $\Pi(\alpha, \beta, \gamma)$ is always $n + n + n - (n+1)$; that is, $2n - 1$.

* See for Latin translation the preceding note.

† The prefixes of any such terms (say K) may be conceived as made up of two parts, an arbitrary constant, as e and $(K-e)$; e will disappear spontaneously from the final derivee.

1st. Let any two of the indices of derivation be taken zero, then it is easily seen that all the terms in $\Pi(\alpha, \beta, \gamma)$ vanish, and consequently the secondary derivative equations obtained upon this hypothesis become mere identities, and are of no use.

2nd. Let any *one* of them become zero.

It is manifest, from the doctrine of simple equations, that $\Pi(\alpha, \beta, \gamma)$ may be made equal to

$$\left\{\lambda U + \mu V + \nu W\right\} \frac{1}{x^\alpha},$$

or

$$\left\{\lambda' U + \mu' V + \nu' W\right\} \frac{1}{x^\beta},$$

or

$$\left\{\lambda'' U + \mu'' V + \nu'' W\right\} \frac{1}{x^\gamma},$$

upon the understanding that

$$\begin{array}{lll} \lambda = G'H'' - G''H', & \mu = H'F'' - H''F', & \nu = F'G'' - F''G', \\ \lambda' = G''H - GH'', & \mu' = H''F - HF'', & \nu' = F''G - FG'', \\ \lambda'' = GH' - G'H, & \mu'' = HF' - H'F, & \nu'' = FG' - F'G. \end{array}$$

The three rows of coefficients will be respectively of the degrees

$$(n-\beta)+(n-\gamma), \quad (n-\gamma)+(n-\alpha), \quad (n-\alpha)+(n-\beta).$$

Thus if any one of the indices α, β, γ be zero, $\Pi(\alpha, \beta, \gamma)$ becomes identical with $\lambda^\epsilon U + \mu^\epsilon V + \nu^\epsilon W$, where the multipliers of U, V, W are of $2n - (\alpha + \beta + \gamma)$ dimensions, that is of $(n-1)$ dimensions, and may accordingly be put under the form

$$\Sigma A x^r y^{r'} z^{r''} U + \Sigma B x^s y^{s'} z^{s''} V + \Sigma C x^t y^{t'} z^{t''} W,$$

that is to say, becomes a *linear function* of the augmentatives, and therefore if combined with them in the process of linear elimination would *give rise* to the identity $0 = 0$.

Hence we must reject all such secondary derivatives as have zero for one of the indices of derivation. But all others, it may be shown, will be *linearly* independent of one another, and of the augmentees previously found. Hence, besides $3\frac{n(n+1)}{2}$ equations of augment of the degree $2n-1$, we shall have of the same degree so many equations of derivation as there are ways of stowing away between three lockers $(n+1)$ things, under the condition that no locker shall ever be left *empty*, that is $\frac{n(n-1)}{2}$*.

Thus, then, in all we have $n\frac{n-1}{2} + 3\frac{n(n+1)}{2} = \frac{2n(2n+1)}{2}$ equations, which is exactly equal to the number of arguments to be eliminated. Hence

* Vide page 76 for the Latin version.

the final derivee can be obtained by the usual explicit rule of permutation, and moreover will be *its lowest form,* for it will contain in each term $\frac{n(n+1)}{2}$ prefixes belonging to the augmentatives of U, and a like number pertaining to those of V and of W, as well as $n\frac{n-1}{2}$ belonging to the secondary derivatives, each prefix in any one of which is triliteral, containing a prefix drawn out of those belonging to each of the proposees.

Thus every member containing $n\frac{n+1}{2}+n\frac{n-1}{2}$, that is n^2 of the original prefixes belonging to U, V, W, singly and respectively, the final derivee evolved by this process will be in its lowest terms; as was to be proved.

CASE A. *Indices all equal.*

Method 2.

It is remarkable that we may vary the method just given by making

$$r+r'+r''=n-2, \quad s+s'+s''=n-2, \quad t+t'+t''=n-2.$$

The augmentatives will thus be of the degree $2n-2$.

Furthermore, we must make $\alpha+\beta+\gamma=n+2$. It will still be possible to satisfy by integer multipliers the equations

$$U=x^\alpha F+y^\beta F'+z^\gamma F'',$$
$$V=x^\alpha G+y^\beta G'+z^\gamma G'',$$
$$W=x^\alpha H+y^\beta H'+z^\gamma H'',$$

[these it will be useful in future to term the *equations,* x^α, y^β, z^γ being the *arguments,* and F, G, H, &c. the *factors* of decomposition] for otherwise calling the indices of x, y, z in any original argument a, b, c, their sum or n would be not greater than $(n+2)-3$, that is $(n-1)$, which is absurd.

For the same reasons as in the last case no index of augmentation must be made zero: the degree of each will be $(n-\alpha)+(n-\beta)+(n-\gamma)$, that is $(2n-2)$, and their number $\frac{(n+1)n}{2}$; the number of augmentatives will be $\frac{3(n-1)n}{2}$ linearly uninvolved, each of the degree $2n-2$, and therefore containing $\frac{(2n-1)2n}{2}$ arguments.

Now $$\frac{(n+1)n}{2}+\frac{3(n-1)n}{2}=\frac{(2n-1)2n}{2}.$$

Hence the final derivee may be found, and it will be in its *lowest terms*, for every member will contain $\frac{3(n-1)n}{2}$ letters due to the augmentative, and $\frac{3(n+1)n}{2}$ due to the partial derivative equations; in all then there will be $3n^2$ letters in each term.

This second method being applied to three quadratic equations of the most general form, leads to the problem of eliminating between six simple equations which lies within the limits of practical feasibility, and it is my intention to register the final derivee upon the pages of some one of our scientific Transactions as a standing monument for the guidance of hereafter coming explorers*.

Scholium to Case A.

If we attempt to carry forward these processes to quaternary systems, it becomes necessary to make

$$\alpha+\beta+\gamma+\delta=(r-2)n+1$$

or else

$$\alpha+\beta+\gamma+\delta=(r-2)n+2,$$

where r is the number of proposees.

Now if the factors in the equations of decomposition are all integer, one of the indices of derivation must be not greater than the corresponding index in any of the original arguments, which may easily be shown to be always impossible for a system of equations, *complete* in *all* their terms, whenever their number r is greater than three, if $\alpha+\beta+\gamma+\delta=(r-2)n+2$; but if $\alpha+\beta+\gamma+\delta=(r-2)n+1$ only possible for the case of $n=2$.

Particular method applicable to *four Quadratics.*

Let $U=0$, $V=0$, $W=0$, $Z=0$, be four quadratic equations existing between x, y, z, t.

Make

$$\begin{array}{llll} xU=0, & xV=0, & xW=0, & xZ=0, \\ yU=0, & yV=0, & yW=0, & yZ=0, \\ zU=0, & zV=0, & zW=0, & zZ=0, \\ tU=0, & tV=0, & tW=0, & tZ=0. \end{array}$$

* Elimination between *two* quadratics leads to a final derivee made up of *seven* terms only; the final derivee of *three* quadratics is made up of at least several *thousand*; nay, I believe I may safely say, several *myriads* of terms!

Also write
$$U = x^2F + yF' + zF'' + tF''' = 0,$$
$$V = x^2G + yG' + zG'' + tG''' = 0,$$
$$W = x^2H + yH' + zH'' + tH''' = 0,$$
$$Z = x^2K + yK' + zK'' + tK''' = 0.$$

By eliminating linearly we get
$$\Sigma \{F\Sigma G' (H''K''' - H'''K'')\} = 0,$$
which will be of the third degree, since the factors represented by the *unmarked* letters F, G, H, K are of zero, and all the rest of *unit* dimensions.

Similarly we may obtain other equations, so that besides the *sixteen* augmentatives already written down, we have four secondary derivatives, namely,
$$\Pi(2\ 111) = 0, \quad \Pi(12\ 11) = 0, \quad \Pi(11\ 21) = 0, \quad \Pi(111\ 2) = 0.$$
Thus we have *twenty* equations and as many arguments to eliminate, since a perfect cubic function of four letters contains twenty terms.

The final derivee will contain $16 + 4.4$ letters, that is 32, 8 or 2^3 belonging to each system of original prefixes in each member, and will therefore be in its lowest terms: for one of the canons of form teaches us, *à priori*, that every member of the derivee deduced from any number of assumed equations must contain in each member as many prefixes belonging to one equation of the system as there are units in the product of the indices of all the rest taken together.

Corollary to Case A.

Either of the two methods given as applicable to this case enables us to determine integer values of X, Y, Z, which shall satisfy the equation
$$XU + YV + ZW = Fx^p y^q z^r,$$
where F is the final derivee and $p + q + r = 3n - 2$. For by the doctrine of simple equations we know how to express F in terms of the linear functions, out of which it is obtained by permutation, that is we are able to assign values of A, B, C, and their antitypes, as also of L and its antitype, which shall satisfy the equation
$$\Sigma (Ax^r y^{r'} z^{r''} U) + \Sigma (Bx^s y^{s'} z^{s''} V) + \Sigma (Cx^t y^{t'} z^{t''} W) + \Sigma \{L\Pi(\alpha, \beta, \gamma)\} = Fx^f y^g z^h, \qquad (1)$$
where A, B, C, as well as L and all the quantities formed after them, are made up of integer combinations of the original prefixes.

Now the functions $\Pi(\alpha, \beta, \gamma)$ may be expressed in three ways in terms of U, V, W, as has been already shown.

We may therefore suppose these functions to be divided into three groups, and make

$$\Sigma L \Pi (\alpha\beta\gamma) = \Sigma \frac{QU + Q'V + Q''W}{x^\alpha} + \Sigma \frac{RU + R'V + R''W}{x^\beta} + \Sigma \frac{SU + S'V + S''W}{x^\gamma}. \quad (2)$$

And it is evident that the equations (1) and (2) lead immediately to the equation

$$XU + YV + ZW = Fx^{a+f} y^{b+g} z^{c+h},$$

if we call a, b, c the *greatest values* attributed respectively to α, β, γ.

Now if we suppose the first method to be followed,

$$f + g + h = 2n - 1.$$

And it will always be possible to make a, b, c of what values we please *subject* to the condition of $a + b + c = n - 1$; for *one* at least of the indices of derivation in $\Pi(\alpha, \beta, \gamma)$ must be not greater than its correspondent among a, b, c; otherwise $\alpha + \beta + \gamma$ would be not less than $(a + b + c) + 3$; but

$$\alpha + \beta + \gamma = n + 1$$

$$a + b + c = n - 1,$$

which is absurd.

Hence we can satisfy $XU + YV + ZW = Fx^p y^q z^r$, p, q, r being subject to the condition of $p + q + r = 3n - 2$, but otherwise arbitrary.

Moreover, we can *not* do so if $p + q + r$ be *less* than $3n - 2$, for that would require $a + b + c$ to be less than $n - 1$. Now if two of the indices of derivation, as α and β, be made equal to $a + 1$, $b + 1$ respectively, the third $\gamma = (n + 1) - (a + b + 2) = (n - 1) - (a + b)$, and is therefore greater than c: so that $\alpha + \beta + \gamma$ for this case becomes greater than $a + b + c$, and the method falls to the ground.

In fact, I have discovered a theorem which lets me know this, *à priori*, a law which serves as a staff to guide my feet from falling into error in devising linear methods of solution, and the importance of which all candid judges who have studied the general theory of elimination cannot fail to recognize. To wit, if $X_1, X_2, X_3 \ldots X_n$ be n integer complete polynomial functions of n letters $x_1, x_2 \ldots x_n$, and severally of the degree $b_1, b_2, b_3 \ldots b_n$; then it is always possible to satisfy the identity

$$P_1X_1 + P_2X_2 + P_3X_3 + \ldots + P_nX_n = Fx_1^{a_1}x_2^{a_2}x_3^{a_3} \ldots x_n^{a_n},$$

if $\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n$ be equal to or greater than $b_1+b_2+b_3+\ldots+b_n-n+1$, but otherwise *not**.

This again is founded immediately upon a simple proposition, of which I have obtained a very interesting and instructive demonstration, shortly to appear, and which may be enumerated thus: "*The number of augmentees of the same degree that can be formed, linearly independent of one another, out of any number of polynomial functions of as many variables, may be either equal to or less than the number of distinct arguments contained in such augmentees, but* never *greater. The latter will be the case when the index of the augmentees diminished by unity is less than the sum of the indices of the original unaugmented polynomials each so diminished; the former, when the aforesaid index is equal to or greater than the aforesaid sum.*"

To return to the particular case of finding X, Y, Z to satisfy

$$XU+YV+ZW=Fx^py^qz^r.$$

This has been already done according to the first method; if we employ the *second* method of elimination we shall have

$$f+g+h=2n-2.$$

But, now since $\alpha+\beta+\gamma=n+2$, we shall easily see by the same method as above, that the least value of $a+b+c$ {where a, b, c denote respectively the greatest values of α, β, γ, appearing in the denominator of the fractional *forms* used to express $\Pi(\alpha,\beta,\gamma)$}, will be one greater than before, or n; so that $f+g+h+a+b+c$ will still be equal to $3n-2$, as we might, *à priori*, by virtue of our rule, have been assured.

Ternary Systems.

Case B. *Two of the indices equal; the third* less *by a unit.*

Let $U=0$, $V=0$, $W=0$, be the three given equations severally of the degree n, n, $(n-1)$.

* Hence it is apparent, that in applying the method of multipliers, a curious and important distinction exists between the cases of there being two equations, and there being a greater number to eliminate from: for in the first case the element of arbitrariness needs never to appear; in the *latter* it cannot possibly be excluded from appearing in the multipliers.

This will explain how it comes to pass that the method of the text may be employed to give *various* solutions of the $XU+YV+ZW=Fx^py^qz^r$; thus not only can p, q and r be variously made up of $(f+a)$, $(g+b)$, $(h+c)$, but also $\Pi(\alpha,\beta,\gamma)$ when two of the indices (α, β suppose) are each not greater than the assigned greatest values a, b may be made to figure indifferently either under the form

$$\frac{\lambda U+\mu V+\nu W}{x^\alpha} \text{ or that of } \frac{\lambda' U+\mu' V+\nu' W}{x^\beta}.$$

Make $r+r'+r''=n-2,\quad s+s'+s''=n-2,\quad t+t'+t''=n-1,$ by multiplying U into $x^r y^{r'} z^{r''}$, V into $x^s y^{s'} z^{s''}$, W into $x^t y^{t'} z^{t''}$, we obtain augmentees each of the same, namely, the $(2n-2)$th degree.

The number of these is

$$\frac{(n-1)n}{2}+\frac{(n-1)n}{2}+\frac{n(n+1)}{2}.$$

Again, make $$\alpha+\beta+\gamma=n+1.$$

It will still be possible, as before, to form equations of decomposition in which x^α, y^β, z^γ are the arguments, and affected with *integer* factors. For if we look to W *even*, all its arguments are of the form $x^a y^b z^c$, where $a+b+c=(n-1)$, and each of these cannot be less than its correspondent, for that would be to say that $(n-1)$ is not greater $(n+1)-3$, *à fortiori*, U and V can be decomposed in the manner described. Thus, then, we shall obtain as many secondary derivees as in the last case (Method 1), that is, $\frac{n(n-1)}{2}$ {since $\alpha+\beta+\gamma$ is still equal to $(n+1)$}, as before. Moreover, each of these will be of $(n-\alpha)+(n-\beta)+(n-1-\gamma)$, that is of $2n-2$ dimensions.

Altogether, therefore, we have

$$\left\{\frac{(n-1)n}{2}+\frac{(n-1)n}{2}+\frac{n(n+1)}{2}\right\}+\frac{(n-1)n}{2}$$

linear independent equations of the degree $2n-2$, and the number of arguments to eliminate is $\frac{(2n-1)2n}{2}$. Now these two numbers are equal. Thus we obtain a final derivee containing of U's coefficients $\frac{(n-1)n}{2}+\frac{(n-1)n}{2}$, an equal number of V's, but of W's $\frac{n(n+1)}{2}+\frac{(n-1)n}{2}$; now $n(n-1)$, $n(n-1)$ and n^2 exactly express the number that ought to appear of each of these respectively: hence the final derivee is clear *of irrelevant factors.*

Ternary Systems.

Case C. *Two of the indices equal; the third one* greater *by a unit.*

Here, calling n the highest index, the augmentees must each be made of the degree $(2n-3)$, their number will evidently be

$$\frac{(n-2)(n-1)}{2}+\frac{(n-1)n}{2}+\frac{(n-1)n}{2},$$

making the sum of the indices of derivation now, as before, equal to $(n+1)$; it will be still possible to form integer equations of decomposition, which will give rise to augmentatives of the degree $(n-\alpha)+(n-1)-\beta+(n-1)-\gamma$, that is, of $(2n-3)$ dimensions. The total number of equations, what with augmentatives and secondary derivatives, will be

$$\left\{\frac{(n-2)(n-1)}{2}+\frac{(n-1)n}{2}+\frac{(n-1)n}{2}\right\}+\frac{n(n-1)}{2}=\frac{4n^2-4n+2}{2}=\frac{(2n-2)(2n-1)}{2},$$

that is, is equal to the exact number of distinct arguments contained between them.

Also the final derivative will contain in each member

$$\frac{(n-2)(n-1)}{2}+\frac{n(n-1)}{2},$$

that is, $(n-1)(n-1)$, letters belonging to the first equation, and

$$\frac{(n-1)n}{2}+\frac{n(n-1)}{2},$$

that is, $n(n-1)$ belonging to those of the second and of the third, and will therefore be in its *lowest terms.*

Corollary to Cases B and C.

It is not necessary, after all that has been already said, to do more than just point out that the processes applicable to these cases enable us to determine X, Y, Z, which satisfy the equation

$$XU+YV+ZW=Fx^fy^gz^h,$$

where $$f+g+h=3n-3 \text{ for Case B},$$

and $$f+g+h=3n-4 \text{ for Case C}.$$

16.

MEMOIR ON THE DIALYTIC METHOD OF ELIMINATION. PART I.

[*Philosophical Magazine*, XXI. (1842), pp. 534—539*.]

THE author confines himself in this part to the treatment of two equations, the final and other derivees of which form the subject of investigation.

The author was led to reconsider his former labours in this department of the general theory by finding certain results announced by M. Cauchy in *L'Institut*, March Number of the present year, which flow as obvious and immediate consequences from Mr Sylvester's own previously published principles and method.

Let there be two equations in x,

$$U = ax^n + bx^{n-1} + cx^{n-2} + ex^{n-3} + \&\text{c.} = 0,$$

$$V = \alpha x^m + \beta x^{m-1} + \gamma x^{m-2} + \&\text{c.} \qquad = 0,$$

and let $n = m + \iota$, where ι is zero or any positive value (as may be).

Let any such quantities as $x^r U$, $x^s V$, be termed augmentatives of U or V.

To obtain the derivee of a degree s units lower than V, we must join s augmentatives of U with $s + \iota$ of V. Then out of $2s + \iota$ equations

$$x^0 U = 0, \quad x^1 U = 0, \quad x^2 U = 0, \ldots\ldots x^{s-1} U = 0,$$

$$x^0 V = 0, \quad x^1 V = 0, \quad x^2 V = 0, \ldots\ldots x^{\iota+s-1} V = 0,$$

we may eliminate linearly $2s + \iota - 1$ quantities.

Now these equations contain no power of x higher than $m + \iota + s - 1$; accordingly, all powers of x, superior to $m - s$, may be eliminated, and the derivee of the degree $(m - s)$ obtained in its prime form.

Thus to obtain the final derivee (which is the derivee of the degree zero), we take m augmentatives of U with n of V, and eliminate $(m + n - 1)$ quantities, namely,

$$x, \; x^2, x^3, \ldots\ldots \text{ up to } x^{m+n-1}.$$

* Reprinted from *Proc. Roy. Irish Acad.*, Vol. II. (1840—1844), p. 130.

This process, founded upon the dialytic principle, admits of a very simple modification. Let us begin with the case where $\iota = 0$, or $m = n$. Let the augmentatives of U be termed $U_0, U_1, U_2, U_3, \ldots$ and of V, $V_0, V_1, V_2, V_3, \ldots$, the equations themselves being written

$$U = ax^n + bx^{n-1} + cx^{n-2} + \&c.$$

$$V = a'x^n + b'x^{n-1} + c'x^{n-2} + \&c.$$

It will readily be seen that

$$a'U_0 - aV_0,$$

$$(b'U_0 - bV_0) + (a'U_1 - aV_1),$$

$$(c'U_0 - cV_0) + (b'U_1 - bV_1) + (a'U_2 - aV_2), \&c.$$

will be each linearly independent functions of $x, x^2, \ldots x^{m-1}$, no *higher* power of x remaining. Whence it follows, that to obtain a derivee of the degree $(m - s)$ in its prime form, we have only to employ the s of those which occur first in order, and amongst them eliminate $x^{m-1}, x^{m-2}, \ldots x^{m-s+1}$. Thus, to obtain the *final* derivee, we must make use of n, that is, the entire number of them.

Now, let us suppose that ι is not zero, but $m = n - \iota$. The equation V may be conceived to be of n instead of m dimensions, if we write it under the form

$$0x^n + 0x^{n-1} + 0x^{n-2} + \ldots + 0x^{m+1} + \alpha x^m + \beta x^{m-1} + \&c. = 0,$$

and we are able to apply the same method as above; but as the first ι of the coefficients in the equation above written are zero, the first ι of the quantities

$$(a'U_0 - aV_0), \quad (b'U_0 - bV_0) + (a'U_1 - aV_1), \&c.$$

may be read simply

$$-aV_0, \quad -bV_0 - aV_1, \quad -cV_0 - bV_1 - aV_2, \&c.$$

and evidently their office can be supplied by the simple augmentatives themselves,

$$V_0 = 0, \quad V_1 = 0, \quad V_2 = 0 \ldots V_{\iota-1} = 0;$$

and thus ι letters, which otherwise would be *irrelevant*, fall out of the several derivees.

The author then proceeds with remarks upon the general theory of simple equations, and shows how by virtue of that theory his method contains a solution of the identity

$$X_r U + Y_r V = D_r$$

where D_r is a derivee of the rth degree of U and V, and accordingly, X_r of the form

$$\lambda + \mu x + \nu x^2 + \ldots + \theta x^{m-r-1},$$

and Y_r of the form

$$l + mx + \ldots + tx^{n-r-1},$$

and accounts *à priori* for the fact of not more than $(n-r)$ simple equations being required for the determination of the $(m+n-2r)$ quantities λ, μ, ν, &c. l, m, n, &c., by exhibiting these latter as *known* linear functions of no more than $(n-r)$ unknown quantities left to be determined.

Upon this remarkable relation may be constructed a method well adapted for the expeditious computation of numerical values of the different derivees.

He next, as a point of curiosity, exhibits the values of the secondary functions,

$$a'U_0 - aV_0,$$

$$b'U_0 - bV_0 + a'U_1 - aV_1,$$

$$c'U_0 - cV_0 + b'U_1 - bV_1 + a'U_2 - aV_2, \text{ \&c.}$$

under the form of symmetric functions of the roots of the equations $U=0$, $V=0$, by aid of the theorems developed in the London and Edinburgh *Philosophical Magazine*, December 1839*, and afterwards proceeds to a more close examination of the final derivee resulting from two equations each of the same (any given) degree.

He conceives a number of cubic blocks each of which has two numbers, termed its *characteristics*, inscribed upon one of its faces, upon which the value of such a block (itself called an *element*) depends.

For instance, the value of the *element*, whose *characteristics* are r, s, is the difference between two products: the one of the coefficient rth in order occurring in the polynomial U, by that which comes sth in order in V; the other product is that of the coefficient sth in order of the polynomial U, by that rth in order of V; so that if the degree of each equation be n, there will be altogether $\frac{1}{2}n(n+1)$ such elements.

The blocks are formed into squares or flats (*plafonds*) of which the number is $\frac{n}{2}$ or $\frac{n+1}{2}$, according as n is even or odd. The first of these contains n blanks in a side, the next $(n-2)$, the next $(n-4)$, till finally we reach a square of four blocks or of one, according as n is even or odd. These flats are laid upon one another so as to form a regularly ascending pyramid, of which the two diagonal planes are termed the planes of separation and symmetry respectively. The former divides the pyramid into two halves, such that no element on the one side of it is the same as that of any block in the other. The plane of symmetry, as the name denotes, divides the pyramid into two exactly *similar* parts; it being a rule, that *all elements lying in any given line of a square (plafond) parallel to the plane of separation are identical*; moreover, the sum of the characteristics is the same, for *all* elements lying *anywhere* in a *plane* parallel to that of separation.

[* p. 40 above. Ed.]

All the terms in the final derivee are made up by multiplying n elements of the pile together, under the sole restriction, that no two or more terms of the said product shall lie in any one plane out of the two *sets* of planes perpendicular to the sides of the squares. The *sign* of any such product is determined by the places of either set of planes parallel to a side of the squares and to one another, in which the elements composing it may be conceived to lie.

The author then enters into a disquisition relating to the *number* of terms which will appear in the final derivee, and concludes this first part with the statement of two general canons, each of which affords as many tests for determining whether a prepared combination of coefficients can enter into the final derivee of any number of equations as there are units in that number, but so connected as together only to afford double that number, *less* one, of independent conditions.

The first of these canons refers simply to the number of letters drawn out of *each of the given equations* (supposed homogeneous); the second to what he proposes to call the *weight* of every term in the derivee in respect to *each of the variables* which are to be eliminated.

The author subjoins, for the purpose of conveying a more accurate conception of his Pyramid of derivation, examples of the mode in which it is constructed.

When $n = 1$ there is one flat, viz.

1, 2

When $n = 2$ there is one flat, viz.

2, 3	2, 4
2, 4	3, 4

Let $n = 3$, there will be two flats:

2, 3

1, 2	1, 3	1, 4
1, 3	1, 4	2, 4
1, 4	2, 4	3, 4

Let $n = 4$, there will still be two flats only:

2, 3	2, 4
2, 4	3, 4

1, 2	1, 3	1, 4	1, 5
1, 3	1, 4	1, 5	2, 5
1, 4	1, 5	2, 5	3, 5
1, 5	2, 5	3, 5	4, 5

Let $n = 5$, there will be three flats:

3, 4

2, 3	2, 4	2, 5
2, 4	2, 5	3, 5
2, 5	3, 5	4, 5

1, 2	1, 3	1, 4	1, 5	1, 6
1, 3	1, 4	1, 5	1, 6	2, 6
1, 4	1, 5	1, 6	2, 6	3, 6
1, 5	1, 6	2, 6	3, 6	4, 6
1, 6	2, 6	3, 6	4, 6	5, 6

Let $n = 6$, there will be three flats:

3, 4	3, 5
3, 5	4, 5

2, 3	2, 4	2, 5	2, 6
2, 4	2, 5	2, 6	3, 6
2, 5	2, 6	3, 6	4, 6
2, 6	3, 6	4, 6	5, 6

1, 2	1, 3	1, 4	1, 5	1, 6	1, 7
1, 3	1, 4	1, 5	1, 6	1, 7	2, 7
1, 4	1, 5	1, 6	1, 7	2, 7	3, 7
1, 5	1, 6	1, 7	2, 7	3, 7	4, 7
1, 6	1, 7	2, 7	3, 7	4, 7	5, 7
1, 7	2, 7	3, 7	4, 7	5, 7	6, 7

Thus the work of computation reduces itself merely to calculating $n\frac{n+1}{2}$ elements, or the $n(n+1)$ cross-products out of which they are constituted, and combining them factorially after that law of the pyramid, to which allusion has been already made.

17.

ELEMENTARY RESEARCHES IN THE ANALYSIS OF COMBINATORIAL AGGREGATION.

[*Philosophical Magazine*, XXIV. (1844), pp. 285—296.]

THE ensuing inquiries will be found to relate to combination-systems, that is, to combinations viewed in an aggregative capacity, whose species being given, we shall have to discover rules for ranging or evolving them in classes amenable to certain prescribed conditions. The question of numerical amount will only appear incidentally, and never be made the primary object of investigation*.

The number of things combined will be termed the modulus of the system to which they belong. The elements taken singly, or combined in twos, threes, &c., will be denominated accordingly the monadic, duadic. triadic elements, or simply the monads, duads, or triads of the system.

Let us agree to denote by the word syntheme† any aggregate of combinations in which all the monads of a given system appear once, and once only.

It is manifest that many such synthemes totally diverse in every term may be obtained for a given system to any modulus, and for any order of combination.

Let us begin with considering the case of duad synthemes. Take the modulus 4 and call the elements a, b, c, d.

(ab, cd), (ac, bd), (ad, cb) constitute three perfectly independent synthemes, and these three synthemes include between them all the duad elements, so that no more independent synthemes can be obtained from them.

* The present theory may be considered as belonging to a part of mathematics which bears to the combinatorial analysis much the same relation as the geometry of position to that of measure, or the theory of numbers to computative arithmetic; number, place, and combination (as it seems to the author of this paper) being the three intersecting but distinct spheres of thought to which all mathematical ideas admit of being referred.

† From σὺν and τίθημι.

Again, let a, b, c, d, e, f be the monads; we can write down five independent synthemes, to wit,

$$\left.\begin{matrix} ab, & cd, & ef \\ ad, & cf, & eb \\ ac, & de, & fb \\ af, & bd, & ce \\ ae, & df, & bc \end{matrix}\right\}.$$

We can write no more than these without repeating duads which have already appeared*.

We propose to ourselves this problem:—*A system to any even† modulus being given, to arrange the whole of its duads‡ in the form of synthemes*; or in other words, *to evolve a Total of duad synthemes to any given even modulus* §.

When the modulus is odd, as before remarked, the formation of a duad syntheme is of course impossible, for any number of duads must necessarily contain an even number of monadic elements; but there is nothing to prevent us from forming in *all* cases what may be termed a bisyntheme or diplotheme, that is, an aggregate of combinations, where each element occurs twice and no more.

For instance, if the elements be called after the letters of the alphabet, we have $\begin{pmatrix} ab, & bc, & cd, & de, & ea \\ ac, & ce, & eb, & bd, & da \end{pmatrix}$, the bisynthematic total to modulus 5; and in

* Such an *aggregate* of synthemes may be therefore termed a Total.

† The modulus must be even, as otherwise it is manifest no single syntheme can be formed. We shall before long extend the scope of our inquiry so as to take in the case of odd moduli.

‡ Triadic systems will be treated of hereafter.

§ It is scarcely necessary to advert here to the fact of the problem being in general indeterminate and admitting of a great variety of solutions.

When the modulus is four there is only one synthematic arrangement possible, and there is no indeterminateness of any kind; from this we can infer, *à priori*, the reducibility of a biquadratic equation; for using ϕ, f, F to denote rational symmetrical forms of function, it follows that

$$F\left\{\begin{matrix} f\{\phi(a, b), & \phi(c, d)\} \\ f\{\phi(a, c), & \phi(b, d)\} \\ f\{\phi(a, d), & \phi(b, c)\} \end{matrix}\right\} \text{ is itself a rational symmetric function of } a, b, c, d.$$

Whence it follows that if a, b, c, d be the roots of a biquadratic equation, $f\{\phi(a, b), \phi(c, d)\}$ can be found by the solution of a cubic: for instance, $(a+b)\times(c+d)$ can be thus determined, whence immediately the sum of any two of the roots comes out from a quadratic equation.

To the modulus 6 there are fifteen different synthemes capable of being constructed; at first sight it might be supposed that these could be classed in natural families of three or of five each, on which supposition the equation of the sixth degree could be depressed; but on inquiry this hope will prove to be futile, not but what natural affinities do exist between the totals; but in order to separate them into families each will have to be taken twice over, or in other words, the fifteen synthemes to modulus 6 being reduplicated subdivide into six natural families of five each. Again, it is true that the triads to modulus 6 (just like the duads to modulus 4) admit of being thrown into but one synthematic total, but then this will contain ten synthemes, a number greater than the modulus itself.

like manner

$$\left.\begin{matrix} ab, & bc, & cd, & de, & ef, & fg, & ga \\ ac, & ce, & eg, & gb, & bd, & df, & fa \\ ad, & dg, & gc, & cf, & fb, & be, & ea \end{matrix}\right\}\text{ the total to modulus 7.}$$

In general, if n be the modulus, the number of duads is $n\frac{n-1}{2}$; n being even, $\frac{n}{2}$ duads go to each syntheme, and therefore the total contains $(n-1)$ of these. If n be odd, then, since always n duads go to a bisyntheme, the number of such in the total is $\frac{n-1}{2}$.

Before proceeding to the solution of the problem first proposed, let us investigate the theory of diplothematic arrangement. Here we shall find another term convenient to employ. By a cyclotheme, I designate a fixed arrangement of the elements in one or more circles, in which, although for typographical purposes they are written out in a straight line, the last term is to be viewed as contiguous and antecedent to the first; the recurrence may be denoted by laying a dot upon the two opened ends of the circle; $\dot{a}.b.c.d.\dot{e}$ will thus denote a cyclotheme to modulus 5; $\dot{a}.b.c.d.e.f.g.h.\dot{k}$ the same to modulus 9; so also is $\dot{a}.b.\dot{c}$, $\dot{d}.e.\dot{f}$, $\dot{g}.h.\dot{k}$ a cyclotheme of another species to the same modulus. In general the number of terms will be alike in each division of a cyclotheme.

Now it is evident that every cyclotheme, on taking together the elements that lie in conjunction, may be developed into a diplotheme. Thus

$$\dot{1}.2.\dot{3} = 12,\ 23,\ 31,$$

$$\dot{1}.2.3.\dot{4} = 12,\ 23,\ 34,\ 41,$$

$$(\dot{1}.2.\dot{3};\ \dot{4}.5.\dot{6};\ \dot{7}.8.\dot{9}) = \begin{pmatrix} 12, & 23, & 31 \\ 45, & 56, & 64 \\ 78, & 89, & 97 \end{pmatrix}.$$

Hence we shall derive a rule for throwing the duads of any system into bisynthemes.

Let $m = 3$, we have simply $\dot{a}b\dot{c}$,

$m = 5$, we write $\dot{a}.b.c.d.\dot{e}$,

$\dot{a}.c.e.b.\dot{d}$,

the second being derived from the first by omitting every alternate term; similarly below, the lines are derived each from its antecedent.

$m = 7$, we have $\dot{a}.b.c.d.e.f.\dot{g}$,

$\dot{a}.c.e.g.b.d.\dot{f}$,

$\dot{a}.e.b.f.c.g.\dot{d}$.

A very little consideration will serve to prove that in this way, m being a *prime number*, $\frac{m-1}{2}$ cyclothemes may be formed, such that no element will ever be found more than once in contact on either side with any other; whence the rule for obtaining the diplothematic total to any prime-number modulus is apparent.

For example, to modulus 7 the total reads thus:—

$$\left.\begin{array}{ll} \text{1st.} & ab,\ bc,\ cd,\ de,\ ef,\ fg,\ ga \\ \text{2nd.} & ac,\ ce,\ eg,\ gb,\ bd,\ df,\ fa \\ \text{3rd.} & ae,\ eb,\ bf,\ fc,\ cg,\ gd,\ da \end{array}\right\},$$

and no more remains to be said on this special case.

Let us now return to the theory of even moduli, and show how to apply what has been just done to constructing a synthematic total to a modulus which is the double of a prime number.

Suppose the modulus to be six, the number of synthemes is five. Let the six elements, a, b, c, d, e, f, be taken in three parts, so that each part contains two of them; let these parts be called A, B, C, where A denotes ab, B, cd, and C, ef.

Now the duads will evidently admit of a distinction into two classes, those that lie in one part, and those that lie between two; thus ab, cd, ef will be each *unipartite* duads, the rest will be *bipartite*.

The unipartite duads may be conveniently formed into a syntheme by themselves; it only remains to form the four remaining bipartite duad synthemes.

Write the parts in cyclothematic order, as below:

$$\dot{A}B\dot{C}.$$

It will be observed that each part may be written in two positions; thus

$$\begin{array}{llll} A \text{ may be expressed by} & \begin{matrix} a \\ b \end{matrix} & \text{or by} & \begin{matrix} b \\ a \end{matrix}, \\ B \quad ,, \quad ,, & \begin{matrix} c \\ d \end{matrix} & ,, & \begin{matrix} d \\ c \end{matrix}, \\ C \quad ,, \quad ,, & \begin{matrix} e \\ f \end{matrix} & ,, & \begin{matrix} f \\ e \end{matrix}. \end{array}$$

Now we may form a cyclic table of positions as below:

$\dot{A}$	B	$\dot{C}$
1	1	1
1	2	2
2	1	2
2	2	1

Here the numbers in each horizontal line denote the synchronic positions of the parts.

On inspection it will be discovered that A will be found in each of its two positions, with B in each of its two; similarly B with C, and C with A. In fact the four permutations, 11, 12, 21, 22, occur, though in different orders, in any two assigned vertical columns.

Now develope the preceding table, and we have

$$\dot{a}c\dot{e} \quad \dot{a}d\dot{f} \quad \dot{b}c\dot{f} \quad \dot{b}d\dot{e},$$
$$bdf \quad bce \quad ade \quad acf;$$

and these being read off (the *superior* of each antecedent with the *inferior* of each consequent*) must manifestly give the four independent bipartite synthemes which we were in quest of, *videlicet*

$$(ad,\ cf,\ eb),\quad (ac,\ de,\ fb),\quad (bd,\ ce,\ fa),\quad (bc,\ df,\ ea);$$

these four, together with the syntheme first described (ab, cd, ef), constitute a duad synthematic total to modulus 6.

Before proceeding further let us take occasion to remark that the foregoing table of positions may evidently be extended to any odd number of terms by repetition of the second and third places, as seen in the annexed tables of position.

$$\dot{1}\,.\,1\,.\,1\,.\,1\,.\,\dot{1} \qquad \dot{1}\,.\,1\,.\,1\,.\,1\,.\,1\,.\,1\,.\,\dot{1},$$
$$\dot{1}\,.\,2\,.\,2\,.\,2\,.\,\dot{2} \qquad \dot{1}\,.\,2\,.\,2\,.\,2\,.\,2\,.\,2\,.\,\dot{2}\dagger,$$
$$\dot{2}\,.\,1\,.\,2\,.\,1\,.\,\dot{2} \qquad \dot{2}\,.\,1\,.\,2\,.\,1\,.\,2\,.\,1\,.\,\dot{2},$$
$$\dot{2}\,.\,2\,.\,1\,.\,2\,.\,\dot{1} \qquad \dot{2}\,.\,2\,.\,1\,.\,2\,.\,1\,.\,2\,.\,\dot{1}.$$

Now let 10 be the modulus.

As before divide the elements into five parts, which call A, B, C, D, E.

The unipartite duads fall into a single syntheme; the eight remaining bipartite synthemes may be found as follows:—

Arrange in cyclothemes $\left(\frac{n-1}{2} \text{ in number}\right)$ the odd modulus system A, B, C, D, E. We have thus

$$\dot{A}BCD\dot{E},$$
$$\dot{A}CEB\dot{D}.$$

* Any other *fixed* order of successive conjunction would answer equally well.

† It will not fail to be borne in mind that in operating with these tables only *contiguous* elements are taken in conjunction: the first with the second, the second with the third, the third with the fourth, &c., and the last with the first; no two terms but such as lie together are in any manner conjugated with one another.

Let each cyclotheme be taken in the four positions given in the table above, we have thus 2×4, that is, eight arguments.

$$\dot{a}\,b\,c\,d\,\dot{e}\,.\,\dot{a}\,\beta\,\gamma\,\delta\,\dot{\epsilon}\,.\,\dot{\alpha}\,b\,\gamma\,d\,\dot{\epsilon}\,.\,\dot{\alpha}\,\beta\,c\,\delta\,\dot{e},$$

$$\alpha\,\beta\,\gamma\,\delta\,\epsilon,\quad \alpha\,b\,c\,d\,e,\quad a\,\beta\,c\,\delta\,e,\quad a\,b\,\gamma\,d\,\epsilon,$$

$$\dot{a}\,c\,e\,b\,\dot{d}\,.\,\dot{a}\,\gamma\,\epsilon\,\beta\,\dot{\delta}\,.\,\dot{\alpha}\,c\,\epsilon\,b\,\dot{\delta}\,.\,\dot{\alpha}\,\gamma\,e\,\beta\,d,$$

$$\alpha\,\gamma\,\epsilon\,\beta\,\delta,\quad \alpha\,c\,e\,b\,d,\quad a\,\gamma\,e\beta\,d,\quad a\,c\,\epsilon\,b\,\delta.$$

And each of these arguments will furnish one bipartite syntheme, by reading off, as before, the *superior* of each antecedent with the *inferior* of each consequent; and the least reflection will serve to show that the same duad can never appear in two distinct arguments.

In like manner, if the modulus be 14 and seven parts be taken, the bipartite synthemes, twelve in number, may be expressed symbolically thus:

$$\left\{\begin{array}{l} \ \dot{1}.1.1.1.1.1.\dot{1} \\ +\ \dot{1}.2.2.2.2.2.\dot{2} \\ +\ \dot{2}.1.2.1.2.1.\dot{2} \\ +\ \dot{2}.2.1.2.1.2.\dot{1} \end{array}\right\} \times \left\{\begin{array}{l} \ \dot{A}.B.C.D.E.F.\dot{G} \\ +\ \dot{A}.C.E.G.B.D.\dot{F} \\ +\ \dot{A}.E.B.F.C.G.\dot{D} \end{array}\right\}.$$

Nay more, from the above table, if we agree to name the elements $\begin{matrix} A_1 & B_1 \\ A_2 & B_2 \end{matrix}$, &c., we can at once proceed to calculate each of the twelve synthemes in question by an easy algorithm. For instance,

$$(\dot{1}.2.2.2.2.2.\dot{2}) \times (\dot{A}.C.E.G.B.D.\dot{F})$$

$$= (A_1C_1,\ C_2E_1,\ E_2G_1,\ G_2B_1,\ B_2D_1,\ D_2F_1,\ F_2A_2).$$

And again

$$(\dot{2}.1.2.1.2.1.\dot{2}) \times (\dot{A}.E.B.F.C.G.\dot{D})$$

$$= A_2E_2,\ E_1B_1,\ B_2F_2,\ F_1C_1,\ C_2G_2,\ G_1D_1,\ D_2A_1;$$

each figure occurring once unchanged as an antecedent and once changed as a consequent.

If it were thought worth while it would not be difficult, by using numbers instead of letters, to obtain a general analytical formula, from which all similarly constituted synthemes to any modulus might be evolved.

But the rule of proceeding must be now sufficiently obvious; the modulus being $2p$, we divide the elements into p classes; these may be arranged into $\frac{p-1}{2}$ distinct forms of cyclothematic arrangement, and each of the cyclothemes taken in four positions, thus giving $4 \times \frac{p-1}{2}$, that is, $2p - 2$ bipartite synthemes, the whole number that can be formed to the given modulus $2p$.

I shall now proceed to the theory of bipartite synthemes to the modulus $2m \times p$, by which it is to be understood that we have p parts each containing $2m$ terms, and p is at present supposed to be a prime number; the total number of synthemes to the modulus $2mp$ being $2mp - 1$, and $2m - 1$ of these evidently being capable of being made unipartite; the remainder, $2mp - 2m$, that is, $(p-1)\,2m$, will be the number of bipartites to be obtained*:

$$2m\,(p-1) = \frac{p-1}{2} \times 4m;$$

$\frac{p-1}{2}$ denotes the total number of cyclothemes to modulus p; $4m$, as will be presently shown, the number of lines or syzygies in the *Table of position.*

To fix our ideas let the modulus be 4×3, and let A, B, C be three parts:

$$\left.\begin{matrix} a_1\, a_2\, a_3\, a_4 \\ b_1\, b_2\, b_3\, b_4 \\ c_1\, c_2\, c_3\, c_4 \end{matrix}\right\} \text{ their constituents respectively.}$$

Give a *fixed* order to the constituents of each part, then each of them may be taken in four positions; thus A may be written

$$\begin{gathered} a_1\, a_2\, a_3\, a_4, \\ a_2\, a_3\, a_4\, a_1, \\ a_3\, a_4\, a_1\, a_2, \\ a_4\, a_1\, a_2\, a_3. \end{gathered}$$

Assume some particular position for each, as, for instance,

$$\begin{gathered} a_1\, b_1\, c_1, \\ a_2\, b_2\, c_2, \\ a_3\, b_3\, c_3, \\ a_4\, b_4\, c_4, \end{gathered}$$

and read off by coupling the first and third vertical places of each antecedent with the second and fourth respectively of each consequent; we have accordingly,

$$\begin{gathered} a_1b_2,\ b_1c_2,\ c_1a_2, \\ a_3b_4,\ b_3c_4,\ c_3a_4. \end{gathered}$$

It is apparent that the same combinations will recur if any two contiguous parts revolve simultaneously through two steps; or in other words, that $A_rB_s = A_{r+2}B_{s+2}$, where μ is any number, odd or even.

* In general, if there be π parts of μ terms each, and $\mu\pi$ be even, the number of bipartite synthemes is $(\pi - 1)\,\mu$, as is easily shown from dividing the whole number of bipartite duads by the semi-modulus.

Symbolically speaking, therefore, as regards our table of position,

$$r : s = r + 2 : s + 2,$$

or more generally,

$$= r + 2 \pm 4i : s + 2 \pm 4i.$$

So that

$$\begin{array}{ll} 1:1 = 3:3, & 2:1 = 4:3, \\ 1:2 = 3:4, & 2.2 = 4:4, \\ 1:3 = 3.1, & 2.3 = 4:1, \\ 1:4 = 3.2, & 2.4 = 4:2. \end{array}$$

There are therefore no more than eight independent unequivalent permutations to every pair of parts. Now inspect the following table of position:—

$$\begin{array}{ll} \dot{1}.1.\dot{1}, & \dot{2}.1.\dot{2}, \\ 1.2.3, & 2.2.4, \\ 1.3.2, & 2.3.1, \\ 1.4.4, & 2.4.3. \end{array}$$

It will be seen that in the first and second, second and third, third and first places, all the eight independent permutations occur under different *names*; the law of formation of such and similar tables will be explained in due time; enough for our present object to see how, by means of this table, we are able to obtain the bipartite synthemes to the given modulus 4×3; the number according to our formula is $2 \times 4 \times \frac{3-1}{2} = 8$, and they may be denoted symbolically as follows:—

$$(\dot{A}.B.\dot{C})\begin{pmatrix} 1.1.1 + 1.2.3 + 1.3.2 + 1.4.4 \\ + 2.1.2 + 2.2.4 + 2.3.1 + 2.4.3 \end{pmatrix}.$$

Each of the eight terms connected by the sign of + gives a distinct syntheme; for example, let us operate on

$$\dot{A}.B.\dot{C} \times (2.3.1).$$

$2.3.1$ denotes $2.3,\ 3.1,\ 1.2$.

2.3 gives rise to $2(3+1) + (2+2).(3+3) = 2.4 + 4.2$.

3.1 gives rise to $3(1+1) + (3+2).(1+3) = 3.2 + 1.4$.

1.2 gives rise to $1(2+1) + (1+2).(2+3) = 1.3 + 3.1$.

The syntheme in question is therefore

$$A_2B_4,\ A_4B_2,\ B_3C_2,\ B_1C_4,\ C_1A_3,\ C_3A_1,$$

and so on for all the rest, the rule being that

$$r : s = r(s+1) + (r+2)(s+3).$$

Now, as before, it is evident that if we look only to contiguous terms, the above table of position may be extended to any number of odd terms, simply by repetition of the second and third figures in each syzygy; and hence the rule for obtaining the bipartite synthemes to the modulus $4 \times p$ is apparent. For instance, let $p = 7$, there will be $8 \times \frac{7-1}{2}$, that is, 8×3 of them denoted as follows:—

$$\left\{\begin{matrix} \dot{A}.B.C.D.E.F.\dot{G} \\ +\dot{A}.C.E.G.B.D.\dot{F} \\ +\dot{A}.E.B.F.C.G.\dot{D} \end{matrix}\right\} \times \left\{\begin{matrix} 1.1.1.1.1.1.1+2.1.2.1.2.1.2 \\ +1.2.3.2.3.2.3+2.2.4.2.4.2.4 \\ +1.3.2.3.2.3.2+2.3.1.3.1.3.1 \\ +1.4.4.4.4.4.4+2.4.3.4.3.4.3 \end{matrix}\right\}.$$

As an example of the mode of development, let us take the term

$$\dot{A}.E.B.F.C.G.\dot{D} \times \dot{2}.4.3.4.3.4.\dot{3},$$

$$\dot{2}.4.3.4.3.4.\dot{3} = (2:4,\ 4:3,\ 3:4,\ 4:3,\ 3:4,\ 4:3,\ 3:2)$$

$$= \left(\begin{matrix}2.1\\+4.3\end{matrix}\right\} + \left.\begin{matrix}4.4\\2.2\end{matrix}\right\} + \left.\begin{matrix}3.1\\1.3\end{matrix}\right\} + \left.\begin{matrix}4.4\\2.2\end{matrix}\right\} + \left.\begin{matrix}3.1\\1.3\end{matrix}\right\} + \left.\begin{matrix}4.4\\2.2\end{matrix}\right\} + \left.\begin{matrix}3.3\\1.1\end{matrix}\right\}\Bigg),$$

$$\dot{A}.E.B.F.C.G.\dot{D} = A.E,\ E.B,\ B.F,\ F.C,\ C.G,\ G.D,\ D.A,$$

and the product

$$= \begin{pmatrix} A_2E_1, & E_4B_4, & B_3F_1, & F_4C_4, & C_3G_1, & G_4D_4, & D_3A_3 \\ A_4E_3, & E_2B_2, & B_1F_3, & F_2C_2, & C_1G_3, & G_2D_2, & D_1A_1 \end{pmatrix}.$$

Let the modulus be 6×3; as before, give a *fixed* cyclic order to the constituents of each part, and each will admit of being exhibited in six positions.

Write similarly as before,

$$\begin{matrix} \dot{a}_1\, b_1\, \dot{c}_1, \\ a_2\, b_2\, c_2, \\ a_3\, b_3\, c_3, \\ a_4\, b_4\, c_4, \\ a_5\, b_5\, c_5, \\ a_6\, b_6\, c_6, \end{matrix}$$

and take the odd places of each antecedent with the even places of each consequent; it will now be seen that

$$r:s = r+2:s+2 = r+4:s+4,$$

and the number of independent permutations is $\frac{6.6}{3} = 2.6$; and so in general, if there be $2m$ constituents in a part, the number of independent permutations is $\frac{2m.2m}{\frac{m}{2}} = 4m$.

The rule for the formation of the table will be apparent on inspection. I suppose only three parts, as the rule may always be extended to any number by reiteration of the second and third terms. The table will be found to resolve itself naturally into four parts, each containing m lines.

Let $m = 1$, we have

1.1.1	2.1.2
1.2.2	2.2.1

$m = 2$, we have

1.1.1	2.1.2
1.2.3	2.2.4
1.3.2	2.3.1
1.4.4	2.4.3

$m = 3$, we have

1.1.1	2.1.2
1.2.3	2.2.4
1.3.5	2.3.6
1.4.2	2.4.1
1.5.4	2.5.3
1.6.6	2.6.5

$m = 4$, we have

1.1.1	2.1.2
1.2.3	2.2.4
1.3.5	2.3.6
1.4.7	2.4.8
1.5.2	2.5.1
1.6.4	2.6.3
1.7.6	2.7.5
1.8.8	2.8.7

So that x, going through all its values from 1 to m, the general expression for the four parts is

$$\Sigma \begin{Bmatrix} 1\,.\,x\,(2x-1) + 1\,(m+x)\,2x \\ +\,2\,.\,x\,.\,2x \quad + 2\,(m+x)\,(2x-1) \end{Bmatrix}.$$

To show the use of this formula, let us suppose that we have seven parts, each containing ten terms, the general expression for the bipartite duad synthemes is

$$\begin{Bmatrix} \dot{A}.B.C.D.E.F.\dot{G} \\ +\,A.C.E.G.B.D.F \\ +\,A.E.B.F.C.G.D \end{Bmatrix} \times \Sigma \begin{Bmatrix} 1\,.\,x\,(2x-1)\,x\,(2x-1)\,x\,(2x-1) \\ +\,2\,.\,x\,.\,2x\,.\,x\,.\,2x\,.\,x\,.\,2x \\ +\,1\,(5+x)\,2x\,(5+x)\,2x\,(5+x)\,2x \\ +\,2\,(5+x)(2x-1)\,(5+x)(2x-1)\,(5+x)\,(2x-1) \end{Bmatrix}.$$

Make, for example, $x = 3$, one of the synthemes in question out of the twelve corresponding to this value will be

$$A.C.E.G.B.D.F \times 2.3.6.3.6.3.6.$$

Here

$$A.C.E.G.B.D.F = AC,\ CE,\ EG,\ GB,\ BD,\ DF,\ FA,$$

$\dot{2}.3.6.3.6.3.\dot{6} =$

$$= \left.\begin{array}{r} 2.4 \\ +\ 4.6 \\ +\ 6.8 \\ +\ 8.10 \\ +10.2 \end{array}\right\} \left.\begin{array}{r} 3.7 \\ +5.9 \\ +7.1 \\ +9.3 \\ +1.5 \end{array}\right\} \left.\begin{array}{r} 6.4 \\ +\ 8.6 \\ +10.8 \\ +\ 2.10 \\ +\ 4.2 \end{array}\right\} \left.\begin{array}{r} 3.7 \\ +5.9 \\ +7.1 \\ +9.3 \\ +1.5 \end{array}\right\} \left.\begin{array}{r} 6.4 \\ +\ 8.6 \\ +10.8 \\ +\ 2.10 \\ +\ 4.2 \end{array}\right\} \left.\begin{array}{r} 3.7 \\ +5.9 \\ +7.1 \\ +9.3 \\ +1.5 \end{array}\right\} \left.\begin{array}{r} 6.3 \\ +\ 8.5 \\ +10.7 \\ +\ 2.9 \\ +\ 4.1 \end{array}\right\}$$

and the product

$$= A_2C_4,\ C_3E_7,\ E_6G_4,\ G_3B_7,\ B_6D_4,\ D_3F_7,\ F_6A_3$$
$$A_4C_6,\ C_5E_9,\ E_8G_6,\ G_5B_9,\ B_8D_6,\ D_5F_9,\ F_8A_5,$$
$$\&c. \qquad \&c. \qquad \&c.$$

To *prove* the rule for the table of formation, it will be sufficient to show that no two contiguous duads ever contain the same or *equivalent* permutations; the equation of equivalence it will be remembered is

$$r : s = r + 2i \pm 2m : s + 2i \pm 2m.$$

Now, as regards the first and second terms, it is manifest that $1 : x$ cannot be equivalent, either to $1 : x'$ nor to $2 : x$, nor to $2 : x'$, where x' is any number differing from x.

Similarly, as regards the last and first terms, $x : 1$ cannot be equivalent to $x' : 1$, nor to $x : 2$, nor to $x' : 2$; therefore there is no danger as far as the first term is concerned, either as antecedent or consequent.

Again, it is clear that $x : (2x - 1)$ cannot interfere with $x' : 2x'$, nor $(m + x) : 2x$ with $(m + x') : (2x' - 1)$; neither can $(2x - 1) : x$ with $2x' : x'$, nor $2x : (m + x)$ with $(2x' - 1) : (m + x')$.

Again, if possible, let

$$x : (2x - 1) = (m + x') : (2x' - 1);$$

then

$$m + x' - x = 2i,$$

and

$$2x' - 2x = 2i,$$

therefore

$$2m = 2i,$$

or

$$m = i,$$

which is impossible, since $+i$ is the difference between two indices, each less than m.

Similarly,

$$m + x : 2x \text{ cannot} = x' : 2x',$$

and *vice versâ* with the terms changed

$$2x : (m + x) \text{ cannot} = 2x' : x',$$

and

$$(2x - 1) : x \text{ cannot} = (2x' - 1) : (m + x'),$$

which proves the rule for the table of formation.

So much for the bipartite duad synthemes. As regards the unipartite synthemes little need be said, for every part may be treated as a separate system, and as each will produce an equal number of synthemes, these being taken one with another, will furnish just as many unipartite synthemes of the whole system as there are synthemes due to each part. Thus then the synthematic resolution of the modulus $2m \times p$ may be made to depend on the synthematization of $2m$ and the cyclothematization of p. This has been already shown (whatever m may be) for the case of p being a prime number; but I proceed now to extend the rule to the more general case of p being any number whatever.

18.

ON THE EXISTENCE OF ABSOLUTE CRITERIA FOR DETERMINING THE ROOTS OF NUMERICAL EQUATIONS.

[*Philosophical Magazine*, XXV. (1844), pp. 442—445.]

I WISH to indicate in this brief notice a fact which I believe has escaped observation hitherto, that there exist, certainly in some cases, and probably in all, infallible criteria for determining whether a given equation has all its roots rational or not.

In the equation of the second degree it is enough, in order that this may be the case, that the expression for the square of the difference of the roots shall be a perfect square; in other words, if $x^2 - px + q = 0$ have its roots *rational*, $p^2 - 4q$ must be not only a *positive* number (the condition of the roots being real), but that number must also be a complete square. In this case it is further evident that p must be either prime to q, or if not, the greatest common measure of p^2 and q must be a perfect square; but this condition is contained in the former, which is a sufficient criterion in itself.

If we now consider the equation of the third degree,

$$x^3 - px^2 + qx - r = 0,$$

one condition is, that the product of the squared differences shall be a perfect square; in other words, the equation cannot have all its roots rational unless

$$p^2q^2 - 4q^3 - 18pqr - 4p^3r - 27r^2$$

be a positive square number.

This remark is made at the end of the second supplement of Legendre's *Theory of Numbers*, and is indeed self-evident; and in like manner one condition may be obtained for an equation of any degree which is to have all its roots rational; but this is far from being the sole condition required.

In the equation of the third degree, however, one other condition, conjoined with that above expressed, will serve to determine infallibly whether all the roots are rational or not.

To obtain this condition, let us suppose that by making $3x = y + p$ we obtain the equation

$$y^3 - Qx - R = 0.$$

Calling the three roots of this new equation α, β, γ (all of which it is evident must be rational if those of the first equation are so), we have

$$\alpha + \beta + \gamma = 0,$$

$$Q = -(\alpha\beta + \alpha\gamma + \beta\gamma) = \alpha^2 + \alpha\beta + \beta^2,$$

$$R = \alpha\beta\gamma.$$

From the last two equations it is easily seen that if k be any prime factor common to Q and R, k^2 will be contained in Q, and k^3 in R; or, in other words, k will be a common measure of α, β, γ.

We have therefore a *second condition*, that $9q - 3p^2$ shall be a negative quantity, which is either prime to $2p^3 - 9qp + 27r$, or else so related to it, that the greatest common measure of the cube of the first and the square of the second is a perfect sixth power.

I now proceed to show the converse, that if these two conditions be both satisfied (and it will appear in the course of the inquiry that the first does *not* involve the second), the roots cannot help being all rational.

It is evident that the two conditions in question are tantamount to supposing that the roots of the proposed equation are linearly connected with those of another $z^3 - Qz - R = 0$ (by virtue of the assumption $3x = kz + p$), where Q may be considered as *prime* to R; and where $4Q^3 - 27R^2$ is a perfect square.

Let now $4Q^3 - 27R^2 = D^2$, then $D^2 + 27R^2 = 4Q^3$, or $D^2 + 3(3R)^2 = 4Q^3$.

Here, as Q is prime to R, D can have no common measure but 3, with $3R$.

Firstly, let Q be prime to $3R$.

Then putting $f^2 + 3g^2 = Q^3$, the complete solution of the equation immediately preceding is contained in the two systems:

1st. $D = 2f$, $3R = 2g$.

2nd. $D = (f \pm 3g)$, $3R = f \mp g$,

and for both systems,

$$f \pm g\sqrt{(-3)} = \{h \pm 3k\sqrt{(-3)}\}^3.$$

The second system must therefore be rejected, for g evidently contains 3, and therefore $f = 3R \pm g$ will contain 3, and therefore D and therefore Q will do the same, contrary to supposition.

Hence

$$\sqrt[3]{\left[\frac{R}{2} \pm \sqrt{\left\{-\left(\frac{Q^3}{27} - \frac{R^2}{4}\right)\right\}}\right]}$$

$$= \sqrt[3]{\left\{\frac{R}{2} \pm \frac{D}{2}\sqrt{\left(-\frac{1}{27}\right)}\right\}}$$

$$= \sqrt[3]{\left\{\frac{g}{3} + f\sqrt{\left(-\frac{1}{27}\right)}\right\}}$$

$$= \mp \frac{1}{3\sqrt{(-3)}} \sqrt[3]{\{f \pm g\sqrt{(-3)}\}}$$

$$= -K \pm \frac{h}{3\sqrt{(-3)}} = \lambda \pm \mu\sqrt{(-3)};$$

and the three roots of the equation being

$$\begin{cases} \{\lambda + \mu\sqrt{(-3)}\} + \{\lambda - \mu\sqrt{(-3)}\}, \\ \dfrac{1 \pm \sqrt{(-3)}}{2}\{\lambda + \mu\sqrt{(-3)}\} + \dfrac{1 \mp \sqrt{(-3)}}{2}\{\lambda - \mu\sqrt{(-3)}\}, \end{cases}$$

will evidently be all rational, which of course includes the necessity of their being also integer.

Again, secondly, if we suppose that Q does contain 3, D^2 will contain 27, and consequently D will contain 9; and we shall have

$$R^2 + 3\left(\frac{D}{9}\right)^2 = 4\left(\frac{Q}{3}\right)^3.$$

Here R being prime to $\frac{D}{9}$, it may be shown, as in the last case, that the complete solution is

$$\frac{R}{2} \pm \frac{D}{18}\sqrt{(-3)} = \{h \pm k\sqrt{(-3)}\}^3,$$

consequently

$$\sqrt[3]{\left\{\frac{R}{2} \pm \sqrt{\left(\frac{R^2}{4} - \frac{Q^3}{27}\right)}\right\}} = h \pm k\sqrt{(-3)};$$

and the three roots of the equation are

$$2h, \quad h - 3k, \quad h + 3k$$

respectively, and are therefore all rational.

Here it may be observed that the condition of R being an even number, which we know, *à priori*, is the case when all the roots are rational, is involved in the two more general conditions already expressed. It will now be evident that the first condition by no means involves the second, as it is perfectly easy to satisfy the equation $f^2 + 3g^2 = Q^3$ without supposing anything relative to k, the common measure of f, g, Q, except that it be itself of the form $\lambda^2 + 3\mu^2$, which will give

$$\left(\frac{f}{k}\right)^2 + 3\left(\frac{g}{k}\right)^2 = (\lambda^2 + 3\mu^2)(r^2 + 3s^2)^3,$$

an equation which can be solved in rational terms for all values of λ, μ, r, s; and consequently the product of the squares of the differences of the roots may be a square, and at the same time the roots themselves may be irrational*.

I believe it will be found on inquiry that the equation $x^n - qx + r = 0$ will always have two *rational* roots if

$$(n-1)^{n-1} . q^n - n^n . r^{n-1}$$

be a complete square, provided that q be prime to r.

Furthermore, viewing the striking analogy of the general nature of the conditions of rationality already obtained, to those which serve to determine the reality of the roots of equations, I am strongly of opinion that a theorem remains to be discovered, which will enable us to pronounce on the existence of integer, as Sturm's theorem on that of possible roots of a complete equation of any degree: the analogy of the two cases fails however in this respect, that while imaginary roots enter an equation in pairs, irrational roots are limited to entering in groups, each containing *two* or MORE.

* Thus then it appears that the *total* rationality of the roots of the equation $x^3 - qx - r = 0$ may be determined by a direct method without having recourse to the method of divisors to determine the roots themselves; the two conditions being that $4q^3 - 27r^2$ shall be a perfect *square*, and the greatest common measure of q^3 and r^2 a perfect *sixth* power.

19.

AN ACCOUNT OF A DISCOVERY IN THE THEORY OF NUMBERS RELATIVE TO THE EQUATION $Ax^3 + By^3 + Cz^3 = Dxyz$.

[*Philosophical Magazine*, XXXI. (1847), pp. 189—191.]

First General Theorem of Transformation.

If in the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz, \qquad (1)$$

A and B are equal, or in the ratio of two cube numbers to one another, and if $27ABC - D^3$ (which I shall call the Determinant) is free from all single or square prime positive factors of the form $6n+1$, but without exclusion of *cubic* factors of such form, and if A and B are each odd, and C the double or quadruple of an odd number, or if A and B are each even and C odd, then, I say, the given equation may be made to depend upon another of the form

$$A'u^3 + B'v^3 + C'w^3 = D'uvw;$$

where

$$A'B'C' = ABC,$$

$$D' = D,$$

$$uvw = \text{some factor of } z.$$

The following are some of the consequences which I deduce from the above theorem. In stating them it will be convenient to use the term Pure Factorial to designate any number into the composition of which no single or square prime positive factor of the form $6n+1$ enters.

The equations

$$x^3 + y^3 + 2z^3 = Dxyz,$$

$$x^3 + y^3 + 4z^3 = Dxyz,$$

$$2x^3 + 2y^3 + z^3 = Dxyz,$$

are insoluble in integer numbers, provided that the Determinant in each case is a Pure Factorial.

The equation

$$x^3 + y^3 + Az^3 = 9Bxyz$$

is insoluble in integer numbers, provided that the Determinant, for which in this case we may substitute $A - 27B^3$, is a pure factorial whenever A is of the form $9n \pm 1$, and equal to $2p^{3i\pm 1}$ or $4p^{3i\pm 1}$, p being any prime number whatever.

I wish however to limit my assertion as to the insolubility of the equations above given. The theorem from which this conclusion is deduced does not preclude the possibility of two of the three quantities x, y, z being taken positive or negative *units*, either in the given equation itself or in one or the other of those into which it may admit of being transformed. Should such values of two of the variables afford a particular solution, then instead of affirming that the equations are insoluble, I should affirm that the *general solution* can be obtained by equations in finite differences*.

Second General Theorem of Transformation.

The equation

$$f^3x^3 + g^3y^3 + h^3z^3 = Kxyz \tag{2}$$

may always be made to depend upon an equation of the form

$$Au^3 + Bv^3 + Cw^3 = Duvw,$$

where

$$ABC = R^3 - S^3,$$

$$D = 3R;$$

and $uvw =$ some factor of $fx + gy + hz$.

$$R \text{ representing } K + 6fgh,$$

$$S \quad \text{,,} \quad K - 3fgh.$$

* Take for instance the equation $x^3+y^3+2z^3=9xyz$. The Determinant 27.25 is a Pure Factorial: consequently if the solution be possible, since in this case the transformed must be identical with the given equation, this latter must be capable of being satisfied by making x and y positive or negative units. Upon trial we find that $x=1$, $y=1$, $z=2$ will satisfy the equation. I believe, but have not fully gone through the work of verification, that these are the only possible values (prime to one another) which will satisfy the equation. Should they not be so, my method will infallibly enable me to discover and to give the law for the formation of all the others.

Here, then, under any circumstances, is an example, the first on record, of the complete resolution of a numerical equation of the third degree between three variables.

I have not leisure to show the consequences of this theorem of transformation in connexion with the one first given, but shall content myself with a single numerical example of its applications:

$$x^3 + y^3 + z^3 = -6xyz$$

may be made to depend on the equation

$$u^3 + v^3 + w^3 = 0,$$

and is therefore insoluble.

It is moreover apparent that the Determinant of equation (2) transformed is in general $-27R^3$, and is therefore always a Pure Factorial, and consequently the equation

$$f^3x^3 + g^3y^3 + h^3z^3 = Kxyz$$

will be itself insoluble, being convertible into an insoluble form, provided that $K + 6fgh$ is divisible by 9, and provided further that $(K + 6fgh)^3 - (K - 3fgh)^3$ belongs to the form m^3Q, where Q is of the form $9n \pm 1$, and also of one or the other of the two forms $2p^{3i\pm1}$, $4p^{3i\pm1}$, p being any prime number whatever.

Pressing avocations prevent me from entering into further developments or simplifications at this present time.

It remains for me to state my reasons for putting forward these discoveries in so imperfect a shape. They occurred to me in the course of a rapid tour on the continent, and the results were communicated by me to my illustrious friend M. Sturm in Paris, who kindly undertook to make them known on my part to the Institute.

Unfortunately, in the heat of invention I got confused about the law of oddness and evenness, to which the coefficients of the given equation are in the first theorem *generally* (in order for the successful application of my method as far as it is yet developed) required to be subject. I stated this law erroneously, and consequently drew erroneous conclusions from my Theorems of Transformation, which I am very anxious to seize the earliest opportunity of correcting. I venture to flatter myself that as opening out a new field in connexion with Fermat's renowned Last Theorem, and as breaking ground in the solution of equations of the third degree, these results will be generally allowed to constitute an important and substantial accession to our knowledge of the Theory of Numbers.

20.

ON THE EQUATION IN NUMBERS $Ax^3 + By^3 + Cz^3 = Dxyz$, AND ITS ASSOCIATE SYSTEM OF EQUATIONS.

[*Philosophical Magazine,* XXXI. (1847), pp. 293—296.]

In the last Number of this *Magazine* I gave an account of a remarkable transformation to which the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz$$

is subject when certain conditions between the coefficients A, B, C, D are satisfied; which conditions I shall begin by expressing with more generality and precision than I was enabled to do in my former communication.

1. Two of the quantities A, B, C are to be to one another in the ratio of two cubes.

2. $27ABC - D^3$ must contain no positive prime factor whatever of the form $6n+1$. I erred in my former communication in not excluding cubic factors of this form.

3. If 2^m is the highest power of 2 which enters into ABC, and 2^n the highest power of 2 which enters into D, then either m must be of the form $3n \pm 1$, or if not, then m must be greater than $3n$.

These three conditions being satisfied, the given equation can always be transformed into another,

$$A'u^3 + B'v^3 + C'w^3 = D'uvw,$$

where

$$A'B'C' = ABC, \quad D' = D, \quad uvw = \text{a factor of } z.$$

The consequence of this is, as stated in my former paper, that wherever A, B, C, D, besides satisfying the conditions above stated, are taken so as likewise to satisfy the condition,—firstly, of ABC being equal to $2^{3m\pm1}$, or secondly, of ABC being equal to $2^{3m\pm1} . p^{3n\pm1}$, provided in the second case that ABC is of the form $9m \pm 1$, and that D is divisible by 9, p being in both cases a prime, then the given equation will be *generally* insoluble. And I am now enabled to add that the *only* solution of which it will in any case admit, is the solitary one found by making two of the terms Ax^3, By^3, Cz^3 equal to one another; so that, for instance, if the given equation should be of the form

$$x^3+y^3+ABCz^3=Dxyz,$$

then the above conditions being satisfied, the one solitary solution of which the equation can possibly admit, is $x=1$, $y=1$,

$$Az^3-Dz+2=0,$$

which may or may not have possible roots. I call this a *solitary* or singular solution, because it exists alone and no other solution can be deduced from it; whereas in general I shall show that any one solution of the equation

$$Ax^3+By^3+Cz^3=Dxyz$$

can be made to furnish an infinity of other solutions independent of the one supposed given, that is, not reducible thereto by expelling a common factor from the new system of values of x, y, z deduced from the given system.

The following is the Theorem of Derivation in question:

Let

$$A\alpha^3+B\beta^3+C\gamma^3=D\alpha\beta\gamma.$$

Then if we write

$$F=A\alpha^3,\qquad G=B\beta^3,\qquad H=C\gamma^3,$$

and make

$$x=F^2G+G^2H+H^2F-3FGH,$$

$$y=FG^2+GH^2+HF^2-3FGH,$$

$$z=\frac{1}{D}\{F^3+G^3+H^3-3FGH\},$$

or

$$=\alpha\beta\gamma\{F^2+G^2+H^2-FG-FH-GH\},$$

we shall have

$$x^3+y^3+ABCz^3=Dxyz.$$

I am hence enabled to show that whenever $x^3+y^3+Az^3=Dxyz$ is insoluble, there will be a whole family of allied equations equally insoluble. For instance, because $x^3+y^3+z^3=0$ is insoluble in integer numbers, I know likewise that

$$x^6+y^6+z^6=x^3y^3+x^3z^3+y^3z^3$$

$$x^6+y^6+z^6=x^3y^3+x^3z^3-2y^3z^3$$

are each equally insoluble.

In fact

$$\begin{aligned}(x^3+y^3+z^3)\times(x^6+y^6+z^6-x^3y^3-x^3z^3-y^3z^3)&\\ \times(x^6+y^6+z^6-x^3y^3-x^3z^3+2y^3z^3)&\\ \times(x^6+y^6+z^6-y^3z^3-y^3x^3+2x^3z^3)&\\ \times(x^6+y^6+z^6-x^3z^3-z^3y^3+2y^3x^3)&\\ =u^3+v^3+w^3,&\end{aligned}$$

where u, v, w are rational integral functions of x, y, z.

Hence each of the factors must be incapable of becoming zero*.

As a particular instance of my general theory of transformation and elevation, take the equation

$$x^3+y^3+2z^3=Mxyz.$$

Then, with the exception of the singular or solitary solution $x=1$, $y=1$, of which I take no account, I am able to affirm that for all values of M between 7 and -6, both inclusive, with the exception of $M=-2$, the equation is insoluble in integer numbers.

Take now the equation where $M=-2$, namely

$$x^3+y^3+2z^3+2xyz=0.$$

One particular solution of this is

$$x=1, \quad y=-1, \quad z=1.$$

Another, which I shall call the second†, is

$$x=1, \quad y=3, \quad z=-2.$$

From the first solution I can deduce in succession the following:

$$\begin{array}{lll} x=11, & y=5, & z=-7, \\ x=-793269121, & y=1179490001, & z=-1189735855, \\ \&c. & \&c. & \&c. \end{array}$$

From the second,

$$\begin{array}{lll} x=-10085, & y=8921, & z=-8442, \\ x=\&c. & y=\&c. & z=\&c. \end{array}$$

As another example, take the equation

$$x^3+y^3+6z^3=6xyz.$$

* It is however sufficiently evident from their intrinsic form, which may be reduced to $\frac{1}{4}(M^2+3N^2)$, that this impossibility exists for all the factors except the first.

† See Postscript.

One solution of the transformed equation

$$u^3+2v^3+3w^3=6uvw$$

is evidently

$$u=1, \quad v=1, \quad w=1.$$

Hence I can deduce an infinite series of solutions of the given equation, of which the first in order of ascent will be

$$x=5, \quad y=7, \quad z=3.$$

Again, the lowest possible solution in integers of the equation

$$x^3+y^3+6z^3=0$$

will be

$$x=17, \quad y=37, \quad z=-21.$$

The equation

$$x^3+y^3+9z^3=0$$

admits of the solutions

$$x=1, \quad y=2, \quad z=-1,$$

$$x=-271, \quad y=919, \quad z=-438.$$

I trust that my readers will do me the justice to believe that I am in possession of a strict demonstration of all that has been here advanced without proof. Certain of the writer's friends on the continent have, in their comments upon one of his former papers which appeared in this *Magazine*, complimented his powers of divination at the expense of his judgment, in rather gratuitously assuming that the author of the Theory of Elimination was unprovided with the demonstrations, which he was too inert or too beset with worldly cares and distractions to present to the public in a sufficiently digested form. The proof of whatever has been here advanced exists not merely as a conception of the author's mind, but fairly drawn out in writing, and in a form fit for publication.

P.S. It must not be supposed that the two primary or basic solutions above given of the equation

$$x^3+y^3+2z^3+2xyz=0,$$

namely,

$$x=1, \quad y=-1, \quad z=1,$$

$$x=1, \quad y=3, \quad z=-2,$$

are independent of one another. The second may be derived from the first, as I shall show in a future communication. In fact there exist *three* independent processes, by combining which together, one particular solution may be made to give rise to an infinite series of infinite series of infinite series of correlated solutions, which it may possibly be discovered contain between them the *general* complete solution of the equation

$$x^3+y^3+Az^3=Dxyz.$$

21.

ON THE *GENERAL* SOLUTION (IN CERTAIN CASES) OF THE EQUATION $x^3 + y^3 + Az^3 = Mxyz$, &c.

[*Philosophical Magazine*, XXXI. (1847), pp. 467—471.]

I shall restrict the enunciation of the proposition I am about to advance to much narrower limits than I believe are necessary to the truth, with a view to avoid making any statement which I may hereafter have occasion to modify. Let us then suppose in the equation

$$x^3 + y^3 + Az^3 = Mxyz$$

that A is a *prime* number, and that $27A - M^3$ is *positive*, but exempt from positive prime factors of the form $6i + 1$. Then I say, and have succeeded in demonstrating, that all the possible solutions in integer numbers of the given equation may be obtained by explicit processes from one particular solution or system of values of x, y, z, which may be called the Primitive system.

This system of roots or of values of x, y, z is that system in which the value of the greatest of the three terms x, y, $A^{\frac{1}{3}}.z$ (which may be called the Dominant) is the least possible of all such dominants. I believe that in general the system of the least *Dominant* is identical with the system of the least *Content*, meaning by the latter term the product of the three terms out of which the *Dominant* is elected. I proceed to show the law of derivation.

To express this simply, I must premise that I shall have to employ such an expression as $S' = \phi(S)$ to indicate, not that a certain quantity, S', is a function of S, but that a certain system of quantities disconnected from one another, denoted by S', are severally functions of a certain other system of quantities denoted by S; and, as usual, I shall denote $\phi\phi S$ by $\phi^2 S$, $\phi\phi^2 S$ by $\phi^3 S$, and so forth.

Let now P be the Primitive system of solution of the equation

$$x^3 + y^3 + Az^3 = Mxyz,$$

P denoting a certain system of values of and written in the order of the letters x, y, z, which may always be found by a limited number of trials (provided that the equation admits of any solution). That this is the case is obvious, since we have only to give the Dominant every possible value from the integer next greatest to $A^{\frac{1}{3}}$ upwards, and combine the values of x^3, y^3, Az^3 so that none shall ever exceed at each step the cube of such dominant, and we must at last, if there *exist any solution*, arrive at the System of the Least Dominant.

Now, every system of solution is of one or the other of two characters. Either x and y must be odd and z even, or x and y must be one odd and the other even and z odd. That all three should be odd is inconsistent with the given conditions as to A being odd and M even; and if all three were even, by driving out the common factor we should revert to one or the other of the foregoing cases.

The systems of solution where z is even may be termed Reducible, those where z is odd Irreducible. Let ϕ denote a certain symbol of transformation hereafter to be explained.

Then the Reducible systems of the first order may be expressed by

$$\phi P,\ \phi^2 P,\ \phi^3 P,\ \textit{ad infinitum};$$

or in general by $\phi^{n_1}P$, n_1 being absolutely arbitrary. I will anticipate by stating that the function ϕ involves no *variable* constants; that is to say, $\phi(S)$ may be found explicitly from S without any reference to the particular equation to which S belongs. Let now ψ denote another symbol of transformation, also hereafter to be defined, and differing from ϕ insofar as it does involve as *constants* the three values of x, y, z contained in P: then the general representations of Irreducible systems of the first order will be denoted by $\psi\phi^{n_1}P$.

It is proper to state here that the symbol ψ is ambiguous; and $\psi\phi^{n_1}P$, when P and n_1 are given, will have two values, according to the way in which the terms represented by P are compared with x, y, z in the given equation

$$x^3+y^3+Az^3=Mxyz;$$

for it is obvious that if $x=a$, $y=b$, $z=c$ satisfies the equation, so likewise will

$$x=b, \quad y=a, \quad z=c.$$

Each however of these values of $\psi\phi^{n_1}P$ gives a solution of the kind above designated.

Proceeding in like manner as before, the Reducible system of the second order may be designated by $\phi^{n_2}.\psi\phi^{n_1}.P$, the Irreducible by $\psi\phi^{n_2}.\psi\phi^{n_1}.P$; and in general *every possible system* of values of x, y, z satisfying the proposed equation, in which z is even, is comprised under the form

$$\phi^{n_r}.\psi\phi^{n_{r-1}}.\psi\ldots\phi^{n_2}.\psi\phi^{n_1}.P;$$

and every possible system of such values, in which z is odd, is comprised under the form

$$\psi\phi^{n_r}.\psi\phi^{n_{r-1}}.\psi...\psi\phi^{n_1}.P:$$

the quantities $n_1, n_2 ... n_r$ being of course all independent of one another, and unlimited in number and value.

Thus then we may be said to have the general solution of the given equation in the same sense as an arbitrary sum of terms, each of a certain form, is in certain cases accepted as the complete solution of a partial differential equation.

As regards the value of the symbols ψ and ϕ, ϕ indicates the process by which a, b, c becomes transformed into α, β, γ, the relations between the two sets of elements being contained in the following equations:

$$a' = a^3, \quad b' = b^3, \quad c' = Ac^3,$$

$$\alpha = a'^2b' + b'^2c' + c'^2a' - 3a'b'c',$$

$$\beta = a'b'^2 + b'c'^2 + c'a'^2 - 3a'b'c',$$

$$\gamma = abc\,\{a'^2 + b'^2 + c'^2 - a'b' - a'c' - b'c'\}.$$

Next, as to the effect of the Duplex symbol ψ. Let e, g, ι be the elements of the Primitive system P: ι being the value of z and e, g of x and y taken in either mode of combination, each with each, which satisfy the proposed equation

$$x^3 + y^3 + Az^3 = Mxyz.$$

Let l, m, n represent any system S,

λ, μ, ν represent any system $\psi(S)$,

ψS has two values, which we may denote by $\psi'S$, $'\psi S$ respectively, and accentuating the elements λ, μ, ν accordingly to correspond, we shall have

$$\lambda' = 3gm\,(gl - em) + 3A\iota n\,(\iota l - en) - M\,(g\iota l^2 - e^2lm),$$

$$\mu' = 3A\iota n\,(\iota m - gl) + 3el\,(em - gl) - M\,(e\iota m^2 - g^2lm),$$

$$\nu' = 3el\,(en - \iota l) + 3gm\,(gn - \iota m) - M\,(egn^2 - \iota^2lm):$$

we have then

$$\psi'S \equiv \lambda', \mu', \nu',$$

and in like manner

$$'\psi S \equiv {}'\lambda, {}'\mu, {}'\nu,$$

$'\psi S$ being derived from $\psi'S$ by the mere interchange of e and g one with the other.

I have stated that every possible solution of the proposed equation comes under one or the other of the orders, infinite in number and infinite to the power of infinity in variety of degree, above given: this is not strictly true, unless we understand that all systems of solution are considered to be equivalent which differ only in a multiplier common to all three terms of each; that is to say, which may be rendered identical by the expulsion of a common factor. So that $m\alpha$, $m\beta$, $m\gamma$ as a system is treated as identical with α, β, γ, which of course substantially it is; and it should be remarked that there is nothing to prevent the operations denoted by ϕ and ψ introducing a common factor into the systems which they serve to generate, and the latter in particular will have a strong tendency so to do.

I believe that this theorem may be extended with scarcely any modification to the case where A, instead of being a prime, is any power of the same, and to suppositions still more general. I believe also that, subject to certain very limited restrictions, the theorem *may* prove to apply to the case where the determinant $27A-M^3$ becomes negative.

The peculiarity of this case which distinguishes it from the former, is that it admits of all the three variables x, y, z in the equation

$$x^3+y^3+Az^3=Mxyz$$

having the same sign, which is impossible when the determinant is positive; or in other words, the curve of the third degree represented by the equation $Y^3+X^3+1=\frac{M}{A^{\frac{1}{3}}}XY$ (in which I call the coefficient of XY the characteristic), which, as long as the quantity last named is less than 3, is a single continuous curve extending on both sides to infinity, as soon as the characteristic becomes equal to 3 assumes to itself an isolated point, the germ of an oval or closed branch, which continues to swell out (always lying apart from the infinite branch) as the characteristic continues indefinitely to increase.

I ought not to omit to call attention to the fact that the theorem above detailed is always applicable to the case of the equation

$$x^3+y^3+Az^3=0,$$

when A is *any* power of a prime number *not* of the form $6i+1$; in other words, the above always belongs to the class of equations having Monogenous solutions, which for the sake of brevity may be termed themselves Monogenous Equations*.

* Thus the equation $x^3+y^3+9z^3=0$ alluded to by Legendre is Monogenous, and the Primitive system of solution is $x=1$, $y=2$, $z=-1$, from which every other possible solution in Integers may be deduced.

On the probable existence of such a class of equations I hazarded a conjecture at the conclusion of my last communication to this *Magazine*. As I hope shortly to bring out a paper on this subject in a more complete form, I shall content myself at this time with merely stating a theorem of much importance to the completion of the theory of insoluble and of Monogenous equations of the third degree; to wit, that the equation in integers

$$a\,(x^3 + y^3 + z^3) + c\,(x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2) + exyz = 0$$

may always be transformed so as to depend upon the equation

$$fu^3 + gv^3 + hw^3 = (6a - e)\,uvw,$$

wherein $$fgh = ae^2 - (c^2 + 3a^2)\,e + 9a^2 - 3ac^2 - 2c^3.$$

By means of the above theorem, among other and more remarkable consequences, we are enabled to give a theory of the irresoluble and monogenous cases of the equation

$$x^3 + y^3 + m^3z^3 = Mxyz,$$

when m is some power of 2, or of certain other numbers.

22.

ON THE INTERSECTIONS, CONTACTS, AND OTHER CORRELATIONS OF TWO CONICS EXPRESSED BY INDETERMINATE COORDINATES.

[*Cambridge and Dublin Mathematical Journal*, v. (1850), pp. 262—282.]

LET $U=0$, $V=0$ be two homogeneous equations of the second degree with real coefficients, between the same three variables ξ, η, ζ.

The direct and most general mode of determining the intersections of the conics expressed by these equations would be to make

$$a\xi + b\eta + c\zeta = t,$$

$$a'\xi + b'\eta + c'\zeta = u:$$

eliminating ξ, η, ζ between the four equations in which they appear, there results a biquadratic equation between t and u. The nature of the intersections will depend upon the nature of the roots of this biquadratic; and thus the conditions may be expressed analytically, which will represent the several cases of all the intersections being real or all imaginary, or one pair real and the other imaginary. These analytical conditions will depend upon the signs of certain functions of the coefficients of the given and the *assumed* equations being of an assigned character; my endeavour has been to obtain conditions of a character perfectly symmetrical and free from the coefficients arbitrarily introduced.

In this research I have only partially succeeded, but the method employed, and some of the collateral results, will, I think, be found of sufficient interest to justify their appearance in the pages of this *Journal*.

Adopting Mr Cayley's excellent designation, let the four points of intersection of the two conics be called a quadrangle. This quadrangle will have three pairs of sides; the intersections of each pair, from principles of analogy, I call the vertices of the quadrangle. Then, inasmuch as the four sets of ratios $\xi : \eta : \zeta$, corresponding with the four sets of the ratio $t : u$, must be so related that we may always make

$$\frac{\xi_1}{\zeta_1} = a + b\sqrt{(-1)}, \quad \frac{\eta_1}{\zeta_1} = c + d\sqrt{(-1)},$$

$$\frac{\xi_2}{\zeta_2} = a - b\sqrt{(-1)}, \quad \frac{\eta_2}{\zeta_2} = c - d\sqrt{(-1)},$$

$$\frac{\xi_3}{\zeta_3} = \alpha + \beta\sqrt{(-1)}, \quad \frac{\eta_3}{\zeta_3} = \gamma + \delta\sqrt{(-1)},$$

$$\frac{\xi_4}{\zeta_4} = \alpha - \beta\sqrt{(-1)}, \quad \frac{\eta_4}{\zeta_4} = \gamma - \delta\sqrt{(-1)},$$

we may easily draw the following conclusions.

If all the four points of the quadrangle of intersection are real, the three vertices and the three pairs of sides are all real. If only two points of the quadrangle are real, one vertex and one of the three pairs of sides will be real; the other two vertices and two pairs of sides being imaginary. If all four points of the quadrangle are unreal, one pair of sides will be real and the other two pairs imaginary, as in the last case; but all the three vertices will remain real, as in the first case. Hence we have a direct and simple criterion for distinguishing the case of *mixed* intersection from intersection wholly real or wholly imaginary; namely, that the cubic equation of the roots of which the coordinates of the vertices are real linear functions shall have *a pair of imaginary roots.* This is the sole and unequivocal condition required.

The equation in question is, or ought to be, well known to be the determinant in respect to ξ, η, ζ of $\lambda U + \mu V$. In fact, if we write

$$U = a\xi^2 + b\eta^2 + c\zeta^2 + 2a'\eta\zeta + 2b'\zeta\xi + 2c'\xi\eta,$$

$$V = \alpha\xi^2 + \beta\eta^2 + \gamma\zeta^2 + 2\alpha'\eta\zeta + 2\beta'\zeta\xi + 2\gamma'\xi\eta,$$

$$\lambda U + \mu V = (a\lambda + \alpha\mu)\,\xi^2 + \&\text{c.} = A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta,$$

the ratios of the coordinates ξ, η, ζ of the vertex of $\lambda U + \mu V$ may easily be shown to be identical with

$$AB - C^2 : C'A' - B'B : B'C' - A'A,$$

and will be real or imaginary as $\lambda : \mu$ is one or the other.

If then the cubic equation in $\lambda : \mu$, namely, $\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V) = 0$, has a pair of imaginary roots, that is, if $\underset{\lambda\mu}{\square}\,\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ is a positive quantity, the intersections of U and V are of a mixed kind, that is, the two conics have two real points in common.

I may remark here, *en passant*, that if we form the biquadratic equation in t and u, $\phi(t, u) = 0$ from the equations

$$U = 0,$$
$$V = 0,$$
$$a\xi + b\eta + c\zeta = t,$$
$$a'\xi + b'\eta + c'\zeta = u,$$

and if any reducing cubic of this equation be $P(\theta, \omega) = 0$, the determinant of $P(\theta, \omega)$ must, from what has been shown above, be identical with $\underset{\lambda\mu}{\square}\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ multiplied by some *squared function* of the extraneous coefficients

$$a, b, c; \; a', b', c'.$$

If $\square\square(\lambda U + \mu V)$ is a negative quantity, it remains to distinguish between the cases of the conics intersecting really in four points or not at all.

The most obvious mode of proceeding to distinguish between purely real and purely imaginary intersections would be as follows. Let λ_1, μ_1; λ_2, μ_2; λ_3, μ_3, be the three sets of values of λ, μ which satisfy the equation

$$\square(\lambda U + \mu V) = 0$$

and make

$$\begin{array}{lll} A_1 = a\lambda_1 + \alpha\mu_1, & A_2 = a\lambda_2 + \alpha\mu_2, & A_3 = a\lambda_3 + \alpha\mu_3, \\ C_1 = c\lambda_1 + \gamma\mu_1, & C_2 = c\lambda_2 + \gamma\mu_2, & C_3 = c\lambda_3 + \gamma\mu_3, \\ B_1' = b'\lambda_1 + \beta'\mu_1, & B_2' = b'\lambda_2 + \beta'\mu_2, & B_3' = b'\lambda_3 + \beta'\mu_3, \\ A_1C_1 - B_1'^2 = e_1, & A_2C_2 - B_2'^2 = e_2, & A_3C_3 - B_3'^2 = e_3. \end{array}$$

Now if the equation

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta = 0$$

represent a pair of straight lines, it may be thrown into the form

$$Au^2 + \frac{AC - B'^2}{A} v^2 = 0,$$

where u and v are linear functions of ξ, η, ζ, and the straight lines will be real or imaginary, according as $B'^2 - AC$ is positive or negative; hence one or else all of the quantities e_1, e_2, e_3, will be necessarily negative, and the intersections will be all real or all imaginary, according as all three are negative or only one is so. A cubic equation in e may be formed containing e_1, e_2, e_3 as its roots by eliminating between the equations

$$e = AC - B'^2; \quad \square(\lambda U + \mu V) = 0,$$

and the conditions for the reality of the intersections will be that all four coefficients of this cubic shall be of the same sign, which in reality amount only to two, since the first and last must in all cases have the same sign.

The same objection however of want of symmetry and consequent irrelevancy and complexity attaches to this as much as to the method originally proposed. The following treatment of the question relieves the objection of want of symmetry as far as the coefficients of the same equation are concerned, but in its practical application necessitates an arbitrary and therefore unsymmetrical election to be made between the two sets of coefficients appertaining to the two equations. It is however, I think, too curious and suggestive to be suppressed.

I observe that if the four intersections are all real, an imaginary conic cannot be drawn through them; for the equation to an imaginary conic may always be reduced to the form $Ax^2 + By^2 + Cz^2 = 0$, where A, B, C are all positive and can therefore have at utmost one real point. Consequently the case of total non-intersection is distinguishable from that of complete intersection by the peculiarity that in the one case μ may be so taken that $U + \mu V = 0$ shall represent an imaginary conic, that is, $U + \mu V$ will be a function whose sign never changes for real values of ξ, η, ζ, whereas in the latter case no value of μ will make $U + \mu V = 0$ the equation to an imaginary conic, and therefore $U + \mu V$ will have values on both sides of zero. On the other hand, it is obvious that an infinite number of real as well as unreal conics may be drawn through four imaginary points of intersection. Consequently if we make $U + \mu V = 0$ (supposing the intersections of U and V to be imaginary), there will be a range or ranges of values of μ consistent, and another range or ranges of values of μ inconsistent with real values of ξ, η, ζ; in other words, $U \pm \mu V = 0$ treated as an equation between the four variables ξ, η, ζ, μ, will give one or more maxima or minima values of μ in the case supposed, but no such values when the intersections are two or all of them real.

To determine these values of μ, let $d\mu = 0$; then we have

$$\frac{d}{d\xi}(U - \mu V) = 0,$$

$$\frac{d}{d\eta}(U - \mu V) = 0,$$

$$\frac{d}{d\zeta}(U - \mu V) = 0,$$

that is
$$\square_{\xi\eta\zeta}(U - \mu V) = 0.$$

In order that any value of μ found from this equation may be a maximum or minimum, Lagrange's condition requires that

$$\left(h\frac{d}{d\xi} + k\frac{d}{d\eta} + l\frac{d}{d\zeta}\right)^2 \mu$$

may be a function of unchangeable sign.

Now
$$\frac{dU}{d\xi} = \mu \frac{dV}{d\xi} + V \frac{d\mu}{d\xi},$$

therefore since $d\mu = 0$,
$$\frac{d^2U}{d\xi^2} = \mu \frac{d^2V}{d\xi^2} + V \frac{d^2\mu}{d\xi^2}.$$

Hence
$$\frac{d^2\mu}{d\xi^2} = \frac{1}{V}\left(\frac{d}{d\xi}\right)^2 \{U - \mu V\};$$

similarly
$$\frac{d}{d\xi} \cdot \frac{d}{d\eta} = \frac{1}{V} \frac{d}{d\xi} \cdot \frac{d}{d\eta} \{U - \mu V\},$$

&c. &c. &c.

Making now as before
$$U = a\xi^2 + b\eta^2 + \text{\&c.},$$
$$V = \alpha\xi^2 + \beta\eta^2 + \text{\&c.},$$
$$a - \mu\alpha = A, \quad b - \mu\beta = B, \text{ \&c.},$$

the condition for μ, a root of $\square \{U - \mu V\} = 0$, giving μ a maximum or minimum, may be expressed by saying that
$$Ah^2 + Bk^2 + Cl^2 + 2A'kl + 2B'hl + 2C'hk$$
shall be unchangeable in sign for all real values of h, k, l.

The above quantity, by virtue of the equation $\square = 0$, is always the product of two linear functions. Hence we see, as above indicated, that if all these pairs are real, that is, if all the points of intersection of U and V are real, there is no maximum or minimum value of μ; but if only one pair be real and the other two pairs be imaginary, that is, if all the four intersections are imaginary, then two of the values of μ, namely those corresponding to the imaginary pairs, are real maxima or minima values of μ, but the third is illusory.

Now I shall show that if $V = 0$ is a *real* conic, but the intersections of U and V are all unreal, the value of μ which makes $U + \mu V$ the product of real linear functions of ξ, η, ζ, is always one or the other *extreme* of the three values of μ which satisfy the equation
$$\square (U - \mu V) = 0.$$

Assume as the three axes of coordinates the three lines joining the vertices of the quadrangle each with each, the two non-intersecting conics may evidently be written under the form
$$U = c(x^2 + y^2) - e(y^2 + z^2) = 0,$$
$$V = -\gamma(x^2 + y^2) + \epsilon(y^2 + z^2) = 0;$$

these equations being only other modes of writing

$$U = Ax^2 + By^2 + Cz^2,$$

$$V = A'x^2 + B'y^2 + C'z^2,$$

in which A, B, C; A', B', C' will be real, because by hypothesis $\square(U+\mu V)=0$ has all its roots real.

Hence x, y, z are linear functions of ξ, η, ζ, and consequently, by a simple inference from a theorem of Prof. Boole*, the roots of $\underset{\xi\eta\zeta}{\square}\{U+\mu V\}$ are identical with those of

$$\underset{xyz}{\square}\{U+\mu V\} = 0.$$

These latter are evidently $\frac{c}{\gamma}$, $\frac{e}{\epsilon}$, $\frac{c-e}{\gamma-\epsilon}$; the third of which is the one which makes $U + \mu V$ the product of two *real* linears, for we have

$$\gamma U + cV = (c\epsilon - \gamma e)(y^2 + z^2),$$

$$\epsilon U + eV = (\epsilon c - e\gamma)(x^2 + y^2),$$

$$(\gamma - \epsilon)\, U + (c - e)\, V = (c\epsilon - e\gamma)(z^2 - x^2)\dagger.$$

Now
$$\frac{c}{\gamma} - \frac{c-e}{\gamma-\epsilon} = \frac{e\gamma - c\epsilon}{\gamma(\gamma-\epsilon)},$$

$$\frac{e}{\epsilon} - \frac{c-e}{\gamma-\epsilon} = \frac{e\gamma - c\epsilon}{\epsilon(\gamma-\epsilon)};$$

and ϵ, γ are supposed to have the same sign, as otherwise V would be an unreal conic; hence the ascending or descending order of magnitudes of the three values of λ follows the scale $\frac{c}{\gamma}$, $\frac{e}{\epsilon}$, $\frac{c-e}{\gamma-\epsilon}$, as was to be shown.

Imagine now lengths reckoned on a line corresponding to all values of μ from $-\infty$ to $+\infty$, and mark off upon this line by the letters A, B, C, the lengths corresponding with the three roots of $\square(U+\mu V)=0$. Then observing that when $\mu = \pm\infty$, $U + \mu V$ is of the same nature as V, and is therefore a possible conic by hypothesis, and agreeing to understand by a possible and impossible region of μ, a range of values for which $U+\mu V$ corresponds to a possible and impossible conic respectively, one or the other of the annexed schemes will represent the circumstances of the case supposed:

$-\infty$ ☜ Poss. Reg. A Imposs. Reg. B Poss. Reg. C Poss. Reg. ☞ $+\infty$

$-\infty$ ☜ Poss. Reg. A Poss. Reg. B Imposs. Reg. C Poss. Reg. ☞ $+\infty$

But in either scheme it is essential to observe that the *middle* root of $\square(U+\mu V)=0$ divides a possible from an impossible region; and therefore

* See Postscript.

† $z^2 - x^2 = 0$ of course represents a *real* pair of lines.

if we can find n, ν, any two values lying between the first and second and second and third roots of the above equation arranged in order of their magnitude, one of the two equations $U+\nu V=0$, $U+nV=0$, will represent a possible and the other an impossible conic: one such couple of values may always be found by taking the roots of the quadratic equation

$$\frac{d}{d\mu}\,\square\,\{U+\mu V\}=0.$$

Hence calling the two roots thereof m and M, we see (which is in itself a theorem) that one at least of the conics $U+mV=0$, $U+MV=0$, must be a possible conic, provided only that $V=0$ be a possible conic: if both $U+mV$ and $U+MV$ are possible conics, the intersections of U and V are all real, and if not, not*. The criteria for distinguishing possible from impossible conics being well known need not be stated in this place.

We may of course proceed analogously by forming the two conics $lU+V$, $LU+V$, where l and L are roots of $\dfrac{d}{d\lambda}\,\square\,\{\lambda U+V\}=0$ upon the supposition of $U=0$ being a possible conic.

If either of the two U and V be not possible, their intersections are of course impossible, and the question is already decided.

It will be seen as pre-indicated that this method only fails in symmetry because of the choice between the couples m, M, and l, L. But moreover a perfect method for the discrimination of the two cases of *unmixed* intersection one from the other should (perhaps?) require the application of only a single test (in lieu of the two conditions which the above method supposes), over and above the condition which expresses the fact of the intersections being so unmixed. Such more perfect method I have not yet been able to achieve.

Another interesting question of intersections remains to be discussed, namely, supposing the two conics are known to be non-intersecting, how are we to ascertain if they are external to one another, or if one contains the other? In order to settle this point we must first establish a criterion for determining whether a given *point* is internal or external to a given conic; the point being in general said to be external when two real tangents can be drawn from it to the curve, and internal when this cannot be done.

* It must be well observed however that the possibility of the conics $U+mV$ and of $U+MV$ does not imply the reality of the intersections unless the conic V is known to be possible.

For if V be impossible ϵ and γ have opposite signs, and therefore $\dfrac{c-e}{\gamma-\epsilon}$ is intermediate between $\dfrac{c}{e}$ and $\dfrac{\gamma}{\epsilon}$, and the scheme for μ will be as here annexed:

$-\infty$ Impossible. A Possible. B Possible. C Impossible. $+\infty$

so that $U+mV$ and $U+MV$ will both represent possible conics.

Let now

$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

be the equation to any conic: l, m, n the coordinates of any point. Let

$$A = bc - a'^2, \quad B = ca - b'^2, \quad C = ab - c'^2,$$

$$A' = aa' - b'c', \quad B' = bb' - c'a', \quad C' = cc' - a'b'.$$

Then the reciprocal equation to the conic is

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta = 0,$$

and in making $l\xi + m\eta + n\zeta = 0$, the ratios of ξ, η, ζ must be real if the tangents drawn from l, m, n are real: this will be found to imply that the determinant

$$\begin{vmatrix} A, & C', & B', & l \\ C', & B, & A', & m \\ B', & A', & C, & n \\ l, & m, & n, & 0 \end{vmatrix}$$

shall be negative*. This determinant may be shown† to be equal to the product of the determinant

$$\begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix}$$

by the quantity

$$al^2 + bm^2 + cn^2 - 2a'mn - 2b'ln - 2c'lm,$$

that is, equal to $\phi(l, m, n) \times \square$.

Hence l, m, n is internal or external to $\phi(x, y, z)$ according as $\phi(l, m, n)$ and $\square\phi$ have the same or contrary sign.

If $\phi(l, m, n) = 0$, the point lies on the conic, and the point is neither internal nor external; if $\square\phi = 0$, the conic becomes a pair of straight lines, and no point can be said either to be within or without such a system. Hence our criterion fails, as it *ought to do*, just in the very two cases where the distinction vanishes. I believe that this criterion is here given for the first time.

* See theorem of the "Diminished Determinant" in Postscript to this paper.

† As we know *à priori* by virtue of a theorem given by M. Cauchy, and which is included as a particular case in a theorem of my own, relating to Compound Determinants, that is, Determinants of Determinants, which will take its place as an immediate consequence of my fundamental Theorem given in a Memoir about to appear. The well-known rule for the multiplication of Determinants is also a direct and simple consequence from my theorem on Compound Determinants, which indeed comprises, I believe, in one glance, all the heretofore existing Doctrine of Determinants.

To return to the two non-intersecting conics. Let us again throw them under the form

$$U = (x^2 + y^2) - e^2 (z^2 + y^2),$$
$$V = k (x^2 + y^2) - k\epsilon^2 (z^2 + y^2),$$

e and ϵ being real, that is, U and V being both functions corresponding to possible conics. Suppose U external to V; then *any point* in U is an external point to V.

Take in U either of the two points represented by the equations $y = 0$, $x^2 = e^2 z^2$; substituting these values of y and x, V becomes $k(e^2 - \epsilon^2) z^2$, and $\square V$ becomes $-k^3\epsilon^2(1 - \epsilon^2)$; therefore $(1 - \epsilon^2)(e^2 - \epsilon^2)$ must be positive, that is, ϵ^2 must be one of the extremes of the three values 1, e^2, ϵ^2. In like manner, if V is external to U, e will be also one of the extremes of the same three quantities; and hence, if the two conics are mutually external, unity will be the middle magnitude of the group e^2, 1, ϵ^2.

Now the three roots of $\square(V + \lambda U) = 0$, are

$$\lambda = -k, \quad \lambda = -k\frac{\epsilon^2}{e^2}, \quad \lambda = -k\frac{1 - \epsilon^2}{1 - e^2}.$$

Hence if U and V be without one another, or, as it may be termed, are extra-spatial, the third value of λ will be of a different sign from the first two; but if the two conics be co-spatial, that is, if one includes the other, all the three values of λ will have the same sign. Hence we have the following elegant criterion of co-spatiality of two possible conics expressed by the equations $U = 0$, $V = 0$, between indeterminate coordinates ξ, η, ζ; the coefficients of the cubic function $\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ must give only changes or only continuations of sign.

If this test be not satisfied, it will remain to determine which of the two conics contains, and which is contained by the other. Let U contain V, then the order of magnitudes will be 1, e^2, ϵ^2; therefore $k\frac{1 - \epsilon^2}{1 - e^2}$ is greater than k, and therefore $k\frac{1 - \epsilon^2}{1 - e^2}$, which is that root of the equation $\square(V + \lambda U) = 0$ which is always one or the other of the extremes, is the *greatest* of the three. Hence the scheme for the impossible and possible regions of λ will be as below:

$-\infty$ ☜ Poss. A Imposs. B Poss. C Poss. ☞ $+\infty$

Hence if the two roots of $\frac{d}{d\lambda}\{V + \lambda U\} = 0$ be l and L, and of the two conics $V + lU = 0$, $V + LU = 0$, the former be the possible, and the latter the impossible one, U contains V or is contained in it according as l is greater or less than L.

Observe that if U and V be non-cospatial, so that the three values of μ in $\square(U+\mu V)=0$ have not all the same sign and consequently zero lies between the greatest and least of them, it will not be necessary to make trial of the characters of the two curves $U+mV=0$, and $U+MV=0$, in order to ascertain whether U and V intersect or not; for it will be sufficient to find which of the two quantities m and M substituted for μ in $\square(U+\mu V)$ causes it to have the opposite sign to $\square(U+0V)$, that is, $\square U$, and this one of the two it is, if either, which will make $U+\mu V$ an impossible conic, and will thus alone serve to determine whether the intersections of U and V are unreal, or the contrary.

It might be a curious question to consider whether, in a certain sense, conics not both possible may not be said to lie one within or without the other. Upon general logical grounds, I think it not improbable that two impossible conics might be discovered *each to contain the other;* but this is an inquiry which I have not had leisure to enter upon.

I have thus far supposed the roots of $\square(\lambda U+V)=0$ to be all distinct from one another. I now approach the discussion of the contact of two conics, in which event two or more of the roots will be equal. The condition for simple contact is evidently $\underset{\lambda\mu}{\square}\ \underset{\xi\eta\zeta}{\square}(\lambda U+\mu V)=0$.

The unpaired value of λ in $\square(\lambda U+V)$ makes $\lambda U+V$ an impossible pair of lines, and therefore, in the scheme for λ drawn as above, will separate the possible from the impossible region.

Whether the conics intersect in two real or two unreal points, besides the point of contact, will be known at once by ascertaining whether $U+\mu V=0$ represents two real or two imaginary lines. If the latter, the two curves lie *dos-à-dos* or one within the other, according as the successions of sign in $\square(\lambda U+V)$ are all of the same kind or not; if they be all of the same kind, one will include the other, namely, U will include V if the equal roots are greater, and be included in it if they be less than the unequal one. This last conclusion however, it should be observed, is inferred upon the principle of continuity, by making two values of λ approach indefinitely near to one another, but cannot be strictly deduced from the equations given for U and V applicable to the general case, in which the axes of coordinates are the three axes joining the vertices; since these latter, in the case supposed, reduce to two only, and consequently such representation of U and V becomes illusory.

If all three values of λ are equal, the three vertices come together, and hence the two conics will have three consecutive points in common, that is, will have the same circle of curvature. On this supposition the two curves cut at the point of contact, and all four points of intersection are of course real.

The classification of contacts between two conics may be stated as follows:

Simple contact = one case.

Second degree contact = two cases, namely, common curvature or double contact.

Third degree contact = one case, namely, contact in four consecutive points.

These four cases of course correspond to the several suppositions of there being two equal roots, three equal roots, two pairs of equal roots, or four equal roots in the biquadratic equation obtained between two variables by elimination performed in any manner between the given equations in the two conics.

The first species and the first case of the second species have been already disposed of. I proceed to assign the conditions appertaining to the second case of the second species, when U and V have a double contact.

Let A, A', B, B' be the two pairs of coincident points in which the conics are supposed to meet; either pair of lines AB, $A'B'$, and AB', $A'B$, becomes a coincident pair. Hence such a value of μ can be found as will make $U+\mu V$ the *square of a linear function of* ξ, η, ζ. If therefore we make $U+\mu V = W$, and form the determinant

$$\left|\begin{matrix} \dfrac{d^2W}{d\xi^2}, & \dfrac{d^2W}{d\xi d\eta}, & \dfrac{d^2W}{d\xi d\zeta}, & p \\[2ex] \dfrac{d^2W}{d\eta d\xi}, & \dfrac{d^2W}{d\eta^2}, & \dfrac{d^2W}{d\eta d\zeta}, & q \\[2ex] \dfrac{d^2W}{d\zeta d\xi}, & \dfrac{d^2W}{d\zeta d\eta}, & \dfrac{d^2W}{d\zeta^2}, & r \\[2ex] p, & q, & r, & 0 \end{matrix}\right|$$

$$= Ap^2 + Bq^2 + Cr^2 + 2Fqr + 2Grp + 2Hpq,$$

where all the coefficients are quadratic functions of μ, and make

$$A = 0,\ B = 0,\ C = 0,\ F = 0,\ G = 0,\ H = 0,$$

each of these six equations in μ will have one and the same root in common.

It is, however, enough to select any three; if these vanish together for any value of μ, the remaining three must also vanish. This is a simple application of a general law* which will appear in a forthcoming memoir on "Determinants and Quadratic Forms," of which this paper is to be considered as an accidental episode.

* For statement of this law called the Homaloidal Law, see *Philosophical Magazine* of this month "On Certain Additions, &c." [p. 150 below. Ed.]

Take now any three of the six equations which for the sake of generality call $P=0$, $Q=0$, $R=0$. The hypothesis of double contact requires that P and Q, Q and R, R and P shall have a factor in common; but these conditions are not sufficiently explicit for our present object, since P, Q, R might be of the form

$$\kappa(\lambda-a)(\lambda-b), \quad \kappa'(\lambda-b)(\lambda-c), \quad \kappa''(\lambda-c)(\lambda-a),$$

and would thus satisfy the conditions above stated, without P, Q, R having a common factor. A sufficient criterion is that $fQ+gR$ and P shall have a common factor for all values of f and g.

Let then the resultant of $fQ+gR$ and P be

$$Lf^2+Mfg+Ng^2,$$

we must have

$$L=0, \quad M=0, \quad N=0,$$

where L is the resultant of P and Q,

N " " " R and Q;

and M is a new function, which if we call $Q=\phi(\lambda)$, $R=\psi(\lambda)$, and suppose a and b to be the two roots of $P=0$, is easily seen to be equal to

$$\phi a \,.\, \psi b+\phi b \,.\, \psi a.$$

This I call the connective of $P\,.\,Q$ and $P\,.\,R$.

L, M, N may conveniently be denoted by the forms

$$P\,.\,Q, \quad P\,.\,R, \quad Q\,.\,P\,.\,R.$$

We may now take more generally

$$aP+bQ+cR,$$

$$\alpha P+\beta Q+\gamma R,$$

which will have a factor in common for all values of a, b, c, α, β, γ.

I am indebted to Mr Cayley for the remark that the resultant of these two functions is a new quadratic function, which, according to my notation just given, may be put under the form

$$PQ(a\beta-b\alpha)^2+QR(b\gamma-c\beta)^2+RP(c\alpha-a\gamma)^2$$

$$+PRQ(b\gamma-c\beta)(c\alpha-a\gamma)+QPR(c\alpha-a\gamma)(a\beta-b\alpha)+RQP(a\beta-b\alpha)(b\gamma-c\beta).$$

Ternary systems of the six coefficients formed upon the type of (PQ, PQR, QR), I call *complete* systems, because the three functions included in such a system equated severally to zero, imply that the remaining three coefficients are all zero. Such a system as (PQ, QR, RP) I term an *incomplete* ternary system as not drawing with it the like implication. Probably (?) we should find on investigation that PRQ, QPR, RQP, would also be an incomplete system, but that systems formed after the type of PRQ, RQ, RQP are complete. This however is only matter of conjecture, as I have been too much occupied with other things to enter upon the inquiry. The distinct types of ternary systems are altogether six in number, namely, four of a symmetrical species,

$$\begin{array}{ccc} PQ, & QR, & RP, \\ PRQ, & QPR, & RQP, \\ PQ, & PQR, & QR, \\ PRQ, & RQ, & RQP; \end{array}$$

and two of an unsymmetrical species, namely,

$$\begin{array}{ccc} PQ, & PQR, & PR, \\ PRQ, & RQ, & QPR.* \end{array}$$

If instead of confining ourselves to three out of the six original quantities, A, B, C; F, G, H, we take them all into account, and write down the resultant of

$$aA + bB + cC + fF + gG + hH,$$

$$\alpha A + \beta B + \gamma C + \phi F + \chi G + \eta H;$$

we shall obtain a quadratic function of 15 variables (not however all independent) having 120 coefficients, all of which must be zero. It would be extremely interesting to determine how many *complete* ternary groups can be formed out of these 120 terms.

It will be recollected that we have assigned as the condition of contact in three consecutive points, that a certain cubic equation shall have all its roots real. Now, as well remarked by Mr Cayley, we cannot express this fact by less than three equations in integral terms of the coefficients. Thus if the cubic be written

$$a\lambda^3 + 3b\lambda^2 + 3c\lambda + d = 0,$$

we have as one of such ternary systems,

$$U = ac - b^2 = 0, \quad V = bd - c^2 = 0, \quad W = bc - ad = 0.$$

The significant parts of these equations are of course, however, capable of being connected by integral multipliers U', V', W', such that

$$U'U + V'V + W'W = 0.$$

* PQ, QR, RP, may be compared in a general way with the angles, and PRQ, QPR, RQP, with the sides of a triangle.

Any number of functions U, V, W so related, I call *syzygetic* functions, and U', V', W' I term the *syzygetic multipliers**. These in the case supposed are c, a, b, respectively.

In like manner it is evident that the members of any group of functions, more than two in number, whose nullity is implied in the relation of double contact, whether such group form a complete system or not, must be in syzygy.

Thus PQ, PQR, QR, must form a syzygy; nor is there any difficulty in assigning a system of multipliers to exhibit such syzygy. Calling $P = \phi(\lambda)$, $R = \psi(\lambda)$, a and b the two roots of $Q = 0$, I have found that

$$\{(\psi a)^2 + (\psi b)^2\} PQ - (\phi a . \psi a + \phi b . \psi b) PQR + \{(\phi a)^2 + (\phi b)^2\} QR = 0.$$

Again, if we take the *incomplete* system

$$(PQ), \quad (QR), \quad (RP),$$

it will be found that

$$L(QR) + M(RP) + N(PQ) = 0,$$

provided that, calling a, b; c, d; e, f, the roots of $P = 0$, $Q = 0$, $R = 0$, respectively, we make

$$L = (k_0 + k_1 a + k_2 a^2 + k_3 a^3 + k_4 a^4) \frac{(a-c)(a-d)(a-e)(a-f)}{a-b}$$

$$+ (k_0 + k_1 b + k_2 b^2 + k_3 b^3 + k_4 b^4) \frac{(b-c)(b-d)(b-e)(b-f)}{b-a},$$

$$M = (k_0 + k_1 c + k_2 c^2 + k_3 c^3 + k_4 c^4) \frac{(c-a)(c-b)(c-d)(c-e)(c-f)}{c-d}$$

$$+ (k_0 + k_1 d + k_2 d^2 + k_3 d^3 + k_4 d^4) \frac{(d-a)(d-b)(d-c)(d-e)(d-f)}{d-c},$$

$$N = (k_0 + k_1 e + k_2 e^2 + k_3 e^3 + k_4 e^4) \frac{(e-a)(e-b)(e-c)(e-d)}{e-f}$$

$$+ (k_0 + k_1 f + k_2 f^2 + k_3 f^3 + k_4 f^4) \frac{(f-a)(f-b)(f-c)(f-d)}{f-e};$$

k_0, k_1, k_2, k_3, k_4 being quite arbitrary, and L, M, N, although presented in a fractional form, being essentially integral.

This fact of L, M, N constituting a system of multipliers to the syzygy QR, RP, PQ, is easily demonstrated; for

$$QR = (c-e)(c-f)(d-e)(d-f),$$

$$RP = (e-a)(e-b)(f-a)(f-b),$$

$$PQ = (a-c)(a-d)(b-c)(b-d).$$

* There will be in general various such systems of multipliers.

Hence $$L(QR)+M(RP)+N(PQ)$$

$$=(a-c)(a-d)(a-e)(a-f)(b-c)(b-d)(b-e)(b-f)(c-e)(c-f)(d-e)(d-f)$$

$$\times\Sigma\frac{k_0+k_1a+k_2a^2+k_3a^3+k_4a^4}{(a-b)(a-c)(a-d)(a-e)(a-f)}=0.$$

My theory of elimination enables me to explain exactly the nature of L, M, N, and the *reason* of their appearance as syzygetic factors.

Let L_r, M_r, N_r signify what L, M, N become, when all the k's except k_r are taken zero. Then the theory given by me in the *Philosophical Magazine* for the year 1838, or thereabouts†, shows that $L_0\lambda+L_1$ is the *prime derivee* of the first degree between the two equations P and $Q\times R$, or, in other words, will be the remainder integralized of $\frac{QR}{P}$.

In like manner $M_0\lambda+M_1$, $N_0\lambda+N_1$ are the integralized remainders of $\frac{RP}{Q}$ and of $\frac{PQ}{R}$ respectively.

If now the resultant of P, Q and of Q, R are each zero, but the resultant of P and R is not zero, it will be evident that P, Q, R must be of the form

$$f(\lambda+a)(\lambda+c),\quad g(\lambda+c)(\lambda+d),\quad h(\lambda+d)(\lambda+b);$$

and therefore $P\times R$ will contain Q, and consequently we must have

$$M_0=0,\quad M_1=0.$$

More generally, if we write

$$Q=0,$$
$$\lambda Q=0,$$
$$\lambda^2Q=0,$$
$$P\times R=0,$$

and eliminate dialytically, that is, treating λ^4, λ^3, λ^2, λ as distinct quantities, we shall obtain*

$$\lambda^4:\lambda^3:\lambda^2:\lambda:1::M_4:M_3:M_2:M_1:M_0;$$

and therefore when $P\times R$ contains Q,

$$M_0=0,\quad M_1=0,\quad M_2=0,\quad M_3=0,\quad M_4=0.$$

* This cannot be obtained directly from what is stated in the paper referred to, although contained in the general theory of derivation there given. The arbitrary functions which enter into the expression for the general derivees have been in that paper evaluated only for the prime derivees, which however are only particular phenomena, with reference to the general results of Dialytic Elimination. Hereafter I may give a more general exposition of this remarkable, although ignored or neglected theory. The prime derivees of fx and $f'x$ are Sturm's Functions, cleared of quadratic factors, and are expressed by virtue of the general theorems there laid down as functions of x and of symmetrical functions of the roots of fx. [† p. 40 above. Ed.]

In like manner, when $Q \times P$ contains R,

$$N_0 = 0, \quad N_1 = 0, \quad N_2 = 0, \quad N_3 = 0, \quad N_4 = 0;$$

and when $R \times Q$ contains P,

$$L_0 = 0, \quad L_1 = 0, \quad L_2 = 0, \quad L_3 = 0, \quad L_4 = 0.$$

Accordingly, we see from the equation

$$L(QR) + M(RP) + N(PQ) = 0,$$

that if $QR = 0$, $RP = 0$; but PQ *not* $= 0$, then $N = 0$; and therefore

$$N_0 = 0, \quad N_1 = 0, \quad N_2 = 0, \quad N_3 = 0, \quad N_4 = 0,$$

and so in like manner for the remaining corresponding two suppositions*.

Before proceeding to consider the remaining case of the highest species of contact, I must observe that besides the equations involved in the condition that A, B, C; F, G, H, or, which is the same thing, that any three of them shall all have a factor in common, we must have $\square(U + \lambda V)$ containing the square of such common factor. In the memoir before adverted to a general theorem will be given and proved, which shows that this latter condition is involved in the former one; in fact, more generally (but still only as a particular case) that when U and V are quadratic functions of n letters, but $U + \epsilon V$ admits of being represented as a complete function of $(n-2)$ quantities only, which are themselves linear functions of the n letters, then $\square(U + \lambda V)$, which is of course a function of λ of the nth degree, will contain the factor $(\lambda - \epsilon)^2$.

When the two conics have four consecutive points in common, the characters of double-point contact and of contact in three consecutive points must exist simultaneously; and consequently the factor common to A, B, C; F, G, H, will enter not as a binary but as a ternary factor into $\square(U + \lambda V)$. This gives the extra condition required. As an example take the two conics,

$$U = \frac{y^2}{1-k} + x^2 - z^2 = 0,$$

$$V = y^2 + x^2 - 2kxz + (2k-1)z^2 = 0,$$

$$U + \lambda V = \left(\frac{1}{1-k} + \lambda\right) y^2 + (1+\lambda)x^2 - \{1 + \lambda(1-2k)\} z^2 - 2k\lambda xz.$$

* Since we are able to assign the values of the syzygetic multipliers in the equations

$$\begin{aligned} L\ \ (PQ) + M\ \ (QR) + N\ \ (RP) &= 0, \\ L'\ (PQ) + M'\ (PQR) + N'\ (QR) &= 0, \\ L''\ (QR) + M''\ (QRP) + N''\ (RP) &= 0, \\ L'''\ (RP) + M'''\ (RPQ) + N'''\ (PQ) &= 0, \end{aligned}$$

it follows that we may eliminate between these four equations any three of the six quantities (PQ), (PRQ), &c., and thus express any one of them in terms of any two others: this method, however, is not practically convenient. I may probably hereafter return to this subject.

The complete determinant of $U+\lambda V$ is then

$$\frac{-1}{1-k}\{1+(1-k)\lambda\}\{(1+\lambda)^2-2k\lambda(1+\lambda)+k^2\lambda^2\}=-\frac{1}{1-k}\{1+(1-k)\lambda\}^3.$$

A, B, C are the determinants of $U+\lambda V$, when $x=0$, $y=0$, $z=0$, respectively. Thus

$$A=\left(\frac{1}{1-k}+\lambda\right)(1+\lambda),$$

$$B=\left(\frac{1}{1-k}+\lambda\right)\{1+\lambda(1-2k)\},$$

$$C=k^2\lambda^2-(1+\lambda)\{1+\lambda(1-2k)\}=\lambda^2(1-k)^2-2\lambda(1-k)-1;$$

$\lambda=-\dfrac{1}{1-k}$ makes $A=0$, $B=0$, $C=0$, and the factor $\lambda+\dfrac{1}{1-k}$ enters cubed into $\square(U+\lambda V)$.

Hence the two conics have a contact of the third order.

This is easily verified; for if we pass from general to Cartesian and rectangular coordinates, and make z unity; $U=0$ will represent an ellipse with centre at the origin, eccentricity $\sqrt{k}$, and mean focal distance 1, and $V=0$ the circle of curvature at the extremity of the axis major*.

I had intended to have added some other remarks connected with the present discussion, and also to have appended an *à posteriori* proof of the propositions relative to the reality and otherwise of the vertices and chordal pairs of intersection which I have, at the commencement of this paper, deduced quite legitimately, but in a manner not at first sight perhaps easily intelligible, from the general principles of conjugate forms; but this discussion has run on already to a length so much greater than I had anticipated and than the importance of the inquiry may seem to justify, that I must reserve for a future number of the *Journal* what further matter I may have to communicate concerning it.

Postscript.—As I have alluded to Professor Boole's theorem relative to Linear Transformations, it may be proper to mention my theorem on the subject, which is of a much more general character, and includes Mr Boole's (so far as it refers to *Quadratic* Functions) as a corollary to a particular case. The demonstration will be given in the forthcoming memoir above alluded to.

Let U be a quadratic function of any number of letters $x_1, x_2 \ldots x_n$, and let any number r of linear equations of the general form

$${}_1a_r x_1 + {}_2a_r x_2 + \ldots\ldots + {}_na_r x_n = 0,$$

* We have thus discussed all the four cases of biconical contact: for an exactly parallel discussion of the theory of contact of a plane with the curve of double curvature in which two surfaces of the second order intersect, see the paper in the *Philosophical Magazine* for this month, before referred to. [p. 148 below. Ed.]

be instituted between them: and by means of these equations let U be expressed as a function of any $(n-r)$ of the given letters, say of $x_{r+1}, x_{r+2} \ldots\ldots x_n$, and let U, so expressed, be called M. Let

$$ {}_1a_r x_1 + {}_2a_r x_2 + \ldots\ldots + {}_na_r x_n $$

be called L_r. Then the determinant of M in respect to the $(n-r)$ letters above given is equal to the determinant of

$$ U + L_1 x_{n+1} + L_2 x_{n+2} + \ldots\ldots + L_r x_{n+r}, $$

considered as a function of the $(n+r)$ letters

$$ x_1 x_2 \ldots\ldots x_{n+r}, $$

divided by the square of the determinant

$$ \begin{vmatrix} {}_1a_1, & {}_2a_1 & \ldots\ldots & {}_ra_1 \\ {}_1a_2, & {}_2a_2 & \ldots\ldots & {}_ra_2 \\ \ldots & \ldots & \ldots & \ldots \\ {}_1a_r, & {}_2a_r & \ldots\ldots & {}_ra_r \end{vmatrix}. $$

This I call the theorem of Diminished Determinants.

If now we have U a function of r letters, and V of r other letters, and V is derived from U by linear transformations, that is, by r equations connecting the $2r$ letters; then, since U may be considered as a function of *all the* $2r$ letters with abortive coefficients for all the terms where any of the second set of r letters enter, we may apply our theorem of diminished determinants to the question so considered, and the result may be found to represent Mr Boole's theorem in a form rather more general and symmetrical, but substantially identical with that given by Mr Boole.

Thus suppose $\frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$ say P, and $\frac{1}{2}\alpha u^2 + \beta uv + \frac{1}{2}\gamma v^2$ say Q, are mutually transformable by virtue of the linear equations

$$ lx + my = \lambda u + \mu v, $$
$$ l'x + m'y = \lambda' u + \mu' v, $$

P may be considered as a function of x, y, u, v, and Q as the value of P, when we eliminate x and y by virtue of the two linear equations

$$ L_1 = lx + my - \lambda u - \mu v = 0, $$
$$ L_2 = l'x + m'y - \lambda' u - \mu' v = 0; $$

we have therefore by our theorem the determinant of Q equal to the squared reciprocal of the determinant $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$ multiplied by the determinant

$$ \begin{vmatrix} a, & b, & 0, & 0, & l, & l' \\ b, & c, & 0, & 0, & m, & m' \\ 0, & 0, & 0, & 0, & -\lambda, & -\lambda' \\ 0, & 0, & 0, & 0, & -\mu, & -\mu' \\ l, & m, & -\lambda, & -\mu, & 0, & 0 \\ l', & m', & -\lambda', & -\mu', & 0, & 0 \end{vmatrix}, $$

which last determinant is evidently equal to the determinant of P multiplied by the square of the determinant $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}$. Whence we see that the determinant of Q divided by the square of $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}$, is equal to the determinant of P divided by the square of $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$. There is also another way more simple, but less direct, by means of which the theorem of diminished determinants may be made to yield Mr Boole's theorem of transformation*. Some unavowed use has been made in the foregoing pages of this former theorem, one of the highest importance in the analytical and geometrical theory of quadratic functions. It has been nearly a year in my possession, and I trust and believe that I am committing no act of involuntary misappropriation in announcing it as a result of my own researches.

* Namely, by considering P and Q as *each* derived from some common function of x, y, u, v, w, by means of the equations $L_1=0$, $L_2=0$; the law of Diminished Determinants will then indicate the determinants of P and Q, each under the form of fractions having the *same numerator*, but whose denominators will be $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}^2$ and $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$ respectively.

23.

AN INSTANTANEOUS DEMONSTRATION OF PASCAL'S THEOREM BY THE METHOD OF INDETERMINATE COORDINATES.

[*Philosophical Magazine*, XXXVII. (1850), p. 212.]

THE new analytical geometry consists essentially of two parts—the one determinate, the other indeterminate.

The determinate analysis comprehends that class of questions in which it is necessary to assume *independent* linear coordinates, or else to take cognizance of the equations by which they are connected if they are not independent. The indeterminate analysis assumes at will any number of coordinates, and leaves the relations which connect them more or less indefinite, and reasons chiefly through the medium of the general properties of algebraic forms, and their correspondencies with the objects of geometrical speculation. Pascal's theorem of the mystic hexagon, and the annexed demonstration of its fundamental property, belong to this branch of the subject, and afford an instructive and striking example of the application of the pure method of indeterminate coordinates.

Let x, y, z, t, u, v be the sides of a hexagon inscribed in the conic U. Let the hexagon be divided by a new line ϕ in any manner into two quadrilaterals, say $xyz\phi$, $tuv\phi$.

Then $$ay\phi + bxz = U = \alpha u\phi + \beta tv;$$

therefore $$(ay - \alpha u)\phi = \beta tv - bxz;$$

therefore $ay - \alpha u$ and ϕ are the diagonals of the quadrilateral $txvz$.

By construction, ϕ is the diagonal joining x, v (that is, the intersection of x and v) with z, t; and thus we see that $ay - \alpha u$ is the line joining t, x with v, z; but this line passes through y, u. Therefore x, t; y, u; z, v lie in one and the same right line. Q.E.D.

24.

ON A NEW CLASS OF THEOREMS IN ELIMINATION BETWEEN QUADRATIC FUNCTIONS.

[*Philosophical Magazine*, XXXVII. (1850), pp. 213—218.]

In a forthcoming memoir on determinants and quadratic functions, I have demonstrated the following remarkable theorem as a particular case of one much more general, also there given and demonstrated.

Let U and V be respectively quadratic functions of the same $2n$ letters, and let it be supposed possible to institute n such linear equations between these letters as shall make U and V both simultaneously become identically zero*. Then the determinant of $\lambda U + \mu V$, which is of course a function of λ and μ of the $2n$th degree, will become the *square* of a function of λ and μ of the nth degree; and conversely, if this determinant be a perfect square, U and V may be made to vanish simultaneously by the institution of n linear equations between the $2n$ letters†.

Let now P and Q be respectively quadratic functions of three letters only, say x, y, z; and let

$$U = P + (lx + my + nz)t,$$

$$V = Q + k(lx + my + nz)t.$$

The determinant of $\lambda U + \mu V$ in respect to x, y, z, t is easily seen to be $(\lambda + k\mu)^2 \times$ the determinant of

$$\lambda P + \mu Q + (lx + my + nz)t$$

in respect to x, y, z, t. Hence if we call

$$\lambda P + \mu Q + (lx + my + nz)t = W,$$

and make $\underset{xyzt}{\square} W$ a squared function of λ, μ or which is the same thing, if

$$\underset{\lambda\mu}{\square}\,\underset{xyzt}{\square}\{W\} = 0,$$

* In the more general theorem above alluded to, the number of letters is any number m, the number of linear equations being any number not exceeding $\frac{m}{2}$.

† When $n=1$, we obtain a theorem of elimination between two quadratics, which has been already given by Professor Boole.

U and V will vanish simultaneously when two linear relations are instituted between the quantities (all or some of them) x, y, z, t.

In order that this may be the case, it will be seen to be sufficient that

$$P = 0, \quad Q = 0, \quad (lx + my + nz) = 0,$$

shall coexist; for then two equations between x, y, z of which $lx+my+nz=0$ will be one, will suffice to make U and V each identically zero. Hence we have the following theorem:

$$\underset{\lambda\mu}{\square}\ \underset{xyzt}{\square}\{\lambda U + \mu V + (lx + my + nz)\, t\}$$

is a factor of the resultant of

$$P = 0, \quad Q = 0, \quad lx + my + nz = 0.$$

A comparison of the orders of the resultant and the determinant shows that they must be identical, *à-ci-près,* of a numerical factor, which, if the resultant be taken in its *general* lowest terms, may no doubt be easily shown to be unity.

As an illustration of our theorem, let

$$P = xy + yz + zx,$$

$$Q = cxy + ayz + bzx.$$

Then

$$\underset{xyzt}{\square}\{\lambda P + \mu Q + (lx + my + nz)\, t\} = \begin{vmatrix} 0, & \lambda + c\mu, & \lambda + b\mu, & l \\ \lambda + c\mu, & 0, & \lambda + a\mu, & m \\ \lambda + b\mu, & \lambda + a\mu, & 0, & n \\ l, & m, & n, & 0 \end{vmatrix}$$

$$= n^2(\lambda + c\mu)^2 + m^2(\lambda + b\mu)^2 + l^2(\lambda + a\mu)^2$$

$$- 2lm(\lambda + b\mu)(\lambda + a\mu) - 2mn(\lambda + c\mu)(\lambda + b\mu) - 2nl(\lambda + a\mu)(\lambda + c\mu)$$

$$= \lambda^2\{n^2 + m^2 + l^2 - 2lm - 2mn - 2nl\}$$

$$+ 2\lambda\mu\{cn^2 + bm^2 + al^2 - lm(a + b) - mn(b + c) - nl(c + a)\}$$

$$+ \mu^2\{c^2n^2 + b^2m^2 + a^2l^2 - 2ablm - 2bcmn - 2canl\}.$$

And we thus obtain, finally,

$$\underset{\lambda\mu}{\square}\ \underset{xyzt}{\square}\{\lambda P + \mu Q + (lx + my + nz)\, t\}$$

$$= (n^2 + m^2 + l^2 - 2lm - 2mn - 2nl)$$

$$\times (c^2n^2 + b^2m^2 + a^2l^2 - 2ablm - 2bcmn - 2canl)$$

$$- \{(cn^2 + bm^2 + al^2 - lm(a + b) - mn(b + c) - nl(c + a)\}^2$$

$$= -4lmn\{(a - b)(a - c)\, l + (b - a)(b - c)\, m + (c - a)(c - b)\, n\}.$$

Now to obtain the resultant of

$$xy + yz + zx = 0,$$

$$cxy + azy + bxz = 0,$$

$$lx + my + nz = 0,$$

we need only find the four systems *in their lowest terms* of $x : y : z$, which satisfy the first two equations, and multiply the four linear functions obtained by substituting these values of x, y, z in the fourth: the product will contain the resultant of the system affected with some numerical factor. In the present case, the four systems of x, y, z are

$$x = 0, \quad y = 0, \quad z = 1,$$

$$y = 0, \quad z = 0, \quad x = 1,$$

$$z = 0, \quad x = 0, \quad y = 1,$$

$$x = (a - b)(a - c), \quad y = (b - a)(b - c), \quad z = (c - a)(c - b),$$

and accordingly the product of

$$lx_1 + my_1 + nz_1,$$

$$lx_2 + my_2 + nz_2,$$

$$lx_3 + my_3 + nz_3,$$

$$lx_4 + my_4 + nz_4,$$

becomes

$$lmn\,\{(a - b)(a - c)\,l + (b - a)(b - c)\,m + (c - a)(c - b)\,n\},$$

agreeing with the result obtained by my theorem,—a *special* numerical factor 4, arising from the peculiar form of the equations, having disappeared from the resultant.

A geometrical demonstration may be given of the theorem which is instructive in itself, and will suggest a remarkable extension of it to functions containing more than three letters; the equation

$$\underset{xyzt}{\square}\,\{\lambda U + \mu V + (lx + my + nz)\,t\} = 0,$$

which is a quadratic equation in $\lambda : \mu$, may easily be shown to imply that the conic $\lambda U + \mu V$ is touched by the straight line

$$lx + my + nz = 0.$$

And we thus see that in general two conics,

$$\lambda U + \mu V = 0,$$

passing through the intersections of two given conics,

$$U = 0, \quad V = 0,$$

may be drawn to touch a given line. If, however, the given line passes through any of the four points of intersection, in such case only one conic can be drawn to touch it; accordingly

$$\Box\ \Box\ \{\lambda U + \mu V + (lx + my + nz)\, t\}$$

must be zero when l, m, n are so taken as to satisfy this condition, that is, if

$$lx_1 + my_1 + nz_1 = 0,$$

or $$lx_2 + my_2 + nz_2 = 0,$$

or $$lx_3 + my_3 + nz_3 = 0,$$

or $$lx_4 + my_4 + nz_4 = 0,$$

whence the theorem.

Now suppose U and V to be each functions of four letters, x, y, z, t; when

$$\underset{xyztu}{\Box}\ \{\lambda U + \mu V + (lx + my + nz + pt)\, u\} = 0,$$

the conoid $\lambda U + \mu V$ touches the plane

$$lx + my + nz + pt = 0\,;$$

and $\Box = 0$ being a cubic equation, in general three such conoids can be drawn.

Considerations of analogy make it obvious to the intuition, that in the particular case of two of these becoming coincident, the given plane

$$lx + my + nz + pt$$

must be a tangent plane to those two coincident conoids at one of the points where it meets the intersections of $U = 0$, $V = 0$; that is

$$lx + my + nz + pt = 0$$

will pass through a tangent line to, or in other words, may be termed a tangent plane to the intersections. Hence the following analytical theorem, derived from supposing q, r, s, t to be proportional to the areas of the triangular faces of the pyramid cut out of space by the four coordinate planes to which x, y, z, t refer. As these planes are left indefinite, q, r, s, t are perfectly arbitrary.

Theorem. The resultant of

1. 2. $\left.\begin{matrix} U = 0 \\ V = 0 \end{matrix}\right\}$, where U and V are functions of x, y, z, t;

3. $$lx + my + nz + pt = 0\,;$$

4. $$\begin{vmatrix} \dfrac{dU}{dx}, & \dfrac{dU}{dy}, & \dfrac{dU}{dz}, & \dfrac{dU}{dt} \\ \dfrac{dV}{dx}, & \dfrac{dV}{dy}, & \dfrac{dV}{dz}, & \dfrac{dV}{dt} \\ l, & m, & n, & p \\ q, & r, & s, & t \end{vmatrix} = 0\,;$$

which system, it will be observed, consists of three quadratic functions, and one linear function of x, y, z, t, *contains* the factor

$$\underset{\lambda\mu}{\square}\,\underset{xyzt}{\square}\{\lambda U + \mu V + (lx + my + nz + pt)\,u\}.$$

This last quantity is of the 4×3th, that is, the 12th order in respect of the coefficients in U and V combined; of the 4×2th, that is, the 8th order in respect of l, m, n, p; and of the zero order in respect of q, r, s, t.

The resultant which contains it is of the $(4 + 4 + 2.4)$th, that is, 16th order in respect to the coefficients in U and V; of the $(4 + 8)$th, that is, the 12th, in respect of l, m, n, p; and of the 4th in respect of q, r, s, t. Hence the special (and, as far as the geometry of the question is concerned, the unnecessary, I may not say extraneous or irrelevant) factor which enters into the resultant is of the 4th order in respect to the combined coefficients of U and V*; and of the same order in respect to l, m, n, p, and in respect to q, r, s, t.

I have not yet succeeded in divining its general value.

In the very particular example, of the system,

$$\alpha x^2 + \beta y^2 = 0,$$

$$cz^2 + dt^2 = 0,$$

$$lx + my + nz + pt = 0,$$

$$\begin{vmatrix} \alpha x, & \beta y, & 0, & 0 \\ 0, & 0, & cz, & dt \\ l, & m, & n, & p \\ q, & 0, & 0, & 0 \end{vmatrix} = 0,$$

I find that the double determinant is

$$c^2d^2\alpha^2\beta^2(cp^2 + dn^2)^2(m^2\alpha + l^2\beta)^2,$$

and the resultant is

$$q^4c^2d^2\alpha^2\beta^4(cp^2 + dn^2)^4(m^2\alpha + l^2\beta)^2,$$

giving as the special factor

$$q^4\beta^2(cp^2 + dn^2)^2.$$

I believe that the theorem which I have here given for determining the condition that $lx + my + nz + pt$ shall be a tangent plane to the intersection of two conoids U and V, namely, that the determinant of

$$\lambda U + \mu V + (lx + my + nz + pt)\,u$$

shall have two equal roots, is altogether novel.

* And consequently of the second in respect to the separate coefficients of each.

What is the meaning of all three roots of this determinant becoming equal, that is, of only one conoid being capable of being drawn through the intersection of U and V to touch the plane

$$lx + my + nz + pt\,?$$

Evidently (*ex vi analogiæ*) that this plane shall pass through three consecutive points of the curve of intersection, that is, that it shall be the osculating plane to the curve.

If we return to the intersection of two co-planar conics, and if we suppose a line to be drawn through two of the points of intersection, the conics capable of being drawn through the four points of intersection to touch the line, besides becoming coincident, evidently degenerate each into a pair of right lines. It would seem, therefore, by analogy, that if a plane be drawn including any two tangent lines to the curve of intersection of two surfaces of the second degree, this should be touched by two coincident cones drawn through the curve of intersection, and consequently every such double tangent plane to the intersection of two conoids (and it is evident that one or more of these can be taken at every point of the curve) must pass through one of the vertices of the four cones in which the intersection may also be considered to lie; and it would appear from this, that in general four double tangent planes admit of being drawn to the curve, which is the intersection of two conoids, at each point thereof. At particular points a tangent plane may be drawn passing through more than one of the vertices, and then of course the number of double tangent planes that can be drawn will be lessened. These results, indicated by analogy, become immediately apparent on considering the curve in question as traced upon any one of the four containing cones. For the plane drawn through a tangent at any point, and the vertex of the cone being a tangent plane to the cone, must evidently touch the curve again where it meets it. We thus have an additional confirmation of the analogy between a point of intersection of two curves and the tangent at any point of the intersection of two surfaces.

I might extend the analytical theorems which have been established for functions of three and four to functions of a greater number of variables; but enough has been done to point out the path to a new and interesting class of theorems at once in elimination and in geometry, which is all that I have at present leisure or the disposition to undertake.

25.

ADDITIONS TO THE ARTICLES*, "ON A NEW CLASS OF THEOREMS," AND "ON PASCAL'S THEOREM."

[*Philosophical Magazine*, XXXVII. (1850), pp. 363—370.]

FIRST *addition*.—I have alluded in the second of the above articles to a more general theorem, comprising, as a particular case, the theorem there given for the simultaneous evanescence of two quadratic functions of $2n$ letters, on n linear equations becoming instituted between the letters.

In order to make this generalization intelligible, I must premise a few words on the Theory of Orders, a term which I have invented with particular reference to quadratic functions, although obviously admitting of a more extended application. A linear function of all the letters entering into a function or system of functions under consideration I call an order of the letters, or simply an order. Now it is clear that we may always consider a function of any number of letters as a function of as many orders as there are letters; but in certain cases a function may be expressed in terms of a fewer number of orders than it has letters, as when the general characteristic function of a conic becomes that of a pair of crossing lines or a pair of coincident lines, in which event it loses respectively one and two orders, and so for the characteristic of a conoid becoming that of a cone, a pair of planes or two coincident planes, in which several events, a function of four letters becomes that of only three orders, or two orders, or one order, respectively. When a function may be expressed by means of r orders less than it contains letters, I call it a function minus r orders. I now proceed to state my theorem.

Let U and V be functions each of the same m letters, and suppose that the determinant in respect of those letters of $U + \mu V$ contains i pairs of

[* pp. 138, 139 above. ED.]

equal linear factors of μ; then it is possible, by means of i linear equations instituted between the letters, to make U and V each become functions of *the same* $m-2i$ orders; and conversely, if by i equations between the letters U and V may be made functions of *the same* $m-2i$ orders, the determinant of $U+\mu V$ considered as a function of μ will contain i square factors.

Thus when $m=2n$ and $i=n$, U and V will each become functions of zero orders, that is, will both disappear, provided that on the institution of a certain system of n linear equations, among the letters of which U and V are functions, the determinant of $(U+\mu V)$ is a perfect square,—which is the theorem given in the article referred to.

So for example if U and V be quadratic functions of four letters, and therefore the characteristics of two conoids, $\square\,(U+\mu V)$ being a perfect square, expresses that these conoids have a straight line in common lying upon each of their surfaces.

If U and V be quadratic functions of three letters only, and admit therefore of being considered as the characteristics of two conics, $\square\,(U+\mu V)$ containing a square factor, is indicative of these conics having a common tangent at a common point, that is, of their touching each other at some point; for it is easily shown that the disappearance of two orders from any quadratic function by virtue of one linear function of its letters being zero, indicates that the line, plane, &c. of which the linear function is the characteristic is a tangent to the curve, surface, &c. of which the quadratic function is the characteristic.

I pass now to a generalization of the theorem which shows how to express, under the form of a double determinant, the resultant of one linear and two quadratic homogeneous functions of three letters (which I should have given in the original paper, had I not there been more intent upon developing an ascending scale than of expatiating upon a superficial ramification of analogies), and which constitutes my *Second addition* to that paper, to wit—

If U and V be homogeneous quadratic, and $L_1, L_2 \ldots L_n$ homogeneous linear functions of $(n+2)$ letters $x_1, x_2 \ldots x_{n+2}$, the determinant of the entire system of $n+2$ functions is equal to

$$\underset{\lambda,\,\mu}{\square}\ \underset{x_1,\,x_2\ldots x_{n+2}\,t_1,\,t_2\ldots t_n}{\boxed{\qquad\qquad\qquad}}\ \{\lambda U+\mu V+L_1t_1+L_2t_2+\ldots+L_nt_n\};$$

the demonstration is precisely similar to the analytical one given in the September Number* for the particular case of $n=1$.

When $n=0$, we revert to Mr Boole's theorem of elimination between U and V already adverted to. The proof, it will be easily recognized, does not require the application of the more general theorem relative to the simul-

[* p. 140 above.]

taneous depression of orders of two quadratic functions, but only the limited one before given, which supplies the conditions of their simultaneous disparition. I now proceed to develope more particularly certain analogies between the theory of the mutual contacts of two conics, and that of the tangencies to the intersection of two conoids.

But here again I must anticipate some of the results which will be given in my forthcoming memoir on Determinants and Quadratic Functions, by explaining what is to be understood by minor determinants, and the relation in which they stand to the complete determinant in which they are included. This preliminary explanation, and the statement of the analogies above alluded to, will constitute my *Third* and last *addition.*

Imagine any determinant set out under the form of a square array of terms. This square may be considered as divisible into lines and columns. Now conceive any one line and any one column to be struck out, we get in this way a square, one term less in breadth and depth than the original square; and by varying in every possible manner the selection of the line and column excluded, we obtain, supposing the original square to consist of n lines and n columns, n^2 such minor squares, each of which will represent what I term a First Minor Determinant relative to the principal or complete determinant. Now suppose two lines and two columns struck out from the original square, we shall obtain a system of $\left\{\frac{n(n-1)}{2}\right\}^2$ squares, each two terms lower than the principal square, and representing a determinant of one lower order than those above referred to. These constitute what I term a system of Second Minor Determinants; and so in general we can form a system of rth minor determinants by the exclusion of r lines and r columns, and such system *in general* will contain

$$\left\{\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}\right\}^2$$

distinct determinants.

I say "*in general*"; because if the principal determinant be totally or partially symmetrical in respect to either or each of its diagonals, the number of distinct determinants appertaining to each system of minors will undergo a material diminution, which is easily calculable.

Now I have established the following law:—

The whole of a system of rth minors being zero, implies only $(r+1)^2$ equations, that is, by making $(r+1)^2$ of these minors zero, all will become zero; and this is true, no matter what may be the dimensions or form of the complete determinant. But furthermore, if the complete determinant be formed from a quadratic function, so as to be symmetrical about one of its diagonals, then $\frac{1}{2}(r+1)(r+2)$ only of the rth minors being zero, will serve to imply that all these minors are zero. Of course, in applying these theorems, care must be taken that the $(r+1)^2$ or $\frac{1}{2}(r+1)(r+2)$ selected equations must be mutually non-implicative, and shall constitute independent conditions.

In the application I am about to make of these principles, we shall have only to deal with a system of *first* minors and of a *symmetrical* determinant. If three of these properly selected be zero, from the foregoing it appears that all must be zero.

Now let U and V be characteristics of two conics, that is, let each be a function of only three letters, it may be shown (see my paper* in the *Cambridge and Dublin Mathematical Journal* for November, 1850) that the different species of contacts between these two conics will correspond to peculiar properties of the compound characteristic $U+\mu V$.

If the determinant of this function have two equal roots, the conics simply touch; if it have three equal roots, the conics have a single contact of a higher order, that is, the same curvature; if its six first minors become zero simultaneously for the same value of μ, the conics have a double contact. If the same value of μ, which makes all these first minors zero, be at the same time not merely a double root (as of analytical necessity it always must be) but a treble root of

$$\square(U+\mu V)=0,$$

then the conics have a single contact of the highest possible order short of absolute coincidence, that is, they meet in four consecutive points.

The parallelism between this theory and that of two quadratic functions P, Q, and one linear function L† of four letters, say x, y, z, t, is exact‡. For let $P+Lu+\mu Q$ be now taken as our compound characteristic (a function, it will be observed, of five letters, x, y, z, t, u); if its determinant have two equal roots, L has two consecutive points in common with the intersection of P and Q, that is, passes through a tangent to that intersection; if it have three equal roots, L has three consecutive points in common with the said intersection, that is, is an osculating plane thereto; if its fifteen first minors admit of all being made simultaneously zero, L has a double contact with the intersection of P and Q, that is, it is a tangent plane to some one of the four cones of the second order containing this intersection;

[* p. 119 above.]

† Observe that $P=0$, $Q=0$, $L=0$ now express the equations to two conoids and a plane respectively.

‡ This parallelism may be easily shown analytically to imply, and be implied, in the geometrical fact, that the contact of the plane L with the intersection of the two surfaces P and Q, is of exactly the same kind as the contact (which must exist) between the two conics which are the intersections of P and Q respectively with the plane L.

if the same linear function of μ which enters into all these first minors be contained cubically in the complete determinant, then the plane L passes through four consecutive points of the intersection of P and Q, and the points where it meets the curve will be points of contrary plane flexure; and, as it seems to me, at such points the tangential direction of the curve must point to the summit of one or other of the four cones above alluded to*. In assigning the conditions for L being a double tangent plane to the intersection of P and Q, we may take any three independent minors at pleasure equal to zero. One of these may be selected so as to be clear of the coefficients of L; in fact, the determinant of $P+\mu Q$ will be a first minor of $P+\mu Q+Lu$; μ may thus be determined by a biquadratic equation; and then, by properly selecting the two other minors, we may obtain two equations in which only the first powers of the coefficients of x, y, z, t in L appear, and may consequently obtain L under the form of

$$(ae+\alpha)\,x+(be+\beta)\,y+(ce+\gamma)\,z+(de+\delta)\,t,$$

where a, α; b, β; c, γ; d, δ will be known functions of any one of the four values of μ. The point of contact being given will then serve to determine e, and we shall thus have the equation to each of the four double tangent planes at any given point fully determined.

In the foregoing discussions I have freely employed the word *characteristic* without previously defining its meaning, trusting to that being apparent from the mode of its use. It is a term of exceeding value for its significance and brevity. The characteristic of a geometrical figure† is the function which, equated to zero, constitutes the equation to such figure. Plücker, I think, somewhere calls it the line or surface function, as the case may be. Geometry, analytically considered, resolves itself into a system of rules for the construction and interpretation of characteristics. One more remark, and I have done. A very comprehensive theorem has been given at the commencement of this commentary, for interpreting the effect of a complete determinant of a linear function of two quadratic functions $(U+\mu V)$, having

* If this be so, then we have the following geometrical theorem :—"*The summit of one of the four cones of the second degree which contain the intersections of two surfaces of the second order drawn in any manner respectively through two given conics lying in the same plane, and having with one another a contact of the third degree, will always be found in the same right line, namely in the tangent line to the two given conics at the point of contact.*"

† More generally, the *characteristic* of any fact or existence is the function which, equated to zero, expresses the condition of the actuality of such fact or existence.

Perhaps the most important pervading principle of modern analysis, but which has never hitherto been articulately expressed, is *that*, according to which we infer, that when one fact of whatever kind is implied in another, the characteristic of the first must contain as a factor the characteristic of the second; and that when two facts are mutually involved, their characteristics will be powers of the same integral function.

The doctrine of characteristics, applied to dependent *systems* of facts, admits of a wide development, logical and analytical.

one or more pairs of equal factors $(e+\epsilon\mu)$. But here a far wider theory presents itself, of which the aim should be to determine the effect and meaning of this determinant, having any amount and distribution of multiplicity whatsoever among its roots. Nor must our investigations end at that point; but we must be able to determine the meaning and effect of common factors, one or more entering into the successive systems of *minor* determinants derived from the complete determinant of $U+\mu V$.

Nor are we necessarily confined to two, but may take several quadratic functions simultaneously into account.

Aspiring to these wide generalizations, the analysis of quadratic functions soars to a pitch from whence it may look proudly down on the feeble and vain attempts of geometry proper to rise to its level or to emulate it in its flights.

The law which I have stated for assigning the number of independent, or to speak more accurately, non-coevanescent determinants belonging to a given system of minors, I call the Homaloidal law, because it is a corollary to a proposition which represents analytically the indefinite extension of a property common to lines and surfaces to all loci (whether in ordinary or transcendental space) of the first order, all of which loci may, by an abstraction derived from the idea of levelness common to straight lines and planes, be called Homaloids. The property in question is, that neither two straight lines nor two planes can have a common segment; in other words, if n independent relations of rectilinearity or of coplanarity, as the case may be, exist between triadic groups of a series of $n+2$, or between tetradic groups of a series of $n+3$ points respectively, then every triad or tetrad of the series, according to the respective suppositions made, will be in rectilinear or in plane order. So, too, if n independent relations of *coincidence* exist between the duads formed out of $n+1$ points, every duad will constitute a coincidence.

This homaloidal law has not been stated in the above commentary in its form of greatest generality. For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding to which may be termed determinants of the pth order. We have, then, the following proposition. The number of uncoevanescent determinants constituting a system of the pth order derived from a given matrix, n terms broad and m terms deep, may equal, but can never exceed the number

$$(n-p+1)(m-p+1).$$

Remark on PASCAL'S *and* BRIANCHON'S *Theorems.*

I omitted to state, in the September Number of the *Journal**, that the demonstration there given by me for Pascal's, applied equally to Brianchon's theorem. This remark is of the more importance, because the fault of the analytical demonstrations hitherto given of these theorems has been, that they make Brianchon's consequence of Pascal's, instead of causing the two to flow simultaneously from the application of the same principles. No demonstration can be held valid in *method*, or as touching the essence of the subject-matter, in which the indifference of the duadic law is departed from. Until these recent times, the analytic method of geometry, as given by Descartes, had been suffered to go on halting as it were on one foot. To Plücker was reserved the honour of setting it firmly on its two equal supports by supplying the complementary system of coordinates. This invention, however, had become inevitable, after the profound views promulgated by Steiner, in the introduction to his Geometry, had once taken hold of the minds of mathematicians. To make the demonstration in the article referred to apply, *totidem literis*, to Brianchon's theorem (recourse being had to the correlative system of coordinates), it is only needful to consider U as the characteristic of the tangential envelope of the conic, x, y, z, t, u, v as the characteristics of the six points of the *circumscribed* hexagon, ϕ the characteristic of the point in which the line x, v meets the line z, t; $ay - \alpha u$ will then be shown to characterize the point in which t, x meets v, z; and thus we see that y, u; t, x; v, z, the three pairs of opposite sides of the hexagon, will meet in one and the same point, which is Brianchon's theorem.

[* p. 138 above.]

26.

ON THE SOLUTION OF A SYSTEM OF EQUATIONS IN WHICH THREE HOMOGENEOUS QUADRATIC FUNCTIONS OF THREE UNKNOWN QUANTITIES ARE RESPECTIVELY EQUATED TO NUMERICAL MULTIPLES OF A FOURTH NON-HOMOGENEOUS FUNCTION OF THE SAME.

[*Philosophical Magazine*, XXXVII. (1850), pp. 370—373.]

LET U, V, W be three homogeneous quadratic functions of x, y, z, and let ω be any function of x, y, z of the nth degree, and suppose that there is given for solution the system of equations

$$U = A\omega,$$
$$V = B\omega,$$
$$W = C\omega.$$

Theorem. The above system can be solved by the solution of a cubic equation, and an equation of the nth degree.

For let D be the determinant in respect to x, y, z of

$$fU + gV + hW,$$

then D is a cubic function of f, g, h. Now make

$$D = 0, \quad Af + Bg + Ch = 0;$$

the ratios of $f : g : h$ which satisfy the last two equations can be determined by the solution of a cubic equation, and there will accordingly be three systems of f, g, h which satisfy the same, as

$$f_1, \quad g_1, \quad h_1,$$
$$f_2, \quad g_2, \quad h_2,$$
$$f_3, \quad g_3, \quad h_3.$$

Now $D = 0$ implies that $fU + gV + hW$ breaks up into two linear factors; accordingly we shall find

$$(l_1x + m_1y + n_1z)(\lambda_1x + \mu_1y + \nu_1z) = 0,$$
$$(l_2x + m_2y + n_2z)(\lambda_2x + \mu_2y + \nu_2z) = 0,$$
$$(l_3x + m_3y + n_3z)(\lambda_3x + \mu_3y + \nu_3z) = 0,$$

in which the several sets of l, m, n; λ, μ, ν can be expressed without difficulty in terms of the several values of $\sqrt{f}, \sqrt{g}, \sqrt{h}$.

Let the above equations be written under the form

$$PP' = 0,$$
$$QQ' = 0,$$
$$RR' = 0.$$

Since the given equations are perfectly general, it is readily seen that the equations

$$(P = 0,\ P' = 0),\quad (Q = 0,\ Q' = 0),\quad (R = 0,\ R' = 0),$$

will severally represent pairs of opposite sides of a quadrangle expressed by general coordinates x, y, z; so that one of the two functions R, R' will be a linear function of P and Q and also of P' and Q', and the other will be a linear function of P and Q' and also of P' and Q*.

In order to solve the equations, we need only consider two such pairs as $PP' = 0$, $QQ' = 0$; we then make

$$P = 0,\quad Q = 0,$$
or
$$P = 0,\quad Q' = 0,$$
or
$$P' = 0,\quad Q = 0,$$
or
$$P' = 0,\quad Q' = 0.$$

Any one of these four systems will give the ratios of $x:y:z$; and then, by substitution in any one of the given equations, we obtain the values of x, y, z by the solution of an ordinary equation of the nth degree. The number of systems x, y, z is therefore always $4n$.

The equations connected with the solution of Malfatti's celebrated problem, "In a given triangle to inscribe three circles such that each circle touches the remaining two circles and also two sides of the triangle," given by Mr Cayley in the November Number for 1849 of the *Cambridge and Dublin Mathematical Journal*, to wit,

$$by^2 + cz^2 + 2fyz = \theta^2 a\,(bc - f^2) = A,$$
$$cz^2 + ax^2 + 2gzx = \theta^2 b\,(ca - g^2) = B,$$
$$ax^2 + by^2 + 2hxy = \theta^2 c\,(ab - h^2) = C,$$

come under the general form which has just been solved. It so *happens*, however, that in this particular case

$$\left.\begin{array}{lll} f_1, & g_1, & h_1 \\ f_2, & g_2, & h_2 \\ f_3, & g_3, & h_3 \end{array}\right\},$$

* Were it not for this being the case, the number of solutions would be n times the number of ways of obtaining duads out of three sets of two things, excluding the duads forming the sets, that is, the number of solutions would be $12n$ in place of $4n$, the true number.

become respectively

$$\left.\begin{matrix} 0, & \dfrac{1}{B}, & -\dfrac{1}{C} \\ -\dfrac{1}{B}, & 0, & \dfrac{1}{C} \\ -\dfrac{1}{C}, & \dfrac{1}{B}, & 0 \end{matrix}\right\},$$

and the cubic equation is resolved without extraction of roots.

It follows from my theorem that the eight intersections of three concentric surfaces of the second order can be found by the solution of one cubic and one quadratic equation; and in general, if we have ϕ, ψ, θ any three quadratic functions of x, y, z, and $\phi=0$, $\psi=0$, $\theta=0$ be the system of equations to be solved, provided that we can by linear transformations express ϕ, ψ, θ under the form of

$$U-aw,$$
$$V-bw,$$
$$W-cw,$$

U, V, W being homogeneous functions, and w a non-homogeneous function of three new variables, x', y', z', we can find the eight points of intersection of the three surfaces, of which U, V, W are the characteristics, by the solution of one cubic and one quadratic. But (as I am indebted to Mr Cayley for remarking to me) that this may be possible, implies the coincidence of the vertices of one cone of each of the systems of four cones in which the intersections of the three surfaces taken two and two are contained.

I may perhaps enter further hereafter into the discussion of this elegant little theory. At present I shall only remark, that a somewhat analogous mode of solution is applicable to two equations,

$$U=aP^2,$$
$$V=bP^2,$$

in which U, V are homogeneous quadratic functions, and P some non-homogeneous function of x, y.

We have only to make the determinant of $fU+gV$ equal to zero, and we shall obtain two systems of values of f, g, wherefrom we derive

$$l_1x+m_1y=\pm\sqrt{(af_1+bg_1)}\,P,$$
$$l_2x+m_2y=\pm\sqrt{(af_2+bg_2)}\,P,$$

from which x and y may be determined.

27.

ON A PORISMATIC PROPERTY OF TWO CONICS HAVING WITH ONE ANOTHER A CONTACT OF THE THIRD ORDER.

[*Philosophical Magazine*, XXXVII. (1850), pp. 438, 439.]

If two conics have with one another a contact of the third order, that is, if they intersect in four consecutive points, it will easily be seen that their *characteristics* referred to coordinate axes in the plane containing them must be of the relative forms x^2+yz, $k(y^2+x^2+yz)$ respectively, y characterizing their common tangent at the point of contact*.

Hence if we take planes of reference in space, and call t the characteristic of the plane of the conics, the equations to any two conoids drawn through them respectively will be of the relative forms

$$U = x^2 + yz + tu = 0,$$

$$V = y^2 + x^2 + yz + tv = 0.$$

Using W to denote $V - U$, and (W) to denote what W becomes when ey is substituted for t, we see that W and (W) are of the respective forms y^2+tw and $y\theta$; showing that the former is the characteristic of a cone which will be cut by any plane $t-ey$ drawn through the line (t, y) in a pair of right lines; or, in other words, that one of the cones containing the intersection of the two variable conoids (V and U) will have its vertex in the *invariable line* which is the common tangent to the two fixed conics: this proves the theorem stated by me hypothetically in a foot-note in one of my papers in the last number of the *Magazine*†. The steps of the geometrical proof there hinted at are as follows.

* These relative or conjugate forms are taken from a table which I shall publish in a future number of this *Magazine*, exhibiting the conjugate characteristics in their simplest forms, correspondent to all the various species of contacts possible between lines and surfaces of the second degree. This table is as important to the geometer as the fundamental trigonometrical formulæ to the analyst, or the multiplication table to the arithmetician; and it is surprising that no one has hitherto thought of constructing such.

[† p. 149 above.]

The four consecutive points in which the two conics intersect will be consecutive points in the curve of intersection of the two variable conoids. This curve lies in each of four cones of the second degree. Every double tangent plane to it passes through the vertex of one amongst these. The plane containing four, that is, two (consecutive) pairs of consecutive points, is a double tangent plane, and will therefore pass through a vertex ; but four consecutive points of a curve of the fourth order described upon a cone, and lying in one tangent plane thereto, can only be *conceived* generally as disposed in the form of an ʃ, of which the belly part will point to the vertex; or, in other words, at any point where two consecutive osculating planes coincide so that the *spherical* curvature vanishes, the linear curvature will also vanish, that is, there will be a point of inflexion at which, of course, the tangent line must pass through the vertex of the cone. This is the assumption felt to be true, but stated by me hypothetically in the paper referred to, because a ready demonstration did not at the moment occur to me. The legitimacy of this inference is now vindicated by the above analytical demonstration.

The methods of general and correlative coordinates and of determinants combined possess a perfectly irresistible force (to which I can only compare that of the steam-hammer in the physical world) for bringing under the grasp of intuitive perception the most complicated and refractory forms of geometrical truth.

28.

ON THE ROTATION OF A RIGID BODY ABOUT A FIXED POINT.

[*Philosophical Magazine,* XXXVII. (1850), pp. 440—444.]

In the *Cambridge and Dublin Mathematical Journal* for March 1848, an article by Professor Stokes, of the University of Cambridge, is ushered in with the words following:—

"The most general *instantaneous** motion of a rigid body moveable in all directions about a fixed point consists in a motion of rotation about an axis passing through that point. This elementary proposition is sometimes assumed as self-evident, and sometimes deduced as the result of an analytical process. It ought hardly *perhaps* to be assumed, but it does not seem desirable to refer to a long algebraical process for the demonstration of a theorem so simple. Yet I am not aware of a geometrical proof anywhere published which might be referred to."

The learned and ingenious professor is indubitably right, and might have trusted himself to assert less hesitatingly the necessity of demonstrating this proposition, which possesses none of the characters of a self-evident truth; but it is to be regretted that he should have stated it in such a form as naturally to lead the incautious reader to mistake the nature and grounds of its existence, which consist in this fact—that any kind of displacement of a body moveable about a fixed axis, whether instantaneous and infinitesimal, or secular and finite, is capable of being effected by a single rotation about a single axis.

The annexed simple proof of this capital law has the advantage of affording a rule for compounding into one any two (and therefore any number of) rotations given in direction, magnitude and *order of succession.*

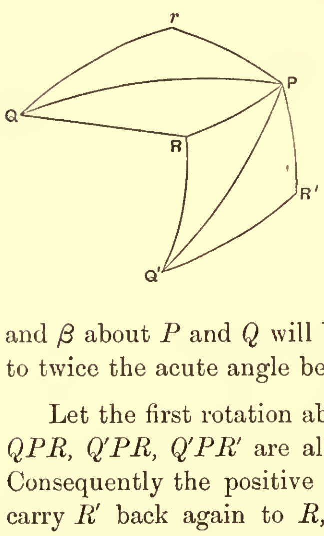
* The italics do not exist in the original.

It will somewhat conduce to simplicity if we fix our attention upon a spherical surface rigidly connected with the rotating body, and having its centre at the fixed point thereof. When the positions of two points in this are given, the position of the body is completely determined.

Now evidently two points A, B may be brought respectively to $A'B'$ (if $AB = A'B'$) by two rotations; the first taking place about a pole situated anywhere in the great circle bisecting AA' at right angles, the second about A', the position into which it is brought by the first rotation. This view leads us to consider the effect of two rotations taking place successively about two axes fixed in *the rotating body*. Or again, we may make the plane $A'B'$ revolve into the position AB round a pole taken at the node in which the two planes intersect, and then the points A, B swing into their new positions A', B' by means of a rotation about the pole of the great circle, of which $A'B'$ forms a part. This mode of effecting the displacement naturally suggests the consideration of the effect of rotations taking place successively about two axes fixed in *space*.

First, then, let us study the effect of the combination of a rotation (α) having P for its pole, followed by another (β), of which Q is the pole, P and Q being points in the surface of the revolving sphere.

In drawing the annexed figure, I have supposed that the two rotations are of the same kind, each tending, when a spectator is standing with his head to the respective poles and his feet to the centre, to make a point at his right-hand pass *in front of his face* towards his left-hand. Let now PQ revolve through $\frac{\alpha}{2}$ positively into the position of PR, and through $\frac{\beta}{2}$ negatively into that of QR.



Then I say that the two impressed rotations α and β about P and Q will be equivalent to a single rotation about R, equal to twice the acute angle between QR, RP.

Let the first rotation about P bring Q to Q' and R to R'; it is clear that QPR, $Q'PR$, $Q'PR'$ are all equal triangles. Therefore $R'Q'R = 2PQR = \beta$. Consequently the positive rotation β about Q' (the new position of Q) will carry R' back again to R, its original position. Hence the actual motion which results from the successive rotations combined being consistent with R remaining at rest, must be equivalent to a single rotation about R.

To find its magnitude, let the second rotation carry P to P'*; then the angular displacement PRP' (which is the required rotation of the whole

* The reader is requested to fill in the point P' and join $P'R$.

body) is equal to twice the acute angle between $Q'R$, RP, which is the same as that between QR, RP, as was to be shown. Thus we see that the semi-rotations about three poles (considered as the angular points of a spherical triangle), which, taken in order, would bring the sphere back to its first undisturbed position, are equal to the included angles at such poles respectively.

If in our figure the order of the rotations had been reversed, PQr, QPr would have been taken respectively equal to PQR, QPR, but on the opposite side of PQ, and r would have been the resultant pole, the resultant rotation remaining in amount the same as before.

If either of the rotations had been negative, the resultant pole would be found in QR produced, namely, at the intersection of rQ or rP with PQ.

Calling the resultant rotation γ, we have always

$$\sin\frac{\alpha}{2} : \sin\frac{\beta}{2} : \sin\frac{\gamma}{2} :: \sin QR : \sin RP : \sin PQ.$$

When the component rotations are infinitesimal in amount, R and r will come together in QP; the order of succession of the rotations will be indifferent, and we shall have

$$\alpha : \beta : \gamma :: \sin\frac{\alpha}{2} : \sin\frac{\beta}{2} : \sin\frac{\gamma}{2} :: \sin QR : \sin RP : \sin PQ,$$

which gives the rule for the parallelogrammatic composition of two simultaneously impressed rotations*.

If, next, we consider the effect of rotations about two poles, P and Q, fixed in space (supposing, as above, that they take place first about P and then about Q), we must take QPr equal to half the *contrary* of the rotation about P, and PQr to half the *direct* rotation about Q (the angle being now taken positive which was on the first supposition negative, and *vice versâ*); so that, retaining the original figure, the first rotation will bring r to R, and the second carry R back to r; showing that r is the resultant pole, and that† $P'rP$, the resultant rotation, will be double the acute angle between Qr, rP, as in the former case.

To popular apprehension the important doctrine of uniaxial rotation may be made intelligible by the following mode of statement. Take a pocket-globe, open the case and roll about the sphere within it in any manner whatever; then closing the case, there will unavoidably remain two points on the terrestrial surface touching the same two points on the celestial surface as they were in apposition with before the sphere was so turned about in its case.

* Compare Mr Airy's *Tracts*, Art. "On Precession and Nutation."

† P' is not expressed in the figure given.

It is right to bear in mind that the whole of this doctrine is comprised in, and convertible with, the following easy geometrical proposition relative to arcs of great circles on any spherical surface, including the plane as an extreme case.

"The arcs joining the extremities (each with each in *either* order) of two other equal arcs, subtend equal angles at either of the points of intersection of two great circles bisecting at right angles the first-named connecting arcs*."

The spherico-triangular mode of compounding rotations given in the above simple disquisition may easily be made the parent of a whole brood of geometrical consequences, which, however, I must leave to the ingenuity and care of those who have a turn for this kind of invention.

But I ought not to omit to invite attention to a remarkable form, which may be imparted to the theorems above stated for the composition of finite rotations, or rather to a theorem which may be derived from them by an obvious process of inference.

Let P, Q, $R \ldots X$, Z be any number of points on a sphere capable of moving about its centre, joined together by arcs of great circles so as to form a spherical polygon. Imagine any number of rotations to take place about these points in succession as poles. It matters not which is considered the first pole of rotation, but the *order* of the circulation must be supposed given, as, for instance, $PQR \ldots XZ$, or $QR \ldots XZP$, or $R \ldots XZPQ$, &c. This will be one order; the reverse order would be $PZX \ldots RQ$, or $QPZX \ldots R$, &c.

I shall suppose the circulation to be of the kind first above written. Now we may make two hypotheses:—

1. That the poles are fixed in space.

2. That they are fixed in the rotating body.

In the first case, let the rotations about the given poles $P, Q, R, S \ldots X, Z$ be double the amounts which would serve to transport PQ to QR, QR to $RS \ldots XZ$ to ZP respectively.

In the second case, let the rotations be double the amounts which would carry PZ to $ZX \ldots SR$ to RQ, RQ to QP respectively. Then, on either supposition, the sum of the combined rotations is zero; or, to use a more convenient and suggestive form of expression, if the poles of rotation form a closed spherical polygon whose angles are respectively equal to the semi-rotations about the poles, the resultant rotation is zero.

* This proposition will be seen to be immediately demonstrable, by the comparison of equal triangles, when viewed as the converse of this other. "The arcs (or right lines) joining the correspondent extremities of the bases of two similar isosceles spherical (or plane) triangles having a common vertex, are equal to each other."

This proposition is immediately derivable from the fundamental one relative to three poles, given above, by dividing the polygon into triangles by arcs, joining any one of the poles with all the rest, or (as pointed out to me by my eminent friend Prof. W. Thomson) it becomes apparent as a particular case of a more general proposition, on representing the motion about the successive axes as effected by two equal pyramids having a common vertex at the centre of motion, of which the one is fixed in space, and the other is fixed in the revolving body and rolls over the first, so that the corresponding equal faces are successively brought into coincident apposition.

P.S. To find the pole of rotation whereby PQ may be brought into the position $P'Q'$, we may use the following simple construction.

Measure off from O the node of the great circles (or right lines) containing PQ and $P'Q'$, two distances in the proper direction upon each (four distinct assumptions may be made), say OR and OS equal to one another and to the difference between OP and OP', then the pole of rotation required, say E, is the centre of the circle described about ROS, and the amount of rotation is the angle subtended by OR or OS at E. The writer of this paper suggests that *axis of displacement* would be a convenient term for designating the line whereby any finite change in the position of a body moveable about a fixed centre may be brought about; a geometrical theory of rotation leading to the investigation of a very curious species of correlation, now opens upon the view, the general object of which may be stated as follows:

"Given upon a sphere or plane any curve considered as the locus of successive poles of instantaneous rotation, and the ratio of the rotation about each pole to its distance from the one that follows*, to construct the curve of the poles of displacement, and to determine the amount of rotation corresponding to each such pole."

The discussion of this question offers a fine field for the exercise of geometrical taste and skill.

* Which by analogy may be termed the "density of rotation."

29.

ON THE INTERSECTIONS OF TWO CONICS.

[*Cambridge and Dublin Mathematical Journal,* VI. (1851), pp. 18—20.]

LET the two conics be written

$$U = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

$$V = \alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha' yz + 2\beta' zx + 2\gamma' xy = 0,$$

and make

$$U + \lambda V = Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy.$$

In my paper in the last number of the *Journal**, I showed that the case of intersection of the two conics in two points was distinguishable from all other cases by the equation $\square(U + \lambda V) = 0$ having two imaginary roots. When all the roots are real, the curves either intersect in four points or not at all.

On the former supposition,

$$-C'^2 + AB, \quad -A'^2 + BC, \quad -B'^2 + CA,$$

which are quadratic functions of λ, will be negative for all three values of λ. On the contrary supposition, one value of λ will make all these three quantities negative, but the other two values with each make them all three positive.

Hence we obtain a symmetrical criterion (which I strangely omitted to state in my former paper) by forming the quantity

$$A'^2 + B'^2 + C'^2 - AB - AC - BC.$$

A cubic equation

$$Ly^3 + My^2 + Ny + P = 0$$

may be then constructed, of which the three values of the above function corresponding to three values of λ will be the roots.

The condition for *real* intersection is that L, M, N, P should be all of the same sign. The conics being supposed real, L and P are necessarily in both cases of the same sign. The condition is therefore satisfied if either $L, M,$

[* p. 119 above.]

N, or M, N, P be of the same sign, and is consequently equivalent to the condition that $\frac{M}{L}$ and $\frac{N}{L}$ shall be both positive, or $\frac{N}{P}$ and $\frac{M}{P}$ both positive. It does not appear to be possible in the nature of the question to find a criterion for distinguishing between the two cases, dependent on the sign of one single function of the coefficients.

The case of double contact, abstraction being made of binary intersection, is a sort of intermediary state between intersection in four points and non-intersection; and accordingly, as shown in my former paper for this case, the two equal values of λ will make the three quantities

$$AB - C'^2, \quad BC - A'^2, \quad CA - B'^2$$

all real; so that two of the values of y corresponding to the equal values of λ are zero, and the criterion becomes nugatory as it ought to do.

Again, when the two conics do not intersect, I distinguished two cases according as they lie each without, or one within the other, that is, according as they have four common tangents or none.

But, as Mr Cayley has well remarked to me, a similar distinction exists when the conics intersect in four points; in that case also they may have four common tangents or not any: when they intersect in two points they have necessarily two and only two common tangents. There is no difficulty in separating these four cases.

Let the conics be written

$$(U) = \xi^2 + \eta^2 - \zeta^2,$$
$$(V) = A\xi^2 + B\eta^2 - C\zeta^2,$$

(U) and (V) being what U and V become when the coordinates are changed from x, y, z to ξ, η, ζ.

A, B, C are the three values of λ in the equation

$$\square(V - \lambda U) = 0.$$

If the curves intersect $A - C$, $B - C$ must have different signs, that is, C must be an intermediary quantity between A and B.

Again, the tangential equations to the conics expressed by the correlative system of coordinates will be

$$\xi_1^2 + \eta_1^2 - \zeta_1^2 = 0,$$
$$\frac{\xi_1^2}{A} + \frac{\eta_1^2}{B} - \frac{\zeta_1^2}{C} = 0;$$

and that these may have four real systems of roots,

$$\frac{1}{A} - \frac{1}{C}, \quad \frac{1}{C} - \frac{1}{B}$$

must have the same sign; and consequently, as $A - C$ and $C - B$ are supposed to have the same sign, A and B, and therefore all three A, B, C, have the same sign. We have therefore the following rule:

Let the equation in λ, namely, $\square(U+\lambda V)=0$, be called $\theta=0$, and the equation in y, above given, $\omega=0$. By an equation being congruent or incongruent, understand that its roots have all the same sign or not all the same sign.

Then ω congruent, θ congruent, implies that the intersections and common tangents are both real; ω congruent, θ incongruent, implies that the intersections are real, but the common tangents imaginary; ω incongruent, θ congruent, implies that the intersections and common tangents are both imaginary; ω incongruent, θ incongruent, implies that the intersections are imaginary, but the common tangents real.

In like manner, as the cases of contact of lines are limiting cases to those which relate to the relative configurations of their points of intersection, so the cases of contact of surfaces are limiting cases in which the characters which usually separate the different forms of their curve of intersection exist blended and indistinguishable. The first step therefore to the study of the particular species of the curve of the fourth degree*, in which two surfaces of the second degree intersect, is to obtain the analytical and geometrical characters of their various species of contact. Accordingly I have made an enumeration of these different species, no less than 12 in number, many of them highly curious and I believe unsuspected, which the reader may consult in the *Philosophical Magazine* for February, 1851†.

By the aid of these landmarks, I have little doubt, should time and leisure permit, of mapping out a natural arrangement of the principal distinctions of form between that class at least of lines in space of the fourth order which admit of being considered the complete intersection of two surfaces.

* I have found that the 16 points of spherical flexure in this curve are the four sets of four points in which it meets the four faces of the pyramid whose summits are the vertices of the four cones of the second degree in which the curve is completely contained, which 16 points reduce to 4 when the two surfaces have an ordinary contact, and to 1 when they have a cuspidal contact: of course in the case of contact the pyramid above described in a manner folds up and vanishes, as there are no longer 4 distinct vertices. I have found also that when the factors of $\square(U+\lambda V)$, (U and V being the characteristics of the two surfaces) are all unreal, the points of flexure are all unreal. When two factors are real and two imaginary, two of the faces of the pyramid (namely, its two real faces) will each contain one (and only one) pair of real points of flexure, and the other two planes none; and lastly, when the factors of $\square(U+\lambda V)$ are all real, then either all the points of flexure are imaginary, or else all the eight contained in a certain two of the pyramidal faces are real: and these two cases admit of being distinguished by a method analogous in its general features to that whereby I have shown in the text above how to distinguish between the cases of 4 real and 4 imaginary points of intersection of two conics. Where the two surfaces have an ordinary contact, the curve of intersection, it is well known, has a double point; and where the surfaces have a higher contact, the curve has a cusp. Thus in the fact of the 16 flexures reducing to 4 and to 1 in these respective cases, we see a beautiful analogy to what takes place with the 9 flexures of a plane curve of the third degree, which contract to 3 and 1, according as the curve has a double point or a cusp.

[† p. 219 below.]

30.

ON CERTAIN GENERAL PROPERTIES OF HOMOGENEOUS FUNCTIONS.

[*Cambridge and Dublin Mathematical Journal,* VI. (1851), pp. 1—17.]

LET χ denote the operation

$$x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \dots + x_n \frac{d}{da_n},$$

and A the operation

$$a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} + \dots + a_n \frac{d}{dx_n}:$$

and now suppose that ω, a homogeneous function of ι dimensions of $a_1, a_2 \dots a_n$, and not of any of the quantities $x_1, x_2 \dots x_n$, is subjected to the successive operations indicated by $A^s \chi^r$.

We have

$$A^s \chi^r \omega = A^{s-1} A \chi^r \omega,$$

$$A\chi^r \omega = \left(a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} + \dots + a_n \frac{d}{dx_n}\right)\left(x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \dots + x_n \frac{d}{da_n}\right)^r \omega$$

$$= r\left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + \dots + a_n \frac{d}{da_n}\right) \chi^{r-1} \omega$$

$$= r(\iota - r + 1) \chi^{r-1} \omega,$$

for $\chi^{r-1}\omega$ is of $(r-1)$ dimensions, lower than ω (which is of ι dimensions) in $a_1, a_2 \dots a_n$.

Hence

$$A^s \chi^r \omega = r(\iota - r + 1) A^{s-1} \chi^{r-1} \omega$$

$$= \text{\&c.} = \{r(r-1) \dots (r-s+1)\}$$
$$\{(\iota - r + 1)(\iota - r + 2) \dots (\iota - r + s)\} \chi^{r-s} \omega. \quad (1)$$

Now in the expression

$$\chi^r\omega\,(a_1,\ a_2 \ldots a_n),$$

suppose that we write

$$x_1 = u_1 + a_1\epsilon,$$

$$x_2 = u_2 + a_2\epsilon,$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$x_n = u_n + a_n\epsilon,$$

we have, by Taylor's theorem,

$$\chi^r\omega = U^r\omega + AU^r\omega\,\epsilon + A^2U^r\omega\,\frac{\epsilon^2}{1\,.\,2} + \ldots + A^rU^r\omega\,\frac{\epsilon^r}{1\,.\,2\,.\,3\ldots r},$$

where $U^r\omega$ denotes what $\chi^r\omega$ becomes, on substituting u's for x's, and A now represents

$$u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_n\frac{d}{da_n}.$$

This expansion stops spontaneously at the $(r+1)$th term, because $\chi^r\omega$ is only of r dimensions in $x_1,\ x_2 \ldots x_n$.

Applying now theorem (1), we obtain

$$\chi^r\omega = U^r\omega + r\,(\iota - r + 1)\,U^{r-1}\omega\epsilon + \tfrac{1}{2}r\,(r-1)\,\{(\iota - r + 1)\,(\iota - r + 2)\}\,U^{r-2}\omega\epsilon^2$$
$$+ \ldots + \{(\iota - r + 1)\,(\iota - r + 2)\ldots\iota\}\,\omega\epsilon^r. \qquad (2)$$

In using this theorem in the course of the ensuing pages, it will be found convenient to assign to ϵ a specific value, and I shall suppose it equal to $\dfrac{x_n}{a_n}$; this gives

$$u_1 = x_1 - \frac{a_1}{a_n}\,x_n,$$

$$u_2 = x_2 - \frac{a_2}{a_n}\,x_n,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_n = x_n - \frac{a_n}{a_n}\,x_n$$
$$= 0.$$

And inasmuch as the U symbol now contains $a_1,\ a_2, \ldots a_n$, so that UU^r no longer equals U^{r+1}, I shall write U_r for U^r. Theorem (2) will thus assume the form

$$\chi^r\omega = U_r\omega + r(\iota - r + 1)\,U_{r-1}\omega\frac{x_n}{a_n} + \tfrac{1}{2}r\,(r-1)\,(\iota - r + 1)\,(\iota - r + 2)\,U_{r-2}\omega\left(\frac{x_n}{a_n}\right)^2$$
$$+ \ldots + \{(\iota - r + 1)\ldots\iota\}\,\omega\left(\frac{x_n}{a_n}\right)^r, \qquad (3)$$

where U_r for all values of r denotes what

$$\left(x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \dots + x_{n-1} \frac{d}{da_{n-1}}\right)^r \omega$$

becomes, on substituting $u_1, u_2, \dots u_{n-1}$ for $x_1, x_2, \dots x_{n-1}$, *after* the processes of derivation have been completed: this it is essential to observe, because $u_1, u_2, \dots u_{n-1}$ now involve $a_1, a_2, \dots a_{n-1}, a_n$. The term $x_n \frac{d}{da_n}$ is omitted from the symbol of linear derivation, because in the substitutions x_n will be replaced by zero.

As an example of this last theorem, take

$$\omega = a^3 + b^3 + c^3 + kabc;$$

then

$$\chi\omega = 3a^2x + 3b^2y + 3c^2z + kbcx + kcay + kabz,$$

$$\chi^2\omega = 6ax^2 + 6by^2 + 6cz^2 + 2kcxy + 2kayz + 2kbzx,$$

$$\chi^3\omega = 6x^3 + 6y^3 + 6z^3 + 6kxyz.$$

$$U_1\omega = 3a^2\left(x - \frac{az}{c}\right) + 3b^2\left(y - \frac{bz}{c}\right) + kbc\left(x - \frac{az}{c}\right) + kca\left(y - \frac{bz}{c}\right),$$

$$U_2\omega = 6a\left(x - \frac{az}{c}\right)^2 + 6b\left(y - \frac{bz}{c}\right)^2 + 2kc\left(x - a\frac{z}{c}\right)\left(y - \frac{bz}{c}\right),$$

$$U_3\omega = 6\left(x - \frac{az}{c}\right)^3 + 6\left(y - \frac{bz}{c}\right)^3,$$

and it will be found that the equations given by theorem (3) are satisfied, namely

$$\chi\omega = U\omega + 3\frac{z}{c}\omega,$$

$$\chi^2\omega = U_2\omega + 2 \,.\, 2\frac{z}{c}U\omega + 2 \,.\, 3\frac{z^2}{c^2}\omega,$$

$$\chi^3\omega = U_3\omega + 3\frac{z}{c}U_2\omega + 3 \,.\, 1 \,.\, 2\frac{z^2}{c^2}U\omega + 1 \,.\, 2 \,.\, 3\frac{z^3}{c^3}\omega.$$

Probably, as this theorem is of rather a novel character, the annexed sketch of a somewhat different course of demonstration may be not unacceptable to my readers.

We have

$$\chi\omega = \left(x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \dots + x_n \frac{d}{da_n}\right)\omega;$$

and by the well-known law for homogeneous functions,

$$\iota\omega = \left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + \dots + a_n \frac{d}{da_n}\right)\omega.$$

Hence

$$\left(\chi - \iota\frac{x_n}{a_n}\right)\omega = \left(u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_{n-1}\frac{d}{da_{n-1}}\right)\omega$$
$$= U\omega.$$

Hence

$$\chi\omega = \left(U + \iota\frac{x_n}{a_n}\right)\omega,$$

$$\chi^2\omega = \left\{U + (\iota - 1)\frac{x_n}{a_n}\right\}\left(U + \iota\frac{x_n}{a_n}\right)\omega,$$

$$\chi^3\omega = \left\{U + (\iota - 2)\frac{x_n}{a_n}\right\}\left\{U + (\iota - 1)\frac{x_n}{a_n}\right\}\left(U + \iota\frac{x_n}{a_n}\right)\omega,$$

&c. = &c.

But in performing the process indicated by the several factors it must be carefully borne in mind that UU_r is not $= U_{r+1}$; this would be the case were it not for the terms $-\frac{a_1}{a_n}x_n$, $-\frac{a_2}{a_n}x_n$, &c., which enter into $u_1, u_2 \ldots u_{n-1}$. But on account of these terms, we have

$$UU_r\omega = \left(u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_{n-1}\frac{d}{da_{n-1}}\right)\left(u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_{n-1}\frac{d}{da_{n-1}}\right)^r$$

$$= U_{r+1}\omega - r\frac{x_n}{a_n}\left\{u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_{n-1}\frac{d}{da_{n-1}}\right\}^{r-1}\omega,$$

for
$$\frac{d}{da_1}u_1 = \frac{d}{da_2}u_2 = \ldots = \frac{d}{da_{n-1}}u_{n-1} = -\frac{x_n}{a_n}.$$

Hence

$$UU_r\omega = U_{r+1}\omega - r\frac{x_n}{a_n}U_r\omega.$$

Let $\frac{x_n}{a_n}$ be called ϵ; we find

$$\chi = U + \iota\epsilon,$$

$$\chi^2 = \{U + (\iota - 1)\epsilon\}(U + \iota\epsilon)$$
$$= UU + (2\iota - 1)\epsilon U + (\iota - 1)\iota\epsilon^2$$
$$= U_2 + 2(\iota - 1)\epsilon U + (\iota - 1)\iota\epsilon^2;$$

$$\chi^3 = \{U + (\iota - 2)\epsilon\}\chi^2$$
$$= UU_2 + 2(\iota - 1)\epsilon UU + (\iota - 1)\iota\epsilon^2 U$$
$$+ (\iota - 2)\epsilon U_2 + 2(\iota - 2)(\iota - 1)\epsilon^2 U + (\iota - 2)(\iota - 1)\iota\epsilon^3$$
$$= U_3 + 3(\iota - 2)\epsilon U_2 + 3(\iota - 2)(\iota - 1)\epsilon^2 U + (\iota - 2)(\iota - 1)\iota\epsilon^3.$$

The same process being continued will lead to results identical with those previously obtained and expressed in theorem (3).

The expansion of χ^r, treated according to this second method, appears to require the solution of the partial equation in differences

$$a_{r+1,\,s+1} = a_{r,\,s+1} + (\iota - 2r)\,a_{r,\,s},$$

$a_{0,s}$ being given as unity for $s = 1$ and as zero for all other values of s.

It is probable however that the solution of this equation might be evaded by some artifice peculiar to the particular case to be dealt with. I do not propose to dwell upon this inquiry, which would be foreign to the object of my present research. It may however not be out of place to make the passing remark, that the equations expressing χ^r in terms of powers of U admit easily of being reverted, as indeed may the more general form

$$\chi_r = u_r + \epsilon_r u_{r-1} + \frac{1}{1.2}\epsilon_r\epsilon_{r-1}u_{r-2} + \&c.$$

which becomes the equation of formula (3), on making

$$\epsilon_r = r(\iota + 1 - r)\frac{x_n}{a_n}, \quad \chi_r = \chi^r\omega, \text{ and } u_r = U_r\omega;$$

for let $$u_r = \epsilon_1\epsilon_2 \ldots \epsilon_r v_r,$$

$$\chi_r = \epsilon_1\epsilon_2 \ldots \epsilon_r y_r,$$

then $$y_r = v_r + v_{r-1} + \frac{v_{r-2}}{1.2} + \frac{v_{r-3}}{1.2.3} + \&c.;$$

whence $$v_r = e^{-\frac{d}{dr}}y_r$$

$$= y_r - y_{r-1} + \frac{y_{r-2}}{1.2} - \frac{y_{r-3}}{1.2.3} + \&c.:$$

and therefore $$u_r = \chi_r - \epsilon_r\chi_{r-1} + \tfrac{1}{2}\epsilon_r\epsilon_{r-1}\chi_{r-2} + \&c.$$

Thus we obtain, from equation (3),

$$U_r\omega = \chi^r\omega - r(\iota - r + 1)\chi^{r-1}\omega\frac{x_n}{a_n} + \&c.$$

As a first application of theorem (3), I shall proceed to show how Joachimsthal's equation to the surface drawn from a given point $(\alpha, \beta, \gamma, \delta)$ through the intersection of two surfaces $\phi(x, y, z, t) = 0$, $\theta(x, y, z, t) = 0$, may be expressed under the *explicit* form of the equation to a cone.

The equation in question is obtained by eliminating λ between

$$\phi\lambda^m + \chi\phi\lambda^{m-1} + \frac{1}{1.2}\chi^2\phi\lambda^{m-2} + \&c. = 0,$$

$$\theta\lambda^m + \chi\theta\lambda^{m-1} + \frac{1}{1.2}\chi^2\theta\lambda^{m-2} + \frac{1}{1.2.3}\chi^3\theta\lambda^{m-3} + \&c. = 0,$$

where

$$\phi = \phi(\alpha, \beta, \gamma, \delta), \quad \theta = \theta(\alpha, \beta, \gamma, \delta), \quad \chi = x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + t\frac{d}{d\delta}.$$

By theorem (3), these two equations, on writing $\frac{x_n}{a_n} = \epsilon$, become

$$\phi\lambda^m + \{U\phi + m\phi\epsilon\}\lambda^{m-1}$$
$$+ \{U^2\phi + 2(m-1)U\phi\epsilon + (m-1)m\phi\epsilon^2\}\frac{\lambda^{m-2}}{1.2} + \&c. = 0,$$

$$\theta\lambda^n + \{U\theta + n\theta\epsilon\}\lambda^{n-1} + (U^2\theta + \&c.)\frac{\lambda^{n-2}}{1.2} + \{U^3\theta + 3(n-2)U^2\theta\epsilon$$
$$+ 3(n-2)(n-1)U\theta\epsilon^2 + (n-2)(n-1)n\epsilon^3\}\frac{\lambda^{n-3}}{1.2.3} + \&c.$$

Now on writing $\lambda = \mu - \epsilon$, these equations take the forms

$$\phi\mu^m + U\phi\mu^{m-1} + U^2\phi\frac{\mu^{m-2}}{1.2} + \&c. = 0,$$

$$\theta\mu^n + U\theta\mu^{n-1} + U^2\theta\frac{\mu^{n-2}}{1.2} + \&c. = 0,$$

as is easily seen by substituting back $\lambda + \epsilon$ in place of μ. Consequently ϵ no longer appears in the coefficients of the terms of the equations between which the elimination is to be performed, and the resultant will accordingly come out as a function only of ϕ, $U\phi$, $U^2\phi$, &c., that is, of α, β, γ, δ, and of

$$x - \frac{\alpha}{\delta}t, \quad y - \frac{\beta}{\delta}t, \quad z - \frac{\gamma}{\delta}t,$$

showing that the equation in x, y, z, t, is of the form of that to a cone, as we know *à priori* it ought to be. Precisely a similar method may be applied to the elucidation of the corresponding theorem for a system of rays drawn from a given point through the locus of the intersection of two curves.

Before entering upon some further and more interesting applications of theorem (3), it will be convenient to explain a nomenclature which has been employed by me on another occasion, and which is almost indispensable in inquiries of the nature we are now engaged upon. Homogeneous functions may be characterized by their degree, by the number of letters which enter into them, and lastly, by the lowest number of linear functions of the letters which may be introduced in place of the letters to represent such functions. Any such linear function I designate as an order, and am now able to discriminate between the number of letters and the number of orders which enter into a given function. The latter number, *generally* speaking, is the same as the former; it can never exceed it, but *may* be any number of units less than it.

I need scarcely observe that a pair of points becoming coincident, a conic becoming a pair of lines, a conoid becoming a cone, and so forth, for the higher realms of space, will be expressed by the homogeneous function of the second order which characterizes such loci*, losing one order, that is, having an order less than the number of letters entering therein. Calling such characteristic $\phi(x, y, z \ldots t)$, it is well known that the condition of such loss of an order is the vanishing of the determinant

$$\begin{vmatrix} \frac{d^2\phi}{dx^2}, & \frac{d^2\phi}{dxdy} & \ldots & \frac{d^2\phi}{dxdt} \\ \frac{d^2\phi}{dydx}, & \frac{d^2\phi}{dy^2} & \ldots & \frac{d^2\phi}{dydt} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^2\phi}{dtdx}, & \frac{d^2\phi}{dtdy} & \ldots & \frac{d^2\phi}{dt^2} \end{vmatrix}.$$

A conoid becoming a pair of planes, a cone becoming a pair of coincident lines, a pair of points becoming indeterminate, will, in like manner, be denoted by their characteristic losing two orders, and so forth, for the higher degrees of degradation. In like manner, in general, a homogeneous function of three letters of any degree losing an order, typifies that the locus to which it is the characteristic will break up into a system of right lines.

Now let ω be a homogeneous function of $\alpha, \beta, \gamma \ldots \delta$, and suppose that we have the equations $\omega = 0$, $\chi\omega = 0$, $\chi^2\omega = 0$, where χ as above

$$= x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + \ldots + t\frac{d}{d\delta}.$$

I say that on eliminating any of the variables $x, y, z \ldots t$ between the second and third of the above equations, the resulting equation will be of one order less than the number of letters, that is, the expulsion of one letter will be attended by the expulsion of *two* orders.

For we have, by theorem (3),

$$\chi\omega = U\omega + 2\frac{x_n}{a_n}\omega = 0,$$

$$\chi^2\omega = U_2\omega + 2\frac{x_n}{a_n}U\omega + 2\left(\frac{x_n}{a_n}\right)^2\omega = 0,$$

and by hypothesis

$$\omega = 0.$$

Hence we have also

$$U\omega = 0,$$

$$U_2\omega = 0;$$

and since $U\omega$, $U_2\omega$ contain one order less than the number of letters in

* If $U=0$ is the equation to any locus, U may be said to *characterize* the same, or to be its characteristic.

ω, the resultant of the elimination between them will contain two orders less than the number of letters in ω; and consequently, whichever of the letters $x, y, z \ldots t$ we eliminate between $\chi\omega = 0$ and $\chi^2\omega = 0$, provided that $\omega = 0$, the resultant equation will contain one order less than the number of letters remaining.

Thus we see how it is that the tangent line to a conic meets it in two coincident points, the tangent plane to a conoid in two intersecting lines, and so forth, for the higher regions of space*. For instance, if we take $\omega(x, y, z, t) = 0$, the equation to a conoid, and $\alpha, \beta, \gamma, \delta$, the coordinates to any point therein, we shall have $\omega(\alpha, \beta, \gamma, \delta) = 0$,

$$\left(x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + t\frac{d}{d\delta}\right)\omega, \text{ that is, } \chi\omega = 0,$$

and $$\omega(x, y, z, t), \text{ that is, } \chi^2\omega = 0,$$

x, y, z, t representing the coordinates of any point in the intersection of the conoid by the tangent plane.

Consequently, by what has been shown above, on eliminating any one of the four letters x, y, z, t, the resultant function of three letters will contain only two orders, and will thus represent a pair of lines, real or imaginary, intersecting one another at $\alpha, \beta, \gamma, \delta$.

The fact which has just been demonstrated (that the resultant of $\chi\omega = 0$, $\chi^2\omega = 0$, loses an order if $\omega = 0$), indicates that on expressing one of the quantities $x, y, z \ldots t$ in terms of the others, by means of the first equation, and then substituting this value in the second, the determinant of the equation so obtained must be zero.

Now by virtue of a theorem which was given by me in a note† to my paper in the preceding number of this *Journal*, this determinant will be equal to the squared reciprocal of the coefficient in the equation $\chi\omega = 0$ of the letter eliminated multiplied by the determinant in respect to $x, y, z \ldots t, \lambda$ of

$$\chi^2\omega + \chi\omega\lambda.$$

This latter determinant is therefore zero; but this determinant is the resultant of the equations

$$\left.\begin{array}{l}
\dfrac{d}{dx}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \text{\&c.}\right)^2\omega + \dfrac{d}{dx}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots\right)\omega = 0, \\[2ex]
\dfrac{d}{dy}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \text{\&c.}\right)^2\omega + \dfrac{d}{dy}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots\right)\omega = 0, \\[2ex]
\qquad \text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.} \\[2ex]
\qquad\qquad \chi\omega = 0, \text{ that is, } \left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots\right)\omega = 0,
\end{array}\right\}.$$

* Thus a tangential section of a hyperlocus of the second degree at any point cuts it in two cones.

[† p. 135 above.]

Thus we obtain the singular law, that the symmetrical determinant

$$\begin{vmatrix} \frac{d}{da}\frac{d}{da}\omega, & \frac{d}{da}\frac{d}{db}\omega, \dots & \frac{d}{da}\frac{d}{dl}\omega, & \frac{d}{da}\omega \\ \frac{d}{db}\frac{d}{da}\omega, & \frac{d}{db}\frac{d}{db}\omega, \dots & \frac{d}{db}\frac{d}{dl}\omega, & \frac{d}{db}\omega \\ \frac{d}{dc}\frac{d}{da}\omega, & \frac{d}{dc}\frac{d}{db}\omega, \dots & \frac{d}{dc}\frac{d}{dl}\omega, & \frac{d}{dc}\omega \\ \dots & \dots & \dots & \dots \\ \frac{d}{dl}\frac{d}{da}\omega, & \frac{d}{dl}\frac{d}{db}\omega, \dots & \frac{d}{dl}\frac{d}{dl}\omega, & \frac{d}{dl}\omega \\ \frac{d}{da}\omega, & \frac{d}{db}\omega, \dots & \frac{d}{dl}\omega, & 0 \end{vmatrix}$$

is zero when ω is zero.

This is easily shown independently by means of a remarkable and I believe novel theorem, relative to homogeneous functions.

If ω be any homogeneous function of ι dimensions of $a, b, c \dots l$, we have (by Euler's theorem already repeatedly applied), remembering that $\frac{d\omega}{da}, \frac{d\omega}{db} \dots \frac{d\omega}{dl}$ are all homogeneous,

$$-\iota\omega + \left(a\frac{d}{da} + b\frac{d}{db} + \dots\dots\dots\dots + l\frac{d}{dl}\right)\omega = 0,$$

$$-(\iota-1)\frac{d\omega}{da} + \left(a\frac{d}{da}\frac{d}{da} + b\frac{d}{da}\frac{d}{db} + \dots + l\frac{d}{da}\frac{d}{dl}\right)\omega = 0,$$

$$-(\iota-1)\frac{d\omega}{db} + \left(a\frac{d}{db}\frac{d}{da} + \dots\dots\dots\dots + l\frac{d}{db}\frac{d}{dl}\right)\omega = 0,$$

&c. &c. &c.

$$-(\iota-1)\frac{d\omega}{dl} + \left(a\frac{d}{dl}\frac{d}{da} + \dots\dots\dots\dots + l\frac{d}{dl}\frac{d}{dl}\right)\omega = 0.$$

Between these equations we may eliminate *all the letters*, $a, b, c \dots l$, and we obtain the equation

$$\begin{vmatrix} \frac{d}{da}\frac{d}{da}\omega, & \frac{d}{da}\frac{d}{db}\omega, \dots & \frac{d}{da}\frac{d}{dl}\omega, & \frac{d}{da}\omega \\ \frac{d}{db}\frac{d}{da}\omega, & \frac{d}{db}\frac{d}{db}\omega, \dots & \frac{d}{db}\frac{d}{dl}\omega, & \frac{d}{db}\omega \\ \frac{d}{dc}\frac{d}{da}\omega, & \frac{d}{dc}\frac{d}{db}\omega, \dots & \frac{d}{dc}\frac{d}{dl}\omega, & \frac{d}{dc}\omega \\ \dots & \dots & \dots & \dots \\ \frac{d}{dl}\frac{d}{da}\omega, & \frac{d}{dl}\frac{d}{db}\omega, \dots & \frac{d}{dl}\frac{d}{dl}\omega, & \frac{d}{dl}\omega \\ \frac{d}{da}\omega, & \frac{d}{db}\omega, \dots & \frac{d}{dl}\omega, & \frac{\iota}{\iota-1}\omega \end{vmatrix} = 0.$$

As a corollary to this theorem, we see that if $\omega=0$ the determinant obtained in the previous investigation becomes zero, agreeing with what has been already shown; in fact the last-named determinant is always equal to

$$\frac{\iota-1}{\iota}\omega\times\begin{vmatrix}\frac{d}{da}\frac{d}{da}\omega, & \frac{d}{da}\frac{d}{db}\omega, \cdots & \frac{d}{da}\frac{d}{dl}\omega\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ \frac{d}{dl}\frac{d}{da}\omega, & \frac{d}{dl}\frac{d}{db}\omega, \cdots & \frac{d}{dl}\frac{d}{dl}\omega\end{vmatrix}.$$

This remarkable theorem, which I have communicated to friends nearly a twelvemonth back, is here, I believe, published for the first time*.

Suppose next that $\omega(x, y, z)$ is the characteristic of a line of any degree, to which a tangent is drawn at the point α, β, γ, using U in a manner correspondent to its previous signification to denote

$$\left(x-\frac{\alpha}{\gamma}z\right)\frac{d}{d\alpha}+\left(y-\frac{\beta}{\gamma}z\right)\frac{d}{d\beta},$$

and understanding $\omega(\alpha, \beta, \gamma)$ by ω, we have for determining the point of intersection, $\omega=0$, $\chi\omega=0$, $\chi^n\omega=0$; and consequently, by aid of our theorem (3), we shall obtain

$$\omega=0,$$

$$U\omega=0,$$

$$U_n\omega+n\frac{z}{\gamma}U_{n-1}\omega+\ldots=0.$$

By means of the two latter equations, we obtain

$$\left(x-\frac{\alpha z}{\gamma}\right)^2 F\left(x-\frac{\alpha z}{\gamma}\right)=0,$$

$$\left(y-\frac{\beta z}{\gamma}\right)^2 G\left(y-\frac{\beta z}{\gamma}\right)=0,$$

* Thus let z be a homogeneous function in x and y of ι dimensions, and let

$$\frac{dz}{dx},\quad \frac{dz}{dy},\quad \frac{d^2z}{dx^2},\quad \frac{d^2z}{dx\,dy},\quad \frac{d^2z}{dy^2},$$

be called p, q, r, s, t; we shall find

$$\begin{vmatrix} r, & s, & p\\ s, & t, & q\\ p, & q, & \frac{\iota}{\iota-1}\omega\end{vmatrix}=0,$$

that is, $$\omega=\frac{\iota-1}{\iota}\,\frac{rq^2-2pqs+tp^2}{rt-s^2}.$$

where F and G are each of only $(n-2)$ dimensions, and serve to determine the intersections of the tangent with the curve, extraneous to the two coincident ones at the point of contact.

Again, suppose that ω is a function of any degree of any number of letters α, β, γ, &c., and that we have given $\omega = 0$, $\chi\omega = 0$, $\chi^2\omega = 0, \ldots \chi^m\omega = 0$; it is evident from our fundamental theorem that these equations may be replaced by

$$\omega = 0, \quad U_1\omega = 0, \quad U_2\omega = 0, \ldots \; U_m\omega = 0;$$

and consequently that the expulsion of $(m-1)$ letters, by aid of the last m of the given equations, will be attended by the disappearance of m orders, or, in other words, the resultant will be minus an order, that is, will have one order less than the number of letters remaining in it.

In applying to space conceptions the preceding theorem, it will be convenient to use a general nomenclature for geometrical species of various dimensions.

Thus we may call a line a monotheme, a surface a ditheme, the species beyond a tritheme, and so on, *ad infinitum.*

A system of points according to the same system of nomenclature would be called a kenotheme.

An n-theme has for its characteristic a homogeneous function of $(n+2)$ letters.

Again, it will be convenient to give a general name to all themes expressed by equations of the first degree. Right lines and planes agree in conveying an idea of levelness and uniformity; they may both be said to be homalous. I shall therefore employ the word homaloid to signify in general any theme of the first degree.

Now let $\omega(x, y, z \ldots t)$ be the characteristic to an n-theme of the nth degree.

The number of letters $x, y, z \ldots t$ is $(n+2)$.

As usual, let ω represent $\omega(\alpha, \beta, \gamma \ldots \delta)$, and suppose

$$\omega = 0, \quad \chi\omega = 0, \quad \chi^2\omega = 0 \ldots \chi^n\omega = 0,$$

and consequently

$$U_1\omega = 0, \quad U_2\omega = 0 \ldots U_n\omega = 0.$$

On eliminating $(n-1)$ letters between the n last equations, the resulting function will be of three letters but of only two orders, and of the $1 \,.\, 2 \,.\, 3 \ldots n$ degree. Hence we see that at every point of an n-theme of the nth degree, and lying in the tangent homaloid thereto, $1 . 2 \ldots . n$ right lines may be drawn coinciding throughout with the n-theme.

Thus one right line can be drawn at each point of a line of the first order lying on the line; two right lines at each point of a surface of the second order lying on the surface; six right lines at each point of a hyperlocus of the third degree, and so forth.

It is obvious that a surface may be treated as the homaloidal section of a tritheme, just as a plane curve may be regarded as a section of a surface. I shall proceed to show upon this view, how we may obtain a theorem given by Mr Salmon for surfaces of the third degree of a particular character from the law just laid down, according to which a tritheme of the third degree admits of six right lines being drawn upon it at every point*.

Let $\omega(x, y, z, t, u)$ be the characteristic of any tritheme of the third degree; $\alpha, \beta, \gamma, \delta, \epsilon$, coordinates to any point in the same. Then $\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, and the equation to the tangent homaloid will be $\chi\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, and the equation to the polar of the second degree to the given tritheme in relation to the assumed point as origin, (that is, the infinite system of homaloids that may be drawn from the point to touch the tritheme), will be $\chi^2\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$.

But the section of any polar through its origin is the polar of the section to the same origin; hence the polar to the intersection of $\omega(x, y, z, t, u) = 0$, with $\chi\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, is the intersection of $\chi\omega = 0$ with $\chi^2 \omega = 0$.

The projections of these intersections upon the space x, y, z, t will be found by eliminating u, and getting the correspondent two equations between x, y, z, t. Hence we see that the projection of the latter intersection upon any space x, y, z, t is a cone; or, in other words, this intersection itself, that is, the polar to the intersection of the tritheme with its tangent homaloid, is a cone; that is to say, the surface of the third degree formed by cutting a tritheme of the third degree by any tangent homaloid has a conical point at the point of contact; so that every surface of the third degree with a conical point may be considered as the intersection of a tritheme of the third degree with any tangent homaloid thereto†.

* The reduction of any equation of the sixth degree to depend upon one of the fifth may be shown by Mr Jerrard's method to be equivalent to drawing a straight line upon a tritheme of the third degree, just as the reduction of the equation of the fifth degree to a trinomial form may be shown to be dependent upon our being able to draw a straight line upon a ditheme of the second degree. Now at every point of a tritheme straight lines may be drawn, but as they keep together in groups of sixes they cannot be found *in general* at a given point without solving an equation of the sixth degree.

† So in like manner a surface of the third degree with more than one conical point may be generated by the intersection of the tritheme with a pluri-tangent plane; and so too we may get other varieties by taking homaloidal sections of trithemes whose characteristics are minus one or more orders.

Hence then we see, as an instantaneous deduction from our general theorem, that at any conical point (when one exists) of a surface of the third degree *six* right lines may be drawn lying completely upon it. This theorem is thus brought into an immediate and natural connexion with the well-known one, that at *every* point in a surface of the second degree, *two* right lines can be drawn lying wholly upon the surface*.

The last geometrical application of the theorem (3) which I shall make, refers to the equations employed by Mr Salmon in No. XXI. (New Series) of this *Journal*, to obtain the locus of the points on any surface at which tangent lines can be drawn passing through four consecutive points. I may remark in passing that these equations may be obtained by rather simpler considerations than Mr Salmon has employed so to do, and without any reference to Joachimsthal's theorem; for if we take ξ, η, ζ, θ, as the coordinates of any point in one of the tangent lines above described, and if we take the first polar to the surface with this point as origin, three out of the four original points will be found in such polar consecutive but distinct; and consequently in the second polar, referred to the same origin, two will continue consecutive but distinct, and consequently one will remain over in the third polar.

Hence writing the equation to the surface $\omega(x, y, z, t) = 0$, and using D to denote $\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz} + \theta \frac{d}{dt}$, we shall evidently have

$$\omega = 0, \tag{1}$$

$$D\omega = 0, \tag{2}$$

$$D^2\omega = 0, \tag{3}$$

$$D^3\omega = 0, \tag{4}$$

as obtained by Mr Salmon. And the same kind of reasoning precisely applies to the theory of points of inflexion in curves; three consecutive points in a right line in this case corresponding to four such in the case above considered.

If now we make

$$\xi - \frac{x}{t}\theta = u,$$

$$\eta - \frac{y}{t}\theta = v,$$

$$\zeta - \frac{z}{t}\theta = w,$$

* If we have an indeterminate system of algebraical equations consisting of one quadratic and another n^c function of three variables, this may be completely resolved by considering the first as an equation to a surface of the second degree, finding at any point thereof the two lines which lie upon the surface, and determining their respective intersections with the surface represented by the second equation. This will require therefore the solution only of a quadratic and an n^c equation. In like manner an indeterminate system of two equations of four variables, one of the third and the other of the nth degree, may be completely resolved (with the aid of the theorem in the text) by means of two equations, one of the sixth and the other of the nth degree.

the equations (2), (3), (4), by our theorem, may be expressed in terms of u, v, w, which being eliminated we obtain an equation between x, y, z, t, which will express the surface whose intersection with the given surface $\omega = 0$ serves to determine the locus of the points in question.

Hence if we proceed in the ordinary manner to eliminate two of the four letters, as ξ and η, between the equations (2), (3), (4), the resultant will be of the form $M \times \phi(\zeta, \theta)$, where M does not contain ξ, η, ζ or θ, and where by the general laws of elimination $\phi(\zeta, \theta)$ will be an integral function of the sixth degree in respect to ζ, θ: and it is manifest that $M \times \phi(\zeta, \theta)$ will be identical with the resultant of (2), (3), (4) expressed in terms of u, v, w, when u and v are eliminated *cy-près* of an integralizing factor, showing that $\phi(\zeta, \theta)$ is w^6 integralized, that is, is equal to $(t\zeta - z\theta)^6$. Consequently as $M\phi$ is of the order $(n-1)\,2.3 + (n-2)\,1.3 + (n-3)\,1.2$, that is, $11n - 18$ in respect to x, y, z, t, it follows that $M = 0$, the equation to the second surface spoken of above, will be of the order $11n - 24$, agreeable to Mr Salmon's showing.

I shall conclude this paper by showing the application of our theorem to the subject propounded by Mr Jerrard and Sir William Hamilton, of systems of equations containing a sufficient number of variable letters for effecting the solution without elevation of degree.

If we have n homogeneous equations containing a sufficient number of letters $a_1, a_2 \dots a_m$ to enable us to express the solution of $(n-1)$ of the equations under the form

$$\begin{aligned} a_1 &= \alpha_1 + \lambda\beta_1, \\ a_2 &= \alpha_2 + \lambda\beta_2, \\ &\dots\dots\dots\dots \\ a_m &= \alpha_m + \lambda\beta_m, \end{aligned}$$

where $\alpha_1, \alpha_2 \dots \alpha_m$, $\beta_1, \beta_2 \dots \beta_m$ are supposed known, and λ is indeterminate, it is evident that by substituting these values in the nth equation, λ may be found by solving an equation of the same degree as that equation contains dimensions of $a_1, a_2 \dots a_m$.

Let us then propose this question: how many letters $a_1, a_2 \dots a_r$ are needed to obtain a linear solution of a system of n equations

$$\phi_1 = 0, \ \phi_2 = 0, \dots \phi_n = 0,$$

of the several degrees $\iota_1, \iota_2 \dots \iota_n$, without elevation of degree; by a linear solution being understood a solution under the form

$$\begin{aligned} a_1 &= \alpha_1 + \lambda\beta_1, \\ a_2 &= \alpha_2 + \lambda\beta_2, \\ &\dots\dots\dots\dots \\ a_r &= \alpha_r + \lambda\beta_r, \end{aligned}$$

where λ is left indeterminate.

Let us suppose that $\alpha_1, \alpha_2 \ldots \alpha_r$, substituted respectively for $a_1, a_2 \ldots a_r$, satisfy the given system of equations. The determination of these values without elevation of degree will, from what has been said before, depend upon the linear solution of a system of equations differing from the given system by the omission of any one of them at pleasure.

Now make

$$D = \alpha_1 \frac{d}{da_1} + \alpha_2 \frac{d}{da_2} + \ldots + \alpha_r \frac{d}{da_r},$$

and then write

$$\left.\begin{array}{l} D\phi_1 = 0,\ D^2\phi_1 = 0 \ldots D^{\iota_1}\phi_1 = 0 \\ D\phi_2 = 0,\ D^2\phi_2 = 0 \ldots D^{\iota_2}\phi_2 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D\phi_n = 0,\ D^2\phi_n = 0 \ldots D^{\iota_n}\phi_n = 0 \end{array}\right\}. \qquad (\theta)$$

The values of $a_1, a_2 \ldots a_r$ derived from this system, say $(a)_1, (a)_2 \ldots (a)_r$, give

$$a_1 = \alpha_1 + \lambda(a)_1,\ a_2 = \alpha_2 + \lambda(a)_2, \ldots a_r = \alpha_r + \lambda(a)_r,$$

a solution under the required form, where λ is left indeterminate.

The solution of this new system without elevation of degree depends on the *linear solution* of all but one of them; this excepted one may be taken the one whose dimensions ι_r are the highest or as high as any of the quantities $\iota_1, \iota_2 \ldots \iota_n$.

Consequently, if we use the symbol $(k_1, k_2 \ldots k_r)$ to denote the number of letters required for the linear solution (without elevation of degree) of k_1 equations of the first degree, k_2 of the second, k_3 of the third, ..., k_r of the rth, it would at first sight appear from the preceding reduction that we must have

$$(k_1, k_2 \ldots k_r) = \{K_1, K_2 \ldots K_{r-1}, K_r'\},$$

where

$$\begin{aligned} K_1 &= k_1 + k_2 + \ldots + k_{r-1} + k_r, \\ K_2 &= k_2 + \ldots + k_{r-1} + k_r, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ K_{r-1} &= k_{r-1} + k_r, \\ K_r' &= k_r - 1. \end{aligned}$$

But now steps in our theorem (3), and shows that the system (θ) may be superseded by another, in which the variables, instead of being $a_1, a_2 \ldots a_n$, will be

$$a_1 - \frac{\alpha_1}{\alpha_n} a_n,\ a_2 - \frac{\alpha_2}{\alpha_n} a_n, \ldots a_{n-1} - \frac{\alpha_{n-1}}{\alpha_n} a_n;$$

consequently the number of really independent variables is only $(n-1)$; we must therefore have

$$(k_1, k_2 \ldots k_r) = 1 + \{K_1, K_2 \ldots K_r'\}.$$

Since the introduction of a new simple equation is equivalent to the requirement of one more disposable letter, we may write the above more symmetrically under the form

$$(k_1, k_2 \dots k_r) = ('K_1, K_2 \dots K_{r-1}, K_r'),$$

where
$$'K_1 = 1 + k_1 + k_2 + \dots + k_r,$$
$$K_r' = \qquad\qquad k_r - 1.$$

By means of this formula of reduction $(k_1, k_2 \dots k_r)$ may be finally brought down to the form (L), and the value of (L) being the number of letters required for the linear solution of a system of L linear equations is evidently $L + 2$.

Thus, to determine the number of letters required for the linear solution of a single quadratic, we write

$$(0, 1) = (2) = 4.$$

For two quadratics, we write

$$(0, 2) = (3, 1) = (5) = 7;$$

for a quadratic and a cubic,

$$(0, 1, 1) = (3, 2) = (6, 1) = (8) = 10;$$

for two cubics,

$$(0, 0, 2) = (3, 2, 1) = (7, 3) = (11, 2) = (14, 1) = (16) = 18.$$

These results coincide with those obtained by Sir William Hamilton in his Report on Mr Jerrard's Transformation of the Equation of the Fifth Degree in the *Transactions* of the British Association. I have much more to say on the subject of the linear solution of a system of indeterminate equations, and am, I believe, able to present the subject in a more general light than has hitherto been done; but my observations on this matter must be deferred until a subsequent communication.

31.

REPLY TO PROFESSOR BOOLE'S OBSERVATIONS ON A THEOREM CONTAINED IN THE LAST NOVEMBER NUMBER OF THE JOURNAL.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 171—174.]

THE restricted space that can be spared for discussion in these pages, necessitates me to compress within the narrowest limit the remarks which I feel bound to make on Mr Boole's extraordinary observations* in the February number of this *Journal*, on my theorem contained in the antecedent number thereof†, which statements I cannot, in the interests of truth and honesty, suffer to pass unchallenged. The object of that theorem was to show how the determinant of the quadratic function resulting from the elimination of any set of the variables between a given quadratic function and a number of linear functions of the same variables, could be represented *without performing* the actual elimination by a fraction, of which the numerator would be constant whichever set of the variables might be selected for elimination, and the denominator the square of the determinant corresponding to the coefficients of the variables so eliminated. The numerator itself is a determinant, obtained by forming the square corresponding to the determinant of the given quadratic function, and bordering it horizontally and vertically with the lines and columns corresponding to the coefficients of all the variables in the given linear equations. An *immediate corollary* from this theorem leads to Mr Boole's. Conversely upon the principle that "tout est dans tout" Mr Boole devotes a page and a half of close print merely to indicate the steps of a method by which from his theorem mine is capable of being deduced, ending with the announcement, that the numerator in question is equal to the quantity

$$\phi_1\phi_2 \dots \phi_r\theta(Q),$$

(the symbols above employed being Mr Boole's own), and concludes with assuring his readers that "he has ascertained that Mr Sylvester's result is reducible to the above form." Mr Sylvester would be very sorry to put his

[* *Cambr. and Dublin Math. Jour.* VI. (1851), pp. 90, 284.] [† p. 135 above.]

result under any such form. Mr Boole could scarcely have reflected upon the effect of his words when he indulged in the remark which follows—"*there cannot be a doubt* that for the discovery of the actual relation in question, the above theorem is far more convenient than Mr Sylvester's." Of the value to be attached to this assertion the annexed comparison of results is submitted as a specimen.

Let the quadratic function be

$$ax^2 + by^2 + cz^2 + dt^2 + 2exy + 2\epsilon zt + 2gxz + 2\gamma yt + 2hyz + 2\eta xt,$$

and the linear functions (taken two in number)

$$lx + my + nz + pt,$$
$$l'x + m'y + n'z + p't.$$

My numerator will be the determinant (hereinafter cited as the *extended* determinant),

$$\begin{vmatrix} a & e & g & \eta & l & l' \\ e & b & h & \gamma & m & m' \\ g & h & c & \epsilon & n & n' \\ \eta & \gamma & \epsilon & d & p & p' \\ l & m & n & p & 0 & 0 \\ l' & m' & n' & p' & 0 & 0 \end{vmatrix}.$$

To find the numerator of Mr Boole's fraction, we must form the symbolical operator

$$\left\{ \begin{aligned} & l^2 \frac{d}{da} + m^2 \frac{d}{db} + n^2 \frac{d}{dc} + p^2 \frac{d}{dd} \\ & \quad + 2lm \frac{d}{de} + 2np \frac{d}{d\epsilon} + 2ln \frac{d}{dg} + 2mp \frac{d}{d\gamma} + 2lp \frac{d}{dh} + 2mn \frac{d}{d\eta} \end{aligned} \right\}$$

$$\times \left\{ \begin{aligned} & l'^2 \frac{d}{da} + m'^2 \frac{d}{db} + n'^2 \frac{d}{dc} + p'^2 \frac{d}{dd} \\ & \quad + 2l'm' \frac{d}{de} + 2n'p' \frac{d}{d\epsilon} + 2l'n' \frac{d}{dg} + 2m'p' \frac{d}{d\gamma} + 2l'p' \frac{d}{dh} + 2m'n' \frac{d}{d\eta} \end{aligned} \right\}$$

and after expanding the determinant hereunder written

$$\begin{vmatrix} a & e & g & \eta \\ e & b & h & \gamma \\ g & h & c & \epsilon \\ \eta & \gamma & \epsilon & d \end{vmatrix},$$

perform the operations above indicated upon the result so obtained.

These are the operations and processes which, on Professor Boole's authority, we are to accept "*as without doubt far more convenient*" than the one simple process of forming, and when necessary, calculating the extended determinant above given. Here for the present I leave the case between Mr Boole and myself to the judgment of the readers of this *Journal.*

In the April Number of the *Philosophical Magazine**, I have shown that the extended determinant serves, not only to represent the full and complete determinant of the reduced quadratic function, but likewise all the minor determinants thereof; the last *set* of which will be evidently no other than the coefficients themselves. For instance, in the example above given, if we wish to find the coefficient of x^2 after z and t have been eliminated, we have only to strike out the line and column $e\ b\ h\ \gamma\ m\ m'$ from the extended determinant; if we wish to find the coefficient of y^2, we must strike out the line and column $a\ e\ g\ \eta\ l\ l'$; to find the coefficient of xy, we must strike out the line $a\ e\ g\ \eta\ l\ l'$ and the column $e\ b\ h\ \gamma\ m\ m'$, or *vice versâ.*

In each of these cases the determinant so obtained is the numerator of the equivalent fraction; the denominator remaining always the same function of the coefficients of transformation as in the original theorem.

Again, if there be taken only one linear equation, and by aid of it x is supposed to be eliminated; and if the reduced quadratic function be called

$$Ly^2 + Mz^2 + Nt^2 + 2Pzt + 2Qyt + 2Rzy,$$

the same extended determinant as before given will serve, when stripped of its outer border, consisting of the line and column $l'\ m'\ n'\ p'$, to produce the various equivalent fractions: thus form the square

$$\begin{matrix} L & R & Q \\ R & M & P \\ Q & P & N. \end{matrix}$$

The numerator of the fraction equivalent to $\begin{vmatrix} L & R \\ R & M \end{vmatrix}$, that is, to $LM - R^2$, may be found by striking out from the form of the extended determinant the line and column $\eta\ \gamma\ \epsilon\ d\ p$; that corresponding to $\begin{vmatrix} L & Q \\ R & P \end{vmatrix}$, that is, $LP - RQ$, will be found by striking out the line $g\ h\ c\ \epsilon\ n$ and the column $\eta\ \gamma\ \epsilon\ d\ p$, or *vice versâ*; and so forth for all the first minor determinants; and similarly the second minors, that is, L, M, N, P, Q, R, may be obtained by striking out in each case a correspondent pair of lines and pair of columns. Thus, to find the numerator of L the same pair of lines and columns, namely, $(g\ h\ c\ \epsilon\ n)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, must be elided. To find the numerator of R, the pair of lines $(g\ h\ c\ \epsilon\ n)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, and the pair of columns $(e\ b\ h\ \gamma\ m)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, or *vice versâ*, will have to be elided; and so forth for the remaining second minors. I may conclude with observing, that the theorem contested by Mr Boole is an immediate corollary from the general Theory of Relative Determinants alluded† to in the "Sketch" inserted in the present number of the *Journal.*

[* p. 241 below.] [† p. 188 below.]

32.

SKETCH OF A MEMOIR ON ELIMINATION, TRANSFORMATION, AND CANONICAL FORMS.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 186—200.]

THERE exists a peculiar system of analytical logic, founded upon the properties of zero, whereby, from dependencies of equations, transition may be made to the relations of functional forms, and *vice versâ*: this I call the logic of characteristics.

The resultant of a given system of homogeneous equations of as many variables, is the function whose nullity implies and is implied by the possibility of their coexistence, that is, is the characteristic of such possibility; but inasmuch as any numerical product of any power of a characteristic is itself an equivalent characteristic, in order to give definiteness to the notion of a resultant, it must further be restricted to signify the characteristic taken in the *lowest form* of which it *in general* admits.

The following very general and important proposition for the change of the independent variables in the process of elimination, is an immediate consequence of the doctrine of characteristics.

Let there be two sets of homogeneous forms of function;

the 1st, $\phi_1, \ \phi_2 \dots \phi_n,$

the 2nd, $\psi_1, \ \psi_2 \dots \psi_n.$

Let the results of applying these forms to any sets of n variables be called

$$(\phi_1), \ (\phi_2) \dots (\phi_n),$$

$$(\psi_1), \ (\psi_2) \dots (\psi_n);$$

then will the resultant (in respect to those variables) of

$$\phi_1 \{(\psi_1), \ (\psi_2) \dots (\psi_n)\},$$

$$\phi_2 \{(\psi_1), \ (\psi_2) \dots (\psi_n)\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\phi_n \{(\psi_1), \ (\psi_2) \dots (\psi_n)\},$$

be the product of powers (assignable by the law of homogeneity) of the separate resultants of the two systems,

$$\{(\phi_1),\ (\phi_2) \dots (\phi_n)\},$$
$$\{(\psi_1),\ (\psi_2) \dots (\psi_n)\}.$$

By means of the doctrine of characteristics the following general problem may be resolved.

Given any number of functions of as many letters, and an inferior number of functions of the same inferior number of letters, obtained by combining, *inter se*, in a known manner, the given functions, to determine the factor by which, the resultant of the reduced system being divided, the resultant of the original system may be obtained.

If in the theorem for the change of the independent variables both sets of forms of funotions be taken linear, we obtain the common rule for the multiplication of determinants: if we take one set linear and the other not, we deduce two rules, namely, That the resultant of a given set of functional forms of a given set of variables, enters as a factor into the resultant,

1st, of linear functions of the given functions of the given variables;

2nd, of the given functions of linear functions of the given variables:

the extraneous factor in each case being a power of what may be conveniently termed the *modulus of transformation*, that is, the resultant of the imported linear forms of functions.

From the second of these rules we obtain the law first stated I believe for functions beyond the second degree by Mr Boole, to wit, that the determinant of any homogeneous algebraical function (meaning thereby the resultant of its first partial differential coefficients) is unaltered by any linear transformations of the variables, except so far as regards the introduction of a power of the modulus of transformation. This is also abundantly apparent from the fact, that the nullity of such determinant implies an immutable, that is, a fixed and inherent, property of a certain corresponding geometrical locus.

There exist (as is now well known) other functions besides the determinant, called by their discoverer (Mr Cayley) hyperdeterminants, gifted with a similar property of immutability. I have discovered a process for finding hyperdeterminants of functions of any degree of any number of letters, by means of a process of Compound Permutation. All Mr Cayley's forms for functions of two letters may be obtained in this manner by the aid of one of the two processes (to wit, that one which will hereafter be called the derivational process), for passing from immutable constants to immutable forms. Such constants and forms, derived from given forms, may be best termed adjunctive; a term slightly varied from that employed by M. Hermite in a more restricted sense.

The two processes alluded to may be termed respectively appositional and derivational. The appositional is founded upon the properties of the binary function $x\xi + y\eta + z\zeta + \ldots$; in which, whether we substitute linear functions of x, y, z, &c., or linear functions of ξ, η, ζ, &c., in place of x, y, z, &c., or ξ, η, ζ, &c., the result is the same.

Consequently, if we apply the form ϕ to $\xi, \eta \ldots \zeta$, and take any constant (in respect to $\xi, \eta \ldots \zeta$) adjunctive to

$$\phi(\xi, \eta \ldots \zeta) + (x\xi + y\eta + \ldots + z\zeta + kt)\, t^{n-1},$$

calling this quantity $\psi(x, y \ldots z, t)$, the form ψ is evidently adjunctive to the form ϕ: and if we expand so as to obtain

$$\psi(x, y \ldots z, t) = \psi_1(x, y \ldots z)\, t^{\alpha} + \psi_2(x, y \ldots z)\, t^{\beta} + \&c.,$$

it is evident ψ_1, ψ_2, &c. will be each separately adjunctive to ϕ. These forms, when ψ is obtained by finding the determinant in respect to $\xi, \eta \ldots \zeta$ of S, are, in fact, identical with Hermite's "formes adjointes."

The derivational mode of generating forms from constants depends upon the property of the operative symbol

$$\chi = \xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots + \zeta \frac{d}{dz},$$

applied to ϕ a function of $x, y \ldots z$; namely, that if in ϕ, in place of these letters, we write linear functions thereof, to wit $x', y' \ldots z'$, we may write

$$\chi = \xi' \frac{d}{dx'} + \eta' \frac{d}{dy'} + \ldots + \zeta' \frac{d}{dz'},$$

where $\xi', \eta' \ldots \zeta'$ will be the same functions of $\xi, \eta \ldots \zeta$ that $x', y' \ldots z'$ are of $x, y \ldots z$.

Suppose now, in the first place, that in regard to $\xi, \eta \ldots \zeta$, $\psi(x, y \ldots z)$ is adjunctive to $\chi^r \phi(x, y \ldots z)$; then is the form ψ adjunctive to the form ϕ, for on changing $x, y \ldots z$ to $x', y' \ldots z'$,

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots + \zeta \frac{d}{dz}\right)^r \phi(x', y' \ldots z')$$

becomes
$$\left(\xi' \frac{d}{dx'} + \eta' \frac{d}{dy'} + \ldots + \zeta' \frac{d}{dz'}\right) \phi(x', y' \ldots z');$$

and consequently $\psi(x, y \ldots z)$ becomes $\psi(x', y' \ldots z')$, multiplied by a power of the modulus of transformation, the modulus of that transformation, be it well observed, whereby $x', y' \ldots z'$ would be replaced by $x, y \ldots z$, and not as in the appositional mode of that converse transformation according to which

$x, y \ldots z$ would be replaced by $x', y' \ldots z'$. It is on account of this converseness of the modes of transformation that the appositional and derivational modes of generating forms cannot except for a certain class of *restricted* linear transformations be combined in a single process. More generally, if instead of a single function $\chi^r \phi(x, y \ldots z)$, we take as many such with different indices to χ as there are variables, and form either the resultant in respect to $\xi, \eta \ldots \zeta$, or any other immutable constant in regard to those variables, (presuming in extension of the hyperdeterminant theory and as no doubt is the case, that such exist), every such resultant or other constant will give a form of function of $x, y \ldots z$ adjunctive to the given form ϕ.

It may be shown that every such resultant so formed will contain ϕ as a factor.

Again, in the former more available determinant mode of generation, if we take the determinant in respect to $\xi, \eta \ldots \zeta$, it may be shown that all the adjunctive functions so obtained will be algebraical derivees of the partial differential coefficients of ϕ in respect to $x, y \ldots z$; that is to say, if these be respectively zero, all such adjunctive functions so derived, as last aforesaid, will be zero, or in other words, each such adjunctive is a syzygetic function of the partial differential coefficients of the primitive function.

To Mr Boole is due the high praise of discovering and announcing, under a somewhat different and more qualified form and mode of statement, this marvel-working process of derivational generation of adjunctive forms. I was led back to it, in ignorance of what Mr Boole had done, by the necessity which I felt to exist of combining Hesse's so-called functional determinant, under a common point of view with the common constant determinant of a function; under pressure of which sense of necessity, it was not long before I perceived that they formed the two ends of a chain of which Hesse's end exists for all homogeneous functions, but the other only when such functions are algebraical.

In fact, if we give to r every value from 2 upwards, the successive determinants in respect to $\xi, \eta \ldots \zeta$ of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \cdots + \zeta \frac{d}{dz}\right)^r \phi(x, y, z),$$

will produce the chain in question, which, when ϕ is algebraical and of n dimensions, comes to a natural termination when $r = n - 1$. The last member of and the number of terms in this chain are identical with the last member of and the number of terms in Sturm's auxiliary functions, when the variables are reduced to two. There is some reason to anticipate that this chain of functions may be made available in superseding Sturm's chain of auxiliaries; and if so, then the fatal hindrance to progress, arising from the unsymmetrical nature of the latter, is overcome, and we shall be able to pass from Sturm's theorem, which relates to the theory of Kenothemes, or Point-systems, to certain corresponding but much higher theories for lines, surfaces, and n-themes generally.

The restriction of space allowed to me in the present number of the *Journal* will permit me only to allude in the briefest terms to the theory of Relative Determinants, which, as it will be seen, plays an important part in the effectuation of the reductions of the higher algebraical functions to their simplest forms. Nor can the effect of the processes to be indicated be correctly appreciated without a knowledge of the circumstances under which the resultant of a *given* system of equations can sink in degree below the resultant of the *general* type of such system. Abstracting from the case when the equations separately, or in combination, subdivide into factors, this lowering of degree, as may be shown by the doctrine of characteristics, can only happen in one of two ways. Either the particular resultant obtained is a rational root of the general resultant, or the general resultant becomes zero for the case supposed, and the particular resultant is of a distinct character from the general resultant, being in fact the characteristic of the possibility not of the given system of equations being merely able to coexist (for that is already supposed), but of their being able to coexist for a certain system of values *other than* a given system or given systems. Such a resultant may be termed a Sub-resultant; the lowest resultant in the former case may be termed a Reduced-resultant. The theory of Sub-resultants is one altogether remaining to be constructed, and is well worthy equally of the attention of geometers and of analysts.

As to the theory of Relative Determinants, the object of this theory is to obtain the determinant resulting from eliminating as many variables as can be eliminated, chosen at pleasure from a set of variables greater in number than the equations containing them; and the mode of effecting this object is through the method of the indeterminate multiplier. To avoid the discussion of the theory of sub-resultants and other particularities, I shall content myself with giving the rule applicable to the case (the only one of which as yet a practical application has offered itself to me in the course of my present inquiries) when all but one of the functions are linear.

If U, L_1, $L_2 \ldots L_m$ be the first an n^c and the others linear functions of n variables, and it be desired to find the determinant of the resultant arising from the elimination of any m out of the n variables, the following is the rule:

Find the determinant, that is, the resultant of the partial differential coefficients in respect to the given variables, and of $\lambda_1, \lambda_2 \ldots \lambda_m$ of

$$U + L_1\lambda_1 + L_2\lambda_2 + \ldots + L_m\lambda_m.$$

This resultant, in its lowest form, will be always a rational $(n-1)$th root of the resultant of the homogeneous system of equations to which the system above given can be referred as its type; and this reduced resultant divided by a power (determinable by the law of homogeneity) of the resultant of $L_1, L_2 \dots L_m$, when all but the selected variables are made zero, will be the resultant determinant required*. As regards what has been said concerning the reducibility of the general typical resultant in the case before us, this is a consequence of, and may be brought into connexion with, the following theorem, which is easily demonstrable by the theory of characteristics. If $Q_1, Q_2 \dots Q_m$ be m homogeneous functions of m variables of the same degree, r of which enter in each equation only as simple powers uncombined with any of the other variables, then the degree of the reduced resultant is equal to the number of the equations multiplied by the $(m-r-1)$th power of the number of units in the degree of each, subject to the obvious exception that when r is m, (there being in fact but *one* step from $r=m-2$ to $r=m$), instead of r, $(r-1)$ must be employed in the above formula. As an example of a sub-resultant as distinguished from a reduced-resultant, I instance the case of three quadratics U, V, W, functions of x, y, z, in each of which no squared power of z is supposed to enter: it may easily be shown by my dialytic method that instead of six equations, between which to eliminate x^2, y^2, z^2, xy, xz, yz, we shall have only 5, the three original ones and two instead of three auxiliaries between which to eliminate x^2, y^2, xy, xz, yz, the *apparent* resultant is accordingly of the 9th instead of the 12th degree. But this is not the true characteristic of the possibility of the coexistence of the given systems, which in fact is zero, as is evidenced by the fact that they always *do* coexist, since they are always satisfiable by only *two* relations between the variables, to wit $x=0$, $y=0$. The apparent resultant is then something different, and what has been termed by the above a Sub-resultant.

I take this opportunity of entering my simple protest against the appropriation of my method of finding the resultant of any set of three equations of degrees equal or differing only by a unit, one from those of the other two, by Dr Hesse, so far as regards quadratic functions, without acknowledgment, four years after the publication of my memoir in the *Philosophical Magazine*: the fundamental idea of Dr Hesse's partial method is identical with that of my general one. Still more unjustifiable is the subsequent use of the *dialytic* principle, by the same author, equally without acknowledgment, and in cases where there is no peculiarity of form of procedure to give even a plausible ground for evading such acknowledgment. It is capable of moral proof that

* The same method applies not only to the Final or Constant Determinant, but likewise to all the Functional Determinants in the chain above described, extending upwards from this to the Hessian, or as it ought to be termed, the first Boolian Determinant.

what I had written on the matter was sufficiently known in Berlin and at Königsberg, at each epoch of Dr Hesse's use of the method.

I now proceed to the consideration of the more peculiar branch of my inquiry, which is as to the mode of reducing Algebraical Functions to their simplest and most symmetrical, or as my admirable friend M. Hermite well proposes to call them, their Canonical forms. Every quadratic function of any number of variables may always be linearly transformed into any other quadratic functions of the same, and that too in an infinite variety of ways; but in every other instance there will be only a limited number of ways, whereby, when possible, one form will admit of being transmuted into any other: and with the sole exception of a cubic function of two letters, such transmutation will never be possible, unless a certain condition, or certain conditions, be satisfied between the constants of the forms proposed for transmutation. The number of such conditions is the number of parameters entering into the canonical form, and is of course equal to the number of terms in the general form of the function diminished by the square of the number of letters. Thus there is one parameter in the canonical form for the biquadratic function of two and the cubic function of three letters, and no parameter in the cubic function of two letters. Hitherto no canonical forms have been studied beyond the cases above cited, but I have succeeded, as will presently be shown, in obtaining methods for reducing to their canonical forms functions with *two* and *four* parameters respectively. Owing to what has been remarked above, the theory of quadratic functions is a theory apart. Simultaneous transformation gives definiteness to that theory, but has no existence for any useful purpose for functions of the higher degrees. Where the theory of simultaneous transformation ends, that of canonical forms properly begins; and in what follows, the case of quadratic forms is to be understood as entirely excluded. Such exclusion being understood, there is no difficulty in assigning the canonical, that is, the simplest and most symmetrical general, form to which every function of two letters admits of being reduced by linear transformations. If the degree be odd, say $2m+1$, the canonical form will be

$$u_1^{2m+1} + u_2^{2m+1} + \ldots + u_{m+1}^{2m+1};$$

if the degree be even, say $2m$, the canonical form will be

$$u_1^{2m} + u_2^{2m} + \ldots + u_m^{2m} + K(u_1 u_2 \ldots u_m)^2,$$

all the u's being linear functions of the two given variables. It is easy to extend an analogous mode of representation to functions of any number of letters. From the above we see that for cubic, biquadratic, and quintic functions of two letters, the canonical forms will be respectively

$$u^3 + v^3, \quad u^4 + v^4 + Ku^2v^2, \quad u^5 + v^5 + w^5,$$

with a linear relation in the last-named case between u, v, w.

First as to the reduction of any 4^c function to Cayley's form

$$u^4 + v^4 + Ku^2v^2.$$

This may be effected in a great variety of ways, of which the following is not the simplest as regards the calculations required, but the most obvious. Let the modulus of transformation, whereby the given biquadratic function, say $F(x, y)$, becomes transmuted into its canonical form, be called M; let the determinant of F be called D_1, and the determinant of the determinant in respect to ξ and η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^2 F(x, y),$$

which latter, for brevity's sake, may be termed the Hessian of F, (although in stricter justice the Boolian would be the more proper designation), be called D_2. Then, by examining the canonical form itself (which is as it were the very *palpitating heart* of the function laid bare to inspection), we shall obtain without difficulty the two equations

$$(1 - 9m^2)^2 = M^{12} D_1 \frac{1}{4^6},$$

$$m^2 (1 - 9m^2)^2 (m^2 - 1)^2 = M^{24} D_2 \frac{1}{12^{12} 4^4}.$$

Eliminating the unknown quantity M, we obtain

$$\frac{m^2 (m^2 - 1)^2}{(1 - 9m^2)^2} = c, \quad \text{or} \quad \frac{m^3 - m}{1 - 9m^2} = c^{\frac{1}{2}},$$

where c is a known quantity.

This *cubic* equation for finding m is of a peculiar form; it being easy to show *à priori*, by going back to the canonical form, that its three roots are m, $\theta(m)$, $\theta^2(m)$, where

$$\theta(m) = \frac{m - 1}{3m + 1},$$

θ being a periodical form of function such that $\theta^3(m) = m$.

This it is which accounts for the simple expression for m, that may be obtained by solving the cubic above given. A better practical mode is to take, instead of the determinant of the given function and its Hessian, the two hyperdeterminants and eliminate as before: a cubic equation having precisely the same properties, and in fact virtually identical with the former, will result. When m and consequently M are found, there is no difficulty whatever, calling the given function F and its Hessian $H(F)$, in forming linear functions of the two, as

$$\left.\begin{array}{l} \phi(m) F + \psi(m) H(F) \\ \phi_1(m) F + \psi_1(m) H(F) \end{array}\right\},$$

which shall be equal to, that is, identical with, $(u^2 + v^2)^2$ and u^2v^2, whence u and v are completely determined.

Another and interesting mode of solution is to take, besides the given function F and its Hessian, either the *second* Hessian or the post-Hessian of the given function, by the post-Hessian understanding the determinant in respect of ξ and η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^3 F:$$

any three of the four functions will be linearly related, and it may be shown that, calling either the second Hessian (that is, the Hessian of the Hessian) or the post-Hessian H', we shall have

$$H'(F) + aH(F) + bF = 0,$$

where a and b will be *rational* and *integer* functions of the coefficients of F, and numerical multiples of two quantities R and S, such that the determinant of F will be equal $R^3 + S^2$; and this, be it observed, without any previous knowledge of the existence of these hyperdeterminants R and S.

If now we go to Hesse's form for a cubic function of three letters, we shall find that precisely similar modes of investigation apply step for step. Calling the function F and its Hessian $H(F)$, and the post-Hessian or second Hessian at choice $H'(F)$, we shall find

$$H'(F) + mSH(F) + nR^2F = 0,$$

where m and n are numerical quantities and $R^3 + S^2$ equal the determinant of F. It is interesting to contrast this equation with the one previously mentioned as applicable to the 4^e functions of two letters, namely,

$$H'(F) + mRH(F) + nSF = 0.$$

In both instances there is no difficulty in assigning the relations between the original R and S, and the R and S of any adjunctive form. All Aronhold's results may be thus obtained and further extended without the slightest difficulty. As regards the equation for finding the parameter in Hesse's canonical form for the cubic of three letters, this will be of the 4th degree in respect to the cube of the parameter, and the roots will be functionally representable as

$$x; \quad \theta(x); \quad \phi(x); \quad \psi(x),$$

where

$$\theta^2(x) = \phi^2(x) = \psi^2(x) = x;$$
$$\theta\phi(x) = \phi\theta(x) = \psi(x),$$
$$\phi\psi(x) = \psi\phi(x) = \theta(x),$$
$$\psi\theta(x) = \theta\psi(x) = \phi(x);$$

owing to which property the equation is soluble under the peculiar form observed by Aronhold.

I pass on now to a brief account of the method, or rather of a method (for I doubt not of being able to discover others more practical), of reducing a function of the 5th degree of two letters (say of x and y) to its canonical form $u^5+v^5+w^5$, subject to the linear relation $au+bv+cw=0$, where the ratios $a:b:c$, and the linear relations between u, v, w and the two given variables are the objects of research. Here I have found great aid from the method of Relative Determinants; and I may notice that the successful application of more compendious methods to the question would be greatly facilitated were there in existence a theory of Relative Hyperdeterminants, which is still all to form, but which I little doubt, with the blessing of God, to be able to accomplish. It may some little facilitate the comprehension of what follows, if c be considered as representing unity.

Calling as before the given quintic function F, the modulus of transformation M, the Hessian and post-Hessian of F, H and H', and its ordinary or constant determinant D, we shall find

$$a^2v^3w^3+b^2w^3u^3+c^2u^3v^3=M^2H,$$

and

$$P_1P_2P_3P_4=M^6H',$$

where

$$\begin{aligned}P_1 &= \quad a^{\frac{3}{2}}vw+b^{\frac{3}{2}}wu+c^{\frac{3}{2}}uv,\\ P_2 &= \quad a^{\frac{3}{2}}vw-b^{\frac{3}{2}}wu-c^{\frac{3}{2}}uv,\\ P_3 &= -a^{\frac{3}{2}}vw+b^{\frac{3}{2}}wu-c^{\frac{3}{2}}uv,\\ P_4 &= -a^{\frac{3}{2}}vw-b^{\frac{3}{2}}wu+c^{\frac{3}{2}}uv;\end{aligned}$$

also $D=M^{20}$ multiplied by the product of the sixteen values of

$$a^{\frac{5}{4}}+b^{\frac{5}{4}}(1)^{\frac{1}{4}}+c^{\frac{5}{4}}(1)^{\frac{1}{4}}.$$

From the above equations it may be shown that H' (a known function of the 8th degree of the given variables x, y) must be capable of being thrown under the form

$$L\{(x-a_1y)(x-a_2y)\times(x-a_3y)(x-a_4y)\\ \times(x-a_5y)(x-a_6y)\times(x-a_7y)(x-a_8y)\},$$

where

$$(a_1-a_2)^2\times(a_3-a_4)^2\times(a_5-a_6)^2\times(a_7-a_8)^2 = \frac{D}{L^2}=K,$$

so that K is a known quantity*. Accordingly the said equation of the 8th degree, considered as an algebraical equation in $\frac{x}{y}$, may by known methods be

* Or in other words, the post-Hessian determinant of a given function in two letters of the second degree, may be divided into four quadratic factors in such a way that the product of the determinants of these several factors shall be equal to the determinant of the given function.

found by means of equations not exceeding the 4th or even the 3rd degree: in fact, to do this it is only necessary to form the equation to the squares of the differences of the roots of $\frac{x}{y}$ in the equation $H' \div y^8 = 0$, which new equation will be of the 28th degree. If we then form two other equations of the 378th degree, one having its roots equal to $\sqrt{K}$ multiplied by the binary products of the twenty-eight roots of the equation last named, the other to $\sqrt{K}$ multiplied by the reciprocal of such binary products, the left-hand members of these two equations expressed under the usual form will have a factor in common, which may be found by the process of common measure and will be of the 6th degree, whose roots consisting of three pairs of reciprocals may be found by the solution of cubics only.

In this way, by means of cubics and quadratics,

$$(a_1 - a_2)^2, \quad (a_3 - a_4)^2, \quad (a_5 - a_6)^2, \quad (a_7 - a_8)^2,$$

can be found, which being known,

$$a_1 a_2, \quad a_3 a_4, \quad a_5 a_6, \quad a_7 a_8,$$

can be determined in pairs by means of quadratics from the equation $H' \div y^8 = 0$. This being supposed to be done, we have

$$P_1 = fL_1,$$
$$P_2 = gL_2,$$
$$P_3 = hL_3,$$
$$P_4 = kL_4,$$

where L_1, L_2, L_3, L_4, are *known* quadratic functions of x and y. To determine the ratios of f, g, h, k, we have three equations* obtained from the identity

$$fL_1 + gL_2 + hL_3 + kL_4 (= P_1 + P_2 + P_3 + P_4) = 0;$$

$f : g : h : k$ being known, $fL_1 : gL_2 : hL_3 : kL_4$ are known ratios.

But
$$P_1 + P_2 = 2a^{\frac{3}{2}} vw,$$
$$P_1 + P_3 = 2b^{\frac{3}{2}} wu,$$
$$P_1 + P_4 = 2c^{\frac{3}{2}} uv.$$

Hence
$$a^{\frac{3}{2}} vw = \lambda P,$$
$$b^{\frac{3}{2}} wu = \lambda Q,$$
$$c^{\frac{3}{2}} uv = \lambda R,$$

where P, Q, R are known quadratic functions of x, y.

* For we must have the coefficients of x^2, xy and y^2 in

$$fL_1 + gL_2 + hL_3 + kL_4,$$

of all them zero.

Hence $a:b:c$ may be found by means of the identical equation

$$a^2w^3v^3+b^2u^3w^3+c^2v^3u^3=H(F),$$

whereby the ratios $a^{-\frac{5}{2}}:b^{-\frac{5}{2}}:c^{-\frac{5}{2}}$ can be obtained without any further extraction of roots, showing that there is but one single true system of ratios $a^5:b^5:c^5$ applicable to the problem; $a:b:c$ being thus found, λ is easily determined, and thus finally u, v, w are found in terms of x and y*.

I have little doubt that a more expeditious mode of solution than the foregoing† will be afforded by an examination of the properties and relations of the *quadratic and cubic forms*, adjunctive to the general quintic functions, and indeed to every $(4n+1)^c$ function of two letters hereinbefore adverted to.

Sufficient space does not remain for detailing the steps whereby the general cubic function of *four* letters may, by aid of equations *not transcending the fifth degree*, be reduced to its canonical form $u^3+v^3+w^3+p^3+q^3$, wherein u, v, w, p, q are connected by a linear equation

$$au+bv+cw+dp+eq=0;$$

the four ratios of whose coefficients $a:b:c:d:e$ give the necessary number $\frac{4.5.6}{1.2.3}-4^2$ parameters furnished by the general rule. Suffice it for the present to say, that the analytical mode of solution depends upon a circumstance capable of the following geometrical statement: Every surface of the 4th degree represented by a function which is the Hessian to any given cubic function whatever of four letters, has lying upon it ten straight lines meeting three and three in ten points, and these ten points are the only points which enjoy the following property in respect to the surface of the 3rd degree denoted by equating to zero the cubical function in question, to wit, that the cone drawn from any one of them as vertex to envelop the surface, will meet it not in a continuous double curve of the 6th degree, but in two curves each of the 3rd degree, lying in *planes* which intersect in the ten lines respectively above named; so that to each of the ten points corresponds one of the ten lines: these ten points and lines are the intersections taken respectively three with three, and two with two, of *a single and unique system* of five principal planes appurtenant to every surface of the 3rd degree, and these planes are no other than those denoted by

$$u=0,\quad v=0,\quad w=0,\quad p=0,\quad q=0.$$

* The problem thus solved may be stated as consisting in reducing the general function $ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5$ to the form

$$(lx+my)^5+(l'x+m'y)^5+(l''x+m''y)^5.$$

† The coefficients in the reducing recurrent equation of the 6th degree in the process above detailed may rise to be of 541632 dimensions in respect to the original coefficients in F.

I have found also by the theory of Sub-resultants, that the analogy between lines and surfaces of the third degree, in regard to the existence of double and conical points, is preserved in this wise: that in the same way as a double point on a curve of the 3rd degree commands the existence of a double point on its Hessian, so does a conical point in a surface of the 3rd degree command over and above the 10 necessary, and so to speak natural conical points, at least one extra, that is to say an 11th conical point on *its* Hessian. And here for the present I must quit my brief and imperfect notice of this subject, composed amidst the interruptions and distractions of an official and professional life.

Observation. It may be somewhat interesting and instructive to my readers, to have a table of the successive scalar* determinants of a quintic function of two letters presented to them at a single glance. Preserving the notation above [page 193], we have the following expressions:

The given function $= u^5 + v^5 + w^5$,

its Hessian $= M^2 (a^2v^3w^3 + b^2w^3u^3 + c^2u^3v^3)$,

its post-Hessian $= M^6 \times$ the product of the *four* forms of

$$a^{\frac{3}{2}} vw + b^{\frac{3}{2}} (1)^{\frac{1}{2}} wu + c^{\frac{3}{2}} (1)^{\frac{1}{2}} uv;$$

its præter-post-Hessian $= M^{12} \times$ the product of the *nine* forms of

$$a^{\frac{4}{3}} v^{\frac{1}{3}} w^{\frac{1}{3}} + b^{\frac{4}{3}} (1)^{\frac{1}{3}} w^{\frac{1}{3}} u^{\frac{1}{3}} + c^{\frac{4}{3}} (1)^{\frac{1}{3}} u^{\frac{1}{3}} v^{\frac{1}{3}},$$

and the final determinant $= M^{20} \times$ the product of the *sixteen* forms of

$$a^{\frac{5}{4}} + (1)^{\frac{1}{4}} b^{\frac{5}{4}} + (1)^{\frac{1}{4}} c^{\frac{5}{4}}.$$

The success of the method applied depends (as above shown) upon the fact of a certain function of the roots of the post-Hessian (which is an octavic function of the variables) being known, which fact *hinges* upon the circumstance that

$$(M^6)^2 \times (M^2)^4 = M^{20}.$$

P.S. I have much pleasure in subjoining the cubical hyperdeterminant of the 12th degree function of two letters, worked out upon the principle of Compound Permutation hinted at in the foregoing pages, for which I am indebted to the kindness and skill of my friend Mr Spottiswoode.

* By which I mean the determinants in respect to ξ, η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^r F(xy).$$

The function being called

$$ax^{12} + 12bx^{11}y + \frac{12.11}{2}cx^{10}y^2 + \&c. \ldots + ly^{12},$$

the following is* its cubical hyperdeterminant:

$$\begin{aligned}
&agm - 6ahl + 15aik + 10aj^2 - 6bfm,\\
&- 24bhk + 30bgl + 20bij - 24cfl + 114cgk,\\
&- 145ci^2 + 50chj + 15cem + 20cgi + 20ch^2,\\
&- 400dgj + 280dhi + 20del + 50dfe + 10d^2k,\\
&+ 385egi - 135e^2k - 290eh^2 + 705fgh,\\
&- 330f^2i - 50g^3.
\end{aligned}$$

Mr Spottiswoode will I hope publish the work itself in the next number of the *Journal*, in which I shall also show how the hyperdeterminants of the cubical function of three letters, Aronhold's S and T, may be similarly obtained.

[* See below, p. 202.]

33.

ON THE GENERAL THEORY OF ASSOCIATED ALGEBRAICAL FORMS.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 289—293.]

THE following brief exposition of the general theory of Associated Forms, as far as it has been as yet developed by the labours or genius of mathematicians, is intended as elucidatory and, to a certain extent, emendative of some of the statements in my paper* on Linear Transformations, in the preceding number of the *Journal.*

In the first place, let a linear equivalent of any given homogeneous function be understood to mean what the function becomes when linear functions of the variables are substituted in place of the variables themselves, subject to the condition of the modulus of transformation (that is, the value of the determinant formed by the coefficients of transformation) being unity.

Secondly, let two square arrays of terms (the determinants corresponding to each of which are unity) be said to be complementary when each term in the one square is equal to the value of what the determinant represented by the other square becomes when the corresponding term itself is taken unity, but all the other terms in the same line and column with it are taken zero. This relation between the two squares is well known to be reciprocal. Thus, for instance,

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}$$

will be said to be reciprocally complementary to one another when the two determinants which they represent are each unity, and when we have

[* p. 184, above.]

$$a = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \beta' & \gamma' \\ 0 & \beta'' & \gamma'' \end{vmatrix} \qquad \alpha = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b' & c' \\ 0 & b'' & c'' \end{vmatrix}$$

$$b = \begin{vmatrix} 0 & 1 & 0 \\ \alpha' & 0 & \gamma' \\ \alpha'' & 0 & \gamma'' \end{vmatrix} \qquad \beta = \begin{vmatrix} 0 & 1 & 0 \\ a' & 0 & c' \\ a'' & 0 & c'' \end{vmatrix}$$

$$b' = \begin{vmatrix} \alpha & 0 & \gamma \\ 0 & 1 & 0 \\ \alpha'' & 0 & \gamma'' \end{vmatrix} \qquad \beta' = \begin{vmatrix} a & 0 & c \\ 0 & 1 & 0 \\ a'' & 0 & c'' \end{vmatrix}$$

&c. &c.

Accordingly, two transformations, say of $F(x, y, z)$ and $G(u, v, w)$ respectively, may be said to be concurrent when in F for x, y, z, we write

$$\begin{aligned} &ax + by + cz, \\ &a'x + b'y + c'z, \\ &a''x + b''y + c''z; \end{aligned}$$

and in G for u, v, w, we write

$$\begin{aligned} &au + bv + cw, \\ &a'u + b'v + c'w, \\ &a''u + b''v + c''w; \end{aligned}$$

but complementary when for u, v, w, we write

$$\begin{aligned} &\alpha u + \beta v + \gamma w, \\ &\alpha' u + \beta' v + \gamma' w, \\ &\alpha'' u + \beta'' v + \gamma'' w; \end{aligned}$$

a, b, c, &c., α, β, γ, &c. being related in the manner antecedently explained.

Two forms, each of the same number of variables, may be said to be associate forms when the coefficients of the one are functions of those of the other; and when it happens that the coefficients of the first are all explicit functions of those of the second, the latter may be termed the originant and the former the derivant.

If now all the linear equivalents of one or of two associated forms are similarly related to corresponding linear equivalents of the other, so that each may be derived from each by the same law, the forms so associated will be said to be concomitant each to the other. This concomitance may be of two kinds, and very probably, in the nature of things, only of the two kinds about to be described.

The first species of concomitance is defined by the corresponding equivalents of the two associated forms being deduced by precisely similar, or, as we have expressed it, concurrent transformations or substitutions, each from its given primitive. The second species of concomitance is defined by the corresponding equivalents being deduced not by similar but by contrary, that is, reciprocal or complementary substitutions. Concomitants of the first kind may be called covariants; concomitants of the second kind may be called contravariants. When of the two associated forms one is a constant, the distinction between co- and contra-variants disappears, and the constant may be termed an invariant of the form with which it is associated*. It follows readily from these definitions that a covariant of a covariant and a contravariant of a contravariant are each of them covariants; but a covariant of a contravariant and a contravariant of a covariant are each of them contravariants; and also that an invariant, whether of a covariant or of a contravariant, is an invariant of the original function†.

It will also readily be seen that as regards functions of two letters a contravariant becomes a covariant by the simple interchange of x, y with $-y$, x, respectively. Covariants are Mr Cayley's hyperdeterminants; contravariants include, but are not coincident with, M. Hermite's formes-adjointes, if we understand by the last-named term such forms as may be derived by the process described by M. Hermite in the third of his letters to M. Jacobi, "Sur différents objets de la Théorie des Nombres," (which process is an extension of that employed for determining the polar reciprocal of an algebraical locus‡). M. Hermite appears, however, elsewhere to have used

* Accordingly an invariant to a given form may be defined to be such a function of the coefficients of the form, as remains absolutely unaltered when instead of the given form any linear equivalent thereto is substituted. Of course if the determinant of the coefficients of the transformations correspondent to the respective equivalents be not taken unity as supposed in this definition, the effect will be merely to introduce as a multiplier some power of the determinant formed by the coefficients of transformation.

† It may likewise be shown that linear equivalents of covariants and contravariants are themselves related to one another as covariants and contravariants respectively, the transformations by which the equivalents are obtained being taken concurrent in the one case and contrary or reciprocal in the other; and of course any algebraic function of any number of covariants is a covariant and of contravariants a contravariant.

‡ This has been further generalized by me in the theorem§ given in the last number of this *Journal*, where I have shown in effect that any invariant in respect to $\xi, \eta \ldots \theta$ of

$$f(\xi, \eta \ldots \theta) + (x\xi + y\eta + \ldots + t\theta + \rho)\rho^{n-1},$$

(f being supposed to be of the degree n) is a contravariant of $f(x, y \ldots t)$. When this invariant is the determinant of f, it may be shown that we obtain M. Hermite's theorem. It is somewhat remarkable that contravariants should have been in use among mathematicians as well in geometry as the theory of numbers (although their character as such was not recognized) before covariants had ever made their appearance. Invariants of course first came up with the theory of the equation to the squares of the differences of the roots of equations, the last term in such equation being an invariant. I believe that I am correct in saying that covariants first made their appearance in one of Mr Boole's papers, in this *Journal*; but Hesse's brilliant application

[§ p. 186 above.]

the term forme-adjointe in a sense as wide as that in which I employ contravariants. For instance, he has given a most remarkable theorem, which admits of being stated as follows:

If we have a function of any number of letters, say of x, y, z, as

$$ax^m + mbx^{m-1}y + mcx^{m-1}z + \frac{m(m-1)}{2}dx^{m-2}y^2 + \&c.,$$

and if I be any invariant of this function, then will

$$\left(x^m \frac{d}{da} + x^{m-1}y\frac{d}{db} + x^{m-1}z\frac{d}{dc} + x^{m-2}y^2\frac{d}{dd}\ \&c.\right)^r I$$

be a "*forme-adjointe*" of the given function. It is perfectly true and admits of being very easily proved, as I shall show in your next number, that this is a contravariant of the given function*; but it is not (as far as I can see) a forme-adjointe in the sense in which the use of that word is restricted in the letter alluded to. If, however, we adopt as the *definition* of *formes-adjointes* generally, that property in regard to their transformées which M. Hermite has demonstrated of the particular class treated of by him in the letter alluded to, then his formes-adjointes become coincident with my contravariants. It will thus be seen that covariants and contravariants form two distinct and coextensive species of associated forms, which divide between them the wide and fertile empire of linear transformations so far as its provinces have been as yet laid open by the researches of analysts. In your next number I propose to enter much more largely into the subject generally. More particularly I shall describe the new method of Permutants, including the theory of Intermutants and Commutants (which latter are a species of the former, but embrace Determinants as a particular case), and their application to the theory of Invariants. I shall also exhibit the connexion between the theory of Invariants and that of Symmetrical Functions, and some remarkable theorems on Relative Invariants†.

Some of your readers may like to be informed that a Supplement to my last paper, under the title of "An Essay on Canonical Forms," has been since published‡; and that I have there given a much simpler method of solution of the problem of the reduction of quintic functions to their canonical form than in the original memoir, and extended the method successfully to the

of one from among the infinite variety of these forms to the discovery of the points of inflexion in a curve of the third order, in other words, to the Canonical Reduction of the Cubic Function of Three Letters, appears to have been the first occasion of their being turned to practical account.

* This is also true if I be taken any *covariant* instead of an invariant of the function.

† It will be readily apprehended that the definitions and conceptions above stated, respecting covariants and contravariants of two single functions, may be extended so as to comprehend systems of functions covariantive or contravariantive to one another.

‡ By Mr George Bell, University Bookseller, Fleet Street. [p. 203 below.]

reduction of all odd-degreed functions to their canonical form. I may take this occasion to state that the Lemma given in Note (B) of the Supplement, upon which this method of reduction is based, is an immediate deduction from the well-known theorem for the multiplication of Determinants.

There is a numerical error in "The Cubical Hyperdeterminant of the Twelfth Degree," worked out after the method of commutants by Mr Spottiswoode, given at the end of my paper in the May Number. The correct result will be stated in the next number of the *Journal*, where I hope also to be able to fix the number of distinct solutions of the problem of reducing a Sextic Function to its canonical form

$$u^6 + v^6 + w^6 + mu^2v^2w^2.$$

For odd-degreed functions there is never more than one solution possible, as shown in the Supplement referred to.

P.S. Since the above was sent to press, I have discovered an uniform mode of solution for the canonical reduction of functions, whether of odd or even degrees. The canonical form however, except for the fourth and eighth degrees, requires to be varied from that assumed in my previous paper. Thus, for the sixth degree the canonical form will be

$$au^6 + bv^6 + cw^6 + muvw\,(v-w)\,(w-u)\,(u-v),$$

where u, v, w are supposed to be connected by the identical equation $u+v+w=0$. And there will be only *two* solutions—a remarkable and most unexpected discovery. For functions of the eighth degree there are five distinct solutions, and in general there is the strongest reason for believing (indeed it may be positively affirmed) that *when the canonical form has been rightly assumed* for a function of the even degree n, the number of solutions will be $\frac{1}{2}(n+2)$ when $\frac{1}{2}n$ is even, but $\frac{1}{4}(n+2)$ when $\frac{1}{2}n$ is odd. It turns out therefore that the theory for functions of the sixth degree is in some respects simpler than for those of the fourth. The investigation into canonical forms here referred to has led me to the discovery of a most unexpected theorem for finding all the invariants of a certain class, belonging to functions of two letters of an even degree.

34.

AN ESSAY ON CANONICAL FORMS, SUPPLEMENT TO A SKETCH OF A MEMOIR* ON ELIMINATION, TRANSFORMATION AND CANONICAL FORMS.

SINCE the above paper was in print I have succeeded in obtaining a canonical representation of the quadratic and cubic functions adjunctive to the general quintic (5th degreed) functions of two letters.

Let F the quintic function of x, y,

$$= u^5 + v^5 + w^5,$$

and

$$au + bv + cw = 0,$$

M being the modulus of the transformation, whereby transition is made from x, y to u, v. Then the quadratic adjunctive is

$$\frac{M^4}{c^4}\{a^4vw + b^4wu + c^4uv\};$$

and the cubic adjunctive is simply

$$\frac{1}{c^3}M^6(abc)^2\,uvw\dagger.$$

Hence we can, in accordance with what I ventured to predict in the preceding sketch, find u, v, w, by means of a simple and practical co-process. To wit, call

$$F = lx^5 + 5mx^4y + 10nx^3y^2 + 10px^2y^3 + 5qxy^4 + ry^5.$$

[* p. 184 above. See p. 201, note ‡.]

† The knowledge of the existence of these *lower* adjunctive forms is mainly a consequence of Mr Cayley's splendid discovery of hyperdeterminant constants. In fact, they are respectively the quadratic and cubic hyperdeterminants in respect to ξ and η of $\frac{1}{1.2.3.4.5}\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^4 F$; x and y being treated as constants.

The fortunate proclaimer of a new outlying planet has been justly rewarded by the offer of a baronetcy and a national pension, which the writer of this wishes him long life and health to enjoy. In the meanwhile, what has been done in honour of the discoverer of a new and inexhaustible region of exquisite analysis?

Form the determinant

$$\begin{vmatrix} lx + my, & mx + ny, & nx + py \\ mx + ny, & nx + py, & px + qy \\ nx + py, & px + qy, & qx + ry \end{vmatrix}.$$

Let this cubic function, by solving it as a cubic equation, be made equal to

$$L(x + fy)(x + gy)(x + hy),$$

then

$$u = k(x + fy), \quad v = l(x + gy), \quad w = m(x + hy).$$

By means of the identity, $F = u^5 + v^5 + w^5$, l^5, m^5, n^5, are known by the solution of linear equations, and thus u, v, w, are determined by solving a cubic equation instead of one of the eighth degree, as in the method first given, and the process of canonising a quintic function is rendered *practically* possible.

For brevity sake let c represent unity. The constant determinant of the cubic adjunctive will be found to be

$$3M^{30}(abc)^{10}.$$

Calling, then, the cubic adjunctive of F, $C(F)$, we have the remarkable equation

$$uvw = \frac{C(F)}{\sqrt[5]{\{\frac{1}{3} \square C(F)\}}}.$$

It may also be shown that if we call the Hessian of F, $H(F)$, we shall have the following equally remarkable equation:

$$\square H(F) = \tfrac{1}{3} \square F \times \square C(F).$$

Again, calling the quadratic adjunctive of F, $Q(F)$, we shall easily find

$$\square Q(F) = M^{10} \begin{Bmatrix} (a^{\frac{5}{2}} + b^{\frac{5}{2}} + c^{\frac{5}{2}}) \\ (a^{\frac{5}{2}} + b^{\frac{5}{2}} - c^{\frac{5}{2}}) \\ (a^{\frac{5}{2}} - b^{\frac{5}{2}} + c^{\frac{5}{2}}) \\ (a^{\frac{5}{2}} - b^{\frac{5}{2}} - c^{\frac{5}{2}}) \end{Bmatrix},$$

or, if we please,

$$= M^{10} \begin{Bmatrix} a^{10} + b^{10} + c^{10} \\ - 2a^5b^5 - 2a^5c^5 - 2b^5c^5 \end{Bmatrix}.$$

When u, v, w are known, a, b, c, which are the resultants of v, w; w, u; u, v respectively are known. But their ratios, or, if we please to say so, the ratios of $a^5 : b^5 : c^5$, may be found independently and very elegantly as follows:—

Let $\quad M^{10} \times$ product of the 4 forms of $a^{\frac{5}{2}} + 1^{\frac{1}{2}} b^{\frac{5}{2}} + 1^{\frac{1}{2}} c^{\frac{5}{2}} = A$,

$M^{20} \times$ product of the 16 forms of $a^{\frac{5}{4}} + 1^{\frac{1}{4}} b^{\frac{5}{4}} + 1^{\frac{1}{4}} c^{\frac{5}{4}} = B$,

$M^{30} \times a^{10} . b^{10} . c^{10} \qquad = C$.

A, B, C are known quantities, being respectively what we have called $\square Q(F)$, $\square(F)$*, $\frac{1}{3}\square C(F)$.

It may easily be shown that

$$B - A^2 = 128M^{20}a^5b^5c^5(a^5 + b^5 + c^5).$$

Hence M^5a^5, M^5b^5, M^5c^5 are the roots of ρ in the cubic equation

$$\rho^3 + \frac{B - A^2}{2^7 C^{\frac{1}{3}}}\rho^2 + \tfrac{1}{4}\left\{\frac{(B - A^2)^2}{2^{14}C} - A\right\}\rho + C^{\frac{1}{3}} = 0.$$

A, B, C, it will be observed, are independent and, as they may be termed, prime or radical adjunctive constants. Hitherto much mystery and uncertainty have attached to the theory of hyperdeterminants, from its having been tacitly assumed that they were always either of lower dimensions than the ordinary determinant, or else algebraical functions of such, and of the determinant. Whereas we now see that, whilst the determinant of a function in two letters of the fifth degree is of eight dimensions, one of its radical or primitive hyperdeterminants is of four, but the other of twelve dimensions. This is a most valuable consequence, and would seem to indicate that the number of radical hyperdeterminants to a function, over and above the common determinant, is always equal to the number of parameters entering into its canonical form. The importance of this ascertainment of an unsuspected third *radical* constant, adjunctive to a quintic function of two letters, in making to march the theory of hyperdeterminants, can hardly be over-estimated.

From the equation last given we are enabled to assign the conditions in order that two functions of the fifth degree may be capable of being linearly transformed either into the other. For if we call F and F' two such linearly equivalent quintic functions, they must be capable each of being thrown under the same form $u^5 + v^5 + (lu + mv)^5$, where l and m shall be the same for each. Consequently we must have the roots of ρ in the same ratio for F and F', which conditions may be expressed by means of the two equations

$$\frac{B - A^2}{C^{\frac{2}{3}}} = \frac{B' - A'^2}{C'^{\frac{2}{3}}},$$

$$\frac{(B - A^2)^2 - 2^{14}AC}{C^{\frac{4}{3}}} = \frac{(B' - A'^2)^2 - 2^{14}A'C'}{C'^{\frac{4}{3}}},$$

* More strictly speaking (and this correction should be supplied throughout in the "Sketch"), B is the negative determinant of $\frac{1}{5}F$. After finding, by the method of characteristics, or any special artifices, the algebraic part of the value of a resultant or determinant, a process frequently of some complexity remains over in assigning its numerical multiplier; this part of the operation being analogous to that which occurs in the Integral Calculus, of determining the constant to be added after the general form of an integral has been determined. In the "Sketch," a correction for the numerical multiplier remains also to be applied to the expressions given for the successive Hessian determinants.

A', B', C', of course representing the same functions of the coefficients of F' as A, B, C, respectively of F.

The two conditions required in their simplest form are accordingly

$$\frac{A}{C^{\frac{1}{3}}} = \frac{A'}{C'^{\frac{1}{3}}},$$

$$\frac{B}{C^{\frac{2}{3}}} = \frac{B'}{C'^{\frac{2}{3}}},$$

or

$$A^3 : B^2 : C :: A'^3 : B'^2 : C',$$

that is to say, *all quintic functions of two letters of which the determinant is to the subduplicate power of the radical hyperdeterminant of the twelfth order and to the sesquiduplicate power of the radical hyperdeterminant of the fourth order in given ratios, are mutually convertible.*

So for the quartic (that is, biquadratic) function of two letters, calling R and S the radical adjunctive constants of the second and third orders, the condition of convertibility between different forms of the same is, that $R^3 : S^2$ shall be a given ratio. And, in general, we may infer that the condition of convertibility between different functions of any degree is, that the several radical adjunctive constants of each raised respectively to such powers as will make them of like dimensions, shall be to one another in given ratios. Of course all cubic functions of two letters, according to this rule, are mutually convertible without any condition, they having but one radical adjunctive constant; and in fact all such functions, being representable as the sum of two cubes of new variables linearly related to those given, are necessarily convertible.

I have further succeeded in obtaining the canonical form of the *quadratic* adjunctive to *any odd degreed* function of two letters, which presents a wonderful analogy to the theory of relative determinants of *quadratic functions of any number* of letters, and constitutes an important step towards the construction of the theory of relative hyperdeterminants.

Let a function of two letters of the odd degree $m\,(=2n-1)$ be thrown under its canonical form,

$$u_1^m + u_2^m + \ldots + u_n^m,$$

and let there exist the $n-2$ equations,

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0, \qquad (1)$$

$$b_1u_1 + b_2u_2 + \ldots + b_nu_n = 0, \qquad (2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$l_1u_1 + l_2u_2 + \ldots + l_nu_n = 0. \qquad (n-2)$$

Then, if M be the modulus of the transformation which converts u_1, u_2 into

x, y, and if, on making $\theta_1, \theta_2 \ldots \theta_n$ disjunctively equal to $1, 2 \ldots n$ we use (θ_{n-1}, θ_n) to denote in general the determinant

$$\begin{vmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_{n-2}} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_{n-2}} \\ \ldots & \ldots \\ l_{\theta_1}, & l_{\theta_2} \ldots l_{\theta_{n-2}} \end{vmatrix},$$

the quadratic adjunctive of $\dfrac{1}{m(m-1)\ldots 2} F$ will be

$$\frac{M^{m-1}}{(1, 2)^{m-1}} \Sigma \{(\theta_r, \theta_s)^{m-1} (u_r \,.\, u_s)\}^*.$$

N.B. By means of this formula, and of the theorem for finding relative determinants of quadratic functions, we can obtain the general canonical form for one set of the biquadratic adjunctive constants (hyperdeterminants of the fourth order in Mr Cayley's language) of any odd degreed function of two letters †.

Thus, for the fifth degree, preserving the notation of the "Sketch," we have the biquadratic adjunctive constant

$$= \begin{vmatrix} 0, & c^4, & b^4, & a \\ c^4, & 0, & a^4, & b \\ b^4, & a^4, & 0, & c \\ a, & b, & c, & 0 \end{vmatrix} \times \frac{M^{10}}{c^{10}}.$$

For the seventh degree, if we suppose the function to be equal to

$$u^7 + v^7 + w^7 + \theta^7,$$

and

$$au + bv + cw + d\theta = 0,$$

$$a'u + b'v + c'w + d'\theta = 0;$$

the biquadratic adjunctive constant will be $\dfrac{M^{14}}{(cd' - c'd)^{14}}$ multiplied by the determinant

$$\begin{vmatrix} 0, & (ab' - a'b)^6, & (ac' - a'c)^6, & (ad' - a'd)^6, & a, & a' \\ (ba' - b'a)^6, & 0, & (bc' - b'c)^6, & (bd' - b'd)^6, & b, & b' \\ (ca' - c'a)^6, & (cb' - c'b)^6, & 0, & (cd' - c'd)^6, & c, & c' \\ (da' - d'a)^6, & (db' - d'b)^6, & (dc' - d'c)^6, & 0, & d, & d' \\ a, & b, & c, & d, & 0, & 0 \\ a', & b', & c', & d', & 0, & 0 \end{vmatrix}.$$

* The condition $m = 2n - 1$ is only necessary in order that $\Sigma_n{}' (u^m)$ may be a canonical, because a possible and determinate, form for any given function of the mth degree. But the theorem in the text, so far as it serves to obtain the quadratic adjunctive of $\Sigma_n{}' (u^m)$, is true for *all* odd values of m, whether greater or less than $2n - 1$.

† See Note (A) of Appendix.

The determinants of the Hessian, the post-Hessian, and the præter-post-Hessian of F will be found (in the case of the quintic function) to be always multiples of powers of the determinant of the given function, and of its cubic adjunctive; and I believe that in general for a function of two letters of any degree the determinants of all the derived forms in the Hessian scale*, will be necessarily algebraical functions of any two of them.

I hope very shortly to accomplish the reduction of functions, as high as the seventh degree of two letters, to their canonical form, and also to present a complete theory of the failing or singular cases of canonical forms.

Since the above was in print I have discovered the following

General Theorem

for reducing a function of two letters of any odd degree to its canonical form.

Let the degree of the function be $(2n-1)$; then its canonical form is

$$u_1^{2n-1} + u_2^{2n-1} + \ldots + u_n^{2n-1},$$

with $(n-2)$ linear relations between $u_1, u_2, \ldots u_n$.

To find $u_1, u_2, \ldots u_n$, proceed as follows. Let the given function of the $(2n-1)$th degree be supposed to be

$$a_1x^{2n-1} + (2n-1)\,a_2x^{2n-2}y + (2n-1)\,\frac{2n-2}{2}\,a_3x^{2n-3}y^2 + \ldots + a_{2n}y^{2n-1}.$$

Form the determinant

$$\left|\begin{array}{llll} a_1x + a_2y, & a_2x + a_3y, & a_3x + a_4y \;\ldots & a_nx + a_{n+1}y \\ a_2x + a_3y, & a_3x + a_4y, & \ldots\ldots\ldots\ldots & a_{n+1}x + a_{n+2}y \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_nx + a_{n+1}y, & a_{n+1}x + a_{n+2}y, & \ldots\ldots\ldots & a_{2n-1}x + a_{2n}y \end{array}\right|.$$

This determinant is a function of x and y of the nth degree, and by resolving an equation of the nth degree, may be decomposed into n factors, say

$$(l_1x + m_1y)\,(l_2x + m_2y)\,\ldots\,(l_nx + m_ny);$$

* I use the term Hessian (more properly speaking the Boolian) Scale, to denote the *determinants* in respect of ξ and η of $\left(\xi\dfrac{d}{dx} + \eta\dfrac{d}{dy} + \&c.\right)^2 F$.

Neither Hesse, however, nor any other writer up to the present time, had thought of constructing, and still less of turning to account, the functions (the first only excepted) which figure in this scale.

we shall then have

$$u_1 = p_1 (l_1 x + m_1 y),$$
$$u_2 = p_2 (l_2 x + m_2 y),$$
$$\dots\dots\dots\dots\dots$$
$$u_n = p_n (l_n x + m_n y),$$

where the l's and m's are known, and the $(2n-1)$th powers of the p's may be found linearly, by means of the identical equation $\Sigma u^{2n-1} = F(x, y)$. Thus for example a function of the seventh degree of two letters may be reduced to its canonical form

$$(lx + my)^7 + (l'x + m'y)^7 + (l''x + m''y)^7 + (l'''x + m'''y)^7,$$

by the resolution of a biquadratic equation. My demonstration of this extraordinary and unexpected consequence rests upon the following lemma*, itself a very beautiful and striking theorem (no doubt capable of much generalisation) in the theory of determinants. Form the rectangular matrix consisting of n rows and $(n+1)$ columns

$$\begin{array}{llll} T_1, & T_2, & T_3 & \dots T_{n+1}, \\ T_2, & T_3, & T_4 & \dots T_{n+2}, \\ T_3, & T_4, & T_5 & \dots T_{n+3}, \\ \dots & \dots & \dots & \dots \\ T_n, & T_{n+1}, & T_{n+2} & \dots T_{2n}, \end{array}$$

where

$$T_i = a_1^{r-i} b_1^{s+i} + a_2^{r-i} b_2^{s+i} + \dots + a_{n-1}^{r-i} b_{n-1}^{s+i}.$$

Then all the $n+1$ determinants that can be formed by rejecting any *one* column at pleasure out of this matrix are identically zero.

In order the better to realise the proof, suppose

$$n = 4, \text{ so that } 2n - 1 = 7.$$

Let

$$F(x, y) = a_1 x^7 + 7a_2 x^6 y + 21 a_3 x^5 y^2 + 35 a_4 x^4 y^3 + 35 a_5 x^3 y^4$$
$$+ 21 a_6 x^2 y^5 + 7 a_7 x y^6 + a_8 y^7.$$

Suppose

$$t^7 + u^7 + v^7 + w^7 = F(x, y) = G(u, v),$$
$$at + bu = v,$$
$$a't + b'u = w.$$

Then, if M is the modulus of transition from x, y to u, v the hyper-

* See Note (B) of Appendix.

determinant, or, to adopt my new expression, the permutant P_4 (meaning thereby)

$$\begin{vmatrix} a_1x+a_2y, & a_2x+a_3y, & a_3x+a_4y, & a_4x+a_5y \\ a_2x+a_3y, & a_3x+a_4y, & a_4x+a_5y, & a_5x+a_6y \\ a_3x+a_4y, & a_4x+a_5y, & a_5x+a_6y, & a_6x+a_7y \\ a_4x+a_5y, & a_5x+a_6y, & a_6x+a_7y, & a_7x+a_8y \end{vmatrix},$$

which is a constant adjunctive in respect to ξ and η of $\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^6 F$, will, according to the principles laid down in the preceding "Sketch," be the product of a power of M multiplied by the corresponding adjunctive constant of $\left(\xi\frac{d}{du}+\eta\frac{d}{dv}\right)^6 G(u, v)$, and is therefore a multiple of the determinant

$$\begin{vmatrix} (1+A_1)t+A_2u, & A_2t+A_3u, & A_3t+A_4u, & A_4t+A_5u \\ A_2t+A_3u, & A_3t+A_4u, & A_4t+A_5u, & A_5t+A_6u \\ A_3t+A_4u, & A_4t+A_5u, & A_5t+A_6u, & A_6t+A_7u \\ A_4t+A_5u, & A_5t+A_6u, & A_6t+A_7u, & A_7t+(1+A_8)u \end{vmatrix},$$

where

$$A_1=a^7+a'^7,\quad A_2=a^6b+a'^6b',\quad A_3=a^5b^2+a'^5b'^2 \ldots A_8=b^7+b'^7.$$

In this determinant the coefficient of u^4 is

$$\begin{vmatrix} A_2, & A_3, & A_4, & A_5 \\ A_3, & A_4, & A_5, & A_6 \\ A_4, & A_5, & A_6, & A_7 \\ A_5, & A_6, & A_7, & 1+A_8 \end{vmatrix}$$

which is numerically equal to

$$A_5\begin{vmatrix} A_3, & A_4, & A_5 \\ A_4, & A_5, & A_6 \\ A_5, & A_6, & A_7 \end{vmatrix} - A_6\begin{vmatrix} A_2, & A_4, & A_5 \\ A_3, & A_5, & A_6 \\ A_4, & A_6, & A_7 \end{vmatrix}$$

$$+A_7\begin{vmatrix} A_2, & A_3, & A_5 \\ A_3, & A_4, & A_6 \\ A_4, & A_5, & A_7 \end{vmatrix} - (1+A_8)\begin{vmatrix} A_2, & A_3, & A_4 \\ A_3, & A_4, & A_5 \\ A_4, & A_5, & A_6 \end{vmatrix}$$

$=0$, because the second factors of the products are all zero by the lemma. Hence the permutant P_4 vanishes when $t=0$, and consequently it contains t as a factor, and in like manner it may be proved to contain u, v, w.

Hence t, u, v, w are the algebraical factors of P_4, and precisely the same proof applies to show in the case of a function in x and y, say F_{2n-1}, of any odd degree $(2n-1)$ whatever, that the corresponding permutant P_n will contain the factors $u_1, u_2 \ldots u_n$ linear functions of x, y, such that

$$u_1^{2n-1} + u_2^{2n-1} + \ldots + u_n^{2n-1} = F_{2n-1}$$

as was to be shown.

Whenever P_n has equal roots, this will denote either (which is the more general case) that the usual canonical form fails and gives place to a singular form, (owing to some of the coefficients of transformation becoming infinite), or, which is the more special supposition, that the canonical form becomes catalectic by one or more of the linear roots* disappearing. Thus in the cubic function, if P_2 has equal roots, and consequently its determinant (which is coincident with that of the function itself) vanish, then the canonical form in general fails; so that, for example, $ax^3 + bx^2y$ cannot in general be exhibited as the sum of two cubes: if, however, certain further relations obtain between the coefficients of F, the canonical form reappears catalectically, the function becoming in fact representable as a single cube. So, again, for the quintic function (referring back to the notation above [page 205]), if P_3 have equal roots, that is if $C=0$, the canonical form fails, unless at the same time $B-A^2=0$, in which case the function becomes the sum of two fifth powers; but if furthermore $A=0$, then this catalectic form again gives place to a singular form, which, on the satisfaction of a further condition between the coefficients, again in its turn gives way before a (bicatalectic, that is) doubly catalectic form, namely, a single fifth power.

It is remarkable, that the form to which Mr Jerrard's method reduces the function of the fifth degree, expressed homogeneously as $ax^5 + bxy^4 + cy^5$, is a singular form, being incapable of being exhibited as the sum of three cubes; such, however, is not the case with the form $ax^5 + bx^3y^2 + cy^5$. It may further be remarked, that although the singly catalectic form of the quintic function is expressible by two conditions only, namely, $C=0$, $B-A^2=0$, it will be indicated by P_3 (which being a cubic function of x and y contains four terms) completely disappearing, so that apparently four conditions would appear to be required or implied. But of course these must be capable of being shown to be non-independent, and to be merely tantamount to the two independent ones, $C=0$, $B-A^2=0$. The theory of the catalectic forms of functions of the higher degrees of two variables presents many strong points of resemblance and of contrast to that of the catalectic forms of quadratic functions of several variables.

One important and immediate corollary from the General Theorem is, that the constants which enter into the linear functions appurtenant to the canonical form of any function of an odd degree form a single and unique system; or, in other words, the canonical forms for such functions are void of

* $u_1, u_2 \ldots u_n$ may be termed the linear roots of the form F_{2n-1}.

multiplicity, a result contrary to what might have been anticipated, and to what we know is the case for the canonical forms of functions of an even degree.

It may further be shown that if we have the $(n-2)$ equations

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0,$$

$$b_1u_1 + b_2u_2 + \ldots + b_nu_n = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$l_1u_1 + l_2u_2 + \ldots + l_nu_n = 0,$$

and call M the modulus of transformation in respect to u_1, u_2, and if we make

$$P_n = Ku_1u_2 \ldots u_n,$$

then

$$\left|\begin{matrix} a_3, & a_4 \ldots a_n \\ b_3, & b_4 \ldots b_n \\ \ldots & \ldots\ldots \\ l_3, & l_4 \ldots l_n \end{matrix}\right.^{\frac{1}{2}n(n-1)} K$$

is equal to the product of the $\frac{1}{2}n(n-1)$ factors of the form

$$\left|\begin{matrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_{n-2}} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_{n-2}} \\ \ldots & \ldots\ldots \\ l_{\theta_1}, & l_{\theta_2} \ldots l_{\theta_{n-2}} \end{matrix}\right|,$$

$\theta_1, \theta_2 \ldots \theta_{n-2}$ being any $(n-2)$ numbers out of the n numbers $1, 2, 3 \ldots n$.

It may hence be shown that

$$u_1u_2 \ldots u_n = \frac{P_n}{\left(\frac{1}{m} \underset{x,y}{\square} P_n\right)^{\frac{1}{2n-1}}} \text{*},$$

m being a number which is a function of n, and which may be shown to be equal to $\underset{x,y}{\square}(x^{n-1}y + xy^{n-1}) \div$ product of the squared differences of the roots of $l^{n-2} = 1$, that is

$$m = \left\{\frac{(n-1)^2 - 1}{-(n-2)}\right\}^{n-2} = (-n)^{n-2},$$

and thus

$$u_1u_2 \ldots u_n = \frac{P_n}{\sqrt[2n-1]{\{(-n)^{-(n-2)} . \underset{x,y}{\square} P_n\}}}.$$

* $\underset{x,y}{\square}$ means the determinant in respect to x and y.

As an example of the mode of finding $u_1, u_2 \dots u_n$, let

$$F = 3x^5 + 20x^3y^2 + 10xy^4,$$

then

$$P_3 = \begin{vmatrix} 3x, & 2y, & 2x \\ 2y, & 2x, & 2y \\ 2x, & 2y, & 2x \end{vmatrix} = 4x^3 - 4y^2x.$$

Hence

$$u = fx, \quad v = g(x+y), \quad w = h(x-y).$$

To find f, g, h, we have $u^5 + v^5 + w^5 = F$, hence

$$f^5 + g^5 + h^5 = 3; \quad g^5 + h^5 = 2; \quad g^5 - h^5 = 1;$$

whence we have

$$F = x^5 + (x+y)^5 + (x-y)^5.$$

Again, we find

$$\square\,(4x^3 - 4y^2x) = -4^4 \times 12,$$

$$\left(\frac{1}{(-3)} \square\, P_3\right)^{\frac{1}{5}} = 4,$$

and accordingly

$$x(x+y)(x-y) = \frac{P_3}{\{(-3)^{-1}.\, \square\, P_3\}^{\frac{1}{5}}},$$

according to the general formula above given.

As a second example let

$$F = 3x^7 + 42x^5y^2 + 70x^3y^4 + 14xy^6 + y^7;$$

then

$$P_4 = \begin{vmatrix} 3x, & 2y, & 2x, & 2y \\ 2y, & 2x, & 2y, & 2x \\ 2x, & 2y, & 2x, & 2y \\ 2y, & 2x, & 2y, & 2x+y \end{vmatrix} = 4(x^3y - xy^3) = 4xy(x-y)(x+y),$$

and accordingly we shall find

$$x^7 + y^7 + (x-y)^7 + (x+y)^7 = F.$$

Moreover

$$\square\,(4x^3y - 4xy^3) = 4^9,$$

and

$$4^{4-2} \qquad = 4^2.$$

Thus

$$\frac{P_4}{tuvw} = \sqrt[7]{\frac{\square\, P_4}{4^{4-2}}},$$

agreeable to the general formula.

As a corollary to our general proposition, it may be remarked, that if F_{2n-1} be a symmetrical function of x, y of the $(2n-1)$th degree, $P_n(F_{2n-1})$ will be also a symmetrical function of x and y, and may therefore be resolved into its factors by solving a *recurring* equation of the nth degree, which may, by well-known methods, be made to depend on the solution of an equation of the $\frac{1}{2}n$th or $\frac{1}{2}(n-1)$th degree, according as n is even or odd.

Hence the reduction of a function of two letters of the degree $4m \pm 1$ to its canonical form as the sum of powers may be made to depend on the solution of an equation of the mth degree; so that, for example, a symmetrical function of x, y, as high as the fifteenth or seventeenth degree, may be reduced by means of a biquadratic equation only.

In a short time I hope to present to the public a complete solution of the canonical forms of functions of two letters of even degrees, and possibly to exhibit some important applications of the principles of the method to the theory of numbers.

APPENDIX.

Note (A).

The permutants (meaning, in Mr Cayley's language, the hyperdeterminants) of $F_{2n+1}(x, y)$ of the fourth dimension in respect to the coefficients of F, may be all obtained by taking the quadratic permutant in respect to x and y of the quadratic permutant in respect of ξ and η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{2l} F_{2n+1}(x, y),$$

l having any integer value from 1 to n.

In extension of a theorem in the foregoing Supplement, which applies only to the case of $l = n$, I am able to state the following more general theorem, in which the same notation is preserved as above [page 207]. The quadratic permutant in respect to ξ and η of

$$\frac{1}{(2n+1)\,2n\,\ldots\,(2n-2l+2)} \left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{2l} F_{2n+1}(x, y),$$

is equal to

$$\frac{M^{2l}}{(1, 2)^{2l}} \Sigma \left\{(\theta_r, \ \theta_s)^{2l} (u_r \,.\, u_s)^{2n+1-2l}\right\}.$$

If now we proceed to form the quadratic permutant of the above sum in respect to x and y, we know *à priori*, by reason of Mr Cayley's invaluable researches, that we shall not get radically distinct results for all values, but only for certain periodically changing values of l.

I have not yet had leisure to seek for an explicit demonstration of this remarkable law, founded upon the above given canonical representation.

NOTE (B).

The lemma, upon which the general method for reducing odd degreed functions to their canonical form is founded, may be stated rather more simply and more generally as follows :—

The determinant

$$\begin{vmatrix} T_{r_1}, & T_{r_2} & \dots T_{r_n} \\ T_{r_1+l_1}, & T_{r_2+l_1} & \dots T_{r_n+l_1} \\ T_{r_1+l_2}, & T_{r_2+l_2} & \dots T_{r_n+l_2} \\ \dots & \dots & \dots \\ T_{r_1+l_{n-1}}, & T_{r_2+l_{n-1}} & \dots T_{r_n+l_{n-1}} \end{vmatrix},$$

where T_θ denotes $A_1 a_1^\theta + A_2 a_2^\theta + \dots + A_m a_m^\theta$ *provided that m is less than n*, is identically zero. In the theorem, as thus stated, there is no substantial loss of generality arising from the omission of the b's.

Thus stated the theorem and its extensions evidently repose upon the same or the like basis as the theory of partial fractions.

NOTE (C), *referring to the original "Sketch."*

The Boolo-Hessian scale of determinants furnishes a very pretty general theorem of geometrical reciprocity in connexion with the doctrine of successive polars. Let $F(x, y, z)$, a cubic homogeneous function of x, y, z equated to zero, express in general a curve of the third degree; then $\left(a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right) F$ will express its first polar in respect to the point a, b, c, that is, the conic which passes through the six points in which the tangents drawn from a, b, c to touch the given curve meet the same.

Again, if we take l, m, n the coordinates of any new point,

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)\left(a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right) F$$

will express the polar, that is the chord of contact of the above conic, in respect to the last named point. If now we eliminate l, m, n between the three equations

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)\left(a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right) F = 0,$$

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)\left(a'\frac{d}{dx} + b'\frac{d}{dy} + c'\frac{d}{dz}\right) F = 0,$$

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)\left(a''\frac{d}{dx} + b''\frac{d}{dy} + c''\frac{d}{dz}\right) F = 0,$$

it is easily seen that the resultant of the elimination is the square of the determinant

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

multiplied by the Hessian of the given function. And, moreover, that if we eliminate x, y, z we shall obtain precisely the same result with the letters l, m, n substituted for x, y, z. Hence it follows, that if we take the doubly infinite system of first polars to a given curve of the third degree, in respect to all the points lying in its plane, and then from any point in the *Hessian* to the given curve, draw pairs of tangents to each conic of the system so generated, then all the chords of contact will meet in one and the same point, which will itself be also a point situated upon the Hessian and conjugate to the former.

So, in general, for a function of any degree of any number of letters, viewed with relation to the doctrine of successive polars, the determinants of the Boolo-Hessian scale take one another up in pairs; namely the first takes up the last but one, the second the last but two, and so on; and consequently, if the degree of the function be odd, that function which (making abstraction of the constant determinant at the end) lies in the middle of the scale pairs with itself, and, in a sense analogous to that above exhibited for a function of the third degree, may be said to be always its own reciprocal.

P.S. I have just discovered the method of reducing functions of two letters of *even degrees* to their canonical form, which will shortly be published in a second Supplement.

At present I offer the annexed theorem (which strikingly contrasts with the law of uniqueness demonstrated of functions of an odd degree) as a foretaste of the enchanting developments with which I hope shortly to present my readers:—

If a given homogeneous function of x and y of the degree $2n$ be supposed to be thrown under its canonical form,

$$u_1^{2n} + u_2^{2n} + \ldots + u_n^{2n} + K(u_1 u_2 \ldots u_n)^2,$$

then will K^n have $n^2 - 1$ in general distinct values, to each of which will correspond a single distinct system of the linear functions of x and y,

$$1^{\frac{1}{n}} u_1, \quad 1^{\frac{1}{n}} u_2, \ \ldots \ 1^{\frac{1}{n}} u_n.$$

35.

EXPLANATION OF THE COINCIDENCE OF A THEOREM GIVEN BY MR SYLVESTER IN THE DECEMBER NUMBER OF THIS JOURNAL, WITH ONE STATED BY PROFESSOR DONKIN IN THE JUNE NUMBER OF THE SAME.

[*Philosophical Magazine*, (Fourth Series) I. (1851), pp. 44—46.]

I WISH to state, without loss of time, that in the theorem given by me* for the composition of two successive rotations about different axes, I have been anticipated by Prof. Donkin in the June Number of your *Journal*.

To my shame I must confess, that, although an occasional contributor to, I am not invariably a constant reader of your valuable miscellany, otherwise I should not have introduced the theorem in question without due acknowledgment of Professor Donkin's claims to whatever merit may attach to the priority of publication. The fact is, that I made out the theorem for myself nine years ago, and had some communication on the subject with Professor De Morgan, who was then writing the seventeenth chapter of his *Differential Calculus*. A recent conversation with this gentleman has brought back to my mind a vivid recollection of the course of that communication. I brought under Professor De Morgan's notice the analytical memoir of Sr Gabrio Pola on the subject in the Memoirs of the Italian Society of Modena, and satisfied myself of the existence of the single axis of displacement by compounding the two rotations in the manner given in my paper, which, for the case of two axes fixed in space, is the same as Professor Donkin's, and for two axes fixed in the rotating body is materially, although not formally the same.

It then occurred to me that a more simple demonstration ought to be deducible from the possibility of always *finding* the point on a sphere, by revolution about which, as a pole, one equal arc could actually be shown to be transportable into the place of another. But in proceeding to work out this idea I fell into a remarkable blunder, in which I have since been followed by more than one able friend to whom I have proposed the question. The

[* p. 158 above.]

blunder was of this kind:—Two arcs have to be drawn, bisecting at right angles the arcs joining the extremities of two equal arcs; the point of intersection of the two bisecting arcs *must* in all cases fall outside the quadrilateral formed by the equal and joining arcs. I supposed it to fall inside. There appears to be a fatal tendency to do so in all who take the subject in hand. In consequence of this error, the cause of which I did not at the moment perceive, I was driven to deny and admit in one breath the same proposition. Mr De Morgan sent me the correct proof after this method (the same as that given by him at page 489 of his *Calculus*), I am inclined to think after I had myself detected my error; but of this I cannot feel certain.

This is the method alluded to by me in the words "it is right to bear in mind, &c.," at the time of writing which all recollection of the same thing having been published by Mr De Morgan had vanished from my memory.

The proof of the triangle of rotations is so simple, that, as Professor Donkin states (in a letter which he has done me the favour of addressing me on the subject) was the case with himself, I thought it incredible that it should not have appeared in some elementary work, and I was therefore at no pains to publish it as my own; nor should I have written at all on the subject, had it not been for the surprise occasioned to my mind by falling in with Professor Stokes's article in the *Cambridge and Dublin Mathematical Journal*, to demonstrate the existence of an instantaneous axis, which proceeds in apparent unconsciousness of the so simply demonstrable law, that any number of rotations of any kind (and therefore those that take place in an instant of time) are representable by a single rotation about a single axis. I shall feel obliged by the early insertion of this explanation, more in justice to myself than to Professor Donkin, whose high and worthily earned reputation, not to speak of the disinterested love of truth for its own sake, apart from personal considerations, which animates the labours of the genuine votary of science, must make him indifferent to whatever credit might be supposed to result from the first authorship or publication of the very simple (however important) theorem in question.

36.

AN ENUMERATION OF THE CONTACTS OF LINES AND SURFACES OF THE SECOND ORDER.

[*Philosophical Magazine*, I. (1851), pp. 119—140.]

It is well known that in general any two homogeneous quadratic functions of the same system of variables may be simultaneously transformed, so as to be expressed each of them as pure quadratic functions of a new system of variables equal in number and linearly connected with the original ones; a pure quadratic function meaning one in which only the squares of the variables are retained.

Every homogeneous quadratic function may be treated as the characteristic* of a locus of the second degree: if the function be of two letters, the locus is a binary system of points in a line wherein the distances of two fixed points from either point of the given system or given multiples of such distances correspond to the variables; if of three letters, the locus is a conic, the distances or given multiples of the distances of every point in which from three given lines in the plane of the conic are represented by the variables; if of four letters, the locus is a surface of the second order, the coordinates being the distances or multiples of the distances of any point therein from four planes drawn in the space in which the surface is contained, and so on for loci of four and higher dimensions.

I propose, however, in the present paper to restrict myself to the theory of the contacts of loci not transcending the limits of vulgar space, by which I mean the space cognizable through the senses†, and shall accordingly be

* According to the definition stated by me in a previous paper, the characteristic of a locus is the function which, equated to zero, constitutes the equation thereto.

† If the impressions of outward objects came only through the sight, and there were no sense of touch or resistance, would not space of three dimensions have been physically inconceivable? The geometry of three dimensions in ordinary parlance would then have been called transcendental. But in very truth the distinction is vain and futile. Geometry, to be properly understood, must be studied under a universal point of view; every (even the most elementary) proposition must be regarded as a fact, and but as a single specimen of an infinite series of homologous facts.

In this way only (discarding as but the transient outward form of a limited portion of an infinite system of ideas, all notion of extension as essential to the conception of geometry, however useful as a suggestive element) we may hope to see accomplished an organic and vital development of the science.

almost exclusively concerned in determining the singular cases of conjugate systems of quadratic forms of two, three, and four letters respectively.

In order that the reduction of any such system, say U and V, to a pure quadratic form may be possible (as it generally is), it is necessary that none of the roots of the complete determinant of $U + \lambda V$ shall be equal; if any relation of equality exist between these roots, the *general* reduction is *generally* no longer possible; under peculiar conditions, however, as will hereafter appear, in spite of the equality of certain of the roots, the irreducibility in its turn will cease, and the ordinary reduction be capable of being effected. It is easily seen, that to every relation of equality between the roots of the determinant of $U + \lambda V$ must correspond a particular species of contact between the loci which U and V characterize. But we should make a great mistake were we to suppose that every such relation of equality corresponded with but one species of contact; for instance, the characteristics of U and V of two conics are functions of three letters, and $\square(U + \lambda V)$ will be a cubic function of λ. Such a function may have two roots, or all its roots equal: this would seem to give but two species of contact, whereas we well know that there are no less than four species of contact possible between two conics. Accordingly we shall find, that, in order to determine the distinctive characters of each species of contact, we must look beyond the complete determinant, and examine into the relations (in themselves and to one another) of the several systems of minor determinants that can be formed from $U + \lambda V$.

By pursuing this method, we may assign *à priori* all the possible species of contact between any two loci of the second degree. How important this method is will be apparent from the fact, that not only have the distinctive characters of the various contacts possible between surfaces of the second order never been determined, but their number and the nature of certain of them have remained until this hour unknown and unsuspected.

The method which we shall pursue is an exhaustive one, and will conduct us by a natural order to a systematic arrangement of all the different modes and gradations of such contacts.

In a paper* in this *Magazine* for November 1850, I explained the decline of minor determinants, and stated a law, called the homaloidal law, concerning them.

If U and V be characteristics of the two loci whose contacts are to be considered, $U + \lambda V$ will be the function, the properties of whose complete determinant, and of the minor systems of determinants belonging to it, will serve to specify the nature of the contact.

It will be remembered, that, whatever be the number of variable letters in any quadratic function U, three of its first minor determinants being zero,

[* p. 150 above.]

makes all the first minors zero; six of its second minors being zero, makes all the second minors zero; and so on for the third, fourth, &c. minor systems according to the progression of the triangular numbers.

It is well known that whatever linear transformations be applied to a quadratic function W, the complete determinant thereof will remain unaltered, except by a multiplier depending upon the coefficients introduced into the equations of transformation; consequently the roots of λ in the equation obtained by making the determinant of $U+\lambda V$ zero remain unaffected by such transformation; and any relation or relations of equality among the roots of the equation $\square(U+\lambda V)=0$ is an immutable property of the system U, V, which is unaffected by linear transformations. Another and more general kind of immutable property (comprehending the above as a particular case), to which I shall have occasion to refer, is the following.

Suppose all the minors of any order of $U+\lambda V$ have a factor $\lambda+\epsilon$ in common; this factor will continue common to the same system of minors when U and V are simultaneously transformed. This is a very important proposition, and easily demonstrated; for if $\lambda+\epsilon$ be a common factor to all the rth minors of $U+\lambda V$, $(U-\epsilon V)$ will have its rth minors zero, and therefore, as explained by me in the paper above referred to, $U-\epsilon V$ will be degraded r orders below U or V. This is clearly a property independent of linear transformation, consequently $\lambda+\epsilon$ will remain a factor of the transformed rth minors.

In like manner it is demonstrable that any number of *distinct* factors $\lambda+\epsilon_1$, $\lambda+\epsilon_2 \ldots$ common to the rth minors of one form of $U+\lambda V$, will remain common factors of any other linearly derived form of the same. It is consequently necessary that each rth minor of one form of any quadratic function W shall be a syzygetic* function of *all* the rth minors of any other form of the same; and consequently a function of λ of any degree, whether all its factors be or *be not distinct*, which is common to the rth minors of one form of $U+\lambda V$, will remain so to the rth minors of any other form of the same.

The law exhibiting the connexion of each rth minor of one form of W (any homogeneous quadratic function) with all the rth minors of any other form of W, will form the subject of a distinct communication.

Finally, to fully comprehend the annexed discussion, the following principle must be apprehended.

* If $A=pL+qM+rN+\&c.$, where p, q, $r \ldots$ do not any of them become infinite when L, M, $N \ldots$ or any of them become zero, A may be termed a syzygetic function of L, M, $N \ldots$.

In the theorem above alluded to, it will be shown (as might be expected) that the syzygy in the case concerned is of the simplest kind, that is, that each rth minor of a quadratic function of any number of letters is a homogeneous linear function of all the rth minors of the same quadratic function linearly transformed.

If any factor K^e enter into all the rth minors of W, and if K^i be the highest power of K common to all the $(r+1)$th minors, then K^{2e-i} will be a common factor to all the $(r-1)$th minors.

Let r be taken unity; it is easily proved* that the complete determinant of *any* square matrix may be expressed by the difference between two products†, each of two first minor determinants divided by a certain second minor determinant. The proposition is therefore demonstrated for this case, and thereby in fact implicitly for every case, inasmuch as the first minors of every rth minor are $(r+1)$th minors of the original matrix. Hence it follows, that if any system of rth minor determinants have a common factor ϵ^i, the complete determinant must contain *at lowest* the factor $\epsilon^{(r+1)i}$, and any system of $(r-s)$th minor determinants thereunto will contain *at lowest* the factor $\epsilon^{(s+1)i}$.

I now proceed to apply these principles to the determination of the relative forms of conjugate quadratic functions representing geometrical loci of the second order. I shall begin with two binary systems of points in a right line.

The general characteristics U and V of two such systems may be thrown under the form

$$\left.\begin{aligned} U &= x^2 + y^2 \\ V &= ax^2 + by^2 \end{aligned}\right\}.$$

When $\square(V+\lambda U)=0$ has its two roots equal, these systems have a point in common. The above forms cease to be applicable, and convert into

$$\left.\begin{aligned} U &= xy \\ V &= ax^2 + bxy \end{aligned}\right\}$$

where $x=0$ represents the common point.

* This will appear in my promised paper on Determinants and Quadratic Functions.

† When the matrix is symmetrical about one of its diagonals (as it is in the case which we are concerned with), one of these products becomes a square. I may take this occasion of hinting, that the theory of quadratic functions merges in a larger theory of binary functions, consisting of the sum of the multiples of binary products formed by combining each of one set of quantities, $x, y, z \dots$ with each of the same number of quantities of another set, as $x', y', z' \dots$. For instance,

$$\begin{aligned} &axx' + bxy' + cxz' \\ &+ a'yx' + b'yy' + c'yz' \\ &+ a''zx' + b''zy' + c''zz' \end{aligned}$$

would be a binary function, and its determinant (no longer, as in a quadratic function, symmetrical about either diagonal) would correspond to the square matrix

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c''. \end{matrix}$$

Almost all the properties of quadratic apply, with slight modifications, to binary functions.

Let U and V now represent two conics. When there is no contact, we have as the types of their characteristics

$$U = x^2 + y^2 + z^2,$$
$$V = ax^2 + by^2 + cz^2.$$

The three roots of $\square\,(V + \lambda U) = 0$ are

$$\lambda = -a, \quad \lambda = -b, \quad \lambda = -c,$$

showing that there are three distinct pairs of lines in which the intersections of U and V are contained, the equations to three pairs being respectively

$$(b-a)\,y^2 + (c-a)\,z^2 = 0,$$
$$(c-b)\,z^2 + (a-b)\,x^2 = 0,$$
$$(a-c)\,x^2 + (b-c)\,y^2 = 0\,;$$

the four points of the intersection being defined by the equations corresponding to the proportions

$$x : y : z :: \sqrt{(b-c)} : \sqrt{(c-a)} : \sqrt{(a-b)}.$$

Now let $\square\,(U + \lambda V)$ have two equal roots; the characteristics assume the form

$$U = x^2 + y^2 + xz,$$
$$V = ax^2 + by^2 + cxz^*.$$

Two of the pairs of lines become identical, that is, two of the four points of intersection coincide.

* We may if we please make $a = b$; for it may be shown that the equations, in their present forms, contain an arbitrariness of 10 degrees; namely, 9 on account of x, y, z being arbitrary linears of ζ, η, θ; 2 on account of the ratios $a : b : c$; together 11 reduced by one degree on account of x, y, z, changed into lx, ly, lz, leaving $U=0$, $V=0$ unaffected. Now the degrees of arbitrariness in two conics, subject to satisfy only one condition, is $2 \times 5 - 1$ or 9. Hence there is one degree of arbitrariness to spare. In fact, on making $a = b$, the axis z becomes the line joining the two points of intersection distinct from the point of contact; x remaining the tangent at the point of contact, and y, strange to say, still arbitrary, subject only to passing through the point of contact; if, however, y be made to pass through the point of contact, and either one of the distinct intersections, this form,

$$U = x^2 + y^2 + xz,$$
$$V = ax^2 + ay^2 + cxz,$$

becomes no longer tenable, but gives place to

$$U = y^2 + yx + xz,$$
$$V = ay^2 + ayx + cxz,$$

where x is the tangent at the point of contact, z the line joining the two intersections with one another, and x, $x + y$ respectively the lines joining either of them with the point of contact; if the multiplier of yx in V in the above be made b instead of a, x remains the tangent as before, y becomes any line through the point of contact, and z any line through one of the distinct intersections. A systematic view of similar modulations of form and the study of the laws of arbitrariness connected with them, as applicable to the general subject-matter of this paper, must be deferred to a subsequent occasion.

This may be termed "Simple Contact." The tangent at the point of contact is $x=0$; this equation making U and V each become of only one order.

The intersections are

$$x=0, \quad y=0, \tag{1}$$

$$x=0, \quad y=0, \tag{2}$$

$$\sqrt{(a-c)}\,x+\sqrt{(b-c)}\,y=0, \quad z=0, \tag{3}$$

$$\sqrt{(a-c)}\,x-\sqrt{(b-c)}\,y=0, \quad z=0. \tag{4}$$

These are obtained by making $V-aU=0$, which gives $x=0$ or $z=0$.

$x=0$ gives $y^2=0$, that is, $y=0$ twice over, and $z=0$ gives

$$(a-c)\,x^2+(b-c)\,y^2=0.$$

The number of conditions to be satisfied in this case is one only.

Next let $\square(U+\lambda V)$ have all its roots equal. This condition will be satisfied (still leaving U and V as general as they can remain consistent with these conditions) by making

$$U=x^2+yz+yx,$$

$$V=ax^2+ayz+byx.$$

Here only one *distinct* pair of lines can be drawn to contain the intersections, showing that three out of the four points come together.

This may be termed "Proximal Contact." The number of affirmative conditions to be satisfied is two, and the contact is therefore entitled of the second degree.

The tangent at the point of contact is $y=0$, and the four intersections become

$$x=0, \quad y=0,$$

$$x=0, \quad y=0,$$

$$x=0, \quad y=0,$$

$$x=0, \quad z=0.$$

These may be obtained from the equation $V-aU=0$, which gives $y=0$ or $z=0$; the former implying concurrently with itself $x^2=0$, and the latter $yz=0$.

Thus we obtain three systems,

$$x=0, \quad y=0,$$

and one

$$x=0, \quad z=0,$$

corresponding to three consecutive points and the single distinct one.

The determinant of $U+\lambda V$ being only of the third degree in λ, we have exhausted the singularities of the system U, V dependent on the form of the complete determinant of $U+\lambda V$.

Let now the first minors of $U+\lambda V$ have a factor in common; this will indicate that $U+\lambda V$ may be made to lose *two* orders by rightly assigning λ, in other words, that the intersections of U and V are contained upon a pair of *coincident* lines. Here it is remarkable that the original forms of U and V reappear, but with a special relation of equality between the coefficients: we shall have, in fact,

$$U = x^2 + y^2 + z^2,$$
$$V = ax^2 + ay^2 + bz^2.$$

This gives the law for double, or, as I prefer to call it, diploidal contact*. By virtue of the Homaloidal law, we know that if three first minors of $U+\lambda V$ be zero, all are zero; we have therefore to express that three quadratic functions of λ have a root in common. This implies the existence of two *affirmative* conditions; the contact of the two conics taken collectively may therefore be still entitled of the second degree, although the contact at each of the two points where it takes place is simple, or of the first degree.

These points are evidently defined by the equation

$$\{x+\sqrt{(-1)}\,y=0,\ z=0\},$$
$$\{x-\sqrt{(-1)}\,y=0,\ z=0\},$$

and the ordinary algebraical solution of the equations $U=0$, $V=0$ would naturally lead to the four systems

$$x+\sqrt{(-1)}\,y=0, \quad z=0,$$
$$x+\sqrt{(-1)}\,y=0, \quad z=0,$$
$$x-\sqrt{(-1)}\,y=0, \quad z=0,$$
$$x-\sqrt{(-1)}\,y=0, \quad z=0;$$

the two tangents at the point of contact are $x+\sqrt{(-1)}\,y=0$, $x-\sqrt{(-1)}\,y=0$, and the coincident pair of lines containing the intersections is $z^2=0$.

* See my remarks† on the conditions which express double contact in the *Cambridge Journal*, Nov. 1850. If n functions, being all zero, be the condition of a fact, but r independent syzygetic equations admit of being formed between these functions, the number of affirmative conditions required is not n, but $(n-r)$; because the fact may be expressed by affirming $(n-r)$ equations and denying certain others. Thus if $P=0$, $Q=0$, $R=0$, $S=0$ express a fact, and

$$PP'+QQ'+RR'+SS'=0,$$
$$PP''+QQ''+RR''+SS''=0,$$

the fact is expressible by affirming $P=0$, $Q=0$, and denying $R'S''-R''S'=0$, for then $P=0$, $Q=0$ will imply $R=0$, $S=0$; or, in like manner, by affirming any other two out of the four necessary equations, and denying the other equations. Observe, however, that all the required equations *may* coexist in the absence of such right of denial.

[† p. 129 above.]

It may at first view appear strange, that whilst no condition is required in order that U and V may be simultaneously metamorphosed into the forms of $x^2+y^2+z^2$, $ax^2+by^2+cz^2$, a, b and c being all unequal, for this metamorphosis to be possible when any two become equal, not one but two conditions must be satisfied. The reason of this is, that the coefficients of transformation, which, as well as a, b, c, are functions of the coefficients of the given quadratic functions, become infinite on constituting between the said coefficients such relations as are necessary for satisfying the equation $a=b$, or $a=c$, or $b=c$, except upon the assumption of some further particular relations between them over and above that implied in such equality.

In the ordinary case of diploidal contact, the first minors having a factor in common, this factor will enter twice into the complete determinant of $U+\lambda V$, but it *may* enter three times: this will indicate, that not only do the four intersections lie on a coincident pair of lines, but furthermore, that there is but one pair of lines of any kind on which they lie.

In the ordinary case of diploidal contact, it will be observed that this latter condition does not obtain; the four intersections lie on a coincident pair of lines; but they lie also on a crossing pair, namely, in the two tangents at the points of contact. In this higher species of diploidal contact, it is clear that the two points of contact, which are ordinarily distinct, come together, and that all four intersections coincide.

This I call *confluent* contact; the forms of U and V corresponding thereto will be

$$U = x^2 + y^2 + xz,$$

$$V = ay^2 + axz;$$

the common tangent at the point of contact being $x=0$, and the four coincident points $x^2=0$, $y^2=0$.

The number of affirmative conditions to be satisfied being three, the contact is to be entitled of the third degree.

Observe, that it is of no use to descend below the first minors in this case; because the second minors, being linear functions of λ, could not have a factor in common, unless $V:U$ becomes a numerical ratio, which would imply that the conics coincided*.

Fortified by the successful application of our general principles to the preceding more familiar cases of contact, we are now in a condition to apply with greater confidence the same *à priori* method to the exhaustion and characterization of all the varied species of contact possible between surfaces

* No-contact and complete coincidence may be conceived as the two extreme cases in the scale of relative conjugate forms.

of the second order; a portion of the subject comparatively unexplored, and never before thought susceptible of reduction to a systematic arrangement.

When there is no contact, we may write

$$U = x^2 + y^2 + z^2 + t^2,$$
$$V = ax^2 + by^2 + cz^2 + dt^2,$$

and the intersection of the surfaces will lie in each of the four cones,

$$(a-d)\,x^2 + (b-d)\,y^2 + (c-d)\,z^2 = 0,$$
$$(a-b)\,x^2 + (c-b)\,z^2 + (d-b)\,t^2 = 0,$$
$$(a-c)\,x^2 + (b-c)\,y^2 + (d-c)\,t^2 = 0,$$
$$(b-a)\,y^2 + (c-a)\,z^2 + (d-a)\,t^2 = 0.$$

Whenever the surfaces are in contact, certain of these cones will coincide with certain others, so that their number will be always less than four. Also, as we shall find in such event, they may degenerate into pairs of intersecting or coincident planes.

Let us begin with considering the cases of contact for which the first minors (and consequently *à fortiori* the minors inferior to the first) have no factor in common.

Here $\square\,(V + \lambda U)$ is a biquadratic function.

If λ have all its roots unequal, we have U and V as above given.

If two roots are equal, the characteristics assume the form

$$\left.\begin{aligned} U &= x^2 + y^2 + z^2 + xt \\ V &= ax^2 + by^2 + cz^2 + dxt \end{aligned}\right\}.$$

The touching plane is $x = 0$; the point of contact is $x = 0$, $y = 0$, $z = 0$; the curve of intersection is one of the fourth degree, with a double point at the point of contact.

There is but one condition to be satisfied, and the contact may be entitled "simple" and of the first degree.

Next let λ have three equal values, the equations become

$$U = x^2 + yz + t^2 + xy,$$
$$V = x^2 + yz + at^2 + bxy.$$

The tangent plane at the point of contact $y = 0$, and the point itself $x = 0$, $y = 0$, $t = 0$. The curve of intersection is a curve of the fourth order, with a cusp at the point of contact. The number of affirmative conditions to be satisfied is two; the contact is of the second degree, and may be termed "proximal" or cuspidal.

Next let $\square(U+\lambda V)$ have two pairs of equal roots, we shall find

$$U = x^2 + xy + zt,$$
$$V = ayz + bxy + czt.$$

The line $x=0$, $z=0$ will be common to both surfaces. The curve of intersection will therefore break up into a right line and a line of the third order.

The former will meet the latter in two points, which will be each of them points of contact. The contact is therefore diploidal; but as there is another species of diploidal contact to which we shall presently come, it will be expedient to characterize each of them by the nature of the intersections of the two surfaces; accordingly this may be termed unilinear-intersection contact, or more briefly, unilinear contact.

The number of affirmative conditions to be satisfied being two, it may be said to be collectively of the second degree, but (obviously?) the contact at each of the two points is of the nature of simple contact.

Lastly, let us suppose that all four roots of $U+\lambda V$ are equal; we shall find, as the most simple expressions of the most general forms of the two surfaces,

$$U = x^2 + xy + yz + zt,$$
$$V = axy + bz^2 + azt.$$

In this case the two points of intersection of the curve of the third degree, and the right line on which the surfaces intersect, come together, so that the right line becomes a tangent to the curve. The number of conditions to be satisfied is three: there is but one point of contact which may be considered as the union of two which have coalesced, and the species may be defined as confluent-unilinear contact.

If we throw the equations to the conoids having an unilinear contact into the form

$$x(x+y) + zt = 0,$$
$$xy + z(y+ct) = 0,$$

we obtain

$$(x+y)(y+ct) - yt = 0,$$

which last equation is no longer satisfied by $x=0$, $z=0$, these systems of roots having been made to disappear by the process of elimination.

The curve of the third degree, in which the two given conoids intersect, may thus be defined as their common intersection with the new conical surface defined by the third of the above equations.

More generally, it is apparent that the three conoids,

$$\left.\begin{aligned} xu - yt &= 0 \\ yv - zu &= 0 \\ zt - xv &= 0 \end{aligned}\right\},$$

in which x, y, z, t, u, v may any of them be considered as a homogeneous linear function of four others, intersect in the same line of the third degree. *Besides* which, the first and second intersect in the right line y, u; the second and third in z, v; the third and first in x, t; each of which lines it is evident is a chord of the common curve of intersection. For instance, $y = 0$, $u = 0$ may be satisfied concurrently with all the above three equations by satisfying the equation $zt - xv = 0$, which, as two linear relations exist originally between the six letters, and two more have been thrown in, becomes a quadratic equation between any two of the letters.

The only case of exception to this reasoning is, when $y = 0$, $u = 0$ can be satisfied concurrently with $z = 0$, $v = 0$, and with $x = 0$, $t = 0$; but in this case the surfaces all become cones; and as there is no longer a curve of the third degree, "Cadit quæstio." Even here, however, the intersection of any two of the surfaces becomes a conic, and two coincident generating lines on the two cones; so that if we take one of these and the conic to represent a degenerate form of a line of the third degree, the remaining straight line passes through a double point of such degenerate form, and the case passes into that of confluent-unilinear contact.

The two double points in the intersection of the two conoids

$$U = x(x + y) + zt = 0,$$

$$V = xy + z(y + ct) = 0,$$

by which I mean the points of intersection of the conic with the right line common to them, are found by making $x = 0$, $z = 0$, and substituting in the derived equation

$$(x + y)(y + ct) - ty = 0,$$

which gives $y = 0$, or $y + (c - 1)t = 0$; so that the two points required are

$$x = 0, \quad y = 0, \qquad z = 0,$$

$$x = 0, \quad y = (1 - c)t, \quad z = 0.$$

It appears also that the entire intersection is contained in each of the two cones,

$$U - V, \text{ that is, } x^2 + z\{(1 - c)t - y\}$$

and

$$cU - V, \text{ that is, } cx^2 + y\{(c - 1)x - z\},$$

the respective vertices of which are at the points above determined.

The equations for confluent-unilinear contact,

$$x(x+y)+z(y+t)=0,$$
$$xy+z(cz+t)=0,$$

give

$$(x+y)(cz+t)-(y+t)y=0;$$

which, on making $x=0$, $z=0$, is satisfied by $y^2=0$; showing that the *confluence* takes place at the point

$$x=0, \quad y=0, \quad z=0.$$

The number of terms in the two equations for ordinary unilinear contact being six, and in those given for confluent unilinears seven, and the empirical rule in all other cases being that the terms tend to diminish and never increase in number as the degree of the contact (expressed by the number of conditions to be satisfied) rises, I am led to suspect that the conjugate system for the latter species of contact may admit of being reduced to some more simple form.

I must state here once for all, that all the *distinct* systems of (at least consecutive) conjugate forms that have been, and will be given, are *mutually* untransformable. This it is which distinguishes *singular* from *particular* forms.

A particular form is included in its primitive; but a singular form is one, which, while it responds to the same conditions as some other *more general* form, is incapable of being expressed as a particular case of the latter, on account of the *additional* condition or conditions which attach to it.

I pass now to the singularities which arise from the first minor determinants of $U+\lambda V$ having a factor in common, the second minors being supposed to be still without a common factor.

When this common factor is linear in respect to λ, let it be supposed to enter not more than twice (twice, we know, by the general principle enunciated at the commencement of this paper, it must enter) into the complete determinant.

Two of the cones containing the intersection of U and V then become coincident, and degenerate each into the same pair of crossing planes. This may be termed biplanar-contact. The characteristics of such contact are

$$U=x^2+y^2+z^2+t^2,$$
$$V=ax^2+ay^2+bz^2+ct^2;$$

the points of contact are two in number, being at the intersection of the two plane conics into which the curve of intersection breaks up. The two planes in which these lie are given by the equation $(b-a)z^2+(c-a)t^2=0$; these intersect in the right line $z=0$, $t=0$, which meets both surfaces in the same two points,

$$z=0, \quad t=0, \quad x+\sqrt{(-1)}\,y=0,$$

$$z=0, \quad t=0, \quad x-\sqrt{(-1)}\,y=0,$$

the two common tangent planes at these points being

$$x+\sqrt{(-1)}\,y=0, \quad x-\sqrt{(-1)}\,y=0$$

respectively.

This, then, is another species of double contact between two conoids, and, as far as I know, the only kind hitherto recognized as such. The number of conditions to be satisfied remains two, as in the former species.

Next suppose that the common factor of the first minor enters three times into the complete determinant instead of *twice* only, as in the last case.

The corresponding characteristics will be found to be

$$U=x^2+zt+y^2+z^2,$$

$$V=ax^2+azt+by^2+cz^2.$$

The intersection of U, V still lies in two planes,

$$(b-a)y^2+(c-a)z^2=0;$$

but the intersection of these two planes,

$$y=0, \quad z=0,$$

meets the surfaces in the two coincident points,

$$y=0, \quad z=0, \quad x^2=0.$$

This, therefore, I call confluent-biplanar contact; the two conics constituting the complete intersection, instead of cutting, touch and at their point of contact the two conoids have a contact of a superior order. The conditions to be satisfied for this case are three in number.

Next suppose that the common factor of the first minors enters only twice into the complete determinant, but that the remaining two factors become equal.

Here the analytical characters of unilinear and biplanar contact are blended; in fact, the intersection consists of a conic and a pair of right lines meeting one another and the conic. The characteristics are

$$U=x^2+y^2+z^2+zt,$$

$$V=ax^2+ay^2+bz^2+czt.$$

The intersection is contained in the two planes

$$z=0, \quad (b-a)z+(c-a)t=0,$$

and consists of the two lines $z=0$, $x^2+y^2=0$, lying in the common tangent plane $z=0$, and the conic

$$\left.\begin{array}{l}(b-a)z+(c-a)t=0\\ (a-c)x^2+(a-c)y^2+(b-c)z^2=0\end{array}\right\}.$$

There are *three* points of contact, namely, the point $x=0$, $y=0$, $z=0$, where the two right lines cut, and $x^2+y^2=0$, $t=0$, $z=0$, where these lines meet the conic. This, then, is a case of triple contact. I distinguish it by the name of bilinear-contact. The number of conditions is still three.

Now all else remaining as before, let the two pairs of equal roots in the complete determinant become identical, or, in other words, let the common factor of the first minors be contained four times in the complete determinant. The characteristics become

$$U=xz+xt+y^2+z^2,$$
$$V=axz+bxt+by^2+bz^2.$$

The intersection becomes the two right lines

$$x=0, \quad y^2+z^2=0,$$

and the conic

$$z=0, \quad x^2+y^2=0.$$

All these meet in the same point,

$$x=0, \quad y=0, \quad z=0;$$

so that instead of contact in three points, the contact takes place about one only, in which the *three* may be conceived as merging. This I call confluent-bilinear contact. It requires the satisfaction of four conditions.

Next let us suppose that the two distinct factors are common to each of the first minors. This will imply the existence of four affirmative conditions.

The complete determinant will of necessity contain each of these factors twice, so that no additional singularity can enter through this determinant. The characteristics assume the form

$$U=x^2+y^2+z^2+t^2,$$
$$V=ax^2+ay^2+bz^2+bt^2.$$

The two surfaces will meet in four straight lines, forming a wry quadrilateral, whose equations are

$$x\pm\sqrt{(-1)}\,y=0,$$
$$z\pm\sqrt{(-1)}\,t=0.$$

These intersect each other in the four points

$$x = 0, \quad y = 0, \quad z^2 + t^2 = 0,$$
$$z = 0, \quad t = 0, \quad x^2 + y^2 = 0,$$

each of which will be a distinct point. This I term quadrilinear contact.

Now let the two factors common to each of the first minors become identical; so that a *squared* function, instead of an ordinary *quadratic* function of λ, is now their common measure.

The factor which enters twice into each of the first minors will enter four times into the complete determinant; the number of conditions to be satisfied is one more than in the preceding case, namely five, and the characteristics become

$$U = x^2 + y^2 + xz + yt,$$
$$V = ax^2 + by^2 + cxz + cyt.$$

Here arises a singularity of form in the intersections utterly unlike anything which has been remarked in the preceding cases. For it will not fail to have been observed, that the intersection in the nine preceding cases was always a line or system of lines of the fourth degree, so as to be cut by any plane in four points.

But in this case, the fact of the first minors having a factor in common, shows that the intersection is contained in two planes (which is of course to be viewed as a degenerate species of cone); and the fact of the complete determinant having all its roots equal, shows that there is but one system of a pair of planes in which the intersection is contained, and no more.

So that the two pairs of planes, into which the wry quadrilateral was divisible in the case immediately preceding, now become a single pair. This can only be explained by two of the opposite sides of the quadrilateral becoming indefinitely near to one another, but still not coinciding in the same planes; so that the actual *visible* or quasi-visible* intersection will be in three right lines, of which the middle one meets each of the two others.

This will further appear by proceeding regularly to solve the equations

$$U = 0, \quad V = 0.$$

$V - cU = 0$ gives $y = \pm kx$, where $k = \sqrt{\left(\dfrac{a-c}{c-b}\right)}$, and therefore $xz + kxt = 0$, or $xz - kxt = 0$; whence we see that the complete intersection is represented by the lines

$$(x = 0, \ y = 0); \quad (z + kt = 0, \ y - kx = 0),$$
$$(x = 0, \ y = 0); \quad (z - kt = 0, \ y + kx = 0),$$

* I use the term quasi-visible, because the intersection may become in part or whole imaginary.

showing that there are but three physically distinct lines, as already premised.

This, then, may be considered as derived from the preceding case of a wry quadrilateral intersection, by conceiving two opposite sides of the quadrilateral to come indefinitely near, but without coinciding.

Let these two lines be called P and P'; take any point in P and any two points in P' indefinitely near to one another and the point first taken, then this indefinitely small plane will be common to both surfaces, and consequently they ought to touch along *every point in the line P*. This is again confirmed by the forms given to U and V. For at any point where the coordinates are 0, 0, ζ, θ the equations to the tangent planes to the two surfaces respectively are

$$\zeta x + \theta y = 0,$$
$$c\zeta x + c\theta y = 0,$$

that is to say, are identical.

Whilst, therefore, certain grounds of geometrical, and still stronger grounds of analytical analogy, might seem to justify this species of contact taking the name of confluent quadrilinear, yet as, in fact, the intersection is trilinear, and as, moreover, the two indefinitely proximate lines must be considered, not as coincident, but as turned away from one another through an indefinitely small angle and out of the same plane, I prefer to take advantage of this striking property of contact *at every point* along a line (a property entirely distinct from any that we have yet considered), and confer upon the species of contact we have been considering the designation of unilinear-indefinite contact.

Where the line of indefinite contact meets the two other lines of the intersection, the contact is of course of a higher order; thus offering a parallel to what takes place in ordinary unilinear contact, in which there is no contact, except only *at two points* of the right line forming part of the complete intersection.

I believe that this kind of contact, which forms a natural family with two others about to be described, and which will close the list, has never before been imagined, and would at first sight have been rejected as impossible.

Having now exhausted the cases of the first class, in which the minors have no factor in common, and the two sections of the second class, in which the second minors have no common factor, but the first minors of $U + \lambda V$ a linear or quadratic function of λ in common, I descend to the third class, in which the second minors, which are quadratic functions of λ, are supposed to have a common factor.

This common factor must enter twice into each of the first minors by virtue of the law previously indicated, and cannot enter more than twice, as otherwise the first minors of $U+\lambda V$ could only differ from one another by a numerical multiplier, which is obviously impossible, except when $U+\lambda V$ is of the form $(k+\lambda)U$, that is, when the two surfaces coincide.

Again, the common factor of the first minor must enter three times into the complete determinant; but there is no reason why it may not enter four times, and thus two cases arise. In the first, the characteristics take the form

$$U = x^2 + y^2 + z^2 + t^2,$$
$$V = ax^2 + ay^2 + az^2 + bt^2.$$

The second determinant having a factor in common, shows that the intersection U, V is contained in a pair of coincident planes; but the complete determinant, having two distinct factors, evidences that these plane intersections, viewed as indefinitely near but still distinct, lie in the same cone, which will be a cone enveloping both the surfaces U and V all along their mutual intersections. This is also seen easily from the forms of U and V; for we have $V - aU = (b-a)t^2$, which proves that the intersection lies in the coincident, or, to speak more strictly, consecutive planes $t^2 = 0$; and at any point $x = \xi$, $y = \eta$, $z = \zeta$, the tangent plane to each surface becomes

$$\xi x + \eta y + \zeta z = 0.$$

As there are six independent, that is, non-necessarily co-evanescent second minors, that the second minor systems shall all have a common factor, implies the satisfaction of five conditions. This species of contact I call curvilineo-indefinite; it is, I believe, the only kind of indefinite contact between two surfaces of the second order hitherto taken account of.

There is still, however, a higher species of contact, *videlicet*, when all the four roots of the complete determinant of $U+\lambda V$ are identical with the root common to each of its second minors. In this case the common enveloping cone becomes identical with the plane (considered as a coincident pair of planes) in which the surfaces intersect.

The characteristics take the form

$$U = x^2 + xy + zt,$$
$$V = \phantom{x^2 +{}} xy + zt.$$

The intersection is contained completely in the common tangent plane $x = 0$, and consists of the *two* right lines,

$$(x = 0, \quad z = 0),$$
$$(x = 0, \quad t = 0).$$

This, the highest and crowning species of contact, I call bilineo-indefinite. It is defined by six conditions.

At each point of the two lines of intersection of U and V there is contact, and a very peculiar species of contact at the intersection of these two lines themselves.

To form a distinct idea of this, let the physical visible or quasi-visible intersection of U, V take place along the two lines L, M; the *rational* intersection must be conceived as made up of the wry quadrilateral, L, M; L', M', in which L is indefinitely near to L', and M to M'. It follows, therefore, that there is contact at the four angles of the quadrilateral; but as there is nothing to fix the relative directions of the diagonal joining the intersection of L and M to that of L' and M', because there is nothing to restrict the position of the latter point, except that it shall lie upon either surface*, it appears that not only is there contact at the junction of the two lines constituting the complete intersection of the two surfaces, but that these surfaces continue to touch at consecutive points taken all round this first, and indefinitely near to it in any direction†.

Bilineo-indefinite (the highest) contact for two conoids is strictly analogous to confluence, the highest species of contact between conics. For this latter may be conceived as an intersection made up of two coincident pairs of coincident points; and the former, as an intersection made up of two coincident pairs of crossing right lines; and a pair of crossing lines is to a plane locus of the second degree what a coincident pair of points is to a rectilinear locus of the same degree.

In the subjoined table I have brought under one point of view the characters and algebraic forms which I call the *condensed forms* corresponding to each species of contact above detailed.

A. *Quadratic loci in a right line.*

Simple contact. One condition.	xy $x^2 + xy$

B. *Quadratic loci in a plane.*

1st Class.

Simple contact. One condition.	$x^2 + y^2 + xz$ $ax^2 + by^2 + cxz$
Proximal contact. Two conditions.	$x^2 + yx + yz$ $ax^2 + byx + ayz$

2nd Class.

Diploidal contact. Two conditions.	$x^2 + y^2 + z^2$ $ax^2 + ay^2 + bz^2$
Confluent contact. Three conditions.	$x^2 + y^2 + xz$ $y^2 + xz$

* This will be better seen by reference to the analogy presented by the case when the two conoids touch all along a curve. The rational intersection is made up of this curve and another indefinitely near it. The two curves, whatever be the position of their node, will lie in the same enveloping cone, so that the position of the node is indeterminate.

† As the two surfaces jut one close into the other at this point, it would perhaps be not improper to designate the contact at such point as umbilical.

C. *Quadratic loci in space.*

1st Class.

Simple contact. One condition.	$\left.\begin{matrix} x^2+y^2+z^2+xt \\ ax^2+by^2+cz^2+dxt \end{matrix}\right\}$
Proximal contact. Two conditions.	$\left.\begin{matrix} x^2+y^2+xt+zt \\ ax^2+by^2+cxt+azt \end{matrix}\right\}$
Unilinear contact. 1st species of diploidal. Two conditions.	$\left.\begin{matrix} x^2+xy+zt \\ ayz+bxy+czt \end{matrix}\right\}$
Confluent-unilinear, or triple contact. Three conditions.	$\left.\begin{matrix} x^2+yz+xy+zt \\ az^2+bxy+bzt \end{matrix}\right\}$

2nd Class, 1st Section.

Biplanar contact. 2nd species of diploidal. Two conditions.	$\left.\begin{matrix} x^2+y^2+z^2+t^2 \\ ax^2+ay^2+bz^2+ct^2 \end{matrix}\right\}$
Confluent-biplanar contact. Three conditions.	$\left.\begin{matrix} x^2+zt+y^2+z^2 \\ ax^2+azt+by^2+cz^2 \end{matrix}\right\}$
Bilinear contact. Three conditions.	$\left.\begin{matrix} x^2+y^2+z^2+zt \\ ax^2+ay^2+bz^2+czt \end{matrix}\right\}$ or $\left\{\begin{matrix} xz+yt \\ axt+byz \end{matrix}\right.$
Confluent-bilinear contact. Four conditions.	$\left.\begin{matrix} xz+xt+y^2+z^2 \\ axz+bxt+by^2+bz^2 \end{matrix}\right\}$

2nd Class, 2nd Section.

Quadrilinear, or quadruple contact. Four conditions.	$\left.\begin{matrix} x^2+y^2+z^2+t^2 \\ ax^2+ay^2+bz^2+bt^2 \end{matrix}\right\}$ or $\left\{\begin{matrix} xy+zt \\ axy+bzt \end{matrix}\right.$
Unilineo-indefinite contact. Five conditions.	$\left.\begin{matrix} x^2+y^2+xz+yt \\ ax^2+by^2+cxz+cyt \end{matrix}\right\}$

3rd Class.

Curvilineo-indefinite contact. Five conditions.	$\left.\begin{matrix} x^2+y^2+z^2+t^2 \\ ax^2+ay^2+az^2+bt^2 \end{matrix}\right\}$
Bilineo-indefinite contact. Six conditions.	$\left.\begin{matrix} x^2+xy+zt \\ xy+zt \end{matrix}\right\}$

Another (and, in a physical sense, more) natural mode of grouping the twelve species of conoidal contact, which, without observing the same lines of demarcation, leaves intact the sequence of the species, is into the three families. The first, or definite-continuous, for which the surfaces touch in a single point, and intersect in an unbroken curve, comprises simple and cuspidal contact.

The second definite-discontinuous, for which the surfaces touch in one, two, three or four points, but intersect in a curve more or less broken up into distinct parts, comprises all the species from the third to the ninth inclusive. The third natural family is that of indefinite contact, and comprises the three last species. It will of course be observed that there are five species of single contact, that is, contact at one point, namely, simple, cuspidal, and the three confluent species, two of double, one of treble, one of quadruple, and three of indefinite contact; the last being distinguishable *inter se*—lineo-indefinite as being special at two points, curvilineo-indefinite as having no speciality, and bilineo-indefinite as being special at one point only.

I might now proceed to discuss more particularly the nature of the contact taken, not collectively, but with reference to each single point where it exists. This, however, must be reserved for a future communication; as also, among other important and curious matter, the ascertainment of the singular forms of quadratic conjugate functions of five or more letters. At present I shall content myself with stating the following general proposition, which naturally suggests itself from a consideration of the cases already considered.

In a conjugate quadratic system of any number of letters, the lowest and also the highest degree of singularity will be always unique; the conditions to be satisfied in the former case being only one in number, and in the latter $\frac{1}{2}r(r-1)$, where r denotes the number of the letters. The first part of this proposition is self-apparent, the latter part may be inferred from the homaloidal law; for the $(r-2)$nd minors will be quadratic functions, and the highest degree of contact will correspond to those having a factor in common, which would involve the satisfaction of $\frac{1}{2}r(r-1)-1$ conditions only; but over and above this, that the complete determinant, instead of containing this common factor, as it needs must, $(r-1)$ times, shall contain it r times: this gives one condition more, making up the entire number to $\frac{1}{2}r(r-1)$.

The total number of different species of singularity for conjugate functions of a given number of letters, can only be expressed by aid of formulæ containing expressions for the number of various ways in which numbers admit of being broken up into a given number of parts.

The computation of this number in particular cases, upon the principle of the foregoing method, is attended with no difficulty.

We have seen that this number for two, three and four letters, is respectively one, four, twelve.

I have found that for five letters the number is twenty-four, for six letters fifty, for seven letters a hundred, and (subject to further examination) for eight letters one hundred and ninety-three. The series, therefore, as far as I have yet traced it, is 1, 4, 12, 24, 50, 100, 193. The last number must not be relied upon at present.

It will be observed, that the foregoing table for the contacts of surfaces of the second order contains no form corresponding to a complete intersection in two non-intersecting lines and an undegenerated conic. In fact, if two such lines form part of the intersection, *at least* one other right line intersecting them both, must go to make up the remaining part. This is easily verified; for it is readily seen that the most general representation of two conoids intersecting in two non-meeting lines will be

$$U = xy + zt,$$
$$V = axy + bzt + cxt + eyz,$$

where the two lines in question are

$$(x = 0, \quad z = 0),$$
$$(y = 0, \quad t = 0).$$

Now it will be found that the first minors of $V + \lambda U$ formed from the above equation will all contain the common factor $(a + \lambda)(b + \lambda) - ce$, showing that the contact is quadrilinear or linear-indefinite, that is bilinear, according as the roots of

$$\lambda^2 + (a + b)\lambda + ab - ce = 0$$

are distinct or equal; which explains how it is that only one species of bilinear contact (that is to say, the case corresponding to the two conoids agreeing in the two right lines in which each is cut by a common tangent plane) comes to find a place in the preceding enumeration.

It may not be uninteresting, under an euristic point of view, to state that the above theory, which, as well in what it accomplishes as in what it suggests (the author cannot but feel conscious), constitutes a substantial accession to analytical science, arose out of a theorem which occurred to him as likely to be true, in the act of reviewing for the press his paper "On Certain Additions" in the last November Number* of this *Magazine*, and which he had only then time to throw into a foot-note as a probable conjecture.

Wishing to subject it to an analytical test, he found it necessary to obtain the *condensed forms* which serve to characterize the confluent contact of

[* p. 148 above.]

conics. In this way he became aware of the great utility of these condensed forms, and of the desideratum to be supplied in obtaining a complete list of them applicable to all varieties of contact. The happy thought then occurred to him of inverting the process which he had applied in the treatment of the contacts of conics, in the November Number* of the *Cambridge and Dublin Mathematical Journal*; for whereas the nature of the contacts was there assumed and translated into the language of determinants, he soon discovered that it was the more easy and secure course to assume the relations of every possible *immutable* kind that could exist between the complete and minor determinants corresponding to the characteristics, by aid of these relations to construct the characteristics, and from the characteristics so obtained, determine the geometrical character of each resulting species of contact. Thus he has been able to effect the very results stated by himself as desiderata at the close of the paper in this *Magazine* above referred to.

Note.—It is proper to remark, that all the condensed forms given in this paper have actually been obtained by the author in the way above pointed out. The limits imposed by the objects to which the *Magazine* is devoted have restricted him from exhibiting the method at full; but any of his readers will be able without difficulty to make it out for himself.

The process consists in finding $U+\lambda V$ by means of solving for each case a problem of position (a kind of chess-board problem) on a square table, containing three places in length and breadth for conics, four places by four for surfaces, and so on (if need be) according to the number of variable letters involved. $U+\lambda V$ being thus determined in form, U and V become readily cognizable. It is right also to add, that some of the condensed forms here set forth have been incidentally noticed and employed by previous authors, as Plücker and Mr Cayley.

The conditions in each case to which the position-problem is subject are immediately deducible from the laws which the complete determinant, and the successive minor systems of determinants of $U+\lambda V$, are required to satisfy.

[* p. 119 above.]

37.

ON THE RELATION BETWEEN THE MINOR DETERMINANTS OF LINEARLY EQUIVALENT QUADRATIC FUNCTIONS.

[*Philosophical Magazine*, I. (1851), pp. 295—305.]

I SHOWED in the preliminary part of my paper on Contacts in the February Number of this *Magazine**, by *à priori* reasoning, that if a quadratic function (U) be linearly converted into another (V), any minor determinant of any order of V must be a syzygetic function of all the minor determinants of U of the same order.

The object of my present communication is to exhibit the syzygy in question, which, as I indicated, is linear; by which I mean that a determinant of the one function is equal to the sum of the pari-ordinal determinants of the other affected respectively with multipliers formed exclusively out of the coefficients of the equations of transformation. In order that a clear enunciation of the theorem in view may be possible, it is necessary to premise a new but simple, and, as experience has proved to me, a most powerful, because natural, method of notation applicable to all questions concerning determinants.

Every determinant is obtained by operating upon a square array of quantities, which, according to the ordinary method, might be denoted as follows:

$$\begin{array}{l}
a_{1,1},\quad a_{1,2}\ldots a_{1,n},\\
a_{2,1},\quad a_{2,2}\ldots a_{2,n},\\
a_{3,1},\quad a_{3,2}\ldots a_{3,n},\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
a_{n,1},\quad a_{n,2}\ldots a_{n,n}.
\end{array}$$

My method consists in expressing the same quantities biliterally as below:

$$\begin{array}{l}
a_1\alpha_1,\quad a_1\alpha_2\ldots a_1\alpha_n,\\
a_2\alpha_1,\quad a_2\alpha_2\ldots a_2\alpha_n,\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
a_n\alpha_1,\quad a_n\alpha_2\ldots a_n\alpha_n,
\end{array}$$

[* p. 221 above.]

where of course, whenever desirable, instead of $a_1, a_2 \dots a_n$, and $\alpha_1, \alpha_2 \dots \alpha_n$, we may write simply $a, b \dots l$, and $\alpha, \beta \dots \lambda$ respectively. Each quantity is now represented by two letters; the letters themselves, taken separately, being symbols neither of quantity nor of operation, but mere umbræ or ideal *elements* of quantitative symbols. We have now a means of representing the determinant above given in a compact form; for this purpose we need but to write one set of umbræ over the other as follows: $\begin{pmatrix} a_1, & a_2 \dots & a_n \\ \alpha_1, & \alpha_2 \dots & \alpha_n \end{pmatrix}$. If we now wish to obtain the algebraic value of this determinant, it is only necessary to take $\alpha_1, \alpha_2 \dots \alpha_n$ in all its $1, 2, 3 \dots n$ different positions, and we shall have

$$\begin{Bmatrix} a_1, & a_2 \dots & a_n \\ \alpha_1, & \alpha_2 \dots & \alpha_n \end{Bmatrix} = \Sigma \pm \{a_1\alpha_{\theta_1} \times a_2\alpha_{\theta_2} \times \dots \times a_n\alpha_{\theta_n}\},$$

in which expression $\theta_1, \theta_2 \dots \theta_n$ represents some order of the numbers $1, 2 \dots n$, and the positive or negative sign is to be taken according to the well-known dichotomous law. Thus, for example,

$$\begin{Bmatrix} abc \\ \alpha\beta\gamma \end{Bmatrix} \text{ will represent } \left.\begin{matrix} a\alpha \times b\beta \times c\gamma \\ + a\beta \times b\gamma \times c\alpha \\ + a\gamma \times b\alpha \times c\beta \\ - a\beta \times b\alpha \times c\gamma \\ - a\alpha \times b\gamma \times c\beta \\ - a\gamma \times b\beta \times c\alpha \end{matrix}\right\}.$$

Although not necessary for our immediate object, it may not be inopportune to observe how readily this notation lends itself to a further natural extension of its application.

$$\begin{Bmatrix} \overline{ab} & \overline{cd} \\ \alpha\beta & \gamma\delta \end{Bmatrix} \text{ will naturally denote}$$

$$\frac{ab}{\alpha\beta} \times \frac{cd}{\gamma\delta} - \frac{ab}{\gamma\delta} \times \frac{cd}{\alpha\beta};$$

that is

$$\begin{Bmatrix} (a\alpha \times b\beta) \\ -(a\beta \times b\alpha) \end{Bmatrix} \times \begin{Bmatrix} (c\gamma \times d\delta) \\ -(c\delta \times d\gamma) \end{Bmatrix} - \begin{Bmatrix} (a\gamma \times b\delta) \\ -(a\delta \times b\gamma) \end{Bmatrix} \times \begin{Bmatrix} (c\alpha \times d\beta) \\ -(c\beta \times d\alpha) \end{Bmatrix}.$$

And in general the compound determinant

$$\begin{Bmatrix} \overline{a_1, b_1 \dots l_1}, & \overline{a_2, b_2 \dots l_2} \dots & \overline{a_r, b_r \dots l_r} \\ \alpha_1, \beta_1 \dots \lambda_1, & \alpha_2, \beta_2 \dots \lambda_2 & \alpha_r, \beta_r \dots \lambda_r \end{Bmatrix}$$

will denote

$$\Sigma \pm \begin{Bmatrix} a_1, & b_1 \dots & l_1 \\ \alpha_{\theta_1}, & \beta_{\theta_1} \dots & \lambda_{\theta_1} \end{Bmatrix} \times \begin{Bmatrix} a_2, & b_2 \dots & l_2 \\ \alpha_{\theta_2}, & \beta_{\theta_2} \dots & \lambda_{\theta_2} \end{Bmatrix} \times \dots \times \begin{Bmatrix} a_r, & b_r \dots & l_r \\ \alpha_{\theta_r}, & \beta_{\theta_r} \dots & \lambda_{\theta_r} \end{Bmatrix},$$

where, as before, we have the disjunctive equation

$$\theta_1,\ \theta_2 \dots \theta_r = 1,\ 2 \dots r.$$

As an example of the power of this notation, I will content myself with stating the following remarkable theorem in compound determinants, one of the most prolific in results of any with which I am acquainted, but which is derived from a more particular case of another vastly more general. The theorem is contained in the annexed equation

$$\left\{\begin{matrix} \overline{a_1,\ a_2 \dots a_r,\ a_{r+1},} & \overline{a_1,\ a_2 \dots a_r,\ a_{r+2}} \dots & \overline{a_1,\ a_2 \dots a_r,\ a_{r+s}} \\ \alpha_1,\ \alpha_2 \dots \alpha_r,\ \alpha_{r+1}, & \alpha_1,\ \alpha_2 \dots \alpha_r,\ \alpha_{r+2} & \alpha_1,\ \alpha_2 \dots \alpha_r,\ \alpha_{r+s} \end{matrix}\right\}$$

$$= \begin{Bmatrix} a_1,\ a_2 \dots a_r \\ \alpha_1,\ \alpha_2 \dots \alpha_r \end{Bmatrix}^{s-1} \times \begin{Bmatrix} a_1,\ a_2 \dots a_r,\ a_{r+1},\ a_{r+2} \dots a_{r+s} \\ \alpha_1,\ \alpha_2 \dots \alpha_r,\ \alpha_{r+1},\ \alpha_{r+2} \dots \alpha_{r+s} \end{Bmatrix}. \qquad (1)$$

It is obvious, that, without the aid of my system of umbral or biliteral notation, this important theorem could not be made the subject of statement without an enormous periphrasis, and could never have been made the object of distinct contemplation or proof.

To return to the more immediate object of this communication, suppose that we have any binary function of two sets of quantities, $x_1,\ x_2 \dots x_n$; $\xi_1,\ \xi_2 \dots \xi_n$, of which the general term will be of the form $c_{r,s} \times x_r \xi_s$; according to the principles of notation above laid down, nothing can be more natural than to represent $c_{r,s}$ by the biliteral group $a_r \alpha_s$; the function in question will then take the form

$$\Sigma a_r \alpha_s \,.\, x_r \xi_s\,;$$

the x's and ξ's denoting quantities, but the a's and α's mere umbræ. The function may then be thrown under the convenient symbolical form

$$(a_1 x_1 + a_2 x_2 + \dots + a_n x_n)$$
$$\times\, (\alpha_1 \xi_1 + \alpha_2 \xi_2 + \dots + \alpha_n \xi_n).$$

So if we confine ourselves to quadratic functions, for which $x_1,\ x_2 \dots x_n$; $\xi_1,\ \xi_2 \dots \xi_n$ become respectively identical, the general symbolical representation of any such will be

$$(a_1 x_1 + a_2 x_2 + \dots + a_n x_n)^2.$$

The complete determinant will be denoted by

$$\begin{Bmatrix} a_1,\ a_2 \dots a_n \\ \alpha_1,\ \alpha_2 \dots \alpha_n \end{Bmatrix},$$

and any minor determinant of the rth order by

$$\begin{Bmatrix} a_1,\ a_2 \dots a_r \\ \alpha_{\theta_1},\ \alpha_{\theta_2} \dots \alpha_{\theta_r} \end{Bmatrix},$$

where $\theta_1, \theta_2 \ldots \theta_r$ are some certain r distinct numbers taken out of the series $1, 2, 3 \ldots r$. Suppose now that we have

$$U = (a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2$$

linearly transformable into

$$V = (b_1y_1 + b_2y_2 + \ldots + b_ny_n)^2,$$

by means of the n equations

$$\left.\begin{array}{l} x_1 = a_1b_1 \,.\, y_1 + a_1b_2 \,.\, y_2 + \ldots + a_1b_n \,.\, y_n \\ x_2 = a_2b_1 \,.\, y_1 + a_2b_2 \,.\, y_2 + \ldots + a_2b_n \,.\, y_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = a_nb_1 \,.\, y_1 + a_nb_2 \,.\, y_2 + \ldots + a_nb_n \,.\, y_n \end{array}\right\};$$

in which equations, be it observed, each coefficient a_rb_s is a single quantity, perfectly independent of the quantities denoted generally by a_ra_s, b_rb_s which enter into U and V. Our object is to be able to express the minor determinant

$$\begin{Bmatrix} b_{k_1}, & b_{k_2} \ldots b_{k_r} \\ b_{l_1}, & b_{l_2} \ldots b_{l_r} \end{Bmatrix},$$

in which the one group of distinct numbers, $k_1, k_2 \ldots k_r$ may either differ wholly from, or agree wholly or in part with the other group of distinct numbers $l_1, l_2 \ldots l_r$, under the form of

$$\Sigma \left\{ \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{pmatrix} \times Q \right\}.$$

The particular value of Q corresponding to each double group, $\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$, may be denoted by $Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$; so that our problem consists in determining the value of $Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$ in the equation

$$\begin{Bmatrix} b_{k_1}, & b_{k_2} \ldots b_{k_r} \\ b_{l_1}, & b_{l_2} \ldots b_{l_r} \end{Bmatrix} = \Sigma \left\{ Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix} \times \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \ldots a_{\phi_r} \end{pmatrix} \right\}.$$

Accordingly I enunciate that

$$Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix} = \begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{l_1}, & a_{l_2} \ldots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{Bmatrix}$$

$$+ \begin{Bmatrix} a_{l_1}, & a_{l_2} \ldots a_{l_r} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{Bmatrix}, \qquad (2)$$

subject to one sole exception in the case of $\theta_1, \theta_2 \ldots \theta_r$ being identical with $\phi_1, \phi_2, \ldots \phi_r$; namely, that for the terms (for such case) of the form

$Q\begin{pmatrix}\theta_1, & \theta_2 \dots \theta_r\\ \theta_1, & \theta_2 \dots \theta_r\end{pmatrix}$, the value to be taken is not that which the general formula would give, namely,

$$2\begin{Bmatrix}a_{k_1}, & a_{k_2} \dots a_{k_r}\\ b_{\theta_1}, & b_{\theta_2} \dots b_{\theta_r}\end{Bmatrix}^2,$$

but the half of this, that is simply the square of

$$\begin{Bmatrix}a_{k_1}, & a_{k_2} \dots a_{k_r}\\ a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r}\end{Bmatrix}.$$

The value of $Q\begin{pmatrix}\theta_1, & \theta_2 \dots \theta_r\\ \phi_1, & \phi_2 \dots \phi_r\end{pmatrix}$, it is obvious, contains only *quantities* of the form $a_r . b_s$, which are coefficients in the equations of transformation, but none of the form $a_r . a_s$ or $b_r . b_s$; showing that the syzygetic connexion between the minor determinants of U and V of the same order is linear, as has been already anticipatively announced.

The problem which I have treated above is only a particular case of a more general one, which may be stated as follows: given

$$U = (a_1x_1 + a_2x_2 + \dots + a_nx_n)^2,$$

and supposing m linear equations to be instituted between x_1, $x_2 \dots x_n$, so that U may be made a function of $(n-m)$ letters only, to express any minor determinant of the reduced form of U without performing the process of elimination between the given equations. Let the given equations be written under the form

$$a_1a_{n+1}x_1 + a_2a_{n+1}x_2 + \dots + a_na_{n+1}x_n = 0,$$
$$a_1a_{n+2}x_1 + a_2a_{n+2}x_2 + \dots + a_na_{n+2}x_n = 0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_1a_{n+m}x_1 + a_2a_{n+m}x_2 + \dots + a_na_{n+m}x_n = 0,$$

and let it be convened (which takes nothing away from the generality of these equations) that $a_{n+r}a_{n+s}$ shall signify zero for all values of r and s *concurrently* greater than zero. Suppose that $x_1, x_2 \dots x_m$, being eliminated, U becomes of the form

$$(b_{m+1}x_{m+1} + b_{m+2}x_{m+2} + \dots + b_nx_n)^2;$$

and suppose that we wish to determine the value of the complete determinant of this last function; it will be found to be

$$\begin{pmatrix}b_{m+1}, & b_{m+2} \dots b_n\\ b_{m+1}, & b_{m+2} \dots b_n\end{pmatrix} = \begin{Bmatrix}a_1, & a_2 \dots a_n, & a_{n+1} \dots a_{n+m}\\ a_1, & a_2 \dots a_n, & a_{n+1} \dots a_{n+m}\end{Bmatrix} \div \begin{pmatrix}a_1, & a_2 & \dots a_m\\ a_{n+1}, & a_{n+2} & \dots a_{n+m}\end{pmatrix}^2,$$

the squared divisor being, as is obvious, a function only of the coefficients of the transforming equations, and depending for its value upon the particular

m quantities selected for elimination. The dividend, on the contrary, is independent of this selection, but involves the coefficients of the function combined with the coefficients of transformation. This is the symbolical representation of the theorem given by me in the postscript to my paper in the *Cambridge and Dublin Mathematical Journal* for November 1850*.

Suppose, now, more generally that we wish to find any minor determinant. The solution is given† by the equation

$$\begin{Bmatrix} b_{\theta_{m+1}}, & b_{\theta_{m+2}} \dots b_{\theta_{m+s}} \\ b_{\phi_{m+1}}, & b_{\phi_{m+2}} \dots b_{\phi_{m+s}} \end{Bmatrix}$$

(wherein the two groups $\theta_{m+1}, \theta_{m+2}, \dots \theta_{m+s}$; $\phi_{m+1}, \phi_{m+2} \dots \phi_{m+s}$ are each of them s differing, or wholly or in part agreeing individuals arbitrarily selected out of the $(n-m)$ numbers $m+1, m+2, \dots n$)

$$= \begin{Bmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_m}, & a_{\theta_{m+1}}, & a_{\theta_{m+2}} \dots a_{\theta_{m+s}} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_m}, & a_{\phi_{m+1}}, & a_{\phi_{m+2}} \dots a_{\phi_{m+s}} \end{Bmatrix} \div \begin{Bmatrix} a_1, & a_2 & \dots a_m \\ a_{n+1}, & a_{n+2} & \dots a_{n+m} \end{Bmatrix}^2. \quad (3)$$

If we make $n = 2\gamma$ and $m = \gamma$, and $a_{\gamma+r}a_{\gamma+s} = 0$ for all positive values of either r or s, and $a_{\gamma-i}a_{n+e} = 0$ for all values of i and e differing from one another, and for equal values $a_{\gamma-e}a_{\gamma+e} = -1$, it will readily be seen that this last theorem reduces to the one first considered; and on careful inspection it will be found, that the solution given of the general question includes within it that presented for the particular case in question. Such inclusion, however, I ought in fairness to state is far from being obvious; and to demonstrate it exactly, and in general terms, requires the aid of methods which my readers would probably find to exceed their existing degree of knowledge or familiarity with the subject.

The theorem above enunciated was in part suggested in the course of a conversation with Mr Cayley (to whom I am indebted for my restoration to the enjoyment of mathematical life) on the subject of one of the preliminary theorems in my paper on Contacts in this *Magazine*.

It is wonderful that a theory so purely analytical should originate in a geometrical speculation. My friend M. Hermite has pointed out to me, that some faint indications of the same theory may be found in the *Recherches Arithmétiques* of Gauss. The notation which I have employed for determinants is very similar to that of Vandermonde, with which I have become acquainted since writing the above, in Mr Spottiswoode's valuable treatise *On the Elementary Theorems of Determinants*. Vandermonde was evidently on the right road. I do not hesitate to affirm, that the superiority of his and my notation over that in use in the ordinary methods is as great and almost as important to the progress of analysis, as the superiority of the notation of the differential calculus over that of the fluxional system. For what is the theory of determinants? It is an algebra upon algebra; a

[* p. 136 above.] [† see p. 251 below.]

calculus which enables us to combine and foretell the results of algebraical operations, in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic. All analysis must ultimately clothe itself under this form*.

I have in previous papers defined a "Matrix" as a rectangular array of terms, out of which different systems of determinants may be engendered, as from the womb of a common parent; these cognate determinants being by no means isolated in their relations to one another, but subject to certain simple laws of mutual dependence and simultaneous deperition. The condensed representation of any such Matrix, according to my improved Vandermondian notation, will be

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ \alpha_1, & \alpha_2 \dots \alpha_m \end{Bmatrix}.$$

To return to the theorems of the text. Theorem (2) admits of being presented in a more convenient form for the purposes of analytical operation, so as to become relieved from all cases of exception appertaining to particular terms.

The limitation to the generality of the expression for Q arises from our treating

$$\begin{Bmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{Bmatrix}$$

as *identical* with its *equal*,

$$\begin{Bmatrix} a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \\ a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \end{Bmatrix}.$$

If, however, we now convene to treat these two forms as distinct, so that in theorem (2)

$$\Sigma \left\{ Q \begin{pmatrix} \theta_1, & \theta_2 \dots \theta_r \\ \phi_1, & \phi_2 \dots \phi_r \end{pmatrix} \times \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{pmatrix} \right\}$$

will contain $\left\{ \dfrac{n(n-1)\dots(n-r+1)}{1.2\dots r} \right\}^2$ terms, then we may write simply

$$Q \begin{pmatrix} \theta_1, & \theta_2 \dots \theta_r \\ \phi_1, & \phi_2 \dots \phi_r \end{pmatrix} = \begin{Bmatrix} a_{k_1}, & a_{k_2} \dots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \dots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{l_1}, & a_{l_2} \dots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \dots b_{\phi_r} \end{Bmatrix},$$

* Perhaps the most remarkable indirect question to which the method of determinants has been hitherto applied is Hesse's problem of reducing a cubic function of 3 letters to another consisting only of 4 terms by linear substitutions—a problem which appears to set at defiance all the processes and artifices of common algebra. I have succeeded in applying a method founded upon this calculus to the linear reduction of a biquadratic function of two letters to Cayley's form $x^4 + mx^2y^2 + y^4$, and of a 5^c function of two letters to the new form $x^5 + y^5 + (ax + by)^5$. This last reduction is effected by means of the properties of a certain other function of the 8th degree connected with the given function of the 5th degree. See a paper on this subject in the forthcoming May Number of the *Cambridge and Dublin Mathematical Journal*. [p. 191 above.]

which equation is subject to no exception for the case of the θ's and ϕ's becoming identical. As regards this theorem, it will not fail to strike the reader that it ought to admit of verification; for that U may be derived from V in the same manner as V from U if we express $y_1, y_2 \dots y_n$ in terms of $x_1, x_2 \dots x_n$, by solving the system of equations (2), which there is no difficulty in doing. In fact, if we write

$$\begin{aligned} y_1 &= \alpha_1\beta_1 x_1 + \alpha_1\beta_2 x_2 + \dots + \alpha_1\beta_n x_n, \\ y_2 &= \alpha_2\beta_1 x_1 + \alpha_2\beta_2 x_2 + \dots + \alpha_2\beta_n x_n, \\ &\dots\dots\dots\dots\dots\dots \\ y_n &= \alpha_n\beta_1 x_1 + \alpha_n\beta_2 x_2 + \dots + \alpha_n\beta_n x_n, \end{aligned}$$

we shall obtain

$$\alpha_r\beta_s = \begin{Bmatrix} a_1, & a_2 \dots a_{r-1}, & a_{r+1}, & a_{r+2} \dots a_n \\ b_1, & b_2 \dots b_{s-1}, & b_{s+1}, & b_{s+2} \dots b_n \end{Bmatrix} \div \begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix}.$$

Accordingly we shall find

$$\begin{Bmatrix} a_{m_1}, & a_{m_2} \dots a_{m_r} \\ a_{p_1}, & a_{p_2} \dots a_{p_r} \end{Bmatrix} = \Sigma \left\{ Q \begin{pmatrix} \psi_1, & \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} \times \begin{pmatrix} b_{\psi_1}, & b_{\psi_2} \dots b_{\psi_r} \\ b_{\omega_1}, & b_{\omega_2} \dots b_{\omega_r} \end{pmatrix} \right\},$$

and

$$Q \begin{pmatrix} \psi_1, & \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} = \begin{pmatrix} \alpha_{m_1}, & \alpha_{m_2} \dots \alpha_{m_r} \\ \beta_{\psi_1}, & \beta_{\psi_2} \dots \beta_{\psi_r} \end{pmatrix} \times \begin{pmatrix} \alpha_{p_1}, & \alpha_{p_2} \dots \alpha_{p_r} \\ \beta_{\omega_1}, & \beta_{\omega_2} \dots \beta_{\omega_r} \end{pmatrix};$$

substituting for the α's and β's their symbolical equivalents given above, and applying the theorem given below, we shall easily obtain

$$Q \begin{pmatrix} \psi_1, & \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} = \begin{pmatrix} a_{m_{r+1}}, & a_{m_{r+2}} \dots a_{m_n} \\ b_{\psi_{r+1}}, & b_{\psi_{r+2}} \dots b_{\psi_n} \end{pmatrix} \times \begin{pmatrix} a_{p_{r+1}}, & a_{p_{r+2}} \dots a_{p_n} \\ \beta_{p_{r+1}}, & \beta_{p_{r+2}} \dots \beta_{p_n} \end{pmatrix}$$
$$\div \begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix}^2.$$

If, now, in the expression

$$\begin{Bmatrix} b_{k_1}, & b_{k_2} \dots b_{k_r} \\ b_{l_1}, & b_{l_2} \dots b_{l_r} \end{Bmatrix} = \Sigma \left\{ \begin{pmatrix} a_{k_1}, & a_{k_2} \dots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \dots b_{\theta_r} \end{pmatrix} \begin{pmatrix} a_{l_1}, & a_{l_2} \dots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \dots b_{\phi_r} \end{pmatrix} \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{pmatrix} \right\},$$

we resubstitute for $\begin{Bmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{Bmatrix}$ its value in the form of

$$\Sigma \left\{ \begin{pmatrix} b_{\omega_1}, & b_{\omega_2} \dots b_{\omega_r} \\ b_{\psi_1}, & b_{\psi_2} \dots b_{\psi_r} \end{pmatrix} Q \right\},$$

we shall obtain $\begin{pmatrix} b_{k_1}, & b_{k_2} \dots b_{k_r} \\ b_{l_1}, & b_{l_2} \dots b_{l_r} \end{pmatrix}$ under the form of

$$\Sigma \left\{ R \begin{pmatrix} \omega_1, & \omega_2 \dots \omega_r \\ \psi_1, & \psi_2 \dots \psi_r \end{pmatrix} \times \begin{pmatrix} b_{\omega_1}, & b_{\omega_2} \dots b_{\omega_r} \\ b_{\psi_1}, & b_{\psi_2} \dots b_{\psi_r} \end{pmatrix} \right\};$$

and $R\begin{pmatrix}\omega_1, & \omega_2 \dots & \omega_r\\ \psi_1, & \psi_2 \dots & \psi_r\end{pmatrix}$ must $=0$, except for the case of $\omega_1, \omega_2 \dots \omega_r$; $\psi_1, \psi_2 \dots \psi_r$ being respectively identical with $k_1, k_2 \dots k_r$; $l_1, l_2 \dots l_r$, for which case $R\begin{pmatrix}k_1, & k_2 \dots & k_r\\ l_1, & l_2 \dots & l_r\end{pmatrix}$ must be unity. I have gone through this calculation and verified the result; in order to effect which, however, the following important generalization of theorem (1) must be apprehended.

Suppose two sets of umbræ,

$$a_1, \; a_2 \dots a_{m+n},$$

$$b_1, \; b_2 \dots b_{m+n},$$

and let r be any number less than m, and let any r-ary combination of the m numbers 1, 2, 3 ... m be expressed by ${}^q\theta_1, {}^q\theta_2 \dots {}^q\theta_m$, where q goes through all the values intermediate between 1 and μ, μ being

$$\frac{m(m-1)\dots(m-r+1)}{1 \, . \, 2 \dots r};$$

then I say that the compound determinant,

$$\begin{vmatrix} a_{{}^1\theta_1}, & a_{{}^1\theta_2} \dots a_{{}^1\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^1\theta_1}, & b_{{}^1\theta_2} \dots b_{{}^1\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n} \end{vmatrix} \quad \begin{vmatrix} a_{{}^2\theta_1}, & a_{{}^2\theta_2} \dots a_{{}^2\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^2\theta_1}, & b_{{}^2\theta_2} \dots b_{{}^2\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n} \end{vmatrix}$$

$$\dots\dots \begin{vmatrix} a_{{}^\mu\theta_1}, & a_{{}^\mu\theta_2} \dots a_{{}^\mu\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^\mu\theta_1}, & b_{{}^\mu\theta_2} \dots b_{{}^\mu\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n} \end{vmatrix},$$

is equal to the following product,

$$\begin{vmatrix} a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{m+1}, & b_{m+2} \dots b_{m+n} \end{vmatrix}^{\mu'}, \quad \begin{vmatrix} a_1, & a_2 \dots a_{m+n} \\ b_1, & b_2 \dots b_{m+n} \end{vmatrix}^{\mu''} \qquad (4)$$

where

$$\mu'' = \frac{(m-1)(m-2)\dots(m-r+1)}{1 \, . \, 2 \dots (r-1)},$$

and

$$\mu' = \frac{(m-1)(m-2)\dots(m-r)}{1 \, . \, 2 \dots r};$$

when $r = 1$, we have the case already given in theorem (2), and of course μ'' is to be taken unity.

This very general theorem is itself several degrees removed from my still unpublished Fundamental Theorem which is a theorem for the expansion of the products of determinants.

Obs. The analogy upon which the extension of the Vandermondian notation from simple to compound determinants is grounded, would be better apprehended if the biliteral symbols of simple quantities were written with the umbral elements disposed vertically, as $\begin{smallmatrix} a \\ b \end{smallmatrix}$, instead of horizontally, as ab; which latter is the method for the purposes of typographical uniformity adopted in the text above. The other mode is, however, much to be preferred, and is what I propose hereafter to adhere to. For my two general umbræ, a, b, Vandermonde uses two *numbers*, one set a-cock upon the other, as 5^4. The objection to the use of numbers is apparent as soon as it becomes necessary to treat of the mutual relations of diverse systems of determinants, and his mode of writing the umbræ militates against the perception of the most valuable algebraical analogies. The one important point in which Vandermonde has anticipated me, consists in expressing a simple determinant by two horizontal rows of umbræ one over the other. But the idea upon which this depends is so simple and natural, that it was sure to reappear in any well-constructed system of notation.

38.

NOTE ON QUADRATIC FUNCTIONS AND HYPER-DETERMINANTS.

[*Philosophical Magazine*, I. (1851), p. 415.]

PERMIT me to correct an error of transcription in the MS. of my paper "On Linearly Equivalent Quadratic Functions" in the last number of the *Magazine*. The theorem [p. 246 above] marked (3), should read as follows:—

$$\begin{Bmatrix} b_{\theta_{m+1}}, & b_{\theta_{m+2}} \dots b_{\theta_{m+s}} \\ b_{\phi_{m+1}}, & b_{\phi_{m+2}} \dots b_{\phi_{m+s}} \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1, & a_2 \dots a_m, & a_{\theta_{m+1}}, & a_{\theta_{m+2}} \dots a_{\theta_{m+s}}, & a_{n+1}, & a_{n+2} \dots a_{n+m} \\ a_1, & a_2 \dots a_m, & a_{\phi_{m+1}}, & a_{\phi_{m+2}} \dots a_{\phi_{m+s}}, & a_{n+1}, & a_{n+2} \dots a_{n+m} \end{Bmatrix}$$

$$\div \begin{Bmatrix} a_1, & a_2 & \dots a_m \\ a_{n+1}, & a_{n+2} & \dots a_{n+m} \end{Bmatrix}^2.$$

I may take this opportunity of mentioning, that by extending to algebraical functions generally a multiliteral system of umbral notation, analogous to the biliteral system explained in the paper above referred to as applicable to quadratic functions, I have succeeded in reducing to a mechanical method of compound permutation the process for the discovery of those memorable forms invented by Mr Cayley, and named by him hyperdeterminants, which have attracted the notice and just admiration of analysts all over Europe, and which will remain a perpetual memorial, as long as the name of algebra survives, of the penetration and sagacity of their author.

39.

ON A CERTAIN FUNDAMENTAL THEOREM OF DETERMINANTS.

[*Philosophical Magazine*, II. (1851), pp. 142—145.]

THE subjoined theorem, which is one susceptible of great extension and generalization, appears to me, and indeed from use and acquaintance (it having been long in my possession) I know to be so important and fundamental, as to induce me to extract it from a mass of memoranda on the same subject; and as an act of duty to my fellow-labourers in the theory of determinants, more or less forestall time (the sure discoverer of truth) by placing it without further delay on record in the pages of this *Magazine*. Its developments and applications must be reserved for a more convenient occasion, when the interest in the New Algebra (for such, truly, it is the office of the theory of determinants to establish), and the number of its disciples in this country, shall have received their destined augmentation. In a recent letter to me, M. Hermite well alludes to the theory of determinants as "That vast theory, transcendental in point of difficulty, elementary in regard to its being the basis of researches in the higher arithmetic and in analytical geometry."

The theorem is as follows:—Suppose that there are two determinants of the ordinary kind, each expressed by a square array of terms made up of n lines and n columns, so that in each square there are n^2 terms. Now let n be broken up in any given manner into two parts p and q, so that $p+q=n$. Let, firstly, one of the two given squares be divided in a given *definite* manner into two parts, one containing p of the n given lines, and the other part q of the same; and secondly, let the other of the two given squares be divided *in every possible way* into two parts, consisting of q and p lines respectively, so that on tacking on the part containing q lines of the second square to the part containing p lines of the first square, and the part containing p lines of the second square to the part containing q of the first, we get back a new couple of squares, each denoting a determinant different from the two given determinants; the number of such new couples will evidently be

$$\frac{n(n-1)\dots(n-p+1)}{1.2\dots p};$$

and my theorem is, that *the product of the given couple of determinants is equal to the sum of the products (affected with the proper algebraical sign) of each of the new couples formed as above described.* Analytically the theorem may be stated as follows.

Let $$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix}, \quad \begin{Bmatrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{Bmatrix},$$

according to the notation heretofore* employed by me in the preceding numbers of this *Magazine*, denote any two common determinants, each of the nth order, and let the numbers $\theta_1, \theta_2 \dots \theta_n$ be disjunctively equal to the numbers $1, 2 \dots n$ and $p+q=n$; then will

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix} \times \begin{Bmatrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{Bmatrix}$$

$$= \Sigma \pm \begin{Bmatrix} a_1, & a_2 & \dots & a_n \\ b_1, & b_2 \dots b_p, & \beta_{\theta_{p+1}}, & \beta_{\theta_{p+2}} \dots \beta_{\theta_n} \end{Bmatrix} \times \begin{Bmatrix} \alpha_1, & \alpha_2 & \dots & \alpha_n \\ \beta_{\theta_1}, & \beta_{\theta_2} \dots \beta_{\theta_p}, & b_{p+1}, & b_{p+2} \dots b_n \end{Bmatrix}.$$

The general term under the sign of summation may be represented by aid of the disjunctive equations

$$\phi_1, \phi_2 \dots \phi_n = 1, 2 \dots n,$$

$$\psi_1, \psi_2 \dots \psi_n = 1, 2 \dots n,$$

under the form of

$$(a_{\phi_1} . b_1 \times a_{\phi_2} . b_2 \times \dots \times a_{\phi_p} . b_p)(a_{\psi_{p+1}} . b_{p+1} \times a_{\psi_{p+2}} . b_{p+2} \times \dots \times a_{\psi_n} . b_n)$$

$$\times (\alpha_{\phi_{p+1}} . \beta_{\theta_{p+1}} \times \alpha_{\phi_{p+2}} . \beta_{\theta_{p+2}} \times \dots \times \alpha_{\phi_n} . \beta_{\theta_n})(\alpha_{\psi_1} . \beta_{\theta_1} \times \alpha_{\psi_2} . \beta_{\theta_2} \times \dots \times \alpha_{\psi_p} . \beta_{\theta_p}).$$

1st. When $\phi_1, \phi_2 \dots \phi_p = \psi_1, \psi_2 \dots \psi_p$, it will readily be seen, that for given values of $\phi_1, \phi_2 \dots \phi_p$, the product of the third and fourth factors becomes *substantially* identical with the general term of the determinant

$$\begin{Bmatrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{Bmatrix},$$

and consequently, making the system $\phi_1, \phi_2 \dots \phi_p$ (or, which is the same thing, its equivalent $\psi_1, \psi_2 \dots \psi_p$) go through all its values, we get back for the sum of the terms corresponding to the equation

$$\phi_1, \phi_2 \dots \phi_p = \psi_1, \psi_2 \dots \psi_p,$$

[* p. 242 above.]

the product of the determinants

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix} \text{ and } \begin{Bmatrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{Bmatrix}.$$

2nd. When we have not the equality above supposed between the ϕ's and the ψ's, let

$$\phi_{p-h} = \psi_{p+k} \text{ and } \phi_{p+\eta} = \psi_{p-\zeta};$$

the corresponding term included under the Σ will contain the factor

$$\alpha_{\phi_{p+\eta}} \cdot \beta_{\theta_{p+\eta}} \times \alpha_{\psi_{p-\zeta}} \cdot \beta_{\theta_{p-\zeta}}.$$

Now leaving $\phi_1, \phi_2 \dots \phi_p$, and $\psi_1, \psi_2 \dots \psi_p$ unaltered, we may take a system of values $\theta_1', \theta_2' \dots \theta_n'$, such that

$$\theta'_{p+\eta} = \theta_{p-\zeta},$$

and

$$\theta'_{p-\zeta} = \theta_{p+\eta},$$

and for all other values of q except $p+\eta$, or $p-\zeta$, $\theta'_q = \theta_q$. The corresponding new value of the general term so formed by the substitution of the θ' for the θ series, will be identical with that of the term first spoken of, but will have the contrary algebraical sign, because the θ' arrangement of the figures 1, 2, 3 ... p is deducible by a single interchange from the θ arrangement of the same, the rule for the imposition of the algebraical sign plus or minus being understood to be, that the term in which

$$\beta_{\theta_{p+1}}, \beta_{\theta_{p+2}} \dots \beta_{\theta_n}; \ \beta_{\theta_1}, \beta_{\theta_2} \dots \beta_{\theta_p}$$

enter into the symbolical forms of the respective derived couples of determinants, has the same sign as, or the contrary sign to, that in which

$$\beta_{\theta'_{p+1}}, \beta_{\theta'_{p+2}} \dots \beta_{\theta'_n}; \ \beta_{\theta'_1}, \beta_{\theta'_2} \dots \beta_{\theta'_p}$$

so enter, according as an odd or an even number of interchanges is required to transform the arrangement

$$\theta_{p+1}, \theta_{p+2} \dots \theta_n; \ \theta_1, \theta_2 \dots \theta_p$$

into the arrangement

$$\theta'_{p+1}, \theta'_{p+2} \dots \theta'_n; \ \theta'_1, \theta'_2 \dots \theta'_p.$$

I have therefore shown that all the terms arising from the expansion of the products included under the sign of summation, for which the disjunctive identity $\phi_1, \phi_2 \dots \phi_p = \psi_1, \psi_2 \dots \psi_p$ does not exist, enter into the final sum in pairs, equal in quantity and differing in sign, which consequently mutually destroy, and that the terms for which the said identity does exist together make up the sum

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix} \times \begin{Bmatrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{Bmatrix};$$

which proves, upon first principles drawn direct from that notion of polar dichotomy of permutation systems which rests at the bottom of the whole theory of the subject, the fundamental, and, as I believe, perfectly new theorem, which it is the object of this communication to establish.

In applying the theorem thus analytically formulized, it is of course to be understood that, under the sign Σ, *permutations* within the separate parts of a given arrangement,

$$\theta_{p+1},\ \theta_{p+2} \dots \theta_n;\ \theta_1,\ \theta_2 \dots \theta_p,$$

are inadmissible, the total number of terms so included being restricted to

$$\frac{n(n-1)\dots(n-p+1)}{1.2\dots p}.$$

The theorem may be extended so as to become a theorem for the expansion of the product of any number of determinants, and adapted so as to take in that far more general class of functions known to Mr Cayley and myself under the new name of commutants, of which determinants present only a particular, and that the most limited instance.

40.

ON EXTENSIONS OF THE DIALYTIC METHOD OF ELIMINATION.

[*Philosophical Magazine*, II. (1851), pp. 221—230.]

THE theory about to be described is a natural extension of the method of elimination presented by me ten years ago (in June, 1841) in the pages of this *Magazine*, which I have been induced to review in consequence of the flattering interest recently expressed in the subject by my friend M. Terquem, and some other continental mathematicians, and because of the importance of the geometrical and other applications of which it admits, and of the inquiries to which it indirectly gives rise. We shall be concerned in the following discussion with systems of homogeneous rational integral functions of a peculiar form, to which for present purposes I propose to give the name of aggregative functions, consisting of ordinary homogeneous functions of the same variables but of different degrees, brought together into one sum made homogeneous by means of powers of new variables entering factorially.

Thus if F, G, $H \ldots L$ be any number of functions of any number of letters $x, y \ldots t$ of the degrees $m, m-\iota, m-\iota' \ldots m-(\iota)$ respectively,

$$F + G\lambda^{\iota} + H\mu^{\iota'} + \ldots + L\theta^{(\iota)}$$

will be an aggregative function of the variables entering into F, G, &c. and of $\lambda, \mu \ldots \theta$. I shall further call such a function binary, ternary, quaternary, and so forth, according to the number of variables contained in the functions F, G, H, &c. thus brought into coalition.

It will be convenient to recall the attention of the reader to the meaning of some of the terms employed by me in the paper above referred to.

If F be any homogeneous function of $x, y, z \ldots t$, the term augmentative of F denotes any function obtained from F of the form

$$x^{\alpha} y^{\beta} z^{\gamma} \ldots t^{\delta} \times F.$$

Again, if we have any number of such functions F, G, $H \ldots K$ of as many variables $x, y, z \dots t$, and we decompose $F, G, H \dots K$ in any manner so as to obtain the equations

$$F = x^a P_1 + y^b P_2 + z^c P_3 + \&c. \dots + t^d (P),$$

$$G = x^a Q_1 + y^b Q_2 + z^c Q_3 + \&c. \dots + t^d (Q),$$

$$H = x^a R_1 + y^b R_2 + z^c R_3 + \&c. \dots + t^d (R),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$K = x^a S_1 + y^b S_2 + z^c S_3 + \&c. \dots + t^d (S),$$

and then form the determinant

$$\begin{vmatrix} P_1, & P_2, & P_3 \dots (P) \\ Q_1, & Q_2, & Q_3 \dots (Q) \\ R_1, & R_2, & R_3 \dots (R) \\ \dots & \dots & \dots \\ S_1, & S_2, & S_3 \dots (S) \end{vmatrix},$$

this determinant, expressed as a function of $x, y, z \dots t$, is what, in the paper referred to, I called a secondary derivee, but which for the future I shall cite by the more concise and expressive name of a *connective* of the system of functions $F, G, H \dots K$ from which it is obtained. One prevailing principle regulates all the cases treated of in this and the antecedent memoir, namely that of forming linearly independent systems of augmentatives or connectives, or both, of the given system whose resultant is to be found, of the same degree one with the other, and equal in number (when this admits of being done) to the number of distinct terms in the functions thus formed. The resultant of these functions, treated as linear functions of the several combinations of powers of the variables in each term, will then be the resultant of the given system clear of all irrelevant factors. If the number of terms to be eliminated exceed the number of the functions, the elimination of course cannot be executed. If the contrary be the case, but the equality is restored by the rejection of a certain number of the equations, the resultant so obtained will vary according to the choice of the equations retained for the purpose of the elimination. The true resultant will not then coincide with any of the resultants so obtained, but will enter as a common factor into them all.

The following simple arithmetical principles will be found applicable and useful for quotation in the sequel:—

(a) The number of terms in a homogeneous function of p letters of the mth degree is

$$\frac{m(m+1)\dots(m+p-1)}{1.2\dots p}.$$

(*b*) The number of augmentatives of the $(m+n)$th degree belonging to a function of p letters of the mth degree is

$$\frac{(n+1)(n+2)\ldots(n+p-1)}{1.2\ldots p}.$$

(*c*) The number of solutions in integers (excluding zeros) of the equation $a_1+a_2+\ldots+a_p=k$ is

$$\frac{(k-1)(k-2)\ldots(k-p+1)}{1.2\ldots(p-1)}.$$

To begin with the case of binary aggregatives. Let

$$\left.\begin{array}{l} F_m(x,y)+F_{m-\iota}(x,y)\lambda^\iota+F_{m-\iota'}(x,y)\mu^{\iota'}+\&\text{c.}\ldots+F_{m-(\iota)}(x,y)\theta^{(\iota)} \\ G_n(x,y)+G_{n-\iota}(x,y)\lambda^\iota+G_{n-\iota'}(x,y)\mu^{\iota'}+\&\text{c.}\ldots+G_{n-(\iota)}(x,y)\theta^{(\iota)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ K_p(x,y)+K_{p-\iota}(x,y)\lambda^\iota+K_{p-\iota'}(x,y)\mu^{\iota'}+\&\text{c.}\ldots+K_{p-(\iota)}(x,y)\theta^{(\iota)} \end{array}\right\} \quad (A)$$

be a system of functions (whose Resultant it is proposed to determine) equal in number to the variables x, y, λ, $\mu\ldots\theta$, and similarly aggregative, that is having only the same powers of λ, μ, &c. entering into them, but of any degrees equal or unequal $m, n\ldots p$. Let the number of the functions be r. Raise each of the given functions by augmentation to the degree s, where

$$s=(m+n+\ldots+p)-(\iota+\iota'+\ldots+(\iota))-1,$$

the number of augmentatives of the several functions will be

$$\begin{array}{l}(s+1)-m,\\ (s+1)-n,\\ \cdots\cdots\cdots\\ (s+1)-p,\end{array}$$

and the total number will therefore be

$$r(s+1)-(m+n+\ldots+p),$$

which $$=(r-1)(m+n+\ldots+p)-r(\iota+\iota'+\ldots+(\iota)).$$

Again, the number of terms to be eliminated will be the sum of the numbers of terms in functions respectively of the sth, $(s-\iota)$th, $(s-\iota')$th, … $(s-(\iota))$th degrees, which are respectively

$$\begin{array}{l}s+1,\\ s+1-\iota,\\ s+1-\iota',\\ \cdots\cdots\cdots\\ s+1-(\iota),\end{array}$$

and the number of these partial functions is $r-1$. Hence the number of terms to be eliminated is

$$(r-1)\{m+n+\&c.+p-(\iota+\iota'+\&c.+(\iota))\}-(\iota+\iota'+\&c.+(\iota))$$
$$=(r-1)(m+n+\&c.+p)-r(\iota+\iota'+\ldots+(\iota)),$$

which is exactly equal to the number of the augmentative functions. Hence the Resultant* of the given functions can be found dialytically by linear elimination, and the exponent of its dimensions in respect to the coefficients of the given functions will be the number

$$(r-1)\Sigma m - r\Sigma\iota,$$

as above found.

The method above given may be replaced by another more compendious, and analogous to that known by the name of Bezout's abridged method for ordinary functions of two letters. As the method is precisely the same whatever the number of the functions employed may be, I shall for the sake of greater simplicity restrict the demonstration to the case of three functions, U, V, W, whose degrees (if unequal, written in ascending order of magnitude) are m, n, p respectively. Let

$$U=F_m(x, y)+F_{m-\iota}(x, y)z^{\iota},$$
$$V=G_n(x, y)+G_{n-\iota}(x, y)z^{\iota},$$
$$W=H_p(x, y)+H_{p-\iota}(x, y)z^{\iota}.$$

Let θ, ω be taken any two numbers which satisfy in integers greater than zero the equation $\theta+\omega=m+1$, and let

$$F_m(x, y)=\phi_{m-\theta}\cdot x^{\theta}+\phi_{m-\omega}\cdot y^{\omega},$$
$$G_n(x, y)=\gamma_{n-\theta}\cdot x^{\theta}+\gamma_{n-\omega}\cdot y^{\omega},$$
$$H_p(x, y)=\eta_{p-\theta}\cdot x^{\theta}+\eta_{p-\omega}\cdot y^{\omega},$$

where the ϕ's, γ's, η's may be always considered rational integer functions of x and y; for every term in each of the functions F_m, G_n, H_p must either contain x^{θ} or y^{ω}, since, if not, its dimensions in x and y would not exceed

$$(\theta-1)+(\omega-1),$$

that is $m-1$, whereas each term is of m conjoined dimensions, at least, in x and y. Hence from the equations

$$U=0,$$
$$V=0,$$
$$W=0,$$

* The Resultant of a system of functions means in general the same thing as the left-hand side of the final equation (clear of extraneous factors) resulting from the elimination of the variables between the equations formed by equating the said functions severally to zero.

by eliminating x^ω, y^θ and z^ι we obtain the connective determinant

$$\begin{vmatrix} \phi_{m-\theta}, & \phi_{m-\omega}, & F_{m-\iota} \\ \gamma_{n-\theta}, & \gamma_{n-\omega}, & G_{n-\iota} \\ \eta_{p-\theta}, & \eta_{p-\omega}, & H_{p-\iota} \end{vmatrix},$$

which will be of the degree

$$m+n+p-(\theta+\omega+\iota),$$

that is of the degree $(n+p-\iota-1)$ in x and y; and the number of such connectives by principle (c) is p.

Again, by augmentation we can raise each of the functions U, V, W to the same degree as the connectives, and by principle (b) the number of such will be

$$n+p-m-\iota,$$
$$p-\iota,$$
$$n-\iota,$$

from U, V, W respectively, together making up the number

$$2n+2p-m-3\iota.$$

Hence in all we have $2n+2p-3\iota$ equations; and the number of terms to be eliminated will be, $n+p-\iota$ arising from F_m, G_n, H_p, and $n+p-2\iota$ from $F_{m-\iota}$, $G_{n-\iota}$, $H_{p-\iota}$; together making up the proper number $2n+2p-3\iota$.

Each connective contains ternary combinations of the coefficients, namely one of the coefficients belonging to that part of U, V, W which contains z^ι, and two coefficients from the other part: the dimensions of the resultant in respect of the coefficients of the former will hence be readily seen to be equal to the number of connectives + the number of terms in the augmentatives into which z^ι enters, that is, will equal $m+n+p-2\iota$; the total dimensions of the resultant in respect to all the coefficients of U, V, W will be

$$3m+(2n+2p-m-3\iota),$$

that is, $$2m+2n+2p-3\iota;$$

and consequently, in respect to the coefficients of F_m; G_n; H_p, will be of

$$(2m+2n+2p-3\iota)-(m+n+p-2\iota),$$

that is, of $m+n+p-\iota$ dimensions. This result, which is of considerable importance, may be generalized as follows.

Returning to the general system (A), for which we have proved that the total dimensions of the resultant are

$$(r-1)(m+n+\ldots+p)-r(\iota+\iota'+\ldots+(\iota)),$$

let the coefficients of the column of partial functions

$$\begin{array}{c} F_m, \\ G_n, \\ \vdots \\ K_p, \end{array}$$

be called the first set; the coefficients of the column

$$\begin{array}{c} F_{m-\iota}, \\ G_{n-\iota}, \\ \vdots \\ K_{p-\iota}, \end{array}$$

the second set, and so forth; then the dimensions in respect of the 1st, 2nd ... $(r-1)$th sets respectively are s, $s-\iota$, $s-\iota'$ ... $s-(\iota)$, where

$$s = m + n + \&c. + p - (\iota + \iota' + \&c. + (\iota)).$$

The important observation remains to be made, that all the above results remain good although any one or more of the indices of dimension of the partial functions in the system (A), as $m-\iota$, $m-\iota'$, $n-\iota$, &c., should become negative, provided that the terms in which such negative indices occur be taken zero, as will be apparent on reviewing the processes already indicated upon this supposition. If we take

$$m = n = \ldots = p, \text{ and } \iota = \iota' = \&c. = (\iota) = m - \epsilon,$$

the exponent of the total dimensions of the resultant becomes

$$\begin{aligned} &(r-1)\,rm - r\,(r-2)\,(m-\epsilon) \\ &= rm + r\,(r-2)\,\epsilon, \end{aligned}$$

when $\epsilon = 0$, this becomes mr, which is made up of $2m$ units of dimension belonging to the coefficients of the first column, and of m belonging to each of the $(r-2)$ remaining columns. Consequently, if we have

$$\begin{aligned} F_m(x,\ y) + \xi\lambda + \xi'\lambda' &= 0, \\ G_m(x,\ y) + \eta\lambda + \eta'\lambda' &= 0, \\ H_m(x,\ y) + \zeta\lambda + \zeta'\lambda' &= 0, \\ K_m(x,\ y) + \theta\lambda + \theta'\lambda' &= 0, \end{aligned}$$

or any other number of equations similarly formed, the result of the elimination is always of m dimensions only in respect of ξ, η, ζ, θ, or of ξ', η', ζ', θ', and of $2m$ in respect of the coefficients in F, G, H, K.

I now proceed to state and to explain some seeming paradoxes connected with the degree of the resultant of such systems of defective functions as have been previously treated of in this memoir, as compared with the degree of the general resultant of a corresponding system of *complete* functions of the same number of variables.

In order to fix our ideas, let us take a system of only three equations of the form

$$\left.\begin{array}{l} F_m(x,\ y)+F_{m-\iota}(x,\ y)\,z^\iota=0 \\ G_n(x,\ y)+\ G_{n-\iota}(x,\ y)\,z^\iota=0 \\ H_p(x,\ y)+H_{p-\iota}(x,\ y)\,z^\iota=0 \end{array}\right\} . \qquad \text{(B)}$$

The resultant of this system found by the preceding method is in all of $2m+2n+2p-3\iota$ dimensions. But in general, the resultant of three equations of the degrees m, n, p is of $mn+mp+np$ dimensions.

Now in order to reason firmly and validly upon the doctrine of elimination, nothing is so necessary as to have a clear and precise notion, never to be let go from the mind's grasp, of the proposition that every system of n homogeneous functions of n variables has a single and invariable Resultant. The meaning of this proposition is, that a function of the coefficients of the given functions can be found, such that, *whenever* it becomes zero, and *never except* when it becomes zero, the functions may be simultaneously made zero for some certain system of ratios between the variables. The function so found, which is sufficient and necessary to condition the possibility of the coexistence of the equality to zero of each of the given functions, is their resultant, and by analogy they may be termed its components. It follows that if R be a resultant of a given system of functions, any numerical multiple of any power of R or of any root of R when (upon certain relations being supposed to be instituted between the coefficients of its components) R breaks up into equal factors, will also be a resultant. This is just what happens in system (B) when $m=n=p=\iota$; the resultant found by the method in the text is of the degree $3m$; the general resultant of the system of three equations to which it belongs is of the degree $3m^2$; the fact being, that the latter resultant becomes a perfect mth power for the particular values of the coefficients which cause its components to take the form of the functions in system (B).

Suppose, however, that we have still $m=n=p$, but ι less than m, $6m-3\iota$ will express the degree of the resultant of system (B); but this is no longer in general an aliquot part of $3m^2$, and consequently the resultant of system (B) that we have found is no longer capable in general of being a root of the general resultant. The truth is, that on this supposition the general resultant is zero; as it evidently should be, because the values $\frac{x}{z}=0$, $\frac{y}{z}=0$ satisfy the equations in system (B), except for the case of $m=\iota$; consequently the resultant furnished in the text, although found by the same process, is something of a different nature from an ordinary resultant; it expresses, not that the system of equations (B) may be capable of coexisting, but that they may be capable of coexisting for values of $\frac{x}{z}, \frac{y}{z}$ other than 0 and 0. This is what I have elsewhere termed a sub-resultant. But there is yet a further case, to which neither of the above considerations will apply. This is when m, n, p are not equal, but $p - \iota = 0$.

On this supposition the degree of the resultant of (B) becomes $2m+2n-p$, which in general will not be a factor of $mn + mp + np$; and in this case it will no longer be true that the values $\frac{x}{z} = 0$, $\frac{y}{z} = 0$ will satisfy the system (B), inasmuch as the last equation therein cannot so be satisfied. Now, calling the general resultant R and the particular resultant R', if R' should break up into factors so as to become equal to $(r')^a \times (s')^b \ldots (t')^c$, it might be the case that R should equal $(r')^\alpha . (s')^\beta \ldots (t')^\gamma$, and there would be nothing in this fact which would be inconsistent with the theory of the resultant as above set forth; but suppose that R' is indecomposable into factors, then it is evident that we must have $R = R' . R''$, and consequently that the existence of such a particular resultant as R' will argue the necessity of the existence of another resultant R''; in other words, the resultant so found cannot be in a strict sense the true and complete resultant for the particular case assumed, and yet the process employed appears to give the complete resultant, or at least it is difficult to see how the wanting factor escapes detection. To make this matter more clear, take a particular and a very simple case, where $m = 2$, $n = 2$, $p = \iota = 1$, so as to form the system of equations

$$\left.\begin{aligned} Ax^2 + Bxy + Cy^2 + (Dx + Ey)\,z &= 0 \\ A'x^2 + B'xy + C'y^2 + (D'x + E'y)\,z &= 0 \\ lx + my + nz &= 0 \end{aligned}\right\}. \qquad \text{(C)}$$

By virtue of my theorem, the degree of the resultant R' is

$$2\,(2 + 2 + 1) - 3 \,.\, 1 = 7,$$

but the resultant R of the system

$$\left.\begin{aligned} Ax^2 + Bxy + Cy^2 + (Dx + Ey)\,z + Fz^2 &= 0 \\ A'x^2 + B'xy + C'y^2 + (D'x + E'y)\,z + F'z^2 &= 0 \\ lx + my + nz &= 0 \end{aligned}\right\}, \qquad \text{(D)}$$

which becomes identical with the former when $F = 0$, $F' = 0$ is of

$$2 \times 2 + 2 \times 1 + 2 \times 1,$$

that is, of 8 dimensions. Hence it is evident that when $F = 0$, $F' = 0$, R must become $R' \times R''$.

It will be found in fact*, that on the supposition of $F=0, F'=0$, R becomes equal to $N \times R'$; and accordingly, besides the portion R' of the resultant of system (C), found by the method in the text, there is another portion N which has dropped through; but it may be asked, is N truly a relevant factor? were it not so, the theory of the resultant would be completely invalidated; but in truth *it is*; for $N=0$ will make the equations in system (C), *considered as a particular case* of system (D), capable of co-existing; the peculiarity, which at first sight prevents this from being obvious, consisting in the fact that the values of $\frac{x}{z}, \frac{y}{z}$ which satisfy the three equations when $N=0$ become *infinite*.

Thus, finally, we have arrived at a clear and complete view of the relation of the particular to the general resultant.

The general resultant may be zero, in which case the particular resultant is something altogether different from an ordinary resultant; or the particular resultant may be a root of the general resultant, or it may be more generally the product of powers of the simple factors, which enter into the composition of the general resultant; or lastly, it may be an incomplete resultant, the factors wanting to make it complete being such as when equated to zero, will enable the components of the resultant to coexist, but not for other than infinite values of certain of the ratios existing between the variables.

Without for the present further enlarging on the hitherto unexplored and highly interesting theory of Particular Resultants, I will content myself with stating one beautiful and general theorem relating to them; to wit, "if $F=0$, $G=0$, &c. be a given system of equations with the coefficients left general, and R be the resultant of F, G, &c., and if now the coefficients in F, G be so taken that R comes to contain as a factor or be coincident with R'^m, then will $R'=0$ indicate that (when the coefficients are so taken as above supposed) $F=0$, $G=0$, &c. will be capable of being satisfied, not, as in general, by one only, but by m distinct systems of values of the variables in F, G, &c., subject of course to the possibility, in special cases, of certain of the systems becoming multiple coincident systems."

I pass on now* to the more recondite and interesting theory of the resultant of Ternary Aggregative Functions, that is to say, functions of the form

$$F_m(x, y, z) + F_{m-\iota}(x, y, z)t^\iota + \&c. \ldots + F_{m-(\iota)}(x, y, z)t^{(\iota)},$$

which will be seen to admit of some remarkable applications to the theory of reciprocal polars.

[* See the Author's remarks below, p. 283.]

41.

ON A REMARKABLE DISCOVERY IN THE THEORY OF CANONICAL FORMS AND OF HYPERDETERMINANTS.

[*Philosophical Magazine,* II. (1851), pp. 391—410.]

In a recently printed continuation* of a paper which appeared in the *Cambridge and Dublin Mathematical Journal,* I published a complete solution of the following problem. A homogeneous function of x, y of the degree $2n+1$ being given, required to represent it as the sum of $n+1$ powers of linear functions of x, y. I shall prepare the way for the more remarkable investigations which form the proper object of this paper, by giving a new and more simple solution of this linear transformation.

Let the given function be

$$a_0x^{2n+1} + (2n+1)\,a_1x^{2n}y + \tfrac{1}{2}(2n+1)(2n)\,a_2x^{2n-1}y^2 + \ldots + a_{2n+1}y^{2n+1},$$

and suppose that this is identical with

$$(p_1x+q_1y)^{2n+1} + (p_2x+q_2y)^{2n+1} + \text{\&c.} + (p_{n+1}x+q_{n+1}y)^{2n+1}.$$

The problem is evidently possible and definite, there being $2n+2$ equations to be satisfied, and $(2n+2)$ quantities p_1, q_1, &c. for satisfying the same.

In order to effect the solution, let

$$q_1 = p_1\lambda_1,$$

$$q_2 = p_2\lambda_2,$$

$$\text{\&c.} = \text{\&c.}$$

$$q_{n+1} = p_{n+1}\lambda_{n+1},$$

[* p. 203 above.]

we have then

$$\begin{aligned}
&p_1^{2n+1} + p_2^{2n+1} + \dots + p_{n+1}^{2n+1} = a_0,\\
&p_1^{2n+1}\lambda_1 + p_2^{2n+1}\lambda_2 + \dots + p_{n+1}^{2n+1}\lambda_{n+1} = a_1,\\
&p_1^{2n+1}\lambda_1^2 + p_2^{2n+1}\lambda_2^2 + \dots + p_{n+1}^{2n+1}\lambda^2_{n+1} = a_2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&p_1^{2n+1}\lambda_1^n + p_2^{2n+1}\lambda_2^n + \dots + p_{n+1}^{2n+1}\lambda^n_{n+1} = a_n,\\
&p_1^{2n+1}\lambda_1^{n+1} + p_2^{2n+1}\lambda_2^{n+1} + \dots + p_{n+1}^{2n+1}\lambda_{n+1}^{n+1} = a_{n+1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&p_1^{2n+1}\lambda_1^{2n+1} + p_2^{2n+1}\lambda_2^{2n+1} + \dots + p_{n+1}^{2n+1}\lambda_{n+1}^{2n+1} = a_{2n+1}.
\end{aligned}$$

Eliminate $p_1, p_2 \dots p_{n+1}$ between the 1st, 2nd, 3rd ... $(n+2)$th equations, and it is easily seen that we obtain

$$a_{n+1} - a_n \Sigma\lambda_1 + a_{n-1}\Sigma\lambda_1\lambda_2 \dots \pm a_0\lambda_1\lambda_2 \dots \lambda_{n+1} = 0.$$

Again, eliminating in like manner $p_1^{2n+1}\lambda_1$, $p_2^{2n+1}\lambda_2 \dots p_{n+1}^{2n+1}\lambda_{n+1}$ between the 2nd, 3rd ... $(n+3)$th equations, we obtain

$$a_{n+2} - a_{n+1}\Sigma\lambda_1 + \dots \mp a_1\lambda_1\lambda_2 \dots \lambda_{n+1} = 0;$$

and proceeding in the same way until we come to the combination of the $(n+1)$th ... $(2n+2)$th equations, and writing

$$\begin{aligned}
\Sigma\lambda_1 &= s_1,\\
\Sigma\lambda_1\lambda_2 &= s_2,\\
&\dots\dots\dots\\
\lambda_1\lambda_2 \dots \lambda_{n+1} &= s_{n+1},
\end{aligned}$$

we find

$$\begin{aligned}
&a_{n+1} - a_n s_1 + a_{n-1} s_2 \dots \pm a_0 s_{n+1} = 0,\\
&a_{n+2} - a_{n+1} s_1 + a_n s_2 \dots \mp a_1 s_{n+1} = 0,\\
&a_{n+3} - a_{n+2} s_1 + a_{n+1} s_2 \dots \pm a_2 s_{n+1} = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_{2n+1} - a_{2n} s_1 + a_{2n-1} s_2 \dots + a_n s_{n+1} = 0\,*.
\end{aligned}$$

Hence it is obvious that

$$(x + \lambda_1 y)(x + \lambda_2 y) \dots (x + \lambda_{n+1} y)$$

is a constant multiple of the determinant

$$\begin{vmatrix}
x^{n+1}, & -x^n y, & x^{n-1}y^2 & \dots & \pm y^{n+1}\\
a_{n+1}, & a_n, & a_{n-1} & \dots & a_0\\
a_{n+2}, & a_{n+1}, & a_n & \dots & a_1\\
\dots & \dots & \dots & \dots & \dots\\
a_{2n+1}, & a_{2n}, & a_{2n-1} & \dots & a_n
\end{vmatrix}.$$

* These equations in their simplified form arise from the ordinary result of elimination, in this case containing as a factor the product of the differences of the quantities $\lambda_1, \lambda_2, \dots \lambda_{n+1}$.

Hence $\lambda_1, \lambda_2 \ldots \lambda_{n+1}$ are known, and consequently

$$p_1, \; p_2 \ldots p_{n+1}, \; q_1, \; q_2 \ldots q_{n+1}$$

are known, by the solution of an equation of the $(n+1)$th degree.

Thus suppose the given function to be

$$F = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + 10fy^5$$
$$= (p_1x + q_1y)^5 + (p_2x + q_2y)^5 + (p_3x + q_3y)^5,$$

we shall have, by an easy inference from what has preceded,

$$(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$$

equal to a numerical multiple of the determinant

$$\begin{vmatrix} x^3, & -x^2y, & xy^2, & -y^3 \\ d, & c, & b, & a \\ e, & d, & c, & b \\ f, & e, & d, & c \end{vmatrix}.$$

The solution of the problem given by me in the paper before alluded to presents itself under an *apparently* different and rather less simple form. Thus, in the case in question, we shall find according to that solution,

$$(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$$

equal to a numerical multiple of the determinant

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy \\ bx + cy, & cx + dy, & dx + ey \\ cx + dy, & dx + ey, & ex + fy \end{vmatrix}.$$

The two determinants, however, are in fact identical, as is easily verified, for the coefficients of x^3 and y^3 are manifestly alike; and the coefficient of x^2y in the second form will be made up of the three determinants,

$$\begin{vmatrix} a, & b, & d \\ b, & c, & e \\ c, & d, & f \end{vmatrix}, \quad \begin{vmatrix} a, & c, & c \\ b, & d, & d \\ c, & e, & e \end{vmatrix}, \quad \begin{vmatrix} b, & b, & c \\ c, & c, & d \\ d, & d, & e \end{vmatrix},$$

of which the latter two vanish, and the first is identical with the coefficient of x^2y in the first solution. The same thing is obviously true in regard of the coefficients of xy^2 in the two forms, and a like method may be applied to show that in all cases the determinant above given is identical with the determinant of my former paper, namely

$$\begin{vmatrix} a_0x + a_1y, & a_1x + a_2y & \ldots & a_nx + a_{n+1}y \\ a_1x + a_2y, & a_2x + a_3y & \ldots & a_{n+1}x + a_{n+2}y \\ \cdots & \cdots & \cdots & \cdots \\ a_nx + a_{n+1}y, & a_{n+1}x + a_{n+2}y & \ldots & a_{2n}x + a_{2n+1}y \end{vmatrix}.$$

Thus, then, we see that for odd-degreed functions, the reduction to their canonical form of the sum of $(n+1)$ powers depends upon the solution of one single equation of the $(n+1)$th degree, and can never be effected in more than one way.

This new form of the resolving determinant affords a beautiful criterion for a function of x, y of the degree $2n+1$ being composed of n instead of, as in general, $(n+1)$ powers. In order that this may be the case, it is obvious that two conditions must be satisfied; but I pointed out in my supplemental paper on canonical forms, that all the coefficients of the resolving determinant must vanish, which appears to give far too many conditions. Thus, suppose we have

$$ax^7 + 7bx^6y + 21cx^5y^2 + 35dx^4y^3 + 35ex^3y^4 + 21fx^2y^5 + 7gxy^6 + hy^7.$$

The conditions of catalecticism, that is, of its being expressible under the form of the sum of three (instead of, as in general, four) seventh powers, requires that all the coefficients of the different powers of x and y must vanish in the determinant

$$\begin{vmatrix} y^4, & -y^3x, & y^2x^2, & -yx^3, & x^4 \\ a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g \\ d, & e, & f, & g, & h \end{vmatrix};$$

in other words, we must have five determinants,

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}, \quad \begin{vmatrix} a, & c, & d, & e \\ b, & d, & e, & f \\ c, & e, & f, & g \\ d, & f, & g, & h \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c, & e \\ b, & c, & d, & f \\ c, & d, & e, & g \\ d, & e, & f, & h \end{vmatrix},$$

$$\begin{vmatrix} a, & b, & d, & e \\ b, & c, & e, & f \\ c, & d, & f, & g \\ d, & e, & g, & h \end{vmatrix}, \quad \begin{vmatrix} b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \\ e, & f, & g, & h \end{vmatrix},$$

all separately zero. But by my homaloidal law*, all these five equations amount only to $(5-4)(5-3)$, that is, to 2. I may notice here, that a theorem substantially identical with this law, and another absolutely identical with the theorem of compound determinants given by me in this *Magazine*, and afterwards generalized in a paper also published† in this *Magazine*, entitled

[* p. 150 above.] [† p. 241 above.]

"On the Relations between the Minor Determinants of Linearly Equivalent Quadratic Forms," have been subsequently published as original in a recent number of M. Liouville's journal.

The general condition of mere singularity, as distinguished from catalecticism, that is, of the function of the degree $2n+1$, being incapable of being expressed as the sum of $n+1$ powers, is that the resolving resultant shall have two equal roots; in other words, that its determinant shall be zero.

Mr Cayley has pointed out to me a very elegant mode of identifying the two forms of the resolving resultant, which I have much pleasure in subjoining. Take as the example a function of the fifth degree, we have by the multiplication of determinants,

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ x, & y, & 0, & 0 \\ 0, & x, & y, & 0 \\ 0, & 0, & x, & y \end{vmatrix}$$

$$= \begin{vmatrix} y^3, & a, & b, & c \\ 0, & ax+by, & bx+cy, & cx+dy \\ 0, & bx+cy, & cx+dy, & dx+ey \\ 0, & cx+dy, & dx+ey, & ex+fy \end{vmatrix},$$

which dividing out each side of the equation by y^3, immediately gives the identity required, and the method is obviously general.

Turn we now to consider the mode of reducing a biquadratic function of two letters to its canonical form, *videlicet*

$$(fx+gy)^4+(hx+ky)^4+6m\,(fx+gy)^2(hx+ky)^2.$$

Let the given function be written

$$ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4.$$

Let $\qquad g=f\lambda_1,\ \ k=h\lambda_2,\ \ mf^2h^2=\mu,\ \ \lambda_1+\lambda_2=s_1,\ \ \lambda_1\lambda_2=s_2,$

then we have

$$\begin{aligned} &f^4+h^4+6\mu=a,\\ &4f^4\lambda_1+4h^4\lambda_2+6\mu\,(2s_1)=4b,\\ &6f^4\lambda_1^2+6h^4\lambda_2^2+6\mu\,(s_1^2+2s_2)=6c,\\ &4f^4\lambda_1^3+4h^4\lambda_2^3+6\mu\,(2s_1s_2)=4d,\\ &f^4\lambda_1^4+h^4\lambda_2^4+6\mu s_2^2=e. \end{aligned}$$

Eliminating f and h between the first, second and third; the second, third and fourth; and the third, fourth and fifth equations successively, we obtain

$$as_2 - bs_1 + c - \mu(8s_2 - 2s_1^2) = 0,$$
$$bs_2 - cs_1 + d - \mu(4s_1s_2 - s_1^3) = 0,$$
$$cs_2 - ds_1 + e - \mu(8s_2^2 - 2s_1^2s_2) = 0.$$

Let now $$(2s_1^2 - 8s_2)\mu = \nu,$$

and we shall have $$as_2 - bs_1 + (c+\nu) = 0,$$
$$bs_2 - \left(c - \frac{\nu}{2}\right)s_1 + d = 0,$$
$$(c+\nu)s_2 - ds_1 + e = 0.$$

Hence ν will be found from the cubic equation

$$\begin{vmatrix} a, & b, & c+\nu \\ 2b, & 2c-\nu, & 2d \\ c+\nu, & d, & e \end{vmatrix} = 0,$$

that is, $$\nu^3 - \nu(ae - 4bd + 3c^2) + \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0,$$

in which equation it will not fail to be noticed that the coefficient of ν^2 is zero, and the remaining coefficients are the two well-known hyperdeterminants, or, as I propose henceforth to call them, the two Invariants of the form

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4;$$

be it also further remarked that

$$\nu = 8(\tfrac{1}{4}s_1^2 - s_2)\mu,$$

in which equation the coefficient of 8μ is the Determinant or Invariant of

$$x^2 + s_1xy + s_2y^2.$$

When ν is thus found, s_1, s_2, and μ, being given by the equations in terms of ν, are known, and by the solution of a quadratic λ_1, λ_2 become known in terms of s_1, s_2, and f, h in terms of λ_1, λ_2, μ, and the problem is completely determined. The most symmetrical mode of stating this method of solution is to suppose the given function thrown under the form

$$(fx+gy)^4 + (f_1x+g_1y)^4 + 6\epsilon(fx+gy)^2(f_1x+g_1y)^2.$$

Then writing

$$(fx+gy)(f_1x+g_1y) = Lx^2 + Mxy + Ny^2,$$

$-\nu$, the quantity to be found by the solution of the cubic last given, becomes

$$8\epsilon\left(LN-\frac{M^2}{4}\right).$$

I shall now proceed to apply the same method to the reduction of the function

$$a_0x^8+8a_1x^7y+28a_2x^6y^2+56a_3x^5y^3+70a_4x^4y^4+56a_5x^3y^5$$
$$+28a_6x^2y^6+8a_7xy^7+a_8y^8,$$

under the form of

$$(p_1x+q_1y)^8+(p_2x+q_2y)^8+(p_3x+q_3y)^8+(p_4x+q_4y)^8$$
$$+70\epsilon(p_1x+q_1y)^2(p_2x+q_2y)^2(p_3x+q_3y)^2(p_4x+q_4y)^2.$$

It will be convenient to begin, as in the last case, by taking

$$q_1=p_1\lambda_1,\quad q_2=p_2\lambda_2,\quad q_3=p_3\lambda_3,\quad q_4=p_4\lambda_4,$$
$$\epsilon p_1^2p_2^2p_3^2p_4^2=m,$$

and

$$(x+\lambda_1y)(x+\lambda_2y)(x+\lambda_3y)(x+\lambda_4y)=x^4+s_1x^3y+s_2x^2y^2+s_3xy^3+s_4y^4=U,$$

we shall then have nine equations for determining the nine unknown quantities of the general form

$$p_1^8\lambda_1^\iota+p_2^8\lambda_2^\iota+p_3^8\lambda_3^\iota+p_4^8\lambda_4^\iota+M_\iota m=a_\iota,$$

where ι has all values from 0 to 8 inclusive, and where

$$M_\iota=70.\frac{1.2\ldots\iota.1.2\ldots(8-\iota)}{1.2\ldots8}$$

multiplied into the coefficient of $y^\iota x^{8-\iota}$ in U^2.

Taking these nine equations in consecutive fives, beginning with the first, second, third, fourth, fifth, and ending with the fifth, sixth, seventh, eighth, ninth, we obtain the five equations following:—

$$a_0s_4-a_1s_3+a_2s_2-a_3s_1+a_4s_0-mN_1=0,$$
$$a_1s_4-a_2s_3+a_3s_2-a_4s_1+a_5s_0-mN_2=0,$$
$$a_2s_4-a_3s_3+a_4s_2-a_5s_1+a_6s_0-mN_3=0,$$
$$a_3s_4-a_4s_3+a_5s_2-a_6s_1+a_7s_0-mN_4=0,$$
$$a_4s_4-a_5s_3+a_6s_2-a_7s_1+a_8s_0-mN_5=0,$$

where

$$N_1=M_0s_4-M_1s_3+M_2s_2-M_3s_1+M_4,$$
$$N_2=M_1s_4-M_2s_3+M_3s_2-M_4s_1+M_5,$$
$$N_3=M_2s_4-M_3s_3+M_4s_2-M_5s_1+M_6,$$
$$N_4=M_3s_4-M_4s_3+M_5s_2-M_6s_1+M_7,$$
$$N_5=M_4s_4-M_5s_3+M_6s_2-M_7s_1+M_8.$$

Developing now U^2, we have

$$M_0 = 70, \quad M_1 = \frac{35}{2} s_1, \quad M_2 = 5s_2 + \frac{5}{2} s_1^2, \quad M_3 = \frac{5}{2} s_3 + \frac{5}{2} s_1 s_2,$$

$$M_4 = 2s_4 + 2s_1 s_3 + s_2^2, \quad M_5 = \frac{5}{2} s_1 s_4 + \frac{5}{2} s_2 s_3, \quad M_6 = 5 s_2 s_4 + \frac{5}{2} s_3^2,$$

$$M_7 = \frac{35}{2} s_3 s_4, \quad M_8 = 70 s_4^2.$$

Hence

$$\begin{aligned} N_1 &= 72 s_4 - 18 s_1 s_3 + 6 s_2^2, \\ N_2 &= 18 s_1 s_4 - \frac{9}{2} s_1^2 s_3 + \frac{3}{2} s_1 s_2^2, \\ N_3 &= 12 s_2 s_4 - 3 s_1 s_2 s_3 + s_2^3, \\ N_4 &= 18 s_3 s_4 - \frac{9}{2} s_1 s_3^2 + \frac{3}{2} s_2^2 s_3, \\ N_5 &= 72 s_4^2 - 18 s_1 s_3 s_4 + 6 s_2^2 s_4. \end{aligned}$$

Hence we have

$$N_1 = 72I, \quad N_2 = 72I \frac{s_1}{4}, \quad N_3 = 72I \frac{s_2}{6}, \quad N_4 = 72I \frac{s_3}{4}, \quad N_5 = 72 I s_4,$$

where it will be observed that I is the quadratic invariant of U.

Making now

$$72mI = \nu,$$

we shall have the five following equations:—

$$\begin{aligned} a_0 s_4 - a_1 s_3 + a_2 s_2 - a_3 s_1 + (a_4 - \nu) &= 0, \\ a_1 s_4 - a_2 s_3 + a_3 s_2 - \left(a_4 + \frac{\nu}{4}\right) s_1 + a_5 &= 0, \\ a_2 s_4 - a_3 s_3 + \left(a_4 - \frac{\nu}{6}\right) s_2 - a_5 s_1 + a_6 &= 0, \\ a_3 s_4 - \left(a_4 + \frac{\nu}{4}\right) s_3 + a_5 s_2 - a_6 s_1 + a_7 &= 0, \\ (a_4 - \nu) s_4 + a_5 s_3 - a_6 s_2 - a_7 s_1 + a_8 &= 0; \end{aligned}$$

so that the problem reduces itself to finding ν, which is found from the equation of the fifth degree:—

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3, & a_4 - \nu \\ a_1, & a_2, & a_3, & a_4 + \frac{\nu}{4}, & a_5 \\ a_2, & a_3, & a_4 - \frac{\nu}{6}, & a_5, & a_6 \\ a_3, & a_4 + \frac{\nu}{4}, & a_5, & a_6, & a_7 \\ a_4 - \nu, & a_5, & a_6, & a_7, & a_8 \end{vmatrix} = 0,$$

ν, it will be observed, being 72 times the quadratic invariant of

$$(p_1x+q_1y)(p_2x+q_2y)(p_3x+q_3y)(p_4x+q_4y),$$

the function being supposed to be thrown under the form of

$$\Sigma(p_1x+q_1y)^8+70\epsilon(p_1x+q_1y)^2(p_2x+q_2y)^2(p_3x+q_3y)^2(p_4x+q_4y)^2.$$

It is obvious that in the equation for finding ν, all the coefficients being functions of the invariable quantities p_1, q_1, &c., and ϵ, must be themselves invariants of the given function; so that the determinant last given will present under one point of view four out of the six invariants belonging to a function of the eighth degree, and these four will be of the degrees 2, 3, 4, 5 respectively*.

I shall now proceed to generalize this remarkable law, and to demonstrate the existence and mode of finding $2n$ consecutively-degreed independent invariants of any homogeneous function of the degree $4n$, and of $n+1$ consecutively-even-degreed independent invariants of any homogeneous function of the degree $4n+2$; a result, whether we look to the fact of such invariants existing, or to the simplicity of the formula for obtaining them, equally unexpected and important, and tending to clear up some of the most obscure, and at the same time interesting points in this great theory of algebraical transformations.

In the first place, let me recall to my readers in the simplest form what is meant by an invariant† of a homogeneous function, say of two variables x and y. If the coefficients of the function $f(x, y)$ be called $a, b, c \ldots l$, and if when for x we put $lx+my$, and for y, $nx+py$, where $lp-mn=1$, the coefficients of the corresponding terms become $a', b' \ldots l'$; and if

$$I(a, b \ldots l)=I(a', b' \ldots l'),$$

then I is defined to be an invariant of f.

Let now $f(x, y)$ be a homogeneous function in x, y of the 2ιth degree, and write

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^\iota f(x, y)+\lambda(\eta x-\xi y)^\iota=P,$$

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^\iota f(lx+my, nx+py)+\lambda(\eta x-\xi y)^\iota=P',$$

where ξ and η are independent of x, y, and $lp-mn=1$.

Let $$x'=lx+my,$$

$$y'=nx+py,$$

then $$\xi\frac{d}{dx}+\eta\frac{d}{dy}=\xi\frac{dx'}{dx}\frac{d}{dx'}+\xi\frac{dy'}{dx}\frac{d}{dy'}+\eta\frac{dx'}{dy}\frac{d}{dx'}+\eta\frac{dy'}{dy}\frac{d}{dy'},$$

* The reasoning in this paragraph seems of doubtful conclusiveness. It may be accepted, however, as a fact of observation confirmed and generalized by the subsequent theorem, that the coefficients are invariants.

† *Olim*, Hyperdeterminant, Constant derivative.

and if we now write

$$l\xi + m\eta = \xi',$$
$$n\xi + p\eta = \eta',$$

we find
$$\xi \frac{d}{dx} + \eta \frac{d}{dy} = \xi' \frac{d}{dx'} + \eta' \frac{d}{dy'}.$$

Again, from the equations between x', y', x, y, we find

$$x = \frac{px' - my'}{pl - mn} = px' - my',$$
$$y = \frac{ly' - nx'}{pl - mn} = ly' - nx';$$

therefore
$$\eta x - \xi y = (p\eta + n\xi)\, x' - (m\eta + l\xi)\, y' = \eta' x' - \xi' y'.$$

Hence
$$P' = \left(\xi' \frac{d}{dx'} + \eta' \frac{d}{dy'}\right)^{\iota} f(x', y') + \lambda\, (\eta' x' - \xi' y')^{\iota}.$$

Again,
$$\frac{d}{d\xi} = l \frac{d}{d\xi'} + n \frac{d}{d\eta'},$$
$$\frac{d}{d\eta} = m \frac{d}{d\xi'} + p \frac{d}{d\eta'}.$$

Hence

$$\left(\frac{d}{d\xi}\right)^{\iota} P' = l^{\iota} \left(\frac{d}{d\xi'}\right)^{\iota} P' + \iota l^{\iota-1} n \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' + \&\text{c.} + n^{\iota} \left(\frac{d}{d\eta'}\right)^{\iota} P',$$

$$\left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' = l^{\iota-1} m \left(\frac{d}{d\xi'}\right)^{\iota} P' + \{l^{\iota-1} p + (\iota - 1)\, l^{\iota-2} mn\} \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' + \&\text{c.}$$
$$+ n^{\iota-1} p \left(\frac{d}{d\eta'}\right)^{\iota} P',$$

....

$$\left(\frac{d}{d\eta}\right)^{\iota} P' = m^{\iota} \left(\frac{d}{d\xi'}\right)^{\iota} P' + \iota m^{\iota-1} p \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P + \&\text{c.} + p^{\iota} \left(\frac{d}{d\eta'}\right)^{\iota} P'.$$

But P' being of ι dimensions in ξ' and η', and also in x and y, each of the equations above written will be of ι dimensions in x and y, and of no dimensions in ξ', η'; in fact, the successive terms of the right-hand members of the above $\iota + 1$ equations will be multiples of the $(\iota + 1)$ quantities

$$(x')^{\iota},\quad (x')^{\iota-1} y',\quad (x')^{\iota-2} y'^{\iota} \ldots (y')^{\iota}.$$

Consequently a linear resultant may be taken of

$$\left(\frac{d}{d\xi}\right)^{\iota} P',\quad \left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' \ldots \left(\frac{d}{d\eta}\right)^{\iota} P',$$

treating x'^{ι}, $x'^{\iota-1} y' \ldots y'^{\iota}$ as independent, and as quantities to be eliminated; and this, according to a well-known principle of elimination, will prove the linear resultant of the foregoing equations to be equal to the linear resultant of

$$\left(\frac{d}{d\xi'}\right)^{\iota} P', \quad \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' \ldots \left(\frac{d}{d\eta'}\right)^{\iota} P',$$

multiplied by the determinant

$$\left|\begin{array}{llll} l^{\iota}, & \iota n l^{\iota-1}, & \tfrac{1}{2}\iota(\iota-1) n^{\iota} l^{\iota-2}, \ldots, & n^{\iota} \\ l^{\iota-1}m, & l^{\iota-1}p + (\iota-1) mnl^{\iota-2}, & \ldots, & n^{\iota-1}p \\ \cdots & \cdots & \cdots & \cdots \\ m^{\iota}, & \iota m^{\iota-1}p, & \tfrac{1}{2}\iota(\iota-1) m^{\iota} p^{\iota-2}, \ldots, & p^{\iota} \end{array}\right|.$$

This last written determinant may be shown from the method of its formation to be equal to $(lp - mn)^{\frac{\iota(\iota+1)}{2}}$, that is, to unity, because $lp - mn = 1$. Again, since

$$x'^{\iota} = l^{\iota}x^{\iota} + \iota l^{\iota-1} m x^{\iota-1} y + \&\text{c.} + m^{\iota} y^{\iota},$$

$$x'^{\iota-1}y' = l^{\iota-1} n x^{\iota} + (l^{\iota-1} n + (\iota-1) l^{\iota-2} mn) x^{\iota-1} y + \&\text{c.} + m^{\iota-1} p y^{\iota},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y'^{\iota} = n^{\iota} x^{\iota} + \iota n^{\iota-1} p x^{\iota-1} y + \ldots + p^{\iota} y^{\iota},$$

the resultant of $\left(\frac{d}{d\xi}\right)^{\iota} P' \ldots \left(\frac{d}{d\eta}\right)^{\iota} P'$, obtained by treating $x^{\iota}, x^{\iota-1}y \ldots y^{\iota}$ as the eliminables, will be equal to the resultant of the same functions when $x'^{\iota}, x'^{\iota-1}y' \ldots y'^{\iota}$ are taken as the eliminables* multiplied by a power of the determinant

$$\left|\begin{array}{ll} l^{\iota}, & \ldots, m^{\iota} \\ l^{\iota-1}n, & \ldots, m^{\iota-1}p \\ \cdots & \cdots \\ n^{\iota}, & \ldots, p^{\iota} \end{array}\right|,$$

which determinant, like the last, is unity. Thus, then, we have succeeded in showing that the resultant obtained by eliminating $x^{\iota}, x^{\iota-1}y \ldots y^{\iota}$ between

$$\left(\frac{d}{d\xi}\right)^{\iota} P', \quad \left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' \ldots \left(\frac{d}{d\eta}\right)^{\iota} P'$$

is equal to the resultant obtained by eliminating $(x')^{\iota}, x'^{\iota-1}y' \ldots y'^{\iota}$ between

$$\left(\frac{d}{d\xi'}\right)^{\iota} P', \quad \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' \ldots \left(\frac{d}{d\eta'}\right)^{\iota} P';$$

* For the statement of the general principle of the change of the variables of elimination, see my paper in the March Number, 1851, of the *Camb. and Dub. Math. Jour.* [p. 186 above].

or, which is evidently the same thing, the resultant obtained by eliminating $x^\iota, x^{\iota-1}y \ldots y^\iota$ between

$$\left(\frac{d}{d\xi}\right)^\iota P, \quad \left(\frac{d}{d\xi}\right)^{\iota-1}\frac{d}{d\eta}P \ldots \left(\frac{d}{d\eta}\right)^\iota P;$$

that is to say, this last resultant remains absolutely unaltered in value when for x, y we write respectively

$$lx + my,$$
$$nx + py,$$

provided that $lp - mn = 1$.

Hence by definition this resultant is an invariant $f(x, y)$, and λ being arbitrary, all the separate coefficients of the powers of λ in this resultant must also be invariants. I proceed to express this resultant in terms of λ and the coefficients of (x, y). Let $\varpi = 1.2.3 \ldots \iota$ and

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^\iota P = \left(\frac{d}{dx}\right)^\iota f + \lambda(-y)^\iota \qquad = E_1,$$

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^{\iota-1}\frac{d}{d\eta}P = \left(\frac{d}{dx}\right)^{\iota-1}\frac{d}{dy}f + \lambda(-y)^{\iota-1}x \qquad = E_2,$$

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^{\iota-2}\left(\frac{d}{d\eta}\right)^2 P = \left(\frac{d}{dx}\right)^{\iota-2}\left(\frac{d}{dy}\right)^2 f + \lambda(-y)^{\iota-2}x^2 = E_3,$$

....

$$\frac{1}{\varpi}\left(\frac{d}{d\eta}\right)^\iota P = \left(\frac{d}{dy}\right)^\iota f + \lambda x^\iota \qquad = E_{\iota+1};$$

and

$$f(x, y) = a_0 x^{2\iota} + 2\iota a_1 x^{2\iota-1}y + \tfrac{1}{2}(2\iota)(2\iota-1)a_2 x^{2\iota-2}y^2 + \&\text{c.} + a_{2\iota}y^{2\iota}.$$

We find, writing $\sigma\lambda$ for λ, where $\sigma = 2\iota(2\iota-1)\ldots(\iota+1)$,

$$\frac{1}{\sigma}E_1 = a_0x^\iota + \iota a_1 x^{\iota-1}y + \tfrac{1}{2}\iota(\iota-1)a_2x^{\iota-2}y^2 \ldots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_{\iota-2}x^2y^{\iota-2} + \iota a_{\iota-1}xy^{\iota-1} + a_\iota y^\iota + \lambda(-y)^\iota,$$

$$\frac{1}{\sigma}E_2 = a_1x^\iota + \iota a_2x^{\iota-1}y + \tfrac{1}{2}\iota(\iota-1)a_3x^{\iota-2}y^2 \ldots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_{\iota-1}x^2y^{\iota-2} + \iota a_\iota xy^{\iota-1} + a_{\iota+1}y^\iota + \lambda(-y)^{\iota-1}x,$$

$$\frac{1}{\sigma}E_3 = a_2x^\iota + \iota a_3x^{\iota-1}y \ldots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_\iota x^2y^{\iota-2} + \iota a_{\iota+1}xy^{\iota-1} + a_{\iota+2}y^\iota + \lambda(-y)^{\iota-2}x^2,$$

....

$$\frac{1}{\sigma}E_{\iota+1} = a_\iota x^\iota + \&\text{c.} + \lambda x^\iota;$$

accordingly, by eliminating

$$x^\iota, \quad \iota x^{\iota-1}y, \quad \tfrac{1}{2}\iota(\iota-1)x^{\iota-2}y^2 \ldots y^\iota,$$

we obtain as the required resultant*,

$$\begin{vmatrix} a_\iota \pm \lambda, & a_{\iota-1}, & a_{\iota-2}, & \ldots\ a_0 \\ a_{\iota+1}, & a_\iota \mp \dfrac{\lambda}{\iota}, & a_{\iota-1}, & \ldots\ a_1 \\ a_{\iota+2}, & a_{\iota+1}, & a_\iota \pm \dfrac{\lambda}{\frac{1}{2}\iota(\iota-1)}, & \ldots\ a_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_{2\iota}, & a_{2\iota-1}, & \ldots\ldots\ldots, & \ldots\ a_\iota + \lambda \end{vmatrix}.$$

Inasmuch as all the coefficients of λ in this expression are invariants of $f(x, y)$, and there are no invariants of the first order, it is clear that the coefficient of λ^ι must be always zero, which is easily verified.

Again, if ι is odd, the determinant remains unaltered if we write $-\lambda$ for λ; hence when $f(x, y)$ is of the degree $4\epsilon + 2$, all the coefficients of the odd powers of λ disappear. Thus, then, our theorem at once demonstrates that a function of x, y of the degree 4ϵ has 2ϵ invariants of all degrees from 2 up to $2\epsilon+1$ inclusive, and that a function of x, y of the degree $4\epsilon+2$ has $\epsilon+1$ invariants whose degrees correspond to all the even numbers in the series from 2 to $2\epsilon+2$.

But in order that the proposition, as above stated, may be understood in its full import and value, it is necessary to show that these invariants are independent of one another, which is usually a most troublesome and difficult task in inquiries of this description, but which the peculiar form of our grand determinant enables us to accomplish with extraordinary facility. In order to make the spirit of the demonstration more apparent, take the case of a function of the twelfth degree, whose coefficients, divided by the successive binomial numbers 1, 12, $\dfrac{12.11}{2}$, &c. may be called

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ m.$$

* Mr Cayley has made the valuable observation, that λ (given by equating to zero the above determinant) may be defined by means of the equation

$$\left(\frac{d}{dx}\frac{d}{d\eta} - \frac{d}{dy}\frac{d}{d\xi}\right)^\iota \{f(x, y) \times \phi(\xi, \eta)\} = \lambda\phi(x, y),$$

ϕ being itself a certain rational integral form of a function of the ιth degree, the ratio of whose coefficients would be given by virtue of the above equations as functions of λ and the coefficients of $f(x, y)$.

Our grand determinant then takes the form

$$\begin{vmatrix} g+\lambda, & f, & e, & d, & c, & b, & a \\ h, & g-\frac{\lambda}{6}, & f, & e, & d, & c, & b \\ i, & h, & g+\frac{\lambda}{15}, & f, & e, & d, & c \\ j, & i, & h, & g-\frac{\lambda}{20}, & f, & e, & d \\ k, & j, & i, & h, & g+\frac{\lambda}{15}, & f, & e \\ l, & k, & j, & i, & h, & g-\frac{\lambda}{6}, & f \\ m, & l, & k, & j, & i, & h, & g+\lambda \end{vmatrix}.$$

Here it will be observed that

a and m appear only 1 time.		
b and l	...	2 times.
c and k	...	3 ...
d and j	...	4 ...
e and i	...	5 ...
f and h	...	6 ...
g	...	7 ...

Let now the coefficients be called

$$H_2, H_3, H_4, H_5, H_6, H_7,$$

H_2 and H_3 manifestly are independent.

Again, if possible, let $H_4 = pH_2^2$, then a and m would appear twice in H_4, contrary to the rule.

Hence H_4 is independent of H_2, H_3.

For a similar reason H_5 cannot depend on H_2, H_3.

Again, if possible, let

$$H_6 = pH_2^3 + qH_2H_4 + rH_3^2,$$

H_2^3 will contain b^6l^6, which by the rule cannot appear in H_2H_4 or in H_3^2.

Hence $p = 0$.

Also H_4 will contain $b^2l^2 \times$ the coefficient of λ^3 in

$$\left(g+\frac{\lambda}{15}\right)\left(g-\frac{\lambda}{20}\right)\left(g+\frac{\lambda}{15}\right),$$

which is not zero. And H_2 also contains bl; hence H_2H_4 will contain b^3l^3. But H_3 will evidently not contain b^3 or l^3, or b^2l or bl^2, nor can H_6 contain b^3l^3; hence $q=0$. Finally, H_3^2 will contain c^6 and k^6, but H_6 can only contain as to these letters the combination c^3k^3; hence $r=0$.

Consequently H_6 does not depend on H_2, H_4, H_3. As regards H_2, H_3, H_4, H_5, H_6 not vanishing, this may be made at once apparent by making all the letters but g vanish; the H's then become identical with the coefficients of

$$(g+\lambda)^2\left(g-\frac{\lambda}{6}\right)^2\left(g+\frac{\lambda}{15}\right)^2\left(g-\frac{\lambda}{20}\right),$$

none of which are zero except that of λ^6. The same or a similar demonstration may be extended to H_7 and easily generalized; hence, then, this most unexpected and surprising law is fully made out*.

To return to the subject of canonical forms, I have not found the method so signally successful in its application to the 4th and 8th degrees, conduct to the solution of other degrees, such as the 6th, 12th, or 16th, of all of which I have made trial; possibly another canonical form must be substituted to meet the exigency of these cases†; and it may be remarked in general, that if we have a function of the $(2n)$th degree, the canonical form assumed may be taken,

$$\Sigma\,(p_1x+q_1y)^{2n}+V;$$

where V, in lieu of being the squared product of

$$(p_1x+q_1y),\,(p_2x+q_2y),\,\ldots,(p_nx+q_ny),$$

* This demonstration, however, does not extend to show that the coefficients of the powers of λ may not possibly be dependents, that is, explicit functions of one another combined with other invariants not included among their number, or of these latter alone. For example, in the case of the 12th degree, we know by Mr Cayley's law that there must be two invariants of the 4th order. Our determinant gives only one of these. Call the other one K_4; by the above reasoning it is not disproved but that we may have

$$H_6=pH_2^3+qH_2H_4+rH_3^2+sH_2K_4.$$

I believe, however, that the H's may be demonstrated without much difficulty to be primitive or fundamental invariants. The law of Mr Cayley here adverted to admits of being stated in the following terms:—The number of independent invariants of the 4th order belonging to a function of x, y of the nth degree is equal to the number of solutions in integers (not less than zero) of the equation $2x+3y=n-3$. *Vide* his memorable paper (in which several numerical errors occur against which the reader should be cautioned) "On Linear Transformations," vol. I. *Cambridge and Dublin Mathematical Journal*, new series. There is no great difficulty in showing, by aid of the doctrine of symmetrical functions, that there can never be *more* than one quadratic or one cubic invariant, and in what cases there is one or the other, or each, to any given function of two variables. The general law, however, for the number of invariants of any order other than 2, 3, 4 remains to be made out, and is a great desideratum in the theory of linear transformations.

† See the Postscript [p. 283] for a verification of this conjecture.

may be any hyperdeterminant, or (as I shall in future call such functions) covariant of this product, understanding $P(x, y)$ to be a covariant of $f(x, y)$ when $P(lx+my, nx+py)$ stands in precisely the same relation to $f(lx+my, nx+py)$ as $P(x, y)$ to $f(x, y)$, provided only that $lp-mn=1$. For the relation and distinction between covariants and contravariants, see a short article of mine* in the *Cambridge and Dublin Mathematical Journal* for this month. In endeavouring to apply the method of the text to the Sextic Function

$$ax^6+6bx^5y+15cx^4y^2+20dx^3y^3+15ex^2y^4+6fxy^5+gy^6,$$

thrown under the form

$$\Sigma(px+qy)^6+20\epsilon U^2,$$

where

$$U=(p_1x+q_1y)(p_2x+q_2y)(p_3x+q_3y)=s_0x^3+s_1x^2y+s_2xy^2+s_3y^3,$$

I obtain the following equations:

$$as_3-bs_2+cs_1-ds_0=\epsilon(162s_0^2s_3-54s_0s_1s_2+12s_1^3),$$
$$bs_3-cs_2+ds_1-es_0=\epsilon(54s_0s_1s_3+6s_1^2s_2-36s_0s_2^2),$$
$$cs_3-ds_2+es_1-fs_0=\epsilon(-54s_0s_2s_3-6s_1s_2^2+36s_3s_1^2),$$
$$ds_3-es_2+fs_1-gs_0=\epsilon(-162s_0s_3^2+54s_1s_2s_3+12s_2^3).$$

In these equations, if we call the quantities multiplied by ϵ respectively L, M, N, P, we shall find

$$s_3L-\frac{1}{3}s_2M-\frac{1}{3}s_1N+s_0P=0,$$

and

$$s_3L-s_2M-s_1N+s_0P=I;$$

where I denotes the determinant, or, as I shall in future call such function (in order to avoid the obscurity and confusion arising from employing the same word in two different senses), the Discriminant†, which is the biquadratic (and of course sole) invariant of the cubic function

$$s_0x^3+s_1x^2y+s_2xy^2+s_3y^3.$$

The reduction of the function of the fourth degree to its canonical form may be effected very easily by means of the properties of the invariants of

[* p. 200 above.]

† "Discriminant," because it affords the *discrimen* or test for ascertaining whether or not equal factors enter into a function of two variables, or more generally of the existence or otherwise of multiple points in the locus represented or characterized by any algebraical function, the most obvious and first observed species of singularity in such function or locus. Progress in these researches is impossible without the aid of clear expression; and the first condition of a good nomenclature is that different things shall be called by different names. The innovations in mathematical language here and elsewhere (not without high sanction) introduced by the author, have been never adopted except under actual experience of the embarrassment arising from the want of them, and will require no vindication to those who have reached that point where the necessity of some such additions becomes felt.

the canonical form, as I have shown in the *Cambridge and Dublin Mathematical Journal.* Accordingly I have endeavoured to ascertain whether the reduction of the sixth degree might not be effected by a similar method.

If we start with the form $ax^6+by^6+cz^6+90mx^2y^2z^2$, where $x+y+z=0$, which is only another mode of representing the canonical form previously given, we shall find that there are four independent invariants, of the second, fourth, sixth and tenth degrees. Calling these H_2, H_4, H_6, H_{10}, and writing s_1, s_2, s_3 for $a+b+c$, $ab+ac+bc$, abc it will be found, after performing some extremely elaborate computations, that

$$H_2 = s_2 - 270m^2,$$

$$H_4 = 6ms_3 + 45m^2s_2 + 216m^3s_1 + 891m^4,$$

$$\begin{aligned} H_6 = 4s_3^2 &+ 120s_2s_3m - \{684s_2^2 + 432s_1s_3\}\, m^2 \\ &+ (13 \,.\, 27 \,.\, 64s_3 - 64 \,.\, 81s_1s_2)\, m^3 + 8 \,.\, 81 \,.\, 169s_2m^4 \\ &+ 7 \,.\, 128 \,.\, 729s_1m^5 + 16 \,.\, 729 \,.\, 239m^6. \end{aligned}$$

H_{10} is too enormously long to attempt to compute; but we can easily prove its independent existence by making $m=0$, in which case the (determinant, or, to use the new term proposed, the) discriminant of $ax^6+by^6+cz^6$ becomes the product of the twenty-five forms of the expression

$$(ab)^{\frac{1}{5}} + (ac)^{\frac{1}{5}} \,.\, 1^{\frac{1}{5}} + (bc)^{\frac{1}{5}} \,.\, 1^{\frac{1}{5}}\,*.$$

Now in general the value of such a product for $\alpha^{\frac{1}{5}}+\beta^{\frac{1}{5}}.\,1^{\frac{1}{5}}+\gamma^{\frac{1}{5}}.\,1^{\frac{1}{5}}$ is obviously of the form

$$(\alpha+\beta+\gamma)^5 + \alpha\beta\gamma\,\{f(\alpha+\beta+\gamma)^2 + g(\alpha\beta+\alpha\gamma+\beta\gamma)\};$$

for when $\alpha=0$ or $\beta=0$ or $\gamma=0$, the product must become respectively $(\beta+\gamma)^5, (\gamma+\alpha)^5$ and $(\alpha+\beta)^5$. Moreover, without caring to calculate f, g†, it is enough for our present purpose to satisfy ourselves that g cannot be zero, as then the product would have a factor $(\alpha+\beta+\gamma)^2$. Hence, then, on putting

* Such a product in the language of the most modern continental analysis is, I believe, termed a Norm. If we suppose the general function of x, y of the 4th degree thrown under the form $Au^4+Bv^4+Cw^4$, where $u+v+w=0$, and the general function of x, y, z of the 3rd degree thrown under the form $Au^3+Bv^3+Cw^3+D\theta^3$, where $u+v+w+\theta=0$, the theory of norms will afford an instantaneous and, so to speak, intuitive demonstration of the respective related theorems, and the discriminant (*aliter* determinant) of each such function is decomposable into the sum of a square and a cube. Each of these forms is indeterminate, in either case there being but two relations fixed between the coefficients A, B, C; A, B, C, D; and we may easily establish the following singular species of algebraical porism. In the first case

$$(ABC)^2 : (AB+AC+BC)^3,$$

and in the second case

$$(ABCD)^3 : (\Sigma A^2B^2C^2 - 2ABCD\Sigma AB)^2$$

are *invariable ratios.*

† $f=-625$, $g=3125$.

$\alpha = bc$, $\beta = ac$, $\gamma = ab$, we see that the discriminant, when m is 0, will be of the form

$$s_2^5 + f s_2^2 s_3^2 + g s_3^3 s_1.$$

But when m is 0, H_4 vanishes, and there is no term s_1 or s_3 in H_2. Hence evidently the discriminant H_{10} just found cannot be dependent on H_2, H_4, or H_6; nor is it possible to make

$$H_{10} + pH_2^5 + qH_2^2 H_6,$$

that is,

$$(p+1) s_2^5 + f s_2^2 s_3^2 + g s_3^3 s_1$$

a perfect square on account of g not vanishing; so there is no H_5 upon which H_{10} can depend. Hence, admitting, as there seems every reason to do, that the number of invariants of a function of x, y of the degree m is $m-2$, we find that the four invariants in the case of the first degree are respectively of the second, fourth, sixth, and tenth dimensions, a determination in itself, as a step to the completion of the theory of invariants, of no minor importance.

But it seems hopeless by means of these forms to arrive at the desired canonical reduction. The forms, however, of H_2, H_4, H_6 are very remarkable as not rising above the first, first and second degrees respectively in s_1, s_2, s_3. Also H_4 vanishes when $m = 0$ and H_4 has been obtained by putting

$$ax^6 + by^6 + cz^6 + 90mx^2y^2z^2$$

under the form of

$$Ax^6 + 6Bx^5y + 15Cx^4y^2 + 20Dx^3y^3 + 15Ex^2y^4 + 6Fxy^5 + Gy^6,$$

and taking the determinant

$$\begin{vmatrix} A & B & C & D \\ B & C & D & E \\ C & D & E & F \\ D & E & F & G \end{vmatrix}.$$

Consequently *in general* the vanishing of the above-written determinant will express the condition that a function of the sixth degree may be decomposable into three sixth powers. This also is true more generally. If $F(x, y)$ be a function of $2i$ dimensions, the vanishing of the resultant in respect to $x^i, x^{i-1}y \ldots y^i$ (taken dialytically) of

$$\left(\frac{d}{dx}\right)^i F, \ \left(\frac{d}{dx}\right)^{i-1} \frac{d}{dy} F \ldots \left(\frac{d}{dy}\right)^i F$$

will indicate that F admits of being decomposed into i powers of linear functions of x, y*.

In consequence of the greater interest, at least to the author, of the preceding investigations, I have delayed the insertion of the promised continuation of my paper on extensions of the dialytic method, which will

* Such a function so decomposable may be termed meio-catalectic. Meio-catalecticism for even-degreed functions is the analogue of singularity for odd-degreed functions.

appear in a subsequent Number. I take this opportunity of correcting a trifling slip of the pen which occurs towards the end* of the paper alluded to. The values of $\frac{x}{z}$ and $\frac{y}{z}$ become zero, and not infinite, when $N = 0$; and the antepenultimate paragraph should end with the words "an incomplete resultant." The theorem also, in the last paragraph but one, should be stated more distinctly as subject to an important exception as follows.

Whenever the resultant of a system of equations $F = 0$, $G = 0$, &c. contains a factor R'^m, this will indicate that, on making $R' = 0$, the given system of equations will admit of being satisfied by m algebraically distinct systems of values of the variables, except in those cases where there is a singularity in the forms of F, G, &c., taken either separately, or in partial combination with one another. An example will serve to make the meaning of the exception apparent. Let F, G, H denote three quadratic equations in x and y, so that $F = 0$, $G = 0$, $H = 0$ may be conceived as representing three conic sections. Let R be the resultant of F, G, H, and suppose the relations of the coefficients in F, G, H to be such that $R = R'^2$; then $R' = 0$ will imply the existence of one or the other of the three following conditions: namely, either that the three conics have a chord in common, which is the most general inference; or, which is less general, that two of the conics touch one another; or, which is the most special case of all, that one of the conics is a pair of right lines.

So, again, if we have two equations in x, and their resultant contains F^2, this may arise either from one of the functions containing a square factor, or from their being susceptible, on instituting one further condition, namely of $F = 0$, of having a quadratic factor in common between them.

P.S. The conjecture made in the preceding pages has been since confirmed by the discovery of a modification in the canonical form applicable to functions of the sixth degree, which simplifies the theory in a remarkable manner. Assume $f(x, y)$, a function of the sixth degree, as equal to

$$au^6 + bv^6 + cw^6 \pm muvw\,(u - v)\,(v - w)\,(w - u),$$

where u, v, w, linear functions of x and y, satisfy the equation

$$u + v + w = 0;$$

then will the product of uvw be capable of being determined by means of the solution of a quadratic equation, of the square root of whose roots the coefficients of uvw will be known linear functions. Thus by an affected quadratic, a pure quadratic, and a cubic equation, the values of u, v, w may be completely ascertained. The discussion of this theory, and of a general inverse method for assigning the true (in the sense of the most manageable) Canonical Form for functions of any even degree, will form the subject of a subsequent communication.

[* p. 264 above.]

42.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

[*Cambridge and Dublin Mathematical Journal*, VII. (1852), pp. 52—97.]

PART I. GENERATION OF FORMS*.

SECTION I. *On Simple Concomitance.*

THE primary object of the Calculus of Forms is the determination of the properties of Rational Integral Homogeneous Functions or systems of functions: this is effected by means of transformation; but to effect such transformation experience has shown that forms or form-systems must be contemplated not merely as they are in themselves, but with reference to the ensemble of forms capable of being derived from them, and which constitute as it were an unseen atmosphere around them. The first part of this essay will therefore be devoted to the theory of the external relations of forms or form-systems; the second part to the analysis of forms: that is to say, the first part will treat of the Generation and affinities, and the second part of the Reduction and equivalences of forms.

In its most crude and absolute, or, so to speak, archetypal condition a Rational Integral Homogeneous Function may be regarded as a linear function of several distinct and perfectly independent classes of variables.

* It may be well at the outset to give notice to my readers of the exact meaning to be attached to the following terms:

1. The linear-transformations are supposed to be always taken such that the modulus, that is, the determinant of the coefficients of transformation, is unity; or, as it may be phrased, the transformations are uni-modular.

2. The word Determinant is restricted in all cases to signify the alternate function formed in the usual manner from a group of quantities arranged in square order.

3. The word *Discriminant* (typified by the prefix-symbol □) is used to denote the determinant (usually but most perplexingly so called) of a homogeneous function of variables.

4. The resultant of two or more homogeneous functions of as many variables is the left-hand side of the final equation (in its *complete form* and free from extraneous factors) which results from eliminating the variables between the equations obtained by making each of the functions zero.

The first step towards the limitation of this very general but necessary conception consists in imagining the total number of classes to become segregated into groups, and certain correspondences to obtain between the variables of a class in any group with some the variables in each other class of the same group. The investigations in this and the subsequent section will be confined exclusively to the theory of functions where the several classes of variables, if more than one, all belong to a single group, so that the variables in one class have each their respective correspondents in the remaining classes. Such a group may again be conceived to become subdivided into sets each of the same number of variables, and the corresponding variables in the different sets to become absolutely identical. This leads to the conception of a homogeneous function of related classes of variables of various degrees of exponency in respect to the several classes. The relation of the different classes, if containing the same number of variables (in which case the relation may be termed Simple) will be understood to be defined by their being simultaneously subject to similar or contrary operations of linear substitution; so that, for example, if x, y, z; ξ, η, ζ are two such classes, when x, y, z are replaced by $ax+by+cz$, $a'x+b'y+c'z$, $a''x+b''y+c''z$, respectively, ξ, η, ζ will be, according to the *species* of the relation, subject to be at the same time replaced either by $a\xi+b\eta+c\zeta$, $a'\xi+b'\eta+c'\zeta$, $a''\xi+b''\eta+c''\zeta$, or otherwise by $\alpha\xi+\beta\eta+\gamma\zeta$, $\alpha'\xi+\beta'\eta+\gamma'\zeta$, $\alpha''\xi+\beta''\eta+\gamma''\zeta$, where

$$\alpha = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b' & c' \\ 0 & b'' & c'' \end{vmatrix} \qquad \beta = \begin{vmatrix} 0 & 1 & 0 \\ a' & 0 & c' \\ a'' & 0 & c'' \end{vmatrix} \qquad \gamma = \begin{vmatrix} 0 & 0 & 1 \\ a' & b' & 0 \\ a'' & b'' & 0 \end{vmatrix}$$

&c. &c. &c.*

On the former supposition the related classes x, y, z, ξ, η, ζ will be said to be cogredient, and on the latter supposition contragredient †. If now we have one or more functions of classes of variables so related ‡, such function or system of functions may have associated with it a concomitant, also made up of distinct but related classes of variables, such classes being capable of being either greater or fewer in number than the classes of the given function or system of functions.

In the primitive function or system, as also in the concomitant, the related classes may be all of the same species, or some of one and the others of the contrary species. Even if we limit ourselves to the conception of a

* See my paper in the previous number of this *Journal* [p. 199 above.]

† The germ of the notion of contragredience will be found in the immortal *Arithmetic* of the great and venerable Gauss.

‡ The relation here spoken of will be observed to be of a *dynamical* character, not referring to the systems as they are in themselves, but to the movements to which they are simultaneously subject.

primitive function or system of functions with only one class of variables, its concomitant may be composed of various classes of variables, in respect to some of which it will be covariant with, and in respect to the others contravariant to, the primitive function or system*. This is an immense and most important extension of the conception of a concomitant given in my preceding paper in this *Journal*, and will be shown to have the effect of reducing the whole existing theory under subjection to certain simple abstract and universal laws of operation.

The relation of concomitance is purely of form. A being a given form, B is its concomitant, when A' being derived from A by simultaneous substitutions impressed upon the class of variables or upon each of the classes (if there be more than one) in A, and B' from B by corresponding (coincident or contrary) substitutions impressed upon the class or classes of variables in B, B' is capable of being derived from A' after the same law as B from A; or, as it may be otherwise expressed, "functions are concomitant when their correlated linear derivatives are homogeneous in point of form †."

This definition implies that one at least of the forms must be the most general possible of its kind: in a secondary but very important sense, however, functions obtained by impressing particular values or relations upon the quantities entering into the primitive and its associate form, will still be called concomitant. Thus x^3-y^3 will be termed a concomitant to x^3+y^3, not that we can affirm that $(ax+by)^3-(cx+dy)^3$:

that is $\qquad (a^3-c^3)\,x^3+3\,(a^2b-c^2d)\,x^2y+3\,(ab^2-cd^2)\,xy^2+(b^3-d^3)\,y^3,$

treated as a function of x and y, can be derived from $(ax+by)^3+(cx+dy)^3$,

that is $\qquad (a^3+c^3)\,x^3+3\,(a^2b+c^2d)\,x^2y+3\,(ab^2+cd^2)\,xy^2+(b^3+d^3)\,y^3,$

when $ad-bc=1$ by the same law as (x^3-y^3) from (x^3+y^3), for the elements for forming such comparison are wanting, but because x^3+y^3 and x^3-y^3 are the correspondent particular values respectively assumed by

$$ax^3+3bx^2y+3cxy^2+dy^3,$$

and its concomitant

$$(ad^2+2c^3-3bcd)\,x^3-(6b^2d-3c^2b-3acd)\,x^2y$$
$$+(6ac^2-3cb^2-3cba)\,xy^2-(a^2d+2b^3-3bca)\,y^3,$$

when $\qquad a=1,\quad b=0,\quad c=0,\quad d=1.$

With the aid of this extended signification of the term concomitant (whether it be a covariant or contravariant) we can in all cases speak (as otherwise we in general could not) of the concomitant of a concomitant. The relation

* And of course the concomitant may be an invariant to its originant in respect of one or more systems of variables entering into the former.

† Or, more generally, it may be said that concomitance consists in the persistence of morphological affinity.

between systems of variables has been stated to be Simple (whether they be cogredient or contragredient) when each variable in one system corresponds with some one in each other. Compound relation arises as follows:—Suppose $x, y; \xi, \eta$ two independent systems of two variables each, and that the system of four variables u, v, w, t is subject to linear variations imitating, in the way of cogredience or contragredience, those to which $x\xi, x\eta, y\xi, y\eta$ are subject; then u, v, w, t may be said to be cogredient or contragredient to the continued systems $x, y; \xi, \eta$. If $x, y; \xi, \eta$ be themselves cogredient, then a system of only three variables u, v, w, may be cogredient or contragredient in respect to $x\xi$, $x\eta + y\xi$, $y\eta$, and if $x, y; \xi, \eta$ be coincident, u, v, w may be similarly related to x^2, xy, y^2. The illustration may easily be generalized, and it will be seen in the sequel that its conception of compound-relation between systems of a differing number of variables will greatly extend the power and application of the methods about to be developed. Without having recourse to a formal definition, it is obvious that the notion of a concomitant conveyed in my former paper in this *Journal* lends itself without difficulty to the most general supposition which can be made of functions between which any number of systems of related variables are distributed, whatever such relation be, whether simple or compound, and whether of cogredience or of contragredience. The proposition stated in my last paper relative to a concomitant of the concomitant of a function being a concomitant of the original still applies to concomitants in the wider sense in which we now understand that term, and the species of each system of variables in the second concomitant with respect to the species or either species (if there be systems of both kinds in the primitive) will be determined upon the general principle which determines the effect of concurrence and contrariety being made to operate each upon itself or one in either order upon the other.

The highest law and the most powerful in its applications which I have yet discovered in the theory of concomitants may be expressed by affirming that when several related classes of variables are present in any concomitant, a new concomitant, *derived from the former by treating one or any number of these classes as independent of the remaining classes*, will still be a concomitant of the primitive. I shall quote this hereafter as the Law of Succession. This law, to which I have been led up inductively, requires an extended examination and a rigorous proof. It is the keystone of the subject, and any one who should suppose that it is a self-evident proposition (as from the simplicity of the enunciation it might be supposed to be) will commit no slight error.

If $\phi(x, y \dots z)$ be any homogeneous form of function of $x, y, \dots z$, every homogeneous sum in the expansion by Taylor's theorem of

$$\phi(u+u', \ v+v' \dots w+w'),$$

which in fact, on making $u'=x$, $v'=y \ldots w'=z$, becomes identical (to a numerical factor *près*) with $\left(u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)^{\iota}\phi$, is what I have elsewhere termed an Emanant, and by a partial method I had demonstrated that every invariant of such an emanant in respect to $u, v \ldots w$, in which $x, y \ldots z$ are treated as constants, or *vice versâ*, would give a covariant of ϕ. The reason of this is now apparent. For it may easily be shown* that every emanant is in fact itself a covariant of the function to which it belongs with respect to each of the related classes of variables which enter into it, or is as it may be termed a double covariant. The law of Succession shows therefore that a concomitant to an emanant from which one of the classes has disappeared will be a covariant of the primitive in respect to the remaining class.

In applying the law of Succession, great use can be made of a function of two classes of letters which may be termed a Universal Mixed Concomitant; this is $x\xi+y\eta+\ldots+z\zeta$, which has the property of remaining unaltered when any linear substitution (for which the modulus is unity) is impressed upon $x, y \ldots z$, and the contrary one upon $\xi, \eta \ldots \zeta$†.

If $f(x, y)$ be any function of x, y, of the degree m, $f+\lambda(x\xi+y\eta)^m$ will

* To demonstrate this it is only necessary to observe that if $u, v, \ldots w, u', v', \ldots w'$ be cogredient with themselves and with $x, y, \ldots z$,

$$\phi(u+\lambda u',\ v+\lambda v',\ \ldots\ w+\lambda w')$$

will evidently be a concomitant of $\phi(x, y, \ldots z)$; and, λ being arbitrary, the coefficients of the different powers of λ must be separately concomitants of $\phi(x, y, \ldots z)$, but these coefficients are the emanants of ϕ. Q. E. D.

† Thus, if

$$\begin{aligned} x&=ax'+by'+cz', & \xi&=(gn-hm)\xi'+(hl-fn)\eta'+(fm-gl)\zeta',\\ y&=fx'+gy'+hz', & \eta&=(-nb+mc)\xi'+(-lc+na)\eta'+(-ma+lb)\zeta',\\ z&=lx'+my'+nz', & \zeta&=(bh-cg)\xi'+(cf-ah)\eta'+(ag-bf)\zeta', \end{aligned}$$

then

$$x\xi+y\eta+z\zeta=\begin{pmatrix} a & b & c\\ f & g & h\\ l & m & n\end{pmatrix}\times(x'\xi'+y'\eta'+z'\zeta')$$
$$=x'\xi'+y'\eta'+z'\zeta'.$$

When the coefficients of transformation correspond to the direction-cosines between one system of rectangular axes and another, the reciprocal system is identical with the direct system; so that x, y, z; ξ, η, ζ, on this particular supposition, may be regarded indifferently as contragredient or as cogredient; accordingly they may be made identical, and then $x^2+y^2+z^2$ remains invariable, which is the well-known characteristic of orthogonal transformation. It may be observed here that there exists a special theory of concomitance limited to such species of linear transformations, which may be termed Conditional Concomitance, and I have found in several cases that the invariants of conditional concomitants turn out to be absolute invariants of the primitive. Much more important is the remark that there exists a theory of universal concomitants for an indefinite number instead of merely two systems of variables, as used in the text. In the sequel it will be seen that the application of this universal concomitant (like the touch of an enchanter's wand) serves to transmute covariants into contravariants, and back again, and causes single invariants to germinate and fructify into complete connected systems of forms.

be a mixed concomitant of f, it being evident that every function of concomitants of a function is itself a concomitant of the same.

Suppose now

$$f = ax^m + mbx^{m-1}y + \tfrac{1}{2}m(m-1)\,cx^{m-2}y^2 + \&c.,$$

the concomitant becomes

$$(a + \lambda\xi^m)\,x^m + m\,(b + \lambda\xi^{m-1}\eta)\,x^{m-1}y + \tfrac{1}{2}m\,(m-1)\,(c + \lambda\xi^{m-2}\eta^2) + \&c.$$

Consequently if P be any concomitant of f, P' obtained from P by writing $a + \lambda\xi^m$, $b + \lambda\xi^{m-1}\eta$, &c. for a, b, &c., will still be a concomitant of f; and by Taylor's theorem P' evidently equals

$$P + \left(\xi^m \frac{d}{da} + \xi^{m-1}\eta \frac{d}{db} + \&c.\right) P$$

$$+ \frac{1}{1\,.\,2}\left(\xi^m \frac{d}{da} + \xi^{m-1}\eta \frac{d}{db} + \&c.\right)^2 P$$

$$+ \&c.$$

If we take P an invariant of f, we have M. Hermite's theorem* for $f(x, y)$, and precisely the same demonstration applies to the general case of $f(x, y \ldots z)$. P' is, by virtue of the general rule, a contravariant of f in respect to $\xi, \eta \ldots \zeta$: if P be taken a function containing one single system, and is also a contravariant to f in respect to that system, P' will be a double contravariant; and if we make the two systems in P' identical, we have the extension of M. Hermite's theorem alluded to by me in one of the notes† to my last paper, wherein I have stated that "I may be taken any *covariant* of the function": as regards the purpose of that statement, the word covariant was used in error for contravariant.

The preceding method may be viewed as a particular application of the general principle, that if $U_1, U_2 \ldots U_m$ be any m functions (whether concomitants any of them of the others or not), then any concomitant of $\lambda_1 U_1 + \lambda_2 U_2 + \ldots + \lambda_m U_m$ being expressed as a function of $\lambda_1, \lambda_2 \ldots \lambda_m$, every coefficient in such expression will be a concomitant of the system $U_1, U_2 \ldots U_m$. Thus, for example, if U and V be two quadratic functions of n variables $x, y \ldots z$, the *discriminant* $\square\,(\lambda U + \mu V)$ will contain $n+1$ terms, of which the coefficients of the first and last will be $\square\, U$ and $\square\, V$; and every one of the $(n+1)$ coefficients will be a concomitant (of course an invariant) of U and V. These $(n+1)$ invariants will in fact constitute the fundamental scale of invariants to the system U and V, and every other invariant of U

* This theorem was first stated to me by Mr Cayley, who, I understand, derived it from M. Eisenstein, under the form of a theorem of covariants, which of course it becomes on interchanging x, y with $-y, x$. But as a theorem of covariants it could not be extended to functions of more than two variables. M. Hermite appears to have discovered this theorem, under its more eligible form, subsequently to, but independently of, M. Eisenstein.

[† p. 201 above, note *.]

and V will be an explicit rational function of the $(n+1)$ terms of the scale. In connexion with this principle may be stated another relative to any system of homogeneous functions of a greater number of variables of the same class, namely, that if any set of the variables one less in number than the number of the functions be selected at will, and any invariant of a given kind be taken of the resultant of the functions in respect to the variables selected, all such invariants so formed will have an integral factor in common, and this common factor will be an invariant of the given system of functions.

It will be convenient to speak hereafter of systems for which the march of the linear substitutions is coincident as cogredient, and those for which the march is contrary as contragredient systems.

Suppose m cogredient classes of m variables, the determinant formed by writing the $m \times m$ quantities in square order will evidently be a universal covariant. Thus, take the two systems x, y; ξ, η. $x\eta - y\xi$ is a universal covariant, and evidently therefore F, which I use to denote

$$\phi(x, y) \times \phi(\xi, \eta) + \lambda(x\eta - y\xi)^m,$$

will be a covariant to $\phi(x, y)$. Let $\phi(x, y)$ be of m dimensions; any invariant of F will be an invariant of ϕ; thus, let the two systems x, y; ξ, η be treated as perfectly independent, and take the discriminant of F (viewed as a function of x, y; ξ, η), that is the resultant of the four functions $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{d\xi}, \frac{dF}{d\eta}$; this resultant will be an invariant of ϕ; and λ being arbitrary, all the coefficients of its different powers will be invariants of ϕ. We thus fall upon another theorem of M. Hermite, namely that if $\lambda = \frac{\phi(x, y) \times \phi(\xi, \eta)}{(x\xi - y\eta)^m}$, the coefficients of the equation which will give the minimum values of λ are invariants of ϕ. So more generally, any invariant of $f(x, y, \xi, \eta) - \lambda(x\xi - y\eta)^m$, f being of the degree m in x, y and in ξ, η, will be an invariant of f; and among other invariants may be taken the discriminant obtained by treating x, ξ, y, η as absolutely unrelated.

If f be a function of various classes each containing n covariables, and if not less than n of these classes be covariable classes, and after selecting at will any n of such systems, as $x_1, y_1 \ldots z_1$; $x_2, y_2 \ldots z_2$; ...... $x_n, y_n \ldots z_n$, the symbolical determinant

$$\begin{vmatrix} \frac{d}{dx_1}, & \frac{d}{dy_1} & \cdots & \frac{d}{dz_1} \\ \frac{d}{dx_2}, & \frac{d}{dy_2} & \cdots & \frac{d}{dz_2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d}{dx_n}, & \frac{d}{dy_n} & \cdots & \frac{d}{dz_n} \end{vmatrix}$$

be expanded and written equal to D, then $D^\iota f$ will be a concomitant of f; and, more generally, by selecting different combinations of the covariable systems n and n together in every way possible, and forming corresponding symbols of operation $E, F \dots H$, we shall have $D^\iota . E^{\iota'} \dots H^{(\iota)} . f$, for all values of $\iota, \iota' \dots (\iota)$, a covariant of f in respect to the classes so combined. This explains and contains the whole pith and marrow of Mr Cayley's simple but admirable method of obtaining covariants and invariants (or, as termed by their author, hyperdeterminants) to a function ϕ_1 of a single system $x_1, y_1 \dots z_1$; he forms similar functions $\phi_2 \dots \phi_\mu$ of $x_2, y_2 \dots z_2; \dots x_\mu, y_\mu \dots z_\mu$, and uses the product $\phi_1 \times \phi_2 \times \dots \times \phi_\mu$ as a function f of μ systems: the multiple covariant obtained by operating thereupon becomes a simple covariant on identifying the different classes of covariables introduced in the procedure.

Section II. *On Complex Concomitance.*

We have hitherto been engaged in considering only a particular case of concomitance, the true idea of which relates not to an individual associated form (as such), but to a complex of forms capable of degenerating into an individual form. Such a complex may be called a Plexus. A plexus of forms is concomitant to a given form or combination of forms under the following circumstances.

If (O) be the originant, meaning thereby the primitive form or system of forms, and P the concomitant plexus made up of the μ forms $P_1, P_2 \dots P_\mu$, and if, when by duly related linear substitutions, O becomes O', the plexus P becomes P', made up of the forms $P'_1, P'_2 \dots P'_\mu$, and if the plexus $'P$ formed from O' after the same law as P from O be made up of the forms $'P_1, 'P_2 \dots 'P_\mu$, then will each form in either of the plexuses $'P, P'$ be a linear function of all the forms in the other plexus, and the connecting constants in every such linear function will be functions of the coefficients of the substitution whereby O and P have become transformed into O' and P'.

A function forming part of a concomitant plexus may be termed a concomitantive. Concomitantives therefore usually have a joint relation to a common plexus and a concomitant is only another name for an unique concomitantive. Every plexus contains a definite number of concomitantives; in place of any one of these may be substituted an arbitrary linear function of all the rest, but the total number of independent forms *sufficient and necessary* to make the complete plexus respond to the requirements of the definition will remain constant.

If now we combine together the whole number of functions contained in one or more plexuses concomitant to any given originant, all of the same degree relative to any given selected system or systems of variables, and if the number of the concomitantives so combined be exactly equal to the number of terms in each, arranged as a function of the selected class or classes of variables, then the dialytic resultant (obtained by treating each combination of the selected variables as an independent variable, and forming a determinant in the usual manner), will be a concomitant to the given originant. This, which is only the partial expansion of some much higher law, may be termed the "Law of Synthesis."

Let f be any function of a single class of variables $x_1, x_2 \dots x_n$. Let χ represent any product of these variables or of their several powers of any given degree r; the number of different values of χ will be μ, where

$$\mu = \frac{n(n+1)\dots(n+r-1)}{1.2\dots r},$$

and $\chi_1 f, \chi_2 f \dots \chi_\mu f$ will form a covariantive plexus to f.

Again, let ϑ represent any product of the degree r of the symbols

$$\frac{d}{dx_1}, \frac{d}{dx_2} \dots \frac{d}{dx_n};$$

$\vartheta_1 f, \vartheta_2 f \dots \vartheta_\mu f$ will also form a covariant plexus to f.

The coefficients of connexion between the forms of either plexus depend in an analogous manner upon the coefficients of the substitution supposed to be impressed upon the variables, with the sole difference that every coefficient taken from the line r and column s of the determinant of substitution which appears in any coefficient of connexion of the one plexus is replaced by the coefficient taken from the line s and the column r in the corresponding coefficient of connexion for the other plexus.

Let $f(x, y)$ be any function of x, y of the degree $2m$; then

$$\left(\frac{d}{dx}\right)^m, \left(\frac{d}{dx}\right)^{m-1}\frac{d}{dy}, \dots\dots \left(\frac{d}{dy}\right)^m$$

will form a covariantive plexus; thus, suppose

$$f(x, y) = a_1 x^{2m} + 2ma_2 x^{2m-1}y + \dots + a_{2m+1}y^{2m};$$

omitting numerical factors, the plexus will be composed of the $(m+1)$ lines following:

$$\begin{array}{l} a_1 x^m \quad + ma_2 x^{m-1}y \quad + \dots + a_{m+1}y^m, \\ a_2 x^m \quad + ma_3 x^{m-1}y \quad + \dots + a_{m+2}y^m, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m+1}x^m + ma_{m+2}x^{m-1}y + \dots + a_{2m+1}y^m, \end{array}$$

and consequently, by the law of synthesis, the determinant

$$\begin{vmatrix} a_1, & a_2 \dots\dots a_{m+1} \\ a_2, & a_3 \dots\dots a_{m+2} \\ \dots\dots & \dots\dots\dots\dots \\ a_{m+1}, & a_{m+2} \dots a_{2m+1} \end{vmatrix}$$

is an invariant of f.

When this determinant is zero, I have proved in my paper* on Canonical Forms, in the *Philosophical Magazine* for November last, that f is resoluble into the sum of m powers of linear functions of x and y. I shall hereafter refer to a determinant formed in this manner from the coefficients of f as its catalecticant. Mr Cayley was, I believe, the first to observe that all catalecticants† are invariants.

Again, more generally, let $f(x, y, \xi, \eta)$ be a function of the mth degree of x, y, and of a like degree in respect of ξ, η, which are supposed to be cogredient with x and y; then

$$f(x, y, \xi, \eta) + \lambda(x\eta - y\xi)^m$$

(say F) will be a concomitant of f; and therefore if we take the system

$$\left(\frac{d}{dx}\right)^m F,\ \left(\frac{d}{dx}\right)^{m-1}\frac{d}{dy}F \ldots\ldots \left(\frac{d}{dy}\right)^m F,$$

which will be functions of ξ and η alone, and take their resultant, this resultant will be an invariant of f. As a particular case of this theorem, let

$$f = \left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^m \phi,$$

where ϕ is supposed to be a function of x and y only and of $2m$ dimensions, f is a concomitant of ϕ, and therefore the invariant of f, obtained in the manner just explained, will be an invariant of ϕ. Thus then we have an instantaneous demonstration of the theorem given‡ by me in the paper of the *Philosophical Magazine* before named, namely, if

$$\phi(x, y) = a_1x^{2m} + 2ma_2x^{2m-1}y + \ldots + a_{2m+1}y^{2m},$$

say, in order to fix the ideas, $= ax^6 + 6bx^5y + 15cx^4y^2 + \ldots + gy^6$; then the determinant

$$\begin{vmatrix} a, & b, & c, & d+\lambda \\ b, & c, & d-\frac{1}{3}\lambda, & e \\ c, & d+\frac{1}{3}\lambda, & e, & f \\ d-\lambda, & e, & f, & g \end{vmatrix},$$

(and the analogously formed determinant for the general case) will be an invariant of ϕ. The general determinant so formed is peculiarly interesting, because it furnishes when equated to zero the one sole equation necessary to be solved in order to be able to effect the reduction of $\phi(x, y)$ to its canonical form, and gives the means, irrespective of any other view of the theory of invariants, of determining completely and absolutely the condition

[* See p. 282 above.]

† But the catalecticant of the biquadratic function of x, y was first brought into notice as an invariant by Mr Boole; and the discriminant of the quadratic function of x, y is identical with its catalecticant, as also with its Hessian. Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant.

[‡ p. 277 above.]

of the possibility of two given functions of the same degree of x, y being linearly transformable one into the other. This theorem will be obtained in a more general manner in the following section. I only pause now to make the very important observation, that not only is the determinant an invariant, but every minor system* of determinants that can be formed from it (there are of course m such systems) is an invariantive plexus to the given function ϕ.

The form under which this theorem presents itself suggests a theorem vastly more general and of peculiar interest, as showing a connexion between the theory of functions of a certain degree and of a certain number of variables with other functions of a lower degree but of a greater number of variables. Here again, under a different aspect, is reproduced the great principle of dialysis, which, originally discovered in the theory of elimination, in one shape or another pervades the whole theory of concomitance and invariants.

Let ϕ represent any function of the degree pq (of any number, or, to fix the ideas, say of three variables x, y, z); let the general term of ϕ be represented by

$$\frac{pq\,(pq-1)\ldots 1}{(1\,.\,2\ldots\alpha)(1\,.\,2\ldots\beta)(1\,.\,2\ldots\gamma)}(\alpha,\beta,\gamma)\,x^\alpha y^\beta z^\gamma,$$

where $\alpha+\beta+\gamma=pq$, and (α, β, γ) represents a portion of the coefficient of $x^\alpha y^\beta z^\gamma$.

Let

$$\frac{1\,.\,2\ldots p}{(1\,.\,2\ldots r)(1\,.\,2\ldots s)(1\,.\,2\ldots t)}\,x^r y^s z^t=\theta_{r,\,s,\,t},$$

where $r+s+t=p$, so that there are as many θ's as there are modes of

* These minor systems mean as follows:—the system of rth minors comprises all the distinct determinants that can be got by striking out from the square array (which I call the Matrix) from which the complete determinant is formed, any r lines and any r columns selected at will. The last, or mth minor, is of course a system consisting of the coefficients of $\phi(x, y)$, and it is evident that if $\phi(x, y \ldots z)$ be any function of any number of variables $x, y \ldots z$, the coefficients will form an invariantive plexus to ϕ.

The following remark as to the changes undergone by the coefficients of ϕ when the variables undergo any substitution, is not without interest and importance for the theory.

Let

$$\begin{array}{ll} x \text{ become } & fx+f'y+\ldots+(f)\,z, \\ y\ \ldots\ldots\ldots & gx+g'y+\ldots+(g)\,z, \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ z\ \ldots\ldots\ldots & hx+h'y+\ldots+(h)\,z. \end{array}$$

Then the coefficient of the highest power of x becomes

$$\phi(f, g \ldots h),$$

and the coefficient of the term containing $y^r \ldots z^s$ becomes

$$\left(f'\frac{d}{df}+g'\frac{d}{dg}+\ldots+h'\frac{d}{dh}\right)^r\times\text{\&c.}\times\left\{(f)\frac{d}{df}+(g)\frac{d}{dg}+\ldots+(h)\frac{d}{dh}\right\}^s\phi(f, g \ldots h).$$

subdividing p into three integral parts (zeros being admissible); that is $\frac{1}{6}(p+1)(p+2)(p+3)$. Then any product such as $x^\alpha y^\beta z^\gamma$ may be divided in a variety of ways into the product of q of these θ's, and it may be shown that the entire quantity

$$\frac{pq(pq-1)\ldots 1}{(1.2\ldots\alpha)(1.2\ldots\beta)(1.2\ldots\gamma)}(x^\alpha y^\beta z^\gamma)$$
$$=\Sigma\left\{\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_r)}(\theta_{\mu_1}{}^{m_1}\theta_{\mu_2}{}^{m_2}\ldots\theta_{\mu_r}{}^{m_r})\right\},$$

where $m_1+m_2+\ldots+m_r=q$. Consequently ϕ may be represented under the form of a function of the degree q of $\frac{1}{6}(p+1)(p+2)(p+3)$ (say ι) variables $\theta_1, \theta_2\ldots\theta_\iota$, and its general term will be of the form

$$\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_r)}(\alpha,\beta,\gamma)\{\theta^{m_1}\theta^{m_2}\ldots\theta^{m_r}\},$$

where α, β, γ are the indices respectively of x, y, z, when the last factor is expressed as a function of these variables*. Now if ϑ be used to denote this new representation of ϕ when $\theta_1, \theta_2\ldots\theta_\iota$ are treated as absolutely independent variables, and if we attach to it any universal concomitant, as $(x\xi+y\eta+z\zeta)^p$ admitting of being written under the form $\omega(\theta_1, \theta_2\ldots\theta_\iota)$, wherein the coefficients will be functions of ξ, η, ζ; then any invariant to ϑ and ω, treated as two systems of ι variables, will be a concomitant to ϕ, the original function in x, y, z†. ϑ and ω may be termed respectively, for facility of reference, the Particular and Absolute functions. Thus, for example, we take ϕ a function of x, y of the degree $4n$, say

$$a_1x^{4n}+4na_2x^{4n-1}y+\&c.+a_{4n+1}y^{4n},$$

and make $p=2n$, $q=2$, so that ϑ becomes a quadratic function of $(2n+1)$ variables obtained by making $x^{2n}=\theta_1$, $x^{2n-1}y=\theta_2\ldots y^{2n}=\theta_{2n+1}$‡, and the concomitant ω, formed from $(\xi x+\eta y)^{2n}$, becomes

$$\theta_1\xi^{2n}+2n\theta_2\xi^{2n-1}\eta+\ldots+\theta_{2n+1}\eta^{2n};$$

then if we take R the quadratic invariant of ω, that is

$$R=\theta_1\theta_{2n+1}-2n\theta_2\theta_{2n}\,\&c.\pm\frac{1.2.3\ldots(2n)}{(1.2\ldots n)^2}\tfrac{1}{2}(\theta_{n+1})^2,$$

* See Note (1) in Appendix. [p. 322 below.]

† In fact ϑ is a concomitant to ϕ, and ω to a power of the universal concomitant; the θ's forming a system of variables cogredient with the compound system $x^{r_1}y^{s_1}z^{t_1}$, $x^{r_2}y^{s_2}z^{t_2}$, &c.: and it must be well observed that the same substitutions which render ϑ and ω respectively identical with ϕ and a power of the universal concomitant, would render an infinite number of other functions also coincident with the same; but none of these other functions would be concomitants. Herein we see the importance of the definition and conception of compound relation; the θ system being compound by relation with the x, y, z system, after the manner of cogredience.

‡ A slight variation upon the method as above explained for the general case has been here introduced inadvertently by writing $x^{2n-1}y=\theta_2$, &c., in lieu of $2nx^{2n-1}y=\theta_2$, &c., which, as it does not in any degree affect the reasoning, I have not deemed it worth while to alter.

it will readily be seen that the determinant of $\vartheta + \lambda R$, treated as a quadratic function of $(2n+1)$ variables, will give an invariant of ϕ, and this will be the same as that obtained by the particular method above given. Thus, suppose

$$\phi(x, y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Let
$$x^2 = \theta_1, \quad 2xy = \theta_2, \quad y^2 = \theta_3,$$
$$\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + e\theta_3^2,$$
$$\omega = (x\xi + y\eta)^2 = x^2\theta_1 + xy\theta_2 + y^2\theta_3,$$
$$R = \theta_1\theta_3 - \frac{\theta_2^2}{4}.$$

Then Λ the discriminant of $\vartheta + 2\lambda R$ in respect to θ_1, θ_2, θ_3

$$= \begin{vmatrix} a, & b, & c+\lambda \\ b, & c - \frac{1}{2}\lambda, & d \\ c+\lambda, & d, & e \end{vmatrix},$$

and I may remark that the relations between the several transformees of the invariantive plexuses formed by the minor determinant systems of Λ (in this, and in general for the case of an evenly-even index) may be found by treating $\vartheta + 2\lambda R$ as a quadratic function of the variables (in this case θ_1, θ_2, θ_3) and applying the rule given by me in the *Philosophical Magazine* in my* paper "On the relation between the Minor Determinants of linearly-equivalent Quadratic Forms."† This second method, however, is not immediately applicable to the case of indices oddly even, that is of the form $4n+2$, to which the first method applies, equally as to the case $4n$; for if we make $2n+1 = p$ and $q = 2$, ω being of an odd degree, has no quadratic invariant; it has however a quadratic covariant, which will be of the second degree in respect to $\theta_1, \theta_2 \ldots \theta_{p+1}$ as well as in respect to ξ, η; and if we call this R and take the discriminant of $\vartheta + \lambda R$ in respect to the variables $\theta_1, \theta_2 \ldots \theta_{p+1}$, we shall obtain, as I am indebted to a remark of my valued friend M. Hermite for bringing under my notice, a very beautiful and interesting function of λ, of which all the coefficients will be contravariants of ϕ. Thus, let

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6,$$

[* p. 241 above.]

† Moreover, upon the supposition made in the text, the particular and absolute functions ϑ and ω may be treated in all respects as if they were functions characterizing quadratic loci, and any singularity in their relation will correspond to and denote a singularity in the given function ϕ to which ϑ refers. Thus, for instance, if ϕ be a function of x, y of the eighth degree, ϑ and ω will be quadratic functions of five letters each. Quadratic loci have no other singularity of relation than what corresponds to different species of contact. The number of contacts between loci, characterized by 5 letters, is 24 (see my paper‡ in the *Philosophical Magazine*, "On the contacts of lines and surfaces of the second order"). Consequently this mode of representing ϑ and ω will give rise to the discovery and specification of 24 different kinds of singularity in ϕ, and the analytical characteristics of each of them. But there of course may, and in fact will, exist other singularities in ϕ besides those which have their correspondencies in the relations of these quadratic concomitants. [‡ p. 237 above.]

make $$x^3 = \theta_1, \quad 3x^2y = \theta_2, \quad 3xy^2 = \theta_3, \quad y^3 = \theta_4,$$
so that
$$\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + 2d\theta_1\theta_4 + g\theta_4^2 + 2f\theta_3\theta_4 + e\theta_3^2 + 2e\theta_4\theta_2,$$
$$\omega = (x\xi + y\eta)^3 = \theta_1\xi^3 + \theta_2\xi^2\eta + \theta_3\xi\eta^2 + \theta_4\eta^3,$$
$$R = \begin{vmatrix} 3\theta_1\xi + \theta_2\eta, & \theta_2\xi + \theta_3\eta \\ \theta_2\xi + \theta_3\eta, & \theta_3\xi + 3\theta_4\eta \end{vmatrix},$$
$$-R = \xi^2\theta_2^2 + \eta^2\theta_3^2 + \xi\eta\theta_2\theta_3 - 9\xi\eta\theta_1\theta_4 - 3\xi^2\theta_1\theta_3 - 3\eta^2\theta_2\theta_4.$$
Consequently the discriminant in respect to $\theta_1, \theta_2, \theta_3, \theta_4$ of $\vartheta - 2\lambda R$ becomes
$$\begin{vmatrix} a, & b, & c - 3\lambda\xi^2, & d - 9\lambda\xi\eta \\ b, & c + 2\lambda\xi^2, & d + \lambda\xi\eta, & e - 3\lambda\eta^2 \\ c - 3\lambda\xi^2, & d + \lambda\xi\eta, & e + 2\lambda\eta^2, & f \\ d - 9\lambda\xi\eta, & e - 3\lambda\eta^2, & f, & g \end{vmatrix}.$$
If this determinant be expanded as a function of λ, all the coefficients of the various powers of λ will be contravariants to the given function ϕ. The term involving λ^4 is zero. Let ξ become $-y$ and η become x, then the remaining terms (abstraction made of the powers of λ) become covariants of ϕ. The first term (the coefficient of λ^3) becomes ϕ itself; the last term is the catalecticant, and thus we see, in general, that for functions of x and y of an oddly-even degree, a whole series of covariants may be interpolated between the function and its catalecticant, the dimensions in respect of the coefficients of ϕ in arriving at each step increasing by 1 unit and the degree in respect of the variables diminishing by 2 units. This is consequently a much simpler and more available scale than one with which I have been long previously acquainted, and which applies alike to functions of any even degree.

Thus, let $\phi(x, y)$ be of $2k$ dimensions; form all the even emanants of ϕ, which will be all of the form $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{2\iota} \phi$, and take their respective catalecticants in respect to ξ and η. We shall in this way obtain a regular scale of covariants interpolated between the Hessian of ϕ (corresponding to $\iota = 1$) and the catalecticant of ϕ (corresponding to $\iota = k$). If ϕ be of the degree $2k + 1$, we shall have an analogous scale interpolated between the Hessian of ϕ and its canonizant; the latter term denoting the function which is the product of the $k + 1$ linear functions of x and y, the sum of whose $(2k + 1)$th powers is identically equal to ϕ*.

By means of the Theory of the Plexus we may obtain various representa-

* See Note (2) in Appendix. [p. 322 below.]

tions of the same invariant; thus, for example, if we take F a function of x, y of the fifth degree and form its Hessian H, that is

$$\begin{vmatrix} \frac{d^2F}{dx^2}, & \frac{d^2F}{dxdy} \\ \frac{d^2F}{dydx}, & \frac{d^2F}{dy^2} \end{vmatrix},$$

this will be a function of the sixth degree in x, y, and of the two orders in the coefficients. If we combine the two plexuses

$$\frac{dF}{dx}, \frac{dF}{dy}; \quad \frac{d^2H}{dx^2}, \frac{d^2H}{dxdy}, \frac{d^2H}{dy^2},$$

we shall have five equations between which x^4, x^3y, x^2y^2, xy^3, y^4 may be eliminated dialytically; the resultant will be of the $2+3.2$, that is the eighth order in the coefficients, and of the form $\square F - I_4^2$, where $\square F$ and I_4 are respectively the determinant and quintic invariant of F, each affected with a proper numerical multiplier (the "$B - A^2$" of my supplemental* essay on canonical forms) which, as Mr Cayley has remarked, may also be represented by the resultant of P; $\frac{dQ}{dx}$; $\frac{dQ}{dy}$ where P and Q are respectively the quadratic and cubic invariants in respect to ξ and η of $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^4 F$.

It will be well at this point to recapitulate in brief a method of elimination applicable to certain systems of functions published by me many years since in the *Philosophical Magazine*, and to compare this method with that afforded by the theory of the plexus for finding an invariant for each of the very same systems, possessing all the external characters, formed in a precisely similar manner to, and not impossibly identical *with*, the resultant of every such system. I shall devote my first moments of leisure to the ascertainment of this last most important point, as to the identity or otherwise of the plexus-invariant with the resultant. Take the case of three functions of x, y, z (say ϕ, ψ, ω) each of the same degree n; to fix the ideas, suppose $n=3$: there are two purely algebraical processes (modifications of the same method and leading to identical results) by which the resultant of ϕ, ψ, ω may be found. I shall call these processes the first and second respectively.

First process: Write

$$\begin{aligned} \phi &= x^2P + yQ + zR, \\ \psi &= x^2P' + yQ' + zR', \\ \omega &= x^2P'' + yQ'' + zR'', \end{aligned}$$

decompositions which may be effected in an infinite variety of manners, so that P, Q, R shall be integer functions of x, y, z; take the linear resultant of ϕ, ψ, ω, in respect to x^2, y, z, which call $H_{2,1,1}$; this will evidently be

[* p. 205 above.]

of $9-4$, that is, of 5 dimensions. Form analogously the functions $H_{1,2,1}$, $H_{1,1,2}$; $H_{2,1,1}$, $H_{1,2,1}$, $H_{1,1,2}$ constitute an auxiliary system of functions which vanish when ϕ, ψ, ω vanish together; combine this auxiliary system with the augmentative system

$$\begin{array}{cccccc} x^2\phi, & y^2\phi, & z^2\phi, & xy\phi, & yz\phi, & zx\phi, \\ x^2\omega, & y^2\omega, & z^2\omega, & xy\omega, & yz\omega, & zx\omega, \\ x^2\psi, & y^2\psi, & z^2\psi, & xy\psi, & yz\psi, & zx\psi. \end{array}$$

We shall thus have in all $3+3\times 6$, that is, 21 functions into which the 21 terms x^5, x^4y, x^4z, &c. enter linearly: the linear resultant of these 21 functions is the resultant of ϕ, ψ, ω, clear of all extraneousness.

Second process: Write

$$\begin{aligned} \phi &= x^3P + yQ + zR, \\ \psi &= x^3P' + yQ' + zR', \\ \omega &= x^3P'' + yQ'' + zR'', \end{aligned}$$

and, as before, take the linear resultant $H_{3,1,1}$, which will however be of $9-5$, that is, of only 4 dimensions.

Again, take

$$\begin{aligned} \phi &= x^2L + y^2M + zN, \\ \psi &= x^2L' + y^2M' + zN', \\ \omega &= x^2L'' + y^2M'' + zN'', \end{aligned}$$

and form the determinant $H_{2,2,1}$; we shall thus have the auxiliary system

$$H_{3,1,1},\quad H_{1,3,1},\quad H_{1,1,3},\quad H_{2,2,1},\quad H_{2,1,2},\quad H_{1,2,2}.$$

Let this be combined with the augmentative system

$$x\omega,\ y\omega,\ z\omega;\quad x\phi,\ y\phi,\ z\phi;\quad x\psi,\ y\psi,\ z\psi.$$

Between these $6+9$, that is, 15 functions, the 15 terms x^4, x^3y, x^3z, &c. may be linearly eliminated, and the resultant thus obtained will be precisely the same as that got by the preceding process.

Here we have 6 auxiliaries and 6 augmentatives; the auxiliaries are of three dimensions in respect to the coefficients of ϕ, ψ, ω; the augmentatives of one dimension only; in the former process there were 3 auxiliaries and 18 augmentatives, $6\times 3+9=27=3\times 3+18$.

Now let this method be compared with the following:

First process: Take the 18 augmentatives $x^2\phi$, $x^2\omega$, $x^2\psi$, &c. as in the first process of the algebraical method above explained; but in place of the 3 auxiliaries therein given, take another system of 9 as follows:

Write the determinant

$$\begin{vmatrix} \frac{d\phi}{dx}, & \frac{d\phi}{dy}, & \frac{d\phi}{dz} \\ \frac{d\psi}{dx}, & \frac{d\psi}{dy}, & \frac{d\psi}{dz} \\ \frac{d\omega}{dx}, & \frac{d\omega}{dy}, & \frac{d\omega}{dz} \end{vmatrix} = R;$$

$\frac{dR}{dx}, \frac{dR}{dy}, \frac{dR}{dz}$ form a concomitantive plexus; the 18 augmentatives form another; the linear resultant of these two plexuses will be an invariant of ϕ, ψ, ω, and of precisely the same dimensions as the resultant last found; if they are not identical it will be indeed a matter of exceeding wonder, and even more interesting than if they should be proved so to be.

Second process: Combine the augmentative plexus

$$x\omega,\ y\omega,\ z\omega;\quad x\phi,\ y\phi,\ z\phi;\quad x\psi,\ y\psi,\ z\psi,$$

with the differential plexus

$$\frac{d^2R}{dx^2}, \frac{d^2R}{dxdy}, \frac{d^2R}{dy^2}, \frac{d^2R}{dydz}, \frac{d^2R}{dz^2}, \frac{d^2R}{dzdx},$$

we thus obtain a linear resultant in a manner precisely similar to that afforded by the second process of our algebraical method.

In general, if ϕ, ψ, ω be of the degrees n, n, n, as there are two algebraical varieties of the linear method for finding the resultant, so are there two varieties of the concomitantive method for finding the resembling invariant. In both methods the augmentatives are identical; the only difference being in the auxiliary system.

In the first process the augmentative system will be got by operating upon each of the functions ϕ, ψ, ω, with the multipliers x^{n-1}, y^{n-1}, z^{n-1}, and the other homogeneous products of x, y, z; the auxiliary system by operating upon R with the symbolical multipliers $\left(\frac{d}{dx}\right)^{n-2}$, $\left(\frac{d}{dy}\right)^{n-2}$, $\left(\frac{d}{dz}\right)^{n-2}$, and the other homogeneous products of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ of the degree $n-2$.

In the second process the augmentative system is formed by the aid of the multipliers x^{n-2}, y^{n-2}, z^{n-2}, &c., and the auxiliary system by aid of

$$\left(\frac{d}{dx}\right)^{n-1}, \left(\frac{d}{dy}\right)^{n-1}, \left(\frac{d}{dz}\right)^{n-1}, \text{\&c.}$$

For the particular case of $n=2$ the first process of the concomitantive method is merely an application under its most symmetrical form of the first process of the general algebraical method. The second process of the concomitantive method for this same case (at least when ϕ, ψ, ω are the partial differential coefficients of the same function of the third degree) has been shown by Dr Hesse to give the resultant, so that for this case, at all events, we know that each concomitantive auxiliary must be a linear function of the augmentatives and the algebraical auxiliaries.

Again, if we go to the system where ϕ, ψ, ω are of the respective degrees n, n, $n+1$. In the algebraical method (for applying which there are no longer two, but one only process), the augmentative system is obtained by multiplying ϕ by the homogeneous products of x^{n-1}, $x^{n-1}y$, $x^{n-1}z$, &c., ψ by the like products, and ω by the homogeneous products x^{n-2}, $x^{n-2}y$, &c. The auxiliary system is made up of functions of the general form

$$H_{p,q,r} \text{ where } p+q+r=n+2,$$

$H_{p,q,r}$ being the determinant obtained by writing

$$\begin{aligned} \phi &= Lx^p + My^q + Nz^r, \\ \psi &= L'x^p + M'y^q + N'z^r, \\ \omega &= L''x^p + M''y^q + N''z^r. \end{aligned}$$

And in like manner for the case of ϕ, ψ, ω, being of the respective degrees n, n, $n-1$, the augmentative system is obtained by affecting ϕ, ψ each with multipliers x^{n-2}, $x^{n-2}y$, &c., and ω with the multipliers x^{n-1}, $x^{n-1}y$, &c.

The number of functions (for either case) in the augmentative and auxiliary plexuses thus obtained will be found to be exactly equal to the number of terms in each such function, as shown by me in the paper alluded to. Let this be compared with the transcendental method (I use this word at this point in preference to concomitantive, because in fact the algebraical and differential auxiliary systems are both alike concomitantive plexuses to ϕ). For the case of n, n, $n+1$, the Jacobian determinant R of ϕ, ψ, ω will be of the degree $3n-2$, and the system $\left(\frac{d}{dx}\right)^{n-1} R$, $\left(\frac{d}{dx}\right)^{n-2}\left(\frac{d}{dy}\right) R$, &c. combined with the augmentative systems

$$\begin{aligned} &x^{n-2}\omega, \quad x^{n-3}y\omega, \quad \text{\&c.} \\ &x^{n-1}\phi, \quad x^{n-2}y\phi, \quad \text{\&c.} \\ &x^{n-1}\psi, \quad x^{n-2}y\psi, \quad \text{\&c.} \end{aligned}$$

will give an invariant resembling (at least in generation and form) if not identical with the resultant of ϕ, ψ, ω. For the case of ϕ, ψ, ω being of the degrees n, n, $n-1$, the Jacobian R is of the degree $3n-4$ and

$$\left(\frac{d}{dx}\right)^{n-2} R, \quad \left(\frac{d}{dx}\right)^{n-3} \frac{d}{dy} R, \text{ \&c.}$$

is the system which, combined with the augmentative systems

$$x^{n-2}\phi,\quad x^{n-3}y\phi,\ \&c.$$
$$x^{n-2}\psi,\quad x^{n-3}y\psi,\ \&c.$$
$$x^{n-1}\omega,\quad x^{n-2}y\omega,\ \&c.$$

will produce the resembling invariant.

Finally, for the last and more special case which the algebraical method applies to, namely of $\phi, \psi, \omega, \theta$, four quadratic functions of x, y, z, t, there can be here little doubt (upon the first impression) that in place of the algebraically obtained plexus

$$H_{2,1,1,1},\quad H_{1,2,1,1},\quad H_{1,1,2,1},\quad H_{1,1,1,2},$$

may be substituted the differential plexus

$$\frac{dR}{dx},\ \frac{dR}{dy},\ \frac{dR}{dz},\ \frac{dR}{dt},$$

which, combined with the augmentatives

$$x\phi,\ x\psi,\ x\omega,\ x\theta;\quad y\phi,\ y\psi,\ y\omega,\ y\theta;\quad z\phi,\ z\psi,\ z\omega,\ z\theta;\quad t\phi,\ t\psi,\ t\omega,\ t\theta,$$

will render possible the dialytic elimination of the 20 homogeneous products

$$x^3,\quad x^2y,\quad x^2z,\quad x^2t,\quad xyz,\quad y^3,\quad \&c.\ \&c.^*$$

Upon precisely the same principles may be verified instantaneously the method given by Hesse (without demonstration) for finding the polar reciprocal of lines of the third and fourth orders, at least to the extent of seeing that the functions obtained by his methods are contravariants (of the right degree and order) of the function from which they are derived. The polar reciprocal to a *surface* of the third degree may be obtained in the same manner.

Let $\phi(x, y, z, t)$ be the characteristic of such a surface. If we form a differential plexus of the first emanant of ϕ taken together with the concomitant $w = x\xi + y\eta + z\zeta + t\theta$, by operating with

$$\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz},\ \frac{d}{dt}\ \text{upon}\ \left(\xi'\frac{d}{dx} + \eta'\frac{d}{dy} + \zeta'\frac{d}{dz} + \theta'\frac{d}{dt}\right)(\phi + \lambda w),$$

and combining this plexus with $x\xi' + y\eta' + z\zeta' + t\theta'$, the resultant taken in respect to $\xi', \eta', \zeta', \theta'$ (say R) will (according to the law of synthesis) be a

* Subsequent reflection induces me to reject as very improbable the (at first view likely) conjecture of the identity of the resultant with the invariant which simulates its form, except in the proved cases of three quadratic functions and the strongly resembling case of four quadratic functions last adverted to in the text above. Did this identity obtain, analogy would indicate that the catalecticant of the Hessian of two homogeneous functions of the same degree in x, should be identical with their resultant, which is easily demonstrated to be false, except when the functions are of the third degree.

contravariant to the system $\phi + \lambda w$ and w, and therefore to ϕ, because w is itself a concomitant to ϕ. R is of the third degree in x, y, z, t, as also in the coefficients of ϕ. If we form a differential plexus of $R + \mu w$ analogous to that formed above with $\phi + \lambda w$, and combine these two plexuses with the augmentative system xw, yw, zw, tw, there will be $4 + 4 + 4$, that is, 12 functions containing the 12 terms $x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt, \lambda, \mu$, and the dialytic resultant, which will be found to be a contravariant of the twelfth degree in ξ, η, ζ, θ, and of the twelfth order in respect of the coefficients of ϕ, will be (there can be little doubt) the polar reciprocal to the characteristic ϕ.

A few remarks upon the analytical character of a polar reciprocal may be not out of place here. If ϕ be any homogeneous function of the degree m of any number (n) of variables ($x, y \ldots z$), the object of the theory of polar reciprocals is to discover what is the relation between $\xi, \eta \ldots \zeta$ expressed in the simplest terms such that, when this equation is satisfied, $\xi x + \eta y + \ldots + \zeta z = 0$ will be tangential to $\phi = 0$. In order for this to take effect it is necessary that when any one of the variables z is expressed in terms of the others $\ldots y, x$, and this value established in ϕ, the discriminant of ϕ, so transformed, should be zero. Consequently the characteristic of the polar reciprocal to ϕ is that rational integral function which is common to all the discriminants obtained by expressing ϕ (by aid of the equation $\xi x + \eta y + \ldots + \zeta z$) as a function of any $(n-1)$ of the variables. Let I_x be any invariant whatever of the order r of ϕ_x (meaning by this last symbol what ϕ becomes when x is eliminated), and $I_y \ldots I_z$ the corresponding invariants when $y \ldots z$ respectively are eliminated; I_x will evidently be of the form $\frac{E_x}{(\xi)^{mr}}$, the numerator being an integer of r dimensions in the coefficients of ϕ and of mr dimensions in respect of $\xi, \eta \ldots \zeta$; and by the fundamental definition of invariants it may easily be shown that

$$I_x : I_y : \ldots : I_z :: \frac{1}{\xi^{\frac{mr}{n-1}}} : \frac{1}{\eta^{\frac{mr}{n-1}}} : \ldots : \frac{1}{\zeta^{\frac{mr}{n-1}}}\text{*},$$

and therefore

$$\frac{E_x}{\xi^p} = \frac{E_y}{\eta^p} = \ldots = \frac{E_z}{\zeta^p}, \quad \text{where } p = \frac{m(n-2)r}{n-1}.$$

Consequently all these quotients must be essentially integer, and any one of them will be of the order r in respect of the coefficients of ϕ and of the

* We see indirectly from this, that for a function of $(n-1)$, say γ, variables of the degree m, an invariant of the order r must be subject to the condition that $\frac{mr}{\gamma} =$ an integer. This is easily shown upon independent grounds; when $\gamma = 2$, $\frac{mr}{\gamma}$ must be not merely an integer but an *even integer*, and doubtless some analogous law applies to the general case.

degree $\frac{mr}{n-1}$ in respect of $\xi, \eta \ldots \zeta$. Consequently the polar characteristic of ϕ, which is the common factor of the *discriminants* of $I_x, I_y \ldots I_z$ (for which species of invariant r evidently is equal to $(n-1)(m-1)^{n-2}$, the function being in fact the discriminant of a function of the mth degree of $(n-1)$ variables), will be of the order $(n-1)(m-1)^{n-2}$ in respect of the coefficients of ϕ and of the degree $m(m-1)^{n-2}$ in respect of the contragredients $\xi, \eta \ldots \zeta$.

As to what relates to the reciprocity which exists between ϕ and its polar reciprocal ψ, this is included in a much higher theory of elimination, one proposition of which may be enunciated somewhat to the effect following, namely that if ϕ be a homogeneous function of $x, y \ldots z$, and ω of $x, y \ldots z, u, v \ldots w$, and if, by aid of the equations

$$\phi = 0,$$

$$\frac{d\phi}{dx} + \lambda \frac{d\omega}{dx} = 0,$$

$$\frac{d\phi}{dy} + \lambda \frac{d\omega}{dy} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d\phi}{dz} + \lambda \frac{d\omega}{dz} = 0,$$

$x, y \ldots z$ be eliminated and the resultant be called ψ, then the effect of performing a similar operation upon ψ, ω, with respect to $u, v \ldots w$, as that just above indicated for the system ϕ, ω, with respect to $x, y \ldots z$, will be to give a resultant, one factor of which will be the primitive function ϕ over again. There is some reason for supposing that polar reciprocals, which are scarcely ever (if ever, except indeed for quadratic functions), the simplest contravariants to a given function, may be expressed algebraically by means of the simpler contravariants, in the same way as discriminants admit (in many, if not in all cases, with the same exception as above) of being represented as algebraical functions of invariants of a lower order or simpler form.

I close this section with the remark that every complete and unambiguous system of functions of the constants in a given form or set of forms *characteristic** of any singularity absolute or relative in such form or forms must

* I repeat here that a function or system of functions which severally equated to zero express unequivocally and completely the existence of any position or negation, is termed the characteristic of such position or negation. Thus for example the resultant of a group of equations is the characteristic of the possibility of their coexistence. The discriminant of a function of two variables is the characteristic of its possession of two equal factors; the catalecticant is the characteristic of its decomposability into the sum of a defined number of powers of linear functions of the variables, &c.

constitute an invariantive plexus or set of invariantive plexuses. The system unambiguously characteristic of a singularity of an order n will (except when $n = 1$) almost universally consist of far more than n functions, subject of course to the existence of syzygetic* relations between any $(n+1)$ of such functions. The existence of multiple roots of a function of two variables is a specific, but by no means a peculiar case of singularity, and requires, for its complete and systematic elucidation, to be treated in connexion with the general theory of the subject.

SECTION III. *On Commutants.*

The simplest species of commutant is the well-known common determinant.

If we combine each of the n letters $a, b \dots l$ with each of the other n, $\alpha, \beta \dots \lambda$, we obtain n^2 combinations which may be used to denote the terms of a determinant of n lines and columns, as thus:

$$\begin{array}{l} a\alpha, \quad a\beta \dots a\lambda, \\ b\alpha, \quad b\beta \dots b\lambda, \\ \dots\dots\dots\dots\dots\dots \\ l\alpha, \quad l\beta \dots l\lambda. \end{array}$$

It must be well understood that the single letters of either set are mere umbræ, or shadows of quantities, and only acquire a real signification when one letter of one set is combined with one of the other set. Instead of the inconvenient form above written, we may denote the determinant more simply by the matrix

$$\begin{array}{l} a, \quad b, \quad c \dots l, \\ \alpha, \quad \beta, \quad \gamma \dots \lambda; \end{array}$$

and to find the expanded value of such a matrix the rule is evidently to take one of the lines in all its $1, 2, 3 \dots n$ different forms, arising from the permutations of the letters (or umbræ) which it contains; and then form the product of the n quantities formed by the combination of the respective pairs of letters in the same vertical column, affecting such product with the sign of $+$ or $-$ according to the rule, that all products corresponding to arrangements of the terms subject to the permutation derivable from one another by an even number of interchanges are of the same, and by an odd number of interchanges of a contrary sign. If both lines are permuted and a similar rule applied, with the additional circumstance that the sign of the products

* Rational integer functions which admit of being multiplied severally by other rational integer functions such that the sum of the products is identically zero, are said to be "syzygetically related."

is made to depend on the product of the algebraical signs due to the respective arrangements in the two lines of umbræ, it is evident that the result will be the same as when only one line is put into motion, save and except that a numerical factor $1 . 2 . 3 \ldots n$ will affect each term. If the two sets of umbræ $a, b, c \ldots l$; $\alpha, \beta, \gamma \ldots \lambda$ be taken identical, and if it be convened that the order of the combination of any two letters shall not affect the value of the quantity thereby denoted, $\begin{matrix} a, b, c \ldots l \\ a, b, c \ldots l \end{matrix}$ will denote a symmetrical determinant.

If instead of two lines of umbræ, three or more be taken, the same principle of solution will continue to be applicable. Thus, if there be a matrix of any even number r of lines each of n umbræ,

$$\begin{matrix} a_1, & b_1 \ldots l_1, \\ a_2, & b_2 \ldots l_2, \\ \ldots & \ldots \\ a_r, & b_r \ldots l_r, \end{matrix}$$

the first may be supposed to remain stationary, and the remaining $(r-1)$ lines each be taken in $1, 2 \ldots n$ different orders; every order in each line will be accompanied by its appropriate sign $+$ or $-$; and each different grouping in each line will give rise to a particular grouping of the letters read off in columns. The value of the commutant expressed by the above matrix will therefore consist of the sum of $(1 . 2 \ldots n)^{r-1}$ terms, each term being the product of n quantities respectively symbolized by a group of r letters and affected with the sign $+$ or $-$ according as the number of negative signs in the total of the arrangements of the lines (from the columnar reading off of which each such term is derived) is even or odd.

For example, the value of

$$\begin{matrix} a, & b, \\ c, & d, \\ e, & f, \\ g, & h, \end{matrix}$$

will be found by taking the $(1 . 2)^3$ arrangements, as below,

$$\begin{matrix} a, b, & a, b, & a, b, & a, b, & a, b, & a, b, & a, b, & a, b, \\ c, d, & d, c, & c, d, & d, c, & c, d, & d, c, & c, d, & d, c, \\ e, f, & e, f, & f, e, & f, e, & e, f, & e, f, & f, e, & f, e, \\ g, h, & g, h, & g, h, & g, h, & h, g, & h, g, & h, g, & h, g. \end{matrix}$$

The signs of c, d; e, f; g, h being supposed $+$, those of d, c; f, e and h, g will be each $-$. Consequently the sum of the terms will be expressed by

$$aceg \times bdfh - adeg \times bcfh - acfg \times bdeh + adfg \times bceh$$
$$- aceh \times bdfg + adeh \times bcfg + acfh \times bdeg - adfh \times bceg.$$

Commutants thus formed may be termed total commutants, because the entire of each line is made to pass through all its possible forms of arrangement. In total commutants it is necessary that the number of lines r be even; for if taken odd, on making all the r lines to change, instead of obtaining $1.2\ldots n$ lines, the result obtained when all but one are made to change, it will be found that the latter will be repeated $\frac{1}{2}(1.2\ldots n)$ times with the sign +, and $\frac{1}{2}(1.2\ldots n)$ times with the sign −, so that the algebraical sum of the terms will be zero. Moreover the commutants of the species above described, besides being total, are simple, inasmuch as all the umbræ to be termed consist of single letters.

My first proposition in the application of the theory of commutants to that of forms is as follows:

Let ϕ be a function homogeneous and linear in respect to an even number r of any systems whatever of variables, as

$$x_1,\ y_1\ldots t_1;\quad x_2,\ y_2\ldots t_2;\quad x_r,\ y_r\ldots t_r.$$

Form the commutant

$$\begin{array}{l} \dfrac{d}{dx_1},\ \dfrac{d}{dy_1}\cdots\dfrac{d}{dt_1}, \\ \dfrac{d}{dx_2},\ \dfrac{d}{dy_2}\cdots\dfrac{d}{dt_2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{d}{dx_r},\ \dfrac{d}{dy_r}\cdots\dfrac{d}{dt_r}. \end{array}$$

Let the general term of this commutant, expanded, be called

$$F_{\theta_1}\times F_{\theta_2}\times\ldots\times F_{\theta_r},$$

then is

$$\Sigma F_{\theta_1}.\phi\times F_{\theta_2}.\phi\times\ldots\times F_{\theta_r}.\phi$$

a covariant or invariant*, as the case may be, of ϕ.

Be it observed that the march of the substitution for the different sets of variables in the above proposition is supposed to be perfectly independent. All the systems but one may undergo linear transformation, or they may all undergo distinct and disconnected transformations at the same time, and the proposition still continue applicable. It will however evidently be no less applicable should the march of substitution for any of the systems become cogredient or contragredient to that of any other systems.

If we suppose ϕ to be a function of an even degree r of a single system of n variables $x, y\ldots t$, so that the r systems x_1, y_1, &c., x_2, y_2, &c. ... x_r, y_r, &c. become identical, we can at once infer from the above scheme the existence and mode of forming an invariant to ϕ of the order n. This last appears

[* See below, p. 324.]

for the case $n=2$, and ought, for all other values of n, to have been known* to the author of the immortal discovery of invariants, termed by him hyperdeterminants, in the sense which, according to the nomenclature here adopted, would be conveyed by the term hyperdiscriminants.

Before proceeding to discuss the theory of compound total commutants, or enlarging upon that of partial commutants, I shall make an interesting application of the preceding general proposition to the discovery of Aronhold's S and T, the two invariants respectively of the fourth and sixth orders appertaining to a homogeneous cubic function (say F) of three variables x, y, z. These may be termed respectively H_4 and H_6. As to H_6 a theoretically possible but eminently prolix and ungraceful method immediately presents itself, namely to take $F^2 = G$, and after forming the commutant with six lines,

$$\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \end{matrix}$$

to operate with the 6^5 ternary products of which this is made up upon G: the result being an invariant of G, will be so to F, and being of the third degree in respect to the coefficients of G, will be of the sixth in respect to those of F. It will evidently therefore be H_6, or at least a numerical multiple of H_6, the form of which, inasmuch as the only other invariant is H_4, we know in form to be unique. But the general theorem affords another and probably the

* That this was not known explicitly to and should have escaped the penetration of the sagacious author of the theory, and those who had studied his papers, must be attributed to the imperfection of the notation heretofore employed for denoting the coefficients of a homogeneous polynomial function. The umbral method of denoting such a function ϕ of the degree r under the form of $(ax+by+\ldots+cz)^r$, which is equivalent to, but a more compendious and independent mode of mentally conceiving and handling the representation

$$\left(x\frac{d}{dx}+y\frac{d}{dy}+\ldots\ldots+z\frac{d}{dz}\right)\phi,$$

exhibits the true internal constitution of such functions, and necessarily leads to the discovery of their essential properties and attributes.

most practically compendious* solution as regards H_6, of which the question admits.

Let G† represent the mixed concomitant to F formed by the bordered determinant

$$\begin{vmatrix} \frac{d^2F}{dx^2}, & \frac{d^2F}{dxdy}, & \frac{d^2F}{dxdz}, & \xi \\ \frac{d^2F}{dydx}, & \frac{d^2F}{dy^2}, & \frac{d^2F}{dydz}, & \eta \\ \frac{d^2F}{dzdx}, & \frac{d^2F}{dzdy}, & \frac{d^2F}{dz^2}, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}.$$

G is a function of the second order as to x, y, z, and of the like order in respect to ξ, η, ζ, which two systems will be respectively cogredient and contragredient in respect to the x, y, z system in F. In other words, which is all we need to look to, G is a concomitant of F, and so also will be

$$G + \lambda (x\xi + y\eta + z\zeta)^2,$$

which may be termed H. Form now the commutant

$$\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, \end{matrix}$$

this being applied to H will give an invariant (the fact that the march of the substitutions for the systems x, y, z; ξ, η, ζ is contrary, being completely immaterial to the applicability of the general theorem above given);

* Having since this was printed been favoured with a view of some of the proof-sheets of Mr Salmon's most valuable Second Part of his *System of Analytical Geometry* (about to appear, and which is calculated, in my opinion, to awaken a higher idea of and excite a new taste for geometrical researches in this country), I find that I am mistaken in this point; the less symmetrical method operated with by Mr Salmon being decidedly the shortest for practically obtaining S and T in the general case. Symmetry, like the grace of an eastern robe, has not unfrequently to be purchased at the expense of some sacrifice of freedom and rapidity of action.

† G is the mixed concomitant to the given cubic function, which is halfway (so to speak) between it and its polar reciprocal. In fact, when the operation is repeated upon G, which was executed upon the given function to obtain G (that is, when we *border* the Hessian of G in respect to x, y, z, vertically and horizontally with the column and line ξ, η, ζ) the determinant thereby represented becomes the polar reciprocal to the given function.

the commutant so formed will be a cubic function of λ, in which the coefficient of λ^3 is a numerical quantity, that of λ^2 is zero, that of λ is H_4 and the constant term is H_6.

Thus for example let $F = x^3 + y^3 + z^3 + 6mxyz$, then

$$G = \begin{vmatrix} x, & mz, & my, & \xi \\ mz, & y, & mx, & \eta \\ my, & mx, & z, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}$$

and therefore

$$H = \Sigma \{(\lambda - m^2) x^2\xi^2 + (\lambda + m^2) 2yz\eta\zeta + yz\xi^2 - 2mx^2\eta\zeta\},$$

the Σ implying the sum of similar terms with reference to the interchanges between x, ξ; y, η; z, ζ.

In developing the commutant above, the first line may be kept in a fixed position; for the sake of brevity, $(x), (y), (z)$; $(\xi), (\eta), (\zeta)$ may be written in the place of

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \quad \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta},$$

and it will readily be seen that the only effective arrangements will be as underwritten:

$$\begin{array}{ccc}
(x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\
(x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\
(\xi)(\eta)(\zeta) & (\eta)(\zeta)(\xi) & (\zeta)(\xi)(\eta) \\
(\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi)
\end{array}$$

$$\begin{array}{cccccc}
(x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\
(x)(z)(y) & (x)(z)(y) & (\zeta)(\eta)(\xi) & (z)(y)(x) & (y)(x)(z) & (y)(x)(z) \\
(\xi)(\eta)(\zeta) & (\xi)(\zeta)(\eta) & (\xi)(\eta)(\zeta) & (\zeta)(\eta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\xi)(\zeta) \\
(\xi)(\zeta)(\eta) & (\xi)(\eta)(\zeta) & (\zeta)(\eta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\xi)(\zeta) & (\xi)(\eta)(\zeta)
\end{array}$$

$$\begin{array}{cccccc}
(x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\
(x)(z)(y) & (x)(z)(y) & (z)(x)(y) & (z)(y)(x) & (y)(x)(z) & (y)(x)(z) \\
(\eta)(\zeta)(\xi) & (\zeta)(\eta)(\xi) & (\xi)(\zeta)(\eta) & (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) & (\xi)(\zeta)(\eta) \\
(\zeta)(\eta)(\xi) & (\eta)(\zeta)(\xi) & (\zeta)(\xi)(\eta) & (\xi)(\zeta)(\eta) & (\xi)(\zeta)(\eta) & (\eta)(\zeta)(\xi)
\end{array}$$

$$\begin{array}{cccccc}
(x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\
(y)(z)(x) & (z)(x)(y) & (y)(z)(x) & (y)(z)(x) & (z)(x)(y) & (z)(x)(y) \\
(\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) \\
(\xi)(\zeta)(\eta) & (\eta)(\zeta)(\xi) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) & (\xi)(\eta)(\zeta)
\end{array}$$

The signs of the four lines in each of these arrangements are two alike, and two contrary to the signs of the correspondent lines in the first arrangement; hence the effective sign is the same for all, and the result, after rejecting from each term the common factor -16, is seen, from inspection, to be

$$4(\lambda-m^2)^3-8m^3+6(\lambda-m^2)(\lambda+m^2)^2-12m(\lambda+m^2)+2(\lambda+m^2)^3+1,$$

which is equal to

$$12\lambda^3+0\,.\,\lambda^2-12(m-m^4)\lambda+1-20m^3-8m^6;$$

here the coefficients $m-m^4$ and $1-20m^3-8m^6$ are the two invariants (Aronhold's S and T) for the canonical form operated upon; and it will be observed that

$$(1-20m^3-8m^6)^2+64(m-m^4)^3=(1+8m^3)^3,$$

which is easily proved to be the discriminant of

$$x^3+y^3+z^3+6mxyz.$$

It may however be observed, that this is not the discriminant of the function in λ just found, as reasons of analogy* might have suggested it probably would be: in order that this might be the case, the coefficient of λ^3 should be 4 instead of 12, and of λ, $m-m^4$ instead of m^4-m. There is ground for supposing that another function of λ may be found by a different method, in which this relation will take effect.

The theorem above given for simple total commutants admits of an interesting application to the general case of a function F of the nth degree, in respect to each of two independent systems of two variables x, y; ξ, η. Let F be symbolically represented by $(ax+by)^n(\alpha\xi+\beta\eta)^n$, so that $a^n\alpha^n$ represents the coefficient of $x^n\xi^n$, $na^{n-1}b\alpha^n$ of $x^{n-1}y\xi^2$, &c. &c.; then the commutant

$$\begin{array}{ll} a,\ b, & (1) \\ a,\ b, & (2) \\ \cdots & \\ a,\ b, & (n) \\ \alpha,\ \beta, & (1) \\ \alpha,\ \beta, & (2) \\ \cdots & \\ \alpha,\ \beta, & (n) \end{array}$$

will represent a quadratic invariant of F, which will contain $(n+1)^2$ coefficients. By expanding this commutant we obtain a general expression for the invariant under a very interesting form.

* The biquadratic function of x, y having only one parameter, and therefore two invariants, its theory possesses striking analogies to the theory of the cubic function of three letters. The function in λ which gives these invariants for the first-named function, according to the method given in the first section, has the same discriminant as the function itself.

I now proceed to give the general theorem for compound total commutants as applicable to the discovery of invariants.

Let there be a function of m disconnected classes of systems of variables; let the systems in the same class be supposed all distinct but cogredient with one another. The function is supposed to be linear in respect to each system in each class, and the number of systems is the same for all the classes, and the number of variables the same in each system. This function may then be represented symbolically under the form

$$\begin{aligned}&({}^1a_1\,.\,{}^1x_1+{}^1b_1\,.\,{}^1y_1+\ldots+{}^1l_1\,.\,{}^1t_1)\,({}^1a_2\,.\,{}^1x_2+{}^1b_2\,.\,{}^1y_2+\ldots+{}^1l_2\,.\,{}^1t_2)\\ &\qquad\qquad\ldots({}^1a_n\,.\,{}^1x_n+{}^1b_n\,.\,{}^1y_n+\ldots{}^1l_n\,.\,{}^1t_n)\\ &\times({}^2a_1\,.\,{}^2x_1+{}^2b_1\,.\,{}^2y_1+\ldots+{}^2l_1\,.\,{}^2t_1)\,({}^2a_2\,.\,{}^2x_2+{}^2b_2\,.\,{}^2y_2+\ldots+{}^2l_2\,.\,{}^2t_2)\\ &\qquad\qquad\ldots({}^2a_n\,.\,{}^2x_n+{}^2b_n\,.\,{}^2y_n+\ldots{}^2l_n\,.\,{}^2t_n)\\ &\times\text{\&c.}\\ &\times({}^pa_1\,.\,{}^px_1+{}^pb_1\,.\,{}^py_1+\ldots+{}^pl_1\,.\,{}^pt_1)\,({}^pa_2\,.\,{}^px_2+{}^pb_2\,.\,{}^py_2+\ldots+{}^pl_2\,.\,{}^pt_2)\\ &\qquad\qquad\ldots({}^pa_n\,.\,{}^px_n+{}^pb_n\,.\,{}^py_n+\ldots{}^pl_n\,.\,{}^pt_n).\end{aligned}$$

In this expression the $x, y \ldots t$'s are all real, but the $a, b \ldots l$'s all umbral; in fact, ${}^fa_g$, ${}^fb_g$, &c. may be understood to denote $\dfrac{d}{d^fx_g}$, $\dfrac{d}{d^fy_g}$, &c.

The n systems of variables in each of the sets above written are supposed to be cogredient *inter se.*

Take the symbolical product of the first set, first making *for the moment*

$${}^1x_1={}^1x_2=\ldots{}^1x_n=x,\ \text{\&c. \&c.},\quad {}^1t_1={}^1t_2=\ldots{}^1t_n=t\,;$$

and let the coefficients of the several terms

$$x^n,\ x^{n-1}y\ldots\text{\&c.},$$

be called $\qquad {}^1A_1,\ {}^1A_2\ldots{}^1A_\mu,$

where μ is the number of terms contained in a homogeneous function of the nth degree of the m variables $x, y \ldots t$. In like manner proceed with each of the lines, and then write down the commutant

$$\begin{matrix}{}^1A_1, & {}^1A_2\ldots{}^1A_\mu,\\ {}^2A_1, & {}^2A_2\ldots{}^2A_\mu,\\ \ldots\ldots & \ldots\ldots\\ {}^pA_1, & {}^pA_2\ldots{}^pA_\mu.\end{matrix}$$

This commutant is an invariant of F: it will of course be remembered that, unless p is even, the commutant vanishes.

Thus, for example, take two sets of two systems of two variables: in all four systems,

$$x, y;\quad \xi, \eta\ :\ p, q;\quad \phi, \psi,$$

each couple of systems on either side of the colon (:) being cogredient *inter se*: and let F be symbolically represented by

$$(ax+by)(\alpha\xi+\beta\eta)(lp+mq)(\lambda\phi+\mu\psi);$$

then the invariant given by the theorem will be the commutant

$$\begin{array}{l} a\alpha;\ \ a\beta+\alpha b;\ \ b\beta, \\ l\lambda;\ \ l\mu+\lambda m;\ \ m\mu. \end{array}$$

The six positions of this are as below written (the first three being positive and the second three negative)

$$\begin{array}{lll} a\alpha;\ a\beta+\alpha b;\ b\beta, & a\alpha;\ a\beta+\alpha b;\ b\beta, & a\alpha;\ a\beta+\alpha b;\ b\beta, \\ l\lambda;\ l\mu+\lambda m;\ m\mu, & l\mu+\lambda m;\ m\mu;\ l\lambda, & m\mu;\ l\lambda;\ l\mu+\lambda m, \\ & & \\ a\alpha;\ a\beta+\alpha b;\ b\beta, & a\alpha;\ a\beta+\alpha b;\ b\beta, & a\alpha;\ a\beta+\alpha b;\ b\beta, \\ l\mu+\lambda m;\ l\lambda;\ m\mu, & l\lambda;\ m\mu;\ l\mu+\lambda m, & m\mu;\ l\mu+\lambda m;\ l\lambda. \end{array}$$

If we write F under its explicit form,

$$\begin{aligned} &Ax\xi p\phi+Bx\xi p\psi+Cx\xi q\phi+Dx\xi q\psi \\ &+A'x\eta p\phi+B'x\eta p\psi+C'x\eta q\phi+D'x\eta q\psi \\ &+A''y\xi p\phi+B''y\xi p\psi+C''y\xi q\phi+D''y\xi q\psi \\ &+A'''y\eta p\phi+B'''y\eta p\psi+C'''y\eta q\phi+D'''y\eta q\psi, \end{aligned}$$

we have identically the relations following,

$$\begin{array}{llll} a\alpha l\lambda=A, & a\alpha l\mu=B, & a\alpha m\lambda=C, & a\alpha m\mu=D, \\ a\beta l\lambda=A', & a\beta l\mu=B', & a\beta m\lambda=C', & a\beta m\mu=D', \\ b\alpha l\lambda=A'', & b\alpha l\mu=B'', & b\alpha m\lambda=C'', & b\alpha m\mu=D'', \\ b\beta l\lambda=A''', & b\beta l\mu=B''', & b\beta m\lambda=C''', & b\beta m\mu=D''', \end{array}$$

and the commutant expanded becomes

$$\begin{aligned} &A(B'+C''+C'+B'')D'''+(B+C)(D'+D'')A'''+D(A'+A'')(B'''+C''') \\ &-(B+C)(A'+A'')D'''-A(D'+D'')(B'''+C''')-D(B'+C''+C'+B'')A'''. \end{aligned}$$

In the foregoing the x's in the several lines were *for the moment* taken identical, in order the more easily to explain the law of formation of the quantities A. But suppose that they become actually identical for the same line. F then becomes a function of the nth degree in respect to each of p systems of variables, and may be represented symbolically under the form

$$({}^1a^1x + {}^1b^1y + \ldots + {}^1l^1t)^n \times ({}^2a^2x + {}^2b^2y + \ldots + {}^2l^2t)^n$$
$$\ldots \times ({}^pa^px + {}^pb^py + \ldots + {}^pl^pt)^n.$$

We may still further limit the generality of the theorem by supposing

$$\begin{aligned} {}^1x &= {}^2x = \ldots {}^px = x, \\ {}^1y &= {}^2y = \ldots {}^py = y, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ {}^1t &= {}^2t = \ldots {}^pt = t; \end{aligned}$$

F then becomes $\qquad (ax + by + \ldots + lt)^{np}$.

Accordingly, as many different factors as can be found contained an even number of times in the exponent of the function, so many invariants can be formed immediately from a function of any number of variables m by the method of total commutation.

If one of these factors be called n, the commutant corresponding thereto will be of the order

$$\frac{(n+1)(n+2)\ldots(n+m-1)}{1\,.\,2\ldots(m-1)}$$

in respect to the coefficients. Thus take $m = 2$, so that

$$F = (ax + by)^{np}.$$

The general form of such a commutant will be found by taking A_1, $A_2 \ldots A_{n+1}$ the coefficients of the several combinations of x, y in $(ax + by)^n$, from which the numerical coefficients n, $\frac{1}{2}n(n-1)$, &c. may be rejected, as only introducing a numerical factor into the result; the commutant will therefore be expressed by means of the form

$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \ldots;\ b^n, \qquad (1)$$

$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \ldots;\ b^n, \qquad (2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \ldots;\ b^n. \qquad (p)$$

If $p = 2$, the compound commutant

$$a^n;\ a^{n-1}b;\ \ldots;\ b^n,$$
$$a^n;\ a^{n-1}b;\ \ldots;\ b^n,$$

will easily be seen to be only another form for the catalecticant of $(ax+by)^{2n}$. Thus, let $n=2$,

$$(ax+by)^4 = Ax^4 + 4Bx^3y + 6Cx^2y^2 + 4Dxy^3 + Ey^4\,;$$

so that $$a^4=A,\quad a^3b=B,\quad a^2b^2=C,\quad ab^3=D,\quad b^4=E.$$

The commutant (which is of the form of the matrix to an ordinary determinant, with the exception that the umbræ enter compoundly instead of simply into the several terms separated by the marks of punctuation), will be

$$\begin{array}{ccc} a^2; & ab\,; & b^2, \\ a^2; & ab\,; & b^2: \end{array}$$

this, written in the six forms

$$\left.\begin{array}{ccc} a^2; & ab; & b^2 \\ a^2; & ab; & b^2 \end{array}\right\} \left.\begin{array}{ccc} a^2; & ab; & b^2 \\ a^2; & b^2; & ab \end{array}\right\} \left.\begin{array}{ccc} a^2; & ab; & b^2 \\ ab; & a^2; & b^2 \end{array}\right\}$$

$$\left.\begin{array}{ccc} a^2; & ab; & b^2 \\ b^2; & ab; & a^2 \end{array}\right\} \left.\begin{array}{ccc} a^2; & ab; & b^2 \\ ab; & b^2; & a^2 \end{array}\right\} \left.\begin{array}{ccc} a^2; & ab; & b^2 \\ b^2; & a^2; & ab \end{array}\right\}$$

gives the expression

$$a^4 \times a^2b^2 \times b^4 - a^4 \times (ab^3)^2 - b^4 \times (a^3b)^2 - (a^2b^2)^3 + 2a^3b \times ab^3 \times a^2b^2\,;$$

that is $$ACE - AD^2 - EB^2 - C^3 + 2BCD.$$

One important observation may here be made of a fact which otherwise might easily escape attention, which is, that commutants, where the same terms simple or compound are found in all or several of the lines, in general give rise to products, some of them equal and with the same sign, and others equal but with the *contrary* sign.

This last phenomenon does not manifest itself in commutants appertaining to functions of two variables of the two particular and different species which first and most naturally present themselves, namely where there are only two lines or only two columns*—I believe that it displays itself in every other case of commutantives to functions of two variables. Thus it is that algebraical expressions derived from given functions disguise their symmetry; to make which come to light it becomes necessary to add terms of contrary sign to such expressions. As an example, the reader is invited to develope the cubic invariant of a function of x and y, symbolically expressed by $(ax+by)^8$, where

$$a^8 = A,\quad a^7b = B \ldots ab^7 = H,\quad b^8 = I,$$

* These commutants give respectively the quadrinvariant and the catalecticant, the former of which alone was formerly recognised by Mr Cayley as a commutant.

by means of the commutant

$$\begin{matrix} a^2, & ab, & b^2, \\ a^2, & ab, & b^2, \\ a^2, & ab, & b^2, \\ a^2, & ab, & b^2.* \end{matrix}$$

Suppose F to be the general even-degreed function of two variables of the degree $2np$.

Let $$H = \left(\xi \frac{d}{dy} - \eta \frac{d}{dx}\right)^{np} F + \lambda (x\xi + y\eta)^{np},$$

and express H umbrally under the form

$$(ax + by)^{np} (\alpha\xi + \beta\eta)^{np}.$$

* [See p. 346 below.] The number of terms resulting from the independent permutation of each of the 3 linear lines is 6^3, that is 216; but the actual result is (using small letters instead of large) $P - Q$, where

$$P = aei + 3ag^2 + 12beh + 3c^2i + 24cf^2 + 24d^2g + 15e^3,$$
$$Q = 4afh + 4bid + 8bgf + 22ceg + 8chd + 36def,$$

so that the effective number of permutations is only 164. The difference between this and 216 divided by 216 may be termed the Index of Demolition, which we see in this case is $\frac{52}{216}$ or $\frac{13}{54}$; that is, somewhat less than $\frac{1}{4}$. For the cubic invariant of the function of the fourth degree this index is zero, all the permutations being effective. If we take the cubic invariant of the function $ax^{12} + 12bx^{11}y + 66cx^{10}y^2 + \&c. + my^{12}$ under the form $P - Q$, we shall find

$$P = 6ahl + 10ajj + 6bfm + 54bhk + 54cfl + 155cii + 10ddm + 430dgj$$
$$+ 155eek + 520ehh + 520ffl + 280ggg,$$
$$Q = agm + 15aik + 30bgl + 50bij + 15cem + 4cgk + 150chj + 30del + 210dfk$$
$$+ 250dhi + 230efj + 555egl + 660fgh.$$

The number of terms in P and Q is of course the same, and will be found to be 2200 for each; so that out of the 6^5, that is 7776 permutations of the 5 lower rows, only 4400 are effective, and the index of demolition becomes $\frac{3300}{7776}$, that is $\frac{275}{648}$, or rather greater than $\frac{5}{12}$. The Index of Demolition thus goes on constantly increasing as the degree of the function rises; probably (?) it converges either towards $\frac{1}{2}$ or else towards unity. In arranging the terms it will be found most convenient to adopt, as I have done above, the dictionary method of sequence. The computations are greatly facilitated by the circumstance of the effect of any arrangement of each of the 5 lower lines not being altered when these *lines* are permuted with one another; this gives rise to the subdivision of the 7776 permutations into groups as follows: 6 of 120 identical terms, 60 of 60, 36 of 20, 60 of 30, 24 of 20, 30 of 10, 30 of 5, and 6 of 1. So that the total number of permutational arrangements to be constructed is only 252. Other methods of abridging the labour will readily suggest themselves to the practical computer. The total number of the *groups* of terms is of course always known *à priori*, and, for instance, in the case before us, must be equal to the number of ways in which $\frac{1}{2}(12 \times 3)$, that is the number 18, can be divided into 3 parts, none of which is to exceed the number 12, that is 25; for the cubic invariant of the function of the eighth degree of two variables it is the number of ways in which 12 can be divided into 3 parts, of which none shall exceed 8, and so forth, zeros being always understood to be admissible; and of course in general for an invariant of the order r to a function of the degree n of i variables, the number of distinct terms is in general the number of ways in which $\frac{nr}{i}$ can be divided into r parts, of which none shall exceed n, subject however always to the possibility in particular cases of a diminution in consequence of some of the groups assuming zero for their coefficient.

The commutant

$$\begin{array}{lr} a^n, \quad a^{n-1}b \ldots b^n, & (1) \\ a^n, \quad a^{n-1}b \ldots b^n, & (2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a^n, \quad a^{n-1}b \ldots b^n, & (p) \\ \alpha^n, \quad \alpha^{n-1}\beta \ldots \beta^n, & (1) \\ \alpha^n, \quad \alpha^{n-1}\beta \ldots \beta^n, & (2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \alpha^n, \quad \alpha^{n-1}\beta \ldots \beta^n, & (p) \end{array}$$

will be a function of λ, and all the several coefficients will be invariants of F*.

When $p = 1$ we obtain the Λ given in the preceding section, and originally published by me in the *Philosophical Magazine* for the month of November, 1851. The Λ obtained on this supposition has for its coefficients a series of independent invariants, commencing with the catalecticant and closing with the quadratic invariant. When p has any other value, we observe a similar series commencing with a commutantive invariant of a lower order than the catalecticant, but always closing with the quadratic invariant. Thus, for example, when $2np=8$, we may obtain by the preceding theorem three different quadratic functions; one giving the invariants of the orders 5, 4, 3, 2, the second those of the orders 3, 2, the third the invariant of the order 2.

In this case the invariants of the same order given by the different Λ's are the same to numerical factors *près*. Whether this is always necessarily the case is a point reserved for further examination.

The commutants applied in the preceding theorems have been called by me total commutants, because the total of each line of umbræ is permuted in every possible manner. If the lines be divided into segments, and the permutation be local for each segment instead of extending itself over the whole line, we then arrive at the notion of partial commutants, to which I have also (in concert with Mr Cayley) given the distinctive name of Intermutants. In order to find the invariants of functions of odd degrees, the theory of total commutants requires the process of commutation to be applied, not immediately to the coefficients of the proposed function, but to some derived concomitant form. I became early sensible of this imperfection, and stated to the friend above named, to whom I had previously

* By substituting the symbols $\frac{d}{dx}, \frac{d}{dy}$, &c. in place of the *umbræ* a, b, &c., the theorem is easily stated for covariants generally. But in applying the commutantive method to obtain covariants, or rather in the statement of the results flowing from each application, it is never necessary to go beyond the case of invariants, because the commutantive covariants of any given homogeneous function are always identical with commutantive invariants of emanants of the same function.

imparted my general method of total commutation, my conviction of the existence of a qualified or restricted method of permutation, whereby the invariants of the cubic function, for instance, of two and of three letters would admit, without the aid of a derived form, of being represented. Many months ago, when I was engaged in this important research, and had made some considerable steps towards the representation of the invariant, that is, the discriminant of the cubic function of x and y, under the form of a single permutant, I was surprised by a note from the friend above alluded to, announcing that he had succeeded in fixing the form of the permutant of which I was at that moment in search. It is with no intention of complaining of this interference on the part of one to whose example and conversation I feel so deeply indebted, (and the undisputed author of the theory of Invariants,) that I may be permitted to say that, independent of the intervention of this communication, I must inevitably have succeeded in shaping my method so as to furnish the form in question; and that with greater certainty, after my theory of commutants had furnished me with the precedent of permutable forms giving rise to terms identical in value but affected with contrary signs. As I have understood that Mr Cayley is likely to develop this part of the subject in the present number of the *Journal*, it will be the less necessary for me to enter at any length into the theory of partial commutants on the present occasion.

The method of partial commutation is a simple but most important corollary from that of total commutation hereinbefore explained. To fix the ideas, conceive a class of p cogredient systems, and that there are qr such classes perfectly independent. Proceed to divide these qr classes in any manner whatever into r sets, each containing q classes; and form the symbol of the total commutant corresponding to each such set. Now let these commutants be placed side by side against one another, and transpose the terms in each compound line thus formed once for all, but in any arbitrary manner. Then permute in every possible way all those symbols in each line, *inter se*, which belong to the same class, and operate with the symbols thus produced by reading off the vertical columns and attending to the rule of the + and − signs, as in the case of a total commutant; the result will be a commutant of the form operated upon. For instance, let $p=1$, $q=3$, $r=2$, and let the number of variables in each system be 2. Form the commutant operators

$$\begin{array}{cc|cc} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{d\xi}, & \frac{d}{d\eta}, \\[1ex] \frac{d}{dp}, & \frac{d}{dt}, & \frac{d}{d\phi}, & \frac{d}{d\theta}, \\[1ex] \frac{d}{dr}, & \frac{d}{ds}, & \frac{d}{d\rho}, & \frac{d}{d\sigma}. \end{array}$$

Interchange *in any manner* but *once for all* the symbols in each line, as thus:

$$\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{d\xi},\ \frac{d}{d\eta},$$
$$\frac{d}{d\phi},\ \frac{d}{dp},\ \frac{d}{dt},\ \frac{d}{d\theta},$$
$$\frac{d}{ds},\ \frac{d}{d\rho},\ \frac{d}{dr},\ \frac{d}{d\sigma}.$$

Now permute, *inter se*, the variables of each system, as

$$\frac{d}{dx},\ \frac{d}{dy};\ \frac{d}{dp},\ \frac{d}{dt},\ \&c.;$$

the total number of the operative forms resulting will be $(1\,.\,2)^6$, and the sum of the $(1\,.\,2)^6$ quantities, half positive and half negative, formed after the type of

$$\Sigma\left\{\begin{array}{l} \dfrac{d}{dx}\dfrac{d}{d\phi}\dfrac{d}{ds}U\times\dfrac{d}{dy}\dfrac{d}{dp}\dfrac{d}{d\rho}U \\ \times\dfrac{d}{d\xi}\dfrac{d}{d\theta}\dfrac{d}{dr}U\times\dfrac{d}{d\eta}\dfrac{d}{dt}\dfrac{d}{d\sigma}U \end{array}\right\},$$

U being supposed to be a function homogeneous in

$$x, y;\ \xi, \eta;\ p, t;\ \phi, \theta;\ r, s;\ \rho, \sigma,$$

will be a covariant of U.

The proof of the truth of this proposition is contained in what is shown in the Notes of the Appendix for total commutants, it being only necessary to make the systems which are independent vary consecutively, and then apply the inference to the supposition of their varying simultaneously.

It may be extended to the more general supposition of classes of an unequal number of cogredient systems of unequal numbers of variables in each, the only condition apparently required being that the number of distinct terms shall be the same in each line of the final commutantive operator. The important remark to be made is, that in applying this theorem there is nothing to prevent any of the systems being made *identical*; or, in other words, a given function of one system of variables may be regarded as a function of as many different, although coincident, sets as we may choose to suppose. Thus, suppose

$$U = Ax^2 + 2Bxy + Cy^2,$$

we may take the partial commutant formed of the two total commutant operators

$$\frac{d}{dx},\ \frac{d}{dy},$$
$$\frac{d}{dx},\ \frac{d}{dy},$$

combined with itself. If we write them in the same order,

$$\begin{array}{cccc} \dfrac{\dot{d}}{dx}, & \dfrac{\dot{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\acute{d}}{dy}, \\[2ex] \dfrac{\dot{d}}{dx}, & \dfrac{\dot{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\acute{d}}{dy}, \end{array}$$

(where I use the dots and dashes to distinguish those in the same line which are considered as belonging to the same class, and therefore as permutable, *inter se*), we shall evidently obtain $4\{AC-B^2\}^2$; if we commence with a permutation, so as to have the form of operation

$$\begin{array}{cccc} \dfrac{\dot{d}}{dx}, & \dfrac{\dot{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\acute{d}}{dy}, \\[2ex] \dfrac{\dot{d}}{dx}, & \dfrac{\acute{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\dot{d}}{dy}, \end{array}$$

it will be found that we obtain $2\{AC-B^2\}^2$.

Again, suppose that we have

$$U = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3.$$

If we write

$$\begin{array}{cccc} \dfrac{\dot{d}}{dx}, & \dfrac{\dot{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\acute{d}}{dy}, \\[2ex] \dfrac{\dot{d}}{dx}, & \dfrac{\dot{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\acute{d}}{dy}, \\[2ex] \dfrac{\dot{d}}{dx}, & \dfrac{\dot{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\acute{d}}{dy}, \end{array}$$

the value of the commutant would come out zero; but if we make a permutation, and write

$$\begin{array}{cccc} \dfrac{\dot{d}}{dx}, & \dfrac{\dot{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\acute{d}}{dy}, \\[2ex] \dfrac{\dot{d}}{dx}, & \dfrac{\dot{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\acute{d}}{dy}, \\[2ex] \dfrac{\dot{d}}{dx}, & \dfrac{\acute{d}}{dy}, & \dfrac{\acute{d}}{dx}, & \dfrac{\dot{d}}{dy}, \end{array}$$

the operation indicated by the above performed upon U, will give a multiple of the discriminant of U.

In like manner we may represent Aronhold's Sextic Invariant of the form $(x, y, z)^3$ by means of the partial commutant

$$\begin{array}{cccccc}
\frac{\dot{d}}{dx}, & \frac{\dot{d}}{dy}, & \frac{\dot{d}}{dz}, & \frac{\acute{d}}{dx}, & \frac{\acute{d}}{dy}, & \frac{\acute{d}}{dz}, \\
\frac{\acute{d}}{dx}, & \frac{\dot{d}}{dy}, & \frac{\dot{d}}{dz}, & \frac{\dot{d}}{dx}, & \frac{\acute{d}}{dy}, & \frac{\acute{d}}{dz}, \\
\frac{\acute{d}}{dx}, & \frac{\acute{d}}{dy}, & \frac{\dot{d}}{dz}, & \frac{\dot{d}}{dx}, & \frac{\dot{d}}{dy}, & \frac{\acute{d}}{dz}.
\end{array}$$

If we make

$$V=\left(\xi'\frac{d}{d\xi}+\eta'\frac{d}{d\eta}+\zeta'\frac{d}{d\zeta}\right)\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\zeta\frac{d}{dz}\right)^2(x, y, z)^3,$$

and use H to signify the determinant

$$\begin{vmatrix} x, & y, & z \\ \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \end{vmatrix},$$

which is evidently an universal triple covariant, and make

$$W=V+\lambda H,$$

and apply to W the partial commutantive symbol

$$\begin{array}{cccccc}
\frac{\dot{d}}{dx}, & \frac{\dot{d}}{dy}, & \frac{\dot{d}}{dz}, & \frac{\acute{d}}{dx}, & \frac{\acute{d}}{dy}, & \frac{\acute{d}}{dz}, \\
\frac{\acute{d}}{d\xi}, & \frac{\dot{d}}{d\eta}, & \frac{\dot{d}}{d\zeta}, & \frac{\dot{d}}{d\xi}, & \frac{\acute{d}}{d\eta}, & \frac{\acute{d}}{d\zeta}, \\
\frac{\acute{d}}{d\xi'}, & \frac{\acute{d}}{d\eta'}, & \frac{\dot{d}}{d\zeta'}, & \frac{\dot{d}}{d\xi'}, & \frac{\dot{d}}{d\eta'}, & \frac{\acute{d}}{d\zeta'},
\end{array}$$

we shall obtain a function of λ of which all the odd powers and the second power will disappear, and such that the coefficients of λ^2 and the constant term will be Aronhold's S and T, and the discriminant of the entire function in respect to λ^2 (if not for the distribution assigned to the dots and dashes in the foregoing, at least for some other distribution) may not improbably be the discriminant of the given function $(x, y, z)^3$.

NOTES IN APPENDIX.

(1) [p. 295 above.] More generally, in as many ways as the number n can be divided into parts, in so many ways can a given function of one set of variables be as it were *unravelled* so as to furnish concomitant forms.

For instance, the form $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ has for a concomitant

$$aux + buy + bvx + cvy + cwx + dwy,$$

where u, v, w are cogredient with x^2, $2xy$, y^2; and also

$$auu'x + buu'y + buv'x + bvu'x + cvv'x + cvu'y + cuv'y + dvv'y,$$

where u, v; u', v' are cogredient with each other and with x and y; and the proposition in the text may be best derived from this more general theorem by dividing the index into equal parts, forming as many systems as there are such parts, and then identifying the systems so formed.

(2) [p. 297 above.] The following additional example will illustrate the power of this method.

Let $\phi = (x, y, z)^4$ be the general function of the fourth degree. Form by *unravelment* the concomitant form $(u, v, w, p, q, r)^2$ (say P) where u, v, w, p, q, r are cogredient with x^2, y^2, z^2, $2zy$, $2xz$, $2yx$.

Again, the universal concomitant $(x\xi + y\eta + z\zeta)^2$ will have for its concomitant

$$u\xi^2 + v\eta^2 + w\zeta^2 + p\eta\zeta + q\zeta\xi + r\xi\eta,$$

where ξ, η, ζ are contragredient to x, y, z. Now take the reciprocal polar of this last form with respect to ξ, η, ζ; that is,

$$\Sigma\,(vw - \tfrac{1}{4}p^2)\,x_1^2 + 2\Sigma\,(\tfrac{1}{4}qr - \tfrac{1}{2}pu)\,y_1z_1 \text{ (say } G),$$

where x_1, y_1, z_1, being contragredient to ξ, η, ζ, will be cogredient with x, y, z. $P + \lambda G$ is a quadratic function of the six variables u, v, w, p, q, r, and its discriminant will give a function of λ of the sixth degree, all of whose even coefficients will be covariants of ϕ. If we replace x_1, y_1, z_1 by x, y, z, these even coefficients will be respectively (understanding that *order* refers to the dimensions *quoad* the coefficients of ϕ and *degree* to the dimensions *quoad* x, y, z) as follows:

Of order	6	degree	0,
"	5	"	2,
"	4	"	4,
"	3	"	6,
"	2	"	8,
"	1	"	10,
"	0	"	12.

The two last coefficients must evidently be identically zero. It is *possible* that some of the others may be so too: as regards the one of the third order and sixth degree, this is of the same form as, and may be identical with, the Hessian of ϕ; as regards the one of the fourth order and fourth degree, this may be ϕ itself multiplied by the cubic invariant (which the theory of Section III. proves to exist) of ϕ. But the covariants of the fifth order and second degree, and of the second order and eighth degree, if they are not identically zero, and if the latter is not ϕ^2 (which a trial or two of some very simple cases will easily establish one way or the other) are probably irreducible forms. The existence of a correlated conic section to a curve of the fourth order, if established, would be particularly interesting, and its geometrical meaning would well deserve being elicited.

(3) [p. 303 above.] If any form (f) of the degree n be written symbolically,

$$(a_1x_1 + a_2x_2 + \ldots + a_\iota x_\iota)^n,$$

where $x_1, x_2 \ldots x_\iota$ are real but $a_1, a_2 \ldots a_\iota$ umbral, and if I_r be any invariant of the order r in respect of the real coefficients of (f), it is easily seen by reason of I_r remaining unaltered when $x_1, x_2 \ldots x_\iota$ become respectively $f_1x_1, f_2x_2 \ldots f_\iota x_\iota$, provided that $f_1, f_2 \ldots f_\iota = 1$, that each term in I_r expressed by means of the umbræ, must contain an equal number of times $a_1, a_2 \ldots a_\iota$, so that each such term will contain $\frac{nr}{\iota}$ of each of them, of course differently subdivided and grouped; hence we have the universal condition that $\frac{nr}{\iota}$ must be an integer; but this is less stringent than the actual condition, which is that $\frac{nr}{\iota}$ must be an integer of a certain form; for instance, as before observed, when $\iota = 2$, $\frac{nr}{\iota}$ must be an *even* integer.

(4) [p. 307 above.] To prove the theorem given in the text for total simple commutants it is only necessary to bear in mind that whenever two columns in any total commutant become identical, the commutant vanishes. To fix the ideas, take the commutant formed of lines similar to $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, written

under one another; let there be r such lines, the total number of terms will be $(1.2.3)^r$: the $1.2.3$ positions of the line written above will correspond to $(1.2.3)^{r-1}$ several groupings of the remaining lines. Now when x, y, z undergo a unimodular linear substitution, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will undergo a related substitution not coincident with that of x, y, z, but still unimodular; let x, y, z change, all the other systems remaining fixed, and suppose $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ to become respectively

$$f\frac{d}{dx}+g\ \frac{d}{dy}+h\ \frac{d}{dz},$$

$$f'\frac{d}{dx}+g'\ \frac{d}{dy}+h'\ \frac{d}{dz},$$

$$f''\frac{d}{dx}+g''\frac{d}{dy}+h''\frac{d}{dz},$$

then each of the $(1.2.3)^{r-1}$ groups of the terms arising from the permutation of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will subdivide into 27 groups, of which we may reject those in which any of the terms $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$ occurs twice or three times; accordingly there will be left only the six effective orders of permutations,

$$\left(f\frac{d}{dx}, g'\frac{d}{dy}, h''\frac{d}{dz}\right); \left(f\frac{d}{dx}, h'\frac{d}{dz}, g''\frac{d}{dy}\right); \text{ \&c.}$$

consequently each of the $(1.2.3)^{r-1}$ groups gives rise to 6 times 6 products whose sum will be $\begin{vmatrix} f'', & g'', & h'' \\ f', & g', & h' \\ f, & g, & h \end{vmatrix}$ × the sum of the 6 products corresponding to the permutations of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$; and therefore, the transformations being unimodular, the sum of the products corresponding to the entire $(1.2.3)^r$ permutations remains constant when x, y, z change. In like manner, all the systems may change one after the other, and consequently all of them at the same time without affecting the value of the commutant: and in like manner for the general case. Q.E.D.

(5) [p. 312 above.] The truth of the proposition relative to compound commutants and the mode of the demonstration will be apparent from the subjoined example.

Let the function be supposed to be

$$(ax+by)(a'x'+b'y')(\alpha\xi+\beta\eta)(\alpha'\xi'+\beta'\eta'),$$

where x, y; x', y' are cogredient and ξ, η; ξ', η' cogredient; the a, b, α, β, &c. are of course mere umbræ. Now take the compound commutant

$$\begin{array}{ccc} aa', & ab'+a'b, & bb', \\ \alpha\alpha', & \alpha\beta'+\alpha'\beta, & \beta\beta'. \end{array}$$

Let x, y; x', y' undergo a linear substitution, and, accordingly,

$$\begin{array}{lll} \text{let } a & \text{become} & fa+gb, \\ a' & \text{,,} & fa'+gb', \\ b & \text{,,} & ha+kb, \\ b' & \text{,,} & ha'+kb', \end{array}$$

f, g, h, k being of course actual and not umbral; then the above commutant will be easily seen to decompose into 6 others, which will be equal to the original commutant multiplied by the determinant

$$\begin{vmatrix} f^2, & 2fg, & g^2 \\ fh, & fk+gh, & gk \\ h^2, & 2hk, & k^2 \end{vmatrix},$$

which is equal to $(fk-gh)^3$, that is $=1$.

And so in general, which shows, as in the preceding note, that all the classes of cogredient systems may be transformed successively one after the other, and therefore simultaneously, without altering the value of the commutant.

(6) In the last May Number* of the *Journal*, Mr Boole, to whose modest labours the subject is perhaps at least as much indebted as to any one other writer, has given a theorem†, (14) p. 94, the excellent idea contained in which there is no difficulty in shaping so as to render it generalizable by aid of the theory of contravariants. It may be regarded in some sort a pendant or reciprocal to the Eisenstein-Hermite theorem, presented by me under a wider aspect in the First Section of this paper.

[* *Camb. and Dub. Math. Journ.* Vol. VI. (1851), pp. 87—106.]

† Mr Boole applied his theorem to obtain the cubic invariant of $(x, y)^4$, say $\phi(x, y)$, by operating upon its Hessian with $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right)$. More generally, when $\phi(x, y)=(x, y)^{2n}$, the catalecticant of the antepenultimate emanant of ϕ is also of the degree $2n$; and this, when operated upon by $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right)$, will give an invariant of the order $n+1$, which is probably identical with the catalecticant of ϕ itself. There exists a most interesting transformation of the catalecticant of any emanant of a function of any degree in x, y, whether even or odd, under the form of a determinant some of the lines of which contain combinations only of x and y, without any of the coefficients, and all the rest the coefficients only of the given function without x or y. The Hessian being the catalecticant of the second emanant is of course included within this statement.

Let $\phi(x, y \ldots z)$ have any contravariant $\theta(x, y \ldots z)$; then will

$$\phi\left(\frac{d}{dx}, \frac{d}{dy} \cdots\cdots \frac{d}{dz}\right) . \theta(x, y \ldots z)$$

be a contravariant of ϕ. For orthogonal transformations the terms contravariant and covariant coincide, and the theorem for this case appears to have been known to Mr Boole, see (15), same page. More generally, if ψ and θ be any two concomitants of ϕ, the algebraical product $\psi\theta$ will also be a concomitant of ϕ, provided that the systems of variables in ψ and θ have all distinct names, or that those which bear the same names are cogredient with one another. If this proviso does not hold good, the product in question will evidently be no longer a concomitant of ϕ. Let however Ψ denote what ψ becomes, and ϑ what θ becomes, when in place of the variables $x, y \ldots z$ of every two contragredient synonymous systems in ψ and θ we write $\frac{d}{dx}, \frac{d}{dy} \cdots\cdots \frac{d}{dz}$, then will $\vartheta\psi$ and $\Psi\theta$ be each of them concomitants of ϕ, the synonymous systems becoming cogredient with ψ in the one case and with θ in the other.

(7) There is one principle of *paramount* importance which has not been touched upon in the preceding pages, which I am very far from supposing to exhaust the fundamental conceptions of the subject, (indeed, not to name other points of enquiry, I have reason to suppose that the idea of contragredience itself admits of indefinite extension through the medium of the reciprocal properties of commutants; the particular kind of contragredience hereinbefore considered having reference to the reciprocal properties of ordinary determinants only).

The principle now in question consists in introducing the idea of *continuous* or *infinitesimal* variation into the theory. To fix the ideas, suppose C to be a function of the coefficients of $\phi(x, y, z)$, such that it remains unaltered when x, y, z become respectively fx, gy, hz, provided that $fgh = 1$. Next, suppose that C does not alter when x becomes $x + ey + \epsilon z$, when e and ϵ are indefinitely small: it is easily and obviously demonstrable that if this be true for e and ϵ indefinitely small, it must be true for *all* values of e and ϵ. Again, suppose that C alters neither when x receives such an infinitesimal increment, y and z remaining constant, nor when y nor z separately receive corresponding increments, z, x and x, y in the respective cases remaining constant; it then follows from what has been stated above that this remains true for finite increments to x or y or z separately; and hence it may easily be shown that C will remain constant for any *concurrent* linear transformations of x, y, z, when the modulus is unity. This all-important principle enables us at once to fix the form of the symmetrical functions of the roots of $\phi\left(\frac{x}{y}, 1\right)$ which represent invariants of $\phi(x, y)$ when the coefficient of the highest power of x is made unity. It also *instantaneously* gives the necessary and *sufficient* conditions to which an invariant of any given order of any homogeneous function whatever is subject, and thereby reduces the problem of discovering invariants to a definite form. But as these conditions coincide with those which have been stated to me as derived from other considerations by the gentleman whose labours in this department are *concomitant* with my own, I feel myself bound to abstain from pressing my conclusions until he has given his results to the press.

(8) By aid of the general principle enunciated in Note (6) above, we can easily obtain Aronhold's S and T. Let U be the given cubic function of x, y, z, and let $G(x, y, z;\ \xi, \eta, \zeta)$ be the polar reciprocal in respect to ξ, η, ζ of $\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\zeta\frac{d}{dz}\right)^2 U$, then $G(\xi, \eta, \zeta;\ x, y, z)$ as well as the former G will be a concomitant to U, but the homonymous systems of variables in the two G's will be contragredient; and, accordingly,

$$G\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz};\ \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}\right).G(\xi, \eta, \zeta;\ x, y, z)$$

will be a concomitant to U; this concomitant is readily seen to be an invariant of the fourth order; that is, Aronhold's S. Again, from S, by means of the Eisenstein-Hermite theorem, we may derive a form $K(x, y, z)$ of the third degree in x, y, z, and whose coefficients will be of three dimensions; and, accordingly, if the Hessian of U be called $H(U)$,

$$K\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right).H(U)$$

will be a Sextic Invariant of U, that is, Aronhold's T.

43.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

[*Cambridge and Dublin Mathematical Journal*, VII. (1852), pp. 179—217.]

PART I. SECTION IV. *Reciprocity, also Properties and Analogies of certain Invariants, &c.*

IT will hereafter be found extremely convenient to represent all systems of variables cogredient with the original system in the primitive form by letters of the Roman, and all contragredient systems by letters of the Greek alphabet; the rules for concomitance may then be applied without paying any regard to the distinction between the direction of the march of the substitutions, the variables at the close of each operation as it were telling their own tale in respect of being cogredients or contragredients. This distinction has not (as it should have) been uniformly observed in the preceding sections; as, for instance, in the notation for emanants which have been derived by the application of the symbol $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \&c.\right)^2$, instead of the more appropriate one $\left(x' \frac{d}{dx} + y' \frac{d}{dy} + \&c.\right)^2$.

The observations in this section will refer exclusively to points of doctrine which have been started in the preceding sections in such order as they more readily happen to present themselves. And, first, as to some important applications of the reciprocity method referred to in Notes (6) and (8) of the Appendix [pp. 325, 327 above].

The practical application of this method will be found greatly facilitated by the rule that x, y, z, &c. may always in any combination of concomitants be replaced respectively by $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c., and *vice versâ*. I shall apply this prolific principle of reciprocity to elucidate some of the properties and relations of Aronhold's S and T, and certain other kindred forms. This S and T are the quartinvariant and sextinvariant respectively of a cubic of three variables. I give the names of s and t to the quadrinvariant and cubinvariant of the quartic function of two variables. Furthermore, whoever will consider attentively the remarks made in Section II. of the foregoing relative to reciprocal polars, will apprehend without any difficulty that to every invariant of a function of any degree of any number of variables will correspond a contravariant of a function of the same degree of variables one more in number, and that between such invariants, whatever relations exist expressed independently of all other quantities, precisely the same relations must exist between the corresponding contravariants. Thus, then, to s and t the two invariants of $(x, y)^4$ will correspond two contravariants σ and τ of $(x, y, z)^4$, and to S and T the two invariants of $(x, y, z)^3$ will correspond Σ and ϑ two contravariants of $(x, y, z, t)^3$. Calling r the resultant of $(x, y)^4$, R the resultant of $(x, y, z)^3$, ρ the polar reciprocal, or, more briefly, the reciprocant of $(x, y, z)^4$, and (R) the reciprocant of $(x, y, z, t)^3$, we have the following equations (presuming that all the quantities are previously affected with the proper numerical multipliers), namely

$$r = s^3 + t^2, \qquad \rho = \sigma^3 + \tau^2,$$

$$R = S^3 + T^2, \quad (R) = \Sigma^3 + \vartheta^2.$$

I propose in this First Annotation to point out the remarkable analogies which exist between the modes of generating the four pairs of quantities s, t, &c., the functions severally corresponding to which I shall call u, ω, U, Ω. The Hessian corresponding to any of these functions will be denoted by an H prefixed, and when we have to consider, not the pure Hessian, but the matrix formed from it by adding a vertical and horizontal border of variables, the same in number but contragredient to the variable of the function (as, for instance, the Hessian of u bordered with ξ, η horizontally and vertically, or of U with ξ, η, ζ), then I shall denote the result by the ruled symbol $\bar{H}$, and if there be occasion to add two borders, as $\xi, \eta, \zeta;\ \xi', \eta', \zeta'$, both repeated in the horizontal and vertical directions, the result will be typified by the doubly ruled $\bar{\bar{H}}$.

Now, in the first place, as observed by me in Note (8) of the Appendix in the last number; if we call the coefficients of U (10 in number) a, b, c, d, &c., we have

$$S = \bar{H}\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta};\ \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right\}\bar{H}\{x, y, z;\ \xi, \eta, \zeta\},$$

also

$$T = \frac{dS}{da}\frac{d^3H}{dx^3} + \frac{dS}{db}\frac{d^3H}{d^2x\,dy} + \frac{dS}{dc}\frac{d^3H}{d^2x\,dz} + \&c.$$

I will now add the further important relation

$$S^2 = \frac{dT}{da}\frac{d^3H}{dx^3} + \frac{dT}{db}\frac{d^3H}{d^2x\,dy} + \frac{dT}{dc}\frac{d^3H}{d^2x\,dz} + \&c.^*,$$

* It will be found hereafter convenient to designate contravariants formed in this manner from invariants as *Evects* of such invariants or contravariants, and according to the number of times that such process of derivation is applied, 1st, 2nd, 3rd, &c. evects. Such evects form a peculiar class, and when considered generally, without reference to the base to which they refer, they may be termed evectants. Evectants will be again distinguishable according as their base is an invariant simply or a contravariant. Perhaps the terms pure and affected evectants may serve to mark this distinction.

so that it will be observed if all the derivatives of S are zero, T is zero, and *vice versâ.*

Precisely in the same way, using h and $\bar{\mathrm{h}}$ to denote respectively the Hessian of u and the same bordered with ξ, η, we have

$$s = \bar{\mathrm{h}}\left(\frac{d}{d\xi}, \frac{d}{d\eta}; \frac{d}{dx}, \frac{d}{dy}\right)\bar{\mathrm{h}}(x, y; \xi, \eta),$$

$$t = \frac{ds}{da}\frac{d^4h}{dx^4} + \frac{ds}{db}\frac{d^4h}{dx^3dy} + \frac{ds}{dc}\frac{d^4h}{dx^2dy^2} + \&c.$$

$$s^2 = \frac{dt}{da}\frac{d^4h}{dx^4} + \frac{dt}{db}\frac{d^4h}{dx^3dy} + \frac{dt}{dc}\frac{d^4h}{dx^2dy^2} + \&c.$$

Again, taking $(\bar{\bar{H}})$ the second bordered Hessian of Ω; that is, Ω bordered as well horizontally as vertically with the double lines and columns ξ, η, ζ, θ; $\xi', \eta', \zeta', \theta'$,

$$\Sigma = (\bar{\bar{H}})\left(\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}, \frac{d}{d\theta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{dt}; \xi', \eta', \zeta', \theta'\right)$$
$$\times (\bar{\bar{H}})(x, y, z, t; \xi, \eta, \zeta, \theta; \xi', \eta', \zeta', \theta'),$$

$$\vartheta = \frac{d\Sigma}{da}\frac{d^3\bar{H}}{dx^3} + \frac{d\Sigma}{db}\frac{d^3\bar{H}}{dx^2dy} + \frac{d\Sigma}{dc}\frac{d^3\bar{H}}{dx^2dz} + \frac{d\Sigma}{dd}\frac{d^3\bar{H}}{dx^2dt} + \&c.,$$

$$\Sigma^2 = \frac{d\vartheta}{da}\frac{d^3\bar{H}}{dx^3} + \frac{d\vartheta}{db}\frac{d^3\bar{H}}{dx^2dy} + \&c.$$

In like manner again

$$\sigma = (\bar{\bar{h}})\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \xi', \eta', \zeta'\right\}$$
$$\times \bar{\bar{h}}\{x, y, z; \xi, \eta, \zeta; \xi', \eta', \zeta'\},$$

$$\tau = \frac{d\sigma}{da}\frac{d^4(\bar{h})}{dx^4} + \&c.,$$

$$\sigma^2 = \frac{d\tau}{da}\frac{d^4\bar{h}}{dx^4} + \&c.,$$

σ and τ are the same quantities as are calculated by Mr Salmon, in his inestimable work *On Higher Plane Curves,* but are there expressed under the names of S and T, with the sole difference that in place of x, y, z, used by Mr Salmon, the contragredient variables ξ', η', ζ' are used in the expressions above. Mr Salmon has also pointed out to me that σ may be obtained by operating with

$$\left(\xi^4\frac{d}{da} + \xi^3\eta\frac{d}{db} + \xi^3\zeta\frac{d}{dc} + \&c.\right)$$

directly upon I a cubic invariant of the function u, or $(x, y, z)^4$. This I is no other than the simple commutant obtained by operating upon u with the commutantive symbol formed by taking four times over the line $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, agreeable to the remark made in the third section that every function of an even degree of n variables possesses an invariant of the nth order in extension of Mr Cayley's observation that every such function of two variables possesses a quadrinvariant, that is an invariant of the second order.

I need hardly remark that σ is of 2 dimensions in the coefficients and of 4 in the contragredient variables, τ of 3 in the coefficients and of 5 in the contragredients, Σ of 4 in the constants and 4 in the contragredients, ϑ of 6 in the constants and 6 in the contragredients, or that the single-bordered Hessians of u and U and the double-bordered Hessians of ω and Ω are each of them quadratic in respect of the x &c. as well as of the ξ &c. systems.

If the right numerical factors be attributed to S, T, Aronhold has shown that

$$H\{H(U)\}+T.H(U)+S^2U=0,$$

and in my paper in the last May Number*, I gave the equation

$$h\{h(u)\}+s.h(u)+tu=0.$$

I think it highly probable that it will be found that the analogous equations obtain, namely

$$\bar{H}\{\bar{H}(\Omega)\}+\vartheta.\bar{H}(\Omega)+\Sigma^2\Omega=0,$$

$$\bar{h}\{\bar{h}(\omega)\}+\sigma.\bar{h}(\omega)+\tau\omega=0.$$

These remarkable equations, if verified (of which I can scarcely doubt), will be most powerful aids to the dissection of the forms ω, Ω, and thereby to the detection of the fundamental properties of curves of the fourth and surfaces of the third degree, of which at present so little is known. It will have been observed that in the preceding developments the contravariants of ω and Ω were derived in precisely the same way from ω and Ω as the corresponding invariants of u and U from u and U, with the sole difference that the Hessian used in the two latter cases is replaced by a single-bordered Hessian in the two former cases, and a single-bordered Hessian in the two latter by a double-bordered Hessian in the two former. The analogies are not even yet stated exhaustively; for it will be remembered (as shown in the third section), that T and S can be derived directly and concurrently by means of operating with the commutantive symbol

$$\left.\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \end{matrix}\right\} \text{upon } \bar{H}(U)+\lambda(x\xi+y\eta+z\zeta)^2,$$

[* p. 192 above.]

which gives a result of the form $m(\lambda^3 + S\lambda + T)$, m being a number; and I conjecture that if

$$\begin{array}{cccc} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, & \frac{d}{dt}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, & \frac{d}{dt}, \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, & \frac{d}{d\theta}, \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, & \frac{d}{d\theta}, \end{array}$$

be made to operate upon

$$\overline{\overline{H}}\Omega + \lambda (x\xi + y\eta + z\zeta + t\theta)^2,$$

and the result be put under the form

$$m(\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D),$$

that A will be zero, B and C will be respectively Σ and ϑ, and perhaps D (a contravariant, if it effectively exist, of 8 dimensions in the coefficients of Ω, and of a like number in the contragredients ξ', η', ζ', θ'), also zero. But of the evanescence of D I do not speak with any degree of assurance.

Mr Salmon has made an excellent observation to the effect that if we call (σ) what σ becomes when ξ', η', ζ' are replaced by $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$, $(\sigma)\,h(\omega)$ will represent a covariant to ω of $3+2$, that is, 5 dimensions in the coefficients, and of $6-4$, that is, of 2 dimensions in x, y, z, $h(\omega)$ being of 3 and 6 dimensions in these respectively, and σ of 2 and 4 dimensions respectively in the same. Now these resulting dimensions 5 and 2 precisely agree with the form especially noticed by me in Note* (2) of the Appendix, where it was derived as one of a group by the method of unravelment. There can be little doubt that these two conics each of them indissolubly connected with every curve of the fourth degree are identical. The form $(\sigma)\,h(\omega)$ enables us to prove readily (thanks to Mr Salmon's calculation of σ, given in his *Higher Plane Curves*, under the name of S) that this is a *bonâ fide* existent conic.

For if we take a particular case of ω, say

$$\omega = a_1x^4 + b_2y^4 + c_3z^4 + 6dy^2z^2,$$

we find

$$h(\omega) = \begin{vmatrix} a_1x^2, & 0, & 0 \\ 0, & b_2y^2 + dz^2, & dyz \\ 0, & dyz, & c_3z^2 + dy^2 \end{vmatrix}$$

$$= a_1(b_2c_3 + d^2)\,x^2y^2z^2 + a_1b_2dx^2y^4 + a_1c_3dx^2z^4,$$

[* p. 323 above.]

and σ becomes

$$a_1 d\eta'^2\zeta'^2,$$

and consequently (σ) is

$$a_1 d\left(\frac{d}{dy}\right)^2\left(\frac{d}{dz}\right)^2,$$

and therefore

$$(\sigma)\,h\,(\omega) = 4a_1^2 d\,(b_2 c_3 + d^2)\,x^2,$$

the conic here reducing to a pair of coincident straight lines. This example demonstrates that the conic is in general actually existent.

As I have said so much upon S and T it may not be irrelevant to state in this place how I obtained the conditions for U, the characteristic of the curve of the third degree becoming the characteristic of a conic and a straight line, that is breaking up into a linear and a quadratic factor, which Mr Salmon has inserted in the notes to his work above referred to. When U is of this form it may obviously by linear transformations be expressed by $ax^3 + 6dxyz$, but when starting with the general form,

$$a_1 x^3 + b_2 y^3 + c_3 z^3 + \&\text{c.} + 6Dxyz,$$

we form two contravariants from S and T, to wit

$$\left(\xi^3\frac{d}{da_1} + \eta^3\frac{d}{db_2} + \zeta^3\frac{d}{dc_3} + \&\text{c.} + \xi\eta\zeta\frac{d}{dD}\right)S,\ \text{say } S',$$

$$\left(\xi^3\frac{d}{da_1} + \eta^3\frac{d}{db_2} + \zeta^3\frac{d}{dc_3} + \&\text{c.} + \xi\eta\zeta\frac{d}{dD}\right)T,\ \text{say } T',$$

and then make $a_1 = a$, $D = d$, and all the other coefficients zero, it will easily be seen on examining the forms of S and T, given by Mr Salmon, that (S) and (T) (the evectants of S and T) become respectively

$$4d^3\xi\eta\zeta,\quad 31d^5\xi\eta\zeta;$$

we have therefore $(T) + \lambda\,(S) = 0$: and (T) and (S), although contravariantive to their primitive U, are covariantive with one another, so that $(T) + \lambda\,(S) = 0$ is a persistent relation unaffected by linear transformations; it follows therefore that when U is of, or reducible to, the form supposed,

$$\frac{dS}{da_1} : \frac{dS}{db_2} : \frac{dS}{dc_3} : \&\text{c.} : \frac{dS}{dD}$$

$$= \frac{dT}{da_1} : \frac{dT}{db_2} : \frac{dT}{dc_3} : \&\text{c.} : \frac{dT}{dD},$$

which is the criterion given in the note referred to*.

I am also able to obtain these equations more directly by another method founded upon a New View of the Theory of Elimination, an account of which,

* Mr Salmon has remarked that the two evectants (S) and (T) intersect in the nine cuspidal points of the polar reciprocal to the curve.

however, I must reserve for another occasion, but which, I may mention, serves to fix not merely the conditions, as in the ordinary restricted theory, that a given set of equations may be simultaneously satisfiable by some one system of values of the variables, but the *conditions* that such set of equations may be simultaneously satisfiable by any given number of distinct systems of variables.

Mr Salmon has remarked to me to the effect that if in τ we write $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, in place of the contragredients, and call τ so altered (τ), then $(\tau)\, h(\omega)$ will be an invariant of 6 dimensions in the coefficients of ω. This sextinvariant I have little doubt is identical with that obtained by operating upon ω with the commutantive symbol

$$\left.\begin{matrix} \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\frac{d}{dx} \end{matrix}\right\}.$$

This, like every other commutant of 2 lines only, is of course capable of being expressed under the form of an ordinary determinant, and the remark is not without interest, as showing how the proposition known with respect to quadratic functions of any number of variables, namely of every such having an invariantive determinant, lends itself to the general case of functions of any even degree of any number of variables which also have always an invariantive determinant attached to them, of which the terms are simple coefficients of such functions. The only peculiarity (if it be one) of quadratic functions in this respect being that they have each but one invariant of such form and no other. In the case before us, if we write

$$\omega = a_1x^4 + b_2y^4 + c_3z^4 + 4a_2x^3y + 4a_3x^3z + 4b_1y^3x + 4b_3y^3z + 4c_1z^3x + 4c_2z^3y$$
$$+ 6dy^2z^2 + 6ez^2x^2 + 6fx^2y^2 + 12lx^2yz + 12mxy^2z + 12nxyz^2,$$

the sextinvariant in question becomes representable under the form of the determinant

$$\begin{vmatrix} a_1, & a_2, & f, & l, & e, & a_3 \\ a_2, & f, & b_1, & m, & n, & l \\ f, & b_1, & b_2, & b_3, & d, & m \\ l, & m, & b_3, & d, & c_2, & n \\ e, & n, & d, & c_2, & c_3, & c_1 \\ a_3, & l, & m, & n, & c_1, & e \end{vmatrix} *.$$

* This determinant is identical with the determinant formed by taking the second differential coefficients of the function and arranging in the usual manner the coefficients of the several powers and combinations of powers of the variables treated as if they were independent quantities.

Before quitting the subject of S and T the two invariants of the cubic function of 3 variables, or, as it may be termed, of the cubic curve, it may not be amiss to give the complete table which I have formed corresponding to all the singular cases which can befall such curve, which will be seen below to be eight in number; it is of the highest importance to push forward the advanced posts of geometry, and for this purpose to obtain the same kind of absolute power and authority over, and clear and absolute knowledge of, the properties and affections of cubic forms as have been already attained for forms of the second degree.

Let $$U = ax^3 + 4bx^2y + 4cx^2z + \&c.$$

(1) When U has one double point $S^3 + T^2 = 0$.

(2) When U has two double points, that is becomes a conic and right line

$$\frac{dS}{da}\frac{dT}{db} - \frac{dS}{db}\frac{dT}{da} = 0, \ \&c. \ \&c.$$

(3) When U has a cusp $S = 0$, $T = 0$.

(4) When U has two coincident double points, that is, is a conic and a tangent line thereto, which comprises the two preceding cases in one,

$$\frac{dT}{da} = 0, \quad \frac{dT}{db} = 0, \ \&c.$$

and also *therefore* $$S = 0.$$

(5) When U becomes three right lines forming a triangle

$$\frac{d^2S}{da\,db}\frac{d^2T}{dc\,de} - \frac{d^2T}{da\,db}\frac{d^2S}{dc\,de} = 0, \ \&c.$$

where a, b, c, e each represent any of the coefficients arbitrarily chosen, whether distinct or identical.

Another, and lower in degree system of equations, may be substituted for the above, obtained by affirming the equality of the ratios between the coefficients of U and the corresponding coefficients of its Hessian.

(6) When U represents a pencil of three rays meeting in a point

$$\frac{dS}{da} = 0, \quad \frac{dS}{db} = 0, \ \&c.$$

and also *therefore* $$T = 0.$$

Also in place of this system may be substituted the system obtained by taking all the coefficients of the Hessian zero.

(7) When U becomes a line, and two other coincident lines,

$$\frac{dS}{da}=0, \quad \frac{dS}{db}=0, \text{ \&c.}$$

and also

$$\frac{d^2T}{da^2}=0, \quad \frac{d^2T}{da\,db}=0, \text{ \&c.}$$

I have not ascertained whether this second system necessarily implies the first; I rather think that it does not. In the preceding case also it would be interesting to show the direct algebraical connexion between the system formed by the coefficients of the Hessian and the system consisting of the first derivatives of S.

(8) When U becomes a perfect cube representing three coincident right lines

$$\frac{d^2S}{da^2}=0, \quad \frac{d^2S}{da\,db}=0, \text{ \&c.}$$

and

$$\frac{d^2T}{da^2}=0, \quad \frac{d^2T}{da\,db}=0, \text{ \&c.}$$

The first of these systems of equations necessarily implies the equations $\frac{dT}{da}=0$, $\frac{dT}{db}=0$, &c., as is obvious from the equation

$$T=\frac{dS}{da}\frac{d^3H}{dx^3}+\frac{dS}{db}\frac{d^3H}{dx^2dy}+\text{ \&c.}$$

but not necessarily the second and lower system $\frac{d^2T}{da^2}=0$, &c. above written.

So if we take

$$u = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$$

when 2 roots are equal

$$s^3 + t^2 = 0,$$

when 2 pairs of roots are equal

$$\frac{ds}{da}\frac{dt}{db}-\frac{ds}{db}\frac{dt}{da}=0, \text{ \&c.,}$$

when 3 roots are equal $\quad s=0, \quad t=0,$

and when all 4 roots are equal

$$\frac{dt}{da}=0, \quad \frac{dt}{db}=0, \text{ \&c.}$$

Before closing this Section I may make a remark, in reference to the sextic invariant of ω, which admits of being extended to all commutants formed by operating upon the function with a commutantive symbol obtained by writing over one another lines consisting of powers of $\frac{d}{dx}$, $\frac{d}{dy}$, &c. and their combinations (to which, in the Third Section, I gave the name of *compound* commutants, a qualification which, for reasons that will hereafter be adduced, I think it advisable to withdraw). The remark I have to make is this, namely that the invariant obtained by operating upon ω with

$$\left.\begin{matrix} \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\frac{d}{dx} \end{matrix}\right\},$$

is precisely the same as may be obtained by operating with

$$\left.\begin{matrix} \frac{d}{du}, & \frac{d}{dv}, & \frac{d}{dw}, & \frac{d}{dp}, & \frac{d}{dq}, & \frac{d}{dr} \\ \frac{d}{du}, & \frac{d}{dv}, & \frac{d}{dw}, & \frac{d}{dp}, & \frac{d}{dq}, & \frac{d}{dr} \end{matrix}\right\}$$

upon the concomitant quadratic function to ω obtained by the method of unravelment, as in Note (2) of the Appendix [p. 322 above]; and so, in general, every commutant obtained by operating upon a function of any number of variables of the degree $2mp$ with a symbol consisting of $2p$ lines in which the mth powers of $\frac{d}{dx}$, $\frac{d}{dy}$, &c. and their mth combinations occur, will be identical with the commutant obtained by operating with a symbol also of $2p$ lines, in which only the simple powers occur of $\frac{d}{du}$, $\frac{d}{dv}$, &c. (where u, v, &c. are cogredient with x^p, $x^{p-1}y$, &c.), upon a function of u, v, &c., formed by the method of unravelment from the given function.

Finally, before quitting the subject of reciprocity, I may state, it follows from the general statement made at the commencement of this Section, that inasmuch as

$$(x\xi + y\eta + z\zeta + \&c.)^2$$

is a universal concomitant form, so also must

$$\left(\frac{d}{d\xi}\frac{d}{dx} + \frac{d}{d\eta}\frac{d}{dy} + \frac{d}{d\zeta}\frac{d}{dz} + \&c.\right)^2$$

be a universal concomitant symbol of operation; accordingly it is certain that any concomitant in which x, y, z, &c., ξ, η, ζ, &c. enter, operated upon with this symbol, will remain a concomitant: in several cases which I have examined, the effect of this operation will be to produce an evanescent form, but I see no ground for supposing that this is other than an accidental, or at all events for supposing that it is a necessary and universal consequence of the operation. It may also be observed that in the case of as many cogredient sets of variables as variables in each set, as for instance 3 sets of 3 variables each, the determinant which may be formed by arranging them in regular order, as

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix},$$

is evidently a universal concomitant, and moreover an equivocal concomitant, possessing the property of remaining a concomitant when the variables are respectively but simultaneously exchanged for their contragredients ξ, η, ζ; ξ', η', ζ'; ξ'', η'', ζ''; which shows also that in place of the variables may be written the differential operators

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \frac{d}{dx'}, \frac{d}{dy'}, \frac{d}{dz'}; \frac{d}{dx''}, \frac{d}{dy''}, \frac{d}{dz''}:$$

a remark which leads us to see the exact place in the general theory occupied by Mr Cayley's method of generating covariants given in the concluding paragraph of the First Section [p. 290 above]. I may likewise add, that inasmuch as $(x'\xi + y'\eta + z'\zeta + \&c.)^2$ is a universal concomitant,

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy} + \&c.\right)^r$$

will be so too, by virtue of the general law of interchange, which conducts immediately to the theory of emanation, showing that this last symbol, operating upon any function, furnishes covariants thereunto for any integer value of z.

One additional interesting remark presents itself to be made concerning U, the cubic function of x, y, z, which is, that calling as before T its sextic invariant, and a, $3b$, $3c$, d, &c. the coefficients, the formula

$$\left(\xi^3\frac{d}{da} + \xi^2\eta\frac{d}{db} + \xi^2\zeta\frac{d}{dc} + \xi\eta\zeta\frac{d}{dd} + \&c.\right)^2 T$$

will give the polar reciprocal, or, as it has been agreed to term it, the reciprocant of U. I believe the remark of the probability of this being the case originated with myself, but Mr Cayley first verified it by actual calculation, using for that purpose the value of T, given by Mr Salmon in his work *On the Higher Plane Curves*, already frequently alluded to, which is an indispensable manual equally for the objects of the higher special geometry as for the new or universal algebra, being in fact a common ground where the two sciences meet and render mutual aid.

Mr Salmon also observed, that the first evect of T, namely

$$\left(\xi^3\frac{d}{da} + \xi^2\eta\frac{d}{db} + \&c.\right) T,$$

was identical in form with what may be termed the first devect of the polar reciprocal, that is, the result of operating upon the polar reciprocal with what U becomes when $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, are substituted in the stead of x, y, z. And inasmuch as, by Euler's law,

$$\left\{a\left(\frac{d}{d\xi}\right)^3 + 3b\left(\frac{d}{d\xi}\right)^2\frac{d}{d\eta} + \&c.\right\} \times \left\{\xi^2\frac{d}{da} + \xi^2\eta\frac{d}{db} + \&c.\right\} T$$

$$= 6\left\{a\frac{d}{da} + b\frac{d}{db} + \&c.\right\} T = 36T,$$

it follows that T is the second *devect* of the polar reciprocal, or at least identical with it in point of form. But, since the preceding matter was printed, I have discovered in the course of a most instructive and suggestive correspondence with Mr Salmon, the principle upon which these and similar identifications depend, thereby dispensing with the necessity for the excessively tedious labour of verification which, even in the simple example before us, would be found to extend over several pages of work.

The theory in which this principle is involved will be given, along with other very important matter, in the next number of the *Journal.*

Supplementary Observations on the Method of Reciprocity.

It has been observed, that ξ, η, &c. may always be inserted in place of $\frac{d}{dx}, \frac{d}{dy}$, &c., and *vice versâ,* in a concomitant form, without destroying its concomitance. Accordingly, instead of the evector symbol

$$\xi^3\frac{d}{da} + \xi^2\eta\frac{d}{db} + \&c.,$$

we may employ

$$\left(\frac{d}{dx}\right)^3\frac{d}{da} + \left(\frac{d}{dx}\right)^2\frac{d}{dy}\frac{d}{db} + \&c.;$$

and operating with this upon any concomitant, the result will be a concomitant. Hence we see, for example, that if we take the concomitant SH formed by the product of the invariant S and the covariant H,

$$\left\{\left(\frac{d}{dx}\right)^3\frac{d}{da} + \left(\frac{d}{dx}\right)^2\frac{d}{dy}\frac{d}{db} + \&c.\right\} SH$$

will be a covariant; in fact this will be found to be T, the difference between this and the expression before given for T, namely

$$\left(\frac{d}{dx}\right)^3 H\frac{dS}{da} + \left(\frac{d}{dx}\right)^2\frac{d}{dy}H\frac{dS}{db} + \&c.,$$

being

$$S \times \left\{\frac{d}{da}\left(\frac{d}{dx}\right)^3 H + \frac{d}{db}\left(\frac{d}{dx}\right)^2\frac{d}{dy}H + \&c.\right\},$$

which is zero, there being no invariant to $(x, y, z)^3$ of the 3rd degree in a, b, c, &c., as the factor multiplied by S would be were it not evanescent. The same observation may be extended to analogous equations given previously.

I have chiefly, however, made the above observation with a view to making more clear the enunciation of the theorem which I am now about to state, the most important perhaps in its application of any yet brought to light on the subject, but the consequences of which, as I have but quite recently discovered it, must be reserved for a future number of the *Journal*.

Let any function of any number of variables be supposed to have for its coefficients the letters a, b, &c. affected with the ordinary binomial or multinomial coefficients; and let another function be taken identical with the former in all respects, except in the circumstance that all their numerical multipliers are suppressed. Let this function or form be termed the respondent to the primitive: furthermore, by the inverse of any form understand what that form becomes when, in place of x, y, z, &c., ξ, η, ζ, &c.,

$$\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz},\ \text{\&c.},\ \frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta},\ \text{\&c.},$$

are respectively substituted (and so for all the systems of the variables), and likewise at the same time similar substitutions are made of $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}$, &c., in place of a, b, c, &c.; then we have this grand and simple law—*The inverse of any concomitant to a respondent is a concomitant to its primitive.* When the inverse of any concomitant to the respondent is made to operate upon the same concomitant of the primitive, it will be found that the result is a power of the universal concomitant. If the concomitant to the respondent be an invariant thereof, the rule indicates that on merely replacing in the respondent a, b, c, &c. by $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}$, &c., the result operating on any invariant or other concomitant of the primitive, leaves it still an invariant or other concomitant. For instance, if we take the function

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5,$$

which has three invariants L, M, N, of the degrees 4, 8, 12, respectively: and if we call λ, μ, ν what L, M, N become when, in place of a, b, c, d, e, f respectively, we write

$$\frac{d}{da},\ \frac{1}{5}\frac{d}{db},\ \frac{1}{10}\frac{d}{dc},\ \frac{1}{10}\frac{d}{dd},\ \frac{1}{5}\frac{d}{de},\ \frac{d}{df},$$

we shall find that

$$\lambda M = L,\quad \mu N = L,$$

and

$$\lambda N = \text{a linear function of } M \text{ and } L^2.$$

Again, if in the case of any function of x, y, z, &c., we take, instead of any other concomitant to the respondent, the respondent itself, its inverse gives the symbol of operation

$$\left(\frac{d}{da}\right)\left(\frac{d}{dx}\right)^3 + \frac{d}{db}\left(\frac{d}{dx}\right)^2\left(\frac{d}{dy}\right) + \&c.,$$

just previously treated of. If again, in the case of a function of x, y, say

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n,$$

we take the inverse of the polar reciprocal of the respondent, we get the operator

$$\frac{d}{da}\left(\frac{d}{d\eta}\right)^n - \frac{d}{db}\left(\frac{d}{d\eta}\right)^{n-1}\frac{d}{d\xi} + \&c.;$$

and replacing $\frac{d}{d\eta}$, $\frac{d}{d\xi}$ by y, x, we find that

$$y^n\frac{d}{da} - y^{n-1}x\frac{d}{db} + \&c.,$$

operating on any concomitant, leaves it still a concomitant, which is M. Eisenstein's theorem before adverted to, only generalized by the introduction of any concomitant in lieu of the discriminant.

This extraordinary theorem of respondence will be found on reflection to favour the notion of treating the coefficients of a general function as themselves a system of variables, in a manner contragredient to the terms to which they are affixed.

Finally, there is yet another mode of applying the principle of reciprocity, which must be carefully distinguished from any previously stated in these pages.

I have said that in place of the quantitative symbols of one alphabet, as x, y, z, &c., we may always substitute the operation symbols $\frac{d}{d\xi}$, $\frac{d}{d\eta}$, $\frac{d}{d\zeta}$, &c. of the opposite alphabet. But now I say, in place of the quantitative symbols x, y, z, &c. occurring in the concomitant to any form f, may be substituted the quantities (observe, no longer operative symbols but quantities) $\frac{dF}{d\xi}$, $\frac{dF}{d\eta}$, $\frac{dF}{d\zeta}$, &c., F being itself any concomitant to f. Thus, for instance, taking F identical with f, we see that $f\left(\frac{df}{d\xi}, \frac{df}{d\eta}, \frac{df}{d\zeta}, \&c.\right)$ is concomitant to f: or again, if f be a function of x, y only, say $f(x, y)$, taking F the polar reciprocal of f, that is $f(-\eta, \xi)$, we see that $f\left(-\frac{df}{dy}, \frac{df}{dx}\right)$ will be a concomitant to f: this concomitant, by the way it may be observed, will always contain f as a factor, because when $f=0$, $x\frac{df}{dx}+y\frac{df}{dy}=0$. Possibly it may be true that, when f is a function of any number of variables x, y, z, &c., and $F(\xi, \eta, \zeta, \text{\&c.})$ its polar reciprocal,

$$f\left(\frac{dF(x, y, z, \text{\&c.})}{dx}, \frac{dF(x, y, z, \text{\&c.})}{dy}, \text{\&c.}\right),$$

which is a concomitant to f, contains f as a factor; but I have not had time to see how this is. It is rather singular that Mr Cayley and Professor Borchardt of Berlin have both independently made to me the observation that, when $f(x, y)$ is taken a cubic function of x and y, $f\left(\frac{df}{dy}, \frac{-df}{dx}\right)$ is equal to the product of f by the first evectant of the discriminant of f. The general consideration of the consequences of this new and important application of the idea of reciprocity must be reserved for a future section.

SECTION V. *Applications and Extension of the Theory of the Plexus.*

If
$$\phi = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

we can obtain, by operating catalectically with x', y' upon

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^2\phi, \quad \left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^4\phi,$$

the two concomitants

$$\begin{vmatrix} ax^2+2bxy+cy^2, & bx^2+2cxy+dy^2 \\ bx^2+2cxy+dy^2, & cx^2+2dxy+ey^2 \end{vmatrix}, \qquad (1)$$

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}, \qquad (2)$$

the one in fact being the Hessian, the other the catalecticant of ϕ itself. Again, if

$$\phi = ax^5 + 5bx^4y + 10cx^3y^2 + \dots + fy^5,$$

by operating catalectically with x', y' upon the second and fourth emanants, as in the last case, we obtain the two covariants

$$\begin{vmatrix} ax^3+3bx^2y+3cxy^2+dy^3, & bx^3+3cx^2y+3dxy^2+ey^3 \\ bx^3+3cx^2y+3dxy^2+ey^3, & cx^3+3dx^2y+3exy^2+fy^3 \end{vmatrix}, \qquad (1)$$

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \\ bx+cy, & cx+dy, & dx+ey \\ cx+dy, & dx+ey, & ex+fy \end{vmatrix}, \qquad (2)$$

which are in fact the Hessian and canonizant respectively of ϕ. So in general, for a function of x, y of the degree 2ι or $2\iota+1$, we can obtain ι covariantive forms, the first being the Hessian, and the last the catalecticant on the first supposition and the canonizant on the second: calling the index of the function for either case n, the forms appearing in this scale will be of the degree $(r+1)$ in the constants, and of the degree $(r+1)(n-2r)$ in x and y.

It has previously* been intimated that all these determinants admit of a remarkable transformation.

This transformation may be expressed more elegantly by dealing not directly with the covariant forms as above given, but with their polar reciprocants obtained immediately by writing ξ for $-y$ and η for x.

(1) Suppose $\qquad \phi = ax^3 + 2bx^2y + 3cxy^2 + dy^3$;

$$\begin{vmatrix} a, & 2b, & c \\ b, & 2c, & d \\ \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix}$$

will be found to be the reciprocant of its Hessian.

(2) Let $\qquad \phi = ax^4 + 4bx^3y + \dots + ey^4$;

the reciprocant of its Hessian will be found to be

$$\begin{vmatrix} a, & 3b, & 3c, & d \\ b, & 3c, & 3d, & e \\ \xi^2, & 2\xi\eta, & \eta^2, & \\ & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix}.$$

(3) Let $\qquad \phi = ax^5 + 5bx^4y + \dots + fy^5$;

the reciprocant of its Hessian will be

$$\begin{vmatrix} a, & 4b, & 6c, & 4d, & e \\ b, & 4c, & 6d, & 4e, & f \\ \xi^2, & 2\xi\eta, & \eta^2, & & \\ & \xi^2, & 2\xi\eta, & \eta^2, & \\ & & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix};$$

[* p. 325 above, note †].

and the reciprocant of its canonizant is

$$\begin{vmatrix} a, & 3b, & 3c, & d \\ b, & 3c, & 3d, & e \\ c, & 3d, & 3e, & f \\ \xi^3, & 3\xi^2\eta, & 3\xi\eta^2, & \eta^3 \end{vmatrix}.$$

The numerical coefficients in this and in the first case are inserted for the sake of uniformity, but it will of course be readily observed that when there is but one line of ξ and η, that the numerical coefficients being the same for each column may be rejected without affecting the form of the result.

So again, if

$$\phi = ax^6 + 6bx^5y + \ldots + gy^6,$$

the reciprocant of the Hessian is

$$\begin{vmatrix} a, & 5b, & 10c, & 10d, & 5e, & f \\ b, & 5c, & 10d, & 10e, & 5f, & g \\ \xi^2, & 2\xi\eta, & \eta^2, & & & \\ & \xi^2, & 2\xi\eta, & \eta^2, & & \\ & & \xi^2, & 2\xi\eta, & \eta^2, & \\ & & & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix},$$

and the reciprocant of the second form in the scale, which comes between the Hessian and the catalecticant, is

$$\begin{vmatrix} a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g \\ \xi^3, & \xi^2\eta, & \xi\eta^2, & \eta^3, & \\ & \xi^3, & \xi^2\eta, & \xi\eta^2, & \eta^3 \end{vmatrix};$$

and so in general. The rule of formation is sufficiently plain not to need formulating in general terms. It is easy to see that all these forms are concomitants to the function from which they are formed; for example, take

$$\phi = ax^6 + 6bx^5y + \ldots + gy^6;$$

then

$$\left(\frac{d}{dx}\right)^2\phi, \quad \frac{d}{dx}\frac{d}{dy}\phi, \quad \left(\frac{d}{dy}\right)^2\phi$$

form a plexus.

So likewise if we take $\psi = (x\xi + y\eta)^4$,

$$\frac{d\psi}{d\xi}, \; \frac{d\psi}{d\eta}$$

form a plexus. But ψ and ϕ are concomitantive, ψ being a universal concomitant. Hence we may combine together these two plexuses, that is

$$\left.\begin{array}{l} ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \\ bx^4 + 4cx^3y + 6dx^2y^2 + 4exy^3 + fy^4 \\ cx^4 + 4dx^3y + 6ex^2y^2 + 4fxy^3 + gy^4 \end{array}\right\},$$

$$\left.\begin{array}{l} \xi^3x^4 + 3\xi^2\eta x^3y + 3\xi\eta^2x^2y^2 + \eta^3xy^3 \\ \qquad\quad \xi^3x^3y + 3\xi^2\eta x^2y^2 + 3\xi\eta^2xy^3 + \eta^3y^4 \end{array}\right\},$$

and, by the principle of the plexus, x^4, x^3y, x^2y^2, xy^3, y^4 may be eliminated dialytically, and the resultant will be the determinant last given, which is therefore a contravariant to ϕ.

The manner in which I was led to notice this singular transformation is somewhat remarkable.

In the supplemental part of my essay *On Canonical Forms* [p. 203 above], my method of solution of the problem of throwing the quintic function of two variables under the form $u^5 + v^5 + w^5$, led me to see that u, v, w are the three factors of

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy \\ bx + cy, & cx + dy, & dx + ey \\ cx + dy, & dx + ey, & ex + fy \end{vmatrix};$$

the more simple mode of the solution of the same problem, given by me in the *Philosophical Magazine* for the month of November last [p. 266 above], led to

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ y^3, & -xy^2, & x^2y, & -x^3 \end{vmatrix},$$

as the product of the same three factors; whence the identity of the two forms becomes manifest. In the paper last named I gave two proofs, one my own, the other Mr Cayley's, of a like kind of identity for the canonizant of any odd-degreed function of x, y in general. The proof of the identity of the corresponding forms in the much more general proposition above indicated [p. 325 above, footnote †] must be reserved until more pressing and important matters are disposed of. In the footnote referred to I ought to have added, in order to make the sense more clear, that the degree of the catalecticant there referred to in respect of the coefficients would be n.

I regret to think that there are many other typographical errors in the earlier sections; the most unfortunate of these is in the note at page [316], in the values of P and Q belonging to the cubic commutant dodecadic function of x and y, the corrected values of which will be given in my next communication. I ought also to observe, in correction of the remark made in the footnote to page [302], that it follows as a consequence of a recent paper by Dr Hesse in *Crelle's Journal,* that the method given by me in the text applied (according to what I have there termed the 1st process for obtaining an invariant resembling the resultant) to a system of three cubic equations (in which application only the 1st powers of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ enter) produces for that case also, as well as for the cases specified in the note, not a counterfeit resemblance of, but the actual resultant itself.

Returning to the theory of the plexus of which I am about to enunciate a most important extension, I beg to refer my readers to the last paragraph, p. [291], in the last number of the *Journal,* where I have shown how to form, under certain conditions, a determinant by combining together various concomitants and eliminating dialytically one set of the variables, which determinant will be concomitantive to the concomitants out of which it is formed, and of course also therefore to their common original.

Now the extension of this theorem, to which I wish to call attention, is this, that not only such determinant as a whole is a concomitant to such original, but every minor system of determinants that can be formed out of it will form a concomitantive plexus complete within itself to the same original. But, much more generally, it should be observed that there is no occasion to begin with a square determinant; it is sufficient to have a rectangular array of terms formed by taking the several terms of one plexus or of several plexuses combined, provided that they are of the same degree in respect to the variables (or to the selected system of variables, if there be several systems), and forming out of such rectangular array any minor system of determinants at will. Every such system will be a concomitantive plexus. The simple illustrations which follow will make my meaning clear.

Suppose

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 21dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6.$$

I have previously remarked, in the foregoing sections, that $a, b, c, d, e, f, g,$ the coefficients form an invariantive plexus to ϕ; so also we know that the catalecticant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

is an invariant to ϕ. But we are now able to couple together these facts and see the law which is contained between them; for if we take

$$\left(\frac{d}{dx}\right)^{\iota}\phi,\ \left(\frac{d}{dx}\right)^{\iota-1}\frac{d}{dy}\phi \ldots \left(\frac{d}{dy}\right)^{\iota}\phi,$$

ι being any number, as for instance, if we take $\iota = 3$, we shall have as a plexus

$$\begin{gathered} ax^3 + 3bx^2y + 3cxy^2 + dy^3, \\ bx^3 + 3cx^2y + 3dxy^2 + ey^3, \\ cx^3 + 3dx^2y + 3exy^2 + fy^3, \\ dx^3 + 3ex^2y + 3fxy^2 + gy^3; \end{gathered}$$

accordingly not only is the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

an invariant, but also the system obtained by striking out any one line and one column, being what I term the first minors, will be an invariantive plexus, so too will the system of second minors

$$ac - b^2,\ bd - c^2,\ ce - d^2,\ ad - bc,\ ae - bd,\ be - cd, \text{ \&c.}$$

form an invariantive plexus, as well as the last minors, that is, the simple terms a, b, c, d, e, f, g. Again, we might have taken the plexus

$$\left(\frac{d}{dx}\right)^2\phi,\ \frac{d}{dx}\frac{d}{dy}\phi,\ \left(\frac{d}{dy}\right)^2\phi,$$

which would give the array

$$\begin{matrix} a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g; \end{matrix}$$

but the minor systems of determinants herein comprised will be found to be identical with those last considered, with the exception that the highest system, containing a single determinant only, will now be wanting. So in general it will easily be seen that a similar method in general, when ϕ is of 2ι dimensions, will lead to $\iota + 1$ invariantive plexuses comprising the given coefficients grouped together at one extremity of the scale, and the catalecticant alone at the other; and if ϕ is of $2\iota + 1$ dimensions, there will still be $\iota + 1$ such plexuses, commencing with the coefficients as one group and ending with a system of combinations of the $(\iota + 1)$th degree in regard to the coefficients, which system accordingly takes the place of the catalecticant of the former case, which for this case is non-existent.

As a profitable example of the application of this law of synthesis, in its present extended form, let it be required to determine the conditions that a function of x, y of the fifth degree may have three equal roots. In general, let $\phi = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5$, then ϕ has a quadratic and cubic covariant of which I have written at large in my supplemental essay above referred to, being in fact the s and t (that is the quadrinvariant and cubinvariant) in respect to x', y' (x, y being treated as constants) of

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi.$$

Let these covariants respectively be called

$$Ax^2 + 2Bxy + Cy^2 = u,$$

$$\alpha x^3 + 3\beta x^2y + 3\gamma xy^2 + \delta y^3 = v;$$

then

$$\left.\begin{matrix} Ax + By \\ Bx + Cy \end{matrix}\right\}$$

forms a plexus, and

$$\left.\begin{matrix} \alpha x^2 + 2\beta xy + \gamma y^2 \\ \beta x^2 + 2\gamma xy + \delta y^2 \end{matrix}\right\}$$

will form another.

Now when $a = 0$, $b = 0$, $c = 0$, ϕ will have three equal roots, and

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi$$

becomes

$$6dy \,.\, x'^2y'^2 + 4\,(dx + ey)\,x'y'^3 + (ex + fy)\,y'^4,$$

of which the quadrinvariant in respect to x', y' is easily seen to be d^2y^2 and the cubinvariant d^3y^3. Accordingly the grouping

$$\begin{matrix} A, & B \\ B, & C \end{matrix} \text{ becomes } \left.\begin{matrix} 0, & 0 \\ 0, & d^2 \end{matrix}\right\},$$

and the grouping

$$\begin{matrix} \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{matrix} \text{ becomes } \left.\begin{matrix} 0, & 0, & 0 \\ 0, & 0, & d^3 \end{matrix}\right\}.$$

Accordingly, we see that the determinant $\begin{vmatrix} A, & B \\ B, & C \end{vmatrix}$ and all the first minors of $\begin{vmatrix} \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}$, that is $\alpha\gamma - \beta^2$, $\beta\delta - \gamma^2$, $\alpha\delta - \beta\gamma$, become zero; but the former single quantity $\begin{vmatrix} A, & B \\ B, & C \end{vmatrix}$ being an invariant, and this last system being an invariantive plexus, all the quantities so affirmed to be zero will remain zero, notwithstanding any linear transformations to which ϕ may be subjected; thus then we obtain an immediate proof of the theorem that when a function of x and y of the fifth degree contains three equal roots the determinant of its quadratic covariant, which in fact is its sole quart-invariant, and the first minors of its cubinvariant will be all separately zero. This theorem may be made still more stringent; for by combining

$$Ax^2 + 2Bxy + Cy^2,$$

$$\alpha x^2 + 2\beta xy + \gamma y^2,$$

$$\beta x^2 + 2\gamma xy + \delta y^2,$$

it becomes manifest that in the case supposed all the first minor determinants of

$$\begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}$$

will be zero, showing in addition to the theorem last enunciated that also

$$A : B : C :: \alpha : \beta : \gamma :: \beta : \gamma : \delta.$$

It is curious and instructive to remark that this last set of equations, stringent as *they appear*, and far more than enough to express a duplex condition, are not sufficient to imply unequivocally the existence of three equal roots, unless we have also $AC - B^2 = 0$; for suppose ϕ to take the form $ax^5 + fy^5$ (b, c, d, e all vanishing); then it will easily be seen that

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0,$$

$$A = 0, \quad B = af, \quad C = 0.*$$

* If we take L, M, N a system of fundamental invariants to ϕ, of which all the other invariants of ϕ are rational integer functions, then $L = \begin{vmatrix} A, & B \\ B, & C \end{vmatrix}$ and the simplest forms for M and N are

$$M = \begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix} \quad \text{and} \quad N = \begin{vmatrix} \alpha, & 2\beta, & \gamma & \\ & \alpha, & 2\beta, & \gamma \\ \beta, & 2\gamma, & \delta, & \\ & \beta, & 2\gamma, & \delta \end{vmatrix},$$

where L and N are the discriminants of the quadratic and cubic covariants of ϕ respectively, and a linear function of M, L^2 is the discriminant of ϕ itself (L, M, N being of 4, 8, and 12 dimensions respectively in the coefficients of ϕ).

For many purposes of the calculus of forms it is desirable to have the command of cases for which any two out of these three invariants may be made to vanish without the third vanishing; and it will be found that when ϕ is of the form $y^2(cx^3+fy^3)$, $L=0$, $M=0$; when ϕ is of the form $y(bx^4+fy^4)$, $N=0$, $L=0$; and when ϕ is of the form ax^5+ey^5, $M=0$, $N=0$; and of course when ϕ is of the form $y^3(dx^2+fy^2)$, $L=0$, $M=0$, $N=0$; it being obviously true in general, as remarked by Mr Cayley, that when not less than half the roots of a function of two variables are equal, all its invariants must vanish together.

Consequently we shall still have all the first minors of

$$\begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}$$

zero, although there is not even so much as a pair of equal roots in ϕ; $AC - B^2$ however, it will be observed, is not zero in this supposition.

The theory of Hessians, simple or bordered, may be regarded as one among the infinite diversity of applications of the principle of the plexus. Let U, V, W, &c. be any number of concomitants having the common system of variables $x, y \dots z$. Let χ represent

$$x'\frac{d}{dx} + y'\frac{d}{dy} + \dots + z'\frac{d}{dz},$$

and take

$$\chi^2 U + \lambda\chi V + \text{\&c.} + \mu\chi W = S;$$

then

$$\frac{dS}{dx'}, \frac{dS}{dy'} \dots \frac{dS}{dz'}$$

forms a plexus; and this, combined with χV, &c. ... χW, enables us to eliminate dialytically x', y', z', $\lambda \dots \mu$. The result is a Hessian of U, bordered with

$$\frac{dV}{dx}, \frac{dV}{dy} \dots \frac{dV}{dz}$$

horizontally and vertically, and also with

$$\frac{dW}{dx}, \frac{dW}{dy} \dots \frac{dW}{dz},$$

&c. &c.

similarly dispersed; which Hessian, so bordered, is thus seen to be a concomitant to U, $V \dots W$. The Hessian, as ordinarily bordered with ξ, $\eta \dots \zeta$, is derived by taking for V the universal concomitant

$$x\xi + y\eta + \dots + z\zeta,$$

and for W (if there be a double border)

$$x\xi' + y\eta' + \dots + z\zeta',$$

and so forth.

If V be taken identical with U, the resulting form, consisting of U bordered with $\frac{dU}{dx}, \frac{dU}{dy} \dots \frac{dU}{dz}$, has been shown* in my paper "On certain general Properties of Homogeneous Functions," in this *Journal*, to be equal to the product of the simple Hessian of U and of U itself multiplied by a

[* p. 173 above.]

numerical factor. The theory of the bordered Hessian may be profitably extended by taking

$$S = \chi^{2r} U + \lambda \chi^r V + \ldots + \mu \chi^r W,$$

and combining with $\chi^r V \ldots \chi^r W$ the plexus obtained by operating upon S with the rth powers and products of $\dfrac{d}{dx}, \dfrac{d}{dy} \cdots \dfrac{d}{dz}$, and eliminating dialytically the rth powers and products of $x', y' \ldots z'$. Thus if

$$U = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \text{ and } V = (x\xi + y\eta)^2,$$

we obtain, by taking $S = \chi^4 U + \lambda \chi^2 V$, and proceeding as indicated in the preceding,

$$\begin{vmatrix} a, & b, & c, & \xi^2 \\ b, & c, & d, & \xi\eta \\ c, & d, & e, & \eta^2 \\ \xi^2, & \xi\eta, & \eta^2, & \end{vmatrix}$$

as a concomitant to U. So again, if

$$U = ax^5 + 5bx^4y + \ldots + fy^5,$$

we find

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy, & \xi^2 \\ bx + cy, & cx + dy, & dx + ey, & \xi\eta \\ cx + dy, & dx + ey, & ex + fy, & \eta^2 \\ \xi^2, & \xi\eta, & \eta^2, & \end{vmatrix}$$

a concomitant to U.

These extensions of the ordinary theory of Hessians will be found to be of considerable practical importance in the treatment of forms, for which reason they are here introduced.

Section VI. *On the Partial Differential Equations to Concomitants, Orthogonal and Plagiogonal Invariants, &c.*

In the 7th note of the Appendix to the three preceding sections* I alluded to the partial differential equations by which every invariant may be defined.

This method may also be extended to concomitants generally. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr Cayley who communicated to me the equations which define the invariants of functions of

[* p. 326 above.]

two variables*. The method by which I obtain these equations and prove their sufficiency is my own, but I believe has been adopted by Mr Cayley in a memoir about to appear in *Crelle's Journal.* I have also recently been informed of a paper about to appear in *Liouville's Journal* from the pen of M. Eisenstein, where it appears the same idea and mode of treatment have been made use of. Mr Cayley's communication to me was made in the early part of December last, and my method (the result of a remark made long before) of obtaining these and the more general equations, and of demonstrating their sufficiency, imparted a few weeks subsequently—I believe between January and February of the present year.

The method which I employ, in fact, springs from the very conception of what an invariant means, and does but throw this conception into a concise analytical form.

Suppose, to fix the ideas,

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \ldots + ly^n,$$

and let $I(a, b, c \ldots l)$ be any invariant to ϕ.

Now suppose x to become $x + ey$, but y to remain unchanged; the modulus of the transformation, $\begin{vmatrix} 1, & e \\ 0, & 1 \end{vmatrix}$, being unity, I cannot alter in consequence of this substitution; but the effect of this substitution is to convert ϕ into the form

$$\alpha x^n + n\beta x^{n-1}y + \tfrac{1}{2}n(n-1)\gamma x^{n-2}y^2 + \ldots + \lambda y^n,$$

where $$\alpha = a, \quad \beta = b + ae, \quad \gamma = c + 2be + ae^2, \text{ \&c. \&c.}$$

$$\lambda = l + \ldots + nbe^{n-1} + ae^n.$$

Consequently, if we make

$$\Delta b = ae, \quad \Delta c = 2be + ae^2, \text{ \&c. \&c.},$$

we have by Taylor's theorem, observing that $\Delta a = 0$,

$$\Delta I = \left(\Delta b\frac{d}{db} + \Delta c\frac{d}{dc} + \text{\&c.}\right)I + \frac{1}{1.2}\left(\Delta b\frac{d}{db} + \Delta c\frac{d}{dc} + \text{\&c.}\right)^2 I$$

$$+ \frac{1}{1.2.3}\left(\Delta b\frac{d}{db} + \text{\&c.}\right)^3 I + \text{\&c.} = 0;$$

* It is extremely desirable to know whether M. Aronhold's equations are the same in form as those here subjoined. It is difficult to imagine what else they can be in substance. Should these pages meet the eye of that distinguished mathematician he will confer a great obligation on the author and be rendering a service to the theory by communicating with him on the subject: and I take this opportunity of adding that I shall feel grateful for the communication of any ideas or suggestions relating to this new Calculus from any quarter and in any of the ordinary mediums of language—French, Italian, Latin or German, provided that it be in the Latin character.

and this being true for all the values of e, every separate coefficient of e in ΔI must be zero: hence we obtain n different equations by equating to zero the coefficients of $e, e^2 \dots e^n$ respectively. The first of these equations will be

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \&c.\right)\phi = 0,$$

and it is obvious that this will imply all the rest; for, when e is taken indefinitely small, $I(a, b, c \dots)$ does not alter (when this equation is satisfied) by changing $a, b, c \dots$ into $a', b', c' \dots$; consequently $I(a', b', c', \&c.)$ will not alter, when in place of a', b', c' we write a'', b'', c'', &c., obtained from a', b', c', &c., by the same law as a', b', c', &c., from a, b, c, &c.

Thus we may go on giving an indefinite number of increments, ey to x, without changing the value of I. Consequently, if the equation above written be satisfied, *à priori* all the rest must be so too. But there is not any difficulty in showing the same thing by a direct method*.

For we have

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \&c.\right)I = 0,$$

an identical equation. Hence

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \&c.\right)\left\{\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \&c.\right)I\right\} = 0;$$

hence

$$\left\{\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \&c.\right)\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \&c.\right)\right\}I$$
$$+ \left\{a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \&c.\right\}^2 I = 0,$$

that is

$$\left\{2\left(a\frac{d}{dc} + 3b\frac{d}{dd} + 6c\frac{d}{de} + \&c.\right) + \left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \&c.\right)^2\right\}I = 0;$$

repeating the application of the symbolic operator

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + \&c.\right),$$

* The method above given has the advantage however of being immediately applicable to every species of concomitant, and we learn from it that concomitance, whether absolute or conditional, is sufficiently determined when affirmed to exist for *infinitesimal* variations; it cannot exist for infinitesimal variations without, by *necessary implication*, existing for finite variations also; a most important consideration this in conducing to a true idea of the nature of invariance and the other kinds of concomitance, and in cutting off all superfluous matter from the statement of the conditions by which they are defined.

we obtain

$$\left.\begin{aligned} &1.2.3\left\{a\frac{d}{dd}+4b\frac{d}{de}+10c\frac{d}{df}+\&c.\right\} \\ &+1.2\left\{a\frac{d}{db}+2b\frac{d}{dc}+\&c.\right\}\left\{a\frac{d}{dc}+3b\frac{d}{dd}+\&c.\right\} \\ &+\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}+\&c.\right)^3 \end{aligned}\right\} I=0,$$

and so on; the numerical multipliers of the terms of the several series within the parentheses forming the regular succession of figurate numbers

$$\begin{matrix} 1, & 2, & 3, & \&c. \\ 1, & 3, & 6, & \&c. \\ 1, & 4, & 10, & \&c. \end{matrix}$$

It is easy to see that these equations correspond to the results of making the coefficients of the successive powers of e equal to zero.

I may remark, that the first instance as far as I know on record of this, (as some may regard it rather bold) but in point of fact perfectly safe and legitimate method of differentiating conjointly operator and operand, occurs in a paper by myself in this *Journal*, Feb. 1851, "On certain General Properties of Homogeneous Functions" [p. 165 above]; where I have applied it in operating with

$$\left\{(x_1-a_1e)\frac{d}{da_1}+(x_2-a_2e)\frac{d}{da_2}+\&c.\right\}$$

upon

$$\left\{(x_1-a_1e)\frac{d}{da_1}+(x_2-a_2e)\frac{d}{da_2}+\&c.\right\}^r\omega,$$

which, as I have there noticed, gives the result

$$\left\{(x_1-a_1e)\frac{d}{da_1}+\&c.\right\}^{r+1}\omega$$

$$-re\left\{(x_1-a_1e)\frac{d}{da_1}+\&c.\right\}^r\omega.$$

The equation $\left(a\frac{d}{db}+2b\frac{d}{dc}+\&c.\right)I=0$ is evidently not enough to define I as an invariant; it merely serves to show that I does not alter when in place of x we write $x+ey$, but this is true for any function of the differences of the roots of the form multiplied by a suitable power of a, namely that power which is just sufficient to cause the product to become integer. But if we now, for convenience, write

$$\phi=ax^n+nbx^{n-1}y+\tfrac{1}{2}n(n-1)cx^{n-2}y^2+\dots$$
$$+\tfrac{1}{2}n(n-1)c'x^2y^{n-2}+nb'xy^{n-1}+a'y^n,$$

and form the similar equation from the other side, namely

$$\left(a' \frac{d}{db'} + 2b' \frac{d}{dc'} + 3c' \frac{d}{dd'} + \&c.\right) I = 0,$$

these two equations together will suffice to define any invariant, as I shall proceed to show—these are the two equations alluded to brought under my notice by Mr Cayley. If they coexist, it follows from the method by which I have deduced them that x may be changed into $x + ey$, or y into $y + fx$, without I being altered, e and f having any values whatever: and it is obvious that these substitutions may be performed, not merely alternatively but successively, because the equations between the coefficients are identical equations, and depend only on the form of I.

Let now x become $x + ey$, and then y become $y + fx$; the result of these substitutions is to convert

$$x \text{ into } x + efx + ey,$$

and

$$y \text{ into } fx + y.$$

Finally, let x become $x + gy$; then x is converted into $(1 + ef)(x + gy) + ey$, and y into $y + f(x + gy)$,

that is

$$x \text{ becomes } (1 + ef) x + (eg + efg) y,$$

and

$$y \text{ becomes } fx + (1 + fg) y.$$

The modulus of substitution it is evident, *à priori*, always remains unity, and nothing would be gained by pushing the substitutions any further, as it is clear that we may satisfy the equations

$$1 + ef = p, \quad e + g + efg = q,$$
$$f = p', \quad 1 + fg = q',$$

for all values of p, q, p', q', which satisfy the equation

$$pq' - p'q = 1,$$

and for none other except such values; hence I remains unaltered for any unit-modular linear transformation of x, y, and is therefore an invariant by definition.

If ϕ be taken a function of three variables, x, y, z, and be thrown under the form

$$az^n + (a_1x + b_1y) z^{n-1} + (a_2x^2 + 2b_2xy + c_2y^2) z^{n-2} + \&c.,$$

and I be any invariant of ϕ, by supposing x to become $x + ey$, and giving b_1, b_2, c_2, &c., the corresponding variations, and taking e indefinitely small, we obtain

$$\left\{a_1 \frac{d}{db_1} + \left(a_2 \frac{d}{db_2} + 2b_2 \frac{d}{dc_2}\right) + \left(a_3 \frac{d}{db_3} + 2b_3 \frac{d}{dc_3} + 3c_3 \frac{d}{dd_3}\right) + \&c.\right\} I = 0,$$

$$\left\{b_1 \frac{d}{da_1} + \left(c_2 \frac{d}{db_2} + 2b_2 \frac{d}{da_2}\right) + \&c. \ \&c.\right\} I = 0:$$

and in like manner, by arranging ϕ according to the powers of y and of x, we obtain two other pairs of equations: it is clear, however, that three equations (it would seem any three out of the six) would suffice and imply the other three. The method of demonstration will be the same as in the instance of two variables: First, it can be shown by the method of successive accretions, that I remaining invariable when x receives an indefinitely small increment ϵy, or y an indefinitely small increment ϵz, or z an indefinitely small increment ϵx, it will also remain invariable when these increments are taken of any finite magnitude. Secondly, by eight successive transformations, admissible by virtue of the preceding conclusion, x, y, z may be changed into any linear functions of x, y, z, consistent with the modulus of transformation being unity. And in general for a function of m variables, m partial differential equations similarly constructed (but not however arbitrarily selected) will be necessary and sufficient to determine any invariant: and it is clear that all the general properties of invariants must be contained in and be capable of being educed out of such equations.

The same method enables us also to establish the partial differential equations for any covariant, or indeed any concomitant whatever.

Thus let

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \ldots + nb'xy^{n-1} + a'y^n = 0,$$

and let $K(a, b, c, \&c.;\ x, y, x', y', \&c.;\ \xi, \eta, \&c.)$ represent any concomitant, $x, y;\ x', y'$ being cogredient, and ξ, η, &c. contragredient systems; when x, y become $x+ey, y$, any such system x', y' becomes $x'+ey', y'$; and any such system as ξ, η becomes $\xi, \eta - e\xi$; and taking e indefinitely small, the second coefficients a, b, c, &c. become $a, b+ae, c+2be$, &c. as before; hence the equation to the concomitant becomes

$$\left\{a\frac{d}{db} + 2b\frac{d}{dc} + \ldots - y\frac{d}{dx} - y'\frac{d}{dx'} + \ldots + \xi\frac{d}{d\eta} - \&c.\right\} = 0^*;$$

and in like manner, by changing y into $y+ex$, results the corresponding equation

$$\left\{a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \ldots - x\frac{d}{dy} - x'\frac{d}{dy'} + \ldots + \eta\frac{d}{d\xi} - \&c.\right\} K = 0.$$

These two equations define in a perfectly general manner every concomitant (with any given number of cogredient and contragredient systems) to the form ϕ; and the due number of pairs of similarly constituted equations will serve to define the concomitant to a function of any given number of variables†.

* For we have

$$K(a, b+ae, c+2be, \&c.;\ x, y, \&c.;\ \xi, \eta, \&c.)$$
$$= K(a, b, c, \&c.;\ x, x+ey, \&c.;\ \xi, \eta - e\xi, \&c.;\ \&c.).$$

† *Vide* Note (10) [p. 361 below].

In like manner we may proceed to form the equations corresponding to what may be termed *conditional* concomitants, whether *orthogonal* or *plagiogonal.* The concomitants previously considered may be termed absolute, the linear transformations admissible being independent of any but the one general relation, imposed merely for the purpose of convenience, namely of their modulus being made unity. An orthogonal concomitant is a form which remains invariable, not for arbitrary unit-modular, but for orthogonal transformation, that is for linear substitutions of $x, y \dots z$, which leave unchanged $x^2+y^2+\dots+z^2$: in like manner, a plagiogonal concomitant may be defined of a form which remains invariable for all linear substitutions of $x, y \dots z$, which leave unaltered any given quadratic function of $x, y \dots z$. Thus, let it be required to express the condition of $Q(a, b, c \dots x, y;\ \xi, \eta)$, being an orthogonal concomitant to the form

$$ax^n + nbx^{n-1}y + \dots + nb'xy^{n-1} + a'y^n.$$

Let x become $x+ey$, e being indefinitely small, then y must become $y-ex$, and the variations of $a, b \dots b', a'$ will be the sum of the variations produced by taking separately $x+ey$ for x and $y-ex$ for y. Hence the one sole condition for Q being of the required form becomes

$$\left\{\begin{array}{l} \left(a\dfrac{d}{db} + 2b\dfrac{d}{dc} + \dots - y\dfrac{d}{dx} + \xi\dfrac{d}{d\eta}\right) \\ -\left(a'\dfrac{d}{db'} + 2b'\dfrac{d}{dc'} + \dots - x\dfrac{d}{dy} + \eta\dfrac{d}{d\xi}\right) \end{array}\right\} Q = 0,$$

or, as it may be written, $\theta Q - \omega Q = 0$, where $\theta Q = 0$, $\omega Q = 0$ are the two equations expressing the conditions of Q, being an unconditional or absolute concomitant; and so in general if ϕ be a function of m variables, we may obtain $\frac{1}{2}m(m-1)$ equations of the form $L - M = 0$ for the concomitant, of which however $(m-1)$ only will be independent.

Supposing, again, the substitutions to which x, y are subject to be conditioned by $lx^2 + 2mxy + ny^2$ remaining unalterable, or which is a more convenient and only in appearance less general supposition by $x^2 + 2mxy + y^2$ remaining unalterable, the general type of an infinitesimal system of substitutions will be rendered by supposing x, y to become $(1+me)x + ey$, $-ex + (1-me)y$, respectively, for then $x^2 + 2mxy + y^2$ becomes

$$(1-m^2e^2)x^2 + \{2m + (2m - 2m^2)e^2\}xy + (1-m^2e^2)y^2,$$

which differs from $x^2 + 2mxy + y^2$ only by quantities of the second order of smallness which may be neglected, and ξ and η will therefore become $(1-me)\xi - e\eta$, $-ex + (1+me)y$, respectively: then, as to the coefficients of ϕ, in addition to the variations which they undergo when m is zero, there will be the variations consequent upon x assuming the increment mex, and y

the increment $-mey$: but by making x become $x+mex$, a, b, c, &c., b', a' assume respectively the variations

$$n\,.\,mea,\ (n-1)\,meb,\ \ldots\ meb',\ 0, \text{ respectively};$$

and by making y become $y-mey$, the corresponding variations become

$$0,\ -meb,\ \ldots -(n-1)\,meb',\ -n\,.\,mea', \text{ respectively.}$$

Hence the equation becomes

$$\theta Q-\omega Q+m\,(\lambda Q-\mu Q)=0,$$

where θ and ω have the same signification as before, and where λ denotes

$$na\frac{d}{da}+(n-1)\,b\frac{d}{db}+\ldots+b'\frac{d}{db'}+x\frac{d}{dx}-\xi\frac{d}{d\xi},$$

and μ denotes

$$b\frac{d}{db}+2c\frac{d}{dc}+\ldots\ldots\ldots+na'\frac{d}{da'}-y\frac{d}{dy}+\eta\frac{d}{d\eta}.$$

If there be several systems of x, y or of ξ, η, or of both, the only difference in the equation of condition will consist in putting

$$\Sigma\left(y\frac{d}{dx}\right),\quad \Sigma\left(x\frac{d}{dy}\right),\quad \Sigma\left(x\frac{d}{dx}\right),\quad \Sigma\left(y\frac{d}{dy}\right),$$

$$\Sigma\left(\eta\frac{d}{d\xi}\right),\quad \Sigma\left(\xi\frac{d}{d\eta}\right),\quad \Sigma\left(\xi\frac{d}{d\xi}\right),\quad \Sigma\left(\eta\frac{d}{d\eta}\right),$$

instead of the single quantities included within the sign of definite summation.

Fearing to encroach too much on the limited space of the *Journal*, I must conclude for the present with showing how to integrate the general equation to the orthogonal invariant of ϕ, the general function of x, y.

Beginning with $\phi=ax^2+2bxy+cy^2$, the equation becomes

$$\left\{-2b\frac{d}{da}+(a-c)\frac{d}{db}+2b\frac{d}{dc}+y\frac{d}{dx}-x\frac{d}{dy}\right\}Q=0.$$

Write now

$$\begin{aligned} da&=-2bd\theta, & dx&=yd\theta,\\ db&=(a-c)\,d\theta, & dy&=-xd\theta,\\ dc&=+2bd\theta; \end{aligned}$$

we have then

$$\lambda da+\mu db+\nu dc=d\theta\,\{\mu a+2(\nu-\lambda)\,b-\mu c\}.$$

Let $\qquad \mu=\kappa\lambda,\quad 2(\nu-\lambda)=\kappa\mu,\quad -\mu=\kappa\nu;$

then $\qquad d\log(\lambda a+\mu b+\nu c)=\kappa d\theta;$

or $\qquad \lambda a+\mu b+\nu c=be^{\kappa\theta}.$

To find κ we have the determinant

$$\begin{vmatrix} \kappa, & -1, & 0 \\ 2, & \kappa, & -2 \\ 0, & 1, & \kappa \end{vmatrix} = 0,$$

that is, $$\kappa^3 + 4\kappa = 0,$$

and calling the three roots of this equation κ_1, κ_2, κ_3, we have

$$\kappa_1 = 0, \quad \kappa_2 = 2\iota, \quad \kappa_3 = -2\iota;$$

accordingly we may put

$$\kappa = 0, \quad \lambda = 1, \quad \mu = 0, \quad \nu = 1,$$

or $$\kappa = 2\iota, \quad \lambda = 1, \quad \mu = 2\iota, \quad \nu = -1,$$

or $$\kappa = -2\iota, \quad \lambda = 1, \quad \mu = -2\iota, \quad \nu = -1.$$

Again, $$pdx + qdy = (py - qx)\, d\theta;$$

and putting $-q = ep$, $p = eq$, so that $px + qy = Ee^{e\theta}$,

$$e^2 = -1, \quad e_1 = \iota, \quad e_2 = -\iota;$$

and we may put

$$e = \iota, \quad p = 1, \quad q = -\iota,$$

or $$e = -\iota, \quad p = 1, \quad q = +\iota.$$

Consequently the complete integral of the given partial differential equation is found by writing

$$a + c = l, \qquad x - \iota y = Ee^{\iota\theta},$$
$$a + 2\iota b - c = l' e^{2\iota\theta}, \quad x + \iota y = E' e^{-\iota\theta},$$
$$a - 2\iota b - c = l'' e^{-2\iota\theta}.$$

By means of these five equations, after eliminating θ, we may obtain four independent equations between a, b, c; x, y. Suppose

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0, \quad Q_4 = 0;$$

then $Q = F(Q_1, Q_2, Q_3, Q_4)$ is the complete integral required.

Pursuing precisely the same method for the general case, it will be found that, calling the degree of the given function n when n is even, the equation in κ to be solved will be

$$\kappa(\kappa^2 + 4)(\kappa^2 + 9) \ldots (\kappa^2 + n^2) = 0;$$

and when n is odd (say $2m + 1$), the equation in κ to solve will be

$$(\kappa + 1)(\kappa^2 + 9) \ldots (\kappa^2 + n^2) = 0;$$

and performing the necessary reductions, and calling the roots of the equation, arranged in order of magnitude, $\kappa_1\iota, \kappa_2\iota \ldots \kappa_n\iota$, respectively, it will be found that the equations containing the integral become

$$\left.\begin{array}{l} L_1 = l_1 e^{\kappa_1\iota\theta} \\ L_2 = l_2 e^{\kappa_2\iota\theta} \\ L_3 = l_3 e^{\kappa_3\iota\theta} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ L_{n+1} = l_{n+1} e^{\kappa_{n+1}\iota\theta} \end{array}\right\} \quad \left.\begin{array}{l} x - \iota y = E e^{\iota\theta} \\ x + \iota y = E' e^{-\iota\theta} \end{array}\right\},$$

where $l_1, l_2 \ldots l_{n+1}$; E, E' are arbitrary constants, and where $L_1, L_2 \ldots L_{n+1}$ are the values assumed by the 1st, 2nd ... $(n+1)$th coefficients of the given function ϕ, or

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n,$$

when it is transformed by writing $x + \iota y$ in place of x, and $y + \iota x$ in place of y. ι is of course employed in the foregoing according to the usual notation to represent $\sqrt{(-1)}$. The same method applies to the general theory of plagiogonal concomitants, where the linear substitutions are supposed such as to leave $lx^2 + 2mxy + ny^2$ unaltered in form, and the equations in θ which contain the integral present themselves under a similar aspect. But a more full discussion of these interesting integrals must be reserved until the ensuing number of the *Journal*.

NOTES IN APPENDIX.

(9) The scale of covariants to a function of (x, y) obtained by the method of unravelment [on p. 297 above], may be otherwise deduced in a form more closely analogous to that of the corresponding theorems for the corresponding invariantive scale [on p. 295 above], by a method which has the advantage of exhibiting the scale equally well for the case of functions of the degree $4\iota + 2$ or $4\iota + 4$, the only difference being that in the latter case the coefficients of the odd powers of λ will be found all to vanish, so that the degrees of the covariants will rise by steps of 4 instead of by steps of 2, just conversely to what happens in the invariantive scale; whereas in the invariantive scale alluded to the forms containing odd powers of λ vanish when the degree of the function is of the form $4\iota + 2$, but do not vanish when it is of the form 4ι. This method in the form here subjoined is a slight modification of one suggested to me by my friend Mr Cayley.

Let F be the given function of x, y of the degree $2n$; take the systems x', y'; x_1, y_1 cogredient with one another and with x, y. Then form the concomitant

$$K = \left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^n F + \lambda\,(x'y - y'x)^{n-1}(x'y_1 - y'x_1)(xy_1 - yx_1).$$

Then (by what may be termed the Divellent method, which has been previously applied by me in the *Philosophical Magazine* for Nov. 1851) calling $\theta_0, \theta_1, \theta_2 \dots \theta_n$, the coefficients of

$$x'^n, \; x'^{n-1}, \; y', \; \dots \; y'^n \text{ in } K,$$

we shall have

$$\theta_0 = A_0 x^n + B_0 x^{n-1} y + \dots + L_0 y^n,$$
$$\theta_1 = A_1 x^n + B_1 x^{n-1} y + \dots + L_1 y^n,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\theta_n = A_n x^n + B_n x^{n-1} y + \dots + L_n y^n,$$

the coefficients being functions of the coefficients of f and of quadratic combinations of x_1, y_1, affected with the multiplier λ; and the determinant

$$\begin{vmatrix} A_0, & B_0 \dots L_0 \\ A_1, & B_1 \dots L_1 \\ \dots & \dots \\ A_n, & B_n \dots L_n \end{vmatrix}$$

will give a function of λ in which the coefficients of the several powers of λ will be all zero or covariants of F.

The actual form of this determinant is not here given for want of space and time, but will be exhibited hereafter. Precisely an analogous method applies to obtain the scale to $(x, y, z)^4$ given in Note (2) [p. 322 above]. Calling $F=(x, y, z)^4$, let the systems x', y', z'; x_1, y_1, z_1, be taken cogredient with one another and with x, y, z. Then, using R to express the determinant

$$\begin{vmatrix} x', & y', & z' \\ x, & y, & z \\ x_1, & y_1, & z_1 \end{vmatrix},$$

and making

$$K = \left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right)^2 F + \lambda R,$$

and proceeding as above by the divellent method, we obtain the scale required.

(10) [p. 356 above.] It is obvious that these defining equations ought to give the means of discovering and verifying all the properties of concomitants; but it is very difficult to see how in the present state of analysis many of the general theorems that have been stated, readily admit of being deduced from them.

The comparatively simple but eminently important theory of the evector symbol does however admit of a very pretty verification by aid of these equations. Thus, suppose θ any concomitant; suppose a contravariant to a function F of x, y, say

$$ax^n + nbx^{n-1}y + \dots + nb'xy^{n-1} + a'y^n.$$

Then θ must satisfy the two equations

$$\left(L+\xi\frac{d}{d\eta}\right)\theta=0,\quad \left(L'+\eta\frac{d}{d\xi}\right)\theta=0,$$

where
$$L=a\frac{d}{db}+2b\frac{d}{dc}+\ldots+nb'\frac{d}{da'},$$

$$L'=a'\frac{d}{db'}+2b'\frac{d}{dc'}+\ldots+nb\frac{d}{da}.$$

Now let $\phi=\chi(\theta)$ where

$$\chi=\xi^n\frac{d}{da}+\xi^{n-1}\eta\frac{d}{db}+\xi^{n-2}\eta^2\frac{d}{dc}+\ldots+\eta^n\frac{d}{da'};$$

then
$$L(\chi\theta)=\chi(L\theta)-(\chi L)\theta$$

$$=\chi(L\theta)-\left(\xi^n\frac{d}{db}+2\xi^{n-1}\eta\frac{d}{dc}+\ldots+n\xi\eta^{n-1}\frac{d}{da'}\right)\theta,$$

$$\xi\frac{d}{d\eta}(\chi\theta)=\chi\left(\xi\frac{d}{d\eta}\theta\right)+\left(\xi\frac{d}{d\eta}\chi\right)\theta$$

$$=\chi\left(\xi\frac{d}{d\eta}\theta\right)+\left(\xi^n\frac{d}{db}+2\xi^{n-1}\eta\frac{d}{dc}+\ldots+n\xi\eta^{n-1}\frac{d}{da'}\right)\theta.$$

Hence
$$\left(L+\xi\frac{d}{d\eta}\right)\chi(\theta)=\chi\left\{\left(L+\xi\frac{d}{d\eta}\right)\theta\right\}=\chi(0)=0.$$

Similarly
$$\left(L'+\eta\frac{d}{d\xi}\right)\chi(\theta)=0.$$

Hence if θ is an integral of the two conditioning equations, so also is $\chi(\theta)$. In like manner, if θ be a covariant or any other kind of concomitant of F, it may be proved that its evectant $\chi(\theta)$ is the same.

(11) [p. 331 above.] Very much akin with the supposed equations is the following most remarkable equation, which can be proved to exist. Let ϕ be a function of x and y of the 5th degree. Let P and Q be the quadratic and cubic covariants of ϕ. P is of two dimensions in the coefficients and also in the variables, and Q of three dimensions in both; they are in fact the s and t (in respect to x' and y') of $\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^4\phi$. Then, giving P and Q proper numerical factors, it will be found that

$$H_2\phi+PH\phi+Q\phi=0.$$

I believe that a similar equation connects any function of x and y above the 3rd degree with its first and second Hessians. The proof will be given in a subsequent Section, where also I shall give a complete proof, which occurred to me immediately after sending the preceding note to the press, of the complete Theory of the Respondent by means of the general equations of concomitance.

P.S. Since the preceding was in type, I have ascertained the existence and sufficiency of a general method for forming the polar reciprocal and probably also the discriminant to functions of any degree of three variables by an explicit process of permutation and differentiation. In particular I am enabled to give the actual rule for constructing the polar reciprocal and the discriminant curves of the 4th and 5th degrees. So far as regards the polar reciprocal of curves of the 4th degree M. Hesse has already given a method of obtaining it, but mine is entirely unlike to this, and rests upon certain extremely simple and universal principles of the calculus of forms. The only thing necessary to be done in order to carry on the process to curves of the 6th or higher degrees, is to ascertain the relation of the discriminants of functions of two variables of those respective degrees to such of the fundamental invariants as are of an inferior order to the discriminant.

The theory applies equally well to surfaces and to functions of any number of variables, and may, I believe, without any serious difficulty be extended so as to reduce to an explicit process the general problem of effecting the elimination between functions of any degree and of any number of variables. The method above adverted to will appear in a subsequent Section.

[*Continued pp.* 402 *and* 411 *below.*]

44.

SUR UNE PROPRIÉTÉ NOUVELLE DE L'ÉQUATION QUI SERT A DÉTERMINER LES INÉGALITÉS SÉCULAIRES DES PLANÈTES.

[*Nouvelles Annales de Mathématiques*, XI. (1852), pp. 438—440.]

[Extract.]

6. *Soit le déterminant carré symétrique*

$$\begin{vmatrix} a_{1,1}, & a_{1,2} \dots a_{1,n} \\ a_{2,1}, & a_{2,2} \dots a_{2,n} \\ \dots & \dots \\ a_{n,1}, & a_{n,2} \dots a_{n,n} \end{vmatrix}, \qquad \text{(M)}$$

dans lequel on a, d'après la définition,

$$a_{l,c} = a_{c,l}.$$

Élevant le déterminant à la puissance p, on obtient le déterminant

$$\begin{vmatrix} A_{1,1}, & A_{1,2} \dots A_{1,n} \\ A_{2,1}, & A_{2,2} \dots A_{2,n} \\ \dots & \dots \\ A_{n,1}, & A_{n,2} \dots A_{n,n} \end{vmatrix}; \qquad \text{(N)}$$

et ce déterminant est symétrique aussi par rapport à la diagonale $A_{1,1}$, $A_{2,2} \dots A_{n,n}$.

Retranchant de chaque terme de la diagonale symétrique de (M) *la même quantité* λ, *on obtient le déterminant*

$$\begin{vmatrix} a_{1,1}-\lambda, & a_{1,2} \dots a_{1,n} \\ a_{2,1}, & a_{2,2}-\lambda \dots a_{2,n} \\ \dots & \dots \\ a_{n,1}, & a_{n,2} \dots a_{n,n}-\lambda \end{vmatrix}. \qquad \text{(P)}$$

Développant ce déterminant et ordonnant par rapport à λ, on obtient une expression qui, étant égalée à zéro, donne l'équation

$$\lambda^n - f\lambda^{n-1} + g\lambda^{n-2} + \ldots (-1)^n t = 0, \qquad (1)$$

équation qui a n racines réelles (voir t. x. p. 259).

Retranchant de chaque terme de la diagonale symétrique du déterminant (N) *la quantité μ, et opérant comme ci-dessus, on parvient à l'équation*

$$\mu^n - F\mu^{n-1} + G\mu^{n-2} + \ldots (-1)^n T = 0, \qquad (2)$$

équation qui a aussi n racines réelles. Les racines de cette équation sont les racines de l'équation (1), *élevées chacune à la puissance p.*

Démonstration. Représentons par

$$\rho_1, \quad \rho_2, \quad \rho_3 \cdots \rho_p,$$

les p racines de l'équation $\rho^p - 1 = 0$. Écrivons le déterminant

$$\begin{vmatrix} a_{1,1} - \rho_q\lambda, & a_{1,2} \cdots a_{1,n} \\ a_{2,1}, & a_{2,2} - \rho_q\lambda \cdots a_{2,n} \\ \cdots\cdots & \cdots\cdots \\ a_{n,1} \cdots\cdots & a_{n,n} - \rho_q\lambda \end{vmatrix},$$

et faisons q égal successivement à tous les nombres de la suite $1, 2, 3 \ldots p$, on aura p déterminants; le produit de tous ces déterminants reste évidemment le même dans quelque ordre qu'on prenne ces déterminants, et, d'après les propriétés connues des racines de l'unité, tous les termes en ρ qui ne seront pas élevés à une puissance p disparaîtront, et λ accompagnant toujours ρ, il ne reste donc que des λ^p, et le déterminant-produit sera

$$\begin{vmatrix} A_{1,1} - \lambda^p, & A_{1,2}, & A_{1,3} \cdots A_{1,n} \\ A_{2,1}, & \ldots, & A_{2,2} - \lambda^p \cdots A_{2,n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ A_{n,1}, & A_{n,2}, & \cdots\cdots A_{n,n} - \lambda^p \end{vmatrix}; \qquad (Q)$$

où, faisant abstraction de λ, on a le déterminant (N). Ainsi

$$\mu = \lambda^p.$$

C. Q. F. D.

7. *Application.*

$$n = 2, \text{ et } p = 2;$$

déterminant $\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}$, (M)

élevant ce déterminant au carré, on a

$$\begin{vmatrix} a^2 + b^2, & ab + bc \\ ab + bc, & b^2 + c^2 \end{vmatrix}; \qquad (N)$$

déterminant
$$\begin{vmatrix} a-\lambda, & b \\ b, & c-\lambda \end{vmatrix}, \tag{P}$$

$$\lambda^2-(a+c)\lambda+ac-b^2=0\,; \tag{1}$$

déterminant
$$\begin{vmatrix} a^2+b^2-\mu, & ab+bc \\ ab+bc, & b^2+c^2-\mu \end{vmatrix},$$

$$\mu^2-(a^2+c^2+2b^2)\mu+(ac-b^2)^2=0, \text{ où } \mu=\lambda^2. \tag{2}$$

Faisons
$$n=2, \quad p=3,$$

(M) ne change pas, et l'on a
$$\begin{vmatrix} a^3+2ab^2+b^2c, & a^2b+abc+b^3+bc^2 \\ a^2b+abc+b^3+b^2c, & ab^2+2b^2c+c^3 \end{vmatrix}; \tag{N}$$

le déterminant (P) et l'équation (1) restent les mêmes; mais l'équation (2) devient
$$\mu^2-(a^3+c^3+3ab^2+3cb^2)\mu+(ac-b^2)^3=0,$$

où
$$\mu=\lambda^3,$$

car, λ_1 et λ_2 étant les deux racines de l'équation (1), on a
$$\lambda_1^3+\lambda_2^3=a^3+c^3+3ab^2+3cb^2, \quad \lambda_1^3\lambda_2^3=(ac-b^2)^3.$$

8. M. Sylvester fait observer que son théorème est un cas particulier d'un théorème plus général, démontré par M. Borchardt, pour des déterminants quelconques, et qui devient le théorème démontré ci-dessus, lorsque le déterminant est symétrique (*Journal de Mathématiques*, t. XII. p. 63, 1847).

45.

ON A REMARKABLE THEOREM IN THE THEORY OF EQUAL ROOTS AND MULTIPLE POINTS.

[*Philosophical Magazine*, III. (1852), pp. 375—378.]

In order that the theorem which I propose to state may be the more easily understood, and with the least ambiguity expressed, I shall commence with the case of a homogeneous function of two variables only, x and y.

Let

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \ldots + nb'xy^{n-1} + a'y^n,$$

and let the result of operating with the symbol

$$x^n \frac{d}{da} + x^{n-1}y\frac{d}{db} + \ldots + y^{n-1}x\frac{d}{db'} + y^n\frac{d}{da'},$$

on any function of $a, b, c \ldots b', a'$ be called the Evectant of such function, and the result of repeating this process r times the rth Evectant.

Understand by the multiplicity of the equation the number of equalities between the roots that exist; so that a pair of equal roots will signify a multiplicity 1, two pairs of equal roots, or three equal roots a multiplicity 2; a pair of equal roots and a set of three equal roots, a multiplicity $1 + 2$ or 3, and so on. Now suppose the total multiplicity of ϕ to be m: the first part of the proposition consists in the assertion that the 1st, 2nd, 3rd ... $(m-1)$th Evectants of the discriminant of ϕ, that is of the result of eliminating x and y between $\frac{d\phi}{dx}, \frac{d\phi}{dy}$ (as well as the discriminant itself), will all vanish in whatever way the multiplicity is distributed; the second part of the proposition about to be stated requires that the mode should be taken into account of the manner in which the multiplicity (m) is made up. Suppose, then, that there are r groups of roots, for one of which the multiplicity is m_1, for the second m_2, &c., and for the rth m_r, so that $m_1 + m_2 + \ldots + m_r = m$. Then, I say, that the mth evectant of the determinant of ϕ is of the form

$$(a_1x + b_1y)^{m_1 n}(a_2x + b_2y)^{m_2 n} \ldots (a_r x + b_r y)^{m_r n},$$

where $a_1 : b_1$, $a_2 : b_2 \ldots a_r : b_r$ are the ratios of $x : y$ corresponding to the several sets of equal roots.

This latter part of the theorem for the case of $m = 1$ was discovered inductively by Mr Cayley, by considering the cases when ϕ is a cubic, or a biquadratic function. I extended the theory to functions of any number of variables, and supplied a demonstration, that is for the case of one pair of equal roots. Mr Salmon showed that my demonstration could be applied to the case of two pairs of equal roots, or two double points, &c., and very nearly at the same time I made the like extension to the case of three equal roots, cusps, &c., and almost immediately after I obtained a demonstration for the theorem in its most general form. This demonstration reposes upon a very refined principle, which I had previously discovered but have not yet published, in the Theory of Elimination.

I have here anticipated a little in speaking of the theorem as applicable to curves and other loci.

Suppose $\phi(x, y, z) = 0$ to be the equation to a curve expressed homogeneously.

Let

$$\phi(x, y, z) = ax^n + (na'x^{n-1}y + nb'x^{n-1}z)$$
$$+ \tfrac{1}{2}n(n-1)a''x^{n-2}y^2 + n(n-1)b''x^{n-2}yz + \tfrac{1}{2}n(n-1)c''x^{n-2}z^2,$$
$$+ \&\text{c.} \qquad \&\text{c.},$$

and understand by the evectant of any quantity the result of operating upon it with the symbol

$$x^n \frac{d}{da} + x^{n-1}y\frac{d}{da'} + x^{n-1}z\frac{d}{db'} + x^{n-2}y^2\frac{d}{da''} + \&\text{c.}$$

Suppose, now, the curve to have double points, the $(r-1)$th evectant (and of course all the inferior evectants) of the discriminant of ϕ (meaning thereby the result of eliminating x, y, z between $\dfrac{d\phi}{dx}, \dfrac{d\phi}{dy}, \dfrac{d\phi}{dz}$) will all vanish, and the rth evectant will be of the form

$$(a_1x + b_1y + c_1z)^n \times (a_2x + b_2y + c_2z)^n \ldots \times (a_rx + b_ry + c_rz)^n,$$

where $a_1 : b_1 : c_1$, $a_2 : b_2 : c_2 \ldots a_r : b_r : c_r$ are the ratios of the coordinates at the respective double points. If there be cusps the multiplicity of each such will be 2; and calling the total multiplicity m, to every cusp will correspond a factor of the $2n$th power in the mth evectant; and so on in general for various degrees of multiplicity at the singular points respectively. The like theorem extends to conical and other singular points of surfaces; so that there exists a method, when a locus is given having any degree of multiplicity, of at once detecting the amount and distribution of this multiplicity, and the positions of the one or more singular points. In conclusion I may state, that precisely analogous results (*mutatis mutandis*) obtain, when, in place of a single function having multiplicity, we take the more general supposition of any number of homogeneous functions being subject to the condition of pluri-simultaneity, that is being capable of being made to vanish by each of several different systems of values for the ratios between the variables. Multiplicity in a single function is, in fact, nothing more nor less than pluri-simultaneity existing between the functions derived from it by differentiating with respect to each of the given variables successively. But as I purpose to give these theorems and their demonstration, which I have already imparted to my mathematical correspondents, in a paper destined for reading before the Royal Society, I need not further enlarge upon them on the present occasion.

P.S. In the above statement I have spoken only of cusps of curves which are the precise and unambiguous analogues of three coincident points in point-systems, in order to avoid the necessity of entering into any disquisition as to the species of singularity in curves or other loci corresponding to higher degrees of multiplicity in point-systems, a subject which has not hitherto been completely made out. I may here also add a remark, which gives a still higher interest to the theory, which is (to confine ourselves, for the sake of brevity, to functions of two variables), that if any root of $x:y$, say $a:b$, occur $1+\mu$ times, the total multiplicity of the equation being supposed m, and its degree n, then taking ι any integer number not exceeding μ, the $(m+\iota)$th evectant of the discriminant will contain the factor $(ax+by)^{(\mu-\iota)n}$. So that, for instance, if there be but a single group of equal roots, and they be $1+\mu$ in number, every evectant up to the $(\mu-1)$th inclusive will vanish, and from the μth to the $(2\mu-\iota)$th will contain a power of $(ax+by)^n$.

46.

OBSERVATIONS ON A NEW THEORY OF MULTIPLICITY.

[*Philosophical Magazine*, III. (1852), pp. 460—467.]

In the Postscript to my paper in the last number of the *Magazine*, I mis-stated, or to speak more correctly, I understated the law of Evection applicable to functions having any given amount of distributive multiplicity. The law may be stated more perfectly, and at the same time more concisely, as follows. Every point represented by the coordinates $\alpha_1, \beta_1 \ldots \gamma_1$, for which the multiplicity is m_1, will give rise in *every evectant* * of the discriminant of the function to a factor $(\alpha_1 x + \beta_1 y + \ldots + \gamma_1 z)^{m_1 n}$, n being supposed to be the degree of the function. Hence if there be r such points, for which the several multiplicities are $m_1, m_2 \ldots m_r$, every evectant must contain $(m_1 + m_2 + \ldots + m_r) n$ linear factors; and as the ιth evectant is of the degree ιn, it follows that all the evectants below the $(m_1 + m_2 + \ldots + m_r)$th evectant must vanish completely, and this Evectant itself be contained as a factor in all above it †. When a function of only two variables is in question, there is no difficulty in understanding what property of the function it is which is indicated by the allegation of the existence of multiplicities $m_1, m_2 \ldots m_r$;

* Frequent use being made in what follows of the word Evectant, I repeat that the evectant of any expression connected with the coefficients of a given function (supposed to be expressed in the more usual manner with letters for the coefficients affected with the proper binomial or polynomial numerical multipliers) means the result of operating upon such expressions with a symbol formed from the given function by suppressing all the binomial or polynomial numerical parts of the coefficients to be suppressed, and writing in place of the literal parts of the coefficients a, b, c, &c. the symbols of differentiation $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}$, &c.; in all that follows it is the successive evectants of the discriminant alone which come under consideration. I need hardly repeat, that the discriminant of a function is the result of the process of elimination (clear from extraneous factors) performed between the partial differential quotients of the function in respect to the several variables which it contains, or to speak more accurately, is the characteristic of their coevanescibility.

† The constitution of the quotients obtained by dividing all the other evectants of the discriminant by the first non-evanescent one, presents many remarkable features which remain yet to be fully studied out, and promise a wide extension of the existing theory.

as already remarked, this simply means that there are r distinct groups of equal roots, such groups containing $1+m_1,\ 1+m_2 \ldots 1+m_r$ roots respectively. So for curves and higher loci, the total distributive multiplicity is the sum of the multiplicities at the several multiple points. But the true theory of the higher degrees of multiplicity separately considered at any point remains yet to be elaborated, and will be found to involve the consideration of the theory of elimination from a point of view under which it has never hitherto been contemplated.

Confining our attention for the present to curves, we have a clear notion of the multiplicity 1: this is what exists at an ordinary double point. As well known, its analytical character may be expressed by saying that the function of x, y, z, which characterizes the curve, is capable, when proper linear transformations are made, of being expanded under the form of a series descending according to the powers of z, such that the constant coefficient of the highest power of z, and the linear function of x, y, which is the coefficient of the next descending power of z, may both disappear. Again, when the multiplicity is 2, the third coefficient, which is a quadratic function of x and y, will become a perfect square. This is the case of a cusp, which, as I have said, is the precise analogue to that of three equal roots for a function of two variables. Before proceeding to consider what it is which constitutes a multiplicity 3 for a curve, it will be well to pause for a moment to fix the geometrical characters of the ordinary double point and the cusp.

If we agree to understand by a first polar to a curve the curve of one degree lower which passes through all the points in which the curve is met by tangents drawn from an arbitrary point taken anywhere in its own plane, we readily perceive that at an ordinary double point all the infinite number of first polars which can be drawn to the curve will intersect one another at the double point. Again, at a cusp all these polars will not only all intersect, they will moreover all touch one another at the cusp. Now we may proceed to inquire as to the meaning of a multiplicity of the third degree, which, strange to say, I believe has never yet been distinctly assigned by geometricians.

This is not the case of a so-called triple point, that is a point where three branches of the curve intersect. Supposing $x=0$, $y=0$, to represent such a point, the characteristic of the curve must be reducible to the form

$$(gx^3+hx^2y+kxy^2+ly^3)\,z^{n-3}+\&\text{c.},$$

which, as is well known, involves the existence of four conditions. This, however, would not in itself be at all conclusive against the multiplicity at a triple point being only of the third degree; for it can readily be shown that there may exist singular points of any degree of *singularity* (as measured by the number of conditions necessary to be satisfied in order that such singularity may come into existence), but for which the multiplicity may be as low as we please; as, for instance, if at a double point (which is not a cusp) there be a point of inflexion on one branch or on both, or a point of undulation, or any other singularity whatever, still provided there be no cusps, the multiplicity will stick at the first degree and never exceed it; for only the discriminant itself will vanish on these suppositions, but no evectant of the discriminant. The reason, on the contrary, why a so-called triple point must be said to have a multiplicity of the degree 4, and not merely of the degree 3, springs from the fact that the 1st, 2nd and 3rd evectants of the discriminant all vanish at such a point.

It is clear, then, that there ought to exist a species of multiplicity for which the 1st and 2nd evectants vanish, but not the 3rd. In fact, as at a double point the first polars all merely intersect, but at a cusp have all a contact with one another of the first degree, so we ought to expect that there should exist a species of multiple point such that all the first polars should have with each other a contact of the second degree (or if we like so to say, the same curvature) at that point. When the curve has a triple point, all its first polars will have that point upon them as a double point; and it is not at the first glance, easy *à priori* to say what is the nature of the contact between two curves which intersect at a point which is a double point to each of them: we know upon settled analytical principles, that when one curve having a double point is crossed there by another curve not having a double point, that the two must be said to have with one another, a contact of the 1st degree; and we now learn from our theory of evection, that if each have a double point at the meeting-point, the degree of the contact must from principles of analogy be considered to be of the 3rd degree*. Now, then, we come to the question of deciding definitely what is a multiple point for which the degree of multiplicity is 3. It is, adopting either test, whether of first polar contact or of evection, a cusp situated or having its *nidus*, so to say, at a point of inflexion. In other words, $x = 0$, $y = 0$ will be a point whose multiplicity is intermediate between that of the cusp and that of a so-called triple point, when the characteristic of the curve admits of being written under the form

$$z^{n-2}x^2 + z^{n-3}(gx^3 + hx^2y + ixy^2) + z^{n-4} \text{ \&c.};$$

or in other words, when over and above the vanishing of the constant and linear coefficients, and the quadratic coefficient being a perfect square, as in the case of an ordinary cusp, this square has a factor in common with the next (the cubic) coefficient; or again, in other words, a curve has a point

* This may easily be verified by direct analytical means; as also the more general proposition, that two curves meeting at a point where there are m branches of the one and n branches of the other, must be considered to have mn coincident points in common, that is, if we like so to express it, to have a contact of the degree $mn - 1$.

for which the multiplicity is 3 when its characteristic function admits of being expanded according to the powers of one of the variables, in such a manner that the first coefficient and the second (the linear) coefficient vanish, and that the discriminant of the third and the resultant of the third and fourth are both at the same time zero. This being the case, it may be shown that the first polars will all have with each other a contact of the second degree; and moreover, that all the evectants of the discriminant will have as a common factor a linear function of the variables, raised to a power whose index is three times that of the characteristic function. As, then, there is but one kind of ordinary double point, and but one kind of point with multiplicity 2, so there is one, and only one, kind of point with a multiplicity 3. A cusp is a peculiar double point; a flex-cusp (as for the moment I call the point last above discussed) is a peculiar cusp. This law of unambiguity, however, appears to stop at the third degree. A so-called triple point (which ought in fact to be called a *quintuple* point) is a point for which the multiplicity, as shown above, is of the fourth degree; but it is not the only point of that degree of multiplicity. Without assuming to have exhausted every possible supposition upon which such a degree of multiplicity may be brought into existence, it will be sufficient to take as an example a curve whose characteristic is capable of assuming the form

$$z^{n-2}x^2 + z^{n-3}(gx^3 + hx^2y) + z^{n-4}(kx^4 + lx^3y + mx^2y^2 + nxy^3) + z^{n-5}\ \&c.$$

It may readily be demonstrated that the first polars of this curve have all with one another at the point x, y a contact of a degree exceeding the 2nd, that is of at least the 3rd degree (and, I believe, in general not higher). Now the point x, y is evidently not a triple-branched point, but a cusp with three additional degrees of singularity; so that we have evidence of the existence of a point whose degree of singularity is 5, and whose multiplicity is at least 4, but which is in no sense a modified triple point. It is probably true (but to demonstrate this requires a further advance to be made than has yet been realized in the theory of the constitution of discriminants) that a cusp may be so modified by the *nidus* at which it is posited, as, without ever passing into a triple point, to be capable of furnishing any amount of multiplicity whatever, curiously in this contrasting with an ordinary double point, no amount whatever of extraordinary singularity imparted to which, or so to speak, to its *nidus*, can ever heighten its multiplicity so as to make it surpass the first degree without first converting it into a cusp. I may illustrate the nature of a flex-cusp by what happens to a curve of the third degree. When it breaks up into a conic and a right line, there are two ordinary double points; for the existence of these double points, as for the existence of a cusp, two conditions are required. When, however, the right line and conic touch one another (a *casus omissus* this in the works of the special geometers), the characters of the cusp and the point of inflexion are combined at the point of contact; the multiplicity is of the third degree, and the singularity also of a degree not exceeding this; three conditions only being necessary to be satisfied in order that a given cubic may degenerate into such a form; and it will be found that the discriminant and the first and second evectants thereof vanish for this case, and that the third evectant of the discriminant will be a perfect 9th power; whereas in order that the cubic may have a so-called triple point, that is may degenerate into a trident of diverging rays, four conditions must be satisfied, and it will be found that when this is the case, the first, second, and third evectants of the discriminant will all vanish, and the fourth will be a perfect 12th power of a linear function of the variables. I may mention, by the way, at this place, that the law of a discriminant and the successive evectants up to the mth inclusive, all vanishing, may be expressed otherwise (not in *identical*, but in *equivalent* or *equipollent* terms), by saying that the discriminant and all its derivatives of a degree not exceeding the mth will all vanish—understanding by a derivative of the discriminant any function obtained from the discriminant by differentiating it any specified number of times with respect to the constants of the function to which it belongs, the same constants being repeated or not indifferently*. And very surprising it must be allowed to be, stated as a bare analytical fact, that $(m+1)$ conditions imposed upon the coefficients of a function of any number of variables and of any degree should suffice to make the inordinately greater number of functions which swarm among the derivatives of the mth and inferior degrees of the discriminant each and all simultaneously vanish.

Without pushing these observations too far for the patience of the general reader, it may be remarked by way of setting foot with our new theory upon the almost unvisited region of the singularities of surfaces, that by the light of analogy we may proceed with a safe and firm step as far as multiplicity of the third degree inclusive.

The function characteristic of the surface being supposed to be expressed in terms of the four variables x, y, z, t, and expanded according to descending powers of t, then when x, y, z is an ordinary double point of the first degree of multiplicity, the constant and the linear coefficient disappear; when the point has a multiplicity 2, the discriminant of the quadratic coefficient will be zero, that is this coefficient will be expressible by means of due linear transformations under the form of x^2+y^2; and when the multiplicity is to be of the degree 3, the cubic coefficient will, at the same time that the quadratic coefficient is put under the form x^2+y^2, itself (for the same system of x and y) assume the form of a cubic function of x, y, z, in which the highest power of z, that is z^3, will not appear; or in other words (restoring to x, y, z their

* Or, to speak more simply, the discriminant and its successive *differentials* up to the mth exclusive must all vanish simultaneously.

generality), not only will the first derivatives of the quadratic function be nullifiable simultaneously with each other, but likewise at the same time with the cubic function itself. These three cases will be for surfaces, the analogues so far, but only so far as regards the degree of the multiplicity, to the double point, cusp, and flex-cusp of curves*. The analogue to the so-called triple point of the curves will be a point whose degree of singularity, depending upon the vanishing of the six constants in the third coefficient (which is a quadratic function of x, y, z) at the same time as the three constants in the linear factor, would seem to be but 6 more than for a double point, that is in all $1 + 6$ or 7, but whose multiplicity, as inferred from the nature of the contact of its first polars, which will be of the 7th order, would appear to be 8 (a seeming incongruity which I am not at present in a condition to explain)†; so that there will apparently be 4 steps of multiplicity to interpolate between this case and the case analogous (*sub modo*) to the flex-cusp, last considered. Whether these intervening degrees correspond to singularities of an unambiguous kind, no one is at present in a condition to offer an opinion. I will conclude with a remark, the result of my experience in this kind of inquiry as far as I have yet gone in it, namely that it would be most erroneous to regard it as a branch of isolated and merely curious or fantastic speculation. Every singularity in a locus corresponds to the imposition of certain conditions upon the form of its characteristic; by aid of the theory of evection we are able to connect the existence of these conditions with certain consequences happening to the form of the discriminant, and thereby it becomes possible, upon known principles of analysis, to infer particulars relating to the constitution of the discriminant itself in its absolutely general form, very much upon the same principle as when the values of a function for particular values of its variable or variables are known, the general form of the function thereby itself, to some corresponding extent, becomes known. Thus, for instance, I have by the theory of evection in its most simple application, been led to a representation of the discriminant

* At an ordinary conical point of a surface for which the multiplicity is 1, every section of the surface is a curve with a double point. When the multiplicity is 2, the cone of contact becomes a pair of planes, through the intersection of which any other plane that can be drawn cuts the surface in a section having an ordinary cusp of multiplicity 2, but which themselves cut the surface in sections, having so-called triple points, so that for these two principal sections (which is rather surprising) the multiplicity suddenly jumps up from 2 to 4. All other things remaining unaltered when the multiplicity of the conical point is 3, the cusp belonging to any section of the surface drawn through any intersection of the two tangent planes passes from an ordinary cusp to a flex-cusp.

† So, too, at a so-called quadruple point in a curve, the degree of the contact of the 1st polars is 8, and therefore the multiplicity of the curve at such point is 9; but the number of constants which vanish for this case (namely all those of the cubic coefficient in x, y) over and above what vanish for the case of a so-called triple point is only 4, which is a unit less than the difference between the measures of the multiplicities at the respective points; and this difference continues to increase as we pass on to so-called quintuple and higher multiple points in the curves.

of a function of two variables under a form very different and very much more complete and fecund in consequences than has ever been supposed, or than I had myself previously imagined, to be possible.

According to the opinion expressed by an analyst of the French school, of pre-eminent force and sagacity, it is through this theory of multiplicity, here for the first time indicated, that we may hope to be able to bridge over for the purposes of the highest transcendental analysis, the immense chasm which at present separates our knowledge of the intimate constitution of functions of two from that of three, or any greater number of variables.

It is, as I take pleasure in repeating, to a hint from Mr Cayley*, who habitually discourses pearls and rubies, that I am indebted for the precious and pregnant observation on the form assumed by the first discriminantal evectant of a binary function with a pair of equal roots, out of which, combined with some antecedent reflections of my own, this new theory of multiplicity has taken its rise. The idea of the process of evection, and the discovery of its fundamental property of generating what, in my calculus of forms (*Cambridge and Dublin Mathematical Journal*), I have called *contravariants*, is due to my friend M. Hermite. The polar reciprocals of curves and other loci are contravariants and, as I have recently succeeded in showing, for curves at least, evectants, but of course not discriminantal evectants; and I am already able to give the actual explicit rule for the formation of the polar reciprocal of curves as high as the 5th degree, which with a little labour and consideration can be carried on to the 6th, and in fact to curves of any degree n when once we are acquainted with any mode of determining all such independent invariants of a function of two variables as are of dimensions not exceeding $2(n-1)$ in respect of the coefficients.

By the special geometers (by whom I mean those who, unvisited by a higher inspiration, continue to regard and to cultivate geometry as the science of mere sensible space) this problem has only been accomplished, and that but recently, for curves whose degrees do not exceed the 4th. Mr Salmon has made the happy and brilliant (and by the calculus of forms instantaneously demonstrable) discovery, communicated to me in the course of a most instructive and suggestive correspondence, that *a certain readily ascertainable*

* Mr Cayley's theorem stood thus :—If

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n$$

have two equal roots, and ϖ be its discriminant, then will

$$\left\{y^n \frac{d}{da} - y^{n-1}x\frac{d}{db} \&\text{c.} \pm x^n \frac{d}{da'}\right\} \varpi$$

be a perfect nth power. It will easily be seen that this theorem is convertible into a theorem of evection by interchanging in the result x and y with y and $-x$.

*evectant of every discriminant of any function whatever is an exact power of its polar reciprocal**.

I believe that it may be shown, that, with the sole exception of odd-degreed functions of two variables, the *polar reciprocal itself* (as distinguished from a power thereof) of every function is an evectant, not (of course) of the discriminant, but of some determinable inferior invariant.

P.S. The terms pluri-simultaneous and pluri-simultaneity, used or suggested by me in my last paper in the *Magazine*, may be advantageously replaced by the more euphonious and regularly formed words consimultaneous, consimultaneity. Multiplicity and all its attributes and consequences are included as particular cases in the general conception and theory of consimultaneity, that is of consimultaneous equations, or, which is the same thing, of consimulevanescent functions.

* Namely, for a function of degree n, and variability (that is, having a number of variables) p, the $(n-1)^{p-1}$th evect of the discriminant is the $(n-1)$th power of the polar reciprocal.

47.

A DEMONSTRATION OF THE THEOREM THAT EVERY HOMOGENEOUS QUADRATIC POLYNOMIAL IS REDUCIBLE BY REAL ORTHOGONAL SUBSTITUTIONS TO THE FORM OF A SUM OF POSITIVE AND NEGATIVE SQUARES.

[*Philosophical Magazine*, IV. (1852), pp. 138—142.]

It is well known that the reduction of any quadratic polynomial

$$(1, 1)\,x^2 + 2\,(1, 2)\,xy + (2, 2)\,y^2 + \ldots + (n, n)\,t^2$$

to the form $a_1\zeta^2 + a_2\eta^2 + \ldots + a_n\theta^2$, where $\zeta, \eta \ldots \theta$ are linear functions of $x, y \ldots t$, such that $x^2 + y^2 + \ldots + t^2$ remains identical with $\zeta^2 + \eta^2 + \ldots + \theta^2$ (which identity is the characteristic test of orthogonal transformation), depends upon the solution of the equation

$$\begin{vmatrix} (1, 1) + \lambda, & (1, 2) \ldots\ldots\ldots (1, n) \\ (2, 1), & (2, 2) + \lambda \ldots (2, n) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ (n, 1), & (n, 2) \ldots\ldots\ldots (n, n) + \lambda \end{vmatrix} = 0.$$

The roots of this equation give $a_1, a_2 \ldots a_n$; and if they are real, it is easily shown that the connexions between $x, y \ldots t$; $\zeta, \eta \ldots \theta$, are also real. M. Cauchy has somewhere given a proof of the theorem*, that the roots of λ in the above equation must necessarily always be real; but the annexed demonstration is, I believe, new; and being very simple, and reposing upon a theorem of interest in itself, and capable no doubt of many other applications, will, I think, be interesting to the mathematical readers of this *Magazine*.

* Jacobi and M. Borchardt have also given demonstrations; that of the latter consists in showing that Sturm's functions for ascertaining the total number of real roots expressed by my formulæ (many years ago given in this *Magazine*) are all, in the case of $f(\lambda)$, representable as the sums of squares, and are therefore essentially positive.

Let

$$f(\lambda) = \begin{vmatrix} (1, 1)+\lambda, & (1, 2) \ldots\ldots\ldots\ldots\ldots\ldots & & (1, n) \\ (2, 1), & (2, 2)+\lambda \ldots\ldots\ldots\ldots & & (2, n) \\ (3, 1), & (3, 2), & (3, 3)+\lambda \ldots & (3, n) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ (n, 1), & (n, 2) \ldots\ldots\ldots\ldots\ldots\ldots & & (n, n)+\lambda \end{vmatrix},$$

it is easily proved that $f(\lambda) \times f(-\lambda)$

$$= \begin{vmatrix} [1, 1]-\lambda^2, & [1, 2] \ldots\ldots\ldots & [1, n] \\ [2, 1], & [2, 2]-\lambda^2 \ldots & [2, n] \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ [n, 1], & [n, 2] \ldots\ldots\ldots & [n, n]-\lambda^2 \end{vmatrix},$$

where $\quad [\iota, \epsilon] = (\iota, 1) \times (1, \epsilon) + (\iota, 2) \times (2, \epsilon) + \ldots + (\iota, n) \times (n, \epsilon)$.

If, now, for all values of r and s, $(r, s) = (s, r)$, that is, if $f(0)$ becomes the complete determinant to a symmetrical matrix, then every term $[r, s]$ in the derived matrix becomes a sum of squares, and is essentially positive, and $(-1)^n f(\lambda) \times f(-\lambda)$ assumes the form

$$(\lambda^2)^n - F(\lambda^2)^{n-1} + G(\lambda^2)^{n-2} + \ldots \pm L,$$

where $F, G, \ldots L$ will evidently be all positive; for it may be shown that F will be the sum of the squares of the separate terms, that is, of the last minor determinants of the given matrix, G the sum of the squares of the last but one minors, and so on, L being the square of the complete determinant. For instance, if

$$f(\lambda) = \begin{vmatrix} a+\lambda, & \gamma, & \beta \\ \gamma, & b+\lambda, & \alpha \\ \beta, & \alpha, & c+\lambda \end{vmatrix}$$

$$-f(\lambda) \times f(-\lambda) = \lambda^6 - F\lambda^4 + G\lambda^2 - H,$$

where $\quad F = a^2 + b^2 + c^2 + 2\alpha^2 + 2\beta^2 + 2\gamma^2,$

$$G = (ab-\gamma^2)^2 + (bc-\alpha^2)^2 + (ac-\beta^2)^2$$
$$+ 2(a\alpha - \beta\gamma)^2 + 2(b\beta - \gamma\alpha)^2 + 2(c\gamma - \alpha\beta)^2,$$

$$H = \begin{vmatrix} a, & \gamma, & \beta \\ \gamma, & b, & \alpha \\ \beta, & \alpha, & c \end{vmatrix}^2.$$

Hence it follows immediately that $f(\lambda) = 0$ cannot have imaginary roots; for, if possible, let $\lambda = p + q\sqrt{(-1)}$, and write

$$a+p = a', \quad b+p = b', \quad c+p = c', \quad \lambda + p = \lambda',$$

$f(\lambda)$ becomes
$$\begin{vmatrix} a'+\lambda', & \gamma, & \beta \\ \gamma, & b'+\lambda', & \alpha \\ \beta, & \alpha, & c'+\lambda' \end{vmatrix},$$
or say $\phi(\lambda')$, and the equation $\phi(\lambda')\times\phi(-\lambda')=0$ will be of the form
$$\lambda'^6 - F'\lambda'^4 + G'\lambda'^2 - H' = 0,$$
where F', G', H' are all essentially positive. Hence, by Descartes' rule, no value of λ'^2 can be negative, that is, $(\lambda-p)^2$ cannot be of the form $-q^2$; that is to say, it is impossible for any of the roots of $f(\lambda)=0$ to be imaginary, or, as was to be demonstrated, all the roots are real.

I may take this occasion to remark, that by whatever linear substitutions, orthogonal or otherwise, a given polynomial be reduced to the form $\Sigma A_1\zeta^2$, the number of positive and negative coefficients is invariable: this is easily proved. If now we proceed to reduce the form (expressed under the umbral notation) $(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2$ to the form
$$A_1\zeta_1^2 + A_2\zeta_2^2 + \ldots + A_{n-1}\zeta^2_{n-1} + A_n\zeta_n^2,$$
by first driving out the mixed terms in which x_1 enters, then those in which x_2 enters, and so forth until eventually only x_n of the original variables is left, it may readily be shown that
$$A_1 = \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \div \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_1a_2a_3 \\ a_1a_2a_3 \end{pmatrix} \div \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \ldots\ldots$$
$$\ldots A_n = \begin{pmatrix} a_1a_2\ldots a_n \\ a_1a_2\ldots a_n \end{pmatrix} \div \begin{pmatrix} a_1a_2\ldots a_{n-1} \\ a_1a_2\ldots a_{n-1} \end{pmatrix}.$$
It follows, therefore, that in whatever order we arrange the umbræ $a_1a_2\ldots a_n$, the number of variations and of continuations of sign in the series
$$1, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \ldots \begin{pmatrix} a_1a_2\ldots a_n \\ a_1a_2\ldots a_n \end{pmatrix},$$
will be invariable, and in fact will be the same as the number of positive and negative roots in the generating function in λ above treated of, that is, since all the roots are real, will be the same as the number of variations and continuations in the series formed by the coefficients of the several powers of λ, that is
$$1, \Sigma\begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \Sigma\begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \ldots \begin{pmatrix} a_1a_2\ldots a_n \\ a_1a_2\ldots a_n \end{pmatrix}.$$

The first part of this theorem admits of an easy direct demonstration; for by my theory of compound determinants, given in this *Magazine**, we know that
$$\begin{matrix} \overline{a_1a_2\ldots a_{r-1}a_r|} & \overline{a_1a_2\ldots a_{r-1}a_{r+1}|} \\ a_1a_2\ldots a_{r-1}a_r & a_1a_2\ldots a_{r-1}a_{r+1} \end{matrix}$$
$$= \begin{pmatrix} a_1a_2\ldots a_{r-1} \\ a_1a_2\ldots a_{r-1} \end{pmatrix} \times \begin{pmatrix} a_1a_2\ldots a_{r-1}a_ra_{r+1} \\ a_1a_2\ldots a_{r-1}a_ra_{r+1} \end{pmatrix}.$$

[* Cf. pp. 241, 252 above.]

The first member of this equation is equivalent to

$$\begin{pmatrix} a_1 a_2 \dots a_{r-1} a_r \\ a_1 a_2 \dots a_{r-1} a_r \end{pmatrix} \times \begin{pmatrix} a_1 a_2 \dots a_{r-1} a_{r+1} \\ a_1 a_2 \dots a_{r-1} a_{r+1} \end{pmatrix} - \begin{pmatrix} a_1 a_2 \dots a_{r-1} a_r \\ a_1 a_2 \dots a_{r-1} a_{r+1} \end{pmatrix}^2.$$

Hence it follows, that if the two factors on the right-hand side of the equation have the same sign,

$$\begin{pmatrix} a_1 a_2 \dots a_{r-1} a_r \\ a_1 a_2 \dots a_{r-1} a_r \end{pmatrix} \text{ and } \begin{pmatrix} a_1 a_2 \dots a_{r-1} a_{r+1} \\ a_1 a_2 \dots a_{r-1} a_{r+1} \end{pmatrix}$$

have also the same sign *inter se*, and consequently the two triads

$$\begin{bmatrix} a_1 a_2 \dots a_{r-1} \\ a_1 a_2 \dots a_{r-1} \end{bmatrix}, \quad \begin{bmatrix} a_1 a_2 \dots a_{r-1} a_r \\ a_1 a_2 \dots a_{r-1} a_r \end{bmatrix}, \quad \begin{bmatrix} a_1 a_2 \dots a_{r-1} a_r a_{r+1} \\ a_1 a_2 \dots a_{r-1} a_r a_{r+1} \end{bmatrix},$$

and

$$\begin{bmatrix} a_1 a_2 \dots a_{r-1} \\ a_1 a_2 \dots a_{r-1} \end{bmatrix}, \quad \begin{bmatrix} a_1 a_2 \dots a_{r-1} a_{r+1} \\ a_1 a_2 \dots a_{r-1} a_{r+1} \end{bmatrix}, \quad \begin{bmatrix} a_1 a_2 \dots a_{r-1} a_{r+1} a_r \\ a_1 a_2 \dots a_{r-1} a_{r+1} a_r \end{bmatrix},$$

will in all cases present the same number of changes and continuations, which proves that the contiguous umbræ, a_r, a_{r+1}, may be interchanged without affecting the number of variations and continuations in the entire series; but, as is well known, any one order of elements is always convertible into any other order by means of successive interchanges of contiguous elements, which demonstrates that, in whatever order the elements $a_1, a_2 \dots a_n$ be arranged, the number of continuations and variations in

$$1, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} a_1 a_2 \\ a_1 a_2 \end{pmatrix} \dots \begin{pmatrix} a_1 a_2 \dots a_n \\ a_1 a_2 \dots a_n \end{pmatrix},$$

is invariable. But that the same thing is true (as we know it to be), for the relation between any one of these unsymmetrical series and the symmetrical series (resulting from the method of orthogonal transformation)

$$1, \Sigma \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \Sigma \begin{pmatrix} a_1 a_2 \\ a_1 a_2 \end{pmatrix}, \dots \begin{pmatrix} a_1 a_2 \dots a_n \\ a_1 a_2 \dots a_n \end{pmatrix},$$

is by no means so easily demonstrable in the general case by a direct method, and the attention of algebraists is invited to supply such direct method of demonstration. My knowledge of the fact of this equivalence is, as I have stated, deduced from that remarkable but simple law to which I have adverted, which affirms the invariability of the number of the positive and negative signs between all linearly equivalent functions of the form $\Sigma \pm c_r x^r$ (subject, of course, to the condition that the equivalence is expressible by means of equations into which only real quantities enter); a law to which my view of the physical meaning of quantity of matter inclines me, upon the ground of analogy, to give the name of the Law of Inertia for Quadratic Forms, as expressing the fact of the existence of an invariable number inseparably attached to such forms.

48.

ON STAUDT'S THEOREMS CONCERNING THE CONTENTS OF POLYGONS AND POLYHEDRONS, WITH A NOTE ON A NEW AND RESEMBLING CLASS OF THEOREMS.

[*Philosophical Magazine*, IV. (1852), pp. 335—345.]

THE beautiful and important geometrical theorems of Staudt are, I believe, little, if at all, known to English mathematicians. They originally appeared in *Crelle's Journal* for the year 1843, and have been recently reproduced in M. Terquem's *Nouvelles Annales* for the August Number of the present year.

These theorems may be summed up, in a word, as intended to show the possibility and method of expressing the product of any two polygons or any two polyhedrons as entire functions of the squares of the distances of the angular points of the two figures from one another. The well-known expression for the square of the area of a triangle in terms of the sides (in which, when expanded, only even powers of the lengths of the sides appear), is but a particular case of Staudt's theorem for polygons, for it may be considered as the case of two equal and similar triangles whose angular points coincide. So in like manner, as observed by Staudt, a similar expression in terms of its sides may be found for the square of a pyramid. This expression had, however, been previously given (although, by a strangle negligence, not named for what it was) by Mr Cayley in the *Cambridge Mathematical Journal* for the year 1841*, in his paper on the relations between the mutual distances to one another of four points in a plane and five points in space; the singularly ingenious (and as singularly undisclosed) principle of that paper consisting in obtaining an expression for the volume of a pyramid in terms of its sides, and equating this, or rather its square, to zero as the conditions of the four angular points lying in the same plane.

* Query, Is not this expression for the volume of a pyramid in terms of its sides to be found in some previous writer? It can hardly have escaped inquiry.

The analogous condition for five points in space is virtually deduced by going out into rational space of four dimensions, and equating to zero the expression obtained for the volume of a plupyramid; meaning thereby the figure which stands in the same relation to space of four as a pyramid to space of three dimensions. Mr Cayley's method, if it had been pursued a step further, would have led him to a complete anticipation of the principal part of Staudt's discovery. The method here given is not substantially different from Mr Cayley's, but is made to rest upon a more general principle of transformation than that which he has employed. As to Staudt's own method, it is as clumsy and circuitous as his results are simple and beautiful. Geometry, trigonometry and statics, are laid under contribution to demonstrate relations which will be seen to flow as immediate and obvious consequences from the most elementary principles in the algorithm of determinants. Perhaps, however, M. Staudt's method is as good as could be found in the absence of the application of the method of determinants, the powers of which, even so recently as ten years ago, were not so well understood or so freely applied as at the present day.

The following new but simple theorem, of which I shall have occasion to make use, will be found to be a very useful addition to the ordinary method for the multiplication of determinants. "If the determinants represented by two square matrices are to be multiplied together, *any number of columns may be cut off* from the one matrix, and a corresponding number of columns from the other. Each of the lines in either one of the matrices so reduced in width as aforesaid being then multiplied by each line of the other, and the results of the multiplication arranged as a square matrix and bordered with the two respective sets of columns cut off arranged symmetrically (the one set parallel to the new columns, the other set parallel to the new lines), the complete determinant represented by the new matrix so bordered (abstraction made of the algebraical sign) will be the product of the two original determinants."

Thus $\begin{pmatrix} ab \\ cd \end{pmatrix} \times \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix}$ may be put under any one of the three following forms:—

$$\begin{vmatrix} a\alpha + b\beta, & a\gamma + b\delta \\ c\alpha + d\beta, & c\gamma + d\delta \end{vmatrix}$$

or
$$\begin{vmatrix} a\alpha, & a\gamma, & b \\ c\alpha, & c\gamma, & d \\ \beta, & \delta, & 0 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 2, & 2, & a, & b \\ 2, & 2, & c, & d \\ \alpha, & \beta, & 0, & 0 \\ \gamma, & \delta, & 0, & 0 \end{vmatrix} *.$$

* Any quantities might be substituted instead of 2 in the places occupied by the figure in the above determinant, as such terms do not influence the result; this figure is probably, however, the proper quantity arising from the application of the rule, because (as all who have calculated with determinants are aware) the value of the determinant represented by a matrix of *no places* is not zero but unity.

And in general for two matrices of n^2 terms each, this rule of multiplication will give $(n+1)$ distinct forms representing their products.

Thus, as a further example,

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} \times \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix},$$

besides the first and last forms, will be representable by the two intermediate forms

$$-\begin{vmatrix} a\alpha+b\beta, & a\alpha'+b\beta', & a\alpha''+b\beta'', & c \\ a'\alpha+b'\beta, & a'\alpha'+b'\beta', & a'\alpha''+b'\beta'', & c' \\ a''\alpha+b''\beta, & a''\alpha'+b''\beta', & a''\alpha''+b''\beta'', & c'' \\ \gamma, & \gamma', & \gamma'', & 0 \end{vmatrix}$$

and

$$+\begin{vmatrix} a\alpha, & a\alpha', & a\alpha'', & b, & c \\ a'\alpha, & a'\alpha', & a'\alpha'', & b', & c' \\ a''\alpha, & a''\alpha', & a''\alpha'', & b'', & c'' \\ \beta, & \beta', & \beta'', & 0, & 0 \\ \gamma, & \gamma', & \gamma'', & 0, & 0 \end{vmatrix}.$$

To arrive, for instance, at the latter of these two forms, we have only to write the two given matrices under the respective forms

$$\begin{vmatrix} a, & b, & c, & 0, & 0 \\ a', & b', & c', & 0, & 0 \\ a'', & b'', & c'', & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} \alpha, & 0, & 0, & \beta, & \gamma \\ \alpha', & 0, & 0, & \beta', & \gamma' \\ \alpha'', & 0, & 0, & \beta'', & \gamma'' \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \end{vmatrix}$$

and then apply the ordinary rule of multiplication. So, again, to arrive at the first of the above written two forms, we must write the two given matrices under the respective forms

$$\begin{vmatrix} a, & b, & c, & 0 \\ a', & b', & c', & 0 \\ a'', & b'', & c'', & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \text{ and } -\begin{vmatrix} \alpha, & \beta, & 0, & \gamma \\ \alpha', & \beta', & 0, & \gamma' \\ \alpha'', & \beta'', & 0, & \gamma'' \\ 0, & 0, & 1, & 0 \end{vmatrix}$$

and proceed as before.

This rule is interesting as exhibiting, as above shown, a complete scale whereby we may descend from the ordinary mode of representing the product of two determinants to the form, also known, where the two original determinants are made to occupy opposite quadrants of a square whose places in one of the remaining quadrants are left vacant, and shows us that under one aspect at least this latter form may be regarded as a matrix *bordered* by the two given matrices.

A second but obvious theorem requiring preliminary notice is the following, namely that the value of the determinant to the matrix

$$\begin{matrix} a_{1,1}, & a_{1,2} \dots a_{1,n}, & 1, \\ a_{2,1}, & a_{2,2} \dots a_{2,n}, & 1, \\ \dots\dots & \dots\dots & \dots \\ a_{n,1}, & a_{n,2} \dots a_{n,n}, & 1, \\ 1, & 1, \dots 1, & 0, \end{matrix}$$

is the same as the value of the determinant to the matrix

$$\begin{matrix} A_{1,1}, & A_{1,2} \dots A_{1,n}, & 1, \\ A_{2,1}, & A_{2,2} \dots A_{2,n}, & 1, \\ \dots\dots & \dots\dots & \dots \\ A_{n,1}, & A_{n,2} \dots A_{n,n}, & 1, \\ 1, & 1, \dots 1, & 0, \end{matrix}$$

where in general

$$A_{r,s} = a_{r,s} + h_r + k_s,$$

$h_1, h_2 \dots h_n$ and $k_1, k_2 \dots k_n$ being any two perfectly arbitrary series of quantities. This simple transformation is of course derived by adding to the respective columns in the first matrix the last column (consisting of units) multiplied respectively by h_1, $h_2 \dots h_n$, 0; and to the respective lines, the last line (consisting of units) multiplied respectively by k_1, $k_2 \dots k_n$, 0.

Suppose, now, that we have two tetrahedrons whose volumes are represented respectively by one-sixth of the respective determinants

$$\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}, \qquad \begin{vmatrix} \xi_1, & \eta_1, & \zeta_1, & 1 \\ \xi_2, & \eta_2, & \zeta_2, & 1 \\ \xi_3, & \eta_3, & \zeta_3, & 1 \\ \xi_4, & \eta_4, & \zeta_4, & 1 \end{vmatrix},$$

x_r, y_r, z_r representing the orthogonal coordinates of the point r in one tetrahedron, and ξ_r, η_r, ζ_r the same for the point r in the other.

By the first theorem their product may be represented (striking off the last column only from each matrix) by the matrix

$$\begin{vmatrix} \Sigma x_1\xi_1, & \Sigma x_1\xi_2, & \Sigma x_1\xi_3, & \Sigma x_1\xi_4, & 1 \\ \Sigma x_2\xi_1, & \Sigma x_2\xi_2, & \Sigma x_2\xi_3, & \Sigma x_2\xi_4, & 1 \\ \Sigma x_3\xi_1, & \Sigma x_3\xi_2, & \Sigma x_3\xi_3, & \Sigma x_3\xi_4, & 1 \\ \Sigma x_4\xi_1, & \Sigma x_4\xi_2, & \Sigma x_4\xi_3, & \Sigma x_4\xi_4, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

where, in general, any such term as $\Sigma x_r \xi_s$ represents

$$x_r \xi_s + y_r \eta_s + z_r \zeta_s.$$

Again, by virtue of the second theorem, adding

$$-\tfrac{1}{2}\Sigma x_1^2, \quad -\tfrac{1}{2}\Sigma x_2^2, \quad -\tfrac{1}{2}\Sigma x_3^2, \quad -\tfrac{1}{2}\Sigma x_4^2$$

to the respective lines, and

$$-\tfrac{1}{2}\Sigma \xi_1^2, \quad -\tfrac{1}{2}\Sigma \xi_2^2, \quad -\tfrac{1}{2}\Sigma \xi_3^2, \quad -\tfrac{1}{2}\Sigma \xi_4^2$$

to the respective columns, the above matrix becomes (after a change of signs not affecting the result) the $-\tfrac{1}{8}$th of

$$\begin{vmatrix} \Sigma(x_1-\xi_1)^2, & \Sigma(x_1-\xi_2)^2, & \Sigma(x_1-\xi_3)^2, & \Sigma(x_1-\xi_4)^2, & 1 \\ \Sigma(x_2-\xi_1)^2, & \Sigma(x_2-\xi_2)^2, & \Sigma(x_2-\xi_3)^2, & \Sigma(x_2-\xi_4)^2, & 1 \\ \Sigma(x_3-\xi_1)^2, & \Sigma(x_3-\xi_2)^2, & \Sigma(x_3-\xi_3)^2, & \Sigma(x_3-\xi_4)^2, & 1 \\ \Sigma(x_4-\xi_1)^2, & \Sigma(x_4-\xi_2)^2, & \Sigma(x_4-\xi_3)^2, & \Sigma(x_4-\xi_4)^2, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

or calling the angular points of the one tetrahedron a, b, c, d, and of the other p, q, r, s, 8×36, that is 288 times, their product is representable by $-1 \times$ the determinant

$$\begin{vmatrix} (ap)^2, & (aq)^2, & (ar)^2, & (as)^2, & 1 \\ (bp)^2, & (bq)^2, & (br)^2, & (bs)^2, & 1 \\ (cp)^2, & (cq)^2, & (cr)^2, & (cs)^2, & 1 \\ (dp)^2, & (dq)^2, & (dr)^2, & (ds)^2, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

and of course if p, q, r, s coincide respectively with a, b, c, d, 576 times the square of the tetrahedron *abcd* will be represented under Mr Cayley's form,

$$\begin{vmatrix} 0, & (ab)^2, & (ac)^2, & (ad)^2, & 1 \\ (ba)^2, & 0, & (bc)^2, & (bd)^2, & 1 \\ (ca)^2, & (cb)^2, & 0, & (cd)^2, & 1 \\ (da)^2, & (db)^2, & (dc)^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} *,$$

four out of the sixteen distances vanishing, and the remaining twelve reducing to six pairs of equal distances. The demonstration of Staudt's

* The corresponding quantity to the above determinant for the case of the triangle (hereafter given) is identical with the Norm to the sum of the sides. I have succeeded in finding the Factor (of ten dimensions in respect of the edges), which, multiplied by the above Determinant itself, expresses the Norm to the sum of the Faces, that is, the superficial area of the Tetrahedron.

theorem for triangles is obtained in precisely the same way by throwing the product of the two determinants

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \xi_1, & \eta_1, & 1 \\ \xi_2, & \eta_2, & 1 \\ \xi_3, & \eta_3, & 1 \end{vmatrix}$$

under the form of $-\frac{1}{4}$th of

$$\begin{vmatrix} \Sigma(x_1-\xi_1)^2, & \Sigma(x_1-\xi_2)^2, & \Sigma(x_1-\xi_3)^2, & 1 \\ \Sigma(x_2-\xi_1)^2, & \Sigma(x_2-\xi_2)^2, & \Sigma(x_2-\xi_3)^2, & 1 \\ \Sigma(x_3-\xi_1)^2, & \Sigma(x_3-\xi_2)^2, & \Sigma(x_3-\xi_3)^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}.$$

When the two triangles coincide, calling their angular points a, b, c the above written determinant becomes

$$\begin{vmatrix} 0, & (ab)^2, & (ac)^2, & 1 \\ (ba)^2, & 0, & (bc)^2, & 1 \\ (ca)^2, & (cb)^2, & 0, & 1 \\ 1, & 1, & 1, & \end{vmatrix},$$

or

$$(ab)^4+(ac)^4+(bc)^4-2(ab)^2.(ac)^2-2(ab)^2.(bc)^2-2(ac)^2.(bc)^2,$$

the negative of which is the well-known form expressing the square of four times the area of the triangle abc.

There is another and more general theorem of Staudt for two triangles not in the same plane, which may be obtained with equal facility. In fact, if we start from the determinant

$$\begin{vmatrix} (a\alpha)^2, & (a\beta)^2, & (a\gamma)^2, & 1 \\ (b\alpha)^2, & (b\beta)^2, & (b\gamma)^2, & 1 \\ (c\alpha)^2, & (c\beta)^2, & (c\gamma)^2, & 1 \\ 1, & 1, & 1, & \end{vmatrix},$$

and add to each column respectively the last column multiplied by $e\xi_1^2$, $e\xi_2^2$, $e\xi_3^2$ respectively, we arrive at the form

$$\begin{vmatrix} (a\alpha)^2+e\xi_1^2, & (a\beta)^2+e\xi_2^2, & (a\gamma)^2+e\xi_3^2, & 1 \\ (b\alpha)^2+e\xi_1^2, & (b\beta)^2+e\xi_2^2, & (b\gamma)^2+e\xi_3^2, & 1 \\ (c\alpha)^2+e\xi_1^2, & (c\beta)^2+e\xi_2^2, & (c\gamma)^2+e\xi_3^2, & 1 \\ 1, & 1, & 1, & \end{vmatrix},$$

and considering $\xi_1, \eta_1;\ \xi_2, \eta_2;\ \xi_3, \eta_3$ as the coordinates of α, β, γ, the projections upon the plane of *abc* of a triangle *ABC*, whose plane intersects the former plane in the axis of y, and makes with that plane an angle whose tangent is e, it is easily seen that this determinant is term for term identical with the determinant

$$\begin{vmatrix} (aA)^2, & (aB)^2, & (aC)^2, & 1 \\ (bA)^2, & (bB)^2, & (bC)^2, & 1 \\ (cA)^2, & (cB)^2, & (cC)^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix},$$

which therefore expresses -16 times the product of the triangles *abc* and $\alpha\beta\gamma$, that is $abc \times ABC \times$ cosine of the angle between the two. A similar method, if we ascend from sensible to rational geometry, may be given for expressing in terms of the distances the product of *any* two pyramids (in a hyperspace) by the cosine of the angle included between the two infinite spaces * in which they respectively lie. To pass from the cases which have been considered of two triangles to two polygons, or of two tetrahedrons to two polyhedrons, generally presents no difficulty; and for Professor Staudt's method of doing so, which is simple and ingenious, and does not admit of material improvement, the reader is referred to the memoir in Crelle's *Journal* or Terquem's *Annales* already adverted to. It is, however, to be remarked (and this does not appear to be sufficiently noticed in the memoirs referred to), that whilst the expression for the product of any two polygons in terms of the distances given by Staudt's theorem is unique, that for the product of two polyhedrons given by the same is not so, but will admit of as many varieties of representation as there are units in the product of the numbers respectively expressing the number of ways in which each polygonal face of each polyhedron admits of being mapped out into triangles. I cannot help conjecturing (and it is to be wished that Professor Staudt or some other geometrician would consider this point) that in every case there exists, linearly derivable from Staudt's optional formulæ (but not coincident with any one of them), some unique and best, because most symmetrical, formula for expressing the product of two polyhedrons in terms of the distances of the angular points of the one from those of the other. In conclusion I may observe, that there is a theorem for distances measured on a given straight line, which, although not mentioned by Staudt, belongs to precisely the same class as his theorems for areas in a plane and volumes in space; namely a theorem which expresses twice the rectangle of any two such distances under the form of an aggregate of four squares, two taken positively and two

* In rational or universal geometry, that which is commonly termed infinite space (as if it were something absolute and unique, and to which, by the conditions of our being, the representative power of the understanding is limited), is regarded as a single homaloid related to a plane, precisely in the same way as a plane is to a right line. Universal geometry brings home to the mind with an irresistible force of conviction the truth of the Kantian doctrine of locality.

negatively; that is to say, if A, B, C, D be any four points on a right line $2AB \times CD = AD^2 + BC^2 - AC^2 - BD^2$. I know not whether this theorem be new, but it is one which evidently must be of considerable utility to the practical geometer.

Note on the above.

The fundamental theorem in determinants, published by me in the *Philosophical Magazine* in the course of last year*, leads immediately to a class of theorems strongly resembling, and doubtless intimately connected with, those of Staudt.

Thus for triangles we have by this fundamental theorem

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ \eta_1, & \eta_2, & \eta_3 \\ 1, & 1, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1, & \xi_1, & \xi_2 \\ y_1, & \eta_1, & \eta_2 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_3, & x_2, & x_3 \\ \eta_3, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} + \begin{vmatrix} x_1, & \xi_2, & \xi_3 \\ y_1, & \eta_2, & \eta_3 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_1, & x_2, & x_3 \\ \eta_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} x_1, & \xi_3, & \xi_1 \\ y_1, & \eta_3, & \eta_1 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_2, & x_2, & x_3 \\ \eta_2, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}$$

and consequently, if ABC, DEF be any two triangles,

$$ABC \times DEF = ADE \times FBC + AEF \times DBC + AFD \times BCE.$$

This may be considered a theorem relating to two ternary systems of points in a plane. The analogous and similarly obtainable theorem for two binary systems of points in the same right line is

$$AB \times CD = AC \times DB - AD \times CB.$$

As in applying this last theorem to obtain correct numerical results we must give the same algebraical sign to any two lengths denoted by the two *arrangements* XY, ZT, according as the direction from X to Y is the same as that from Z to T, or contrary to it, so in the theorem for the products of triangles, the areas denoted by any two ternary arrangements XYZ, TUV must be taken with the like or the contrary sign, according as the direction of the rotation XYZ is consentient with or contrary to that of TUV; so that three of the six possible arrangements of XYZ may be used indifferently for one another, but the other three would imply a change of sign. If we

[* See pp. 249, 253 above.]

analyse what we mean by fixing the direction of the rotation of XYZ, and reduce this form of speech to its simplest terms, we easily see that it amounts to ascertaining on which side of B, C lies, that is whether to its right or left, to a spectator stationed at A on a given side of the plane ABC.

Let us now pass to the corresponding theorems for two tetrahedrons put respectively under the forms

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix}, \qquad \begin{vmatrix} \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ \zeta_1, & \zeta_2, & \zeta_3, & \zeta_4 \\ 1, & 1, & 1, & 1 \end{vmatrix}.$$

We may represent this product in either of two ways by the application of our fundamental theorem, namely as

$$\begin{vmatrix} x_1, & \xi_1, & \xi_2, & \xi_3 \\ y_1, & \eta_1, & \eta_2, & \eta_3 \\ z_1, & \zeta_1, & \zeta_2, & \zeta_3 \\ 1, & 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_4, & x_2, & x_3, & x_4 \\ \eta_4, & y_2, & y_3, & y_4 \\ \zeta_4, & z_2, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix} + \text{\&c.}$$

or as

$$\begin{vmatrix} x_1, & x_2, & \xi_1, & \xi_2 \\ y_1, & y_2, & \eta_1, & \eta_2 \\ z_1, & z_2, & \zeta_1, & \zeta_2 \\ 1, & 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_3, & \xi_4, & x_3, & x_4 \\ \eta_3, & \eta_4, & y_3, & y_4 \\ \zeta_3, & \zeta_4, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix} + \text{\&c.}$$

there being four products to be added together in the first expression and six in the latter; and the rule, if we wish that all the products may be additive, being that on removing the sign of multiplication the determinant to the square matrix formed by the Greek letters *in situ* shall always preserve the same sign. Hence we derive two geometrical formulæ concerning the products of polyhedrons, namely

$$\begin{aligned} (1)\quad ABCD \times EFGH &= ABCE \times FGHD - ABCF \times GHED \\ &\quad + ABCG \times HEFD - ABCH \times FGED. \\ (2)\quad ABCD \times EFGH &= ABEF \times GHCD + ABGH \times EFCD \\ &\quad + ABEG \times HFCD + ABHF \times EGCD \\ &\quad + ABEH \times FGCD + ABFG \times EHCD. \end{aligned}$$

These formulæ give rise to an exceedingly interesting observation. In order that they shall be numerically true, we must have a rule for fixing the sign to be given to the solid content represented by any reading off of the four points of a tetrahedron, that is we must have a rule for determining the sign of solid contents of figures situated anywhere in space analogous to that which, as applied to linear distances reckoned on a given right line, is the true foundation of the language of trigonometry, and the condition precedent for the possibility of any system of analytical geometry such as exists, and which, not altogether without surprise, I have observed in the pages of this *Magazine* one of the learned contributors has thought it necessary to vindicate the propriety of importing into his theory of quaternions.

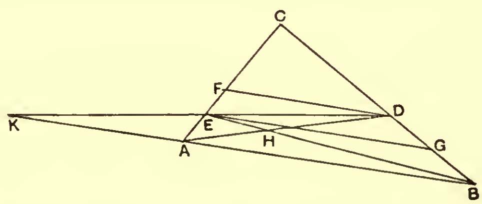
Various rules may be given for fixing the sign of a tetrahedron denoted by a given order of four letters. One is the following: the content of $ABCD$ is to be taken positive or negative, according as to a spectator at A the rotation of BCD is positive or negative. Another, again, is to consider AB and CD as representing, say two electrical currents, and to suppose a spectator so placed that the current AB shall pass through the longitudinal axis of his body from the head towards the feet, and looking towards the other current CD; the sign of the solid content of the tetrahedron (and, indeed, also the effect, in a general sense, of the action of the two currents upon one another) will depend upon the circumstance of this latter current appearing to flow from the right to the left, or contrariwise in respect of the spectator. Last and simplest mode of all, the sign of the solid content of $ABCD$ will depend upon the *nature* (in respect to its being a right-handed or left-handed-screw) of any regular screw-line (whether the common helix or one in which the increase or decrease of the inclination is always in the same direction) terminating at B and C, and so taken that BA shall be the direction of the tangent produced at B, and CD the direction of the tangent produced at C. Inasmuch as of the twenty-four permutations of a quaternary arrangement a defined twelve have one sign, and the other twelve the contrary sign, these various definitions of the direction, or, as it may be termed, polarity, of a tetrahedron corresponding to a given reading, whether as taken each in itself or compared one with another, give rise to, or rather imply a considerable number of interesting theorems included in our intuitions of space, and probably belonging to the, in my belief, inexhaustible class of primary and indemonstrable truths of the understanding.

49.

ON A SIMPLE GEOMETRICAL PROBLEM ILLUSTRATING A CONJECTURED PRINCIPLE IN THE THEORY OF GEOMETRICAL METHOD.

[*Philosophical Magazine,* IV. (1852), pp. 366—369.]

THE following theorem deserves attention as illustrating a principle of geometrical method which will be presently adverted to. It is curious, also, from the fact of its solution being by no means so obvious and self-evident as one would expect from the extreme simplicity of its enunciation. It appeared, and for the first time, it is believed, at the University of Cambridge about a twelvemonth back, where it excited considerable attention among some of the mathematicians of the place. The proposition, as originally presented, was merely to prove that if ABC be a triangle, and if AD and BE drawn bisecting the angles at A and B and meeting the opposite sides in D and E be equal, then the triangle must be isosceles. It is particularly



noticeable that all the geometrical demonstrations yet given of this theorem are indirect. Thus the first and simplest (communicated to me by a promising young geometrician, Mr B. L. Smith of Jesus College, Cambridge), was the following:—Assume one of the angles at DAB to be greater than the corresponding angle EBA; it can easily be shown that, upon this supposition, D will be higher up from AB than E; so that if DF and EG be drawn parallel to AB, DF will be above EG; it is then easily shown that $DF = AF$, $EG = BG$, and consequently DF and AF are each respectively less than EG

and BG; and also DFA, which is the supplement of twice DAB, will be less than EGB, which is the supplement of twice FBA; from which it is readily inferred, by an easy corollary to a proposition of Euclid, that DA will be less than FB, whereas it should be equal to it; so that neither of the half angles at the base can be greater than the other, and the triangle is proved to be isosceles. Another and independent demonstration by the writer of this article is less simple, but has the advantage of lending itself at once to a considerable generalization of the theorem as proposed. Assuming, as above, that DAB is greater than EBA, it is easily seen that DE produced will cut BA at K on the side of it: also if AD and BE intersect in H, it is readily demonstrable, by a suitably constructed apparatus of similar triangles, that

$$AH : BH :: CE : CD.$$

But as HBA is less than HAB, AH is less than BH, and therefore CE is less than CD, and therefore CED is greater than CDE; that is to say, CAB less K is greater than CBA plus K, and therefore DAB less K is greater than EBA, that is ADE is greater than ABE, and therefore the perpendicular from A upon DE is greater than that from E on AB, which is easily proved to be absurd. Hence, as before, the triangle is proved to be isosceles. This proof, it is obvious, remains good for *all cases* in which EB and DA, drawn on either side of the base, divide the angles at the base proportionally, provided that these lines remain equal, and make positive or negative angles with the base not *less* than one-half of the respective corresponding angles which the sides of the triangle are supposed to make with it. The analytical solution of the question, as might be expected, extends the result still further. To obtain this, let

$$BAC = n\,.\,BAD, \qquad ABC = n\,.\,ABE,$$

n for the present being any numerical quantity, positive or negative; calling $BAC = 2n\alpha$, $ABC = 2n\beta$, we readily obtain, by comparison of the equal dividing lines with the base of the triangle,

$$\frac{\sin(2n\alpha + 2\beta)}{\sin 2n\alpha} = \frac{\sin(2n\beta + 2\alpha)}{\sin 2n\beta},$$

or

$$\frac{\sin(2n\alpha + 2\beta)}{\sin(2n\beta + 2\alpha)} = \frac{\sin 2n\alpha}{\sin 2n\beta};$$

and by an obvious reduction,

$$\frac{\tan(n-1)(\alpha-\beta)}{\tan n(\alpha-\beta)} = \frac{\tan(n+1)(\alpha+\beta)}{\tan n(\alpha+\beta)}.$$

When this equation is put under an integer form, it is of course satisfied by making $\alpha = \beta$; on any other supposition than $\alpha = \beta$ it evidently cannot be satisfied by admissible values of the angles for any value of n between

$+1$ and $+\infty$; for on that supposition, since $(\alpha-\beta)$ and $(\alpha+\beta)$ are each less than $\frac{180}{2n}$, the first side of the equation will be necessarily a proper fraction and positive; but the second side, either a positive improper fraction if $(n+1)(\alpha+\beta)$ be less, and a negative proper or a negative improper fraction if $(n+1)(\alpha+\beta)$ be greater than a right angle.

If n be negative, let it equal $-\nu$, then

$$\frac{\tan(\nu+1)(\alpha-\beta)}{\tan\nu(\alpha-\beta)}=\frac{\tan(\nu-1)(\alpha+\beta)}{\tan\nu(\alpha+\beta)};$$

and for the same reason as before, if ν lies between ∞ and 1, this equation cannot be satisfied. Hence the theorem is proved to be true for all values of n, except between $+1$ and -1. For these values it ceases to be true; in fact, for such values for any given values of $(\alpha-\beta)$ there will be always, as it may be easily proved, one or more values of $(\alpha+\beta)$; thus if $n=\frac{1}{2}$, the equation becomes

$$\frac{\tan 3\left(\frac{\alpha+\beta}{2}\right)}{\tan\frac{\alpha+\beta}{2}}=-1;$$

and if $n=-\frac{1}{2}$,

$$\frac{\tan 3\left(\frac{\alpha-\beta}{2}\right)}{\tan\frac{\alpha-\beta}{2}}=-1,$$

showing that $\alpha+\beta=90$ and $\alpha-\beta=\pm 90$ in these respective cases will afford a solution over and above the solution $\alpha=\beta$, which is easily verified geometrically*. It would be an interesting inquiry (for those who have leisure for such investigations) to determine for any given value of n between $+1$ and -1 the superior and inferior limits to the number of admissible values of $\alpha+\beta$ corresponding to any given value of $\alpha-\beta$†.

My reader will now be prepared to see why it is that all the geometrical demonstrations given of this theorem, even in the simplest case of all, namely when $n=2$, are indirect, I believe I may venture to say *necessarily* indirect. It is because the truth of the theorem depends on the necessary non-existence of real roots (between prescribed limits) of the analytical equation expressing the conditions of the question; and I believe that it may be safely taken as an axiom in geometrical method, that whenever this is the case no other

* In the first of these cases, if the base of the triangle is supposed given, the locus of the vertex is a right line and a circle; in the second case, a right line and an equilateral hyperbola.

† When $\pm n$ lies between $\frac{1}{2\iota-1}$ and $\frac{1}{2\iota+1}$ (ι being any positive integer), it is easily seen that the superior limit must be at least as great as ι.

form of proof than that of the *reductio ad absurdum* is possible in the nature of things. If this principle is erroneous, it must admit of an easy refutation in particular instances.

As an example, I throw out (not a challenge, but) an invitation to discover a direct proof, if such exist, of the following geometrical theorem, as simple a one as it is perhaps possible to imagine :—"To prove that if from the middle of a circular arc two chords be drawn, and the remoter segments of these chords cut off by the line joining the end of the arc be equal, the nearer segments will also be equal." The analytical proof depends upon the fact of the equation $x^2 + ax = b^2$ (where a is the given length of each segment, and b the length of the chord of half the given arc) having only one admissible root; and if the principle assumed or presumed to be true be valid, no other form of pure geometrical demonstration than the *reductio ad absurdum* should be applicable in this case. For the converse case, where the nearer segments are given equal, the reducing equation is $a(a+x) = b^2$, indicating nothing to the contrary of the possibility of there being a direct solution, which accordingly is easily shown to exist. The indirect form of demonstration, it may be mentioned, is sometimes liable to be introduced in a manner to escape notice. As, for instance, if it should be taken for granted in the course of an argument, that one triangle upon the same base and the same side of it as another triangle, and having the same vertical angle, must have its vertex lying on the same arc; this would seem to be *immediately* true by virtue of the well-known theorem, that angles in the same circular segment are equal, but in reality can only be *inferred* from it indirectly by showing the impossibility of its lying outside or inside the arc in question. To go one step further, I believe it to be the case, that granted to be true all those fundamental propositions in geometry which are presupposed in the principles upon which the language of analytical geometry is constructed, then that the *reductio ad absurdum* not only is of necessity to be employed, but moreover in propositions of an affirmative character never need be employed, except when as above explained the analytical demonstration is founded on the impossibility or inadmissibility of certain roots due to the degree of the equation implied in the conditions of the question. If this surmise turn out to be correct, we are furnished with *a universal criterion for determining when the use of the indirect method of geometrical proof should be considered valid and admissible and when not**.

* If report may be believed, intellects capable of extending the bounds of the planetary system and lighting up new regions of the universe with the torch of analysis, have been baffled by the difficulties of the elementary problem stated at the outset of this paper, in consequence, it is to be presumed, of seeking a form of geometrical demonstration of which the question from its nature does not admit. If this be so, no better evidence could be desired to evince the importance of such a criterion as that suggested in the text.

50.

ON THE EXPRESSIONS FOR THE QUOTIENTS WHICH APPEAR IN THE APPLICATION OF STURM'S METHOD TO THE DISCOVERY OF THE REAL ROOTS OF AN EQUATION.

[*Hull British Association Report* (1853), Part II., pp. 1—3.]

Many years ago I published expressions for the residues which appear in the application of the process of common measure to fx and $f'x$, and which constitute Sturm's auxiliary functions. These expressions are complete functions of the factors of fx and of differences of the roots of fx, and are therefore in effect functions of the factors exclusively, since the difference between any two roots may be expressed as the difference between two corresponding factors. Having found that in the practical applications of Sturm's theorem the quotients may be employed with advantage to replace the use of the residues, I have been led to consider their constitution; and having succeeded in expressing these quotients (which are of course linear functions of x) under a similar form to that of the residues, that is, as complete functions of the factors and differences of the roots of fx, I have pleasure in submitting the result to the notice of the Mathematical Section of the British Association.

Let $h_1, h_2, h_3 \ldots h_n$ be the n roots of fx.

Let $\zeta(a, b, c \ldots l)$ in general denote the squared product of the differences of $a, b, c \ldots l$.

Let Z_i denote in general $\Sigma\zeta(h_{\theta_1} h_{\theta_2} \ldots h_{\theta_i})$, where $\theta_1, \theta_2 \ldots \theta_i$ indicate any combination of i out of the n quantities $a, b, c, \ldots l$, with the convention that $Z_0 = 1$, $Z_1 = n$; and let (i) denote $\frac{1}{2}\{1 + (-1)^i\}$, being zero when i is odd, and unity when i is even; then I find that the ith quotient Q_i may be written under the form

$$Q_i = {}_iP_1^2(x - h_1) + {}_iP_2^2(x - h_2) + \ldots + {}_iP_n^2(x - h_n),$$

where in general

$${}_iP_e = \frac{Z_{i-1}}{Z_i}\frac{Z^2{}_{i-3}}{Z^2{}_{i-2}}\frac{Z^2{}_{i-5}}{Z^2{}_{i-4}}\cdots\frac{Z^2{}_{(i)}}{Z^2{}_{(i)+1}}$$

$$\times\Sigma\{\zeta(h_{\theta_1}h_{\theta_2}\ldots h_{\theta_{i-1}}) \times (h_e - h_{\theta_1})(h_e - h_{\theta_n})\ldots(h_e - h_{\theta_{-1}})\}.$$

If we suppose $\frac{f'x}{fx}$, by means of the common measure process, to be expanded under the form of an improper continued fraction, the successive quotients will be the values of $Q_1, Q_2 \dots Q_n$ above found, that is

$$\frac{f'x}{fx} = \frac{1}{Q_1 -} \frac{1}{Q_2 -} \frac{1}{Q_3 -} \dots \frac{1}{Q_n};$$

the successive convergents of this fraction will be

$$\frac{1}{Q_1}, \frac{Q_2}{Q_1 Q_2 - 1}, \frac{Q_2 Q_3 - 1}{Q_1 Q_2 Q_3 - Q_1 - Q_3}, \dots, \frac{f'x}{fx}.$$

The numerators and denominators of these convergents will consequently also be functions of the factors exclusively. They are the quantities the sum of the products of which multiplied respectively by fx and $f'x$ produce (to constant factors *près*) the residues. The denominators are expressible very simply in terms of the factors and the differences of the roots; and their values under such forms were published by me about the same time as the values of the residues in the *Philosophical Magazine*; the expression for the numerators is much more complicated, but is given in my paper, "The Syzygetic Relations," &c., in the *Philosophical Transactions*. [p. 429 below.]

By comparing the expression for any quotient with the expressions for the two residues from which it may be derived, we obtain the following remarkable identity: $Z_{i-1} \times Z_i$, that is

$$\Sigma\zeta(h_1 h_2 \dots h_{i-1}) \times \Sigma\zeta(h_1 h_2 \dots h_i) = {}_iP_1^2 + {}_iP_2^2 + {}_iP_3^2 + \dots + {}_iP^2_n.$$

When the roots are all real, we have thus the product of one sum of squares by the product of another sum of squares (the number in each sum depending upon the arbitrary quantity i), brought under the form of a sum of a constant number n of squares, which in itself is an interesting theorem.

The expression above given for Q_i leads to a remarkable relation between the quotients and convergents to $\frac{f'x}{fx}$.

Let it be supposed, as before, that

$$\frac{f'x}{fx} = \frac{1}{Q_1 x -} \frac{1}{Q_2 x -} \frac{1}{Q_3 x -} \dots \frac{1}{Q_n x},$$

and let the successive convergents to this continued fraction be

$$\frac{N_1(x)}{D_1(x)}, \frac{N_2(x)}{D_2(x)}, \frac{N_3(x)}{D_3(x)}, \dots \frac{N_n(x)}{D_n(x)},$$

where the numerators and denominators are not supposed to undergo any reductions, but are retained in their crude forms as deduced from the law

$$N_i = Q_i N_{i-1} - N_{i-2},$$
$$D_i = Q_i D_{i-1} - D_{i-2}.$$

$N_1(x)$ being 1, and $D_1(x)$ being $Q_1(x)$; then it may be deduced from the published results above adverted to that

$$D_i(x) = \frac{Z^2_{i-1} Z^2_{i-3} \dots Z^2_{(i)}}{Z_i^2 Z^2_{i-2} \dots Z^2_{(i)+1}} \{\zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_i})(x - h_{\theta_1})(x - h_{\theta_2}) \dots (x - h_{\theta_i})\}.$$

Hence $$\Sigma\{\zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_{i-1}}) \times (h_e - h_{\theta_1})(h_e - h_{\theta_2}) \dots (h_e - h_{\theta_{i-1}})\}$$
$$= \frac{Z^2_{i-1} Z^2_{i-3} \dots Z^2_{(i-1)+1}}{Z^2_{i-2} Z^2_{i-4} \dots Z^2_{(i-1)}} D_{i-1}(h_e);$$

and we have therefore

$$_iP_e = \frac{Z^3_{i-1}}{Z_i} \frac{Z^4_{i-3}}{Z^4_{i-4}} \frac{Z^4_{i-5}}{Z^4_{i-6}} \dots \frac{Z^4_{(i)}}{Z^4_{(i)+1}} D_{i-1}(h_e),$$

and consequently

$$Q_i = \frac{Z^6_{i-1}}{Z_i^2} \frac{Z^8_{i-3}}{Z^8_{i-4}} \dots \frac{Z^8_{(i)}}{Z^8_{(i)+1}} \Sigma\{(D_{i-1}(h_e))^2 (x - h_e)\},$$

which is the general equation connecting the form of each quotient with that of the denominator to the immediately preceding unreduced convergent in the expansion of $\frac{f'x}{fx}$ under the form of an improper continued fraction.

If instead of the denominator of the unreduced convergents, the denominators of the convergents reduced to their simplest forms be employed, the powers of Z in the constant factor will undergo a diminution. The essential part of this theorem admits of being stated in general terms as follows:—

"If the quotient of an algebraical function of x by its first differential coefficient be expressed under the form of a continued fraction whose successive partial quotients are linear functions of x, any one of these quotients may be found (to a constant factor *près*) by taking the sum of the products formed by multiplying each factor $(x - h)$ of the given function by the square of what the denominator of the immediately antecedent convergent fraction becomes after substituting in it for x the root corresponding to such factor."

P.S. Since the above was read before the British Association, the theory has been extended by the author to comprise the general case of the expansion of any two algebraical functions under the form of a continued fraction, and has been incorporated into the paper in the *Philosophical Transactions* above referred to.

51.

ON A THEOREM CONCERNING THE COMBINATION OF DETERMINANTS.

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 60—62.]

Let ${}^1A$ represent the line of terms ${}^1a_1,\ {}^1a_2, \dots {}^1a_m$,

$$\begin{array}{llll} {}^1B & ,, & ,, & ,, \quad {}^1b_1,\ {}^1b_2, \dots {}^1b_m. \end{array}$$

Let ${}^1A \times {}^1B$ represent $\Sigma\,({}^1a_r \times {}^1b_r)$, where of course there are m terms within the symbol of summation.

Again, let ${}^2A$ represent the line ${}^2a_1,\ {}^2a_2, \dots {}^2a_m$,

$$\begin{array}{lll} {}^2B & ,, & ,, \quad {}^2b_1,\ {}^2b_2, \dots {}^2b_m, \end{array}$$

and let $\begin{vmatrix} {}^1A \\ {}^2A \end{vmatrix} \times \begin{vmatrix} {}^1B \\ {}^2B \end{vmatrix}$ represent $\Sigma \begin{vmatrix} {}^1a_r, & {}^1a_s \\ {}^2a_r, & {}^2a_s \end{vmatrix} \times \begin{vmatrix} {}^1b_r, & {}^1b_s \\ {}^2b_r, & {}^2b_s \end{vmatrix}$,

$\begin{vmatrix} {}^1a_r, & {}^1a_s \\ {}^2a_r, & {}^2a_s \end{vmatrix}$ denoting the determinant $({}^1a_r\,.\,{}^2a_s - {}^1a_s\,.\,{}^2a_r)$,

$\begin{vmatrix} {}^1b_r, & {}^1b_s \\ {}^2b_r, & {}^2b_s \end{vmatrix}$,, ,, ,, $({}^1b_r\,.\,{}^2b_s - {}^1b_s\,.\,{}^2b_r)$,

there being of course $\frac{1}{2}m(m-1)$ terms comprised within the sign of summation; and so, in general, let

$$\begin{vmatrix} {}^1A \\ {}^2A \\ {}^3A \\ \vdots \\ {}^nA \end{vmatrix} \times \begin{vmatrix} {}^1B \\ {}^2B \\ {}^3B \\ \vdots \\ {}^nB \end{vmatrix},\ n \text{ being less than } m,$$

$\left(\begin{array}{l}\text{and where in general } {}^rA \text{ denotes } {}^ra_1, {}^ra_2, \ldots {}^ra_m \\ \qquad\text{and } {}^rB \text{ denotes } {}^rb_1, {}^rb_2, \ldots {}^rb_m\end{array}\right)$ represent

$$\Sigma \begin{vmatrix} {}^1a_{h_1}, & {}^1a_{h_2}, & \ldots & {}^1a_{h_n} \\ {}^2a_{h_1}, & {}^2a_{h_2}, & \ldots & {}^2a_{h_n} \\ \cdots & \cdots & \cdots & \cdots \\ {}^na_{h_1}, & {}^na_{h_2}, & \ldots & {}^na_{h_n} \end{vmatrix} \times \begin{vmatrix} {}^1b_{h_1}, & {}^1b_{h_2}, & \ldots & {}^1b_{h_n} \\ {}^2b_{h_1}, & {}^2b_{h_2}, & \ldots & {}^2b_{h_n} \\ \cdots & \cdots & \cdots & \cdots \\ {}^nb_{h_1}, & {}^nb_{h_2}, & \ldots & {}^nb_{h_n} \end{vmatrix}.$$

Now let r be any integer less than m, and let

$$\mu = \frac{m(m-1)\ldots(m-r+1)}{1.2\ldots r},$$

and, supposing $\theta_1, \theta_2, \ldots \theta_r$ to be r numbers of the set $1, 2, \ldots m$, let $G_1, G_2, \ldots G_\mu$ denote the μ rectangular matrices of the forms

$$\begin{vmatrix} {}^{\theta_1}A \\ {}^{\theta_2}A \\ \cdots\cdots \\ {}^{\theta_r}A \end{vmatrix} \text{ respectively,}$$

and let $H_1, H_2, \ldots H_\mu$ denote the μ rectangular matrices of the forms

$$\begin{vmatrix} {}^{\theta_1}B \\ {}^{\theta_2}B \\ \cdots\cdots \\ {}^{\theta_r}B \end{vmatrix} \text{ respectively.}$$

Now form the determinant

$$\begin{vmatrix} G_1 \times H_1, & G_1 \times H_2, & \ldots & G_1 \times H_\mu \\ G_2 \times H_1, & G_2 \times H_2, & \ldots & G_2 \times H_\mu \\ \cdots & \cdots & \cdots & \cdots \\ G_\mu \times H_1, & G_\mu \times H_2, & \ldots & G_\mu \times H_\mu \end{vmatrix},$$

then, if we give r the successive values $1, 2, 3 \ldots m$ (in which last case the determinant in question reduces to a single term), the values of the determinant above written will be severally in the proportions of

$$K, \ K^m, \ K^{\frac{1}{2}m(m-1)}, \ \ldots \ K^m, \ K;$$

that is to say, the logarithms of these several determinants will be as the coefficients of the binomial expansion $(1+x)^m$.

When we make $r = m$, and equate the determinant corresponding to this value of r with that formed by making $r = 1$, the theorem becomes identical with a theorem previously given by M. Cauchy, for the Product of Rectangular Matrices.

It would be tedious to set forth the demonstration of the general theorem in detail. Suffice it here to say that it is a direct corollary from the formula marked (4) in my paper in the *Philosophical Magazine* for April 1851, entitled "On the Relations between the Minor Determinants of Linearly Equivalent Quadratic Functions*," when that formula is particularized by making

$$\begin{Bmatrix} a_{m+1}, & a_{m+2}, & \dots & a_{m+n} \\ b_{m+1}, & b_{m+2}, & \dots & b_{m+n} \end{Bmatrix}$$

represent a determinant all whose terms are zeros except those which lie in one of the diagonals, these latter being all units, which comes, in fact, to defining that

$$\begin{vmatrix} a_{m+e} \\ b_{m+e} \end{vmatrix} = 1, \text{ and } \begin{vmatrix} a_{m+e} \\ b_{m+e} \end{vmatrix} = 0.$$

The important theorem here referred to is made almost unintelligible by an unfortunate misprint of ${}^q\theta_m$, ${}^1\theta_m$, ${}^2\theta_m$, ${}^\mu\theta_m$, in place of ${}^q\theta_r$, ${}^1\theta_r$, ${}^2\theta_r$, ${}^\mu\theta_r$. I may here take notice of another and still more inexplicable blunder in the same paper, formula (3)†, in the latter part of the equation belonging to which

$$\begin{Bmatrix} a_{\theta_1}, & a_{\theta_2}, & \dots & a_{\theta_m}, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \dots & a_{\theta_{m+s}} \\ a_{\phi_1}, & a_{\phi_2}, & \dots & a_{\phi_m}, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \dots & a_{\phi_{m+s}} \end{Bmatrix}$$

is written in lieu of

$$\begin{Bmatrix} a_1, & a_2, & \dots & a_m, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \dots & a_{\theta_{m+s}} & a_{n+1} & a_{n+2} & \dots & a_{n+m} \\ a_1, & a_2, & \dots & a_m, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \dots & a_{\phi_{m+s}} & a_{n+1} & a_{n+2} & \dots & a_{n+m} \end{Bmatrix}.$$

[* p. 249 above.] [† See pp. 246, 251 above.]

52.

NOTE ON THE CALCULUS OF FORMS.

[See pp. 363 and 411.]

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 62—64.]

ACCIDENTAL causes have prevented me from composing the additional sections on the Calculus of Forms, which I had destined for the present Number of this *Journal*. In the meanwhile the subject has not remained stationary. Among the principal recent advances may be mentioned the following.

1. The discovery of Combinants; that is to say, of concomitants to systems of functions remaining invariable, not only when combinations of the variables are substituted for the variables, but also when combinations of the functions are substituted for the functions; and as a remarkable first-fruit of this new theory of double invariability, the representation of the Resultant of any three quadratic functions under the form of the square of a certain combinantive sextic invariant added to another combinant which is itself a biquadratic function of 10 cubic invariants. When the three quadratic functions are derived from the same cubic function, this expression merges in M. Aronhold's for the discriminant of the cubic. The theory of combinants naturally leads to the theory of invariability for non-linear substitutions, and I have already made a successful advance in this new direction.

2. The unexpected and surprising discovery of a quadratic covariant to any homogeneous function in x, y of the nth degree, containing $(n-1)$ variables cogredient with $x^{n-2}, x^{n-3}y \ldots y^{n-2}$ and possessing the property of indicating *the number of real and imaginary roots* in the given function. This covariant, on substituting for the $(n-1)$ variables the combinations of the powers of x, y with which they are cogredient, becomes the Hessian of the given function*.

* This covariant furnishes, if we please, functions symmetrical in respect to the two ends of an equation for determining the number of its real and imaginary roots. The ordinary Sturmian functions, it is well known, have not this symmetry. As another example of the successful application of the new methods to subjects which have been long before the mathematical world and supposed to be exhausted, I may notice that I obtain without an effort, by their aid, a much more simple, practical, and complete solution of the question of the simultaneous transformation of two quadratic functions, or the orthogonal transformation of one such function, than any previously given, even by the great masters Cauchy and Jacobi, who have treated this question.

3. The demonstration due to M. Hermite of a law of reciprocity connecting the degree or degrees of any function or system of functions with the order or orders of the invariants belonging to the system. The theorem itself was first propounded by me about a twelvemonth back, and communicated to Messrs Cayley, Polignac, and Hermite, as serving to connect together certain phenomena which had presented themselves to me in the theory: unfortunately it appeared to contradict another law too hastily assumed by myself and others as probably true, and I consequently laid aside the consideration of this great law of reciprocality. To M. Hermite, therefore, belongs the honour of reviving and establishing,—to myself whatever lower degree of credit may attach to suggesting and originating,—this theorem of numerical reciprocity, destined probably to become the corner-stone of the first part of our new calculus; that part, I mean, which relates to the generation and affinities of forms*.

4. I may notice that the Calculus of Forms may now with correctness be termed the Calculus of Invariants, by virtue of the important observation that every concomitant of a given form or system of forms may be regarded as an invariant of the given system and of an absolute form or system of absolute forms combined with the given form or system. As regards that particular branch of the theory of invariants which relates to resultants, or, in other words, to the doctrine of elimination, I may here state the theorem alluded to in a preceding Number of the *Journal*, to wit that if R be the resultant of a system of n homogeneous functions of n variables, written out in their complete and most general form (so that by definition $R=0$ is the condition that the equations got by making the n given functions zero, shall be simultaneously satisfiable by one system of ratios), then the condition that these equations may be satisfied by ι distinct systems of ratios between the n variables is $\delta^\iota R=0$, the variation δ being taken in respect to every constant entering into each of the n equations.

* This theorem of numerical reciprocity promises to play as great a part in the Theory of Forms as Legendre's celebrated theorem of reciprocity in that of Numbers. Another demonstration of it, which leaves nothing to be desired for beauty and simplicity, has been since discovered by Mr Cayley, which ultimately rests upon that simple law (essentially although not on the face of it a law of reciprocity) given by Euler, which affirms that the number of modes in which a number admits of being partitioned is the same whether the condition imposed upon the mode of partitionment be that no part shall exceed a given number, or that the number of parts constituting any one partition shall not exceed the same number.

53.

ON THE RELATION BETWEEN THE VOLUME OF A TETRAHEDRON AND THE PRODUCT OF THE *SIXTEEN* ALGEBRAICAL VALUES OF ITS SUPERFICIES.

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 171—178.]

THE area of a triangle is related (as is well known) in a very simple manner to the eight algebraical values of its perimeter: If we call the values of the squared sides of the triangle a, b, c, there will be nothing to distinguish the algebraical affections of sign of the simple lengths so as to entitle one to a preference over the other. The area of the triangle can only vanish by reason of the three vertices coming into a straight line; hence, according to the general doctrine of characteristics, we must have the Norm of $\sqrt{a}+\sqrt{b}+\sqrt{c}$, containing as a factor some root or power of the expressions for the area of the triangle. The Norm in question being representable as $-N^2$ where N is the Norm of $a^{\frac{1}{2}} \pm b^{\frac{1}{2}} \pm c^{\frac{1}{2}}$, which is of four dimensions in the elements a, b, c, and undecomposable into rational factors, we infer that to a numerical factor *près* the square of the area must be identical with the Norm N, and thus, by a logical *coup-de-main*, completely supersede all occasion for the ordinary geometrical demonstration given of this proposition, which in its turn, with certain superadded definitions, would admit of being adopted as the basis of an absolutely pure system of Analytical Trigonometry that should borrow nothing from the methods and results of sensuous or practical geometry. But into this speculation it is not my present purpose to enter: what I propose to do is to extend a similar mode of reasoning to space of three dimensions, and to point out a general theorem in determinants which is involved as a consequence in the generalization of the result of the inquiry when pushed forward into the regions of what may be termed Absolute or Universal Rational Space.

Let F, G, H, K be the four squared areas of the faces of a tetrahedron, and V the volume; then, since V only becomes zero in the case of the four vertices coming into the same plane, which is characterised by the equation

$$\sqrt{F}+\sqrt{G}+\sqrt{H}+\sqrt{K}=0$$

subsisting, we infer that N the Norm of

$$\sqrt{F} \pm \sqrt{G} \pm \sqrt{H} \pm \sqrt{K}$$

must contain a power of V as a rational factor. V^2 is rational and of three dimensions in the squared edges; the Norm above spoken of is of eight dimensions in the same. Consequently there is a rational factor, say Q, remaining, which is of five dimensions in the squared edges, and this factor I now proceed to determine, the other factor V^2 being, as is well known, a numerical product of the determinant

$$\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2, & 1 \\ ba^2, & 0, & bc^2, & bd^2, & 1 \\ ca^2, & cb^2, & 0, & cd^2, & 1 \\ da^2, & db^2, & dc^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}$$

a, b, c, d being the four angular points of the tetrahedron. See *London and Edinburgh Philosophical Magazine,* 1852. [p. 386 above.]

The quantity Q possesses an interest of a geometrical character; for if we call the radii of the eight spheres which can be inscribed in a tetrahedron $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8$, we evidently have $r_1r_2r_3r_4r_5r_6r_7r_8 \times N = (3V)^8$. Hence (R), the product of the eight radii in question, $= \dfrac{3^8V^8}{N} = \dfrac{3^8V^6}{Q}$.

Consequently Q is the quantity which characterises the fact of one or more of *the radii of the inscribed spheres becoming infinite.* For the triangle there exists no corresponding property; this we know *à priori,* and can explain also analytically from the fact that if we call P the product of the radii of the four inscribable circles, ν the Norm of the perimeter, and A the area, we have

$$P\nu = 2^4A^4,$$

and

$$\nu = \frac{2^4A^4}{P} = A^2,$$

which contains no denominator capable of becoming zero, so that as long as the sides remain finite the curvature of the inscribed circles is incapable of vanishing.

To determine N as a function of the edges, and then to discover by actual division the value of $\dfrac{N}{V^2}$, would be the direct but an excessively tedious and almost impracticably difficult process. I have ever felt a preference for the *à priori* method of discovering forms whose properties are known, and never yet have met with an instance where analysis has denied to gentle solicitation conclusions which she would be loth to grant to the application of force. The case before us offers no exception to the truth of this remark. Q is a function of five dimensions in terms of the squared edges: let us begin by finding the value of that part of Q in which at most a certain set of four of these edges make their appearance, and to find which consequently the other two edges may be supposed zero without affecting the result. We may make two distinct hypotheses concerning these two edges; we may suppose that they are opposite, that is non-intersecting edges, or that they are contiguous, that is intersecting edges.

To meet the first hypothesis suppose $ab = 0$, $ce = 0$.

For convenience sake, use F, G, H, K to denote 16 times the square of each area, instead of the simple square of the areas. Call

$$16\,(abc)^2 = K,\quad 16\,(abd)^2 = H,\quad 16\,(acd)^2 = G,\quad 16\,(bcd)^2 = F.$$

Then

$$\begin{aligned}-K &= (ab)^4 + (ac)^4 + (bc)^4 - 2\,(ab)^2(ac)^2 - 2\,(ab)^2(bc)^2 - 2\,(ac)^2(bc)^2\\ &= ac^4 + bc^4 - 2\,(ac)^2(bc)^2.\end{aligned}$$

Similarly,

$$\begin{aligned}-H &= ad^4 + bd^4 - 2\,(ad)^2(bd)^2,\\ -G &= ca^4 + da^4 - 2ca^2da^2,\\ -F &= cb^4 + db^4 - 2cb^2db^2.\end{aligned}$$

Hence one value of $\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K}$ will be

$$\sqrt{(-1)}\,\{(ac^2 - bc^2) + (bd^2 - ad^2) + (da^2 - ac^2) + (bc^2 - bd^2)\} = 0.$$

Hence, on this first supposition, the Norm vanishes. But V^2 does not vanish when $ab = 0$, $cd = 0$, for it becomes, *saving* a numerical factor,

$$\begin{vmatrix} 0, & 0, & ac^2, & ad^2, & 1\\ 0, & 0, & bc^2, & bd^2, & 1\\ ca^2, & cb^2, & 0, & 0, & 1\\ da^2, & db^2, & 0, & 0, & 1\\ 1, & 1, & 1, & 1, & \end{vmatrix},$$

that is

$$\begin{aligned}&(ac^2 . bd^2 - ad^2 . bc^2)(cb^2 + ad^2 - ca^2 - bd^2)\\ &+ (bc^2 - ac^2)(ca^2 . db^2 - cb^2 . da^2)\\ &+ (ad^2 - bd^2)(ca^2 . db^2 - cb^2 . da^2)\\ &= 2\,(ac^2 . bd^2 - ad^2 . bc^2)(ad^2 + bc^2 - ac^2 - bd^2);\end{aligned}$$

and consequently, since N vanishes but V^2 does not vanish, Q vanishes, showing that there is no term in Q but what contains one at least of any two opposite edges as a factor; or, in other words, there is no term in Q of which the product of the square of the product of all three sides of some one or other of the four faces does not form a constituent part.

Next, let us suppose $ab = 0$, $ac = 0$, then

$$\begin{aligned}
K^2 &= 16abc^2 = -bc^4,\\
H^2 &= 16abd^2 = -(ad^2 - bd^2)^2,\\
G^2 &= 16acd^2 = -(ad^2 - cd^2)^2,\\
F^2 &= 16bcd^2 = -bc^4 - bd^4 - cd^4 + 2bc^2 . bd^2 + 2bc^2 . cd^2 + 2bd^2 . cd^2.
\end{aligned}$$

Four of the factors of N will be therefore

$$\{\iota (bc^2 + cd^2 - bd^2) \pm F\}, \quad \{\iota (bc^2 - cd^2 + bd^2) \pm F\},$$

ι denoting $\sqrt{(-1)}$, and the product of these four factors will be

$$\{(bc^2 + cd^2 - bd^2)^2 + F^2\} \times \{(bc^2 - cd^2 + bd^2)^2 + F^2\},$$

which is equal to

$$16bc^4 . bd^2 . cd^2;$$

and similarly, the remaining part of the Norm will be

$$\{(2ad^2 - bd^2 - cd^2 + bc^2)^2 + F^2\} \times \{(2ad^2 - bd^2 - cd^2 - bc^2)^2 + F^2\},$$

that is

$$\begin{aligned}
&\{4ad^4 - 4ad^2 (bd^2 + cd^2 + bc^2) + 4bc^2 . bd^2 + 4bd^2 . cd^2 + 4cd^2 . bc^2\}\\
&\times \{4ad^4 - 4ad^2 (bd^2 + cd^2 - bc^2) + 4bd^2 . cd^2\}.
\end{aligned}$$

Again, since $ac^2 = 0$ and $bc^2 = 0$, V^2 becomes

$$\begin{vmatrix} 0, & 0, & 0, & ad^2, & 1 \\ 0, & 0, & bc^2, & bd^2, & 1 \\ 0, & cb^2, & 0, & cd^2, & 1 \\ da^2, & db^2, & dc^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

which is evidently equal to

$$2bc^2 \begin{vmatrix} 0, & 0, & ad^2, & 1 \\ 0, & cb^2, & cd^2, & 1 \\ da^2, & db^2, & 0, & 1 \\ 1, & 1, & 1, & \end{vmatrix} - bc^4 \begin{vmatrix} 0, & ad^2, & 1 \\ da^2, & 0, & 1 \\ 1, & 0, & \end{vmatrix},$$

$$\begin{aligned}
&= 2bc^2 \{2bc^2 ad^2 + ad^4 - ad^2 bd^2 - cd^2 ad^2 + bd^2 cd^2\} - 2bc^4 ad^2\\
&= 2bc^2 \{ad^4 - ad^2 (bd^2 + cd^2 - bc^2) + bd^2 . cd^2\}.
\end{aligned}$$

Hence, paying no attention to any mere numerical factor, we have found that when $ac=0$ and $bc=0$, Q or $\frac{N}{V^2}$ becomes

$$bc^2 . bd^2 . cd^2 \{ad^4 - ad^2(bd^2 + cd^2 + bc^2) + bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\}.$$

Hence, with the exception of the terms in which five out of the six edges enter, the complete value of Q will be

$$\Sigma(bc^2 . bd^2 . cd^2)\{ad^4 - ad^2(bd^2 + cd^2 + bc^2) + bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\},$$

or more fully expressed, and still abstracting from terms containing five edges,

$$= \Sigma bc^2 . bd^2 . cd^2 \{(ab^4 + ac^4 + ad^4) - (ab^2 + ac^2 + bc^2)(bd^2 + bc^2 + cd^2)$$
$$+ bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\}.$$

It remains only to determine the value of the numerical coefficient affecting each of the six terms of the form

$$ab^2 . ac^2 . ad^2 . bc^2 . bd^2.$$

To find this, let

$$ab^2 = ac^2 = ad^2 = bc^2 = bd^2 = cd^2 = 1;$$

then evidently, since all the squared areas are equal, several of the factors of N will become zero, but V^2 evidently does not become zero for a regular tetrahedron; hence Q becomes zero: and if we call the numerical factor sought for λ, we must have (observing that the Σ includes four parts corresponding to each of the four faces)

$$4\{3 - 9 + 3\} + 6\lambda = 0,$$

therefore $$-12 + 6\lambda = 0, \text{ or } \lambda = 2.$$

Hence the complete value of Q is

$$\Sigma ab^2 . bc^2 . ca^2 \{(da^4 + db^4 + dc^4) - (da^2 + db^2 + dc^2)(ab^2 + bc^2 + ca^2)$$
$$+ ab^2 . bc^2 + bc^2 . ca^2 + ca^2 . ab^2\}$$
$$+ 2\Sigma(ab^2 . bc^2 . cd^2 . da^2 . ac^2);$$

or, which is the same quantity somewhat differently and more simply arranged,

$$Q = \Sigma(ab^2 . bc^2 . ca^2)\{(da^4 + db^4 + dc^4 + da^2 . db^2 + db^2 . dc^2 + dc^2 . da^2)$$
$$+ (ab^2 . bc^2 + bc^2 . ca^2 + ca^2 . ab^2) - (da^2 + db^2 + dc^2)(ab^2 + bc^2 + ca^2)\},$$

and this quantity equated to zero expresses the conditions of a radius of an inscribed sphere becoming infinite. The direct method would have involved, as the first step, the formation of the Norm of a numerator consisting of

$$\surd F \pm \surd G \pm \surd H \pm \surd K,$$

the value of which is

$$\Sigma F^4 - 4\Sigma F^3G + 6\Sigma F^2G^2 + 4\Sigma F^2GH - 40FGHK,$$

and contains $4+6+12$, that is 22 positive terms, and 12, that is 13 negative terms, together 35 terms, each of which might be an aggregate of 6^4 or 1296 quantities, and thus involve in all the consideration of 45360 separate parts, for each of the quantities F, G, H, K being a quadratic function of three of the squared edges, will contain six terms. It is not uninteresting to notice that in addition to the case already mentioned of two opposite edges being each zero, as $ab=0$, $cd=0$, Q will also vanish for the case of $ab=cd$, $bc=ad$; that is for the case of two intersecting edges being each equal in length to the edges respectively opposite to them. This is evident from the fact that on the hypothesis supposed the face $acb=acd$ and the face $bdc=bda$; hence $N=0$, and therefore, V not vanishing, $\frac{N}{V^2}$, that is Q, will vanish.

We may moreover remark that since $ab=0$ and $cd=0$ does not make V vanish, the perpendicular distance of ab from cd, which, multiplied by $ab \times cd$, gives six times the volumes, must on this supposition become infinite. When three edges lying in the same plane all vanish simultaneously, Q vanishes, since one edge at least in every face of the pyramid vanishes, and V also vanishes, as is evident from the expression for V^2, when $ab=0$, $ac=0$, $bc=0$, becoming a multiple of

$$\begin{vmatrix} 0, & 0, & 0, & ad^2, & 1 \\ 0, & 0, & 0, & bd^2, & 1 \\ 0, & 0, & 0, & cd^2, & 1 \\ ad^2, & bd^2, & cd^2, & 0, & 0 \\ 1, & 1, & 1, & 0, & 0 \end{vmatrix},$$

which is evidently zero.

It appeared to me not unlikely, from the situation and look of Q (the characteristic of one of the inscribed spheres becoming infinite), that it might admit of being represented as a determinant, but I have not succeeded in throwing it under that form. I have a strong suspicion that if we take Q' a function corresponding to a tetrahedron $a'b'c'd'$, in the same way as Q corresponds to $abcd$, QQ', and not improbably $\surd(QQ')$, will be found to be (as we know from Staudt's Theorem of $\sqrt{(V^2 . V'^2)}$) a rational integral function of the squares of the distances of the points a, b, c, d from the points a', b', c', d'.

That N should divide out by V^2 is in itself an analytical theorem relating to 6 arbitrary quantities $ab^2, ac^2, ad^2, bc^2, bd^2, cd^2$, which evidently admits of extension to any triangular number 10, 15, &c. of arbitrary quantities. Thus we may affirm, *à priori*, that the norm of

$$\sqrt{L} \pm \sqrt{M} \pm \sqrt{N} \pm \sqrt{P} \pm \sqrt{Q},$$

where (for the sake of symmetry, retaining double letters, as AB, AC, &c., to denote *simple* quantities)

$$Q = \begin{vmatrix} 0, & AB, & AC, & AD, & 1 \\ AB, & 0, & BC, & BD, & 1 \\ AC, & BC, & 0, & CD, & 1 \\ AD, & BD, & CD, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}, \quad P = \begin{vmatrix} 0, & AB, & AC, & AE, & 1 \\ AB, & 0, & BC, & BE, & 1 \\ AC, & BC, & 0, & CE, & 1 \\ AE, & BE, & CE, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

$$N = \&c., \quad M = \&c., \quad L = \&c.,$$

will contain as a factor the determinant

$$\begin{vmatrix} 0, & AB, & AC, & AD, & AE, & 1 \\ AB, & 0, & BC, & BD, & BE, & 1 \\ AC, & BC, & 0, & CD, & CE, & 1 \\ AD, & BD, & CD, & 0, & DE, & 1 \\ AE, & BE, & CE, & DE, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

and a similar theorem may evidently be extended to the case of any $\frac{n(n+1)}{2}$ arbitrary quantities whatever.

54.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

[*Continued from p.* 363 *above.*]

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 256—269.]

SECTION VII. *On Combinants.*

REASONS of convenience have induced me to depart from the plan to which I originally intended to adhere in the development of this theory, and I shall hereafter, from time to time, continue to add sections on such parts of the subject as may chance to be most present to my mind or most urgent upon my attention, without waiting for the exact place which they ought to occupy in a more formal treatise, and without having regard to the separation of the subject into the two several divisions stated at the outset of the first section. The present section will be devoted to a brief and partial exposition of the theory of Combinants*, with a view to the application of this theory to the solution of the problem of throwing the resultant of three general homogeneous quadratic functions under its most simple form, being analogous to that given by Aronhold in the particular case where the three functions are derived from the same cubic, and becoming identical therewith when the coefficients are accommodated to this particular supposition†. I shall confine myself for the present to combinants relating to systems of functions, all of the same degree.

If $\phi_1, \phi_2, \ldots \phi_r$, be homogeneous functions of any number of variables, any invariant or other concomitant of the system which remains unchanged, not only for linear substitutions impressed upon the variables contained within the functions, but also for linear combinations impressed upon the functions themselves, is what I term a Combinant. A Combinant is thus an invariant or other concomitant of a system in its corporate capacity (quâ *system*), being in fact

* Discovered by the Author of this paper in the winter of 1852.

† A similar method will subsequently be applied to the representation of the resultant of two cubic equations as a function of Combinants bearing relations to the quadratic and cubic invariants of a quartic function of x and y, precisely analogous to those which the Combinants that enter into the solution above alluded to bear to the Aronholdian invariants of a cubic function.

common to the whole family of forms designated by $\lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_r\phi_r$, where $\lambda_1, \lambda_2, \ldots \lambda_r$, are arbitrary constants. If the coefficients of $\phi_1, \phi_2, \ldots \phi_r$, be supposed to be written out in r lines (the coefficients of corresponding terms occupying the same place in each line), so as to form a rectangular matrix, any combinantive invariant will be a function of the determinants corresponding to the several squares of r^2 terms each that can be formed out of such matrix, or, as they may be termed, the *full* determinants belonging to such rectangular matrix. If we call any such combinant K, then, over and above the ordinary partial differential equations which belong to it in its character of an invariant, it will be necessary and sufficient, in order to establish its combinantive character, that K shall be subject to satisfy $(r-1)$ pairs of equations of the form

$$\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc}\ldots\right)K = 0,$$

$$\left(a\frac{d}{da'} + b\frac{d}{db'} + c\frac{d}{dc'}\ldots\right)K = 0,$$

where $a, b, c \ldots$; $a', b', c' \ldots$, are respectively lines in the matrix above referred to.

So any combinantive concomitant will be a function of the full determinants of the matrix formed by the coefficients of the given system of forms and of the variables, and will be subject to satisfy the additional differential equations just above written.

It will readily be understood furthermore, that an invariant or other concomitant may be combinantive in respect to a certain number of forms of a system, and not in respect of other forms therein; or more generally, may be combinantive in respect of each, separately considered, of a series of groups into which a given system may be considered to be subdivided, without being so in respect of the several groups taken collectively.

In the fourth section of my memoir [p. 429 below] on a "Theory of the Conjugate Properties of two rational integral Algebraical Functions," recently presented to the Royal Society of London, the case actually arises of an invariant of a system of three functions, which is combinantive in respect only to two of them.

For greater simplicity, let the attention for the present be kept fixed upon combinants which are such in respect of a single group of functions, all of the same degree in the variables. (It will of course have been perceived that when the system is made up of several groups, there would be nothing gained by limiting the groups to be all of the same degree *inter se*; it is sufficient that all of the same group be of the same degree *per se*.)

All such combinants will admit of an obvious and immediate classification. Let us suppose that a combinant is proposed which is in its lowest terms, that is to say, incapable of being expressed as a rational integral algebraical function of combinants of an inferior order. Such a combinant may, notwithstanding this, admit of being decomposed into non-combinantive invariants of inferior dimensions to its own, and in such event will be termed a *complex* combinant; or it may be indecomposable after this method, in which event it will be termed a *simple* combinant. It will presently be shown, that the resultant of a system of three quadratic functions is made up of a complex combinant of twelve dimensions, and of the square of a simple combinant of six dimensions, expressible as a biquadratic function of ten non-combinantive invariants, each of three dimensions in the coefficients. There is an obvious mode of generating complex combinants; according to which they admit of being viewed as invariants of invariants. Supposing $\phi_1, \phi_2, \ldots \phi_r$, to be the functions of the given system, $\lambda_1\phi_1+\lambda_2\phi_2+\ldots+\lambda_r\phi_r$ may conveniently be termed the conjunctive of the system: if now one or more invariants or other concomitants be taken of this conjunctive, there results a derivative function or system of functions of the quantities $\lambda_1, \lambda_2, \ldots \lambda_r$, in which every term affecting any power or combination of powers of the λ series is necessarily an invariant or concomitant of the given system. If now an invariant or other concomitant be taken of the new system in respect to $\lambda_1, \lambda_2, \ldots \lambda_r$, (the original variables (supposing them to enter) being treated as constants), this secondarily derived invariant will be itself an Invariant, or at all events a Concomitant in respect of the original system, and being unaffected by linear substitutions impressed upon the λ system, is by definition a combinant of such system. A similar method will obviously apply if the original system be made up of various groups; each group will give rise to a conjunctive, and one or more concomitants being taken of this system of conjunctives and treated as in the case first supposed, (the only difference being, that there will on the present supposition be several *unrelated* systems instead of a single system of new variables, that is, several λ systems instead of one only) the result, when all the λ systems have been *invariantized out* (that is, made to disappear by any process for forming invariants), will be a combinant in respect to each of the groups, severally considered, of the given system of functions.

Here let it be permitted to me to make a momentary digression, in order to be enabled to avoid for the future the inconvenience of using the phrase "invariant or other concomitant," and so to be enabled at one and the same time to simplify the language and to give a more complete unity to the matter of the theory, by showing how every concomitant may in fact be viewed as a simple invariant, so that the calculus of forms may hereafter admit of being cited, as I propose to cite it, under the name of the Theory of Invariants.

Thus, to begin with the case of *simple* contragredience and cogredience, if $\xi, \eta, \zeta \ldots$ are contragredient to $x, y, z \ldots$, any form containing $\xi, \eta, \zeta \ldots$, which is concomitantive to a given form or system of forms S, which contains $x, y, z \ldots$, may be regarded as concomitantive to the system S', made up of S and the superadded *absolute* form $\xi x + \eta y + \zeta z + \ldots$, say ϑ; where $\xi, \eta, \zeta \ldots$ are treated no longer as variables, but as *constants.* In like manner every system of variables contragredient to $x, y, z \ldots$, or to any other system of variables in S, will give rise to a superadded form analogous to ϑ, the totality of which may be termed S_1; and thus the various systems $\xi, \eta, \zeta \ldots$ will no longer exist as variables in the derived form, but purely as constants. Again, if S contain any system of variables ϕ, ψ, ϑ, &c., contragredient to x, y, z, &c., the system of variables u, v, w, &c., cogredient with x, y, z, &c., may be considered as constants belonging to the superadded form $\phi u + \psi v + \vartheta w \ldots$; but if S do not contain any system contragredient to x, y, z, &c., then u, v, w, &c. may be treated as constants belonging to the superadded system of forms $xv - yu$, $yw - zv$, $zu - xw$, &c.; and so in general any concomitant containing any sets of variables in simple relation, whether of cogredience or contragredience, with any of the sets in the given system S, may in all cases be treated as an *invariant* of the system S', made up of S and a certain superadded system S_1, all the forms contained in which are absolute, by which I mean, that they contain no literal coefficient. The same conclusion may be extended to the case of concomitants containing sets of variables in *compound* relation with the sets in the given system of forms S. Thus, suppose $u_1, u_2, \ldots u_n$, to be in compound relation of cogredience with $x^{n-1}, x^{n-2}y, x^{n-3}y^2, \ldots y^{n-1}$; $u_1, u_2, \ldots u_n$, may be regarded as constants belonging to the superadded form

$$u_1 y^{n-1} - (n-1) u_2 y^{n-2} x + \tfrac{1}{2}(n-1)(n-2) u_3 y^{n-3} x^2 \mp \ldots \pm u_n x^{n-1},$$

say Ω. And thus universally we are enabled to affirm, that a concomitant of whatever nature to a given system of forms, may be reduced to the form of an invariant of a system made up of the given system and a certain other superadded system of absolute forms: without, therefore, abandoning the use of the terms concomitant, cogredience, contragredience, &c., which for many purposes are highly convenient and save much circumlocution, we may regard every concomitant as a disguised invariant, and under the name of the Theory of Invariants comprise the totality of the theory of Concomitance. I have already had occasion to make use of the superadded form Ω in discussing the theory of the Bezoutiant (a quadratic form concomitant to two functions of the same degree in x, y, which plays a most important part in the theory of the relations of their real roots), in the memoir for the Royal Society previously adverted to.

I now return to the question of applying the theory of combinants to the decomposition of the resultant of three general quadratic functions of

x, y, z. It will of course be apparent that every resultant of any system of n functions of the same degree of a single set of n variables is a combinantive invariant of the system. This is an immediate and simple corollary to the theorem given by me in this *Journal*, in May, 1851. Accordingly, in proceeding to analyse the composition of the resultant of three quadratic functions, I may, besides impressing linear combinations upon the variables, impress linear combinations upon the functions themselves, in any way most conducive to simplicity and facility of expression and calculation; and whatever relations shall be proved to exist between the resultant and other combinants for such specific representation, must be universal, and hold good for the functions in their most general form.

(1) The system, by means of linear substitutions impressed upon the variables which enter into the functions, may be made to assume the form

$$x^2 + y^2 + z^2,$$
$$ax^2 + by^2 + cz^2,$$
$$lx^2 + my^2 + nz^2 + 2pyz + 2qzx + 2rxy.$$

(2) By means of linear combinations of the functions themselves the system may evidently be made to take the form

$$(c-a)\,x^2 + (c-b)\,y^2,$$
$$(a-b)\,y^2 + (a-c)\,z^2,$$
$$ky^2 + 2pyz + 2qzx + 2rxy;$$

and finally, by taking suitable multipliers of x, y, z in lieu of x, y, z, it may be made to become

$$\rho\,(x^2 - y^2),$$
$$\sigma\,(y^2 - z^2),$$
$$y^2 + 2fyz + 2gzx + 2hxy.$$

We have thus reduced the number of constants in the system from eighteen to five; and as it will readily be seen that in any combinant of the system in its reduced form ρ and σ can only enter as factors of the simple quantity, $(\rho\sigma)^i$, for all purposes of comparison of the combinants of the system of like dimensions with one another, ρ and σ might admit of being treated as being each unity, and accordingly, practically speaking, we have only to deal with three in place of eighteen constants, a marvellous simplification, and which makes it obvious, *à priori*, or at least affords a presumption almost amounting to and capable of being reduced to certainty, that the number of fundamental combinants of the system, of which all the rest must be explicit rational functions, will be exactly four in number; which, for the canonical form hereinbefore written, on making ρ and σ each unity, will correspond to

$$1,\quad f^2+g^2+h^2,\quad f^2g^2+g^2h^2+h^2f^2,\quad fgh,$$

and will be of the 3rd, 6th, 12th, and 9th degrees respectively. The reason why the squares of f, g, h, instead of the simple terms f, g, h, appear in the 2nd and 3rd of these forms is, because, on changing x into $-x$, y into $-y$, or z into $-z$, two of the quantities f, g, h will change their sign, but the forms representing the invariants of even degrees ought to remain absolutely unaltered for such transformations. I shall in the course of the present section set forth the methods for obtaining these four combinants, which, although of the regularly ascending dimensions 3, 6, 9, 12, belong obviously to two different groups, the one of three dimensions forming a class in itself, and the natural order of the three others being that denoted by the sequence 6, 12, and 9, and not that which would be denoted by the sequence 6, 9, 12, the combinant of the ninth degree being properly to be regarded as in some sort an accidentally rational square root of a combinant of 18 dimensions.

Let now
$$\rho(x^2-y^2)=U,$$
$$\sigma(y^2-z^2)=W,$$
$$y^2+2fyz+2gzx+2hxy=V.$$

The resultant will be found by making
$$x=\pm y,$$
$$z=\pm y,$$

when
$$\left.\begin{matrix} x=+y \\ z=+y \end{matrix}\right\},\quad V=(1+2f+2g+2h)\,y^2,$$
$$\left.\begin{matrix} x=+y \\ z=-y \end{matrix}\right\},\quad V=(1-2f-2g+2h)\,y^2,$$
$$\left.\begin{matrix} x=-y \\ z=+y \end{matrix}\right\},\quad V=(1+2f-2g-2h)\,y^2,$$
$$\left.\begin{matrix} x=-y \\ z=-y \end{matrix}\right\},\quad V=(1-2f+2g-2h)\,y^2.$$

Hence the resultant R
$$=\rho^4\sigma^4(1+2f+2g+2h)(1-2f-2g+2h)(1+2f-2g-2h)(1-2f+2g-2h)$$
$$=(\rho\sigma)^4\{(1+2h)^2-4(f+g)^2\}\{(1-2h)^2-4(f-g)^2\}$$
$$=(\rho\sigma)^4\{(1+4h^2-4f^2-4g^2)^2-(4h-8fg)^2\}$$
$$=(\rho\sigma)^4[1-8(f^2+g^2+h^2)+16\{(f^4+g^4+h^4)-2(g^2h^2+h^2f^2+f^2g^2)\}+64fgh].$$

Let now
$$K=\lambda U+\mu V+\nu W,$$
K being what I term a linear conjunctive of U, V, W. The invariant of K, in respect to x, y, z, will be the determinant
$$\begin{vmatrix} \rho\lambda, & h\mu, & g\mu \\ h\mu, & \mu-\rho\lambda+\sigma\nu, & f\mu \\ g\mu, & f\mu, & -\sigma\nu \end{vmatrix},$$

that is

$$= (2fgh - g^2)\,\mu^3 + \sigma\,(h^2 - g^2)\,\mu^2\nu - \rho\,(f^2 - g^2)\,\mu^2\lambda - \rho\sigma\mu\lambda\nu + \rho^2\sigma\lambda^2\nu - \rho\sigma^2\lambda\nu^2;$$

or, multiplying by 6, we may write

$$I_{x,y,z}K = 6d\lambda\mu\nu + 3b_3\mu^2\nu + 3b_1\mu^2\lambda + 3a_3\lambda^2\nu + 3c_1\lambda\nu^2 + b_2\mu^3,$$

where

$$\begin{aligned} d &= -\rho\sigma, & b_2 &= 12fgh - 6g^2, \\ b_1 &= -2\rho\,(f^2 - g^2), & b_3 &= 2\sigma\,(h^2 - g^2), \\ a_3 &= \rho^2\sigma, & c_1 &= -2\rho\sigma^2, \end{aligned}$$

the notation being accommodated to that employed by Mr Salmon in *The Higher Plane Curves*, λ, μ, ν in IK being correspondent to x, y, z in Mr Salmon's form. If now we employ Mr Salmon's expression for the S (the biquadratic Aronholdian of IK), observing that

$$a_2 = 0, \quad c_2 = 0, \quad a_1 = 0, \quad c_3 = 0,$$

we have the complex combinant

$$S_{\lambda,\mu,\nu}I_{x,y,z}K = d^4 - 2d^2(b_1c_1 + a_3b_3) + da_3b_2c_1 - a_3c_1b_1b_3 + b_1^2c_1^2 + a_3^2b_3^2$$

$$= \rho^4\sigma^4\begin{pmatrix} 1 - 8\,(f^2 + h^2 - 2g^2) + 4\,(12fgh - 6g^2) \\ -16\,(f^2 - g^2)(h^2 - g^2) + 16\,(f^2 - g^2)^2 + (h^2 - g^2)^2 \end{pmatrix}$$

$$= \rho^4\sigma^4\,\{1 - 8\,(f^2 + g^2 + h^2) + 16\,(f^4 + g^4 + h^4 - h^2g^2 - g^2f^2 - f^2h^2) + 48fgh\}.$$

Hence, calling the resultant R, we have

$$\begin{aligned} -3R + 4S_{\lambda,\mu,\nu}I_{x,y,z}K = 1 &- 8\,(f^2 + g^2 + h^2) + 16\,(f^4 + g^4 + h^4) \\ &+ 32\,(f^2g^2 + g^2h^2 + h^2f^2) = \{1 - 4\,(f^2 + g^2 + h^2)\}^2 = P^2. \end{aligned}$$

Let Ω be taken the polar reciprocal to the conjunctive

$$-\lambda U + \mu V + \nu W;$$

and for greater simplicity, as we know, *à priori*, from the fundamental definition of a combinant, which (save as to a factor) must remain unaltered by any linear modification impressed upon the functions to which it appertains, that ρ and σ can enter factorially only in any combinant, let ρ and σ be each taken equal to unity in performing the intermediary operations.

Then

$$\Omega = \begin{vmatrix} -\lambda, & h\mu, & g\mu, & \xi \\ h\mu, & \lambda + \mu + \nu, & f\mu, & \eta \\ g\mu, & f\mu, & -\nu, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}$$

$$= \left\{\begin{array}{l} \xi^2\,(\nu^2 + \nu\mu + \nu\lambda + f^2\mu^2) \\ +\,\eta^2\,(-\lambda\nu + g^2\mu^2) \\ +\,\zeta^2\,(\lambda^2 + \lambda\mu + \lambda\nu + h^2\mu^2) \\ -\,2\eta\zeta\,(f\lambda\mu + hg\mu^2) \\ +\,2\zeta\xi\,\{g\,(\mu\lambda + \mu\nu) + (g - fh)\,\mu^2\} \\ -\,2\xi\eta\,(h\mu\nu + fg\mu^2) \end{array}\right\}.$$

Upon Ω, which is a quadratic function in respect of each of the two unrelated systems ξ, η, ζ; λ, μ, ν, and also in respect of the coefficients in (U, V, W), we may operate with the commutantive symbol

$$\left.\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu}, & \frac{d}{d\nu} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu}, & \frac{d}{d\nu} \end{matrix}\right\},$$

which, for facility of reference, I shall term $8E$.

Considering the first line as stationary, we shall obtain, for the value of $8E(\Omega)$, 216 commutantives, which may be expressed under the following forms:

$$\left|\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d^2}{d\lambda^2},\right. & \frac{d^2}{d\mu^2}, & \left.\frac{d^2}{d\nu^2}\right] \end{matrix}\right|,$$

$$-\left|\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d^2}{d\lambda^2},\right. & \frac{d}{d\mu}\frac{d}{d\nu}, & \left.\frac{d}{d\mu}\frac{d}{d\nu}\right] \end{matrix}\right|,$$

$$-\left|\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\nu},\right. & \frac{d^2}{d\mu^2}, & \left.\frac{d}{d\lambda}\frac{d}{d\nu}\right] \end{matrix}\right|,$$

$$-\left|\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\mu},\right. & \frac{d}{d\lambda}\frac{d}{d\mu}, & \left.\frac{d^2}{d\nu^2}\right] \end{matrix}\right|,$$

$$2\begin{vmatrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\mu}, \right. & \frac{d}{d\mu}\frac{d}{d\nu}, & \left. \frac{d}{d\nu}\frac{d}{d\lambda}\right] \end{vmatrix}.$$

In this expression the first lines may be considered stationary, the second lines are subject to the usual process of commutation, which makes three of the six permutations positive and three negative; and the third or bracketed lines are subject to the simple process which makes all the permutations of the same sign. In the three middle groups two of the terms in the final line are always identical; it will therefore be more convenient to introduce the multiplier 2, and then to consider each such line to represent the three distinct permutations, taken singly.

Let now

$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2}, \frac{d^2}{d\eta^2}, \frac{d^2}{d\zeta^2}\right\}\Omega = (\Omega),$$

$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2}, \frac{d}{d\eta}\frac{d}{d\zeta}, \frac{d}{d\eta}\frac{d}{d\zeta}\right\}\Omega = (\Omega)',$$

$$\frac{1}{8}\left\{\frac{d}{d\xi}\frac{d}{d\zeta}, \frac{d^2}{d\eta^2}, \frac{d}{d\xi}\frac{d}{d\zeta}\right\}\Omega = (\Omega)'',$$

$$\frac{1}{8}\left\{\frac{d}{d\eta}\frac{d}{d\xi}, \frac{d}{d\eta}\frac{d}{d\xi}, \frac{d^2}{d\zeta^2}\right\}\Omega = (\Omega)''',$$

$$\left\{\frac{d}{d\xi}\frac{d}{d\eta}, \frac{d}{d\eta}\frac{d}{d\zeta}, \frac{d}{d\zeta}\frac{d}{d\xi}\right\}\Omega = (\Omega)_1.$$

And let

$$\left[\frac{d^2}{d\lambda^2}, \frac{d^2}{d\mu^2}, \frac{d^2}{d\nu^2}\right] = L,$$

$$\left[\frac{d^2}{d\lambda^2}, \frac{d}{d\mu}\frac{d}{d\nu}, \frac{d}{d\mu}\frac{d}{d\nu}\right] = L',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\nu}, \frac{d^2}{d\mu^2}, \frac{d}{d\lambda}\frac{d}{d\nu}\right] = L'',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\mu}, \frac{d}{d\lambda}\frac{d}{d\mu}, \frac{d^2}{d\nu^2}\right] = L''',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\mu}, \frac{d}{d\mu}\frac{d}{d\nu}, \frac{d}{d\nu}\frac{d}{d\lambda}\right] = L_1.$$

Then, attending to the convention just previously explained, we shall have

$$E(\Omega) = (L - 2L' - 2L'' - 2L''' + 2L_1)$$
$$\times \{(\Omega) - 2(\Omega)' - 2(\Omega)'' - 2(\Omega)''' + 2(\Omega)_1\},$$

a symbolical product, any term in which such as $L'\Omega''$ will mean

$$\left\{\begin{matrix}\left[\dfrac{d^2}{d\lambda^2}, & \dfrac{d}{d\mu}\dfrac{d}{d\nu}, & \dfrac{d}{d\mu}\dfrac{d}{d\nu}\right] \\ \dfrac{d}{d\xi}\dfrac{d}{d\zeta}, & \dfrac{d^2}{d\eta^2}, & \dfrac{d}{d\xi}\dfrac{d}{d\zeta}\end{matrix}\right\}\tfrac{1}{8}\Omega,$$

and a similar interpretation must be extended to each of the 25 partial products; we have then

$$L(\Omega) = 8g^2, \quad -2L'(\Omega) = 0, \quad -2L'''(\Omega) = 0,$$
$$-2L''(\Omega) = -4g^2, \quad 2L_1(\Omega) = -2,$$
$$-2L(\Omega)' = 0, \quad -2L(\Omega)''' = 0,$$
$$4L'(\Omega)' = 0, \quad 4L''(\Omega)''' = 0,$$
$$4L''(\Omega)' = 0, \quad 4L'''(\Omega)''' = 0,$$
$$4L'''(\Omega)' = 8f^2, \quad 4L'(\Omega)''' = 8h^2,$$
$$-2L(\Omega)'' = 0, \quad 4L'(\Omega)'' = 0, \quad 4L''(\Omega)'' = 0, \quad 4L'''(\Omega)'' = 0,$$
$$-4L_1(\Omega)' = 0, \quad -4L_1(\Omega)''' = 0,$$
$$-4L_1(\Omega)'' = 4g^2;$$

and, finally, the five terms comprised in

$$2L(\Omega)_1, \ \ldots, \ 4L_1(\Omega)_1,$$

each $= 0$. All the above equations can be easily verified by direct inspection, it being observed that $8(\Omega)$ represents

$$\nu^2 + \lambda\nu + \mu\nu + f^2\mu^2, \quad -\lambda\nu + g^2\mu^2, \quad \lambda^2 + \lambda\mu + \lambda\nu + h^2\mu^2,$$

that $8(\Omega)'$ represents

$$\nu^2 + \mu\nu + \lambda\nu + f^2\mu^2, \quad -f\lambda\mu - hg\mu^2, \quad -f\lambda\mu - hg\mu^2,$$

that $8(\Omega)''$ represents

$$-\lambda\nu + g^2\mu^2, \quad g(\mu\lambda + \mu\nu) + (g - fh)\mu^2, \quad g(\mu\lambda + \mu\nu) + (g - fh)\mu^2,$$

that $8(\Omega)'''$ represents

$$\lambda^2 + \mu\lambda + \nu\lambda + h^2\mu^2, \quad -h\mu\nu - fg\mu^2, \quad -h\mu\nu - fg\mu^2,$$

and that $(\Omega)_1$ represents

$$-f\lambda\mu - hg\mu^2, \quad g(\mu\lambda + \mu\nu) + (g - fh)\mu^2, \quad -h\mu\nu - fg\mu^2.$$

We have thus

$$E(\Omega) = 8g^2 - 4g^2 - 2 + 8f^2 + 8h^2 + 4g^2$$
$$= 2\{4f^2 + 4g^2 + 4h^2 - 1\}.$$

Hence

$$3R = 4S_{\lambda,\mu,\nu}I_{x,y,z}K - \tfrac{1}{4}\{E\Omega\}^2. \tag{A}$$

If we restore to U, V, W their general values, and make

$$U = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$
$$V = a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy,$$
$$W = a''x^2 + b''y^2 + c''z^2 + 2f''yz + 2g''zx + 2h''xy,$$

and construct the cubic function

$$\begin{aligned}\vartheta = {} & (ax + a'y + a''z)(bx + b'y + b''z)(cx + c'y + c''z) \\ & - (ax + a'y + a''z)(fx + f'y + f''z)^2 - (bx + b'y + b''z)(gx + g'y + g''z)^2 \\ & - (cx + c'y + c''z)(hx + h'y + h''z)^2 \\ & + 2(fx + f'y + f''z)(gx + g'y + g''z)(hx + h'y + h''z),\end{aligned}$$

that is

$$\begin{aligned} & \Sigma\,(abc - af^2 - bg^2 - ch^2 + 2fgh)\,x^3 \\ & + \Sigma\,\{a'bc + ab'c + abc' - (a'f^2 + 2aff') - (b'g^2 + 2bgg') - (c'h^2 + 2chh') \\ & \qquad + 2f'gh + 2fg'h + 2fgh'\}\,x^2y \\ & + \{a'b''c + a'bc'' + a''b'c + a''bc' + ab'c'' + ab''c' - 2a'ff'' - 2af'f'' - 2a''ff' \\ & \qquad - 2b'gg'' - 2bg'g'' - 2b''gg' - 2c'hh'' - 2ch'h'' - 2c''hh' \\ & \qquad + 2f''g'h + 2f'g''h + 2fg'h'' + 2f''gh' + 2f''gh' + 2fg''h'\}\,xyz,\end{aligned}$$

$S_{\lambda,\mu,\nu} I_{x,y,z} K$ in the preceding equation becomes simply the Aronholdian S to ϑ, which may be calculated by Mr Salmon's formula previously quoted.

Ω may be taken equal to the determinant

$$\left|\begin{array}{cccc} ax + a'y + a''z, & hx + h'y + h''z, & gx + g'y + g''z, & \xi \\ hx + h'y + h''z, & bx + b'y + b''z, & fx + f'y + f''z, & \eta \\ gx + g'y + g''z, & fx + f'y + f''z, & cx + c'y + c''z, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{array}\right|.$$

And the cubic commutant of this, obtained by affecting it with the commutantive operator,

$$\left.\begin{array}{ccc} \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz} \\ \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz} \\ \dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta} \\ \dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta} \end{array}\right\}$$

will give $48E(\Omega)$ if each of the four lines of the operator undergoes permutation, or $8E(\Omega)$, if one of the four lines is kept stationary. Thus it falls within the limits of practical possibility to calculate explicity, by the formula (A), the value of the resultant. I give to the S of ϑ the appellation of the Hebrew letter שׁ (*shin*), and to the commutant of Ω the appellation of the Hebrew letter ט (*teth*). These letters are chosen with design; for I shall presently show that when the three given quadratic functions are the differential derivatives of the same cubic function ψ, the ט becomes the Aronholdian T to the cubic function, or, as we may write it, $T\psi$, and the שׁ becomes the Aronholdian S of the Hessian thereto, that is $SH\psi$.

Thus for the first time the true inward constitution of the resultant of three quadratics is brought to light. The methods anteriorly given by me, and the one subsequently added by M. Hesse for finding this resultant, adverted to in Section II., lead, it is true, to the construction of the form, but throw no light upon the essential mode of its composition.

55.

THÉORÈME SUR LES LIMITES DES RACINES REELLES DES ÉQUATIONS ALGÉBRIQUES.

[*Nouvelles Annales de Mathématiques*, XII. (1853), pp. 286—287.]

SOIT

$$f(x) = 0$$

une équation algébrique de degré n, et supposons qu'en opérant sur $f(x)$ et $f'(x)$ comme dans le théorème de M. Sturm, on obtienne les n quotients

$$a_1x + b_1, \quad a_2x + b_2, \quad a_3x + b_3 \dots a_nx + b_n;$$

il faut remarquer seulement qu'on obtient le $n^{\text{ième}}$ quotient, $a_nx + b_n$, en divisant l'avant-dernier résidu par le dernier résidu.

Formons la série de $2n$ quantités

$$\frac{\pm 2 - b_1}{a_1}, \quad \frac{\pm 2 - b_2}{a_2}, \quad \frac{\pm 2 - b_3}{a_3} \dots \frac{\pm 2 - b_n}{a_n};$$

il n'y a aucune racine de l'équation

$$f(x) = 0$$

entre la plus grande de ces quantités et $+\infty$, ni entre la plus petite de ces quantités et $-\infty$ *.

* Prochainement, une démonstration de ce théorème *généralisé*. [p. 424 below.]

56.

NOUVELLE MÉTHODE POUR TROUVER UNE LIMITE SUPÉRIEURE ET UNE LIMITE INFÉRIEURE DES RACINES RÉELLES D'UNE ÉQUATION ALGÉBRIQUE QUELCONQUE.

[*Nouvelles Annales de Mathématiques,* XII. (1853), pp. 329—336.]

1. Lemme. Soient

$$C_1,\ C_2,\ C_3 \ldots C_{r-1},\ C_r$$

une suite de quantités positives, assujetties à cette loi

$$C_1 = \mu_1, \quad C_2 = \mu_2 + \frac{1}{\mu_1}, \quad C_3 = \mu_3 + \frac{1}{\mu_2} \ldots C_i = \mu_i + \frac{1}{\mu_{i-1}} \ldots C_r = \mu_r,$$

où les μ sont des quantités positives quelconques.

Si, dans la fraction continue

$$\frac{1}{q_1 +} \frac{1}{q_2 +} \frac{1}{q_3 +} \cdots \frac{1}{+ q_{r-1} +} \frac{1}{q_r},$$

(les quantités q_1, $q_2 \ldots$ étant des quantités positives ou négatives), on a les inégalités

$$[q_1] > C_1, \quad [q_2] > C_2, \quad [q_3] > C_3 \ldots [q_{r-1}] > C_{r-1}, \quad [q_r] > C_r$$

(les crochets indiquent la racine carrée positive du carré de la quantité que ces crochets renferment), le dénominateur de la fraction continue aura même signe que le produit $q_1 q_2 q_3 \ldots q_{r-1} q_r$.

Démonstration. Posons

$$q_1 = m_1,$$

$$q_2 + \frac{1}{m_1} = m_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q_i + \frac{1}{m_{i-1}} = m_i,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q_r + \frac{1}{m_{r-1}} = m_r;$$

il est aisé de vérifier que les dénominateurs successifs de la fraction continue sont

$$m_1,\quad m_1m_2,\quad m_1m_2m_3,\ \ldots\ldots,\ m_1m_2m_3\ldots m_{r-1}m_r;$$

m_1 a même signe que q_1:

$$\frac{1}{q_1}=\frac{1}{m_1},\quad \frac{1}{[q_1]}<\frac{1}{\mu_1},\quad \frac{1}{m_1}<\frac{1}{\mu_1},\quad [q_2]>\frac{1}{\mu_1},\quad [q_2]>\frac{1}{m_1},\ \text{etc.};$$

donc q_2 a même signe que m_2, et aussi m_1m_2 est de même signe que q_1q_2:

$$m_2>\mu_2+\frac{1}{\mu_1},\quad m_2>\mu_2,\quad \frac{1}{m_2}<\frac{1}{\mu_2},\quad [q_3]>\frac{1}{\mu_2};$$

donc q_3 a même signe que m_3; ainsi $m_1m_2m_3$ est de même signe que $q_1q_2q_3$, et, en continuant, on parvient à démontrer que $m_1m_2m_3\ldots m_{r-1}m_r$, c'est-à-dire le dénominateur de la fraction continue, est de même signe que le produit $q_1q_2q_3\ldots q_{r-1}q_r$.

2. Théorème. *Si $f(x)$ est une fonction algébrique entière de degré n, et si l'on prend arbitrairement une autre $\phi(x)$ algébrique et entière, et d'un degré moindre que n, et qu'on développe la fraction $\frac{\phi(x)}{f(x)}$ en fraction continue*

$$\frac{\phi(x)}{f(x)}=\frac{1}{X_1+}\,\frac{1}{X_2+}\ldots+\frac{1}{X_{r-1}+}\,\frac{1}{X_r},$$

où X_1, $X_2\ldots X_r$ sont des fonctions rationnelles de x, et si l'on forme l'équation

$$(\theta)\qquad (X_1^2-C_1^2)(X_2^2-C_2^2)\ldots(X^2_{r-1}-C^2_{r-1})(X_r^2-C_r^2)=0,$$

la racine réelle supérieure de cette équation sera plus grande, et la racine réelle inférieure de cette équation sera moindre qu'aucune des racines réelles de l'équation

$$f(x)=0;$$

et si toutes les racines de l'équation (θ) sont imaginaires, l'équation

$$f(x)=0,$$

aura aussi toutes ses racines imaginaires.

Démonstration. Tous les quotients de la fraction continue qui suivent le premier quotient, savoir: X_2, $X_3\ldots X_r$, sont *en général* des fonctions linéaires de x, et X_1 sera aussi linéaire, si $\phi(x)$ est de degré $n-1$; les cas particuliers ne changent pas la marche de la démonstration; mais il faut remarquer que lorsque $f(x)$ et $\phi(x)$ ont des racines communes, le dernier quotient aura la forme $\frac{[\chi]}{0}$, $[\chi]$ étant l'avant-dernier terme, et alors, dans l'équation (θ), au lieu de $X_r^2-C_r^2$, on écrit simplement X_r^2.

Soient L la plus grande racine et Λ la plus petite racine de l'équation (θ); alors aucun facteur de (θ) ne peut devenir nul pour des valeurs de x comprises entre $+\infty$ et L, et entre Λ et $-\infty$; donc on aura toujours

$$[X_1] > C_1,$$
$$[X_2] > C_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$[X_{r-1}] > C_{r-1},$$
$$[X_r] > C_r.$$

Or $f(x)$ est évidemment égal au dénominateur de la fraction continue multiplié par un facteur constant. Donc, en vertu du lemme, le dénominateur de la fraction continue est de même signe que le produit $X_1X_2X_3 \ldots X_{r-1}X_r$ pour les valeurs de x comprises entre $+\infty$ et L, et entre Λ et $-\infty$; mais dans ces intervalles la fonction générale X_i n'étant pas comprise entre $+C_i$ et $-C_i$ ne peut devenir nulle, et, par conséquent, ne peut changer de signe; donc le dénominateur de la fraction continue conserve le même signe pour toute valeur de x renfermée entre ces intervalles, et de même $f(x)$; L est donc une limite supérieure et Λ une limite inférieure des racines de l'équation

$$f(x) = 0.$$

Le nombre des racines réelles de l'équation (θ) est évidemment pair, zéro compris; dans ce dernier cas, c'est-à-dire (θ) n'ayant aucune racine réelle, $f(x)$ ne changera donc pas de signe pour des valeurs de x comprises entre $+\infty$ et $-\infty$; autrement toutes les racines de $f(x) = 0$ sont imaginaires. Le théorème est donc complétement démontré.

3. Si $\phi(x)$ est de degré $n-1$, la fraction continue renferme *en général* (sauf les cas où quelques-uns des coefficients deviennent nuls), comme il a été dit plus haut, n quotients linéaires de la forme

$$a_1x - b_1, \quad a_2x - b_2 \ldots a_{n-1}x - b_{n-1}, \quad a_nx - b_n;$$

donc, d'après le théorème, la plus grande et la plus petite des $2n$ quantités

$$\frac{b_1 \pm C_1}{a_1}, \quad \frac{b_2 \pm C_2}{a_2} \cdots \frac{b_{n-1} \pm C_{n-1}}{a_{n-1}}, \quad \frac{b_n \pm C_n}{a_n},$$

sont respectivement une limite supérieure et une limite inférieure des racines de l'équation

$$f(x) = 0.$$

Si l'on prend $(r = n)$

$$\mu_1 = \mu_2 = \ldots = \mu_{n-1} = 1, \quad \mu_n = 2,$$

on vient au théorème énoncé [p. 423].

4. Lors même que les quotients X_1, X_2, etc., ne sont pas linéaires, on n'aura pourtant jamais à résoudre que des équations du premier degré. En effet, soient les $2r$ équations de degré quelconque

$$X_1 - C_1 = 0, \quad X_2 - C_2 = 0 \ldots X_r - C_r = 0,$$
$$X_1 + C_1 = 0, \quad X_2 + C_2 = 0 \ldots X_r + C_r = 0.$$

Il suffit de trouver une quantité l supérieure aux racines de ces équations, et une quantité λ inférieure à ces mêmes racines, l et λ seront des limites pour l'équation

$$f(x) = 0.$$

Si donc une de ces équations est de degré $p > 1$, on applique à cette équation le procédé ci-dessus, en choisissant une fonction $\phi(x)$ de degré $p-1$, et, en agissant ainsi, on arrivera par une sorte de trituration à n'avoir à traiter que des équations du premier degré.

5. On a

$$C_i = \mu_i + \frac{1}{\mu_{i-1}};$$

plus la valeur de μ_i est petite, et plus on aura de chances à resserrer les limites dans les deux fractions $\dfrac{b_i \pm C_i}{a_i}$; par contre, on aura un désavantage sous ce rapport dans les deux fractions suivantes $\dfrac{b_{i+1} \pm C_{i+1}}{a_{i+1}}$; car $C_{i+1} = \mu_{i+1} + \dfrac{1}{\mu_i}$; plus μ_i diminue, et plus C_{i+1} augmente. Cet inconvénient n'a pas lieu pour la dernière fraction; on peut donc prendre $\mu_n = 0$ et $C_n = \dfrac{1}{\mu_{n-1}}$.

6. Il est à remarquer que tous les raisonnements précédents subsistent en renversant la suite des μ et l'écrivant ainsi:

$$\frac{1}{\mu_{r-1}}, \quad \frac{1}{\mu_{r-2}} + \mu_{r-1}, \ \ldots \ \frac{1}{\mu_2} + \mu_1.$$

7. Il y a lieu à des recherches intéressantes sur la forme à donner à $\phi(x)$, et sur les valeurs à donner aux quantités μ pour obtenir les limites les plus resserrées, et je crois être parvenu à démontrer que la forme la plus avantageuse est $f'(x)$, précisément la forme que M. Sturm a adoptée.

8. Dans la réduction en fraction continue de $\dfrac{\phi(x)}{f(x)}$, nous n'avons considéré que des quotients binômes; mais on peut pousser les divisions plus loin et obtenir des quantités de la forme

$$ax + b + \frac{c}{x} + \frac{d}{x^2} + \ldots + \frac{l}{x^r};$$

le reste correspondant sera de la forme

$$a'x^{r+1} + b'x^r + c'x^{r-1} + \ldots + \frac{l}{x^r}.$$

En opérant ainsi, le nombre de termes dans chaque reste ira en diminuant, comme dans le procédé ordinaire, et le dernier reste sera de la forme Cx^μ, μ étant un entier positif ou négatif, et le dernier quotient de la forme $Px^p + Qx^{p-1}$, p étant un entier positif ou négatif; nommant les quotients ainsi obtenus q_1, $q_2 \ldots q_r$, on voit aisément qu'on aura

$$f(x) = Mx^{\pm i}D,$$

où M est une constante, i un nombre entier positif ou négatif dont la valeur dépend de la manière dont on a opéré dans les divisions successives, et D est le dénominateur de la fraction continue

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \ldots + \frac{1}{q_{r-1}} + \frac{1}{q_r}.$$

Donc, si l'on écrit, comme ci-dessus,

$$X = (q_1^2 - C_1^2)(q_2^2 - C_2^2) \ldots (q_r^2 - C_r^2) = 0,$$

nommant L et Λ les racines extrêmes de cette équation, si zéro n'est pas compris entre $+\infty$ et L, ni entre Λ et $-\infty$, la démonstration donnée ci-dessus subsiste encore pour le cas général. Et lors même que zéro est compris entre ces limites, L et Λ restent tout de même les limites pour les racines, abstraction faite de la racine zéro.

57.

ON A THEORY OF THE SYZYGETIC* RELATIONS OF TWO RATIONAL INTEGRAL FUNCTIONS, COMPRISING AN APPLICATION TO THE THEORY OF STURM'S FUNCTIONS, AND THAT OF THE GREATEST ALGEBRAICAL COMMON MEASURE.

[*Philosophical Transactions of the Royal Society of London*, CXLIII. (1853), Part III., pp. 407—548.]

INTRODUCTION.

"How charming is divine philosophy!
Not harsh and crabbed as dull fools suppose,
But musical as is Apollo's lute,
And a perpetual feast of nectar'd sweets,
Where no crude surfeit reigns!"—COMUS.

IN the first section of the ensuing memoir, which is divided into five sections, I consider the nature and properties of the residues which result from the ordinary process of successive division (such as is employed for the purpose of finding the greatest common measure) applied to $f(x)$ and $\phi(x)$, two perfectly independent rational integral functions of x. Every such residue, as will be evident from considering the mode in which it arises, is a syzygetic function of the two given functions; that is to say, each of the given functions being multiplied by an appropriate other function of a given degree in x, the sum of the two products will express a corresponding residue. These multipliers, in fact, are the numerators and denominators to the successive convergents to $\frac{\phi x}{fx}$ expressed under the form of a continued fraction. If now we proceed *à priori* by means of the given conditions as to

* *Conjugate* would imply something very different from *Syzygetic*, namely, a theory of the Invariantive properties of a system of two algebraical functions.

the degree in x of the multipliers and of any residue, to determine such residue, we find, as shown in Art. 2, that there are as many homogeneous equations to be solved as there are constants to be determined; accordingly, with the exception of one arbitrary factor which enters into the solution, the problem is definite; and if it be further agreed that the quantities entering into the solution shall be of the lowest possible dimensions in respect of the coefficients of f and ϕ, and also of the lowest numerical denomination, then the problem (save as to the algebraical sign of *plus* or *minus*) becomes absolutely determinate, and we can assign the numbers of the dimensions for the respective residues and syzygetic multipliers. The residues given by the method of successive division are easily seen not to be of these lowest dimensions; accordingly there must enter into each of them a certain unnecessary factor, which, however, as it cannot be properly called irrelevant, I distinguish by the name of the Allotrious Factor. The successive residues, when divested of these allotrious factors, I term the Simplified Residues, and in Arts. 3 and 4 I express the allotrious factor of each residue in terms of the leading coefficients of the preceding simplified residues of f and ϕ. In Art. 5 I proceed to determine by a direct method these simplified residues in terms of the coefficients of f and ϕ. Beginning with the case where f and ϕ are of the same dimensions (m) in x, I observe that we may deduce, from f and ϕ, m linearly independent functions of x each of the degree $(m-1)$ in x, all of them syzygetic functions of f and ϕ (vanishing when these two simultaneously vanish), and with coefficients which are made up of terms, each of which is the product of one coefficient of f and one coefficient of ϕ. These, in fact, are the very same m functions as are employed in the method which goes by the name of Bezout's abridged method to obtain the resultant to (that is, the result of the elimination of x performed upon) f and ϕ. As these derived functions are of frequent occurrence, I find it necessary to give them a name, and I term them the m Bezoutics or Bezoutian Primaries; from these m primaries m Bezoutian secondaries may be deduced by eliminating linearly between them in the order in which they are generated,—first, the highest power of x between two, then the two highest powers of x between three, and finally, all the powers of x between them all: along with the system thus formed it is necessary to include the first Bezoutian primary, and to consider it accordingly as being also the first Bezoutian secondary; the last Bezoutian secondary is a constant identical with the Resultant of f and ϕ. When the m times m coefficients of the Bezoutian primaries are conceived as separated from the powers of x and arranged in a square, I term such square the Bezoutic square. This square, as shown in Art. 7, is symmetrical about one of its diagonals, and corresponds therefore (as every symmetrical matrix must do) to a homogeneous quadratic function of m variables of which it expresses the determinant. This quadratic function, which plays a great part in the last section and in the theory of real roots, I term the Bezoutiant; it may be regarded as a species of generating function. Returning to the Bezoutic system, I prove that the Bezoutian secondaries are identical in form with the successive simplified residues. In Art. 6 I extend these results to the case of f and ϕ being of different dimensions in x. In Art. 7 I give a mechanical rule for the construction of the Bezoutic square. In Art. 8 I show how the theory of $f(x)$ and $\phi(x)$, where the latter is of an inferior degree to f, may be brought under the operation of the rule applicable to two functions of the same degree at the expense of the introduction of a known and very simple factor, which in fact will be a constant power of the leading coefficient in $f(x)$. In Art. 9 I give another method of obtaining directly the simplified residues in all cases. In Art. 10 I present the process of successive division under its most general aspect. In Arts. 11 and 12 I demonstrate the identity of the *algebraical sign* of the Bezoutian secondaries with that of the simplified residues, generated by a process corresponding to the development of $\frac{\phi x}{fx}$ under the form of an *improper* continued fraction (where the negative sign takes the place of the positive sign which connects the several terms of an ordinary continued fraction). As the simplified residue is obtained by driving out an allotrious factor, the signs of the former will of course be governed by the signs accorded by previous convention to the latter; the convention made is, that the allotrious factors shall be taken with a sign which renders them always *essentially positive* when the coefficients of the given functions are real. I close the section with remarking the relation of the syzygetic factors and the residues to the convergents of the continued fraction which expresses $\frac{\phi x}{fx}$, and of the continued fraction which is formed by reversing the order of the quotients in the first named fraction.

In the second section I proceed to express the residues and syzygetic multipliers in terms of the roots and factors of the given functions; the method becoming as it may be said *endoscopic* instead of being *exoscopic**, as in the first section. I begin in Arts. 14 and 15 with obtaining in this

* These words admit of an extensive and important application in analysis. Thus the methods for resolving an equation (or to speak more accurately, for making one equation depend upon another of a simpler form) furnished by Tschirnhausen and Mr Jerrard (although not so presented by the latter) are essentially exoscopic; on the other hand, the methods of Lagrange and Abel for effecting similar objects are endoscopic. So again, the memoir of Jacobi, "De Eliminatione," hereinafter referred to, takes the exoscopic, and the valuable "Nota ad Eliminationem pertinens" of Professor Richelot in *Crelle's Journal*, the endoscopic view of the subject. In the present memoir (in which the two trains of thought arising out of these distinct views are brought into mutual relation) the subject is treated (chiefly but not exclusively) under its endoscopic aspect in the second, third and fourth sections, and exoscopically in the first and last sections.

way, under the form of a sum or double sum of terms involving factors and roots of f and ϕ, and certain arbitrary functions of the roots in each term, a general representative, or to speak more precisely, a group of general representatives for a conjunctive of any given degree in x to f and ϕ, that is, a rational integral function of x, which is the sum of the products of f and ϕ multiplied respectively by rational integral functions of x, so as to vanish of necessity when f and ϕ simultaneously vanish. This variety of representatives refers not merely to the appearance of arbitrary functions, but to an essential and precedent difference of representation quite irrespective of such arbitrariness.

In Arts. 16, 17, 18, 19, 20, 21, I show how the arbitrary form of function entering into the several terms of any one (at pleasure) of the formulæ that represent a conjunctive of any given degree may be assigned, so as to make such conjunctive identical in form with a simplified residue of the same degree. The form of arbitrary function so assigned, it may be noticed, is a fractional function of the roots, so that the expression becomes a sum or double sum of fractions. I first prove in Arts. 16, 17 that such sum is essentially integral, and I determine the *weight* of its leading coefficient in respect of the roots of f and ϕ (this weight being measured by the number of roots of f and ϕ conjointly, which appear in any term of such coefficient). Now in the succeeding articles I revert to the Bezoutic system of the first section, and beginning with the supposition of m and n being equal, I demonstrate that the most general form of a conjunctive of any degree in x will be a linear function of the Bezoutics, from which it is easy to deduce that the simplified residues of any given degree in x are the conjunctives whose weight in respect of the roots is a *minimum*; so that all conjunctives having that weight must be identical (to a numerical factor *près*), and any integral form of less weight apparently representing a conjunctive must be nugatory, every term vanishing identically. These results are then extended to the case of two functions of unlike degrees. The conclusion is, that the weight of the forms assumed in Arts. 16 and 17 being equal to the minimum weight, they must (unless they were to vanish, which is easily disproved) represent the simplified residues, or which is the same thing, the Bezoutian secondaries.

We thus obtain for each simplified residue a number of essentially distinct forms of representation, but all of which must be identical to a numerical factor *près*, a result which leads to remarkable algebraical theorems.

The number of these different formulæ depends upon the degree of the residue; there being only one for the last or constant residue, two for the last but one, three for the last but two, and so on. The formulæ continue to have a meaning when their degree in x exceeds that of f or ϕ; but then, as although always representing conjunctives, they no longer represent residues, this identity no longer continues to subsist. In Arts. 22, 23, 24, 25, I enter into some developments connected with the general formulæ in question; these, it may be observed, are all expressed by means of fractions containing in the numerator and denominator products of differences; the differences in the numerator products being taken between groups of roots of f and groups of roots of ϕ; and in the denominator between roots of f *inter se* and roots of ϕ *inter se*. A great enlargement is thus opened out to the ordinary theory of partial fractions.

In Art. 26 I find the numerical ratios between the different formulæ which represent (to a numerical factor *près*) the same simplified residue, and in Arts. 27 and 28 I determine the relations of algebraical sign of these formulæ to the simplified residues or Bezoutian secondaries. In Art. 29 I determine the syzygetic multipliers corresponding to any given residue in terms of the factors and roots of the given functions; but the expressions for these, which are closely analogous to those for the residues, cease to be polymorphic. They are obtained separately from the syzygetic equation, and it is worthy of notice, that to obtain the one we use the first of the polymorphic expressions for the residue, and to obtain the other the opposite extremity of the polymorphic scale. In the subsequent articles of this section, by aid of certain general properties of continued fractions, I establish a theorem of reciprocity between the series of residues and either series of syzygetic multipliers.

Section III. is devoted to a determination of the values of the preceding formulæ for the residues and multipliers in the case applicable to M. Sturm's theorem, where ϕx becomes the differential derivative of fx. It becomes of importance to express the formulæ for this case in terms of their roots and factors of fx alone, without the use of the roots and factors of $f'x$, which will of course be functions of the former.

By selecting a proper form out of the polymorphic scale, the fractional terms of the series for each residue in this case become separately integral, and we obtain my well-known formulæ for the simplified residues (Sturm's reduced auxiliary functions) in terms of the factors and the squared differences of partial groups of roots. This is shown in Art. 35. In Art. 36 the multiplier of $f'x$ in the syzygetic equation is expressed by formulæ of equal simplicity, and in a certain sense complementary to the former. This method, however, does not apply to obtaining expressions for the multiplier of fx in the same equation in terms of the roots and factors of fx; for the separate fractions whose sum represents any one of these factors, it will be found, do not admit of being expressed as integral functions of the roots and factors. To obviate this difficulty I look to the syzygetic equation itself, which contains five quantities, namely, the given function, its first differential derivative, the residue of a given degree, and the two multipliers, all of which, except the multiplier of fx, are known, or have been previously determined as rational integral functions of the roots and factors of fx. I use this equation itself for determining the fifth quantity, the multiplier in question. To perform the general operations by a direct method required for this would be impossible; the difficulty is got over by finding, by means of the syzygetic equation, the particular form that the result must assume when certain relations of equality spring up between the roots of fx; and then, by aid of these particular determinations, the general form is demonstratively inferred.

This investigation extends over Arts. 38, 39, 40, 41, 42, 43. It turns out that the expressions for the multipliers of fx are of much greater complexity than for the multipliers of $f'x$ or for the residues. Any such multiplier consists of a sum of parts, each of which, as in the case of the residues and the factors of $f'x$, is affected with a factor consisting of the squared differences of a group of roots; but the other factor, instead of being simply (as for the residues and factors before mentioned) a product of certain factors of fx, consists of the sum of a series of products of sums of powers by products of combinations of factors of fx, each of which series is affected with the curious anomaly of its last term becoming augmented in a certain numerical ratio beyond what it should be in order to be conformable to the regular flow of the preceding terms in the series*.

The fourth section opens with the establishment of two propositions concerning quadratic functions which are made use of in the sequel. Art. 44 contains the proof of a law which, although of extreme simplicity, I do not remember to have seen, and with which I have not found that analysts are familiar: I mean the law of the constancy of signs (as regards the number of positive and negative signs) in any sum of positive and negative squares into which a given quadratic function admits of being transformed by substituting for the variables linear functions of the variables with real coefficients. This constant *number* of positive signs which attaches to a quadratic function under all its transformations, which is a transcendental function of the coefficients invariable for *real* substitutions, may be termed conveniently its *inertia*, until a better word be found. This inertia it is shown in Art. 45, by aid of a theorem identical with one formerly given by M. Cauchy, is measured by the number of combinations of sign in the series of determinants of which the first is the complete determinant of the function, the second, the determinant when one variable is made zero, the next, the determinant when another variable as well as the first is made zero, and so on, until all the variables are exhausted, and the determinant

* The syzygetic multipliers are identical with the numerators and denominators (expressed in their simplest form) of the successive convergents to the continued fraction which expresses $\frac{f'x}{fx}$.

becomes positive unity. In Art. 46 I give some curious and interesting expressions for the residues and syzygetic multipliers, under the form of determinants, communicated to me by M. Hermite; and in Art. 47 I show how, by the aid of the generating function which M. Hermite employs, and of the law of inertia stated at the opening of the section, an *instantaneous* demonstration may be given of the applicability of my formulæ for M. Sturm's functions for discovering the number of real roots of fx, without any reference to the rule of common measure; and moreover, that these formulæ may be indefinitely varied, and give the generating function, out of which they may be evolved, in its most general form. Had the law of inertia been familiar to mathematicians, this constructive and instantaneous method of finding formulæ for determining the number of real roots within prescribed limits would, in all probability, have been discovered long ago, as an obvious consequence of such law. I then proceed in Arts. 48 and 49, to inquire as to the nature of the indications afforded by the successive simplified residues to two general functions f and ϕ; and I find that the succession of signs of these residues serves to determine the number of roots of f or ϕ comprised between given limits, after all pairs of roots of either function contained within the given limits and not separated by roots of the other function have been removed, and the operation, if necessary, repeated *toties quoties* until no two roots of either function are left unseparated by roots of the other; or in other words, until every root finally retained in one function is followed by a root of the other, or else by one of the assigned limits. The system of roots comprised between given limits thus reduced I call the effective scale of intercalations; such a scale may begin with a root of the numerator or of the denominator of $\frac{\phi x}{fx}$; and upon this and the relative magnitudes of the greatest root of ϕx and fx it will depend whether in the series of residues (among which fx and ϕx are for this purpose to be counted) changes will be lost or gained as x passes from positive infinity to negative infinity. In Art. 50 I observe that the theory of real roots of a single function given by M. Sturm's theorem is a corollary to this theory of the intercalations of real roots of two functions, depending upon the well-known law, that odd groups of the limiting function $f'x$ lie between every two consecutive real roots of fx. In Art. 51 I verify the law of reciprocity, already stated to exist between the residues of fx and ϕx, by an *à posteriori* method founded on the theory of intercalations. In Arts. 52, 53, 54, I obtain a remarkable rule, founded upon the process of common measure, for finding a superior and inferior limit in an infinite variety of ways to the roots of any given function. This method stands in a singular relation of contrast to those previously known. All previous methods (including those derived through Newton's Rule) proceed upon the idea of treating the function whose roots are to be limited as made up of the *sum* of parts, each of which retains a constant sign for all values of the variable external to the quantities which are to be shown to limit the roots. My method, on the other hand, proceeds upon the idea of treating the function as the product of *factors* retaining a constant sign for such values of the variable. In Art. 55, the concluding article of the fourth section, I point out a conceivable mode in which the theory of intercalations may be extended to systems of three or more functions.

In Section V. Arts. 56, 57, I show how the *total* number of effective intercalations between the roots of two functions of the same degree is given by the *inertia* of that quadratic form which we agreed to term the Bezoutiant to f and ϕ; and in the following article (58) the result is extended to embrace the case contemplated in M. Sturm's theorem; that is to say, I show, that on replacing the function of x by a homogeneous function of x and y, the Bezoutiant to the two functions, which are respectively the differential derivatives of f with respect to x and with respect to y, will serve to determine by its form or *inertia* the total number of *real* roots and of *equal* roots in $f(x)$. The subject is pursued in the following Arts. 59, 60. The concluding portion of this section is devoted to a consideration of the properties of the Bezoutiant under a purely morphological point of view; for this purpose f and ϕ are treated as homogeneous functions of two variables x, y, instead of being regarded as functions of x alone. In Arts. 61, 62, 63, it is proved that the Bezoutiant is an invariantive function of the functions from which it is derived; and in Art. 64 the important remark is added, that it is an invariant of that particular class to which I have given the name of Combinants, which have the property of remaining unaltered, not only for linear transformations of the variables, but also for linear combinations of the functions containing the variables, possessing thus a character of double invariability. In Arts. 65, 66, I consider the relation of the Bezoutiant to the differential determinant, so called by Jacobi, but which for greater brevity I call the Jacobian. On proper substitutions being made in the Bezoutiant for the m variables which it contains (m being the degree in x, y of f and ϕ), the Bezoutiant becomes identical with the Jacobian to f and ϕ; but as it is afterwards shown, this is not a property peculiar to the Bezoutiant; in fact there exists a whole family of quadratic forms of m variables, lineo-linear (like the Bezoutiant) in respect of the coefficients in f and ϕ, all of which enjoy the same property. The number of individuals of such family must evidently be infinite, because any linear combination of any two of them must possess a similar property; I have discovered, however, that the number of independent forms of this kind is limited, being equal to the number of odd integers not greater than the degree of the two functions f and ϕ. In Arts. 67 and 68, I give the means of constructing the scale of forms, which I term the constituent or fundamental scale, of which all others of the kind are merely numerico-linear combinations. This scale does not directly include the Bezoutiant within it, and it becomes an object of interest to determine the numbers which connect the Bezoutiant with the fundamental forms; this calculation I have carried on (in Arts. 69, 70, 71) from $m=1$ to $m=6$ inclusive, and added an easy method of continuing indefinitely. In this method the numbers in the linear equation corresponding to any value of m are determined successively, and each made subject to a verification before the next is determined, there being always pairs of equations which ought to bring out the same result for each coefficient.

In the next and concluding Art. 72, I remark upon the different directions in which a generalization may be sought of the subject-matter of the ideas involved in M. Sturm's theorem, and of which the most promising is, in my opinion, that which leads through the theory of intercalations. Some of the theorems given by me in this paper have been enunciated by me many years ago, but the demonstrations have not been published, nor have they ever before been put together and embodied in that compact and organic order in which they are arranged in this memoir,—the fruit of much thought and patient toil, which I have now the honour of presenting to the Royal Society.

P.S. In a supplemental part to the third section I have given expressions in terms of the roots of ϕx and fx for the *quotients* which arise in developing $\dfrac{\phi x}{fx}$ under the form of a continued fraction, and some remarkable properties concerning these *quotients*. In a supplemental part to the fourth section I have given an extended theory of my new method of finding limits to the real roots of any algebraical equation. This method, so extended, possesses a marked feature of distinction from all preceding methods used for the same purpose, inasmuch as it admits in every case of the limits being brought up into actual coincidence with the extreme roots, whereas in other methods a wide and arbitrary interval is in general necessarily left between the roots and the limits.

Section I.

On the complete and simplified residues generated in the process of developing under the form of a continued fraction, an ordinary rational algebraical fraction.

Art. 1. Let P and Q be two rational integral functions of x, and suppose that the process of continued successive division leads to the equations

$$\left.\begin{array}{l} P\ - M_0Q\ + R_1 = 0 \\ Q\ - M_1R_1 + R_2 = 0 \\ R_1 - M_2R_2 + R_3 = 0 \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \end{array}\right\}, \tag{1}$$

so that

$$\frac{Q}{P} = \frac{1}{M_0 -}\,\frac{1}{M_1 -}\,\frac{1}{M_2 -}\ \&\text{c.} \tag{2}$$

which is what I propose to call an improper continued fraction, differing from a proper only in the circumstance of the successive terms being connected by negative instead of positive signs.

M_0, M_1, M_2, &c., R_1, R_2, R_3, &c. are, of course, functions of x: the latter we may agree to call the 1st, 2nd, 3rd, &c. residues (in order to avoid the use of the longer term "residues with the signs changed"); and by way of distinction from what they become when certain factors are rejected, we may call R_1, R_2, R_3, &c. the complete residues. Each such complete residue will in general be of the form $\frac{N_\iota \rho_\iota}{D_\iota}$, N_ι and D_ι being integral functions of the coefficients only of P and Q, but ρ_ι an integral function of these coefficients, and of x; ρ_ι may then be termed the ιth simplified residue, and $\frac{N_\iota}{D_\iota}$ the ιth *allotrious* factor. Suppose P to be of m and Q of n dimensions in x, and $m - n = e$, the process of continued division may be so conducted, that all the residues may contain only integer powers of x; and we may upon this supposition make M_0 of e dimensions, and M_1, M_2, M_3, &c. each of one dimension only in x; so that R_1, R_2, R_3, ... will be respectively of $(n-1)$, $(n-2)$, $(n-3)$, &c. dimensions in x.

P and Q are supposed to be perfectly unrelated, and each the most general function that can be formed of the same degree. From (1) we obtain

$$\left.\begin{aligned} R_1 &= M_0Q - P \\ R_2 &= M_1R_1 - Q \\ &= (M_0M_1 - 1)\,Q - M_1P \\ R_3 &= (M_0M_1M_2 + M_0 + M_2)\,Q - (M_1M_2 - 1)\,P \\ \&c. &= \&c. \end{aligned}\right\}, \qquad (3)$$

and in general we shall have

$$R_\iota = Q_\iota Q + P_\iota P, \qquad (4)$$

where it is evident that Q_ι will be of $e + (\iota - 1)$, and P_ι of $(\iota - 1)$ dimensions in x.

Art. 2. Hence it follows that the ratios $P_\iota : Q_\iota : R_\iota$ may be ascertained by the direct application of the method of indeterminate coefficients, for Q_ι will contain $e + \iota$, and P_ι will contain ι disposable constants, making $e + 2\iota$ disposable constants in all. Again, $Q_\iota Q$ and $P_\iota P$ will each rise to the degree $n + e + \iota - 1$ in x; but their sum R_ι is to be only of $n - \iota$ dimensions in x. Hence we have to make $(n + e + \iota - 1) - (n - \iota)$, that is $e + 2\iota - 1$ quantities (which are linear in respect to the given coefficients in P and Q, as well as in respect to the new disposable constants in P_ι and Q_ι) all vanish, that is to say, there will be $e + 2\iota - 1$ linear homogeneous equations to be satisfied by means of $e + 2\iota$ disposable quantities; the ratios of these latter are, therefore, *determinate*, so that we may write

$$\left.\begin{aligned} P_\iota &= \lambda_\iota\,(P_\iota) \\ Q_\iota &= \lambda_\iota\,(Q_\iota) \\ R_\iota &= \lambda_\iota\,(R_\iota) \end{aligned}\right\}; \qquad (5)$$

and when (P_ι), (Q_ι), (R_ι) are taken prime to one another, it is obvious that (R_ι) will be in all of $e + 2\iota$ dimensions in the given coefficients, that is of ι in respect of the coefficients of P, and of $e + \iota$ in respect of those of Q; λ_ι will correspond to what I have previously called the allotrious factor; being in fact foreign to the value of R_ι as determined by means of the equation (4), and arising only from the particular method employed to obtain it through the medium of the system (1): it becomes a matter of some interest and importance to determine the values of this allotrious factor for different values of ι*.

* These are identical with what I termed quotients of succession in the *London and Edinburgh Philosophical Magazine* (December, 1839) [p. 43 above]; but by an easily explicable error of inadvertence, the quantities Q_1, Q_2, &c. therein set out are not as they are therein stated to be,

Art. 3. This may be done by the following method, which is extremely simple, and would admit of a considerable extension in its applications, were it not beside my immediate purpose to digress from the objects set out in the title to the memoir, by entering upon an investigation of the special or singular cases which may arise in the process of forming the continued fraction, when one or more of the leading coefficients in any of the residues vanish; such an inquiry would require a more general character to be imparted to the values of the quotients and residues than I shall for my present purposes care to suppose.

Let us begin with supposing $e=1$, and write

$$\left.\begin{aligned} f &= ax^n + bx^{n-1} + cx^{n-2} + \&c. \\ \phi &= \alpha x^{n-1} + \beta x^{n-2} + \gamma x^{n-3} + \&c. \end{aligned}\right\}. \qquad (6)$$

Let ψ be the first residue of $\frac{f}{\phi}$, and ω of $\frac{\phi}{\psi}$, and therefore of $\frac{\phi}{\alpha^2\psi}$, so that ω is the second residue of $\frac{f}{\phi}$.

Let $\omega = \lambda\,(\omega)$, ω being entirely integer, and λ a function of the coefficients in f and ϕ. If we make $\lambda = \frac{N}{D}$, N and D being integer functions, D will evidently be L^2, where L denotes the first coefficient in the simplified residue $\alpha^2\psi$, and is evidently of two dimensions in α, β, &c., and of one in a, b, &c.; $D\omega$ is therefore of $2\times 2+1$, that is five dimensions in α, β, &c., and of two dimensions in a, b, &c.; but ω (by virtue of what has been observed of the equations in system (5)) is of three dimensions in α, β, &c., and of two in a, b, &c. Hence N is of two dimensions in α, β, &c., and of none in a, b, &c. This enables us at once to perceive that $N=\alpha^2$.

$$\left.\begin{aligned} &\text{For } \psi \text{ is of the form } f-(px+q)\,\phi, \\ &\text{and } \omega \text{ is of the form } \phi-(p'x+q')\,\psi \end{aligned}\right\}; \qquad (7)$$

the quotients of succession or allotrious factors themselves, but the ratios of each such to the one preceding, if in the series; so that—

$$Q_1 \text{ is } \lambda_1$$

$$Q_2 \text{ is } \frac{\lambda_2}{\lambda_1}$$

$$Q_3 \text{ is } \frac{\lambda_3}{\lambda_2}$$

&c. ....

This error is corrected by my distinguished friend M. Sturm (*Liouville's Journal*, t. VIII. 1842, Sur un théorème d'Algèbre de M. Sylvester), who appears, however, to have overlooked that I was obviously well acquainted with the existence and nature of these factors, and their essential character, of being perfect squares in the case contemplated in his memoir and my own. MM. Borchardt, Terquem, and other writers, in quoting my formulæ for M. Sturm's auxiliary functions, have thus been led into the error of alluding to them as completed by M. Sturm.

but $N=0$ makes ω vanish, and therefore, upon this supposition, f and ϕ would appear to have a common algebraical factor ψ, that is to say, N vanishing would appear to imply that the resultant of f and ϕ must vanish, so that N would appear to be contained as a factor in this general resultant, which latter is, however, clearly indecomposable into factors—a seeming paradox—the solution of which must be sought for in the fact, that the equation $N=0$ is *incompatible* with the existence of the usual equations (7) connecting f, ϕ, ψ and ω: but this failure of the existence of the equations (7) (bearing in mind that N has been shown to be a function only of the set of coefficients α, β, &c.), can only happen by reason of α vanishing whenever N vanishes; α must therefore be a root of N, or which is the same thing, N a power of α and hence $N=\alpha^2$.

The same result may be obtained *à posteriori* by actually performing the successive divisions; if the coefficients of any dividend be a, b, c, d, &c., and of the divisor α, β, γ, δ, &c., the first remainder, forming the second divisor, will be easily seen to have for its coefficients—

$$\frac{1}{\alpha^2}\begin{vmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \gamma \end{vmatrix}, \quad \frac{1}{\alpha^2}\begin{vmatrix} a, & b, & d \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \delta \end{vmatrix}, \quad \frac{1}{\alpha^2}\begin{vmatrix} a, & b, & e \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \epsilon \end{vmatrix} \text{ \&c.}$$

Hence the coefficients in the next remainder (making $\begin{vmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \gamma \end{vmatrix} = m$) will be each of the form of the compound determinant,—

$$\frac{1}{m^2}\begin{vmatrix} \alpha, & \beta, & \gamma \\ 0, & \begin{vmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \gamma \end{vmatrix}, & \begin{vmatrix} a, & b, & d \\ 0, & \alpha, & \gamma \\ \alpha, & \beta, & \delta \end{vmatrix} \\ \begin{vmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \gamma \end{vmatrix}, & \begin{vmatrix} a, & b, & d \\ 0, & \alpha, & \gamma \\ \alpha, & \beta, & \delta \end{vmatrix}, & \begin{vmatrix} a, & b, & e \\ 0, & \alpha, & \delta \\ \alpha, & \beta, & \epsilon \end{vmatrix} \end{vmatrix}.$$

The compound determinant above written will be the first coefficient in the remainder under consideration; the subsequent coefficients will be represented by writing f, ϕ; g, γ, &c., respectively in lieu of e, ϵ. Omitting the common multiplier $\frac{1}{m^2}$, the determinant above written is equal to

$$\alpha\left\{\begin{vmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \gamma \end{vmatrix} \times \begin{vmatrix} a, & b, & e \\ 0, & \alpha, & \delta \\ \alpha, & \beta, & \epsilon \end{vmatrix} - \begin{vmatrix} a, & b, & d \\ 0, & \alpha, & \gamma \\ \alpha, & \beta, & \delta \end{vmatrix} \times \begin{vmatrix} a, & b, & d \\ 0, & \alpha, & \gamma \\ \alpha, & \beta, & \delta \end{vmatrix}\right\}$$

$$+\begin{vmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \gamma \end{vmatrix} \times \left\{\beta \begin{vmatrix} a, & b, & d \\ 0, & \alpha, & \gamma \\ \alpha, & \beta, & \delta \end{vmatrix} - \gamma \begin{vmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \gamma \end{vmatrix}\right\}.$$

The last written pair of terms are together equal to

$$\begin{vmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ \alpha, & \beta, & \gamma \end{vmatrix} \times \{-d\beta\alpha^2 + c\gamma\alpha^2 + a\alpha(\beta\delta - \gamma^2)\},$$

which is of the form $\alpha^2 A - a^2\beta^2(\beta\delta - \gamma^2)\alpha$; and the sum of the first written pair is of the form $\alpha^2 B + (a\beta^2 a\beta\delta - a\gamma\beta\, a\gamma\beta)\alpha$. Hence the entire determinant is of the form $\alpha^2(A+B)$, showing that α^2 will enter as a factor into this and every subsequent coefficient in the second remainder, as previously demonstrated above.

It may, moreover, be noticed, that this remainder, when α^2 has been expelled, will for general values of the coefficients be numerically as well as literally in its lowest terms, as evinced by the fact that there exist terms (for example $\alpha a^2\gamma\epsilon$) having ± 1 for their numerical part. The same explicit method might be applied to show, that if the first divisor were e degrees instead of being only one degree in x lower than the first dividend, α^{e+1} would be contained in every term of the second residue: the difficulty, however, of the proof by this method augments with the value of e; but the same result springs as an immediate consequence from the method first given, which remains good *mutatis mutandis* for the general case, as may easily be verified by the reader. Applying now this result to the functions P and Q, supposed to be of the respective degrees n and $n-e$ in x, and calling the coefficients of the leading terms in the successive simplified residues $\alpha_1, \alpha_2, \alpha_3$, &c., and denoting by α the leading coefficient in Q, and as before denoting the successive allotrious factors by λ_1, λ_2, &c., it will readily be seen that

$$\lambda_1 = \frac{1}{\alpha^{e+1}}, \quad \lambda_2\lambda_1 = \frac{1}{\alpha_1^2}, \quad \lambda_3\lambda_2 = \frac{1}{\alpha_2^2}, \quad \lambda_4\lambda_3 = \frac{1}{\alpha_3^2}, \text{ \&c.,}$$

that is

$$\lambda_1 = \frac{1}{\alpha^{e+1}}, \quad \lambda_2 = \frac{\alpha^{e+1}}{\alpha_1^2}, \quad \lambda_3 = \frac{\alpha_1^2}{\alpha^{e+1}\alpha_2^2}, \quad \lambda_4 = \frac{\alpha^{e+1}\alpha_2^2}{\alpha_1^2\alpha_3^2},$$

and in general

$$\left.\begin{aligned} \lambda_{2m+1} &= \frac{1}{\alpha^{e+1}}\,\frac{\alpha_1^2\alpha_3^2\ldots\alpha^2_{2m-1}}{\alpha_2^2\alpha_4^2\ldots\alpha^2_{2m}} \\ \lambda_{2m} &= \alpha^{e+1}\,\frac{1\,.\,\alpha_2^2\alpha_4^2\ldots\alpha^2_{2m-2}}{\alpha_1^2\alpha_3^2\alpha_5^2\ldots\alpha^2_{2m-1}} \end{aligned}\right\}. \qquad (8)$$

Art. 4. Strictly speaking, we have not yet fully demonstrated that the complete allotrious factors are represented by the values above given for λ, but only that these latter are contained as factors in the allotrious factors; we must further prove that there exist no other such factors. This may be shown as follows: it is obvious from the nature of the process that the complete residues will always remain of one dimension in respect of the given coefficients, that is, first of one dimension in the set a, b, c, &c., and of zero dimensions in α, β, γ, &c.; then conversely, of one dimension in α, β, γ, &c., and of zero dimensions in a, b, c, &c., and so on, the residues being evidently required to conform in their dimensions to those of the first dividend and the first divisor alternately. These coefficients then are always of unit dimensions in respect to the given coefficients; whereas it has been shown (Art. 2) that the simplified residues in respect to these coefficients are successively of the dimensions $2+e$, $4+e$, $6+e$, &c.

Let the complete residue corresponding to λ_{2m} be $M\lambda_{2m}\alpha_{2m}$, that is

$$M\frac{\alpha^{e+1}}{\alpha_1^2}\frac{\alpha_2^2}{\alpha_3^2}\frac{\alpha_4^2}{\alpha_5^2}\cdots\frac{\alpha^2_{2m-2}}{\alpha^2_{2m-1}}\alpha_{2m},$$

or say ML; in passing from α_{2q} to α_{2q+1} the dimensions rise 2 units for all values of q except zero, and when $q=0$ the dimensions increase *per saltum* from 1 to $2+e$; hence the total dimensions of L in the joint coefficients will be

$$\{(e+1)-2(e+2)\}-4(m-1)+4m+e=1,$$

and therefore M is of zero dimensions, and λ_{2m} is the complete allotrious factor. In like manner if the complete residue corresponding to λ_{2m+1} be $M\lambda_{2m+1}\alpha_{2m+1}$, that is

$$M\frac{1}{\alpha^{e+1}}\frac{\alpha_1^2}{\alpha_2^2}\frac{\alpha_3^2}{\alpha_4^2}\cdots\frac{\alpha^2_{2m-1}}{\alpha^2_{2m}}\alpha_{2m+1},$$

or say ML, the dimensions of L will be

$$-(e+1)-4m+\{e+2(2m+1)\},\ \text{that is, } 1,$$

and hence, as in the preceding case, M is of zero dimensions, and λ_{2m+1} is the complete allotrious factor.

Art. 5. I proceed to show how the simplified residues may be most conveniently obtained by a direct process, identical with that which comes into operation in applying to the two given functions of x the method familiarly known under the name of Bezout's abridged method of elimination. Let us call the two given functions U and V, and commence with the case where U and V are of equal dimensions (n) in x. The simplified ιth residue will then be a function of $n-\iota$ dimensions in x, and of ι dimensions in respect of each given set of coefficients, and may be taken equal to $V_\iota U + U_\iota V$, where V_ι and U_ι are each of $(\iota-1)$ dimensions in x.

Let

$$U = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n,$$

$$V = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n;$$

we may write in general, m being taken any positive integer not exceeding n,

$$U = (a_0x^m + a_1x^{m-1} + \ldots + a_m)\,x^{n-m} + (a_{m+1}x^{n-m-1} + a_{m+2}x^{n-m-2} + \ldots + a_n),$$

$$V = (b_0x^m + b_1x^{m-1} + \ldots + b_m)\,x^{n-m} + (b_{m+1}x^{n-m-1} + b_{m+2}x^{n-m-2} + \ldots + b_n).$$

Hence

$$(b_0x^m + b_1x^{m-1} + \ldots + b_m)\,U - (a_0x^m + a_1x^{m-1} + \ldots + a_m)\,V$$
$$= {}_mK_1x^{n-1} + {}_mK_2x^{n-2} + {}_mK_3x^{n-3} + \ldots + {}_mK_n, \qquad (9)$$

where if we use (r, s) to denote $a_rb_s - a_sb_r$ for all values of r and s, we have

$${}_mK_1 = (0,\ m+1), \qquad {}_mK_2 = (0,\ m+2) + (1,\ m+1),$$

$${}_mK_3 = (0,\ m+3) + (1,\ m+2) + (2,\ m+1),$$

and in general ${}_mK_i = \Sigma\,(r, s)$, the values of r and s admissible within the sign of summation being subject to the two conditions, one the equality $r+s=m+i$, the other the inequality r *less than* i. By giving to m all the different values from 0 to $m-1$ in succession, and calling

$$b_0x^m + b_1x^{m-1} + \ldots + b_m, \quad a_0x^m + a_1x^{m-1} + \ldots + a_m$$

respectively Q_m and P_m, we have

$$\left.\begin{array}{l} Q_0U - P_0V = K_1x^{n-1} + K_2x^{n-2} + \ldots + K_n \\ Q_1U - P_1V = {}_1K_1x^{n-1} + {}_1K_2x^{n-2} + \ldots + {}_1K_n \\ Q_2U - P_2V = {}_2K_1x^{n-1} + {}_2K_2x^{n-2} + \ldots + {}_2K_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Q_{n-1}U - P_{n-1}V = {}_{n-1}K_1x^{n-1} + {}_{n-1}K_2x^{n-2} + \ldots + {}_{n-1}K_n \end{array}\right\}. \qquad (10)$$

The right-hand members of these n equations I shall henceforth term the Bezoutians to U and V.

The determinant formed by arranging in a square the n sets of coefficients of the n Bezoutians, and which I shall term the Bezoutian matrix, gives, as is well known, the Resultant (meaning thereby the Result in its simplest form of eliminating the variables out) of U and V.

Eliminating dialytically, first x^{n-1} between the first and second, then x^{n-1} and x^{n-2} between the first, second and third, and so on, and finally, all the powers of x between the first, second, third, ... nth of these Bezoutians, and repeating the first of them, we obtain a derived set of n equations, the right-hand members of which I shall term the secondary Bezoutians to U and V, this secondary system of equations being

$$\left.\begin{array}{l} Q_0 U - P_0 V = K_1 x^{n-1} + K_2 x^{n-2} + K_3 x^{n-3} + \ldots + K_n \\ ({}_1K_1 Q_0 - K_1 Q_1) U - ({}_1K_1 P_0 - K_1 P_1) V = L_1 x^{n-2} + L_2 x^{n-3} + \ldots + L_{n-1} \\ \{({}_1K_1\, {}_2K_2 - {}_2K_1\, {}_1K_2) Q_0 + ({}_2K_1 K_2 - K_1\, {}_2K_2) Q_1 + (K_1\, {}_1K_2 - {}_1K_1\, {}_1K_2) Q_2\} U \\ - \{({}_1K_1\, {}_2K_2 - {}_2K_1\, {}_1K_2) P_0 + ({}_2K_1 K_2 - K_1\, {}_2K_2) P_1 + (K_1\, {}_1K_2 - {}_1K_1\, {}_1K_2) P_2\} V \\ = M_1 x^{n-3} + M_2 x^{n-4} + \ldots + M_{n-2} \\ \&c. = \&c. \end{array}\right\} \quad (11)$$

And we can now already without difficulty establish the important proposition, that the successive simplified residues to $\frac{U}{V}$, expanded under the form of an improper continued fraction, abstracting from the algebraical sign (the correctness of which also will be established subsequently), will be represented by the n successive *Secondary Bezoutians* to the system U, V.

For if we write the system of equations (11) under the general form

$$\vartheta_\iota U - H_\iota V = A_\iota x^{n-\iota} + B_\iota x^{n-\iota-1} + \&c.,$$

the degree of ϑ_ι and H_ι in x will be that of $Q_{\iota-1}$ and $P_{\iota-1}$, that is $\iota - 1$; and the dimensions of A_ι, B_ι, &c., in respect of each set of coefficients is evidently ι; consequently, by virtue of Art. 2, $A_\iota x^{n-1} + B_\iota x^{n-2} +$ &c., which is the ιth Bezoutian, will (saving at least a numerical factor of a magnitude and algebraical sign to be determined, but which, when proper conventions are made, will be subsequently proved to be $+1$) represent the ιth simplified residue to $\frac{U}{V}$*, as was to be shown.

Art. 6. More generally, suppose U and V to be respectively of $n + e$ and n dimensions in x.

Let $$U = a_0 x^{n+e} + a_1 x^{n+e-1} + a_2 x^{n+e-2} + \&c.$$

$$V = b_0 x^n + b_1 x^{n-1} + \&c.$$

Making

$$U = (a_0 x^{e+m} + a_1 x^{e+m-1} + \&c. + a_{e+m})\, x^{n-m} + (a_{e+m+1} x^{n-m-1} + \&c. + a_{n+e}),$$

$$V = (b_0 x^m + b_1 x^{m-1} + \ldots + b_m)\, x^{n-m} + (b_{m+1} x^{n-m-1} + \&c. + b_n),$$

we obtain the equation

$$Q_m U - P_{e+m} V = {}_mK_1 x^{n+e-1} + {}_mK_2 x^{n+e-2} + \&c. + {}_mK_{n+e}, \qquad (12)$$

* V is supposed to be taken as the first divisor, and the term residue is used, as hitherto in this paper, throughout in the sense appertaining to the expansion conducted, so as to lead to an *improper* continued fraction, in that sense, in fact, in which it would, more strictly speaking, be entitled to the appellation of *excess* rather than that of *residue*.

where

$$Q_m = (b_0 x^m + \ldots + b_m), \quad P_{e+m} = (a_0 x^{e+m} + \ldots + a_{e+m});$$

$${}_mK_1 = a_0 b_{m+1}\,;\ {}_mK_2 = a_0 b_{m+2} + a_1 b_{m+1}\,;\ \ldots\ {}_mK_e = a_0 b_{m+e} + a_1 b_{m+e-1} + \text{\&c.} + a_e b_m\,;$$

$${}_mK_{e+1} = a_0 b_{m+e+1} + \text{\&c.} + a_{e+1} b_m - a_{e+m+1} b_0\,;\ \text{\&c.} = \text{\&c.}$$

By giving to m every integer value from 0 to $(n-1)$ inclusive, we thus obtain n equations of the form of (12), each of the degree $n+e-1$ in x, and of one dimension in regard to each set of coefficients.

In addition to these equations we have the e equations of the form

$$x^\mu V = b_0 x^{n+\mu} + b_1 x^{n+\mu-1} + \text{\&c.} + b_n x^\mu, \tag{13}$$

in which μ may be made to assume every value from 0 to $(e-1)$ inclusive, and the right-hand side of the equation for all such values of μ will remain of a degree in x not exceeding $n+e-1$, the degree of the equations of the system above described. There will thus be e equations in which only the (b) set of coefficients appear, and n equations containing in every term one coefficient out of each of the two sets.

The total number of equations is of course $n+e$. Between the e equations of the second system (13) and the r occurring first in order of the first system (12), we may eliminate dialytically the $e+r-1$ highest powers of x, and there will thus arise an equation of the form

$$\theta_{r-1} U - \omega_{e+r-1} V = Lx^{n-r} + L'x^{n-r-1} + \text{\&c.} + (L), \tag{14}$$

where θ_{r-1} and ω_{e+r-1} are respectively of the degrees $r-1$ and $e+r-1$ in x, and $L, L' \ldots (L)$ are of r dimensions in the (a) set, and of $(e+r)$ dimensions in the (b) set of coefficients, and consequently $Lx^{n-1} + L'x^{n-r-1} + \ldots + (L)$ must satisfy the conditions necessary and sufficient to prove its being (to a numerical factor *près*) a simplified residue to (U, V).

Thus suppose

$$U = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4,$$

$$V = \qquad\qquad b_0 x^2 + b_1 x + b_2.$$

Then, corresponding to the system of which equation (13) is the type, we have

$$V = b_0 x^2 + b_1 x + b_2,$$

$$xV = b_0 x^3 + b_1 x^2 + b_2 x.$$

Again, to form the system of which equation (12) is the type, we write

$$b_0 U - (a_0 x^2 + a_1 x + a_2) V = b_0 (a_3 x + a_4) - (a_0 x^2 + a_1 x + a_2)(b_1 x + b_2)$$

$$= -a_0 b_1 x^3 - (a_0 b_2 + a_1 b_1) x^2 + (b_0 a_3 - a_1 b_2 - a_2 b_1) x + (b_0 a_4 - a_2 b_2),$$

$$(b_0 x + b_1) U - (a_0 x^3 + a_1 x^2 + a_2 x + a_3) V = (b_0 x + b_1) a_4 - (a_0 x^3 + a_1 x^2 + a_2 x + a_3) b_2$$

$$= -a_0 b_2 x^3 - a_1 b_2 x^2 + (b_0 a_4 - a_2 b_2) x + (b_1 a_4 - b_2 a_3).$$

Combining the two equations of the first system with the first of the second system, we obtain the first simplified residue $Lx+L'$, where

$$-L=\begin{vmatrix} 0, & b_0, & b_1 \\ b_0, & b_1, & b_2 \\ a_0b_1, & a_0b_2+a_1b_1, & a_1b_2+a_2b_1-b_0a_3 \end{vmatrix}$$

and

$$L'=\begin{vmatrix} 0, & b_0, & b_2 \\ b_0, & b_1, & 0 \\ a_0b_1, & a_0b_2+a_1b_1, & a_2b_2-b_0a_4 \end{vmatrix}.$$

By again combining the two equations of the first system with both of the second system, we have the determinant

$$R=\begin{vmatrix} 0, & b_0, & b_1, & b_2 \\ b_0, & b_1, & b_2, & 0 \\ a_0b_1, & a_0b_2+a_1b_1, & a_1b_2+a_2b_1-b_1a_3, & a_2b_2-b_0a_4 \\ a_0b_2, & a_1b_2, & a_2b_2-b_0a_4, & a_0b_2-a_4b_1 \end{vmatrix}$$

which is the last simplified residue, or in other terms, the resultant to the system U, V.

Art. 7. It is most important to observe that the Bezoutian matrix to two functions of the same degree (n) is a symmetrical matrix, the terms similarly disposed in respect to one of the diagonals being equal.

Thus retaining the notation of Art. 5, so that

$$\begin{array}{lll} (0,\ 1)=a\beta-b\alpha, & (1,\ 2)=b\gamma-c\beta, & (2,\ 3)=c\delta-d\gamma, \\ (0,\ 2)=a\gamma-c\alpha, & (1,\ 3)=b\delta-d\beta, & \&\text{c.} \\ (0,\ 3)=a\delta-d\alpha, & \&\text{c.} & \\ \&\text{c.} & & \end{array}$$

when $n=1$ the Bezoutian matrix consists of a single term $(0,\ 1)$;

when $n=2$, it becomes

$$\begin{matrix} (0,\ 1) & (0,\ 2) \\ (0,\ 2) & (1,\ 2); \end{matrix}$$

when $n=3$, it becomes

$$\begin{matrix} (0,\ 1) & (0,\ 2) & (0,\ 3) \\ (0,\ 2) & \begin{pmatrix} (0,\ 3) \\ + \\ (1,\ 2) \end{pmatrix} & (1,\ 3) \\ (0,\ 3) & (1,\ 3) & (2,\ 3); \end{matrix}$$

when $n=4$, it becomes

$$\begin{array}{cccc}
(0,\ 1) & (0,\ 2) & (0,\ 3) & (0,\ 4) \\
(0,\ 2) & \begin{pmatrix}(0,\ 3)\\+\\(1,\ 2)\end{pmatrix} & \begin{pmatrix}(0,\ 4)\\+\\(1,\ 3)\end{pmatrix} & (1,\ 4) \\
(0,\ 3) & \begin{pmatrix}(0,\ 4)\\+\\(1,\ 3)\end{pmatrix} & \begin{pmatrix}(1,\ 4)\\+\\(2,\ 3)\end{pmatrix} & (2,\ 4) \\
(0,\ 4) & (1,\ 4) & (2,\ 4) & (3,\ 4);
\end{array}$$

when $n=5$, it becomes

$$\begin{array}{ccccc}
(0,\ 1) & (0,\ 2) & (0,\ 3) & (0,\ 4) & (0,\ 5) \\
(0,\ 2) & \begin{pmatrix}(0,\ 3)\\+\\(1,\ 2)\end{pmatrix} & \begin{pmatrix}(0,\ 4)\\+\\(1,\ 3)\end{pmatrix} & \begin{pmatrix}(0,\ 5)\\+\\(1,\ 4)\end{pmatrix} & (1,\ 5) \\
(0,\ 3) & \begin{pmatrix}(0,\ 4)\\+\\(1,\ 3)\end{pmatrix} & \begin{pmatrix}(0,\ 5)\\+\\(1,\ 4)\\+\\(2,\ 3)\end{pmatrix} & \begin{pmatrix}(1,\ 5)\\+\\(2,\ 4)\end{pmatrix} & (2,\ 5) \\
(0,\ 4) & \begin{pmatrix}(0,\ 5)\\+\\(1,\ 4)\end{pmatrix} & \begin{pmatrix}(1,\ 5)\\+\\(2,\ 4)\end{pmatrix} & \begin{pmatrix}(2,\ 5)\\+\\(3,\ 5)\end{pmatrix} & (3,\ 5) \\
(0,\ 5) & (1,\ 5) & (2,\ 5) & (3,\ 5) & (4,\ 5),
\end{array}$$

and so forth. Every such square it is apparent may be conceived as a sort of sloped pyramid, formed by the successive superposition of square layers, which layers possess not merely a simple symmetry about a diagonal (such as is proper to a *multiplication* table), but the higher symmetry (such as exists in an *addition* table), evinced in all the terms in any line of terms parallel to the diagonal transverse to the axis of symmetry being alike*. Thus for $n=5$, the three layers or stages in question will be seen to be, the first—

$$\begin{array}{ccccc}
(0,\ 1) & (0,\ 2) & (0,\ 3) & (0,\ 4) & (0,\ 5) \\
(0,\ 2) & (0,\ 3) & (0,\ 4) & (0,\ 5) & (1,\ 5) \\
(0,\ 3) & (0,\ 4) & (0,\ 5) & (1,\ 5) & (2,\ 5) \\
(0,\ 4) & (0,\ 5) & (1,\ 5) & (2,\ 5) & (3,\ 5) \\
(0,\ 5) & (1,\ 5) & (2,\ 5) & (3,\ 5) & (4,\ 5);
\end{array}$$

* A square arrangement having this kind of symmetry, namely, such as obtains in the so-called Pythagorean addition table as distinguished from that which obtains in the multiplication table, may be universally called Persymmetric.

the second—

$$\begin{array}{ccc} (1, 2) & (1, 3) & (1, 4) \\ (1, 3) & (1, 4) & (2, 4) \\ (1, 4) & (2, 4) & (3, 4); \end{array}$$

and the third—

$$(2, 3).$$

In general, when n is odd, say $2p+1$, the pyramid will end with a single term $(p, (p+1))$, and when even, as $2p$, with a square of four terms,

$$\begin{array}{cc} ((p-2), \ (p-1)), & ((p-2), p) \\ ((p-2), p), & ((p-1), p). \end{array}$$

Each stage may be considered as consisting of three parts, a diagonal set of equal terms transverse to the axis of symmetry, and two triangular wings, one to the left, and the other to the right of this diagonal; the terms in each such diagonal for the respective stages will be

$$(0, n), \quad (1, n-1), \quad (2, (n-2)) \dots (p, (p+1)),$$

p being $\frac{n}{2}-1$ when n is even, and $\frac{n-1}{2}$ when n is odd.

If we change the order of the coefficients in each of the two given functions, it will be seen that the only effect will be to make the left and right triangular wings to change places, the diagonals in each stage remaining unaltered. The mode of forming these triangles is an operation of the most simple and mechanical nature, too obvious to need to be further insisted on here.

Art. 8. When we are dealing with two functions of unequal degrees, n and $n+e$, we can still form a square matrix with the coefficients of the two systems of e and n equations respectively, but this will no longer be symmetrical about a diagonal; it is obvious, however, that if we treat the function of the lower degree, as if it were of the same degree as the other function, which we may do by filling up the vacant places with terms affected with zero coefficients, the symmetry will be recovered; and it is somewhat important (as will appear hereafter) to compare the values of the Bezoutian secondaries as obtained, first in their simplest form by treating each of the two functions as complete in itself, and secondly, as they come out, when that of the functions which is of the lower degree is looked upon as a defective form of a function of the same degree as the other. A single example will suffice to make the nature of the relation between the two sets of results apparent.

Take

$$fx = a\ x^4 + b\ x^3 + cx^2 + dx + e,$$

$$\phi x = 0\ x^4 + 0\ x^3 + \gamma x^2 + \delta x + \epsilon.$$

The general method of Art. 7 then gives for the Bezoutian matrix

$$\begin{array}{llll} 0, & a\gamma, & a\delta, & a\epsilon \\ a\gamma, & \begin{pmatrix} a\delta \\ + \\ b\gamma \end{pmatrix}, & \begin{pmatrix} a\epsilon \\ + \\ b\delta \end{pmatrix}, & b\epsilon \\ a\delta, & \begin{pmatrix} a\epsilon \\ + \\ b\delta \end{pmatrix}, & \begin{pmatrix} b\epsilon \\ + \\ c\delta - d\gamma \end{pmatrix}, & c\epsilon - e\gamma \\ a\epsilon, & b\epsilon, & c\epsilon - e\gamma, & d\epsilon - e\delta. \end{array}$$

We shall not affect the value either of the complete determinant, or of any of the minor determinants appertaining to the above matrix, by subtracting the first line of terms, each increased in the ratio of $b : a$, from the second line of terms respectively; the matrix so modified becomes

$$\begin{array}{llll} 0, & a\gamma, & a\delta, & a\epsilon \\ a\gamma, & a\delta, & a\epsilon, & 0 \\ a\delta, & a\epsilon + b\delta, & \begin{pmatrix} b\epsilon \\ + \\ c\delta - d\gamma \end{pmatrix}, & c\epsilon - e\gamma \\ a\epsilon, & b\epsilon, & c\epsilon - e\gamma, & d\epsilon - e\delta. \end{array}$$

Again, adopting the method of Art. 6, we should obtain the matrix

$$\begin{array}{llll} 0, & \gamma, & \delta, & \epsilon \\ \gamma, & \delta, & \epsilon, & 0 \\ \delta, & a\epsilon - b\delta, & \begin{pmatrix} b\epsilon \\ + \\ c\delta - d\gamma \end{pmatrix}, & c\epsilon - e\gamma \\ a\epsilon, & b\epsilon, & c\epsilon - e\gamma, & d\epsilon - e\delta. \end{array}$$

Hence it is apparent that the secondary Bezoutians obtained by the symmetrizing method will differ from those obtained by the unsymmetrical method by a constant factor a^2; and so in general it may readily be shown that the secondary Bezoutians, by the use of the symmetrizing method, will each become affected with a constant irrelevant factor a^ω, where ω is the difference of the degrees of the two functions, and a the leading coefficient of the higher one of the two. When a is taken unity, the Bezoutian secondaries, as obtained by either method, will of course be identical.

Art. 9. There is another method* of obtaining the simplified residues to any two functions U and V of the degrees n and $n + e$ respectively, which,

* Originally given by myself in the *London and Edinburgh Philosophical Magazine*, as long ago as 1839 or 1840 [p. 54 above]; and some years subsequently in unconsciousness of that fact, reproduced by my friend Mr Cayley, to whom the method is sometimes erroneously ascribed, and who arrived at the same equations by an entirely different circle of reasoning.

although less elegant, ought not to be passed over in silence. This method consists in forming the identical equations (of which for greater brevity the right-hand members are suppressed)

$$\begin{aligned}
V &= \&c.\\
xV &= \&c.\\
&\vdots\\
x^{e-1}V &= \&c.\\
U &= \&c.\\
x^eV &= \&c.\\
xU &= \&c.\\
x^{e+1}V &= \&c.\\
x^2U &= \&c.\\
x^{e+2}V &= \&c.\\
\&c. &= \&c.\\
x^{n-1}U &= \&c.\\
x^{e+n-1}V &= \&c.
\end{aligned}$$

If we equate the right-hand members of $(e+2\iota)$ of the above equations to zero, and then eliminate dialytically the several powers of x from $x^{n+e+\iota-1}$ to $x^{n-\iota+1}$ (both inclusive), the result of this process will evidently be of $(e+\iota)$ dimensions in respect of the coefficients in V, of ι dimensions in respect of the coefficients in U and of the degree $x^{n-\iota}$ in x; it will also be of the form

$$(A + Bx + \ldots + Lx^{e-1})\,U + (F + Gx + \ldots + Qx^{e+\iota-1})\,V,$$

and by virtue of Art. 2, must consequently be the ιth simplified residue to the system U, V.

Art. 10. The most general view of the subject of expansion by the method of continued division, consists in treating the process as having reference solely to the two systems of coefficients in U and V, which themselves are to be regarded in the light of generating functions. To carry out this conception, we ought to write

$$U = a_0 + a_1y + a_2y^2 + a_3y^3 + \&c.\ ad\ inf.$$
$$V = b_0 + b_1y + b_2y^2 + b_3y^3 + \&c.\ ad\ inf.,$$

and might then suppose the process of successive division applied to U and V, so as to obtain the successive equations

$$\begin{aligned}
U - M_1V + R_1 &= 0,\\
V - M_2R_1 + R_2 &= 0,\\
R_1 - M_3R_2 + R_3 &= 0,\\
\&c.\ \&c.,&
\end{aligned}$$

M_1, M_2, M_3, &c. being each severally of any degree whatever in y, and in general the degree of y in M_ι being any given arbitrary function $\phi(\iota)$ of ι. The values of the coefficients of the residues R_1, R_2, $R_3 \ldots$, or of these forms simplified by the rejection of detachable factors, become then the distinct object of the inquiry, and will, of course, depend only upon the coefficients in U and V and the nature of the arbitrary continuous or discontinuous function $\phi(\iota)$, which regulates the number of *steps* through which each successive process of division is to be pursued. Following out this idea in a particular case, if we again reduce our two initial functions to the forms previously employed, and write

$$U = a_0 x^n + a_1 x^{n-1} + \text{\&c.}$$

$$V = b_0 x^n + b_1 x^{n-1} + \text{\&c.};$$

and if, instead of making, according to the more usual course of proceeding, the divisions proceed first through one step and ever after through two steps at a time, which is tantamount to making $\phi 1 = 1$, $\phi(1+\omega) = 2$, we push each division through one step only at a time, and no more (so that in fact $\phi(\iota)$ is always 1), we shall have

$$\begin{aligned} U - m_1\ V + R_1 &= 0, \\ V - m_2 x R_1 + R_2 &= 0, \\ R_1 - m_3\ R_2 + R_3 &= 0, \\ R_2 - m_4 x R_3 + R_4 &= 0, \\ \text{\&c. \&c.}, \end{aligned}$$

m_1, m_2, m_3, &c. being functions of the coefficients only of U and V; and it is not without interest to observe (which is capable of an easy demonstration) that the simplified residues contained in R_1, R_2, &c., found according to this mode of development, will be the successive dialytic resultants obtained by eliminating the $(\iota - 1)$th highest powers of x between the ι first of the system of annexed equations (supposed to be expressed in terms of x)

$$\begin{aligned} U &= 0, \\ V &= 0, \\ xU &= 0, \\ xV &= 0, \\ x^2 U &= 0, \\ x^2 V &= 0, \\ \text{\&c.} &= \text{\&c} \\ x^{n-1} U &= 0, \\ x^{n-1} V &= 0. \end{aligned}$$

If we combine together $2i+1$ of the above equations, the highest power of x entering on the left-hand side will be x^{n+i}, and we shall be able to eliminate $2i$ of these factors, leaving x^{n-i} the highest power remaining uneliminated. If we take $2i$, that is i pairs of the equations, the highest power of x appearing in any of them will be x^{n+i-1}, and we shall be able to eliminate between them so as still to leave $x^{n+i-1-(2i-1)}$, that is x^{n-i} as before, the highest power of x remaining uneliminated; and it will be readily seen that such of the simplified residues corresponding to this mode of development as occupy the odd places in the series of such residues, will be identical with the successive simplified residues resulting from the ordinary mode of developing $\frac{U}{V}$ under the form of a continued fraction.

Art. 11. It has been shown that the simplified residues of fx and ϕx resulting from the process of continued division are identical in point of form with the secondary Bezoutians of these functions, but it remains to assign the numerical relations between any such residue and the corresponding secondary.

To determine this numerical relation, it will of course be sufficient to compare the magnitude of the coefficient of any one power of x in the one, with that of the same power in the other; and for this purpose I shall make choice of the leading coefficients in each. In what follows, and throughout this paper, it will always be understood that in calculating the determinant corresponding to any square the product of the terms situated in the diagonal descending from left to right will always be taken with the positive sign, which convention will serve to determine the sign of all the other products entering into such determinant. Now adopting the umbral notation for determinants*, we have, by virtue of a much more general theorem for compound determinants, the following identical equation:—

$$\begin{pmatrix} a_1 a_2 a_3 \ldots a_{m-1} \\ \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_{m-1} \end{pmatrix} \times \begin{pmatrix} a_1 a_2 a_3 \ldots a_{m+1} \\ \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 a_3 \ldots a_{m-1} a_m \\ \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_{m-1} \alpha_m \end{pmatrix} \times \begin{pmatrix} a_1 a_2 \ldots a_{m-1} a_{m+1} \\ \alpha_1 \alpha_2 \ldots \alpha_{m-1} \alpha_{m+1} \end{pmatrix}$$

$$- \begin{pmatrix} a_1 a_2 a_3 \ldots a_{m-1} a_m \\ \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_{m-1} \alpha_{m+1} \end{pmatrix} \times \begin{pmatrix} a_1 a_2 \ldots a_{m-1} a_{m+1} \\ \alpha_1 \alpha_2 \ldots \alpha_{m-1} \alpha_m \end{pmatrix},$$

and consequently

$$\begin{pmatrix} a_1 a_2 a_3 \ldots a_{m-1} \\ \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_{m-1} \end{pmatrix} \times \begin{pmatrix} a_1 a_2 a_3 \ldots a_{m-1} a_m a_{m+1} \\ \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_{m-1} a_m a_{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 a_3 \ldots a_{m-1} a_m \\ \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_{m-1} a_m \end{pmatrix} \times \begin{pmatrix} a_1 a_2 \ldots a_{m-1} a_{m+1} \\ \alpha_1 \alpha_2 \ldots \alpha_{m-1} a_{m+1} \end{pmatrix} - \begin{pmatrix} a_1 a_2 \ldots a_{m-1} a_m \\ \alpha_1 \alpha_2 \ldots \alpha_{m-1} a_{m+1} \end{pmatrix}^2,$$

* See *London and Edinburgh Philosophical Magazine*, April 1851 [p. 242 above].

and consequently when

$$\begin{pmatrix} a_1 a_2 \ldots a_{m-1} a_m \\ \alpha_1 \alpha_2 \ldots \alpha_{m-1} a_{m+1} \end{pmatrix} = 0,$$

$$\begin{pmatrix} a_1 a_2 \ldots a_{m-1} \\ \alpha_1 \alpha_2 \ldots \alpha_{m-1} \end{pmatrix} \text{ and } \begin{pmatrix} a_1 a_2 \ldots a_{m-1} a_m a_{m+1} \\ \alpha_1 \alpha_2 \ldots \alpha_{m-1} a_m a_{m+1} \end{pmatrix}$$

will have different algebraical signs, it being of course understood that all the quantities entering into the determinants thus *umbrally* represented above are supposed to be real quantities. This theorem, translated into the ordinary language of determinants, may be stated as follows:—Begin with any square of terms whether symmetrical or otherwise, say of r lines and r columns: let this square be bordered laterally and longitudinally by the same number r of new quantities symmetrically disposed in respect to one of the diagonals, the term common to the superadded line and column being filled up with any quantity whatever; we thus obtain a square of $(r+1)$ lines and columns; let this be again bordered laterally and longitudinally by $(r+1)$ quantities symmetrically disposed above the same diagonal as that last selected, the place in which this new line and column meet being also filled up with any arbitrary quantity; and proceeding in this manner, let the determinants corresponding to the square matrices thus formed be called D_r, D_{r+1}, $D_{r+2}\ldots$: this series of quantities will possess the property, that no term in it can vanish without the terms on either side of that so vanishing having contrary signs. Thus if we begin with a square consisting of one single term, we may suppose that by accretions formed after the above rule it has been developed into the square (M) below written, and which of course may be indefinitely extended:—

$$\begin{matrix} a, & l, & m, & p, & s, \\ l, & b, & n, & q, & t, \\ m, & n, & c, & r, & u, \\ p, & q, & r, & d, & v, \\ s, & t, & u, & v, & e. \end{matrix} \tag{M}$$

Here D_0, D_1, D_2, D_3, D_4, D_5 will represent the progression

$$1, \quad a, \quad \begin{vmatrix} a, & l \\ l, & b \end{vmatrix}, \quad \begin{vmatrix} a, & l, & m \\ l, & b, & n \\ m, & n, & c \end{vmatrix}, \quad \begin{vmatrix} a, & l, & m, & p \\ l, & b, & n, & q \\ m, & n, & c, & r \\ p, & q, & r, & d \end{vmatrix}, \quad \begin{vmatrix} a, & l, & m, & p, & s \\ l, & b, & n, & q, & t \\ m, & n, & c, & r, & u \\ p, & q, & r, & d, & v \\ s, & t, & u, & v, & e \end{vmatrix}; \tag{Π}$$

so if we use the matrix

$$\begin{matrix} a, & l, & m, & p, & s, \\ l', & b, & n, & q, & t, \\ m, & n, & c, & r, & u, \\ p, & q, & r, & d, & v, \\ s, & t, & u, & v, & e, \end{matrix}$$

the determinants D_1, D_2, D_3, D_4, representing

$$a, \quad \begin{vmatrix} a, & l \\ l', & b \end{vmatrix}, \quad \begin{vmatrix} a, & l, & m \\ l', & b, & n \\ m, & n, & c \end{vmatrix}, \quad \begin{vmatrix} a, & l, & m, & p \\ l', & b, & n, & q \\ m, & n, & c, & r \\ p, & q, & r, & d \end{vmatrix},$$

will possess the property in question; the line and column l, b; l', b not being identical, the first determinant D_0 representing unity must not be included in the progression.

We shall have occasion to use this theorem as applicable to the case of a matrix symmetrical throughout, and we may term the progression (II), above written, a progression of the successive principal determinants about the axis of symmetry of the square matrix (M), and so in general. Now it is obvious that the leading coefficients of the successive Bezoutian secondaries are the successive principal determinants about the axis of symmetry of the Bezoutian squares; they will therefore have the property which has been demonstrated of such progressions; to wit, if the first of them vanishes, the second will have a sign contrary to that of $+1$; if the second vanishes, the third will have a sign contrary to that of the first, and so on.

Art. 12. Now let fx and ϕx be any two algebraical functions of x with the leading coefficients in each, for greater simplicity, supposed positive: and in the course of developing $\frac{\phi x}{fx}$ under the form of an improper continued fraction by the common process of successive division, let any two consecutive residues (the word residue being used in the same conventional sense as employed throughout) be

$$Ax^{\iota} + Bx^{\iota-1} + Cx^{\iota-2} + \&c.$$

$$B'x^{\iota-1} + C'x^{\iota-2} + D'x^{\iota-3} + \&c.$$

The residue next following, obtained by actually performing the division and duly changing the sign of the remainder, will be

$$\left\{\left(\frac{AD'}{B'} - C\right) - \left(\frac{AC'}{B'} - B\right)\frac{C'}{B'}\right\} x^{\iota-2} + \&c.,$$

which is of the form

$$\frac{1}{B'^2}\{B'M - AC'^2\}\,x^{\iota-2} + \&c.$$

Thus the leading coefficients in the complete unreduced residues will be

$$A,\ B',\ \frac{1}{B'^2}\{B'M - AC'^2\},$$

and when reduced by the expulsion of the allotrious factor will become A, B', $B'M - AC'^2$, and consequently, when B' the leading coefficient of one of the simplified residues vanishes, the leading coefficients of the residues immediately preceding and following that one will have contrary signs.

First, let fx and ϕx be of the same degree. As regards the numerical ratio of each Bezoutian secondary to the corresponding simplified residue, it has been already observed that there are always unit coefficients in the latter of these, and the same is obviously true of the former; hence if we call the progression of the leading coefficients of the simplified residues

$$R_1,\ R_2,\ R_3,\ R_4,\ \&c.,$$

and that of the leading coefficients of the Bezoutian secondaries

$$B_1,\ B_2,\ B_3,\ B_4,\ \&c.,$$

we have

$$B_1 = \pm R_1,\quad B_2 = \pm R_2,\quad B_3 = \pm R_3,\quad B_4 = \pm R_4,\quad \&c.$$

It may be proved by actual trial that $B_1 = R_1$ and $B_2 = R_2$. Moreover, since the signs are invariable, and do not depend upon the values of the coefficients, we may suppose $B_2 = 0$ (which may always be satisfied by real values of the quantities of which B_2 is a function); we shall also, therefore, have $R_2 = 0$, and consequently B_3 has the opposite sign to that of B_1, and R_3 the opposite sign to that of R_1, which is equal to B_1: hence when $B_2 = 0$, B_3 and R_3 are equal, and consequently are always equal; in like manner we can prove that R_4 and B_4 have the same sign when R_3 and B_3 vanish, and consequently are always equal, and so on *ad libitum*, which proves that the series B_1, B_2, ... B_n is identical with the series R_1, R_2, ... R_n, and consequently that the Bezoutian secondaries are identical in form, magnitude and algebraical sign with the simplified residues.

Secondly, when fx and ϕx are not of the same degree, it has been shown that the secondaries formed from the non-symmetrical matrix corresponding to this case will be the same as those formed from the *symmetrical* matrix corresponding to fx and Φx (where Φx is ϕx treated by aid of evanescent terms as of the same degree as fx), with the exception merely of a constant multiplier (a power of the leading coefficient of fx) being introduced into each secondary. By aid of this observation, the proposition established for the case of two functions of the same degree may be readily seen to be capable of being extended, from the case of f and ϕ being of equal dimensions in x, to the general case of their dimensions being any whatever.

Art. 13. Before closing this section, it may be well to call attention to the nature of the relation which connects the successive residues of fx and ϕx with these functions themselves, and with the improper continued fractional form into which $\frac{\phi x}{fx}$ is supposed to be developed in the process of obtaining these residues.

If ϕx be of n degrees, and fx of $n + e$ degrees in x, we shall have

$$\frac{\phi x}{fx} = \frac{1}{Q_1 -} \frac{1}{q_2 -} \frac{1}{q_3 -} \cdots \frac{1}{q_n},$$

where Q_1 may be supposed to be a function of x of the degree e, and $q_2, q_3, \ldots q_n$, are all linear functions of x; the total number of the quotients $Q_1, q_2, \ldots q_n$ being of course n when the process of continued division is supposed to be carried out until the last residue is zero. Upon this supposition the last but one residue is a constant, the preceding one a function of x of the first degree, the one preceding that a function of x of the second degree, and so on.

Let us call the residue of the degree ι in x, ϑ_ι; it will readily be seen that the successive complete residues arranged in an ascending order will be

$$\vartheta_0,\ \vartheta_0 q_n,\ \vartheta_0 (q_{n-1}q_n - 1),\quad \vartheta_0 (q_{n-2}q_{n-1}q_n - q_{n-2} - q_n),\ \&c.,$$

being in the ratios

$$1,\ q_n,\ q_{n-1} - \frac{1}{q_n},\ q_{n-2} - \frac{1}{q_{n-1} -} \frac{1}{q_n},\ \&c.$$

Again, we shall have in general

$$\Lambda_\iota f - L_\iota \phi = \vartheta_\iota, \tag{15}$$

Λ_ι being an integral function of x of the degree $n - \iota - 1$, and L_ι an integral function of x of the degree $(n + e) - \iota - 1$; and it is easy to see that the successive convergents to the continued fraction

$$\frac{1}{Q_1 -} \frac{1}{q_2 -} \frac{1}{q_3 -} \&c.$$

have their respective numerators and denominators identical with those of the fractions

$$\frac{\Lambda_{n-1}}{L_{n-1}},\ \frac{\Lambda_{n-2}}{L_{n-2}},\ \frac{\Lambda_{n-3}}{L_{n-3}},\ \&c.$$

Adopting the language which I have frequently employed elsewhere, I call ϑ_ι a syzygetic function, or more briefly a conjunctive of f and ϕ, and Λ_ι and L_ι may be termed the syzygetic factors to ϑ_ι so considered. If we divide each term of the equation (15) by the allotrious factor (M), we have

$$\frac{\Lambda_\iota}{M}f - \frac{L_\iota}{M}\phi = R_\iota,$$

where R_ι is the ιth simplified residue to (f, ϕ); and if we call $\frac{\Lambda_\iota}{M} = \tau_\iota$, and $\frac{L_\iota}{M} = t_\iota$, so as to obtain the equation

$$\tau_\iota f - t_\iota \phi = R_\iota, \tag{16}$$

we see that $\frac{\tau_\iota}{t_\iota}$, the fraction formed by the component factors to any simplified residue of (f, ϕ), will be identical in value (although no longer in its separate terms) with one of the corresponding convergents to $\frac{\phi}{f}$, exhibited under the form of an improper continued fraction. I shall in the next section show how, not only the successive simplified residues, but also the component syzygetic factors of each of them, and consequently the successive convergents, may be expressed in terms of the roots of the two given functions.

Since the preceding section was composed the valuable memoir of the lamented Jacobi, entitled "De Eliminatione Variabilis è duabus Equationibus Algebraicis," *Crelle*, Vol. XVI., has fallen under my notice. That memoir is restricted to the consideration of two equations of the same degree, and the principal results in this section as regards the Bezoutic square and the allotrious factors applicable to that case will be found contained therein. The mode of treatment however is sufficiently dissimilar to justify this section being preserved unaltered under its original form.

Section II.

On the general solution in terms of the roots, of any two given algebraical functions of x, of the syzygetic equation, which connects them with a third function, whose degree in x is given, but whose form is to be determined.

Art. 14. Let f and ϕ be two given functions in x of the degrees m and n respectively in x, and for the sake of greater simplicity let the coefficients of the highest power of x in f and ϕ be each taken unity, and let it be proposed to solve the syzygetic equation

$$\tau_\iota f - t_\iota \phi + \vartheta_\iota = 0, \tag{17}$$

where ϑ_ι is given only in the number of its dimensions in x, which I suppose to be ι; but the forms of τ_ι, t_ι, ϑ_ι are all to be determined in terms of $h_1, h_2 \dots h_m$ the roots of f and $\eta_1, \eta_2 \dots \eta_n$ the roots of ϕ.

I shall begin with finding ϑ_ι; and before giving a more general representation of ϑ_ι, I propose now to demonstrate that we may make

$$\vartheta_\iota = \Sigma \{P_{q_1, q_2 \dots q_\iota} \times (x - h_{q_1})(x - h_{q_2}) \dots (x - h_{q_\iota})\}, \tag{18}$$

where $P_{q_1, q_2 \dots q_\iota}$ is used to denote

$$\left\{\begin{array}{l} (h_{q_{\iota+1}} - \eta_1)(h_{q_{\iota+1}} - \eta_2) \dots (h_{q_{\iota+1}} - \eta_n) \\ \times (h_{q_{\iota+2}} - \eta_1)(h_{q_{\iota+2}} - \eta_2) \dots (h_{q_{\iota+2}} - \eta_n) \\ \times (h_{q_{\iota+3}} - \eta_1)(h_{q_{\iota+3}} - \eta_2) \dots (h_{q_{\iota+3}} - \eta_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \times (h_{q_m} - \eta_1)(h_{q_m} - \eta_2) \dots\dots (h_{q_m} - \eta_n) \end{array}\right\} R(h_{q_1}, h_{q_2} \dots h_{q_\iota}),$$

$R(h_{q_1}, h_{q_2} \dots h_{q_\iota})$ denoting any rational symmetrical function whatever of the quantities preceded by the symbol R, and $q_1, q_2 \dots q_\iota, q_{\iota+1} \dots q_m$ being any permutation of the m indices $1, 2 \dots m$.

Suppose $f = 0$ and $\phi = 0$, then x is equal to one of the series of roots

$$h_1, h_2 \dots h_m,$$

and also to one of the series of roots

$$\eta_1, \eta_2 \dots \eta_n.$$

Suppose then that

$$x = h_\alpha = \eta_\omega,$$

and consider any term of ϑ_ι.

If in any such term α is found in the series $q_1, q_2 \dots q_\iota$, then

$$(x - h_{q_1})(x - h_{q_2}) \dots (x - h_{q_\iota}) = 0.$$

But if not, then x must be found in the complementary series

$$h_{q_{\iota+1}}, h_{q_{\iota+2}} \dots h_{q_m},$$

and consequently $P_{q_1, q_2 \dots q_\iota}$ will contain a factor $h_\alpha - \eta_\omega$ and $P_{q_1, q_2 \dots q_\iota} = 0$; in every case therefore

$$P_{q_1, q_2 \dots q_\iota} \times (x - h_{q_1})(x - h_{q_2}) \dots (x - h_{q_\iota}) = 0.$$

Therefore ϑ_ι as expressed in equation (18) is a syzygetic function of f and ϕ; and we have found a function of the ιth degree in x, and of course expressible by calculating the symmetric functions as a function only of x and of the coefficients of f and ϕ, which will satisfy the equation

$$\tau_\iota f - t_\iota \phi + \vartheta_\iota = 0.$$

It will be remembered that by virtue of Art. 2 we know *à priori* that all the values of ϑ_ι satisfying this equation are identical, save as to an allotrious factor, which is a function only of the coefficients in f and ϕ.

It is clear that we may interchange the h and η, m and n, and thus another representation of a value of ϑ_ι satisfying the equation (17) will be

$$\vartheta_\iota = \Sigma R(\eta_{q_1}, \eta_{q_2} \ldots \eta_{q_\iota}) \times \left\{\begin{matrix} (\eta_{q_{\iota+1}} - h_1)(\eta_{q_{\iota+1}} - h_2) \ldots (\eta_{q_{\iota+1}} - h_m) \\ (\eta_{q_{\iota+2}} - h_1)(\eta_{q_{\iota+2}} - h_2) \ldots (\eta_{q_{\iota+2}} - h_m) \\ (\eta_{q_{\iota+3}} - h_1)(\eta_{q_{\iota+3}} - h_2) \ldots (\eta_{q_{\iota+3}} - h_m) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\eta_{q_n} - h_1)\ \ (\eta_{q_n} - h_2) \ldots\ \ (\eta_{q_n} - h_m) \end{matrix}\right\} (x - \eta_{q_1})(x - \eta_{q_2}) \ldots (x - \eta_{q_\iota}).$$

Art. 15. If we employ in general the condensed notation

$$\begin{bmatrix} l, m, n \ldots p \\ \lambda, \mu, \ldots \nu \end{bmatrix},$$

to denote the product of the differences resulting from the subtraction of each of the quantities $\lambda, \mu \ldots \nu$ in the lower line from all of those in the upper line $l, m, n \ldots p$, the two values above given for ϑ_ι may be written under the respective forms

$$\Sigma R(h_{q_1}, h_{q_2} \ldots h_{q_\iota}) \cdot \begin{bmatrix} h_{q_{\iota+1}}, h_{q_{\iota+2}} \ldots h_{q_m} \\ \eta_1, \quad \eta_2 \ldots \quad \eta_n \end{bmatrix} (x - h_{q_1})(x - h_{q_2}) \ldots (x - h_{q_\iota}),$$

and $$\Sigma R(\eta_{\xi_1}, \eta_{\xi_2} \ldots \eta_{\xi_\iota}) \cdot \begin{bmatrix} \eta_{\xi_{\iota+1}}, \eta_{\xi_{\iota+2}} \ldots \eta_{\xi_n} \\ h_1, \quad h_2 \ldots \quad h_m \end{bmatrix} (x - \eta_{\xi_1})(x - \eta_{\xi_2}) \ldots (x - \eta_{\xi_\iota}),$$

in each of which equations disjunctively and in some order of relation each with each

$$q_1, q_2, q_3 \ldots q_m = 1, 2, 3 \ldots m,$$

and $$\xi_1, \xi_2, \xi_3 \ldots \xi_n = 1, 2, 3 \ldots n.$$

These two forms are only the two extremities of a scale of forms all equally well adapted to express ϑ_ι; for let v and ν be any two integers so taken as to satisfy the equation

$$v + \nu = \iota,$$

and let $R(\overline{\cdots\cdots}; \overline{\cdots\cdots})$, where the dots denote any quantities whatever, be used to denote a rational function which remains unaltered in value when any two of the quantities under either of the two bars are mutually interchanged, then we may write

$$\vartheta_\iota = \Sigma \left\{\begin{matrix} R(\overline{h_{q_1}, h_{q_2} \ldots h_{q_v}}; \overline{\eta_{\xi_1}, \eta_{\xi_2} \ldots \eta_{\xi_\nu}}) \times \begin{bmatrix} h_{q_{v+1}}, h_{q_{v+2}} \ldots h_{q_m} \\ \eta_{\xi_{\nu+1}}, \eta_{\xi_{\nu+2}} \ldots \eta_{\xi_n} \end{bmatrix} \\ \times (x - h_{q_1})(x - h_{q_2}) \ldots (x - h_{q_v}) \times (x - \eta_{\xi_1})(x - \eta_{\xi_2}) \ldots (x - \eta_{\xi_\nu}) \end{matrix}\right\}. \quad (19)$$

For if, as above, we suppose $x = h_\alpha = \eta_\omega$, any term of ϑ_ι in which $q_1, q_2 \dots q_v$ comprise among them α, or in which $\xi_1, \xi_2 \dots \xi_\nu$ comprise among them ω, will vanish by virtue of the factors

$$(x - h_{q_1})(x - h_{q_2}) \dots (x - h_{q_v}) \times (x - \eta_{\xi_1})(x - \eta_{\xi_2}) \dots (x - \eta_{\xi_\nu});$$

but if neither α nor ω is so comprised, then α must be one of the terms in the complementary series $q_{v+1}, q_{v+2}, \dots q_m$, and ω one of the terms in the complementary series $\xi_{\nu+1}, \xi_{\nu+2} \dots \xi_n$, and therefore one of the quantities $h_{q_{v+1}}, h_{q_{v+2}} \dots h_{q_m}$ will equal one of the quantities $\eta_{\xi_{\nu+1}}, \eta_{\xi_{\nu+2}} \dots \eta_{\xi_n}$, and consequently the term of ϑ_ι in question will vanish by virtue of the factor $\begin{bmatrix} h_{q_{v+1}}, h_{q_{v+2}} \dots h_{q_m} \\ \eta_{\xi_{\nu+1}}, \eta_{\xi_{\nu+2}} \dots \eta_{\xi_n} \end{bmatrix}$ vanishing. In either case therefore every term included within the sign of summation vanishes when $x = h_\alpha = \eta_\omega$, that is, whenever $fx = 0$ and $\phi x = 0$. Hence ϑ_ι, as given by equation (19), will satisfy the syzygetic equation $\tau_\iota f - t_\iota \phi + \vartheta_\iota = 0$ for all values of v and ν which make $v + \nu = \iota$, and for all symmetrical forms of the function denoted by the symbol $R(\overline{\cdots\cdots}; \overline{\cdots\cdots})$.

Art. 16. I shall now proceed to show how to assign the arbitrary function whose form is denoted by this symbol in such a manner as to make ϑ_ι become identical with a simplified residue to f and ϕ. To this end I take for $R(h_{q_1}, h_{q_2} \dots h_{q_v}; \eta_{\xi_1}, \eta_{\xi_2} \dots \eta_{\xi_\nu})$ the value

$$R = \frac{\begin{bmatrix} h_{q_1}, h_{q_2} \dots h_{q_v} \\ \eta_{\xi_1}, \eta_{\xi_2} \dots \eta_{\xi_\nu} \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} & \dots h_{q_v} \\ h_{q_{v+1}}, & h_{q_{v+2}} & \dots h_{q_m} \end{bmatrix} \times \begin{bmatrix} \eta_{\xi_1}, & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \dots \eta_{\xi_n} \end{bmatrix}}; \tag{20}$$

we shall then have

$$\vartheta_\iota = \Sigma \frac{\begin{bmatrix} h_{q_1}, h_{q_2} \dots h_{q_v} \\ \eta_{\xi_1}, \eta_{\xi_2} \dots \eta_{\xi_\nu} \end{bmatrix} \times \begin{bmatrix} h_{q_{v+1}} h_{q_{v+2}} \dots h_{q_m} \\ \eta_{\xi_{\nu+1}} \eta_{\xi_{\nu+2}} \dots \eta_{\xi_n} \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} & \dots h_{q_v} \\ h_{q_{v+1}}, & h_{q_{v+2}} & \dots h_{q_m} \end{bmatrix} \times \begin{bmatrix} \eta_{\xi_1}, & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \dots \eta_{\xi_n} \end{bmatrix}}$$

$$\times \{(x - h_{q_1})(x - h_{q_2}) \dots (x - h_{q_v})\} \{(x - \eta_{\xi_1})(x - \eta_{\xi_2}) \dots (x - \eta_{\xi_\nu})\}. \tag{21}$$

I shall first show this sum of fractions is in substance an integral function of the quantities $h_1, h_2 \dots h_m; \eta_1, \eta_2 \dots \eta_n$. For greater conciseness write in general $x - h = E$, $x - \eta = H$; we have then, since

$$h - \eta = H - E, \quad h_{q_r} - h_{q_s} = E_{q_s} - E_{q_r}, \quad \eta_{\xi_r} - \eta_{\xi_s} = H_{\xi_s} - H_{\xi_r},$$

$$\vartheta_\iota = \Sigma \frac{\begin{bmatrix} H_{\xi_1}, H_{\xi_2} \dots H_{\xi_\nu} \\ E_{q_1}, E_{q_2} \dots E_{q_v} \end{bmatrix} \times \begin{bmatrix} H_{\xi_{\nu+1}}, H_{\xi_{\nu+2}} \dots H_{\xi_n} \\ E_{q_{v+1}}, E_{q_{v+2}} \dots E_{q_m} \end{bmatrix}}{\begin{bmatrix} E_{q_{v+1}}, & E_{q_{v+2}} & \dots E_{q_m} \\ E_{q_1}, & E_{q_2} & \dots E_{q_v} \end{bmatrix} \times \begin{bmatrix} H_{\xi_{\nu+1}}, & H_{\xi_{\nu+2}} & \dots H_{\xi_n} \\ H_{\xi_1}, & H_{\xi_2} & \dots H_{\xi_\nu} \end{bmatrix}} . E_{q_1} \dots E_{q_v} H_{\xi_1} \dots H_{\xi_\nu}. \tag{22}$$

On reducing the fractions contained within the sign of summation to a common denominator, ϑ_ι will take the form $\frac{N}{D \,.\, \Delta}$, where D will be the product of the $\frac{1}{2}m(m-1)$ differences of $E_1, E_2 \ldots E_m$ subtracted each from each, and Δ the corresponding product of the differences *inter se* of $H_1, H_2 \ldots H_n$. Hence, unless the sum in question is an integral function of the E's and H's it will become infinite when any two of the E series, or any two of the H series of quantities are made equal. Suppose now $E_1 = E_2$; the terms in (22) which contain $E_1 - E_2$ in the denominator will evidently group themselves into pairs of the respective forms,

$$\frac{(E_1 E_{q_3} \ldots E_{q_v}) \times (H_{\xi_1} H_{\xi_2} \ldots H_{\xi_\nu}) \times \begin{bmatrix} E_1, & E_{q_3} \ldots E_{q_v} \\ H_{\xi_1}, & H_{\xi_2} \ldots H_{\xi_\nu} \end{bmatrix} \times \begin{bmatrix} E_2, & E_{q_{v+2}} \ldots E_{q_m} \\ H_{\xi_{\nu+1}}, & H_{\xi_{\nu+2}} \ldots H_{\xi_n} \end{bmatrix}}{\begin{bmatrix} E_1, & E_{q_3} & \ldots E_{q_v} \\ E_2, & E_{q_{v+1}} & \ldots E_{q_m} \end{bmatrix} \times \begin{bmatrix} H_{\xi_1}, & H_{\xi_2} & \ldots H_{\xi_\nu} \\ H_{\xi_{\nu+1}}, & H_{\xi_{\nu+2}} & \ldots H_{\xi_n} \end{bmatrix}},$$

and

$$\frac{(E_2 E_{q_3} \ldots E_{q_v}) \times (H_{\xi_1} H_{\xi_2} \ldots H_{\xi_\nu}) \times \begin{bmatrix} E_2, & E_{q_3} \ldots E_{q_v} \\ H_{\xi_1}, & H_{\xi_2} \ldots H_{\xi_\nu} \end{bmatrix} \times \begin{bmatrix} E_1, & E_{q_{v+1}} \ldots E_{q_m} \\ H_{\xi_{\nu+1}}, & H_{\xi_{\nu+2}} \ldots H_{\xi_n} \end{bmatrix}}{\begin{bmatrix} E_2, & E_{q_3} & \ldots E_{q_v} \\ E_1, & E_{q_{v+1}} & \ldots E_{q_m} \end{bmatrix} \times \begin{bmatrix} H_{\xi_1}, & H_{\xi_2} & \ldots H_{\xi_\nu} \\ H_{\xi_{\nu+1}}, & H_{\xi_{\nu+2}} & \ldots H_{\xi_n} \end{bmatrix}};$$

the sum of this pair of terms will be of the form

$$\frac{P}{Q}\left\{\frac{E_1}{E_1 - E_2}\,\frac{\begin{bmatrix} E_1 & \\ H_{\xi_1}, & H_{\xi_2} \ldots H_{\xi_\nu} \end{bmatrix} \times \begin{bmatrix} E_2 & \\ H_{\xi_{\nu+1}}, & H_{\xi_{\nu+2}} \ldots H_{\xi_n} \end{bmatrix}}{\begin{bmatrix} E_1 & \\ E_{q_{v+1}}, & E_{q_{v+2}} \ldots E_{q_m} \end{bmatrix}}\right\}$$

$$+\frac{P}{Q}\left\{\frac{E_2}{E_2 - E_1}\,\frac{\begin{bmatrix} E_2 & \\ H_{\xi_1}, & H_{\xi_2} \ldots H_{\xi_\nu} \end{bmatrix} \times \begin{bmatrix} E_1 & \\ H_{\xi_{\nu+1}}, & H_{\xi_{\nu+2}} \ldots H_{\xi_n} \end{bmatrix}}{\begin{bmatrix} E_2 & \\ E_{q_{v+1}}, & E_{q_{v+2}} \ldots E_{q_m} \end{bmatrix}}\right\},$$

where Q, it may be observed, does not contain $H_1 - H_2$, so that $\frac{P}{Q}$ remains finite when $H_1 = H_2$.

The above pair of terms together make up a sum of the form

$$\frac{P}{Q}\,\frac{1}{E_1 - E_2}\,\frac{\phi(E_1, E_2)\,\psi E_2 - \phi(E_2, E_1)\,\psi E_1}{\psi E_1 \times \psi E_2},$$

which, as the numerator of the third factor vanishes when $E_1 = E_2$, remains finite on that supposition. Hence the whole sum of terms in (22) which is made up of such pairs of terms, and of other terms in which $E_1 - E_2$ does not enter, remains finite when $E_1 - E_2 = 0$, and therefore generally when $D = 0$, and similarly when $H_1 - H_2 = 0$, and therefore also when $\Delta = 0$; hence the expression for ϑ_ι in (22) is an integral function of the E and H series of quantities, as was to be proved.

Art. 17. Let us now proceed to determine the dimensions of the coefficient of x^ι, the highest power of x in this value of ϑ_ι, when supposed to be expressed under the form of an integral function (as it has been proved to be capable of being expressed) of $h_1, h_2 \dots h_m$; $\eta_1, \eta_2 \dots \eta_n$; x.

This coefficient is the sum of fractions the numerators of each of which consist of two factors, which are respectively of $v \times \nu$ and of $(m - v) \times (n - \nu)$ dimensions in respect of the two sets of roots taken conjointly, and the denominators of two factors respectively of $v(m - v)$ and $\nu(n - \nu)$ dimensions in respect of the same.

Consequently, the exponent of the total dimensions of the coefficient in question

$$= v\nu + (m - v)(n - \nu) - v(m - v) - \nu(n - \nu)$$

$$= (m - v - \nu) \times (n - v - \nu)$$

$$= (m - \iota)(n - \iota),$$

and thus is seen to depend only on the degree ι in x of ϑ_ι, and not upon the mode of partitioning ι into two parts v and ν, for the purpose of representing ϑ_ι, by means of formula (19).

Art. 18. I shall now demonstrate that every form in this scale (to a numerical factor *près*) is identical with a simplified residue to f, ϕ, of the same degree ι in x. Any such simplified residue is, like ϑ_ι, a syzygetic function, or to use a briefer form of speech a conjunctive of f, ϕ; and if we agree to understand by the "weight" of any function of the coefficients of f and ϕ its joint dimensions in respect of the roots of f and ϕ combined, I shall prove,—first, that any simplified residue of f and ϕ of a given degree in x is that conjunctive, whose weight in respect of the roots of f and ϕ is less than the weight of any other such conjunctive; and second, that ϑ_ι, as determined above (in equation 22), is of the same weight as the simplified residue, and can therefore only differ from it by some numerical factor. For the purpose of comparison of weights, it will of course be sufficient to confine our attention to the coefficients of the highest power in x (or any other, the same for each) of the forms whose weights are to be compared.

Suppose f to be of m dimensions, and ϕ to be of n dimensions in x; and let $m = n + e$.

Suppose $$\Lambda f + L\phi = Ax^{\iota} + Bx^{\iota-1} + \ldots + K, \tag{23}$$
$$\Lambda = \lambda_0 x^q + \lambda_1 x^{q-1} + \ldots + \lambda_q,$$
$$L = l_0 x^{q+e} + l_1 x^{q+e-1} + \ldots + l_{q+e},$$
the number of homogeneous equations to be satisfied by the $q+1$ quantities $\lambda_0, \lambda_1 \ldots \lambda_q$, and the $q+e+1$ quantities $l_0, l_1 \ldots l_{q+e}$ will be $m+q-\iota$, and therefore $q+1$ and $q+e+1$ taken together must be not less than $m+q-\iota+1$, that is $2q+e+2$ must be not less than $q+m-\iota+1$, that is q not less than $m-\iota-e-1$; and if this inequality be satisfied $2q+e+2-(q+m-\iota+1)+1$, that is $q+\iota+e-m+2$ will be the number of arbitrary constants entering into the solution of equation (23).

If q be greater than $(n-1)$, let $q=(n-1)+t$; and let
$$(\Lambda) = \lambda_0 x^{n-1} + \lambda_1 x^{n-2} + \ldots + \lambda_{n-1},$$
$$(L) = l_0 x^{n+e-1} + l_1 x^{n+e-2} + \ldots + l_{e+n-1};$$
and let (Λ), (L) be so taken as to satisfy the equation
$$(\Lambda) f + (L)\,\phi = Ax^{\iota} + Bx^{\iota-1} + \ldots + K;$$
and make $$\Xi = (\Lambda) + (f + gx + \ldots + hx^{t-1})\,\phi,$$
$$X = (L) - (f + gx + \ldots + hx^{t-1}) f,$$
$f, g \ldots h$ being arbitrary constants; then
$$\Xi f + X\phi = (\Lambda) f + (L)\,\phi = Ax^{\iota} + Bx^{\iota-1} + \ldots + K.$$

Now the total number of arbitrary constants in the system (Λ) and (L) will be $n-1+\iota+e-m+2$, that is $\iota+1$; hence the total number of arbitrary constants in Ξ and X will be $\iota+1+t$, that is $q-n+\iota+2$, which is equal to $q+\iota+e-m+2$, the number of arbitrary constants in the most general values of Λ and L. Hence $\{\Lambda = \Xi,\ L = X\}$ is the general solution of the equation $\Lambda f + L\phi = Ax^{\iota} + Bx^{\iota-1} + \ldots + K$; and consequently the most general form of $Ax^{\iota} + Bx^{\iota-1} + \ldots + K$, which is evidently independent of the (t) arbitrary quantities $f, g \ldots h$, will contain the same number of arbitrary constants as enter into the system (Λ) and (L), that is $\iota+1$.

Art. 19. Let us now begin with the case of greater simplicity when $m=n$, that is $e=0$; and let us revert to the system of equations marked (10) in Section I., in which U and V are to be replaced by f and ϕ.

First, let $\iota = n-1$, then $\iota+1$, the number of arbitrary quantities in the conjunctive, is n.

From the system of equations (10) we have, for all values of $\rho_1, \rho_2, \rho_3 \ldots \rho_n$,
$$(\rho_1 Q_0 + \rho_2 Q_1 + \ldots + \rho_n Q_{n-1}) f$$
$$-(\rho_1 P_0 + \rho_2 P_1 + \ldots + \rho_n P_{n-1})\,\phi$$
$$= (\rho_1 K_1 + \rho_2\, {}_1K_1 + \ldots + \rho_n\, {}_{n-1}K_1)\, x^{n-1} + \&\text{c.},$$

and consequently the most *general* value of ϑ_{n-1} in the equation

$$\tau_{n-1}f - t_{n-1}\phi + \vartheta_{n-1} = 0,$$

where $$\vartheta_{n-1} = Ax^{n-1} + Bx^{n-2} + \ldots + L,$$

will be obtained by making

$$\tau_{n-1} = \rho_1 Q_0 + \rho_2 Q_1 + \ldots + \rho_n Q_{n-1},$$

$$t_{n-1} = -\rho_1 P_0 - \rho_2 P_1 \ldots - \rho_n P_{n-1},$$

which solution contains n, that is the proper number of arbitrary constants.

Again, if $\iota = n-2$, $\iota + 1 = n-1$, which will therefore be the number of arbitrary constants in the most general value of ϑ_{n-2} in the equation

$$\tau_{n-2}f - t_{n-2}\phi + \vartheta_{n-2} = 0.$$

This most general value of ϑ_{n-2} is therefore found by making

$$\tau_{n-2} = \rho'_1 Q_0 + \rho'_2 Q_1 + \ldots + \rho'_n Q_{n-1},$$

$$t_{n-1} = -\rho'_1 P_0 - \rho'_2 P_1 \ldots - \rho'_n P_{n-1},$$

where ρ'_1, $\rho'_2 \ldots \rho'_n$ are no longer entirely independent, but subject to the equation

$$\rho'_1 K_1 + \rho'_{2\,1}K_1 + \ldots + \rho'_{n\,n-1}K_1 = 0,$$

so as to leave $(n-1)$ constants arbitrary.

We thus obtain $\vartheta_{n-2} = (\rho'_1 K_2 + \rho'_{2\,1}K_2 + \ldots + \rho'_{n\,n-1}K_2)\,x^{n-2} + \&c.$ In like manner, and for the same reasons, the most general values of ϑ_{n-3} in the equation

$$\tau_{n-3}f - t_{n-3}\phi + \vartheta_{n-3} = 0,$$

will be found by making

$$\tau_{n-3} = \rho''_1\, Q_0 + \rho''_2\, Q_1 + \ldots + \rho''_n\, Q_{n-1},$$

$$t_{n-3} = -\rho''_1\, P_0 - \rho''_2\, P_1 \ldots - \rho''_n\, P_{n-1},$$

where ρ''_1, $\rho''_2 \ldots \rho''_n$ are subject to satisfying the two equations

$$\rho''_1\, K_1 + \rho''_{2\,1}K_1 + \ldots + \rho''_{n\,n-1}K_1 = 0,$$

$$\rho''_1\, K_2 + \rho''_{2\,1}K_2 + \ldots + \rho''_{n\,n-1}K_2 = 0,$$

so as to leave $(n-2)$ constants arbitrary; and we thus obtain

$$\vartheta_{n-3} = (\rho''_1\, K_3 + \rho''_{2\,1}K_3 + \ldots + \rho''_{n\,n-1}K_3)\, x^{n-3} + \&c.,$$

and so on, the number of independent arbitrary constants in ϑ decreasing (as it ought) each time by one unit as the degree of ϑ descends, until finally, if $\tau_0 f - t_0\phi + \vartheta_0 = 0$, ϑ_0 being a constant, the general value for ϑ_0 is found by making

$$\tau_0 = (\rho_1)\, Q_0 + (\rho_2)\, Q_1 + \ldots + (\rho_n)\, Q_{n-1},$$

$$t_0 = -(\rho_1)\, P_0 - (\rho_2)\, P_1 - \ldots - (\rho_n)\, P_{n-1},$$

where (ρ_1), (ρ_2) ... (ρ_n) are subject to satisfy the $(n-1)$ equations

$$(\rho_1)\,K_1 + \&c. = 0,$$
$$(\rho_1)\,K_2 + \&c. = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(\rho_1)\,K_{n-1} + \&c. = 0,$$

which gives $$\vartheta_0 = K_n\,(\rho)_1 + {}_1K_n\,(\rho)_2 + \ldots + {}_{n-1}K_n\,(\rho)_n.$$

Now evidently the lowest weight in respect to the roots of U and V that can be given to $(\rho_1K_1 + \rho_{2\,1}K_1 + \ldots + \rho_{n\;n-1}K_1)\,x^{n-1} + \&c.$, when the multipliers ρ_1, ρ_2 ... ρ_n are absolutely independent, is found by taking

$$\rho_1 = 1,\;\; \rho_2 = 0,\;\; \rho_3 = 0 \ldots \rho_n = 0,$$

which makes the weight of the leading coefficient in ϑ_{n-1}, the same as that of K_1, that is 1.

Again, when one equation,

$$\rho'_1\,K_1 + \rho'_{2\,1}K_1 + \ldots + \rho'_{n\;n-1}K_1 = 0,$$

exists between the (ρ)'s, the lowest weight will be found by making

$$\rho'_1 = {}_1K_1,\quad \rho'_2 = -\,K_1,\quad \rho'_3 = 0,\quad \rho'_4 = 0 \ldots \rho'_n = 0,$$

which makes the weight of the leading coefficient in ϑ_{n-2} depend on

$${}_1K_1\,K_2 - K_1\,{}_1K_2,$$

which is of the weight $1+3$, that is 4, in respect of the roots of f and ϕ.

Similarly, ϑ_{n-3} will have its lowest weight when its leading coefficient is the determinant

$$\begin{vmatrix} K_1, & K_2, & K_3 \\ {}_1K_1, & {}_1K_2, & {}_1K_3 \\ {}_2K_1, & {}_2K_2, & {}_2K_3 \end{vmatrix},$$

the weight of which is $1+3+5=9$; and finally, the lowest weighted value of ϑ_0 is the determinant represented by the complete Bezoutian square; the weight in general of ϑ_{n-i} being $1+3+\ldots+(2i-1)$, that is i^2, or which is the same thing otherwise expressed, the weight of the leading coefficient of the lowest-weighted conjunctive of f and ϕ of the degree ι in x is $(n-\iota)(m-\iota)$*. It will of course have been seen in the foregoing demonstration, that the weight of ${}_rK_s$ [which means $\Sigma\,(a_rb_s - a_sb_r)$, a_r, a_s being the coefficients of x^{n-r}, x^{n-s} in f, and b_r, b_s of the same in ϕ] has been correctly taken to be $r+s$ in respect of the roots of f and ϕ conjoined.

* n and m are supposed equal and $\iota = n - i$.

Art. 20. If now we proceed in like manner with the general case of $m = n + e$, it may be shown, in precisely the same way as in the preceding article, that the most general value of any conjunctive of f and ϕ will be a linear function of e functions,

$$\begin{array}{l} x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n, \\ x^{n+1} + a_1x^n + a_2x^{n-1} + \dots + a_nx, \\ x^{n+2} + a_1x^{n+1} + a_2x^n + \dots + a_nx^2, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ x^{m-1} + a_1x^{m-2} + \&c. \qquad + a_nx^{e-1}, \end{array}$$

and of the n functions,

$$\begin{array}{l} K_1x^{n-1} + K_2x^{n-2} + \dots + K_n, \\ {}_1K_1x^{n-1} + {}_1K_2x^{n-2} + \dots + {}_1K_n, \\ \quad \&c. \qquad \&c. \\ {}_{n-1}K_1x^{n-1} + {}_{n-1}K_2x^{n-2} + \dots + {}_{n-1}K_n, \end{array}$$

and that consequently, if the degree of such conjunctive in x be $(n - i)$, it will be of the lowest weight when it is a linear function of the entire e upper set of functions, and i of the lower set; and consequently, the coefficient of the highest power of x in such conjunctive will be the determinant

$$\left|\begin{array}{ccccc} K_1, & K_2, & K_3 \dots\dots K_i \dots\dots\dots & K_{i+e} \\ {}_1K_1, & {}_1K_2, & {}_1K_3 \dots\dots {}_1K_i \dots\dots\dots & {}_1K_{i+e} \\ {}_2K_1, & {}_2K_2, & {}_2K_3 \dots\dots {}_2K_i \dots\dots\dots & {}_2K_{i+e} \\ \dots\dots & \dots\dots & \dots\dots\dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots\dots\dots & \dots\dots \\ {}_{i-1}K_1, & {}_{i-1}K_2, & {}_{i-1}K_3 \dots {}_{i-1}K_i \dots\dots & {}_{i-1}K_{i+e} \\ 1, & a_1, & a_2 \dots\dots a_{i-1}, & a_i \dots a_{i+e-1} \\ & 1, & a_1 \dots\dots a_{i-1}, & a_i \dots a_{i+e-2} \\ & & 1 \dots\dots a_{i-2}, & a_i \dots a_{i+e-3} \\ \dots\dots & \dots\dots & \dots\dots\dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots\dots\dots & \dots\dots \\ & & & 1 \dots\dots\dots\dots a_i \end{array}\right|,$$

the weight of which is evidently that of

$$K_1 \times {}_1K_2 \times {}_2K_3 \dots \times {}_{i-1}K_i \times (a_i)^e,$$

that is
$$1 + 3 + 5 + \dots + (2i - 1) + ei,$$

that is $i^2 + ei$, or $i(e + i)$, which is $(n - \iota)(m - \iota)$ if $\iota = n - i$.

Hence the weight of the leading coefficient in the lowest-weighted conjunctive of f and ϕ of the degree ι in x is $(m-\iota)(n-\iota)$, m being the degree of f and n of ϕ.

From this we infer that any conjunctive of f and ϕ of the degree ι, of which the leading coefficient is of the weight $(m-\iota)(n-\iota)$, all the coefficients being of course understood to be integral functions of the roots of f and ϕ, must, to a numerical factor *près*, be equivalent to any other of the same weight; and furthermore, any supposed function of x of the ιth degree which possesses the property characteristic of a conjunctive of vanishing when f and ϕ vanish simultaneously, but of which the weight of the leading coefficient would be *less* than $(m-\iota)(n-\iota)$, must be a mere nugatory form and have all its terms *identically zero**.

Art. 21. We have previously shown, Art. 16, that ϑ, as defined by equation (21), is an integral function of the roots f and ϕ, and vanishes when f and ϕ vanish. Moreover, its weight in the roots has been proved to be $(m-\iota)(n-\iota)$, and consequently, if by way of distinguishing the several forms of ϑ_ι we name that one where ι in the equation above cited is supposed to be divided into two parts, v and ν, $\vartheta_{v,\nu}$, we have for all values of v and ν, such that $v+\nu$ is not greater than n, $\vartheta_{v,\nu}$ to a constant numerical factor *près* identical with the $(v+\nu)$th simplified residue to (f, ϕ), so that the form of $\vartheta_{v,\nu}$ depends only upon the value of $v+\nu$.

Art. 22. It must be well borne in mind that this permanency of the value of $\vartheta_{v,\iota-v}$ for different values of v has only been established for the case where ι can be the degree of a residue to f and ϕ, that is to say, when ι is less than the lesser of the two indices m and n. When ι does not satisfy this condition of inequality, the theorem ceases to be true. It is clear that when $m=n$ and $v+\nu=m=n$, $\vartheta_{v,\nu}$, which always remains a conjunctive of f and ϕ, can only be a numerical linear function of f and ϕ; and I have ascertained when $m=n$ on giving to v and ν the respective values successively $(0, n)$, $(1, n-1)$, $(2, (n-2)) \dots (n, 0)$ that

$$\vartheta_{0,n}=f;\quad \vartheta_{1,n-1}=(n-1)f+\phi;\quad \vartheta_{2,n-2}=\frac{(n-1)(n-2)}{1.2}f+(n-1)\phi;\dots$$
$$\vartheta_{n-1,1}=f+(n-1)\phi;\quad \vartheta_{n,0}=\phi.$$

Thus, by way of a simple example, let

$$f=x^2+ax+b=(x-h_1)(x-h_2),$$
$$\phi=x^2+\alpha x+\beta=(x-k_1)(x-k_2),$$

* And more generally it admits of being demonstrated by precisely the same course of reasoning, that the number of arbitrary parameters in a conjunctive of the degree ι, and of the weight $(m-\iota)(n-\iota)+\epsilon$ in the roots, cannot (abstraction being supposed to be made of an arbitrary numerical multiplier) exceed the number ϵ.

$$\vartheta_{0,2} = (x-h_1)(x-h_2)\left\{\frac{\begin{bmatrix} h_1h_2 \\ \cdots \end{bmatrix} \times \begin{bmatrix} \cdots \\ k_1k_2 \end{bmatrix}}{\begin{bmatrix} h_1h_2 \\ \cdots \end{bmatrix} \times \begin{bmatrix} \cdots \\ k_1k_2 \end{bmatrix}}\right\} = (x-h_1)(x-h_2) = f,$$

$$\vartheta_{1,1} = \Sigma\,(x-h_1)(x-k_1)\frac{\begin{bmatrix} h_1 \\ k_1 \end{bmatrix} \times \begin{bmatrix} h_2 \\ k_2 \end{bmatrix}}{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \times \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$$

$$= \Sigma\frac{x-h_1}{h_1-h_2}\Sigma\frac{x-k_1}{k_1-k_2}\{(h_1-k_1)(h_2-k_2)\},$$

that is

$$= \Sigma\frac{x-h_1}{h_1-h_2}\left\{\frac{1}{k_1-k_2}\begin{pmatrix}(x-k_1)(h_1-k_1)(h_2-k_2) \\ -(x-k_2)(h_1-k_2)(h_2-k_1)\end{pmatrix}\right\}$$

$$= \Sigma\frac{x-h_1}{h_1-h_2}\{(h_1-h_2)x + [(k_1+k_2)h_2 - (h_1h_2+k_1k_2)]\}$$

$$= (x-h_1)x + (x-h_2)x - (k_1+k_2)x + (h_1h_2+k_1k_2)$$

$$= \{x^2 - (h_1+h_2)x + h_1h_2\} + \{x^2 - (k_1+k_2)x + k_1k_2\}$$

$$= (x^2+ax+b) + (x^2+\alpha x+\beta)$$

$$= f + \phi;$$

so we find also $\vartheta_{2,0} = \phi$.

Art. 23. The expression $\vartheta_{v,\nu}$, which is universally a conjunctive of f and ϕ, continues algebraically interpretable so long as $v+\nu$ has any value intermediate between 0 and $m+n$; when $v+\nu=0$ we must of course have $v=0$ and $\nu=0$, and $\vartheta_{0,0}$ becomes the resultant of f and ϕ; when $v+\nu=m+n$ we must also have the unique solution $v=m$ and $\nu=n$, and $\vartheta_{m,n}$ becomes necessarily $f\times\phi$, which we thus see stands in a sort of antithetical relation to the resultant of f and ϕ, say (f, ϕ). Nor is it without interest to remark that $f\times\phi=0$ implies that a factor of f or else of ϕ is zero; and $(f, \phi)=0$ implies that if a factor of the one of the functions is zero, so also is a factor of the other, that is that a factor of each or of neither is zero. As ι increases from 0 to n or decreases from $m+n$ to $m-1$, the number of solutions of the equation $v+\nu=\iota$ in the one case, and the number of admissible solutions of the equation $v+\nu=\iota$ in the other case, which is subject to the condition that ν must not exceed n, continues to increase by a unit at each step; there being thus $n+1$ different forms $\vartheta_{v,\nu}$ when $v+\nu=n$, and the same number when $v+\nu=m-1$. For all values of ι intermediate between n and $(m-1)$ (both taken exclusively) it is very remarkable that $\vartheta_{v,\nu}$ will vanish, as I proceed to demonstrate.

Art. 24. The weight of the coefficient of the highest power of $\vartheta_{v,\nu}$ ($v+\nu$ being equal to ι) is $(m-\iota)(n-\iota)$, and consequently, when ι is greater than n, and less than m, $\vartheta_{v,\nu}$ would contain fractional functions of the roots of f and ϕ, if there were in it a power x^ι, but $\vartheta_{v,\nu}$ has been proved to be always an integer function of the roots. Hence the coefficient of x^ι will be zero, and so more generally the first power of x in $\vartheta_{v,\nu}$, of which the coefficient is not zero, will be $x^{\iota-\omega}$, subject to the condition (since evidently the weight of the several coefficients goes on increasing by units as the degree of the terms in x decreases by the same) that ω be not less than $(m-\iota)(\iota-n)$; let then $\omega=(m-\iota)(\iota-n)$, $\vartheta_{v,\nu}$ becomes of the form $Ax^{\iota-\omega}+Bx^{\iota-\omega-1}+\&c.$, where A is of zero dimensions; but this is impossible if $\iota-\omega<n$, for then $Ax^{\iota-\omega}+\&c.$ is a conjunctive of weight lower than the lowest-weighted simplified residue of the degree $\iota-\omega$. Hence ω is not greater than $\iota-n$, that is $(m-\iota)(\iota-n)$ is not greater than $\iota-n$, that is $m-\iota$ cannot be greater than 1, that is ι when intermediate between m and n cannot be less than $m-1$, otherwise $\vartheta_{v,\nu}$ will vanish identically. Moreover, when $\iota=m-1$, $\omega=\iota-n$, and $\iota-\omega=n$, and accordingly $\vartheta_{v,m-1-v}$ is not merely, as we might know, *à priori* an algebraical, but more simply a numerical multiple of ϕ for all values of v. The same is of course true also, m being greater than n, for every form $\vartheta_{v,n-v}$, since this is always a conjunctive of f and ϕ, of which the former is of a degree higher than the ϑ in question, so that the multiplier of f in this conjunctive must be zero*.

Art. 25. To enter into a further or more detailed examination of the values assumed by $\vartheta_{v,\nu}$ for the most general values of m, n, ι, would be to transcend the limits I have proposed to myself in drawing up the present memoir. What we have established is, that to every form of $\vartheta_{v,\iota-v}$ appertaining to a value of ι between 0 and n, there is a sort of conjugate form for which ι lies between $m+n$ and m; that for $\iota=m-1$ or $\iota=n$, $\vartheta_{v,\iota-v}$ becomes a numerical multiplier of ϕ; and that when ι lies in the intermediate region between n and $m-1$, $\vartheta_{v,\iota-v}$ vanishes for all values of v. I pause only for a moment to put together for the purpose of comparison the forms corresponding to ι and to $m+n-\iota$. By Art. 16, making $\iota=v+\nu$,

$$\vartheta_\iota=\Sigma\,(x-h_{q_1})(x-h_{q_2})\dots(x-h_{q_v})\times(x-\eta_{\xi_1})(x-\eta_{\xi_2})\dots(x-\eta_{\xi_\nu})$$

$$\times\frac{\begin{bmatrix}h_{q_1}, & h_{q_2} & \dots & h_{q_v}\\ \eta_{\xi_1}, & \eta_{\xi_2} & \dots & \eta_{\xi_\nu}\end{bmatrix}\times\begin{bmatrix}h_{q_{v+1}}, & h_{q_{v+2}} & \dots & h_{q_m}\\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \dots & \eta_{\xi_n}\end{bmatrix}}{\begin{bmatrix}h_{q_1}, & h_{q_2} & \dots & h_{q_v}\\ h_{q_{v+1}}, & h_{q_{v+2}} & \dots & h_{q_m}\end{bmatrix}\times\begin{bmatrix}\eta_{\xi_1}, & \eta_{\xi_2} & \dots & \eta_{\xi_\nu}\\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \dots & \eta_{\xi_n}\end{bmatrix}}.$$

* It thus appears that if the indices m and n do not differ by at least 3 units, ϑ will have an actual quantitative existence for all values of ι between 0 and $m+n$; or in other words, the failure in the quantitative existence of the forms ϑ_ι only begins to show itself when this difference is 3; thus if $m=n+3$, ϑ_n exists, and ϑ_{n+2} exists, but $\vartheta_{n+1}=0$.

The conjugate form for which $\iota' = m + n - \iota$ and $m - v$, $n - \nu$, $v\nu$ take the places of v, ν and $(m-v)(n-\nu)$, will be got by taking

$$\vartheta_{\iota'} = \Sigma\,(x - h_{q_{v+1}})(x - h_{q_{v+2}}) \ldots (x - h_{q_m}) \times (x - \eta_{\xi_{\nu+1}})(x - \eta_{\xi_{\nu+2}}) \ldots (x - \eta_{\xi_n})$$

$$\times \frac{\begin{bmatrix} h_{q_1}, & h_{q_2} & \ldots & h_{q_v} \\ \eta_{\xi_1}, & \eta_{\xi_2} & \ldots & \eta_{\xi_\nu} \end{bmatrix} \times \begin{bmatrix} h_{q_{v+1}}, & h_{q_{v+2}} & \ldots & h_{q_m} \\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \ldots & \eta_{\xi_n} \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} & \ldots & h_{q_v} \\ h_{q_{v+1}}, & h_{q_{v+2}} & \ldots & h_{q_m} \end{bmatrix} \times \begin{bmatrix} \eta_{\xi_1}, & \eta_{\xi_2} & \ldots & \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \ldots & \eta_{\xi_n} \end{bmatrix}},$$

which it will be perceived are identical, term for term, in the fractional constant factor, and differ only in the linear functions of x, which in ϑ_ι and in $\vartheta_{\iota'}$ are complementary to one another. Our proper business is only with those forms for which $\iota < n$.

Art. 26. It will presently be seen to be necessary to ascertain the numerical relations between $\vartheta_{0,\iota}$ and $\vartheta_{\iota,0}$ when $\iota < n$, and this naturally brings under our notice the inquiry into the numerical relations which exist between the entire series of forms $\vartheta_{v,\iota-v}$ for a given value of ι, corresponding to all values of v between 0 and ι inclusive.

In order to avoid a somewhat oppressive complication of symbols, I shall take a particular numerical example, that is $m = 7$, $n = 6$, $\iota = 4$, and compare the values of $\vartheta_{0,4}$; $\vartheta_{1,3}$; $\vartheta_{2,2}$; $\vartheta_{3,1}$; $\vartheta_{4,0}$, all of which we know to be identical [to a numerical factor *près*] with one another and with the second simplified residue to f and ϕ, that being of the fourth degree in x; our object in the subjoined investigation is to determine the numerical ratios of these several forms of ϑ to one another.

First, let $v = 0$, $\nu = 4$. The leading coefficient $\vartheta_{0,4}$ is

$$\Sigma \frac{\begin{bmatrix} \eta_5\eta_6 \\ h_1h_2h_3h_4h_5h_6h_7 \end{bmatrix}}{\begin{bmatrix} \eta_5\eta_6 \\ \eta_1\eta_2\eta_3\eta_4 \end{bmatrix}}$$

which we know *à priori* (it should be observed) to be essentially an *integral* function of the h and the η system. In this, the term containing η_6^3 will be evidently

$$\Sigma \frac{\begin{bmatrix} \eta_5 \\ h_1h_2h_3h_4h_5h_6h_7 \end{bmatrix}}{\begin{bmatrix} \eta_5 \\ \eta_1\eta_2\eta_3\eta_4 \end{bmatrix}}, \qquad \text{(A)}$$

the η system to which the latter summation relates being now reduced to consist of $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$. In this expression, again, the coefficient of η_5^3 is evidently 1. Hence, therefore, the leading coefficient in $\vartheta_{0,4}$ contains the term $\eta_5^3\eta_6^3$.

Secondly, let $v=1$, $\nu=3$. The leading coefficient in $\vartheta_{1,3}$ becomes

$$\Sigma \frac{\begin{bmatrix}\eta_1\eta_2\eta_3\\ h_1\end{bmatrix} \times \begin{bmatrix}\eta_4\eta_5\eta_6\\ h_2h_3h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_2h_3h_4h_5h_6h_7\\ h_1\end{bmatrix} \times \begin{bmatrix}\eta_4\eta_5\eta_6\\ \eta_1\eta_2\eta_3\end{bmatrix}}.$$

In this, the factor affecting $\eta_6{}^3$ will be

$$\Sigma \frac{\begin{bmatrix}\eta_1\eta_2\eta_3\\ h_1\end{bmatrix} \times \begin{bmatrix}\eta_4\eta_5\\ h_2h_3h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_2h_3h_4h_5h_6h_7\\ h_1\end{bmatrix} \times \begin{bmatrix}\eta_4\eta_5\\ \eta_1\eta_2\eta_3\end{bmatrix}},$$

η_6 being now understood to be eliminated out of the η system included within the above summation. Again, in this latter sum the factor affecting $\eta_5{}^3$ will be

$$\Sigma \frac{\begin{bmatrix}\eta_1\eta_2\eta_3\\ h_1\end{bmatrix} \times \begin{bmatrix}\eta_4\\ h_2h_3h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_2h_3h_4h_5h_6h_7\\ h_1\end{bmatrix} \times \begin{bmatrix}\eta_4\\ \eta_1\eta_2\eta_3\end{bmatrix}}, \qquad \text{(B)}$$

η_5 and η_6 being now both eliminated out of the η system. This last sum can of course only represent a numerical quantity.

So in like manner, again, if $v=2$, $\nu=2$, the coefficient of $\eta_6{}^3\eta_5{}^3$ in $\vartheta_{2,2}$ will be similarly reducible to the form

$$\Sigma \frac{\begin{bmatrix}\eta_1\eta_2\\ h_1h_2\end{bmatrix} \times \begin{bmatrix}\eta_3\eta_4\\ h_3h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_3h_4h_5h_6h_7\\ h_1h_2\end{bmatrix} \times \begin{bmatrix}\eta_3\eta_4\\ h_1h_2\end{bmatrix}}. \qquad \text{(C)}$$

So, again, when $v=3$, $\nu=1$, the coefficient of $\eta_6{}^3\eta_5{}^3$ in $\vartheta_{3,1}$ will be

$$\Sigma \frac{\begin{bmatrix}\eta_1\\ h_1h_2h_3\end{bmatrix} \times \begin{bmatrix}\eta_2\eta_3\eta_4\\ h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_4h_5h_6h_7\\ h_1h_2h_3\end{bmatrix} \times \begin{bmatrix}\eta_2\eta_3\eta_4\\ \eta_1\end{bmatrix}}; \qquad \text{(D)}$$

and finally, the coefficient of $\eta_6{}^3\eta_5{}^3$ in $\vartheta_{0,4}$ will be

$$\Sigma \frac{\begin{bmatrix}\eta_1\eta_2\eta_3\eta_4\\ h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_5h_6h_7\\ h_1h_2h_3h_4\end{bmatrix}}, \qquad \text{(E)}$$

out of all which sums it is to be remembered that η_5 and η_6 are supposed excluded from appearing. All these several coefficients being numbers in disguise, we may determine them by giving any values at pleasure to the terms in the h and η system.

Let now $\eta_1 = h_1$, $\eta_2 = h_2$, $\eta_3 = h_3$, $\eta_4 = h_4$, then in (B) it will readily be seen that all the terms included within the sign of summation vanish identically, except the following, namely,—

$$\frac{\begin{bmatrix}\eta_1\eta_2\eta_3\\ h_4\end{bmatrix}\times\begin{bmatrix}\eta_4\\ h_1h_2h_3h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_1h_2h_3h_5h_6h_7\\ h_4\end{bmatrix}\times\begin{bmatrix}\eta_4\\ \eta_1\eta_2\eta_3\end{bmatrix}},$$

$$\frac{\begin{bmatrix}\eta_1\eta_2\eta_4\\ h_3\end{bmatrix}\times\begin{bmatrix}\eta_3\\ h_1h_2h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_1h_2h_4h_5h_6h_7\\ h_3\end{bmatrix}\times\begin{bmatrix}\eta_3\\ \eta_1\eta_2\eta_4\end{bmatrix}},$$

$$\frac{\begin{bmatrix}\eta_1\eta_3\eta_4\\ h_2\end{bmatrix}\times\begin{bmatrix}\eta_2\\ h_1h_3h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_1h_3h_4h_5h_6h_7\\ h_2\end{bmatrix}\times\begin{bmatrix}\eta_2\\ \eta_1\eta_3\eta_4\end{bmatrix}},$$

$$\frac{\begin{bmatrix}\eta_2\eta_3\eta_4\\ h_1\end{bmatrix}\times\begin{bmatrix}\eta_1\\ h_2h_3h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_2h_3h_4h_5h_6h_7\\ h_1\end{bmatrix}\times\begin{bmatrix}\eta_1\\ \eta_2\eta_3\eta_4\end{bmatrix}}.$$

In each of these expressions the first factor of the numerator is identical in value (by reason of the equations $h_1 = \eta_1$, $h_2 = \eta_2$, $h_3 = \eta_3$, $h_4 = \eta_4$) with $(-)^3 \times$ the second factor of the denominator, and the second factor of the numerator with $(-)^6 \times$ the first factor of the denominator; hence the coefficient of $\eta_5{}^3\eta_6{}^3$ in $\vartheta_{1,3}$ is -4.

In like manner the only effective terms of $\vartheta_{2,2}$ will be

$$\frac{\begin{bmatrix}\eta_1\eta_2\\ h_3h_4\end{bmatrix}\times\begin{bmatrix}\eta_3\eta_4\\ h_1h_2h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_1h_2h_5h_6h_7\\ h_3h_4\end{bmatrix}\times\begin{bmatrix}\eta_3\eta_4\\ \eta_1\eta_2\end{bmatrix}},\qquad \frac{\begin{bmatrix}\eta_3\eta_4\\ h_1h_2\end{bmatrix}\times\begin{bmatrix}\eta_1\eta_2\\ h_3h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_3h_4h_5h_6h_7\\ h_1h_2\end{bmatrix}\times\begin{bmatrix}\eta_1\eta_2\\ \eta_3\eta_4\end{bmatrix}},$$

$$\frac{\begin{bmatrix}\eta_1\eta_3\\ h_2h_4\end{bmatrix}\times\begin{bmatrix}\eta_2\eta_4\\ h_1h_3h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_1h_3h_5h_6h_7\\ h_2h_4\end{bmatrix}\times\begin{bmatrix}\eta_2\eta_4\\ \eta_1\eta_3\end{bmatrix}},\qquad \frac{\begin{bmatrix}\eta_2\eta_4\\ h_1h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\eta_3\\ h_2h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_2h_4h_5h_6h_7\\ h_1h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\eta_3\\ \eta_2\eta_4\end{bmatrix}},$$

$$\frac{\begin{bmatrix}\eta_1\eta_4\\ h_2h_3\end{bmatrix}\times\begin{bmatrix}\eta_2\eta_3\\ h_1h_4h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_1h_4h_5h_6h_7\\ h_2h_3\end{bmatrix}\times\begin{bmatrix}\eta_1\eta_4\\ \eta_2\eta_3\end{bmatrix}},\qquad \frac{\begin{bmatrix}\eta_2\eta_3\\ h_1h_4\end{bmatrix}\times\begin{bmatrix}\eta_1\eta_4\\ h_2h_3h_5h_6h_7\end{bmatrix}}{\begin{bmatrix}h_2h_4h_5h_6h_7\\ h_1h_3\end{bmatrix}\times\begin{bmatrix}\eta_2\eta_3\\ \eta_1\eta_4\end{bmatrix}}.$$

Any other term will necessarily contain in the numerator a factor, whose symbolical representation will contain one of the quantities $\eta_1, \eta_2, \eta_3, \eta_4$, in the upper line, and one of the quantities h_1, h_2, h_3, h_4, having the same subscript index, in the lower line, and which will therefore vanish; the number of effective terms being evidently the number of ways in which four things can be combined 2 and 2 together, and the value of each term is evidently $(-)^{2\,2}\,(-1)^{2\,5}\,1$, so that the entire value of the coefficient of $\eta_5^3\,\eta_6^3$ in $\vartheta_{2,2}$ is $+6$.

Precisely in the same manner, we shall find that the leading coefficient in $\vartheta_{3,1}$ will contain the term $-4\eta_5^3\,\eta_6^3$, the (-1) resulting from the operation $(-1)^{1\,3}\,(-)^{3\,4}$, and in $\vartheta_{4,0}$ the term $+\eta_5^3\,\eta_6^3$, the $+1$ resulting from the operation $(-1)^{4\,3}$. Hence it appears that $\vartheta_{0,4}$; $\vartheta_{1,3}$; $\vartheta_{2,2}$; $\vartheta_{3,1}$; $\vartheta_{4,0}$ are to one another in the ratios of 1; -4; 6; -4; 1; and so in general for any values of m, n, ι (ι being less than m and less than n) it will be found that

$$\vartheta_{0,\iota},\quad \vartheta_{1,\iota-1},\quad \vartheta_{2,\iota-2}\ldots\vartheta_{\iota,0}$$

will be in the ratios of the numbers

$$1;\quad (-1)^{m-1}\,\iota;\quad (-1)^{2(m-2)}\,\iota\,\frac{\iota-1}{2};\quad (-1)^{3(m-3)}\,\iota\,\frac{\iota-1}{2}\,\frac{\iota-2}{3},\ \ldots;\quad (-1)^{\iota(m-\iota)}.$$

Art. 27. The method employed in the preceding investigation will enable us to affix the proper sign and numerical factor to $\vartheta_{0,\iota}$ or $\vartheta_{\iota,0}$, or in general to $\vartheta_{v,\iota-v}$, in order that it may represent the Bezoutian secondary of the degree ι in x. This latter has been already identified with the simplified residue obtained by expanding $\dfrac{\phi x}{fx}$ under the form of an improper continued fraction. For this purpose, it will be sufficient to compare a single term of any such ϑ with the corresponding one in the Symmorphic Bezoutian secondary. Let us first suppose that $m=n$, f and ϕ being of the same degree. A glance at the form of the Bezoutian square will show that if we form the Bezoutian secondary of the degree $(n-i)$ in x, the coefficient of its leading term will contain the term $(-)^{(i-1)\frac{i}{2}}\,(0,\,i)^i$; $(0,\,i)$ as usual denoting the product of the coefficient of x^n in f by the coefficient of x^{n-i} in ϕ, less the product of the coefficient of x^n in ϕ by that of x^{n-i} in f; and as we suppose the first coefficients in f and ϕ to be each 1, if we term the other coefficients last spoken of a_i and α_i respectively, this said coefficient of the leading term of the ith Bezoutian secondary will contain the term $(-)^{(i-1)\frac{i}{2}}\,(\alpha_i-a_i)^i$, and consequently $(-1)^{(i-1)\frac{i}{2}}\,\alpha_i^i$ and $(-)^{i\frac{i+1}{2}}\,a_i^i$.

Now by the like reasoning to that employed in the preceding article, the coefficient of the leading term in $\vartheta_{m-i,0}$, that is

$$\Sigma\,(x-h_{q_{i+1}})\,(x-h_{q_{i+2}})\ldots(x-h_{q_m})\,\frac{\begin{bmatrix} h_{q_1}, & h_{q_2} & \ldots & h_{q_i} \\ \eta_1, & \eta_2 & \ldots & \eta_m \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} & \ldots & h_{q_i} \\ h_{q_{i+1}}, & h_{q_{i+2}} & \ldots & h_{q_m} \end{bmatrix}},$$

will contain the quantity $\Sigma(h_1 h_2 h_3 \dots h_i)^i$, and therefore will contain a term $\{\Sigma(h_1 h_2 h_3 \dots h_i)\}^i$, that is $(-)^{ii} a_i^i$, which is equal to $(-)^i a_i^i$, since $(i-1)i$ is always even. Hence $\vartheta_{m-i,0} = (-)^{i\frac{i-1}{2}} \times$ the corresponding Bezoutian secondary.

Art. 28. The above applies to the case where we have supposed $m = n$. When this equality does not exist we may proceed as follows. Prefix to ϕx, the first coefficient of which is still supposed to be 1, a term ϵx^m, where ϵ is positive and indefinitely small, and let ϕx so augmented be called $\Phi(x)$. Then if $\eta_1, \eta_2 \dots \eta_n$ are the roots of ϕx, $\eta_1, \eta_2 \dots \eta_n$, together with the $(m-n)$ values of $\left(\frac{1}{\epsilon}\right)^{\frac{1}{m-n}}$, will be the roots of $\Phi(x)$.

But it has already been proved that when (as here supposed) the first coefficient of fx is 1, the Bezoutian secondaries to f and ϕ will be identical with those to f and Φ respectively; at least it has been proved that these latter, when $\epsilon = 0$, but the form of Φ is preserved, become identical with the former, and consequently the same is true when ϵ is taken indefinitely small. Now if we call the $(m-n)$ roots of Φ which do not belong to ϕ, η_{n+1}, $\eta_{n+2} \dots \eta_m$, and make

$$\Psi_{m-i,0} = \Sigma(x - h_{q_{i+1}})(x - h_{q_{i+2}}) \dots (x - h_{q_m}) \frac{\begin{bmatrix} h_{q_1}, & h_{q_2} \dots h_{q_i} \\ \eta_1, & \eta_2 \dots \eta_m \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} \dots h_{q_i} \\ h_{q_{i+1}}, & h_{q_{i+2}} \dots h_{q_m} \end{bmatrix}},$$

we have

$$\Psi_{m-i,0} = \Sigma P(h_{q_1}, h_{q_2} \dots h_{q_i}) \begin{bmatrix} h_{q_1}, & h_{q_2} \dots h_{q_i} \\ \eta_{n+1}, & \eta_{n+2} \dots \eta_m \end{bmatrix},$$

where

$$P(h_{q_1}, h_{q_2} \dots h_{q_i}) = (x - h_{q_{i+1}})(x - h_{q_{i+2}}) \dots (x - h_{q_m}) \frac{\begin{bmatrix} h_{q_1}, & h_{q_2} \dots h_{q_i} \\ \eta_1, & \eta_2 \dots \eta_m \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} \dots h_{q_i} \\ h_{q_{i+1}}, & h_{q_{i+2}} \dots h_{q_m} \end{bmatrix}}.$$

But since $\eta_{n+1}, \eta_{n+2} \dots \eta_m$ are infinite in value,

$$\begin{bmatrix} h_{q_1}, & h_{q_2} \dots h_{q_i} \\ \eta_{n+1}, & \eta_{n+2} \dots \eta_m \end{bmatrix} = \{(-\eta_{n+1})(-\eta_{n+2}) \dots (-\eta_m)\}^i \left(\frac{1}{\epsilon}\right)^i.$$

Hence
$$\Psi_{m-i,0} = \left(\frac{1}{\epsilon}\right)^i \Sigma P(h_{q_1}, h_{q_2} \dots h_{q_i})$$

$$= \left(\frac{1}{\epsilon}\right)^i \vartheta_{m-i,0},$$

and
$$\vartheta_{m-i,0} = \epsilon^i \Psi_{m-i,0}.$$

But by what has been shown antecedently, taking account of the fact of the leading coefficient of Φ being ϵ in place of 1, which introduces the factor ϵ^i, we have

$$\epsilon^i \Psi_{m-i,0} = (-)^{(i-1)\frac{i}{2}} B_i',$$

where B_i' is the Bezoutian secondary of the $(m-i-1)$th degree in x to f and ϕ; but B_i' has been proved $= B_i$, the Bezoutian secondary of the same degree to f and ϕ; hence $\vartheta_{m-i,0} = (-)^{i\frac{i-1}{2}} B_i$.

Art. 29. If now we return to the syzygetic equation, $\tau f - t\phi + \vartheta = 0$, ϑ may be treated as known, having in fact been completely determined as a function of the roots, as well in its most general form, as also so as to represent the simplified residues to f and ϕ in the preceding articles; it remains to determine the values of τ and t as functions of the roots corresponding to any allowable form of ϑ, but I shall confine the investigation to the case where ϑ is the lowest-weighted conjunctive or, which is the same thing, a simplified residue to f and ϕ of any given degree in x; each value of $\frac{\tau}{t}$ will then represent one of the convergents to $\frac{\phi}{f}$ when expanded under the form of a continued fraction. If ϑ be of the ιth degree in x, τ is of the degree $(n-\iota-1)$ and t of the degree $(m-\iota-1)$. This being supposed, and calling $n-\iota-1=\nu$, $m-\iota-1=\mu$, I say that t will be represented by G and τ by Γ, where

$$G = (-)^\iota \Sigma (x-h_{q_1})(x-h_{q_2}) \ldots (x-h_{q_\mu}) \frac{\begin{bmatrix} h_{q_1}, & h_{q_2} & \ldots & h_{q_\mu} \\ \eta_1, & \eta_2 & \ldots & \eta_n \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} & \ldots & h_{q_\mu} \\ h_{q_{\mu+1}}, & h_{q_{\mu+2}} & \ldots & h_{q_m} \end{bmatrix}},$$

and τ is an analogous form Γ; $h_1, h_2 \ldots h_m$, as heretofore, being the roots of f, and $\eta_1, \eta_2 \ldots \eta_n$ of ϕ. To fix the ideas and make the demonstration more immediately seizable, give m and n specific values; thus let $m=5$, $n=4$, $\iota=2$, so that $\mu = 5-2-1=2$. Put ϑ under the form $\vartheta_{\iota,0}$, so that ϑ in the case before us

$$= \Sigma (x-h_{q_1})(x-h_{q_2}) \frac{\begin{bmatrix} h_{q_3} h_{q_4} h_{q_5} & \\ \eta_1 \; \eta_2 \; \eta_3 & \eta_4 \end{bmatrix}}{\begin{bmatrix} h_{q_3} h_{q_4} h_{q_5} \\ h_{q_1} h_{q_2} \end{bmatrix}}.$$

Now make $x = h_1$, then $f = 0$, and ϑ becomes

$$\Sigma (h_1 - h_{q_1})(h_1 - h_{q_2}) \frac{\begin{bmatrix} h_{q_3} h_{q_4} h_{q_5} & \\ \eta_1 \; \eta_2 \; \eta_3 & \eta_4 \end{bmatrix}}{\begin{bmatrix} h_{q_3} h_{q_4} h_{q_5} \\ h_{q_1} h_{q_2} \end{bmatrix}},$$

that is
$$\Sigma \frac{\begin{bmatrix} h_1 \\ h_4 h_5 \end{bmatrix} \begin{bmatrix} h_1 h_2 h_3 \\ \eta_1 \eta_2 \eta_3 \eta_4 \end{bmatrix}}{\begin{bmatrix} h_1 h_2 h_3 \\ h_4 h_5 \end{bmatrix}},$$

h_1 being kept constant in the above sum, but h_2, h_3, h_4, h_5 being partitionable in all the six possible ways into two groups, as into h_4, h_5; h_2, h_3 in the term above expressed. This sum is evidently identical with

$$\Sigma \frac{\begin{bmatrix} h_1 h_2 h_3 \\ \eta_1 \eta_2 \eta_3 \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_4 h_5 \end{bmatrix}}, \text{ that is } \begin{bmatrix} h_1 \\ \eta_1 \eta_2 \eta_3 \eta_4 \end{bmatrix} \times \Sigma \frac{\begin{bmatrix} h_2 h_3 \\ \eta_1 \eta_2 \eta_3 \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_4 h_5 \end{bmatrix}}.$$

Again, ϕ becomes
$$\begin{bmatrix} h_1 \\ \eta_1 \eta_2 \eta_3 \eta_4 \end{bmatrix}.$$

Hence $t = \dfrac{\vartheta}{\phi}$ becomes
$$\Sigma \frac{\begin{bmatrix} h_2 h_3 \\ \eta_1 \eta_2 \eta_3 \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_4 h_5 \end{bmatrix}}.$$

But, when $x = h_1$, $\dfrac{G}{(-)^\iota}$ becomes
$$\begin{bmatrix} h_1 \\ h_2 h_3 \end{bmatrix} \frac{\begin{bmatrix} h_2 h_3 \\ \eta_1 \eta_2 \eta_3 \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_1 h_4 h_5 \end{bmatrix}},$$

that is
$$= \frac{\begin{bmatrix} h_1 \\ h_2 h_3 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_1 \end{bmatrix}} \frac{\begin{bmatrix} h_2 h_3 \\ \eta_1 \eta_2 \eta_3 \eta_4 \end{bmatrix}}{\begin{bmatrix} h_2 h_3 \\ h_4 h_5 \end{bmatrix}},$$

$$= (-1)^\iota t.$$

Thus when $x = h_1$, $t = G$. In like manner, when $x = h_2$, or h_3, or h_4, or h_5, t always $= G$; but t and G are both functions of x of the same, namely of only two, dimensions in x. Hence t is identical with G. So in general it may be proved, that whenever $x = h_1$ or h_2 or h_3 ... or h_n, t and G, which are each of only $(m - 1 - \iota)$ dimensions in x, are equal. Hence universally $t = G$, as was to be shown. To find τ we must avail ourselves of the symmorphic, or as we may better say (it being at the opposite extremity of the scale of forms), the antimorphic, value of ϑ represented by $\vartheta_{0,\,\iota}$, taking care to preserve ϑ strictly identical under both forms of representation, in point of sign as well as quantity. That is to say, we must make

$$\vartheta_{0,\iota} = (-)^{\iota(m-\iota)} \Sigma (x-\eta_{q_1})(x-\eta_{q_2}) \dots (x-\eta_{q_\iota}) \frac{\begin{bmatrix} h_1, & h_2 & \dots h_m \\ \eta_{q_{\iota+1}}, & \eta_{q_{\iota+2}} & \dots h_{q_n} \end{bmatrix}}{\begin{bmatrix} \eta_{q_{\iota+1}}, & \eta_{q_{\iota+2}} & \dots \eta_{q_n} \\ \eta_{q_1}, & \eta_{q_2} & \dots \eta_{q_\iota} \end{bmatrix}}$$

$$= (-)^{\omega} \Sigma (x-\eta_{q_1})(x-\eta_{q_2}) \dots (x-\eta_{q_\iota}) \frac{\begin{bmatrix} \eta_{q_{\iota+1}}, & \eta_{q_{\iota+2}} & \dots \eta_{q_n} \\ h_1, & h_2 & \dots h_m \end{bmatrix}}{\begin{bmatrix} \eta_{q_{\iota+1}}, & \eta_{q_{\iota+2}} & \dots \eta_{q_n} \\ \eta_{q_1}, & \eta_{q_2} & \dots \eta_{q_\iota} \end{bmatrix}},$$

where $$\omega = \iota(m-\iota) + m(n-\iota),$$

so that $$(-)^{\omega} = (-)^{m\iota - \iota + mn - m\iota} = (-)^{mn-\iota};$$

and consequently the same reasoning as was applied to t to prove $t = G$, will serve to show that $-\tau = \Gamma$, where

$$\Gamma = (-)^{mn} \Sigma (x-\eta_{\xi_1})(x-\eta_{\xi_2}) \dots (x-\eta_{\xi_\nu}) \frac{\begin{bmatrix} \eta_{\xi_1}, & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \\ h_1, & h_2 & \dots h_m \end{bmatrix}}{\begin{bmatrix} \eta_{\xi_1}, & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \dots \eta_{\xi_n} \end{bmatrix}},$$

or

$$\tau = (-)^{\omega} \Sigma (x-\eta_{\xi_1})(x-\eta_{\xi_2}) \dots (x-\eta_{\xi_\nu}) \frac{\begin{bmatrix} h_1, & h_2 & \dots h_m \\ \eta_{\xi_1}, & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \end{bmatrix}}{\begin{bmatrix} \eta_{\xi_1}, & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \dots \eta_{\xi_n} \end{bmatrix}},$$

where $$\omega = mn - 1 - m\nu = mn - 1 - m(n - \iota - 1)$$

$$= m\iota + m - 1.$$

Art. 30. I have not succeeded in throwing t and τ under any other than the single forms for each above given, and it is remarkable that whilst apparently t and τ admit only of this single representation, ϑ admits of the variety of forms included under the general symbol $\vartheta_{v,\iota-v}$ for a given value of ι; and it ought to be remarked that these forms, although the most perfectly symmetrical and exactly balanced representations, and for that reason possibly the most commodious for the ascertainment of the allotrious factor belonging to them respectively, by no means exhaust the almost infinite variety of modes by which the simplified residues, that is, the hekistobarytic, or if we like so to call them, the prime conjunctives, admit of being represented as functions of the roots of the given functions; for if in Art. 16, instead of writing

$$R = \frac{\begin{bmatrix} h_{q_1}, & h_{q_2} & \dots h_{q_v} \\ \eta_{\xi_1}, & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} & \dots h_{q_v} \\ h_{q_{v+1}}, & h_{q_{v+2}} & \dots h_{q_m} \end{bmatrix} \times \begin{bmatrix} \eta_{\xi_1}, & \eta_{\xi_2}, & \dots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \dots \eta_{\xi_n} \end{bmatrix}},$$

we had made

$$R = \frac{P(h_{q_1}, h_{q_2} \dots h_{q_v}; \eta_{\xi_1}, \eta_{\xi_2} \dots \eta_{\xi_\nu})}{\begin{bmatrix} h_{q_1}, & h_{q_2} & \dots h_{q_v} \\ h_{q_{v+1}}, & h_{q_{v+2}} & \dots h_{q_m} \end{bmatrix} \times \begin{bmatrix} \eta_{\xi_1}, & \eta_{\xi_2} & \dots \eta_{\xi_\nu} \\ \eta_{\xi_{\nu+1}}, & \eta_{\xi_{\nu+2}} & \dots \eta_{\xi_m} \end{bmatrix}},$$

where P represents any function symmetrical in respect of $h_{q_1}, h_{q_2} \dots h_{q_v}$, and also in respect of $\eta_{\xi_1}, \eta_{\xi_2} \dots \eta_{\xi_\nu}$, (the interchanges, that is to say, between one h and another h, or between one η and another η, leaving P unaltered), it might be shown that the value of $\vartheta_{v,\nu}$ resulting from the introduction of this more general value of R would (as for the particular value assumed) always be expressible as an integral function of the roots; and consequently, if P be taken of the same dimensions in the roots as the numerator of R previously assumed, that is $v\nu$, $\vartheta_{v,\nu}$ would continue to be (unless indeed it vanish) identical (to some numerical factor *près*) with the corresponding simplified residue. If, on the other hand, P be taken of less than $v\nu$ dimensions in h and η, we know *à priori* that $\vartheta_{v,\nu}$ must vanish, as otherwise we should have a conjunctive of a weight less than the minimum weight. When P is of the proper *amount* of weight $v\nu$, it is I think probable that another condition as to the *distribution* of the weight will be found to be necessary in order that $\vartheta_{v,\nu}$ may not vanish, namely, that the highest power of any single h in P shall not exceed v, nor the highest power of any single η exceed ν. But as I have not had leisure to enter upon the inquiry, the verification or disproval of this supposed law, and more generally the evolution of the allotrious numerical factor introduced into $\vartheta_{v,\nu}$ by assigning any particular form to P satisfying the necessary conditions of amount and distribution of weight, must be reserved, amongst other points connected with the theory of the remarkable forms (19) Art. 15, as a subject for future investigation.

Art. 31. A property of continued fractions, which, if known, I have not met with in any treatise on the subject (but which has been already cursorily alluded to in these pages), gives rise to a remarkable property of reciprocity connecting τ and t severally with ϑ in the syzygetic equation $\tau f - t\phi + \vartheta = 0$.

Let the successive convergents to the ordinary continued fraction

$$\frac{1}{q_1 +}\,\frac{1}{q_2 +}\,\frac{1}{q_3 +} \dots \frac{1}{q_{i-1} +}\,\frac{1}{q_i}$$

be called

$$\frac{l_1}{m_1}, \frac{l_2}{m_2} \dots \frac{l_{i-1}}{m_{i-1}}, \frac{l_i}{m_i},$$

respectively; it is well known that

$$m_{i-1} l_i - m_i l_{i-1} = (-)^{i-1} 1;$$

but I believe that it has not been observed that this is only the extreme case of a much more general equation, namely

$$m_{i-\rho} l_i - m_i l_{i-\rho} = (-)^{i-\rho} \mu_{\rho-1} *,$$

where μ_1, $\mu_2 \ldots \mu_i$ denote respectively the denominators to the convergents to the continued fractions formed with the quotients taken in a reverse order, that is, the continued fraction

$$\frac{1}{q_i +} \frac{1}{q_{i-1} +} \frac{1}{q_{i-2} +} \cdots + \frac{1}{q_2 +} \frac{1}{q_1}.$$

This is easily proved when $\rho = 1$; μ_0 is of course (as usual) to be considered 1. So more simply for the improper continued fraction,

$$\frac{l_i}{m_i} = \frac{1}{q_1 -} \frac{1}{q_2 -} \cdots \frac{1}{q_{i-1} -} \frac{1}{q_i},$$

of which the convergents are supposed to be

$$\frac{l_1}{m_1}, \quad \frac{l_2}{m_2} \cdots \frac{l_{i-1}}{m_{i-1}}, \quad \frac{l_i}{m_i},$$

and the reverse fraction

$$\frac{1}{q_i -} \frac{1}{q_{i-1}} \cdots \frac{1}{q_2 -} \frac{1}{q_1},$$

of which the convergents are supposed to be

$$\frac{\lambda_1}{\mu_1}, \quad \frac{\lambda_2}{\mu_2} \cdots \frac{\lambda_i}{\mu_i},$$

we have the more simple equation

$$l_i m_{i-\rho} - l_{i-\rho} m_i + \mu_{\rho-1} = 0.$$

And it is well known, or at all events easily demonstrable, that

$$\frac{l_{i-1}}{l_i} = \frac{1}{q_i -} \frac{1}{q_{i-1} -} \frac{1}{q_{i-2}} \cdots \frac{1}{q_2},$$

$$\frac{m_{i-1}}{m_i} = \frac{1}{q_i -} \frac{1}{q_{i-1} -} \frac{1}{q_{i-2}} \cdots \frac{1}{q_2 -} \frac{1}{q_1}.$$

Art. 32. If now we use subscript indices to denote the degree in x of the quantities to which they are affixed, we have the general syzygetic equation

$$K\tau_{n-\iota-1} f_m - K t_{m-\iota-1} \phi_n + K \vartheta_\iota = 0,$$

where K, a constant (which I have given the means of determining in the first section), being rightly assumed, $K\tau_{n-\iota-1}$, $Kt_{m-\iota-1}$ become the numerator and denominator respectively of one of the convergents to $\frac{\phi}{f}$, expressed as

* See *London and Edinburgh Philosophical Magazine*, "On a Fundamental Theorem in the Theory of Continued Fractions," Vol. VI., October, 1853. [See below.]

an improper continued fraction, and $K\vartheta_\iota$ becomes the denominator to one of the convergents to $\frac{t_{m-1}}{f}$, or, which is the same thing, to $\frac{\tau_{n-1}}{\phi}$*. Conversely, it is obvious that if we adopt as our primitive functions cf_m and t_{m-1}, c being the value of K when $\iota=0$, we shall obtain as the general form of our syzygetic equation, bearing in mind that $(m-1)$ now replaces n,

$$cK'\tau_{n-\iota-1}f_m - K'\vartheta_{m-\iota-1}t_{m-1} + K't_\iota = 0;$$

and similarly, if we adopt as our primitive functions τ_{n-1} and $c\phi_n$, we obtain for our general syzygetic equation, observing that $(n-1)$ now replaces m,

$$K'\vartheta_{n-\iota-1}\tau_{n-1} - cK't_{m-\iota-1}\phi_n + K'\tau_\iota = 0;$$

so that (making abstraction of the constant factors and looking merely to the forms of the several functions which enter into the equations) we see that on the first hypothesis, namely of t_{m-1} being substituted for ϕ_n, the conjunctives of each degree in x change places with the second conjunctive factors, that is the original multipliers of ϕ of the same degree in x, and *vice versâ*; and in the second hypothesis, where τ_{n-1} takes the place of f_m, the conjunctives of each degree in x change places with the first conjunctive factors, that is the original multipliers of f of the same degree in x, and *vice versâ*; t_{m-1} and τ_{n-1} being respectively multipliers of ϕ and f, such that the difference of the respective products is independent of x. These results ought to be capable of being verified by aid of our general formulæ for t, τ, ϑ, and as this verification will serve to exhibit in a clearer light the nature of the reciprocity between the conjunctives and the conjunctive factors, it may be not uninteresting to set it out.

Art. 33. As usual, let $h_1, h_2 \dots h_m$ be the roots of fx, and $\eta_1, \eta_2 \dots \eta_n$ the roots of ϕx; the last conjunctive factor to ϕ, which is of the degree $(m-1)$ in x, will be represented, neglecting powers of $(-)$, by t_{m-1}, where

$$t_{m-1} = \Sigma\,(x-h_{q_1})(x-h_{q_2})\dots(x-h_{q_{m-1}})\,\frac{\begin{bmatrix} h_{q_1}, & h_{q_2} & \dots & h_{q_{m-1}} \\ \eta_1, & \eta_2 & \dots & \eta_n \end{bmatrix}}{\begin{bmatrix} h_{q_m} \\ h_{q_1}, \; h_{q_2} \dots h_{q_{m-1}} \end{bmatrix}}.$$

If now we for greater simplicity make $t_{m-1} = t(x)$, and call the roots of t, $\eta'_1, \eta'_2 \dots \eta'_{m-1}$, any such quantity as

$$\begin{aligned}\begin{bmatrix} h_{q_m} \\ \eta'_1, \; \eta'_2 \dots \eta'_{m-1} \end{bmatrix} &= t(h_{q_m}) = (h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\dots(h_{q_m}-h_{q_{m-1}}) \\ &\qquad \times \frac{\phi(h_{q_1})\,\phi(h_{q_2})\dots\phi(h_{q_{m-1}})}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\dots(h_{q_m}-h_{q_{m-1}})} \\ &= \phi(h_{q_1})\,\phi(h_{q_2})\dots\phi(h_{q_{m-1}}) \\ &= R\,\frac{1}{\phi(h_{q_m})},\end{aligned}$$

* Since ι is always supposed less than n (n being the degree of the lower degreed of the two functions f and ϕ), the fact of the last quotient to $\frac{t_{m-1}}{f}$ being wanting to $\frac{\tau_{n-1}}{\phi}$ will not affect the accuracy of the statement in the text above, since this latter will contain as many quotients as can in any case be required for expressing ϑ_ι.

R denoting a constant independent of the root h_{q_m} selected, in fact the resultant of the two functions fx and ϕx, that is to say,

$$\phi(h_1)\,\phi(h_2)\,\phi(h_3)\ldots\phi(h_m).$$

But by our general formulæ the simplified residue to fx and $t(x)$ of the ιth degree in x will be represented by

$$\vartheta'_{\iota,0} = \Sigma\,(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\iota})\left\{\frac{\begin{bmatrix} h_{q_{\iota+1}}, & h_{q_{\iota+2}} \ldots h_{q_m} \\ \eta'_1, & \eta'_2 \ \ldots \eta'_{m-1}\end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} \ldots h_{q_\iota} \\ h_{q_{\iota+1}}, & h_{q_{\iota+2}} \ldots h_{q_m}\end{bmatrix}}\right\};$$

therefore

$$\vartheta'_{\iota,0} = \Sigma\,(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\iota})\times\left\{R^{m-\iota}\frac{\phi(h_{q_{\iota+1}})^{-1}\,\phi(h_{q_{\iota+2}})^{-1}\ldots\phi(h_{q_m})^{-1}}{\begin{bmatrix} h_{q_1}, & h_{q_2} \ldots h_{q_\iota} \\ h_{q_{\iota+1}}, & h_{q_{\iota+2}} \ldots h_{q_m}\end{bmatrix}}\right\}$$

$$= R^{m-\iota-1}\,\Sigma\,(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\iota})\frac{\phi(h_{q_1})\,\phi(h_{q_2})\ldots\phi(h_{q_\iota})}{\begin{bmatrix} h_{q_1}, & h_{q_2} \ldots h_{q_\iota} \\ h_{q_{\iota+1}}, & h_{q_{\iota+2}} \ldots h_{q_m}\end{bmatrix}},$$

or

$$\vartheta'_\iota = R^{m-\iota-1}t_\iota,$$

the relation which was to be obtained. So conversely, in precisely the same manner, calling t'_ι the conjunctive factor of the degree ι in x to $t(x)$ in the syzygetic equation which connects fx and $t(x)$ with a corresponding simplified residue, we have

$$t'_\iota = \Sigma\,(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\iota})\frac{\begin{bmatrix} h_{q_1}, & h_{q_2} \ldots h_{q_\iota} \\ \eta'_1, & \eta'_2 \ldots \eta'_{m-1}\end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} \ldots h_{q_\iota} \\ h_{q_{\iota+1}}, & h_{q_{\iota+2}} \ldots h_{q_m}\end{bmatrix}}$$

$$= R^{\iota-1}\Sigma\,(x-h_{q_1})(x-h_{q_2})\ldots(x-h_{q_\iota})\frac{\phi(h_{q_{\iota+1}})\,\phi(h_{q_{\iota+2}})\ldots\phi(h_{q_m})}{\begin{bmatrix} h_{q_1}, & h_{q_2} \ldots h_{q_\iota} \\ h_{q_{\iota+1}}, & h_{q_{\iota+2}} \ldots h_{q_m}\end{bmatrix}}$$

$$= R^{\iota-1}\vartheta_\iota,$$

the conjugate equation to the one previously obtained*.

And evidently the same reasoning serves to establish the reciprocity, or rather reciprocal convertibility, between the ϑ series and the τ series, when in lieu of the original primitives fx and ϕx we take as our primitives $\tau(x)$ and ϕx, $\tau(x)$ being the function which satisfies the equation

$$\tau(x)fx - t(x)\,\phi x + \vartheta = 0.$$

* M. Hermite, by a peculiar method, first discovered one of these two conjugate relations of reciprocity, applicable to the case of Sturm's theorem, where $\phi x = f'x$, and I am indebted to him for bringing the subject under my notice.

Art. 34. It may be remarked that if $n = m - 1$, the last syzygetic equation being thus $t_{m-1}\phi_{m-1} - \tau_{m-2}f_m - \vartheta_0 = 0$, when t_{m-1} and f_m are taken as the primitives, the corresponding equation will be of the form

$$t'_{m-1}t_{m-1} - \tau'_{m-2}f_m + \vartheta'_0 = 0\,;$$

these two equations must therefore be identical, and consequently $t'_{m-1} = \phi_{m-1}$ (to a numerical factor *près*), so that t_{m-1} and ϕ_{m-1} are reciprocal forms; this is also obvious from the consideration that t'_{m-1} must, by the general law of reciprocity (established above), be a residue to (f_m, ϕ_{m-1}), which the latter function itself may be considered to be. Or the same thing is obvious directly, by writing

$$t_{m-1} = t(x) = \Sigma\,(x - h_{q_1})(x - h_{q_2})\ldots(x - h_{q_{m-1}})\frac{\phi(h_{q_1})\,\phi(h_{q_2})\ldots\phi(h_{q_{m-1}})}{(h_{q_m} - h_{q_1})(h_{q_m} - h_{q_2})\ldots(h_{q_m} - h_{q_{m-1}})},$$

and then making

$$t'_{m-1} = \Sigma\,(x - h_{q_1})(x - h_{q_2})\ldots(x - h_{q_{m-1}})\frac{t(h_{q_1})\,t(h_{q_2})\ldots t(h_{q_{m-1}})}{(h_{q_m} - h_{q_1})(h_{q_m} - h_{q_2})\ldots(h_{q_m} - h_{q_{m-1}})}$$

$$= R^{m-1}\,\Sigma\,(x - h_{q_1})(x - h_{q_2})\ldots(x - h_{q_{m-1}})\frac{\phi(h_{q_1})^{-1}\ldots\phi(h_{q_{m-1}})^{-1}}{(h_{q_m} - h_{q_1})\ldots(h_{q_m} - h_{q_{m-1}})}$$

$$= R^{m-2}\,\Sigma\,(x - h_{q_1})\ldots(x - h_{q_{m-1}})\frac{\phi(h_{q_m})}{(h_{q_m} - h_{q_1})\ldots(h_{q_m} - h_{q_{m-1}})},$$

or finally,

$$t'_{m-1} = R^{m-2}\,\phi,$$

as was to be shown.

Section III.

On the application of the Theorems in the preceding Section to the expression in terms of the roots of any primitive function of Sturm's auxiliary functions, and the other functions which connect these with the primitive function and its first differential derivative.

Art. 35. The formulæ in the preceding Section had reference to the case of two absolutely independent functions and their respective systems of roots: when the functions become so related that the roots of the one system become explicitly or implicitly functions of the roots of the other system, the formulæ will become expressible in terms of these latter alone, and in some cases the terms (of which the sum is always essentially integral) will become separately and individually representable under an integral form. Such, as I shall proceed to show, is the case for two functions, of which one is the differential derivative of the other. When f and ϕ are thus related, so that $\phi = \dfrac{df}{dx}$, calling as before $h_1, h_2 \dots h_m$ the roots of f, and $\eta_1, \eta_2 \dots \eta_{m-1}$ the roots of ϕ, we shall have in general

$$\begin{bmatrix} h_{q_{i+1}} & \\ \eta_1, & \eta_2 \dots \eta_{m-1} \end{bmatrix} = (h_{q_{i+1}} - \eta_1)(h_{q_{i+1}} - \eta_2) \dots (h_{q_{i+1}} - \eta_{m-1})$$

$$= f' h_{q_{i+1}} = \begin{bmatrix} h_{q_{i+1}} & \\ h_{q_1}, & h_{q_2} \dots h_{q_i}, \; h_{q_{i+2}} \dots h_{q_m} \end{bmatrix} = \begin{bmatrix} h_{q_{i+1}} & \\ h_{q_1}, & h_{q_2} \dots h_{q_i} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+1}} & \\ h_{q_{i+2}}, & h_{q_{i+3}} \dots h_{q_m} \end{bmatrix}.$$

Consequently

$$\begin{bmatrix} h_{q_{i+1}}, & h_{q_{i+2}} \dots h_{q_m} \\ \eta_1, & \eta_2 \quad \dots \eta_{m-1} \end{bmatrix} = \begin{bmatrix} h_{q_{i+1}} & \\ \eta_1, & \eta_2 \dots \eta_{m-1} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+2}} & \\ \eta_1, & \eta_2 \dots \eta_{m-1} \end{bmatrix} \times \dots$$
$$\times \begin{bmatrix} h_{q_m} & \\ \eta_1, & \eta_2 \dots \eta_{m-1} \end{bmatrix}$$
$$= \begin{bmatrix} h_{q_{i+1}} & \\ h_{q_1}, & h_{q_2} \dots h_{q_i} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+1}} & \\ h_{q_{i+2}}, & h_{q_{i+3}} \dots h_{q_{m-1}} \end{bmatrix}$$
$$\times \begin{bmatrix} h_{q_{i+2}} & \\ h_{q_1}, & h_{q_2} \dots h_{q_i} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+2}} & \\ h_{q_{i+1}}, & h_{q_{i+3}} \dots h_{q_{m-1}} \end{bmatrix}$$
$$\times \dots\dots\dots$$
$$\times \begin{bmatrix} h_{q_m} & \\ h_{q_1}, & h_{q_2} \dots h_{q_i} \end{bmatrix} \times \begin{bmatrix} h_{q_m} & \\ h_{q_{i+1}}, & h_{q_{i+2}} \dots h_{q_{m-1}} \end{bmatrix}.$$

Hence
$$\frac{\begin{bmatrix} h_{q_{i+1}}, & h_{q_{i+2}} \dots h_{q_m} \\ \eta_1, & \eta_2 \quad \dots \eta_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_{i+1}}, & h_{q_{i+2}} \dots h_{q_m} \\ h_{q_1}, & h_{q_2} \quad \dots h_{q_i} \end{bmatrix}}$$
$$= \begin{bmatrix} h_{q_{i+1}} & \\ h_{q_{i+2}}, & h_{q_{i+3}} \dots h_{q_m} \end{bmatrix} \times \begin{bmatrix} h_{q_{i+2}} & \\ h_{q_{i+1}}, & h_{q_{i+3}} \dots h_{q_m} \end{bmatrix} \times \dots \times \begin{bmatrix} h_{q_m} & \\ h_{q_{i+1}}, & h_{q_{i+2}} \dots h_{q_{m-1}} \end{bmatrix}$$
$$= (-)^{\frac{1}{2}(m-i)(m-i-1)} \zeta(h_{q_{i+1}}, h_{q_{i+2}} \dots h_{q_m}),$$

the ζ denoting the operation of taking the product of the squares of the differences of the quantities which this symbol governs. Hence the Bezoutian secondary to f and f' of the $(m-i-1)$th degree in x, namely

$$(-)^{i\frac{i-1}{2}} \Sigma (x - h_{q_{i+1}})(x - h_{q_{i+2}}) \dots (x - h_{q_m}) \frac{\begin{bmatrix} h_{q_1}, & h_{q_2} \dots h_{q_i} \\ \eta_1, & \eta_2 \; \dots \eta_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} \quad \dots h_{q_i} \\ h_{q_{i+1}}, & h_{q_{i+2}} \dots h_{q_m} \end{bmatrix}},$$

becomes

$$(-)^{i(i-1)} \Sigma \zeta(h_{q_1}, h_{q_2} \dots h_{q_i})(x - h_{q_{i+1}})(x - h_{q_{i+2}}) \dots (x - h_{q_m})$$
$$= \Sigma \zeta(h_{q_1}, h_{q_2} \dots h_{q_i})(x - h_{q_{i+1}})(x - h_{q_{i+2}}) \dots (x - h_{q_m}),$$

since $(-)^{i(i-1)} = 1$; this gives the well-known formulæ (enunciated * by me in the *London and Edinburgh Philosophical Magazine* for 1839) for expressing M. Sturm's auxiliary functions in terms of the roots of the primitive, and which I therein stated were immediately deducible from the general formulæ (also enunciated in the same paper) applicable to any two functions. These more general formulæ appear to have completely escaped the notice of M. Sturm and others, who have used the special formulæ applicable to the case of one function becoming the first differential derivative of the other.

Art. 36. In precisely the same manner, if we form as usual the ordinary syzygetic equation

$$tf'x - \tau fx + \vartheta = 0,$$

we may find the different values of t given by the complementary formulæ; and using t_i to denote the multiplier of the degree i in x, that is appertaining to the *residue* of the degree $(m-i-1)$ in x, we have

$$t_i = \Sigma \frac{\begin{bmatrix} h_{q_1}, & h_{q_2} \ldots h_{q_i} \\ \eta_1, & \eta_2 \ldots \eta_{m-1} \end{bmatrix}}{\begin{bmatrix} h_{q_1}, & h_{q_2} & \ldots h_{q_i} \\ h_{q_{i+1}}, & h_{q_{i+2}} & \ldots h_{q_m} \end{bmatrix}} (x - h_{q_1})(x - h_{q_2}) \ldots (x - h_{q_i})$$

$$= \Sigma\zeta(h_{q_1}, h_{q_2} \ldots h_{q_i})(x - h_{q_1})(x - h_{q_2}) \ldots (x - h_{q_i}).$$

Art. 37. Thus, if we make $i = m-1$,

$$f_1'x = t_{m-1} = \Sigma\zeta(h_{q_1}, h_{q_2} \ldots h_{q_{m-1}})(x - h_{q_1})(x - h_{q_2}) \ldots (x - h_{q_{m-1}}).$$

It is evident from the form of $f_1'x$ that it possesses relative to fx, the same property as $f'x$, I mean the property that when x is indefinitely near to a real root of fx, and is passing from the inferior to the superior side of such root, $\frac{f_1'x}{fx}$ like $\frac{f'x}{fx}$ will pass from being negative to being positive, or in other words, $f_1'x$ and $f'x$ have always the same sign in the immediate vicinity to a real root of fx. Hence it follows that $f_1'x$ might be used instead of $f'x$, to produce, by the Sturmian process of common measure, a series of auxiliary functions, which with fx and $f_1'x$ would form a rhizoristic series, that is a series for determining (as in the manner of M. Sturm's ordinary auxiliaries) the number of real roots of fx comprised within given limits. The rhizoristic series generated by this process will, it is easily seen, be (to a constant factor *près*) the denominators (reckoning $+1$ as the denominator in the zero place) of the successive convergents to $\frac{f'x}{fx}$ thrown under the form

[* p. 45 above.]

of a continued fraction $\frac{1}{q_1 -}\, \frac{1}{q_2 -} \dots \frac{1}{q_{n-1} -}\, \frac{1}{q_n}$; M. Sturm's own rhizoristic series, on the contrary, will be (to a constant factor *près*) the denominators of the convergents to the inverse fraction $\frac{f_1'x}{fx}$, which will be of the form $K\left(\frac{1}{q_n -}\, \frac{1}{q_{n-1} -} \dots \frac{1}{q_2 -}\, \frac{1}{q_1}\right)$; accordingly these two rhizoristic series will be equivalent as regards the number of changes and of combinations of sign (afforded by each) corresponding to any given value of x, of which of course the q's are linear functions. This result agrees with what has been demonstrated by me* by a more general method (in the *London and Edinburgh Philosophical Magazine,* June and July 1853), where it has been proved, by means of a very simple theorem of determinants, that the two series

$$\frac{1}{q_1},\ \frac{1}{q_1 -}\,\frac{1}{q_2},\ \frac{1}{q_1 -}\,\frac{1}{q_2 -}\,\frac{1}{q_3},\ \dots \frac{1}{q_1 -}\,\frac{1}{q_2 -}\,\frac{1}{q_3} \dots \frac{1}{q_n},$$

and

$$\frac{1}{q_n},\ \frac{1}{q_n -}\,\frac{1}{q_{n-1}},\ \frac{1}{q_n -}\,\frac{1}{q_{n-1} -}\,\frac{1}{q_{n-2}},\ \dots \frac{1}{q_n -}\,\frac{1}{q_{n-1} -}\,\frac{1}{q_{n-2}} \dots \frac{1}{q_1},$$

always contain (for real values of $q_1,\ q_2,\ q_3 \dots q_n$) the same number of positive and negative signs.

Art. 38. Having now determined the general values of ϑ and t in the equation $tf'x - \tau fx + \vartheta = 0$ as explicit integral functions of the roots of fx, the more difficult task remains to assign to τ its value similarly expressed. This cannot readily be effected by means of substitutions in the general formulæ, the method we adopted for finding t and ϑ; but all the other quantities except τ in the syzygetic equation being integral functions of the roots, it is evident that τ also must be an integral function of the same, and to obtain it we may use the expression $\tau = \frac{tf'x + \vartheta}{fx}$.

To obtain the general form of τ by direct calculation from this formula would however be found to be impracticable; the mode I adopt therefore to discover the general expression for τ corresponding to different values of ϑ, is to ascertain its value on the hypothesis of particular relations existing between the roots of fx, and then from the particular values of τ thus obtained to infer demonstratively its general form, as will be seen below. The demonstration of τ is unavoidably somewhat long, τ being in fact represented by a double sum of partial symmetrical functions.

Using the subscript indices of each function as the syzygetic equation to denote its degree in x, we have in general

$$t_{m-i-1}f'x - \tau_{m-i-2}fx + \vartheta_i = 0,$$

[* See below pp. 616 and 621.]

where if we make

$$h_1 - x = k_1, \quad h_2 - x = k_2 \ldots h_m - x = k_m,$$

so that

$$h_i - h_\omega = k_i - k_\omega,$$

and therefore

$$\zeta(h_{\theta_1}, h_{\theta_2} \ldots h_{\theta_\rho}) = \zeta(k_{\theta_1}, k_{\theta_2} \ldots k_{\theta_\rho}),$$

we have in effect found

$$\vartheta_i = \Sigma k_{q_1} k_{q_2} \ldots k_{q_i} \zeta(k_{q_{i+1}}, k_{q_{i+2}} \ldots k_{q_m})$$

and

$$t_{m-i-1} = \pm \Sigma k_{q_1} k_{q_2} \ldots k_{q_{m-i-1}} \zeta(k_{q_1}, k_{q_2} \ldots k_{q_{m-i-1}});$$

we have also $f'(x) = (-)^{m-1} \Sigma k_{q_1} k_{q_2} \ldots k_{q_{m-1}}$.

Let us commence with the case where $i = 0$; we have then

$$\vartheta_0 = \zeta(k_1, k_2 \ldots k_m),$$

$$t_{m-1} = \Sigma k_{q_1} k_{q_2} \ldots k_{q_{m-1}} \zeta(k_{q_1}, k_{q_2} \ldots k_{q_{m-1}});$$

we have thus

$$(-)^m \tau_{m-2} k_1 k_2 \ldots k_m = \zeta(k_1, k_2 \ldots k_m)$$
$$- \Sigma k_{q_1} k_{q_2} \ldots k_{q_{m-1}} \times \Sigma k_{q_1} k_{q_2} \ldots k_{q_{m-1}} \zeta(k_{q_1}, k_{q_2} \ldots k_{q_{m-1}}).$$

It may easily be verified that the negative sign interposed between the two parts of the right-hand member of the equation has been correctly taken, for

$$\zeta(k_1, k_2 \ldots k_m) \text{ contains a term } k_1^{2(m-1)} k_2^{2(m-2)} \ldots k^4_{m-2} k^2_{m-1},$$

$$\Sigma k_{q_1} k_{q_2} \ldots k_{q_{m-1}} \text{ contains a term } k_1 k_2 \ldots k_{m-2} k_{m-1},$$

and

$$\Sigma k_{q_1} k_{q_2} \ldots k_{q_{m-1}} \zeta(k_{q_1}, k_{q_2} \ldots k_{q_{m-1}}) \text{ contains a term } k_1^{2m-3} k_2^{2m-5} \ldots k^3_{m-2} k^1_{m-1},$$

and thus the term $k_1^{2(m-1)} k_2^{2(m-2)} \ldots k^4_{m-2} k^2_{m-1}$, which does not contain $k_1 k_2 \ldots k_m$, will (as it ought to do) disappear from the right-hand side of the equation.

Now suppose

$$k_1 = k_2,$$

then

$$\zeta(k_1, k_2 \ldots k_m) = 0,$$

and also

$$\zeta(k_{q_1}, k_{q_2} \ldots k_{q_{m-1}}) = 0,$$

except when one or the other of the two disjunctive equations

$$q_1, q_2, q_3 \ldots q_{m-1} = 1, 3, 4 \ldots m,$$

$$q_1, q_2, q_3 \ldots q_{m-1} = 2, 3, 4 \ldots m,$$

is satisfied (by a disjunctive equation, meaning an equation which affirms the equality of one set of quantities with another set the same in number, each with each, but in some unassigned order).

Hence

$$\Sigma k_{q_1} k_{q_2} \dots k_{q_{m-1}} \zeta(k_{q_1}, k_{q_2} \dots k_{q_{m-1}})$$
$$= 2k_1 k_3 \dots k_m \zeta(k_1, k_3 \dots k_m).$$

Hence when $k_1 = k_2$, $(-)^m \tau_{m-2}$ becomes

$$\frac{2}{k_1} \Sigma k_{q_1} k_{q_2} \dots k_{q_{m-1}} \zeta(k_1, k_3 \dots k_m),$$

that is

$$2\zeta(k_1, k_3 \dots k_m) \{k_1 \Sigma k_{r_3} k_{r_4} \dots k_{r_{m-1}} + 2k_3 k_4 \dots k_m\},$$

the Σ referring to $r_3, r_4 \dots r_m$ supposed to be disjunctively equal to 3, 4 ... m.

Now τ_{m-2} is of $(m-2)$ dimensions in x, and whenever more than one equality exists between the k's, ϑ_0 and t_{m-1} both vanish (in fact every term in each vanishes separately), and therefore τ_{m-2}, which $= \dfrac{\vartheta_0 + t_{m-1} f'x}{k_1 k_2 \dots k_m}$, will vanish.

Hence $(-)^m \tau_{m-2}$ must be always of the form

$$\Sigma\zeta(h_{q_1}, h_{q_2} \dots h_{q_{m-1}}) \times \Psi(k_{q_1}, k_{q_2} \dots k_{q_{m-1}}, k_{q_m}),$$

Ψ denoting some integral function of $(m-2)$ dimensions in respect of the system of quantities $k_{q_1}, k_{q_2} \dots k_{q_m}$. The result above obtained enables us to assign the value of

$$\Psi(k_1, k_3 \dots k_m, k_2),$$

when $k_1 = k_2$, namely

$$k_1 \Sigma(k_{r_3}, k_{r_4} \dots k_{r_{m-1}}) + 2k_3 k_4 \dots k_m.$$

Now for a moment suppose, selecting $(m-1)$ terms $k_1, k_3, k_4 \dots k_m$ out of the m terms of the k series, that

$$\Omega(k_1, k_3, k_4 \dots k_m, k_2) = k_2^{m-2} - k_2^{m-3} S_1(k_1, k_3 \dots k_m) + k_2^{m-4} S_2(k_1, k_3 \dots k_m)$$
$$\pm \dots \mp k_2 S_{m-3}(k_1, k_3 \dots k_m) \pm 2S_{m-2}(k_1, k_3 \dots k_m),$$

where S_1 means that the quantities which it governs are to be simply added together, S_2 denotes that their binary, S_3 that their ternary, and in general S_r that their r-ary products are to be added together.

When $k_1 = k_2$, Ω becomes

$$k_1^{m-2} - k_1^{m-3}\{k_1 + S_1(k_3, k_4 \dots k_m)\} + k_1^{m-4}\{k_1 S_1(k_3, k_4 \dots k_m) + S_2(k_3, k_4 \dots k_m)\}$$
$$- k_1^{m-5}\{k_1 S_2(k_3, k_4 \dots k_m) + S_3(k_3, k_4 \dots k_m)\} \pm \dots$$
$$\pm k_1\{k_1 S_{m-4}(k_3, k_4 \dots k_m) + S_{m-3}(k_3, k_4 \dots k_m)\} \pm 2S_{m-2}(k_3, k_4 \dots k_m),$$

which evidently equals

$$\pm\{2S_{m-2}(k_3, k_4 \dots k_m) + k_1 S_{m-3}(k_3, k_4 \dots k_m)\},$$

that is
$$\pm\{k_1 \Sigma(k_{r_3}, k_{r_4} \dots k_{r_{m-1}}) + 2k_3 k_4 \dots k_m\}.$$

Hence when $k_1 = k_2$, $\Psi = \Omega$, and

$$(-)^m \tau_{m-2} = \Sigma\zeta(h_{q_1}, h_{q_2} \ldots h_{q_{m-1}}) \times \Omega(k_{q_1}, k_{q_2} \ldots k_{q_{m-1}}, k_{q_m});$$

and so in like manner, when k_1 is equal to any one of the $(m-1)$ quantities $k_2, k_3 \ldots k_m$, the form of τ_{m-2} above written will have been correctly assumed. But τ_{m-2} may be treated as a function of $(m-2)$ dimensions in k_1, and consequently any form of $(m-2)$ dimensions in k_1, which fits it for $(m-1)$ different values of k_1, must be its general form, and accordingly we have universally,

$$\begin{aligned}(-)^m \tau_{m-2} = \Sigma\zeta(h_{q_1}, h_{q_2} \ldots h_{q_{m-1}}) \times \{&(x-h_{q_m})^{m-2} \\ &- (x-h_{q_m})^{m-3} S_1(x-h_{q_1}, x-h_{q_2} \ldots x-h_{q_{m-1}}) \\ &+ (x-h_{q_m})^{m-4} S_2(x-h_{q_1}, x-h_{q_2} \ldots x-h_{q_{m-1}}) \pm \&c. \\ &\mp (x-h_{q_m}) S_{m-3}(x-h_{q_1}, x-h_{q_2} \ldots x-h_{q_{m-1}}) \\ &\pm 2S_{m-2}(x-h_{q_1}, x-h_{q_2} \ldots x-h_{q_{m-1}})\}.\end{aligned}$$

Art. 39. With a view to better paving our way to the general form of τ for all values of i, let us pass over the case of $i = 1$ and go at once to the equation

$$t_{m-3} f'x - \tau_{m-4} fx + \vartheta_2 = 0;$$

and to better fix our ideas let $m = 7$, so that the equation becomes

$$t_4 f'x - \tau_3 fx + \vartheta_2 = 0;$$

we have then, preserving the same relation as before, that is, using h to denote any root of fx, and k to denote $h - x$, the equation

$$\begin{aligned}\pm k_1k_2k_3k_4k_5k_6k_7\tau_3 = {}& \Sigma k_{q_1}k_{q_2}\zeta(k_{q_3}k_{q_4}k_{q_5}k_{q_6}k_{q_7}) \\ &- \Sigma k_{q_1}k_{q_2}k_{q_3}k_{q_4}k_{q_5}k_{q_6} \times \Sigma\{k_{q_1}k_{q_2}k_{q_3}k_{q_4}\,\zeta(k_{q_1}k_{q_2}k_{q_3}k_{q_4})\};\end{aligned}$$

now τ_3 will vanish whenever more than three relations of equality exist between the k's, for then each term in *both* of the two sums in the right-hand member of the equation above written will separately vanish; and of course three relations of equality between the same are sufficient to make all the terms in the first of these sums vanish. This relationship between the different k's corresponding to a multiplicity 3 may arise in different ways; the multiplicity 3 may be divided into 3 units corresponding to 3 pairs of equal roots, or into 2 and 1 corresponding one set of 3 equal roots, and a second set of 2 equal roots, or may be taken *en bloc*, which corresponds to the case of one set of 4 equal roots. I shall make the first of these suppositions, which will sufficiently well answer our purpose in the case before us.

Thus I shall suppose

$$k_1 = k_4, \quad k_2 = k_5, \quad k_3 = k_6,$$

then, as above remarked,

$$\zeta(k_{q_3}k_{q_4}k_{q_5}k_{q_6}k_{q_7}) = 0$$

for all values of q_3, q_4, q_5, q_6, q_7, and therefore

$$\Sigma k_{q_1} k_{q_2} \zeta (k_{q_3} k_{q_4} k_{q_5} k_{q_6} k_{q_7}) = 0;$$

also $\Sigma k_{q_1} k_{q_2} k_{q_3} k_{q_4} k_{q_5} k_{q_6}$ becomes

$$k_1 k_2 k_3 \{k_1 k_2 k_3 + 2k_7 (k_1 k_2 + k_1 k_3 + k_2 k_3)\},$$

and $\zeta (k_{q_1} k_{q_2} k_{q_3} k_{q_4})$ vanishes, except for the cases where q_1, q_2, q_3, q_4 represent respectively, q_1 the index 1 or 4, q_2 the index 2 or 5, q_3 the index 3 or 6, and q_4 the index 7.

Hence $$\Sigma k_{q_1} k_{q_2} k_{q_3} k_{q_4} \zeta (k_{q_1} k_{q_2} k_{q_3} k_{q_4}) = 2^3 k_1 k_2 k_3 k_7 \zeta (k_1 k_2 k_3 k_7),$$

and consequently τ_3 becomes

$$\pm 8\zeta (k_1 k_2 k_3 k_7) \times \{k_1 k_2 k_3 + 2k_7 (k_1 k_2 + k_1 k_3 + k_2 k_3)\}.$$

Hence we are able to predict that the general expression for our τ in the case before us will be

$$\begin{aligned}\tau_3 = \mp \Sigma \zeta (k_{q_1} k_{q_2} k_{q_3} k_{q_7}) \\ \times \{(k_{q_4}^3 + k_{q_5}^3 + k_{q_6}^3) - (k_{q_4}^2 + k_{q_5}^2 + k_{q_6}^2)(k_{q_1} + k_{q_2} + k_{q_3} + k_{q_7}) \\ + (k_{q_4} + k_{q_5} + k_{q_6})(k_{q_1} k_{q_2} + k_{q_1} k_{q_3} + k_{q_1} k_{q_7} + k_{q_2} k_{q_3} + k_{q_2} k_{q_7} + k_{q_3} k_{q_7}) \\ - 4 (k_{q_1} k_{q_2} k_{q_3} + k_{q_1} k_{q_2} k_{q_7} + k_{q_1} k_{q_3} k_{q_7} + k_{q_2} k_{q_3} k_{q_7})\}.\end{aligned}$$

For in the first place, the fact that the τ vanishes when more than three relations of equality exist between the k's, proves that we may assume τ_3 of the form

$$\Sigma \zeta (k_{q_1} k_{q_2} k_{q_3} k_{q_7}) \times \phi \{k_{q_1} k_{q_2} k_{q_3} k_{q_7};\ k_{q_4} k_{q_5} k_{q_6}\},$$

the semicolon (;) separating the k's into two groups, in respect of each of which severally ϕ is a symmetrical form. But if in the expression last above written for τ_3 we make

$$k_1 = k_4, \quad k_2 = k_5, \quad k_3 = k_6,$$

it becomes

$$\begin{aligned}\mp 8\zeta (k_1 k_2 k_3 k_7) \times \{(k_1^3 + k_2^3 + k_3^3) - (k_1^2 + k_2^2 + k_3^2)(k_1 + k_2 + k_3 + k_7) \\ + (k_1 + k_2 + k_3)(k_1 k_2 + k_1 k_3 + k_2 k_3 + k_1 k_7 + k_2 k_7 + k_3 k_7) \\ - 4 (k_1 k_2 k_3 + k_1 k_2 k_7 + k_1 k_3 k_7 + k_2 k_3 k_7)\}.\end{aligned}$$

Now in general if

$$\sigma_r = a_1^r + a_2^r + a_3^r + \dots + a_t^r,$$

and $$S_r = \Sigma (a_1 a_2 a_3 \dots a_r),$$

then $$\sigma_r - \sigma_{r-1} S_1 + \sigma_{r-2} S_2 \pm \dots \pm r S_r = 0.$$

Consequently the sum of the terms constituting the second factor in the above expression

$$= (3 - 4) k_1 k_2 k_3 + (2 - 4) k_7 (k_1 k_2 + k_1 k_3 + k_2 k_3).$$

Hence the above expression becomes

$$\pm 8\zeta(k_1k_2k_3k_7)\{k_1k_2k_3 + 2(k_1k_2 + k_1k_3 + k_2k_3)k_7\}.$$

Thus, then, whenever k_1, k_2, k_3 are respectively equal to any three of the quantities k_4, k_5, k_6, k_7, which may take place in twenty-four different ways (twenty-four being the number of permutations of four things), our τ_3 will have been correctly assumed; but $\zeta(k_{q_1}k_{q_2}k_{q_3}k_{q_7})$ being replaceable by $\zeta(h_{q_1}h_{q_2}h_{q_3}h_{q_7})$, the τ_3 may be treated as a cubic function in k_1, k_2, k_3, and arranged according to the powers of k_1, k_2, k_3 will contain only twenty terms; hence, since the assumed form is verified for more than twenty, that is, for twenty-four values of h_1, h_2, h_3, it follows that the assumed form is universally identical with the form of τ, which was to be determined.

Art. 40. Now, again, in order to facilitate the conception of the general proof, let us suppose fx to be of only five dimensions in x, i still remaining 3: it will no longer be possible when we suppose a multiplicity three to prevail among the roots, to conceive this multiplicity to be distributed into three parts, for that would require the existence of three pairs of roots, there being only five. But we may, if we please, make $h_1 = h_2 = h_3$, and $h_4 = h_5$, or else $h_1 = h_2 = h_3 = h_4$, or in any other mode conceive the multiplicity to be divided into two parts, 2 and 1 respectively, or to be taken collectively *en bloc*. As a mode of proceeding the more remote from that last employed, I shall choose the latter supposition. Then we obtain (τ now becoming τ_{5-2-2}, that is τ_1)

$$k_1k_2k_3k_4k_5\tau_1 = \pm \Sigma k_{q_1}k_{q_2}k_{q_3}k_{q_4} \times \Sigma k_{q_1}k_{q_2}\,\zeta(k_{q_1}k_{q_2}),$$

and $\zeta(k_{q_1}k_{q_2})$ will vanish, except in the case where q_1 represents the indices 1 or 2 or 3 or 4, and q_2 the index 5; also

$$\Sigma k_{q_1}k_{q_2}k_{q_3}k_{q_4} = k_{q_1}{}^4 + 4k_1{}^3k_5.$$

Hence our equation becomes

$$k_1{}^4k_5\tau = \pm (k_1{}^4 + 4k_1{}^3k_5)\,4k_1k_5\,\zeta(k_1k_5),$$

and τ becomes $$-4\zeta(k_1k_5)(k_1 + 4k_5).$$

If, now, we assume for the general value of τ in the case before us

$$\tau = \Sigma\zeta(k_{q_1}k_{q_5})\{(k_{q_2} + k_{q_3} + k_{q_4}) - 4(k_{q_1} + k_{q_5})\},$$

when $k_1 = k_2 = k_3 = k_4$, τ becomes

$$\pm 4\zeta(k_1k_5)\{3k_1 - (4k_1 + k_5)\},$$

that is $$\pm 4\zeta(k_1k_5)(k_1 + 4k_5).$$

Hence then for the two systems of values of h_1, h_2, h_3, namely

$$\left.\begin{matrix} h_1 = h_4 \\ h_2 = h_4 \\ h_3 = h_4 \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} h_1 = h_5 \\ h_2 = h_5 \\ h_3 = h_5, \end{matrix}\right.$$

the form of τ will have been correctly assumed. But since the derived form is a linear function of h_1, h_2, h_3, this is not enough to identify the assumed with the general form, since for such verification four systems of values must be taken, four being the number of terms in a function of three variables of the first degree. If, however, we had adopted a separation of the multiplicity three into two parts, and had started with supposing $k_1 = k_2 = k_3$, $k_4 = k_5$, we should have found that τ would have become

$$= 6\zeta(k_1, k_5)(2k_1 + 3k_5).$$

Moreover, when these equalities subsist,

$$k_1k_2k_3k_4 + k_1k_2k_3k_5 + k_1k_2k_4k_5 + k_1k_3k_4k_5 + k_2k_3k_4k_5$$

becomes $2k_1^3k_5 + 3k_1^2k_5^2$, and the common factor $k_1^2k_4$ disappears in the course of the operations for finding τ, and eventually we have to show (in order to support the universality of the previously assumed form for τ) that

$$k_{q_2} + k_{q_3} + k_{q_4} - 4(k_{q_1} + k_{q_5})$$

becomes $-2k_{q_4} - 3k_5$ when

$$k_{q_2} = k_{q_3} = k_{q_1} = k_1,$$

and

$$k_{q_4} = k_{q_5} = k_5,$$

which is evidently true. Hence then τ will have been correctly assumed for the following cases,

$$k_1 = k_2 = k_5 = k_3$$

$$k_1 = k_2 = k_5 = k_4;$$

and also for the cases

$$\left.\begin{array}{l} k_1 = k_2 = k_3 \text{ and } k_5 = k_4 \\ k_1 = k_5 = k_3 \text{ and } k_2 = k_4 \\ k_2 = k_5 = k_3 \text{ and } k_1 = k_4 \end{array}\right\}$$

$$\left.\begin{array}{l} k_1 = k_2 = k_4 \text{ and } k_5 = k_3 \\ k_1 = k_5 = k_4 \text{ and } k_2 = k_3 \\ k_2 = k_5 = k_4 \text{ and } k_1 = k_3 \end{array}\right\},$$

that is, for eight cases in all, whereas four only would have sufficed. Hence, *ex abundantiâ demonstrationis*, the form assumed for τ_1 is in the case before us the general form.

Art. 41. We may now easily write down the general form which τ assumes for all values of i and prove its correctness. If the roots be

$$h_1, h_2, h_3 \ldots h_m,$$

and

$$t_{m-i-1}f'x - \tau_{m-i-2}fx + \vartheta_i = 0,$$

we shall have

$$\pm \tau_{m-i-2} = \Sigma \{\zeta (h_{q_1} h_{q_2} h_{q_3} \dots h_{q_{m-i-1}}) \times [\sigma_{m-i-2} - \sigma_{m-i-3} S_1 + \sigma_{m-i-4} S_2 \mp \&c. + (-)^{m-i-3} \sigma_1 S_{m-i-3} + (-)^{m-i-2} (\sigma_0 + 1) S_{m-i-2}]\},$$

where σ_r denotes in general the sum of the rth *powers* of the $(i+1)$ quantities

$$(x - h_{q_{m-i}}),\ (x - h_{q_{m-i+1}}),\ \dots\ (x - h_{q_m}),$$

and S_r denotes in general the sum of the *products* of the *complementary* $(m - i - 1)$ quantities

$$(x - h_{q_1}),\ (x - h_{q_2}) \dots (x - h_{q_{m-i-1}})$$

combined r and r together. It will of course also be understood that $\sigma_0 = i + 1$, so that $\sigma_0 + 1 = i + 2$.

Art. 42. To prove the correctness of this general determination of the form of τ_{m-i-2}, let us suppose in general that $i + 1$ relations of equality spring up between the m quantities $k_1, k_2 \dots k_m$; we shall then easily obtain (N representing a certain numerical multiplier)

$$\pm Q = N\zeta (k_1, k_2 \dots k_{m-i-1}) \frac{\Sigma k_{q_1} k_{q_2} \dots k_{q_{m-1}}}{k_1^{\mu_1 - 1} k_2^{\mu_2 - 1} \dots k_{m-i-1}^{\mu_{m-i-1}}},$$

$k_1, k_2 \dots k_{m-i-1}$ being what the k system becomes when repetitions are excluded, and being respectively supposed to occur $\mu_1, \mu_2 \dots \mu_{m-i-1}$ times respectively, so that

$$\mu_1 + \mu_2 + \dots + \mu_{m-i-1} = m;$$

the fractional part of the right-hand member of the equation immediately above written will be readily seen to be equivalent to

$$\Sigma \mu_{\theta_{m-i-1}} k_{\theta_1} k_{\theta_2} \dots k_{\theta_{m-i-2}}.$$

To establish the correctness of the assumed form, we must be able, as in the particular cases previously selected, to prove two things; the one, and the more difficult thing to be proved is, that when the series of distinct quantities $k_1, k_2, k_3 \dots k_m$ become converted into μ_1 groups of k_1; μ_2 groups of $k_2, \dots \mu_{m-i-1}$ groups of k_{m-i-1}, then that

$$\Sigma \mu_{\theta_1} k_{\theta_2} k_{\theta_3} k_{\theta_4} \dots k_{\theta_{m-i-1}},$$

or in other terms

$$\Sigma \pm k_{\theta_1} k_{\theta_2} k_{\theta_3} \dots k_{\theta_{m-i-1}} \sum_{m-i-1}^{1} (\mu_\theta),$$

becomes identical with

$$\sigma_{m-i-2} - \sigma_{m-i-3} S_1 \pm \&c. + (-1)^{m-i-2} (\sigma_0 + 1) S_{m-i-2}.$$

The other step to be made, and with which I shall commence, consists in showing that the number of terms in the expression last above written, considered as a function of $(m - i - 2)$th degree of $(i + 1)$ variables, is never greater than the entire number of ways in which $(i + 1)$ quantities out of m quantities may be equated to the remaining $(m - i - 1)$ quantities, namely each of the first set respectively to all the same, or all different, or some the same and some different; in short, in any manner each of the $i+1$ quantities with some one or another (without restriction against repetitions) of the $m-i-1$ remaining quantities. This latter number being in fact the number of ways in which $(m-i-1)$ quantities may be combined $(i+1)$ together with repetitions admissible, by a well-known arithmetical theorem, is $(m-i-1)^{i+1}$, and the first number is $\frac{(i+1)(i+2)\dots(m-2)}{1.2\dots(m-i-2)}$, which is always less than the other. It remains then only to prove the remaining step of the demonstration*.

Art. 43. To fix the ideas let $m=10$, $i=5$, and consider the expression

$$(k_5^3+k_6^3+k_7^3+k_8^3+k_9^3+k_{10}^3)-(k_5^2+k_6^2+k_7^2+k_8^2+k_9^2+k_{10}^2)(k_1+k_2+k_3+k_4)$$
$$+(k_5+k_6+k_7+k_8+k_9+k_{10})(k_1k_2+k_1k_3+k_1k_4+k_2k_3+k_2k_4+k_3k_4)$$
$$-7(k_1k_2k_3+k_1k_2k_4+k_1k_3k_4+k_2k_3k_4).$$

Now suppose the six quantities $k_5, k_6, k_7, k_8, k_9, k_{10}$ to become respectively equal each to some one or another of the four quantities k_1, k_2, k_3, k_4, as for instance, I shall suppose

$$k_5 = k_6 = k_7 = k_1$$
$$k_8 = k_9 = k_2$$
$$k_{10} = k_3.$$

Then $\mu_1 = 4,\ \mu_2 = 3,\ \mu_3 = 2,\ \mu_4 = 1,$

and the formula of Art. 41 becomes

$$(3k_1^3+2k_2^3+k_3^3)-(3k_1^2+2k_2^2+k_3^2)(k_1+k_2+k_3+k_4)$$
$$+(3k_1+2k_2+k_3)(k_1k_2+k_1k_3+k_1k_4+k_2k_3+k_2k_4+k_3k_4)$$
$$-7(k_1k_2k_3+k_1k_2k_4+k_1k_3k_4+k_2k_3k_4)$$
$$=3[\{k_1^3-k_1^2(k_2+k_3+k_4)+k_1\}+k_1\{(k_2k_3+k_2k_4+k_3k_4)+k_4(k_2+k_3+k_4)\}]$$
$$+2\{k_2^3-k_2^2(k_1+k_3+k_4)+k_2\}+k_2\{(k_1k_3+k_1k_4+k_3k_4)+(k_2k_1+k_3)+k_4\}$$
$$+\{k_3^3-k_3^2(k_1+k_2+k_4)+k_3\}+k_3\{(k_1k_2+k_1k_4+k_2k_4)+k_3(k_1+k_2+k_4)\}$$
$$-(k_2k_3k_4+k_1k_3k_4+k_1k_2k_4+k_1k_2k_3)$$
$$=-k_1k_2k_3-2k_1k_2k_4-3k_1k_3k_4-4k_2k_3k_4$$
$$=-k_1k_2k_3k_4\left\{\frac{\mu_1}{k_1}+\frac{\mu_2}{k_2}+\frac{\mu_3}{k_3}+\frac{\mu_4}{k_4}\right\}.$$

* If this first step of the demonstration appear unsatisfactory or subject to doubt, it may be dispensed with, and the result obtained in the succeeding article (the demonstration of which is wholly unexceptionable) being assumed, it may be proved that the formula there obtained on a particular hypothesis must be universally true, in precisely the same way and by aid of the same Lemma in and by aid of which the formula obtained in the Supplement to this section for the simplified quotients to $\frac{f'x}{fx}$ upon a like particular hypothesis is shown to be of universal application, that is, by showing that otherwise a function of $2i-1$ variables would contain a function of $2i$ variables as a factor.

In the above investigation the quantities which with their repetitions make up the k's system, are k_4, k_1, k_2, k_3, *appearing* respectively 1, 2, 3, 4 times, that is to say *repeated* 0, 1, 2, 3 times; 7 is one more than the sum of the repetitions $0+1+2+3$, and the numbers 1, 2, 3, 4 arise from subtracting from 7 the sums $1+2+3$; $0+2+3$; $0+1+3$; $0+1+2$; respectively, so that the remainders 1, 2, 3, 4 denote respectively one more than the number of *repetitions* of k_4, k_1, k_2, k_3, that is, are the number of *appearances* of k_4, k_1, k_2, k_3; and thus with a slight degree of attention to the preceding process the reader may easily satisfy himself that the preceding demonstration (although not so expressed) is in essence universal, and the form of τ as an explicit function of x and of the roots of fx is thus completely established for all values of m and of i.

Supplement to SECTION III.

On the Quotients resulting from the process of continuous division ordinarily applied to two Algebraical Functions in order to determine their greatest Common Measure.

Art. (a)*. We have now succeeded in exhibiting the forms of the numerators and denominators of $\frac{f'x}{fx}$ developed into a continued fraction in terms of the differences of the roots and factors of fx. It remains to exhibit the quotients themselves of this continued fraction under a similar form.

LEMMA. *An equation being supposed of an arbitrary degree n, there exists no function of n and of less than $2i$ of the coefficients*†, *which vanishes for all values of n whenever the n roots reduce in any manner to i distinct groups of equal roots; or in other words, any function of n and the first $2i-1$ coefficients of an equation of the nth degree, which vanishes for all values of n in every case where the roots retain only i distinct names, must be identically zero.*

To render the statement of the proof more simple, let i be taken equal to 3. And let the roots be supposed to reduce to p roots a, q roots b, and

* The articles in this and subsequent sections to which Latin, Greek and Hebrew letters are prefixed, although in strict connexion with the context, are supplementary in the sense of having been supplied since the date when the paper was presented for reading to the Royal Society. All the articles marked with numbers (from 1 to 72), and the Introduction, appeared in the memoir as originally presented to the Society, June 16, 1853.

† In the proposition thus enunciated the coefficient of the highest power of x is supposed to be a numerical quantity.

r roots c. And let s_r in general denote the sum of the rth powers of the roots. Then we have evidently

$$\begin{aligned} p\ \ + q\ \ + r\ \ &= s_0, \\ pa\ + qb\ + rc\ &= s_1, \\ pa^2 + qb^2 + rc^2 &= s_2, \\ pa^3 + qb^3 + rc^3 &= s_3, \\ pa^4 + qb^4 + rc^4 &= s_4, \\ \&\text{c. } \&\text{c., } &\textit{ad infinitum.} \end{aligned}$$

Eliminating p, q, r between the first, second, third and fourth equations, we obtain

$$\begin{vmatrix} 1, & 1, & 1, & s_0 \\ a, & b, & c, & s_1 \\ a^2, & b^2, & c^2, & s_2 \\ a^3, & b^3, & c^3, & s_3 \end{vmatrix} = 0.$$

In like manner eliminating ap, bq, cr between the second, third, fourth and fifth equations, we have

$$\begin{vmatrix} 1, & 1, & 1, & s_1 \\ a, & b, & c, & s_2 \\ a^2, & b^2, & c^2, & s_3 \\ a^3, & b^3, & c^3, & s_4 \end{vmatrix} = 0;$$

and so in general we have for all values of e,

$$\begin{vmatrix} 1, & 1, & 1, & s_e \\ a, & b, & c, & s_{e+1} \\ a^2, & b^2, & c^2, & s_{e+2} \\ a^3, & b^3, & c^3, & s_{e+3} \end{vmatrix} = 0;$$

whence it may immediately be deduced, that, upon the given supposition of there being only three groups of distinct roots, we must have the following infinite system of coexisting equations satisfied, namely,

$$\begin{aligned} s_0t + s_1u + s_2v + s_3w &= 0 \quad \text{say} \quad L_0 = 0, \\ s_1t + s_2u + s_3v + s_4w &= 0 \quad\ \ ,, \quad\ \ L_1 = 0, \\ s_2t + s_3u + s_4v + s_5w &= 0 \quad\ \ ,, \quad\ \ L_2 = 0, \\ s_3t + s_4u + s_5v + s_6w &= 0 \quad\ \ ,, \quad\ \ L_3 = 0, \\ s_4t + s_5u + s_6v + s_7w &= 0 \quad\ \ ,, \quad\ \ L_4 = 0, \\ \&\text{c. } \&\text{c. } \&\text{c.} & \qquad\qquad\quad \&\text{c.}; \end{aligned}$$

and conversely, when this infinite system of equations is satisfied the roots must reduce themselves to three groups of equal roots.

Let now ϕ be any function of $s_0, s_1, s_2 \ldots$ which vanishes when this is the case. Then ϕ must necessarily contain as a factor some derivee of the infinite system of equations above written, that is, some function of $s_0, s_1, s_2 \ldots$ which vanishes when these equations are satisfied, that is, some conjunctive of the quantities $L_0, L_1, L_2, L_3 \ldots$; but it is obviously impossible in any such conjunctive to exclude s_6 from appearing, unless by introducing some other s with an index higher than 6, and consequently ϕ cannot be merely a function of $s_0, s_1, s_2, s_3, s_4, s_5$, nor consequently of n and the first five coefficients; or if such, it is identically zero. And so in general any function of n and only $2i-1$ of the coefficients which vanishes when the roots reduce to i groups of equal roots, must be identically zero; as was to be proved.

Art. (b). It ought to be observed that the preceding reasoning depends essentially upon the circumstance of n being left arbitrary. If n were given the proposition would no longer be true. In fact, on that supposition, the n roots reducing to i distinct roots would imply the existence of $n-i$ conditions between the n roots; and consequently $n-i$ independent equations would subsist between the n coefficients, and functions could be formed of i only of the coefficients, which would satisfy the prescribed condition of vanishing when the roots resolved themselves into i groups of distinct identities.

Art. (c). Let $D_{r_1, r_2 \ldots r_i}$ be used in general to denote the determinant

$$\begin{vmatrix} s_{r_1}, & s_{r_1+1}, & s_{r_1+2} \ldots s_{r_1+i-1} \\ s_{r_2}, & s_{r_2+1}, & s_{r_2+2} \ldots s_{r_2+i-1} \\ \ldots & \ldots & \ldots \\ s_{r_i}, & s_{r_i+1}, & s_{r_i+2} \ldots s_{r_i+i-1} \end{vmatrix};$$

then the simplified ith Sturmian residue R_i may be expressed under the form

$$D_{1,2,3\ldots i}\, x^{n-i-1} - D_{2,3\ldots i+1}\, x^{n-i-2} + D_{3,4\ldots i+2}\, x^{n-i-3} \ldots \pm D_{n-i, n-i-1 \ldots n},$$

which is easily identifiable with the known expression for such residue.

Now obviously the necessary and sufficient condition in order that the n roots may consist of only repetitions of i distinct roots is, that R_i shall be identically zero, that is to say, we must have

$$D_{1,2\ldots i} = 0, \quad D_{2,3\ldots i+1} = 0 \ldots D_{n-i, n-i-1 \ldots n} = 0.$$

But the reasoning of the preceding article shows that although these equations are necessary and sufficient, they are but a selected system of equations of an infinite number of similar equations which subsist*, and that, in fact,

* But *quære* whether any other *sufficient* system can be found of equations so few in number as this system.

whatever be the value of n, we may take $r_1, r_2 \ldots r_i$ perfectly arbitrary and as great as we please, and the equation

$$D_{r_1, r_2 \ldots r_i} = 0$$

must exist by virtue of the existence of the $n-i$ equations last above written.

Art. (d). I now return to the question of expressing the successive quotients of $\frac{f'x}{fx}$ as functions of the differences of the roots and factors; that they must be capable of being so expressed is an obvious consequence of the fact that the numerators and denominators of the convergents have been put under that form, since, if

$$\frac{N_{i-2}}{D_{i-2}}, \ \frac{N_{i-1}}{D_{i-1}}, \ \frac{N_i}{D_i},$$

are any three consecutive convergents of the continued fraction

$$\frac{1}{Q_1 -} \, \frac{1}{Q_2 -} \cdots \frac{1}{Q_i},$$

we must have

$$D_{i-2} N_i - N_{i-2} D_i = Q_i.$$

It would not, however, be easy to perform the multiplications indicated in the above equation, so as to obtain Q_i under its reduced form as a linear function of x. I proceed therefore to find Q_i constructively in the following manner.

Let R_{i-2}, R_{i-1}, R_i be three consecutive residues, $f'x$ counting as the residue in the zero place, then $Q_i = \frac{R_{i-2} - R_i}{R_{i-1}}$, and is of the form $\frac{p}{q} x + \frac{p'}{q'}$.

Now in general if we denote the n roots of fx, where the coefficient of x^n is supposed unity, by $h_1, h_2 \ldots h_n$, and if we use Z_i to denote $\Sigma \zeta (h_{\theta_1}, h_{\theta_2} \ldots h_{\theta_i})$*, with the convention that $Z_1 = n$, $Z_0 = 1$, we have, employing (i) to denote $\frac{1}{2} \{(-1)^i + 1\}$,

$$R_i \ = \frac{Z^2_{i-1} Z^2_{i-3} \ldots Z^2_{(i)}}{Z^2_i \ Z^2_{i-2} \ldots Z^2_{(i)+1}} \Sigma \{\zeta (h_{\theta_1}, h_{\theta_2} \ldots h_{\theta_{i+1}}) (x - h_{\theta_{i+2}}) (x - h_{\theta_{i+3}}) \ldots (x - h_{\theta_n})\},$$

$$R_{i-1} = \frac{Z^2_{i-2} Z^2_{i-4} \ldots Z^2_{(i)+1}}{Z^2_{i-1} Z^2_{i-3} \ldots Z^2_{(i)}} \Sigma \{\zeta (h_{\theta_1}, h_{\theta_2} \ldots h_{\theta_i}) (x - h_{\theta_{i+1}}) (x - h_{\theta_{i+2}}) \ldots (x - h_{\theta_n})\},$$

$$R_{i-2} = \frac{Z^2_{i-3} Z^2_{i-5} \ldots Z^2_{(i)}}{Z^2_{i-2} Z^2_{i-4} \ldots Z^2_{(i)+1}} \ \Sigma \{\zeta (h_{\theta_1}, h_{\theta_2} \ldots h_{\theta_{i-1}}) (x - h_{\theta_i}) (x - h_{\theta_{i+1}}) \ldots (x - h_{\theta_n})\}.$$

* ζ it will be remembered is the symbol of the operation of taking the product of the squares of the differences of the quantities which it governs.

The part of R_{i-1} within the sign of summation is

$$Z_i x^{n-i} - \Sigma\,(h_{\theta_{i+1}} + h_{\theta_{i+2}} + \dots + h_{\theta_n})\,\zeta\,(h_{\theta_1}, h_{\theta_2} \dots h_{\theta_i})\,x^{n-i-1} + \&c.,$$

say

$$Z_i x^{n-i} - Z_i' x^{n-i-1} + \&c.,$$

and the part of R_{i-2} within the sign of summation is

$$Z_{i-1} x^{n-i+1} - Z'_{i-1} x^{n-i} + \&c.,$$

and

$$Z_i^2 \frac{Z_{i-1} x^{n-i+1} - Z'_{i-1} x^{n-i}}{Z_i x^{n-i} - Z_i' x^{n-i-1}} = Z_{i-1} Z_i x + (Z_{i-1} Z_i' - Z_i Z'_{i-1}) + \text{an algebraic fraction.}$$

Hence

$$Q_i = \frac{1}{Z_i^2} \frac{Z^2_{i-3} Z^2_{i-5} \dots Z^2_{(i)}}{Z^2_{i-2} Z^2_{i-4} \dots Z^2_{(i)+1}} \left\{ \frac{Z^2_{i-2} Z^2_{i-4} \dots Z^2_{(i)+1}}{Z^2_{i-1} Z^2_{i-3} \dots Z^2_{(i)}} \right\}^{-1}$$

$$\times \{Z_{i-1} Z_i x + (Z_{i-1} Z_i' - Z_i Z'_{i-1})\}$$

$$= \frac{Z^2_{i-1}}{Z_i^2} \frac{Z^4_{i-3} Z^4_{i-5} \dots Z^4_{(i)}}{Z^4_{i-2} Z^4_{i-4} \dots Z^4_{(i)+1}} T_i,$$

T_i denoting $Z_{i-1} Z_i x + (Z_{i-1} Z_i' - Z_i Z'_{i-1})$.

Art. (*e*). If the process of obtaining the successive quotients and residues be considered, it will easily be seen that each step in the process imports two new coefficients into the quotients, the first quotient containing no literal quotient in the part multiplying x and containing the first literal coefficient in the other part, the second quotient containing two literal coefficients in the one part and three in the other, and in general the ith quotient containing $2i-2$ of the letters in the one part and $2i-1$ of them in the other. Hence T_i being made equal to $L_i x + M_i$, L_i contains $2i-2$ and M_i contains $2i-1$ of the literal coefficients of fx.

Moreover, we have Z_i of the form

$$T_i^2 \frac{P_{i-2} - mP_i}{P_{i-1}},$$

where

$$P_{i-1} = \Sigma\zeta\,(h_{\theta_1}, h_{\theta_2} \dots h_{\theta_i})\,\eta_{\theta_{i+1}} \eta_{\theta_{i+2}} \dots \eta_{\theta_n},$$

$$P_{i-2} = \Sigma\zeta\,(h_{\theta_1}, h_{\theta_2} \dots h_{\theta_{i-1}})\,\eta_{\theta_i} \eta_{\theta_{i+1}} \dots \eta_{\theta_n},$$

and P_i, which is the ith simplified residue, vanishes when the n roots in any manner become reduced to only i distinct groups.

I proceed to show that if we make

$$A_i x + B_i = U_i = A^2_{i,1}(x - h_1) + A^2_{i,2}(x - h_2) + \dots + A^2_{i,n}(x - h_n),$$

where in general

$$A_{i,e} \text{ represents } \Sigma\zeta\,(h_{\theta_1}, h_{\theta_2} \dots h_{\theta_{i-1}})\,(h_e - h_{\theta_1})(h_e - h_{\theta_2}) \dots (h_e - h_{\theta_{i-1}}),$$

then will

$$T_i = U_i.$$

It will be observed that $A_{i,e}$ is identical with what the simplified denominator of the $(i-1)$th convergent becomes when we write h_e in place of x, and consequently, when arranged according to the powers of h_e, will be of the form

$$c_1 h_e^{i-1} + c_2 h_e^{i-2} + \ldots + c_i,$$

where $c_1, c_2 \ldots c_i$ are functions of the coefficients, but containing no more of them than enter into Q_{i-1}, that is, containing only $2i-2$ of them.

Now A_i is made up of terms, each consisting of some binary product of

$$c_1, c_2 \ldots c_i,$$

combined with some term of the series

$$\Sigma h^{2i-2},\ \Sigma h^{2i-3} \ldots \Sigma h^0;$$

and any one of this latter set of terms expressed as a function of the coefficients of fx contains at most $2i-2$ of them.

Hence only $2i-2$ of the coefficients enter into A_i, and in like manner only $2i-1$ of them into B_i.

The number of letters, therefore, in A_i and in B_i is the same as in L_i and in M_i, namely $2i-2$ and $2i-1$ respectively.

Now let the roots consist of only i distinct groups of equal roots, so that T_i becomes $= Z_i^2 \dfrac{P_{i-2}}{P_{i-1}}$.

I shall show that in whatever way the equal roots are supposed to be grouped upon this supposition, there will result the equation

$$T_i = U_i,$$

where
$$T_i = \{\Sigma\zeta(\eta_{\theta_1}, \eta_{\theta_2} \ldots \eta_{\theta_i})\}^2 \frac{P_{i-2}}{P_{i-1}},$$

$$P_{i-2} = \Sigma\,\{\eta_{\theta_i}\eta_{\theta_{i+1}} \ldots \eta_{\theta_n}\,\zeta(\eta_{\theta_1}, \eta_{\theta_2} \ldots \eta_{\theta_{i-1}})\},$$

$$P_{i-1} = \Sigma\,\{\eta_{\theta_{i+1}}\eta_{\theta_{i+2}} \ldots \eta_{\theta_n}\,\zeta(\eta_{\theta_1}, \eta_{\theta_2} \ldots \eta_{\theta_i})\},$$

and
$$H_i = A_1^2\eta_1 + A_2^2\eta_2 + \ldots + A_n^2\eta_n,$$

A_e meaning $\Sigma\,\{(\eta_e - \eta_{\theta_1})(\eta_e - \eta_{\theta_2}) \ldots (\eta_e - \eta_{\theta_{i-1}})\,\zeta(\eta_{\theta_1}, \eta_{\theta_2} \ldots \eta_{\theta_{i-1}})\}$,

and η_ω meaning $x - h_\omega$.

Let the n factors be constituted of m_1 factors η_1, m_2 factors $\eta_2 \ldots m_i$ factors η_i. Then

$$Z_i = \mu\zeta(\eta_1, \eta_2 \ldots \eta_i),$$

where
$$\mu = m_1 m_2 \ldots m_i,$$

$$P_{i-1} = \mu\zeta(\eta_1, \eta_2 \ldots \eta_i)\,\eta_1^{m_1-1}\eta_2^{m_2-1} \ldots \eta_i^{m_i-1},$$

and
$$P_{i-2} = \mu_1\zeta(\eta_2, \eta_3 \dots \eta_i)\,\eta_1^{m_1}\eta_2^{m_2-1}\dots\eta_i^{m_i-1}$$
$$+\,\mu_2\zeta(\eta_1, \eta_3 \dots \eta_i)\,\eta_1^{m_1-1}\,\eta_2^{m_2}\dots\eta_i^{m_i-1}$$
$$+\text{ \&c. \&c.}$$
$$+\,\mu_i\zeta(\eta_1, \eta_2 \dots \eta_{i-1})\,\eta_1^{m_1-1}\eta_2^{m_2-1}\dots\eta_i^{m_i},$$

where
$$\mu_1 = \frac{\mu}{m_1},\ \mu_2 = \frac{\mu}{m_2}\dots\mu_i = \frac{\mu}{m_i}.$$

Hence
$$T_i = \mu^2\zeta(\eta_1, \eta_2 \dots \eta_i)\left\{\frac{\eta_1\zeta(\eta_2, \eta_3 \dots \eta_i)}{m_1} + \frac{\eta_2\zeta(\eta_1, \eta_3 \dots \eta_i)}{m_2} + \dots + \frac{\eta_i\zeta(\eta_1, \eta_2 \dots \eta_{i-1})}{m_i}\right\}.$$

Again, in U_i the term containing η_1 will be
$$m_1\eta_1\{\Sigma(\eta_1-\eta_2)(\eta_1-\eta_3)\dots(\eta_1-\eta_i)\,\zeta(\eta_2, \eta_3 \dots \eta_i)\}^2$$
$$= m_1\eta_1 \times (m_2 m_3 \dots m_i)^2 \times (\eta_1-\eta_2)^2(\eta_1-\eta_3)^2\dots(\eta_1-\eta_i)^2\,\{\zeta(\eta_2, \eta_3 \dots \eta_i)\}^2$$
$$= \frac{\mu^2}{m_1}\eta_1 \times \zeta(\eta_1, \eta_2 \dots \eta_i)\,\zeta(\eta_2, \eta_3 \dots \eta_i).$$

Hence
$$U_i = \mu^2\zeta(\eta_1, \eta_2 \dots \eta_i)\left\{\frac{\eta_1\zeta(\eta_2\eta_3 \dots \eta_i)}{m_1} + \frac{\eta_2\zeta(\eta_1\eta_3 \dots \eta_i)}{m_2} + \text{\&c.}\right\} = T_i.$$

Hence, therefore, $U_i - T_i$ vanishes whenever the roots of fx contain only i distinct groups of equal roots, and it has been shown that U_i and T_i each contain only $2i-1$ of the coefficients of fx, so that $U_i - T_i$ is a function only of n and these $2i-1$ letters, and consequently, by virtue of the Lemma in Art. (a), $U_i - T_i$ is universally zero, that is, U_i is identical with T_i, as was to be proved. In the same manner, as observed in a preceding note [p. 494], the expression given in the antecedent articles for the numerator of the ith convergents, having been verified for the case of the roots consisting of only i distinct groups, could have been at once inferred to be generally true by aid of the Lemma above quoted.

Art. (f). Since the coefficient of x in T_i is $Z_{i-1} \times Z_i$, we deduce the unexpected relation
$$\Sigma\zeta(h_1, h_2 \dots h_{i-1}) \times \Sigma\zeta(h_1, h_2 \dots h_i) = P_1^2 + P_2^2 + \dots + P_n^2,$$
where
$$P_e = \Sigma\{(h_e - h_{\theta_1})(h_e - h_{\theta_2})\dots(h_e - h_{\theta_{i-1}})\,\zeta(h_{\theta_1}, h_{\theta_2}\dots h_{\theta_{i-1}})\}.$$

So that every simplified Sturmian quotient to $\dfrac{f'x}{fx}$, when the n roots of fx are real, will be the sum of n squares. But the equation is otherwise remarkable, in exhibiting the product of the sum of $\dfrac{n(n-1)\dots(n-i+2)}{1.2\dots(i-1)}$ squares by another sum of $\dfrac{n(n-1)\dots(n-i+1)}{1.2\dots i}$ squares under the form of the sum of n squares.

If we denote the ith simplified denominator to the Sturmian convergents to $\frac{f'x}{fx}$ by $D_i x$, and if we call the ith simplified quotient $X_i x$, we have

$$X_i x = \sum_n^1 (D_{i-1} h_e)^2 (x - h_e).$$

If we construct the numerators and denominators of the convergents to

$$\frac{1}{Q_1 -}\,\frac{1}{Q_2 -}\,\frac{1}{Q_3}\cdots\frac{1}{Q_i},$$

according to the general rule for continued fractions, as functions of Q_1, Q_2, Q_3, &c., so that calling the denominators $\Delta_1, \Delta_2, \Delta_3 \ldots \Delta_i$,

$$\Delta_1 = Q,\;\; \Delta_2 = Q_1 Q_2 - 1 \ldots \Delta_i = Q_i \Delta_{i-1} - \Delta_{i-2},$$

we have
$$\Delta_{i-1} x = \frac{Z^2_{i-2} Z^2_{i-4} \ldots Z^2_{(i-1)}}{Z^2_{i-1} Z^2_{i-3} \ldots Z^2_{(i)}} D_{i-1} x,$$

$\Delta_{i-1} x$ being in fact the multiplier of $f'x$ in the equation which connects fx and $f'x$ with the $(i-1)$th *complete* residue, and consequently, retaining $Q(x)$ to designate the complete ith quotient, we have

$$Q_i(x) = \frac{Z^2_{i-1}}{Z_i^2} \frac{Z^4_{i-3} Z^4_{i-5} \ldots Z^4_{(i)}}{Z^4_{i-2} Z^4_{i-4} \ldots Z^4_{(i)+1}} \Sigma \{D_{i-1} h_e\}^2 (x - h_e)$$

$$= \frac{Z^6_{i-1}}{Z_i^2} \frac{Z^8_{i-3} Z^8_{i-5} \ldots Z^8_{(i)}}{Z^8_{i-2} Z^8_{i-4} \ldots Z^8_{(i)+1}} \Sigma \{\Delta_{i-1} h_e\}^2 (x - h_e),$$

which equation gives the connexion between the form of any quotient and that of the immediately preceding convergent denominator of the continued fraction which expresses $\frac{f'x}{fx}$.

Art. (g). I have found that the coefficients of the n factors of fx in the expression above given for the quotients possess the property that the sum of their square roots taken with the proper signs is zero for each quotient except the first (the coefficients for the first being all units), that is $D_i h_1 + D_i h_2 + \ldots D_i h_n = 0$ for all values of i except $i = 1$. Moreover I find that the determinant formed by the n sets of the n coefficients of the factors of fx in the complete set of n quotients is identically zero, that is, the determinant represented by the square matrix

$$\begin{vmatrix} 1, & 1, & 1, & 1, \\ (D_1 h_1)^2, & (D_1 h_2)^2, & (D_1 h_3)^2 & \ldots (D_1 h_n)^2 \\ (D_2 h_1)^2, & (D_2 h_2)^2, & (D_2 h_3)^2 & \ldots (D_2 h_n)^2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (D_{n-1} h_1)^2, & (D_{n-1} h_2)^2, & (D_{n-1} h_3)^2 & \ldots (D_{n-1} h_n)^2 \end{vmatrix} = 0.$$

Art. (h). It should be observed that U_i is the form of the simplified quotients for all the quotients except the nth (that is, the last), for which the simplified form is not U_n, but $U_n \div \zeta(h_1, h_2 \dots h_n)$, which arises from the circumstance of the last divisor, which is the final Sturmian residue, not containing x; it being evidently the case that the division of a rational function of x by another one degree lower, introduces into the integral part of the quotient the square of the leading coefficient of the divisor, *subject to the exception* that when the divisor is of the degree zero, the simple power enters in lieu of the square. The general formula gives for the reduced nth quotient the expression

$$\Sigma\,\{(h_1 - h_2), (h_1 - h_3) \dots (h_1 - h_n)\,\zeta(h_2, h_3 \dots h_n)\}^2 (x - h_1),$$

which equals

$$\zeta(h_1, h_2 \dots h_n)\,\Sigma\zeta(h_2, h_3 \dots h_n)\,(x - h_1).$$

Rejecting the first factor, we have

$$\Sigma\zeta(h_2, h_3 \dots h_n)\,(x - h_1),$$

which is equal to the penultimate residue, which residue is (as it evidently ought to be) identical with the simplified last quotient.

Art. (i). We have thus succeeded in giving a perfect representation of $\dfrac{f'x}{fx}$, that is, of

$$\frac{1}{x - h_1} + \frac{1}{x - h_2} + \dots + \frac{1}{x - h_n},$$

under the form of a continued fraction of the form

$$\frac{1}{m_1(x - e_1) -}\;\frac{1}{m_2(x - e_2) -}\;\cdots\cdots\;\frac{1}{m_n(x - e_n)},$$

where $m_1, m_2 \dots m_n$; $e_1, e_2 \dots e_n$ are all determinate and known functions of $h_1, h_2 \dots h_n$.

We may by means of this identity, differentiating any number of times with respect to x both sides of the equation, obtain analogous expressions for the series

$$\frac{1}{(x - h_1)^t} + \frac{1}{(x - h_2)^t} + \dots + \frac{1}{(x - h_n)^t}.$$

But to do this we must be in possession of a rule for the differentiation of continued fractions whose quotients are linear functions of the variable. I subjoin here the first step only toward such investigation.

Let the denominator of

$$\frac{1}{q_1 -}\;\frac{1}{q_2 -}\;\cdots\cdots\;\frac{1}{q_n},$$

where $q_1, q_2 \cdots q_n$ are any n arbitrary quantities, be denoted by $[q_1, q_2, q_3 \cdots q_n]$, so that the entire fraction will be equal to

$$\frac{[q_2, q_3 \cdots q_n]}{[q_1, q_2, q_3 \cdots q_n]}.$$

Any such quantity as $[q_i, q_{i+1} \cdots q_n]$ may be termed a Cumulant, of which $q_i, q_{i+1} \cdots q_n$ may be severally termed the elements or Components, and the complete arrangement of the elements may be termed the Type. The cumulant corresponding to any type remains unaffected by the order of the elements in the type being reversed, as is evident from any cumulant being in fact representable under the form of a symmetrical determinant, thus, for example, the cumulant $[q_1, q_2, q_3, q_4]$ may be represented by the determinant

$$\begin{vmatrix} q_1, & 1, & 0, & 0 \\ 1, & q_2, & 1, & 0 \\ 0, & 1, & q_3, & 1 \\ 0, & 0, & 1, & q_4 \end{vmatrix},$$

and $[q_4, q_3, q_2, q_1]$ will in like manner be represented by the determinant

$$\begin{vmatrix} q_4, & 1, & 0, & 0 \\ 1, & q_3, & 1, & 0 \\ 0, & 1, & q_2, & 1 \\ 0, & 0, & 1, & q_1 \end{vmatrix},$$

which is equal to the former.

Art. (j). Let it be proposed in general to find the first differential coefficient in respect to x of the fraction

$$\frac{[q_i, q_{i+1} \cdots q_n]}{[q_1, q_2, q_3 \cdots q_n]} = F_i,$$

where each q is a function of one or more variables.

I find that the variation of F_i may be expressed as follows:

$$\begin{aligned} -\delta F_i = \{&\delta\,[q_1, q_2 \cdots q_{i-2}, q_n] + \delta\,[q_1, q_2 \cdots q_{i-2}, q_{n-1}]\,q_n^2 \\ &+ \delta\,[q_1, q_2, q_3 \cdots q_{i-2}, q_{n-2}]\,[q_n, q_{n-1}]^2 + \cdots \\ &+ \delta\,[q_1, q_2, q_3 \cdots q_{i-2}, q_{i-1}]\,[q_n, q_{n-1}, q_{n-2} \cdots q_i]^2\} \\ &\div [q_1, q_2, q_3 \cdots q_n]^2. \end{aligned}$$

Art. (*k*). Suppose $i=2$, and $q_1=a_1x+b_1$, $q_2=a_2x+b_2 \dots q_n=a_nx+b_n$, we shall have by virtue of the above equation,

$$\frac{d}{dx}F_2, \text{ that is } \frac{d}{dx}\left\{\frac{1}{q_1-}\frac{1}{q_2-}\frac{1}{q_3}\cdots\frac{1}{q_n}\right\}$$

$$=-\frac{1}{[q_1, q_2 \dots q_n]^2}\{a_n 1^2+a_{n-1}q_n^2+a_{n-2}[q_n, q_{n-1}]^2+\text{\&c.}$$

$$+a_1[q_n, q_{n-1}, q_{n-2} \dots q_2]^2\}.$$

If we call $F_2=\frac{\phi x}{fx}$ every such quantity as $[q_n, q_{n-1} \dots q_i]$ represents to a constant factor *près* the $(i-1)$th simplified residue (ϕx counting as the first of them) to $\frac{\phi x}{fx}$, and making certain obvious but somewhat tedious reductions, and rejecting the common factor $-\frac{1}{(fx)^2}$, we obtain the expression

$$\frac{C_0R_1^2}{C_1}+\frac{R_2^2}{C_1C_2}+\frac{R_3^2}{C_2C_3}+\dots+\frac{R_n^2}{C_{n-1}C_n}=\phi xf'x-\phi' xfx,$$

where $R_1, R_2 \dots R_n$ represent ϕx and the successive simplified residues to fx, ϕx, while C_i means the coefficient of the highest power of x in R_i, and C_0 the first coefficient in fx*.

Art. (*l*). If we take gx of the same degree as fx, and for greater simplicity make the first coefficients in fx and gx, each of them unity,

* This result may be obtained directly as follows :—

Let fx, ϕx and the $(m-1)$ complete Sturmian residues be called $\rho_0, \rho_1, \rho_2 \dots \rho_n$; let the n complete quotients be called $q_1, q_2 \dots q_n$, and let the allotrious factors to the residues $\rho_2, \rho_3 \dots \rho_n$ be called $\mu_2, \mu_3 \dots \mu_n$; then

$$\rho_0=q_1\rho_1-\rho_2,\ \rho_1=q_2\rho_2-\rho_3,\ \rho_2=q_3\rho_3-\rho_4, \text{ \&c.};$$

hence
$$\rho_1\delta\rho_0-\rho_0\delta\rho_1=\rho_1^2\delta q_1+(\rho_2\delta\rho_1-\rho_0\delta\rho_2)$$
$$=\rho_1^2\delta q_1+\rho_2^2\delta q_2+(\rho_3\delta\rho_2-\rho_2\delta\rho_3)$$
$$=\text{\&c.}$$
$$=\rho_1^2\delta q_1+\rho_2^2\delta q_2+\rho_3^2\delta q_3+\dots+\rho_n^2\delta q_n;$$

but we have in general $\rho_i=\mu_iR_i$,

hence
$$\delta q_i=\frac{C_{i-1}}{C_i}\frac{\mu_{i-1}}{\mu_i}\delta x,$$

and
$$\rho_i^2\delta q_i=\frac{C_{i-1}}{C_i}\mu_{i-1}\mu_iR_i^2\delta x;$$

but it may be easily seen that

$$\mu_{i-1}\mu_i=\frac{1}{C^2_{i-1}}, \text{ except when } i=1, \text{ for which case } \mu_{i-1}\mu_i=1,$$

hence
$$\rho_i^2\delta q_i=\frac{1}{C_{i-1}C_i}R_i^2\delta x, \text{ when } i>1, \text{ and } =\frac{C_0}{C_1}R_1^2\delta x \text{ when } i=1,$$

which proves the theorem in the text.

the successive simplified residues to $\frac{gx}{fx}$ will be identical with the simplified residues to $\frac{-fx+gx}{gx}$ (including amongst them the quantity $gx-fx$ itself), and, since

$$\{fx-gx\}\,g'x-\{fx-gx\}'\,gx=g'xfx-f'xgx,$$

the right-hand side of the equation above written, when the residues, instead of referring to f and ϕ, are made to refer to f and g, taken of the same degree in x, becomes equal to $f'xgx-fxg'x$; and if we now agree to consider f and g as homogeneous functions each of the nth degree in x and 1, the equation becomes

$$\begin{aligned}
&\frac{R_1^2}{C_1}+\frac{R_2^2}{C_1C_2}+\frac{R_3^2}{C_2C_3}+\ldots+\frac{R_n^2}{C_{n-1}C_n}\\
&\quad=g\,(x,\,1)\,\frac{d}{dx}f(x,\,1)-f\,(x,\,1)\,\frac{d}{dx}\,g\,(x,\,1)\\
&\quad=\frac{1}{n}\left(x\,\frac{d}{dx}g+\frac{d}{d1}\,g\right)\left(\frac{d}{dx}f\right)-\frac{1}{n}\left(x\,\frac{d}{dx}f+\frac{d}{d1}\,f\right)\left(\frac{d}{dx}g\right)\\
&\quad=\frac{1}{n}\left\{\frac{df}{dx}\frac{dg}{d1}-\frac{df}{d1}\frac{dg}{dx}\right\}=\frac{1}{n}\,J\,(f,\,g),
\end{aligned}$$

where J indicates the Jacobian of the given functions f and g in respect to the variables x and 1, meaning thereby the so-called *Functional Determinant* of Jacobi to f and g in respect of x and 1, which equation also obviously must continue to hold good when we restore to the coefficients of x^n in f and g their general values.

It may happen that for particular relations between the coefficients of f and g certain of the residues may be wanting, which will be the case when any of the secondary Bezoutics have their first or successive terms affected with the coefficient zero; the equation connecting the residues with the Jacobian will then change its form (as some of the quantities $C_1,\ C_2 \ldots C_n$ will become zero); but I do not propose to enter for the present into the theory of these failing, or as they may more properly be termed, Singular cases in the theory of elimination.

Art. (m). The series last obtained for $J\,(f,\,g)$ leads to a result of much interest in the theory, and of which great use is made in the concluding section of this memoir, namely the identification of the Jacobian (abstraction made of the numerical factor n) with what the Bezoutiant becomes when in place of the n variables in it, $u_1,\ u_2 \ldots u_n$, we write $x^{n-1},\ x^{n-2} \ldots x,\ 1$. Thus suppose f and g to be each of the third degree, and let

$$Ax^2+Hx+G,$$

$$Hx^2+Bx+F,$$

$$Gx^2+Fx+C,$$

be the three primary Bezoutics; if we make

$$x^2 = u, \quad x = v, \quad 1 = w,$$

these may be written under the form

$$\begin{aligned} Au + Hv + Gw &= L, \\ Hu + Bv + Fw &= M, \\ Gu + Fv + Cw &= N, \end{aligned}$$

and if the Bezoutiant be called $\mho$, we have

$$L = \frac{d\mho}{du}, \quad M = \frac{d\mho}{dv}, \quad N = \frac{d\mho}{dw}.$$

The simplified residues to f and g are L, (L, M), (L, M, N), where (L, M) means the result of eliminating u between L and M, and (L, M, N) the result of eliminating u and v between L, M, N: and by a theorem (virtually implied in the direct method* of reducing a quadratic function to the form of a sum of squares), if we call the leading coefficients of these quantities C_1, C_2, C_3, we have

$$\frac{L^2}{C_1} + \frac{(L, M)^2}{C_1C_2} + \frac{(L, M, N)^2}{C_2C_3} = \mho.$$

Hence, when $n=3$, $\frac{1}{3}J(f, g) = \mho$ when in $\mho$, u, v, w are turned into $x^2, x, 1$; and so in general for any values of n, the Bezoutiant correspondingly modified, becomes $\frac{1}{n}J(f, g)$, as was to be shown†.

Art. (n). The expressions obtained for the quotients to $\frac{f'x}{fx}$ may be generalized and extended to the quotients to $\frac{\phi x}{fx}$, where ϕx and fx are two functions of x of any degrees m and n, whose roots are respectively, $k_1, k_2 \ldots k_m$, and $h_1, h_2 \ldots h_n$. If we suppose

$$\frac{\phi x}{fx} = \frac{1}{Q(x) -} \, \frac{1}{q_2(x) -} \, \frac{1}{q_3(x) -} \cdots \frac{1}{q_{m+1}(x)},$$

where $Q(x)$ is of $n - m$ dimensions, and $q_2(x)$, $q_3(x) \ldots q_{m+1}(x)$ each of one dimension in x, it may be proved that on writing

$$\frac{1}{Q(x) -} \, \frac{1}{q_2(x) -} \cdots \frac{1}{q_i(x)} = \frac{N_i(x)}{D_i(x)},$$

* Namely, that of M. Cauchy, adverted to in Section IV. Arts. 44—45. [p. 511 below.]

† Compare Jacobi, *De Eliminatione*, § 2. The general expression for the allotrious factor, I may here incidentally mention, is given under the head Theorem *a*, § 16, which comes quite at the end of the same paper.

we shall have

$$\sum_{\theta=1}^{m}\left\{(N_i k_\theta)^2 \frac{fk_\theta}{\phi' k_\theta}(x-k_\theta)\right\} = Cq_{i+1}(x), \tag{A}$$

$$\sum_{\theta=1}^{n}\left\{(D_i h_\theta)^2 \frac{\phi h_\theta}{f' h_\theta}(x-h_\theta)\right\} = C'q_{i+1}(x), \tag{B}$$

where

$$C \pm C' = 0, \tag{E}$$

$Cq_{i+1}(x)$ being the $(i+1)$th simplified quotient. When $Q(x)$ is a linear function of x, in finding $q_1 x$ from the formula (B), we must take $D_0 x = 1$. The proof of this theorem being generally true, may easily be shown to depend upon its being true in the special case*, when $m = \mu + i$, and $n = \mu + i'$ (m being supposed less than n), and $h_1, h_2 \ldots h_n$ become $l_1, l_2 \ldots l_\mu, h_1, h_2 \ldots h_{i'}$, while $k_1, k_2 \ldots k_m$ become $l_1, l_2 \ldots l_\mu, k_1, k_2 \ldots k_i$; and the truth of the theorem for this special case (if for instance we wish to prove the formula (B)) depends upon the expression

$$\begin{pmatrix} h_1, & h_2 \ldots h_{i'-1} \\ k_1, & k_2 \ldots k_m \end{pmatrix} \div \begin{pmatrix} h_1, & h_2 & \ldots h_{i'-1} \\ h_{i'}, & h_{i'+1} \ldots h_n \end{pmatrix}$$
$$\times \begin{pmatrix} h_1, & h_2 \ldots h_{i'} \\ k_1, & k_2 \ldots k_m \end{pmatrix} \div \begin{pmatrix} h_1, & h_2 & \ldots h_{i'} \\ h_{i'+1}, & h_{i'+2} \ldots h_n \end{pmatrix}$$

being identical with the expression

$$\left\{\begin{pmatrix} h_1, & h_2 \ldots h_{i'-1} \\ k_1, & k_2 \ldots k_m \end{pmatrix} \div \begin{pmatrix} h_1, & h_2 & \ldots h_{i'-1} \\ h_{i'}, & h_{i'+1} \ldots h_n \end{pmatrix} \times (h_{i'} - h_1)(h_{i'} - h_2) \ldots (h_{i'} - h_{i'-1})\right\}$$
$$\times \frac{\begin{pmatrix} h_{i'} & \\ k_1, & k_2 \ldots k_m \end{pmatrix}}{\begin{pmatrix} h_{i'} & \\ h_1, & h_2 \ldots h_{i'-1}\, h_{i'+1} \ldots h_n \end{pmatrix}},$$

as it may readily be shown to be. And the formula (A) may be verified in precisely the same manner. There is no difficulty in finding the values of C and C', which are products of powers, some positive and some negative, of the leading coefficients in the simplified residues, and recognising that they satisfy the equation (E); when ϕx is of one degree below fx this equation is of the form $C + C' = 0$.

Art. (o). When $\phi x = f'x$, this expression for the $(i+1)$th simplified quotient becomes $\Sigma (D_i h)^2 (x-h)$, as previously found; the correlative expression will be

$$-\Sigma (N_i k)^2 \frac{fk}{f''k}(x-k),$$

* By virtue of the Lemma, that when ϕx and fx are two algebraical functions, no function of the coefficients vanishing identically when i roots of fx coincide with i roots of ϕx respectively can be formed, in which there are fewer of the coefficients of f and ϕ respectively than appear in the leading coefficient of the $(n-i+1)$th residue of $\frac{\phi}{f}$.

k being any root of $f'x = 0$, which is equal to the former expression. The general expressions above given for the simplified quantities are of course integral functions of h and k, although given under the form of the sums of fractions, by virtue of the well-known theorem that $\Sigma \frac{\vartheta h}{f'h}$, where ϑ is an integral function of h, and the summation comprises all the roots (h) of $fh = 0$, is always integral.

Art. (p). It will be found that for all values of i greater than unity

$$\sum_{\theta=1}^{m} (N_i k_\theta) \frac{fk_\theta}{\phi' k_\theta} = 0,$$

and that

$$\sum_{\theta=1}^{n} (D_i h_\theta) \frac{\phi h_\theta}{f' h_\theta} = 0.$$

The theorem of Art. (n) is in effect a theorem of cumulants of the form

$$[Q_1(x),\ q_2(x) \dots q_i(x) \dots q_n(x)],$$

where the elements are all independent of one another, and

$$fx = [Q_1(x),\ q_2(x),\ q_3(x) \dots q_n(x)],\quad \phi x = [q_2(x),\ q_3(x) \dots q_n(x)],$$

n being any *number whatever greater* than i; this makes the theorem still more remarkable. The urgency of the press precludes my investigating for the present the more general theorem which must be presumed to exist, whereby q_{i+1} can be connected with $[q_1, q_2, q_3 \dots q_i]$, or $[q_2, q_3 \dots q_i]$, and with $[q_1, q_2, q_3 \dots q_{i+e}]$ and $[q_2, q_3 \dots q_{i+e}]$, when each q represents a function of an *arbitrary* degree in x. The theorem so generalized would comprehend the complete theory of the quotients arising from the process of continued division, without exclusion of the singular cases (at present supposed to be excluded) where one or several consecutive principal coefficients in one or more of the residues, vanish.

Art. (q). The complete statement of two twin theorems suggested by and intimately connected with the biform representation of the quotients $\frac{\phi x}{fx}$, given in the preceding article, is too remarkable to be omitted.

Suppose $\phi x = f'x$, and let the successive convergents to $\frac{f'x}{fx}$ be called

$$\frac{1}{T_1 x},\ \frac{t_1 x}{T_2 x} \cdots \frac{t_{n-2} x}{T_{n-1} x},\ \frac{t_{n-1} x}{T_n x},$$

where the subscript index to t or T indicates the degree in x. Then if we call the roots of fx, $h_1, h_2 \dots h_n$, the theorem already cited in a preceding article, concerning the denominators of the convergents, may be expressed as follows:—

$$\begin{vmatrix} \left(\frac{f'h_1}{\phi h_1}\right)^2, & \left(\frac{f'h_2}{\phi h_2}\right)^2 & \dots & \left(\frac{f'h_n}{\phi h_n}\right)^2 \\ (T_1 h_1)^2, & (T_1 h_2)^2 & \dots & (T_1 h_n)^2 \\ (T_2 h_1)^2, & (T_2 h_2)^2 & \dots & (T_2 h_n)^2 \\ \dots & \dots & \dots & \dots \\ (T_{n-1} h_1)^2, & (T_{n-1} h_2)^2 & \dots & (T_{n-1} h_n)^2 \end{vmatrix} = 0,$$

where it will be observed that the first line of terms consists exclusively of units, since $f'x = \phi x$ by hypothesis.

Correlatively I have ascertained that preserving the same assumption that $\phi x = f'x$, so that consequently $\frac{\phi' k}{fk}$ means $\frac{f''k}{fk}$, the following theorem obtains, namely that if $k_1, k_2 \dots k_{n-1}$ are the $(n-1)$ roots of ϕx,

$$\begin{vmatrix} \left(\frac{\phi' k_1}{fk_1}\right)^2, & \left(\frac{\phi' k_2}{fk_2}\right)^2 & \dots & \left(\frac{\phi' k_{n-1}}{fk_{n-1}}\right)^2 \\ \{t_1(k_1)\}^2, & \{t_1(k_2)\}^2 & \dots & \{t_1(k_{n-1})\}^2 \\ \{t_2(k_1)\}^2, & \{t_2(k_2)\}^2 & \dots & \{t_2(k_{n-1})\}^2 \\ \dots & \dots & \dots & \dots \\ \{t_{n-2}(k_1)\}^2, & \{t_{n-2}(k_2)\}^2 & \dots & \{t_{n-2}(k_{n-1})\}^2 \end{vmatrix} = 0.$$

It may consequently be conjectured, when ϕ and f are independent functions of x and respectively of the degree $n-1$ and n, and $\frac{\phi x}{fx}$ is expanded under the form of a continued fraction, of which, as before, $\frac{1}{T_1}, \frac{t_1}{T_2} \dots \frac{t_{n-1}}{T_n}$ are the successive convergents, that we shall have analogous determinants to the twin forms above given, each separately vanishing, these more general determinants differing only from their model forms in respect of the uppermost line of terms in the one of them, being each multiplied by certain functions of $h_1, h_2 \dots h_n$ respectively (all of which become units when $\phi x = f'x$), and in the other of them by certain functions of $k_1, k_2 \dots k_n$.

The exact form, however, of such functions, and even the possibility of such form being found capable of making the determinants vanish, remains open for further inquiry.

Section IV.

On some further Formulæ connected with M. Sturm's theorem, and on the Theory of Intercalations, whereof that theorem may be treated as a corollary.

Art. 44. As preparatory to some remarks about to be made on the formulæ connected with M. Sturm's theorem, it is necessary to premise two theorems of great importance concerning quadratic functions, one of which, notwithstanding its extreme simplicity, is as far as I know very little (if at all) known, and the other was given in part many years ago by M. Cauchy, but is also not generally known. The former of these two theorems is as follows. If a quadratic homogeneous function of any number of variables be (as it may be in an infinite variety of ways) transformed into a function of a new set of variables, linearly connected by real coefficients with the original set, in such a way that only positive and negative squares of the new variables appear in the transformed expression, the number of such positive and negative squares respectively will be constant for a given function whatever be the linear transformations employed. This evidently amounts to the proposition, that if we have $2n$ positive and negative squares of homogeneous real linear functions of n variables identically equal to zero, the number of positive squares and of negative squares must be equal to one another, so that for example we cannot have

$$u_1^2 + u_2^2 + \dots + u_n^2 + u^2_{n+1} - u^2_{n+2} - u^2_{n+3} - \dots - u^2_{2n}$$

identically zero when n of the variables are linear functions of the remaining n; and this is obviously the case, for if the equation could be identically satisfied we might make

$$u_{n+2} = u_1, \quad u_{n+3} = u_2 \dots u_{2n} = u_{n-1},$$

and we should then be able to find u_{n+1} as a real numerical multiple of u_n, and consequently should have the equation $u_n^2 \{1 + k^2\} = 0$, which is obviously impossible; *à fortiori* we may prove that in the identical equation existing between the sum of an even number of positive and of negative squares of real linear functions of half the number of independent variables, there cannot be *more* than a difference of two (as we have proved that there cannot be that difference) between the number of positive and negative squares. Hence there must be as many of one as of the other; and as a consequence, the number of positive squares or of negative squares in the transform of a given quadratic function of any number of variables effected by any set of real linear substitutions is constant, being in fact some unknown transcendental function of the coefficients of the given function. I quote this law (which I have enunciated before, but of which I for the first time publish the proof) under the name of the law of inertia for quadratic forms.

Art. 45. The other theorem is the following. If any quadratic function be represented in the umbral notation* under the form of

$$(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2,$$

where $a_1, a_2 \ldots a_n$ are the umbræ of the coefficients, and $x_1, x_2 \ldots x_n$ the variables, then by writing

$$\begin{vmatrix} a_1 \\ a_1 \end{vmatrix} x_1 + \begin{vmatrix} a_1 \\ a_2 \end{vmatrix} x_2 + \begin{vmatrix} a_1 \\ a_3 \end{vmatrix} x_3 + \begin{vmatrix} a_1 \\ a_4 \end{vmatrix} x_4 + \ldots + \begin{vmatrix} a_1 \\ a_n \end{vmatrix} x_n = y_1,$$

$$\begin{vmatrix} a_1, a_2 \\ a_1, a_2 \end{vmatrix} x_2 + \begin{vmatrix} a_1, a_2 \\ a_1, a_3 \end{vmatrix} x_3 + \begin{vmatrix} a_1, a_2 \\ a_1, a_4 \end{vmatrix} x_4 + \ldots + \begin{vmatrix} a_1, a_2 \\ a_1, a_n \end{vmatrix} x_n = y_2,$$

$$\begin{vmatrix} a_1, a_2, a_3 \\ a_1, a_2, a_3 \end{vmatrix} x_3 + \begin{vmatrix} a_1, a_2, a_3 \\ a_1, a_2, a_4 \end{vmatrix} x_4 + \ldots + \begin{vmatrix} a_1, a_2, a_3 \\ a_1, a_2, a_n \end{vmatrix} x_n = y_3,$$

&c. &c. &c.

$$\begin{vmatrix} a_1, a_2 \ldots a_n \\ a_1, a_2 \ldots a_n \end{vmatrix} x_n = y_n,$$

$(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2$ will assume the form

$$\begin{vmatrix} a_1 \\ a_1 \end{vmatrix} y_1^2 + \frac{\begin{vmatrix} a_1, a_2 \\ a_1, a_2 \end{vmatrix}}{\begin{vmatrix} a_1 \\ a_1 \end{vmatrix}} y_2^2 + \frac{\begin{vmatrix} a_1, a_2, a_3 \\ a_1, a_2, a_3 \end{vmatrix}}{\begin{vmatrix} a_1, a_2 \\ a_1, a_2 \end{vmatrix}} y_3^2 + \ldots + \frac{\begin{vmatrix} a_1, a_2 \ldots a_{n-1}, a_n \\ a_1, a_2 \ldots a_{n-1}, a_n \end{vmatrix}}{\begin{vmatrix} a_1, a_2 \ldots a_{n-1} \\ a_1, a_2 \ldots a_{n-1} \end{vmatrix}} y_n^2,$$

and consequently the number of positive squares in the reduced form of the given function will always be the number of continuations or permanencies of sign of the series

$$1; \begin{vmatrix} a_1 \\ a_1 \end{vmatrix}; \begin{vmatrix} a_1, a_2 \\ a_1, a_2 \end{vmatrix}; \begin{vmatrix} a_1, a_2, a_3 \\ a_1, a_2, a_3 \end{vmatrix} \ldots \begin{vmatrix} a_1, a_2 \ldots a_n \\ a_1, a_2 \ldots a_n \end{vmatrix},$$

the several terms of this progression being in fact the determinants of what the given function becomes when we obliterate successively all the variables but one, then all but that and another, then all but these two and a third, until finally, the last term is the determinant of the given function with all the variables retained. This comes to saying that if we call the function (suppose of four variables) f, and write down the matrix

$$\begin{matrix} \dfrac{d^2f}{dx_1^2}, & \dfrac{d^2f}{dx_1dx_2}, & \dfrac{d^2f}{dx_1dx_3}, & \dfrac{d^2f}{dx_1dx_4}, \\ \dfrac{d^2f}{dx_2dx_1}, & \dfrac{d^2f}{dx_2^2}, & \dfrac{d^2f}{dx_2dx_3}, & \dfrac{d^2f}{dx_2dx_4}, \\ \dfrac{d^2f}{dx_3dx_1}, & \dfrac{d^2f}{dx_3dx_2}, & \dfrac{d^2f}{dx_3^2}, & \dfrac{d^2f}{dx_3dx_4}, \\ \dfrac{d^2f}{dx_4dx_1}, & \dfrac{d^2f}{dx_4dx_2}, & \dfrac{d^2f}{dx_4dx_3}, & \dfrac{d^2f}{dx_4^2}, \end{matrix}$$

* For an explanation of the umbral notation, see *London and Edinburgh Philosophical Magazine*, April 1851, or thereabouts [p. 243 above].

(where all the terms are of course coefficients of the given function expressed as above for greater symmetry of notation), the inertia of f will be measured by the number of continuations of sign in the series formed of the *successive principal minor coaxal* determinants $\left(\text{in writing which I shall use in general } (r, s) \text{ to denote } \frac{d^2f}{dx_r dx_s}\right)$,

$$1, \quad (1,1), \quad \begin{vmatrix} (1,1), & (1,2) \\ (2,1), & (2,2) \end{vmatrix}, \quad \begin{vmatrix} (1,1), & (1,2), & (1,3) \\ (2,1), & (2,2), & (2,3) \\ (3,1), & (3,2), & (3,3) \end{vmatrix},$$

$$\begin{vmatrix} (1,1), & (1,2), & (1,3), & (1,4) \\ (2,1), & (2,2), & (2,3), & (2,4) \\ (3,1), & (3,2), & (3,3), & (3,4) \\ (4,1), & (4,2), & (4,3), & (4,4) \end{vmatrix},$$

and in like manner in general*.

Art. 46. Reverting now to the simplified Sturmian residues, since by the theory set out in the first Section these differ from the unsimplified complete residues required by the Sturmian method only in the circumstance of their being divested of factors which are necessarily perfect squares and therefore essentially positive, these simplified Sturmians may of course be substituted for the complete Sturmians for the purposes of M. Sturm's theorem. The leading coefficients in these simplified Sturmians, reckoning $f'(x)$ as one of them, will be

$$m\Sigma\zeta(h_1, h_2), \quad \Sigma\zeta(h_1, h_2, h_3) \ldots \zeta(h_1, h_2 \ldots h_m),$$

which it is easily seen, as remarked long ago by Mr Cayley, are the successive principal minor coaxal determinants of the matrix

$$\begin{matrix} \sigma_0, & \sigma_1, & \sigma_2, & \sigma_3 \ldots \sigma_{m-1}, \\ \sigma_1, & \sigma_2, & \sigma_3 & \ldots \sigma_m, \\ \sigma_2, & \sigma_3 & & \ldots \sigma_{m+1}, \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{m-1}, & \sigma_m & & \ldots \sigma_{2m-2}, \end{matrix}$$

* I have given a direct *à posteriori* demonstration in the *London and Edinburgh Philosophical Magazine*, that the number of continuations of sign in any series formed like the above from a symmetrical matrix, is unaffected by any permutations of the lines and columns thereof, which leaves the symmetry subsisting, that is to say (using the umbral notation), if $\theta_1, \theta_2, \theta_3 \ldots \theta_i$ are disjunctively equal, each to each, in any arbitrary order to $1, 2, 3 \ldots i$, the number of continuations of sign in the series

$$1, \begin{vmatrix} a_{\theta_1} \\ a_{\theta_1} \end{vmatrix}, \begin{vmatrix} a_{\theta_1}, a_{\theta_2} \\ a_{\theta_1}, a_{\theta_2} \end{vmatrix}, \begin{vmatrix} a_{\theta_1}, a_{\theta_2}, a_{\theta_3} \\ a_{\theta_1}, a_{\theta_2}, a_{\theta_3} \end{vmatrix}, \ldots, \begin{vmatrix} a_{\theta_1}, a_{\theta_2}, a_{\theta_3} \ldots a_{\theta_i} \\ a_{\theta_1}, a_{\theta_2}, a_{\theta_3} \ldots a_{\theta_i} \end{vmatrix},$$

is irrespective of the order of the natural numbers $1, 2, 3 \ldots i$ in the arrangement $\theta_1, \theta_2, \theta_3 \ldots \theta_i$.

where in general $\sigma_r = h_1^r + h_2^r + \ldots + h_m^r$, and of course $\sigma_0 = m$. M. Hermite has improved upon this remark by observing, what is immediately obvious, that if we use σ_r to denote, not the quantity above written, but

$$\frac{h_1^r}{x-h_1} + \frac{h_2^r}{x-h_2} + \ldots + \frac{h_m^r}{x-h_m},$$

the successive coaxal determinants of the above matrix will become respectively

$$\Sigma\,\frac{1}{x-h_1},\quad \Sigma\left\{\frac{\zeta(h_1, h_2)}{(x-h_1)(x-h_2)}\right\},$$

$$\Sigma\,\frac{\zeta(h_1, h_2, h_3)}{(x-h_1)(x-h_2)(x-h_3)}, \ldots \frac{\zeta(h_1, h_2 \ldots h_m)}{(x-h_1)(x-h_2)\ldots(x-h_m)};$$

that is to say, these successive coaxal determinants, when multiplied up by fx, will become respectively

$$\Sigma\,(x-h_2)(x-h_3)\ldots(x-h_m),\quad \Sigma\zeta(h_1, h_2)\{(x-h_3)(x-h_4)\ldots(x-h_m)\}, \ldots$$
$$\Sigma\zeta(h_1, h_2 \ldots h_m),$$

that is to say, will represent the simplified Sturmian series given by my general formulæ. M. Hermite further remarks, that the matrix formed after this rule will evidently be that which represents the determinant of the quadratic function (which may be treated as a generating function)

$$\Sigma\,\frac{1}{x-h_1}\{u_1 + h_1u_2 + h_1^2u_3 + \ldots + h_1^{m-1}u_m\}^2,$$

in which, since only the squared differences of the terms in the (h) series finally remain in the successive coaxal determinants, we may write $(x-h_1)$, $(x-h_2) \ldots (x-h_m)$ simultaneously in place of $h_1, h_2 \ldots h_m$ without affecting the result; consequently the generating function above may be replaced by the generating function

$$\Sigma\,\frac{1}{x-h_1}\{u_1 + (x-h_1)\,u_2 + (x-h_1)^2\,u_3 + \ldots + (x-h_1)^{m-1}\,u_m\}^2,$$

the corresponding matrix to which becomes

$$\begin{array}{llll}
\Sigma\,\dfrac{1}{x-h_1}, & \theta_0, & \theta_1 \ldots \theta_{m-2}, \\
\theta_0, & \theta_1, & \theta_2 \ldots \theta_{m-1}, \\
\theta_1, & \theta_2, & \ldots \theta_m, \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
\theta_{m-2}, & \theta_{m-1} & \ldots \theta_{2m-3},
\end{array}$$

where θ_i denotes $\Sigma(x-h)^i$, and $\Sigma\frac{1}{x-h_1}=\frac{f'x}{fx}$. Hence every simplified residue is of the form

$$f'x \times \begin{vmatrix} \theta_1, & \theta_2 & \ldots \theta_r \\ \theta_2, & \theta_3 & \ldots \theta_{r+1} \\ \ldots & \ldots & \ldots \\ \theta_r, & \theta_{r+1} & \ldots \theta_{2r-1} \end{vmatrix} + fx \times \begin{vmatrix} 0, & \theta_0, \theta_1 & \ldots \theta_r \\ \theta_0, & \theta_1 & \ldots \theta_{r+1} \\ \ldots & \ldots & \ldots \\ \theta_r, & \theta_{r+1} & \ldots \theta_{2r+1} \end{vmatrix}.$$

The residue in question will be of the degree $m-r-2$ in x, and consequently we have, according to the notation antecedently used for the syzygetic equations

$$t_{r+1} = \begin{vmatrix} \theta_1, & \theta_2 & \ldots \theta_r \\ \theta_2, & \theta_3 & \ldots \theta_{r+1} \\ \ldots & \ldots & \ldots \\ \theta_r, & \theta_{r+1} & \ldots \theta_{2r-1} \end{vmatrix}$$

$$-\tau_r = \begin{vmatrix} 0, & \theta_0, & \theta_1 \ldots \theta_r \\ \theta_0, & \theta_1 & \ldots \theta_{r+1} \\ \theta_1 & & \ldots \theta_{r+2} \\ \ldots & \ldots & \ldots \\ \theta_r, & \theta_{r+1} & \ldots \theta_{2r+1} \end{vmatrix}.$$

Elegant and valuable for certain purposes as are these formulæ for t_{r+1} and τ_r, they are affected with the disadvantage of being expressed by means of formulæ of a much higher degree in the variable x than really appertains to them, the paradox (if it may be termed such) being explained by the circumstance of the coefficients of all the powers of x above the right degree being made up of terms which mutually destroy one another; upon the face of the formulæ, t_{r+1} and τ_r which are in fact only of the degrees $r+1$ and r respectively in x would appear to be of the degree

$$1+3+5+\ldots+(2r-1),$$

that is of the degree r^2.

Art. 47. I may add the important remark, which does not appear to have occurred immediately to my friend M. Hermite when he communicated to me the above most interesting results, that in fact, by virtue of the law of inertia for quadratic forms, we may dispense with any identification of the successive coaxal determinants of the matrix to the generating function

$$\Sigma\,\frac{1}{\rho-h_1}\{u_1+h_1u_2+h_1^2u_3+\ldots+h_1^{m-1}u_m\}^2$$

with my formulæ for the Sturmian functions, and prove *ab initio* in the most simple manner, that the successive ascending coaxal determinants (always of course supposed to be taken about the axis of symmetry) of the matrix to the form above written, or to the more general form (which I shall quote as (G), namely)

$$\Sigma (\rho - h_1)^q \{\phi_1 (h_1) u_1 + \phi_2 (h_1) u_2 + \ldots + \phi_m (h_1) u_m\}^2, \qquad \text{(G)}$$

(where ϕ_1, $\phi_2 \ldots \phi_m$ are absolutely arbitrary integral forms of function with real coefficients), will form a rhizoristic series in regard to fx (that is a series, the difference between the number of the continuations of sign between the successive terms of which corresponding to two different values of ρ will determine the number of real roots of x lying between such two assumed values), provided only that q be an odd positive or negative integer. Nothing can be easier than the demonstration, for whenever ρ is greater than any one of the real roots as h_1:—

Firstly, any pair of imaginary roots will give rise to two terms of the form

$$(l + m\sqrt{-1})^q (v + w\sqrt{-1})^2 \text{ and } (l - m\sqrt{-1})^q (v - w\sqrt{-1})^2,$$

or more simply

$$(L + M\sqrt{-1})(v^2 - w^2 + 2vw\sqrt{-1})$$

and

$$(L - M\sqrt{-1})(v^2 - w^2 - 2vw\sqrt{-1}),$$

where v and w are real linear functions of u_1, $u_2 \ldots u_m$. The sum of which couple will be

$$2\{L(v^2 - w^2) - 2Mwv\} = \frac{2}{L}\{(Lv - Mw)^2 - (L^2 + M^2) w^2\} = p^2 - q^2;$$

so that each such couple combined will for every value of x give rise to one positive and one negative square.

Secondly, any real root of the series h_1, $h_2 \ldots h_m$, when ρ is taken greater than such root, will give rise to a positive square of a real linear function of u_1, $u_2 \ldots u_m$.

Thirdly, any real root of the same series, when ρ is beneath it in value (q being odd), will give rise to the negative of the square of a real linear function of the same. Hence the number of real roots between ρ taken equal to one value (a), and ρ taken equal to any other value (b), will be denoted by the loss of an equal number of positive squares in the reduced form of the expression (G) when ρ is taken a and when ρ is taken b; that is by virtue of Art. 45 will be denoted by the difference of the number of permanencies of sign in the successive minor determinants of the matrix corresponding to the quadratic form (G)* (which we have taken as our

* The *inertia* of the quadratic form (G) is the measure of the number of real roots of fx comprised between ∞ and ρ, and may be estimated in any manner that may be found most convenient. If ρ be made infinity, and $\phi_i h$ be taken equal to h^{i-1}, and the inertia of the corresponding value of (G) be estimated by means of the formulæ in ordinary use by geometers for

generating function) resulting from the substitution respectively of a and b in place of ρ, which gives a theorem equivalent to that of M. Sturm, transformed by my formulæ, when we choose to adopt the particular suppositions

$$q=-1, \quad \phi_1 h=1, \quad \phi_2 h=h, \quad \phi_3 h=h^2, \ldots \phi_m h=h^{m-1}.$$

This method of *constructing* a rhizoristic series to fx by a direct process is deserving of particular attention, because it does not involve the use of the notion of continuous variation, upon which all preceding proofs of Sturm's theorem proceed. It completes the cycle of the Sturmian ideas. Happily this cycle was commenced from the other end, for it would have been difficult to have suspected that the root-expressions for the terms in the rhizoristic series could be identified with the residues, had the former been the first to be discovered, and much of the theory of algebraical common measure laid open by means of this identification would probably have remained unknown.

Art. 48. I proceed now to consider a theorem concerning the relative positions of the real roots of two independent algebraical functions as indicated by the succession of signs presented by their Bezoutian secondaries; this more general theory of intercalations or relative interpositions will be seen to include within it as a corollary the justly celebrated theorem of M. Sturm.

Let the real roots of fx taken in descending order of magnitude be $h_1, h_2 \ldots h_p$, and the real roots of ϕx taken in the like order $\eta_1, \eta_2 \ldots \eta_q$, so that

$$fx=(x-h_1)(x-h_2)\ldots(x-h_p)H,$$

$$\phi x=(x-\eta_1)(x-\eta_2)\ldots(x-\eta_q)K,$$

H and K being functions of x incapable of changing their signs. Now, as in M. Sturm's method, let us inquire what takes place in respect to the sign of $\dfrac{\phi(x)}{f(x)}$, which I shall call the Indicatrix, as x descends the scale of real magnitude from $+\infty$ to $-\infty$. If between $+\infty$ and h_1, i real roots of ϕx are contained, it is obvious that as x travels from $+\infty$ to the superior brink of h_1, the Indicatrix will change its sign from $+$ to $-$ and from $-$ to $+$ altogether i times, so that at the moment when x is about to pass through h_1, it

determining the nature of a surface of the second degree, the criteria of the number of real roots in fx will be, or may be made to be, symmetrical in respect to the two ends of the expression fx. This system of criteria, however, is not so good as that given by the Bezoutiant to the two differential coefficients of $f(x, 1)$ taken with regard to x and 1 respectively, which will also possess the like character of symmetrical indifference, and be one less in number than the former.

will be positive if i is zero or even, and negative if i is odd; but the moment after x has passed through the value h_1, the indicatrix will be negative on the first supposition, and positive on the other supposition. Hence immediately after the passage of x through h_1 the indicatrix will have been once oftener negative than positive on the one supposition, and as often negative as positive on the other. Again, in like manner as x traverses the interval between h_1 and the inferior brink of h_2, if no η or an even number of η's occupy this interval, the sign which the indicatrix had at the beginning of this interval will have been reversed once oftener than restored; but if there be an odd number of η's so interposed, the number of reversals and restorations will have been identical; and so for each successive interval, reckoned from a value for x immediately subsequent to one real root of fx, down to a value immediately subsequent to the next less real root of the same; and it is evident that the effect upon the sign of the indicatrix at the end of every such interval depends, not upon the number of η's grouped together in such interval, but upon the form of the group as regards its being made up of an odd or even number of terms, the first interval being of course understood to extend from $+\infty$ to a value immediately inferior to h_1, and the last from a value immediately inferior to h_p to $-\infty$. Hence as regards the relation of the sign of the indicatrix at the beginning to the sign at the end of every such interval, nothing will be altered by taking away any even number of η's that may be found therein. If we suppose this to be done, we shall then have in some of the intervals one η occurring and in the other intervals no η; that is to say, some of the h's will be separated by single η's, but other h's will come together. Again, by removing any even number of h's not separated by η's (and thus removing an even number of intervals), it is clear that as many changes of sign of the indicatrix will have been done away with from $+$ to $-$ as from $-$ to $+$, and no effect upon the excess of the one kind of changes of sign over the other kind of changes of sign will have been produced. By removing pairs of h's in this manner, it may happen that η's will again be brought together, any even number of which, not separated by h's, may again be removed and then pairs of h's not separated by η's in their turn, and so continually *toties quoties* until at length we must arrive at a reduced system of h's and η's, where no two h's and no two η's come together, or else all the h's and all the η's will have disappeared. Let the scale of h's and η's thus simplified and reduced be called the effective scale of intercalations. The number of h's and the number of η's in any such scale will be equal, or will at most differ from one another by a unit, since at each part of the scale, except at the end, every h is followed by an η and every η by an h. If the scale begins and ends with an h, there will of course be one more h than η; if it begin and end with an η, there will be one more η than h; if it begin with an h or an η and end with an η or h, there will be as many of the one as of the other.

Firstly, suppose the effective intercalation scale to commence with an h; then in passing from $+\infty$ to just beyond the first h the sign of the indicatrix $\frac{\phi x}{fx}$ changes from $+$ to $-$; it changes again from $-$ to $+$ as it passes the first η, then again from $+$ to $-$ as it passes the second h, and so on; that is to say, there will be a change always in the same direction from $+$ to $-$ as x passes from being just greater than to being just less than any h appearing in the effective scale. Secondly, if the effective scale begin with η, the indicatrix will conversely be negative after passing the first and every subsequent η, and change from $-$ to $+$ in the act of passing through the first and every subsequent h. So that on either supposition the changes of sign for the effective scale always take place in the same direction, and the number of h's in the effective scale will be measured by the number of such changes, and consequently will be measured by the difference between the number of times that the indicatrix $\frac{\phi x}{fx}$ changes its sign from $+$ to $-$ as x passes through each in turn of the real roots of fx, and the number of times that in passing through any such root it changes its sign from $-$ to $+$; if the former number be greater than the latter, the effective scale of interpositions will begin with a root of fx; if it be less, the scale will begin with a root of ϕx. If instead of beginning with $+\infty$ and ending with $-\infty$ we begin and end with any two limits, a and b respectively (making abstraction of all roots of fx or of ϕx lying outside these limits, and forming the effective intercalation scale with the roots comprised within these limits exclusively), we shall obviously obtain a similar result, but with the condition that the changes from $+$ to $-$ will be *in excess* if an even number of h's and η's combined be cut off by the superior limit, and the effective scale begin with an h, or if an odd number of h's and η's combined be so cut off and the scale begin with an η; and *in defect* if an odd number of h's and η's combined be so cut off and the scale begin with an h, or an even number be so cut off and the scale begin with an η. If, now, supposing fx to be of n, and ϕx of not more than n, say m dimensions, we form the *signaletic* series fx, ϕx, $B_1, B_2 \ldots B_m$ (where the $B_1, B_2 \ldots B_m$ are the Bezoutian secondaries or simplified successive residues corresponding to $\frac{\phi x}{fx}$ expanded under the form of an improper continued fraction), it may be shown, in the same way as for Sturm's theorem, that whenever $\frac{\phi x}{fx}$ changes from $+$ to $-$ a change of sign will be gained in the series, and when from $-$ to $+$ a change will be lost; and that no change can be gained or lost except as x passes through the successive real roots of fx. Hence the difference between the number of changes of sign in the above signaletic series when x is taken a, and the number of the same when x is taken b, will indicate the number of roots of fx remaining in the effective scale of interpositions formed between such of the roots of fx and of ϕx as lie between a and b; calling the one number $I(a)$ and the other $I(b)$, the sign of $I(b) - I(a)$ depends not on the relative magnitudes of a and b, but upon the manner in which the effective scale commences; if $I(a) - I(b)$ is positive, the effective scale formed between the a and b will commence with a root of fx; if negative, it will commence with a root of ϕx.

Art. 49. In forming the scale of effective interpositions, it is evidently not necessary to go on reducing the h series and the η series separately and alternately; the same result will be effected more expeditiously by eliding simultaneously any even number of h's that come together without being separated by an η, and any even number of η's that come together without being separated by an h, and, repeating this process of simultaneous elision, as often as may be required, until no two h's or η's come together. Thus, for instance, denoting the magnitudes of the series of real roots of f and of ϕ by the distances of h and η points taken along a right line from a fixed point therein, and supposing such series of roots between the limits a and b to be

$$h\,h\,h\,\eta\,\eta\,\eta\,h\,\eta\,\eta\,h\,\eta\,\eta\,\eta\,h\,h\,\eta\,h\,\eta\,h\,h\,h\,h\,h\,\eta\,\eta\,h,$$

our first reduction brings this scale to the form

$$h\,\eta\,h\,h\,\eta\,\eta\,h\,\eta\,h\,h\,;$$

the next reduction brings it to the form

$$h\,\eta\,\eta\,\eta\,h\,\eta\,;$$

and a third and final reduction brings it to the form

$$h\,\eta\,h\,\eta\,;$$

and accordingly we shall find for such an arrangement of the h and η system

$$I(b) - I(a) = \pm 2.$$

Art. 50. If we suppose $\phi x = \dfrac{dfx}{dx}$, by a well-known theorem of algebra, any two consecutive roots of fx will contain between them an odd number of roots of ϕx, and the number of real roots of $f'x$ greater than the greatest root of fx, and the number of real roots of $f'x$ less than the least root of fx will each be even. Hence the effective intercalation scale between any two limits a and b will be formed by merely reducing the η groups to single units, and the number of h's in the scale so formed will be the total number of h's between the limits a and b. Moreover, since such scale commences always with a root of fx, or with an even number of roots of $f'x$ followed by a root of fx, if the number of h's and η's cut off be even, and with a root of $f'x$ or an even number of roots of fx followed by a root of fx, if the number so cut off be odd, it follows that for this case $I(a) - I(b)$, a being the superior limit, will be always positive, and will measure the total number of real roots of fx lying between a and b; this, then, is Sturm's theorem, treated as a corollary to the Theory of Intercalations.

Art. 51. If we write down the last syzygetic equation between fx of m and ϕx of n dimensions, namely

$$\tau_{n-1}(x) fx - t_{m-1}(x)\phi x + \vartheta_0 = 0,$$

it has been shown that the succession of signs in the series formed with fx, ϕx and their successive Bezoutian secondaries will contain the same number of continuations and variations as the series formed with fx, $t_{m-1}(x)$, and their successive Bezoutian secondaries. This indicates that the effective scale of interpositions for fx and ϕx will contain an equal number of roots of fx with the effective scale for fx and $t_{m-1}(x)$; the two scales however will not necessarily be identical, because the roots of ϕx will not necessarily be in the same order relative to the h's in the one scale as those of $t_{m-1}(x)$ relative to the h's in the other scale. This equality is perfectly well explained *à posteriori* by the form of $t_{m-1}(x)$, which by the formula in Section II. will be represented by

$$\Sigma(x-h_{q_1})(x-h_{q_2})\dots(x-h_{q_{m-1}})\frac{\phi h_{q_1}\phi h_{q_2}\dots\phi h_{q_{m-1}}}{(h_{q_m}-h_{q_1})(h_{q_m}-h_{q_2})\dots(h_{q_m}-h_{q_{m-1}})}.$$

Now, whenever x is indefinitely near to any one of the roots of fx, as h_{q_m}, this sum reduces to the simple expression

$$\phi h_{q_1}\phi h_{q_2}\dots\phi h_{q_{m-1}} = \{\phi h_1\phi h_2\dots\phi h_m\}\frac{1}{\phi h_{q_m}},$$

and consequently in the immediate neighbourhood of every real root of fx, ϕx and $t_{m-1}(x)$ will have always the same or always a contrary sign, according as $\phi h_{q_1}\phi h_{q_2}\dots\phi h_{q_m}$ is positive or negative, which will depend upon the relative disposition of the real roots in f and ϕ; in either case the effective scale of interpositions for fx with ϕx and for fx with $t_{m-1}x$ must contain the same number of h's; but the difference will be, that if $\phi h_1\phi h_2\dots\phi h_m$ is positive an h will occupy the first place in each scale, or the second place in each scale; but if negative, then in one scale an h will occupy the first place, and in the other scale the second place.

Art. 52. The same process of common measure or residues which serves to furnish a rhizoristic series for fx or a syrrhizoristic series for fx and ϕx, will serve also to furnish superior and inferior limits to the real roots of any proposed equation. Thus suppose fx to be any rational integral function of

x of the degree n and ϕx any other function of x, which I shall begin with supposing to be of the degree $(n-1)$, and let the successive quotients resulting from the process of finding the greatest common measure of fx, ϕx continued until the last remainder is not a constant but zero, be supposed to be (as they may generally be taken, but subject to cases of exception, which will hereafter be alluded to) n linear functions $q_1, q_2 \ldots q_n$; then we shall have

$$\frac{\phi x}{fx} = \frac{1}{q_1 +}\,\frac{1}{q_2 +} \cdots \frac{1}{q_{n-1} +}\,\frac{1}{q_n},$$

and therefore

$$\phi x = KN,$$

$$fx = KD,$$

where N is the numerator and D the denominator of the continued fraction and K is a constant; the value of this constant is immaterial but is in fact

$$\pm \frac{L_0}{L_1^2}\,\frac{L_2^2}{L_3^2}\,\frac{L_4^2}{L_5^2}\,\&\text{c.},$$

L_0, L_1, L_2, L_3, &c. being the leading coefficients of the last, the last but one, the last but two, &c. of the Bezoutian secondaries to fx and ϕx. Accordingly,

if $n=1$, let $D = q_1 = \mu_1$;

if $n=2$, let $D = q_2 q_1 + 1 = \mu_1 \left\{q_2 + \frac{1}{\mu_1}\right\} = \mu_1\mu_2$;

if $n=3$, let $D = q_3\{q_2 q_1 + 1\} + q_1 = \mu_1\mu_2\left\{q_3 + \frac{1}{\mu_2}\right\} = \mu_1\mu_2\mu_3$;

....

and in general let

$$D = \mu_1\mu_2\mu_3 \ldots \mu_n,$$

where

$$\mu_1 = q_1, \quad \mu_2 = q_2 + \frac{1}{\mu_1}, \quad \mu_3 = q_3 + \frac{1}{\mu_2}, \ \ldots\ \mu_n = q_n + \frac{1}{\mu_{n-1}}.$$

Now suppose x to be so taken that

$$\left.\begin{array}{llll} q_1 & \text{does not lie between} & +1 \text{ and } -1 \\ q_2 & \quad ,, \qquad\qquad ,, & +2 \text{ and } -2 \\ q_3 & \quad ,, \qquad\qquad ,, & +2 \text{ and } -2 \\ q_4 & \quad ,, \qquad\qquad ,, & +2 \text{ and } -2 \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ q_{n-1} & \quad ,, \qquad\qquad ,, & 2 \text{ and } -2 \\ q_n & \quad ,, \qquad\qquad ,, & 1 \text{ and } -1 \end{array}\right\}, \qquad (\omega)$$

where it will be observed that the excluded region lies between $+2$ and -2 for all the intermediate quotients, but between only $+1$ and -1 for the first and last quotients. Then μ_1 is positively or negatively greater than 1, therefore $\frac{1}{\mu_1}$ is a positive or negative fraction; but q_2 is positively or negatively greater than 2; therefore μ_2 will be of the same sign as q_2, and also μ_2 will be positively or negatively greater than 1; therefore $\frac{1}{\mu_2}$ will be a positive or negative fraction; but q_3 is positively or negatively greater than 2; therefore μ_3 will be of the same sign as q_3, and also μ_3 will be positively or negatively greater than 1; and proceeding in this way, we find that all values of μ_i, from $i=1$ to $i=n-1$, will be of the same sign as q_i, and positively or negatively greater than 1. Finally, $\frac{1}{\mu_{n-1}}$ will be a fraction, and therefore, since q_n is positively or negatively greater than 1, $\mu_n = q_n + \frac{1}{\mu_{n-1}}$ will have the same sign as q_n (but of course is not necessarily greater than 1, nor would that condition serve any purpose were it satisfied). We infer consequently, that when the conditions (ω) are satisfied, μ_1, μ_2, $\mu_3 \ldots \mu_n$ will respectively have the same signs as $q_1, q_2 \ldots q_n$; and therefore $D = \mu_1\mu_2\mu_3 \ldots \mu_n$ has the same sign as $q_1q_2q_3 \ldots q_n$. Now suppose

$$q_1 = a_1x + b_1, \quad q_2 = a_2x + b_2 \ldots q_n = a_nx + b_n,$$

and solve the $2n$ equations

$$\begin{array}{llll} a_1x + b_1 = +c_1, & a_2x + b_2 = +c_2 \ldots & a_{n-1}x + b_{n-1} = c_{n-1}, & a_nx + b_n = c_n, \\ a_1x + b_1 = -c_1, & a_2x + b_2 = -c_2 \ldots & a_{n-1}x + b_{n-1} = -c_{n-1}, & a_nx + b_n = -c_n, \end{array}$$

where

$$c_1 = 1, \quad c_2 = 2, \quad c_3 = 2 \ldots c_{n-1} = 2, \quad c_n = 1.$$

Whenever in any one of the n pairs of equations above written the coefficient of x is positive, the upper equation of the pair will bring out the greater value of x; but when the coefficient is negative the lower equation will give the greater value. Take the pair

$$a_ix + b_i = c_i,$$
$$a_ix + b_i = -c_i.$$

If a_i is positive $a_ix + b_i$ will always be positive, and greater than c_i, between $x = \infty$ and $x =$ the greater of the two values of x; if a_i is negative $a_ix + b_i$ will always be negative, and less (that is nearer to $-\infty$) than $-c_i$, for all values of x between the same limits as before. So again it will be seen in like manner, that whether a_i be positive or negative, between $x = -\infty$ and $x =$ the lesser of the two values of x corresponding to the above pair of equations, $a_ix + b_i$ will always retain the same sign, and will be greater than $+c_i$, or less than $-c_i$, according as a_i is negative or positive. If, then, we take the greatest of the *greaters* of the n pairs of values of x, that is the absolute greatest of the $2n$ values, and the least of the *lessers*, that is the absolute least of the same, say L and Λ, then between L and Λ, $q_1, q_2 \dots q_n$ will each always retain an invariable sign, and will then fall without the limits $\pm c_1, \pm c_2, \dots \pm c_{n-1}, \pm c_n$, so that between $+\infty$ and L and between Λ and $-\infty$, $\mu_1\mu_2 \dots \mu_n$, that is a constant multiple of $f(x)$, will retain the same sign as $q_1q_2 \dots q_n$, that is will never change its sign from the beginning to the end of one interval, nor from the beginning to the end of the other; and consequently L and Λ will be a superior and inferior limit respectively to the real roots of fx. It will of course be observed that it is indifferent for the purposes of the foregoing theorem, whether $\frac{\phi x}{fx}$ be expanded under the form of a proper or an improper fraction, that is whether we employ the ordinary or the Sturmian process of successive division; for changing the signs of the residues will only have the effect of changing q_i into $(\pm) q_i$, and the pair of equations $(\pm) q_i = \pm c_i$ remains the same whether the $+$ or the $-$ sign be prefixed to q_i. The result is, that if we form the $2n$ quantities

$$\frac{\pm 1 - b_1}{a_1}, \quad \frac{\pm 2 - b_2}{a_2}, \quad \frac{\pm 2 - b_3}{a_3} \dots \frac{\pm 2 - b_{n-1}}{a_{n-1}}, \quad \frac{\pm 1 - b_n}{a_n},$$

the greatest of them will be a superior, and the least of them an inferior limit to the roots of fx*.

It may be remarked that if the successive dividends in the course of the process be multiplied respectively by $k_1, k_2 \dots k_n$, $\frac{\phi x}{fx}$ will take the form

$$\frac{k_1}{q_1 +} \frac{k_2}{q_2 +} \frac{k_3}{q_3 +} \dots \frac{k_n}{q_n};$$

and if we write

$$a_1 x + b_1 = \pm c_1, \quad a_2 x + b_2 = \pm c_2 \dots a_n x + b_n = \pm c_n$$

and make

$$c_1 = 1, \quad c_2 = 1 + k_2, \quad c_3 = 1 + k_3 \dots c_n = 1 + k_n,$$

the same reasoning as above will show that the greatest and least of the $2n$ quantities

$$\frac{\pm 1 - b_1}{a_1}, \quad \frac{\pm (1 + k_2) - b_2}{a_2} \dots \frac{\pm (1 + k_n) - b_{n-1}}{a_{n-1}}, \quad \frac{\pm 1 - b_n}{a_n},$$

will be a superior and inferior limit to the roots of fx.

For greater simplicity, again, consider $k_1, k_2 \dots k_n$ to be all equal to unity; we may make this addition to the theorem as above stated, namely calling

* For a generalization and improved form of statement of this theorem see Supplement to the present Section.

L_1, Λ_1; $L_2, \Lambda_2 \ldots L_n, \Lambda_n$ the greatest and least values of the terms contained respectively in the series marked below 1, 2, 3 ... n, namely—

$$\frac{\pm 1 - b_1}{a_1}, \quad \frac{\pm 2 - b_2}{a_2}, \quad \frac{\pm 2 - b_3}{a_3} \ldots \frac{\pm 2 - b_{n-1}}{a_{n-1}}, \quad \frac{\pm 1 - b_n}{a_n}, \qquad (1)$$

$$\frac{\pm 1 - b_2}{a_2}, \quad \frac{\pm 2 - b_3}{a_3} \ldots \frac{\pm 2 - b_{n-1}}{a_{n-1}}, \quad \frac{\pm 1 - b_n}{a_n}, \qquad (2)$$

$$\frac{\pm 1 - b_3}{a_3} \ldots \frac{\pm 2 - b_{n-1}}{a_{n-1}}, \quad \frac{\pm 1 - b_n}{a_n}, \qquad (3)$$

....

....

$$\frac{\pm 1 - b_{n-1}}{a_{n-1}}, \quad \frac{\pm 1 - b_n}{a_n}, \qquad (n-1)$$

$$\frac{\pm 1 - b_n}{a_n}, \qquad (n)$$

L_1, Λ_1; $L_2, \Lambda_2 \ldots L_n, \Lambda_n$ will be respectively superior and inferior limits to fx, ϕx and their successive residues. As a corollary, we see, of course, that L and Λ, the superior and inferior limits to the roots of the given function fx, must always lie between $+\infty$ and the greatest root, and between $-\infty$ and the least root, of the arbitrarily assumed function ϕx.

Art. 53. Let us now assume somewhat more generally that ϕx is any number of degrees θ_1 in x lower than fx, which will cause the first quotient q_{θ_1} to be of the degree θ_1 in x; and let us further suppose that ϕx stands in such a relation to fx that the following quotients, $q_{\theta_2}, q_{\theta_3} \ldots q_{\theta_\rho}$, are of the degrees $\theta_2, \theta_3 \ldots \theta_\rho$ in x ($\theta_2, \theta_3 \ldots \theta_\rho$ being supposed not necessarily units, as they would generally be, but any positive integers whatever, as may happen in consequence of one or more of the leading coefficients in any residue vanishing); then

$$\frac{\phi x}{fx} = \frac{1}{q_{\theta_1} +} \frac{1}{q_{\theta_2} +} \frac{1}{q_{\theta_3} +} \ldots + \frac{1}{q_{\theta_\rho}},$$

where $\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_\rho = n$; and consequently fx will be equal to the denominator of the last convergent above written, multiplied by a constant, so that we have now $cfx = m_1 m_2 \ldots m_\rho$, where

$$m_1 = q_{\theta_1}, \quad m_2 = q_{\theta_2} + \frac{1}{m_1} \ldots m_\rho = q_{\theta_{\rho-1}} + \frac{1}{m_{\rho-1}}.$$

And as in the case previously considered, so long as

$$q_{\theta_1}\begin{pmatrix} >1 \\ \text{or} \\ <-1 \end{pmatrix}, \quad q_{\theta_2}\begin{pmatrix} >2 \\ \text{or} \\ <-2 \end{pmatrix}, \quad q_{\theta_3}\begin{pmatrix} >2 \\ \text{or} \\ <-2 \end{pmatrix}, \ldots \; q_{\theta_\rho}\begin{pmatrix} >1 \\ \text{or} \\ <-1 \end{pmatrix},$$

fx will have the same sign as $q_{\theta_1} q_{\theta_2} \ldots q_{\theta_\rho}$.

Let now $q_{\theta_1} = \pm c_1, \quad q_{\theta_2} = \pm c_2 \ldots q_{\theta_\rho} = \pm c_\rho,$
where $c_1 = 1, \quad c_2 = 2 \ldots c_{\rho-1} = 2, \quad c_\rho = 1.$
Consider any pair of the above equations as $q_{\theta_i}^2 - c_i^2 = 0$.

Firstly, suppose all the roots of this equation are impossible; $q_{\theta_i}^2 - c_i^2$ must be positive for all values of x, and q_{θ_i} can never lie between $+c_i$ and $-c_i$; moreover, since upon the hypothesis made, $q_{\theta_i} + c_i$ and $q_{\theta_i} - c_i$ always retain the same sign, namely, that of the coefficient of the highest power of q_{θ_i}, it follows that q_{θ_i} must also always retain the same sign; for if we construct the two curves $y = q_{\theta_i} + c_i$ and $y = q_{\theta_i} - c_i$, these will both lie on the same side of the axis of x, and never cut the axis, consequently the curve $y = q_{\theta_i}$, which lies between them, must also lie on the same side as either of them, and never cut the axis.

Hence, then, if the roots of the equation are all impossible, q_{θ_i} will always retain the same sign, and will never fall within the region bounded on two sides by $+c_i$ and $-c_i$.

Secondly, suppose the equation to have one or more possible roots, and l_i to be the greatest, and λ_i the least (which of course, if there is but one possible root, will be identical). If the leading coefficient of q_{θ_i} is positive, the greatest root (l) of the equation $q_{\theta_i} - c_i = 0$ will exceed the greatest root (l') of the equation $q_{\theta_i} + c_i = 0$; for between $x = \infty$ and $x = l'$, q_{θ_i} must go through all values intermediate between ∞ and $-c_i$; hence there must be a quality l intermediate between l' and $+\infty$, which will make $q_{\theta_i} = c_i$. In like manner, if the leading coefficient of q_{θ_i} is negative, it will be seen that the greatest root of $q_{\theta_i} + c_i = 0$ will exceed that of $q_{\theta_i} - c_i = 0$. Moreover, in the one case q_{θ_i} will be always positive and greater than c_i, and in the other always negative and less than c_i. In every case, therefore, between $+\infty$ and l_i, q_{θ_i} retains the same sign, and does not fall within the region bounded by $+c_i$ and $-c_i$; the same thing may be shown to be true for all values of x between $-\infty$ and λ_i. Hence, then, by the same reasoning as that employed in the preceding article, we are enabled to affirm, that if we form the equation

$$(q_{\theta_1}^2 - 1)(q_{\theta_2}^2 - 4)(q_{\theta_3}^2 - 4) \ldots (q^2_{\theta_{\rho-1}} - 4)(q_{\theta_\rho}^2 - 1) = 0, \qquad (\psi)$$

its greatest root will be a superior limit, and its least root an inferior limit to the roots of the equation $fx = 0$, whatever be the value of the assumed function ϕx; and if the above equation (ψ) has no real root, all the roots of fx will be imaginary.

Art. 54. In the preceding two articles it has been supposed that all the quotients are taken integral functions of x; but the process of successive division may be so conducted as to give rise to quotients of the form

$$ax^i + bx^{i-1} + \ldots + c + \frac{d}{x} + \ldots + \frac{l}{x^{i'}}.$$

Suppose then that we have in general

$$\frac{\phi x}{fx} = \frac{1}{q_1 +} \frac{1}{q_2 +} \cdots + \frac{1}{q_\omega},$$

where $q_1, q_2 \ldots q_\omega$ are each of the general form above written (but of course i and i' being not necessarily the same for any two of the quotients), and suppose that the sum of the degrees in x of $q_1, q_2 \ldots q_\omega$ is $n+t$, where t is essentially (as it must be) positive. Then we shall find, as in the last article, that L and Λ being called the greatest and least roots of

$$(q_1^2 - 1)(q_2^2 - 4) \ldots (q^2_{\omega-1} - 4)(q_\omega^2 - 1),$$

D, the denominator of the last convergent to the continued fraction above written, will never change its sign between $+\infty$ and L, nor between Λ and $-\infty$; but here we shall have

$$fx = Kx^t \times D.$$

Hence $x^t D$ will be invariable in sign within each of these two intervals.

Firstly, let t be even; then fx will be invariable in sign, whatever L and Λ may be for each such interval.

Secondly, let t be odd; then if L is > 0 and $\Lambda < 0$, fx cannot change its sign in either interval; but if L is < 0 or $\Lambda > 0$, fx will change its sign as x passes through zero, but will be invariable for each of the three regions contained between $+\infty$ and L, L and 0, or 0 and Λ (as the case may be), and Λ and $-\infty$; so that universally L and Λ will be a superior and inferior limit to the roots of fx, making abstraction of the roots (if any such there be in fx) whose value is zero.

Art. 55. I shall close this section with offering (for what it is worth) a bare suggestion as to the mode in which the theory of Intercalations may hereafter be found to admit of being extended from a system of two general functions of x, to a system of three general functions of x, y, four general functions of x, y, z, and in general to a system of ϵ general functions of $\epsilon - 1$ variables, or which is the same thing, of ϵ homogeneous functions of ϵ variables. In the case of two functions of x, fx and ϕx, $fx = 0$ and $\phi x = 0$ may be considered to represent two systems of points in a right line; and the theory relates in this case to the relative positions of these two "Kenothemes" or point systems; and of course using x and y to denote the distances of any point in a line from two fixed points therein respectively, instead of fx and ϕx, we may employ two homogeneous functions of x and y, as $f(x, y)$ and $\phi(x, y)$, to denote these two systems of points. So, similarly, if we have three functions of two variables, $f(x, y)$, $g(x, y)$, $h(x, y)$, which I shall suppose to be of the same degree, we may consider the mutual relations of the Monothemes, that is to say, the three plane curves, denoted by the equations $f(x, y) = 0$, $g(x, y) = 0$, $h(x, y) = 0$. Now every two of these will intersect one another in a system of points, which we may call (f, g) for the intersections of f and g, (g, h) for those of g and h, and (h, f) for those of h and f. If we take any two of these systems of intersections, as (f, g) and (g, h), they will both lie upon one of the given curves (g). And by reading off the two systems of points (f, g) and (g, h), arranged according to the order upon which they are disposed upon the curve g, we may, by following the course of such curve, form a scale of effective intercalations for these two systems, and in like manner for the two systems (g, h) and (h, f); (h, f) and (f, g). Now I believe that it will be found that when f, g, h represent any algebraical curves consisting of a single continuous line, either extending to infinity in both directions, or returning to itself (and I have fully satisfied myself of the truth of this for the case of ellipses), each effective scale of intercalation will contain the same number of pairs of points; if, however, the curves consist of more than one branch, as if hyperbolae be considered, such is no longer necessarily the case; from these facts, conjoined with the light thrown upon the subject by its relation to the theory of combinants explained in the succeeding section, I am induced to infer the probability of the truth of the following law (which, for avoidance of further uncertainty, I confine to the case of functions of the same degree), namely, that if f, g, h be three homogeneous functions of x, y, and z of the same degree, and if U, V, W be any three linear functions of f, g, h, and if $U = 0$, $V = 0$, $W = 0$ be treated as the equations to three cones, and if we form an effective scale of the intercalations of the lines of intersection of U and W, and V and W, according to the order in which they are disposed upon W (which seems to require that the lines shall be continuous, in order to admit of a fixed order of reading off the intersections of any two of them upon the third); then, whatever value may have been given to the coefficients in the linear functions, the number of elements remaining in any such scale will (as I conjecture) be constant, and some theory (to be discovered) for three functions, analogous to that of Bezoutian residues for two functions, will serve to determine the number of the elements so remaining. And so, in like manner, but with a difficulty increasing at each step (as at the next step we should have to pass into *quasi-space* of four dimensions), a theory of intercalations may be conjectured to exist for any n general functions of any $(n-1)$ variables.

Development of the method of assigning a superior and inferior limit to the roots of any algebraical equation.

Art. (α). Since the articles in the preceding part of this section on the method of discovering limits to the roots of an algebraical equation were written, the method of which the germ is therein contained has presented itself in a much more fully developed form, which I proceed to exhibit: for greater simplicity I shall suppose ϕx to be of $n-1$, and fx to be of n dimensions in x, and that by means of the ordinary process for common measure (except that as in Sturm's theorem the signs of all the remainders are changed) $\frac{\phi x}{fx}$ has been thrown under the form of the improper continued fraction

$$\frac{1}{q_1 -} \frac{1}{q_2 -} \frac{1}{q_3 -} \cdots \frac{1}{q_n},$$

where $q_1, q_2 \ldots q_n$ are all restricted to signify simple linear functions of x.

Suppose the series $q_1, q_2, q_3 \ldots q_n$ to be resolved into the distinct sequences

$$q_1 q_2 \cdots q_i, \; q_{i+1} q_{i+2} \cdots q_{i'}, \; q_{i'+1} \cdots q_{i''}, \ldots, q_{(i)+1} \cdots q_n,$$

in such a manner that in each sequence, as $q_{i+1}, q_{i+2} \ldots q_{i'}$, the coefficients of x have all the same sign, but that in any two adjoining sequences the coefficients of x have opposite signs, so that for instance in q_i and q_{i+1} the coefficients of x are unlike, as also in $q_{i'}$ and $q_{i'+1}$; there will of course be nothing to preclude any of these sequences becoming reduced to a single term.

The first theorem is, that the greatest and least roots of the product of the cumulants [p. 504 above]

$$[q_1 q_2 \cdots q_i] \times [q_{i+1} q_{i+2} \cdots q_{i'}] \cdots \times [q_{(i)+1} q_{(i)+2} \cdots q_n]$$

are superior and inferior limits to the roots of fx. To prove this theorem I begin with premising the two following lemmas, one virtually and the other expressly contained in the *Philosophical Magazine* for the months of September and October of the present year* [p. 641 below].

* Each of these two lemmata flows readily from the faculty previously adverted to engaged by every cumulant of being representable under the form of a determinant. As to the second lemma, it becomes apparent immediately when the cumulant is so represented, by separating the matrix into two rectangles and expressing the entire determinant according to a well-known rule for the decomposition of determinants as a function of the determinants belonging to these two rectangles taken separately. As to the first lemma, by reason of the cumulant $[\omega_1 \omega_2 \ldots \omega_{i-1} \omega_i \omega_{i+1}]$ being so representable, we know that when $[\omega_1 \omega_2 \ldots \omega_{i-1} \omega_i] = 0$, $[\omega_1 \omega_2 \ldots \omega_{i-1}]$ and $[\omega_1 \omega_2 \ldots \omega_{i+1}]$ must have opposite signs. Suppose, now, that the theorem is true when the number of elements in the type does not exceed i; then the roots of $[\omega_1 \omega_2 \ldots \omega_{i-1}]$, say of ψ_{i-1}, being called $h_1, h_2 \ldots h_{i-1}$, and of $[\omega_1 \omega_2 \ldots \omega_{i-1} \omega_i]$, say of ψ_i, being called $k_1, k_2 \ldots k_i$, these may be arranged in the following order of magnitude $k_1, h_1, k_2, h_2, k_3 \ldots k_{i-1}, h_{i-1}, k_i$; and if the roots of $[\omega_1 \omega_2 \ldots \omega_{i-1} \omega_i \omega_{i+1}]$, say of ψ_{i+1}, be called $l_1, l_2 \ldots l_{i+1}$, from the fact of the leading coefficients in ψ_{i-1} and ψ_{i+1} expanded according to the powers of x having the same sign, it follows that when $x = \infty$, ψ_{i-1} and ψ_{i+1} have the same sign, but they have contrary signs when $x = k_1$; but ψ_{i-1} does not change its sign between $x = \infty$ and $x = k_1$, hence ψ_{i+1} does change its sign between $x = \infty$ and $x = k_1$, and therefore a root of ψ_{i+1} lies between ∞ and k_1; in like manner precisely it may be shown that a root of ψ_{i+1} lies between $-\infty$ and k_i; and since ψ_{i-1} changes its sign between k_1 and k_2, between k_2 and $k_3 \ldots$ and between k_{i-1} and k_i, ψ_{i+1} must likewise change its sign between one and the other extremity of each of these intervals, and hence the roots $l_1, l_2 \ldots l_{i+1}$ are intercalated between $\infty, k_1, k_2 \ldots k_i, -\infty$, or which is the same thing, $k_1, k_2 \ldots k_i$ are respectively intercalated between $l_1, l_2 \ldots l_{i+1}$; consequently, if the theorem is true up to i, it is true for $i+1$, and therefore true universally; but is manifestly true when $i = 2$, for then $x = \pm\infty$ makes $[\omega_1 \omega_2]$, that is, $\omega_1 \omega_2 - 1$ positive; but $\omega_1 = 0$ makes it negative, which proves the theorem contained in Lemma A.

LEMMA A. The roots of the cumulant $[q_1 q_2 \ldots q_i]$, in which each element is a linear function of x, and wherein the coefficient of x for each element has the like sign, are all real, and between every two of such roots is contained a root of the cumulant $[q_1 q_2 \ldots q_{i-1}]$, and *ex converso* a root of the cumulant $[q_2 q_3 \ldots q_i]$; and (as an evident corollary) for all values of ρ and ρ' intermediate between 1 and i the greatest root of $[q_1 q_2 \ldots q_{i-1} q_i]$ will be greater, and the least root of the same will be less, than the greatest and least roots respectively of $[q_\rho q_{\rho+1} \ldots q_{\rho'-1} q_{\rho'}]$.

LEMMA B. For all values of the elements $q_1 q_2 \ldots q_n$, the cumulant

$$[q_1 q_2 \cdots q_{\omega-1} q_\omega q_{\omega+1} q_{\omega+2} \cdots q_n] = [q_1 q_2 \cdots q_{\omega-1} q_\omega] \times [q_{\omega+1} q_{\omega+2} \cdots q_n] \\ - [q_1 q_2 \cdots q_{\omega-1}] \times [q_{\omega+2} \cdots q_n].$$

Thus for example the cumulant $[abcd]$, that is

$$abcd - ab - cd - ad + 1 = [ab] \times [cd] - [a] \times [d] = (ab-1)(cd-1) - ad,$$

and $[abcde]$, that is

$$abcde - abc - abe - ade - cde + a + c + e = [abc]\,[de] - [ab]\,[e],$$

that is $$= (abc - a - c)(de - 1) - (ab - 1)\,e.$$

Art. (β). Also suppose that $q_1 q_2 \ldots q_\omega q_{\omega+1} \ldots q_n$ are all linear functions of x, and that the coefficients of x have all one (say the positive) sign in $q_1, q_2 \ldots q_\omega$, and all the contrary signs in $q_{\omega+1} \ldots q_n$, and let L be not less than the greatest root of $[q_1 q_2 \ldots q_\omega]$ or of $[q_{\omega+1} \ldots q_n]$, and also let Λ be not greater that the least root of each of these same two cumulants; then by Lemma A, L and Λ will also be respectively greater than the greatest, and less than the least roots of $[q_1 q_2 \ldots q_{\omega-1}]$ and of $[q_{\omega+2} \ldots q_n]$. Now the coefficient of the highest power of x in both $[q_1 q_2 \ldots q_\omega]$ and in $[q_1 q_2 \ldots q_{\omega-1}]$ is positive, but as to $[q_{\omega+1} \ldots q_n]$ and $[q_{\omega+2} \ldots q_n]$ is of contrary signs in the two, namely, negative in that one of those cumulants which contains an odd, and positive in that one of the two which contains an even number of elements. Hence by virtue of Lemma B, L and any quantity greater than L substituted for x will make $[q_1 q_2 \ldots q_n]$ to have always the same sign, and in like manner it may be shown that Λ and any quantity less than Λ substituted for x will also cause $[q_1 q_2 \ldots q_n]$ to retain always the same sign. Hence L and Λ are superior and inferior limits to $[q_1 q_2 \ldots q_n]$; and the same reasoning would evidently apply if we had supposed the signs of the coefficients of x in the first partial series of elements to have been negative, and in the other series of elements to have been positive.

The greatest and least roots of $[q_1 q_2 \ldots q_\omega] \times [q_{\omega+1} \ldots q_n]$ evidently satisfy the condition to which L and Λ are subject, and may be taken in place of L and Λ respectively. They will accordingly be superior and inferior limits to the cumulant

$$[q_1 q_2 \cdots q_\omega q_{\omega+1} \cdots q_n].$$

Again, by virtue of Lemma B it may readily be shown that

$$
\begin{aligned}
&[q_1 q_2 \cdots q_{\omega_1},\ q_{\omega_1+1} q_{\omega_2+2} \cdots q_{\omega_2},\ q_{\omega_2+1} \cdots q_n] \\
&\qquad = [q_1 q_2 \cdots q_{\omega_1}] \times [q_{\omega_1+1} q_{\omega_1+2} \cdots q_{\omega_2}] \times [q_{\omega_2+1} \cdots q_n] \\
&\qquad - [q_1 q_2 \cdots q_{\omega_1-1}] \times [q_{\omega_1+2} \cdots q_{\omega_2}] \times [q_{\omega_2+1} \cdots q_n] \\
&\qquad - [q_1 q_2 \cdots q_{\omega_1}] \times [q_{\omega_1+1} \cdots q_{\omega_2-1}] \times [q_{\omega_2+2} \cdots q_n] \\
&\qquad + [q_1 q_2 \cdots q_{\omega_1-1}] \times [q_{\omega_1+2} \cdots q_{\omega_2-1}] \times [q_{\omega_2+2} \cdots q_n];
\end{aligned}
$$

and hence if $q_1, q_2 \ldots q_n$ are all linear functions of x in which the coefficients of x have all the same algebraical sign in any one (taken *per se*) of the three series

$$q_1 q_2 \cdots q_{\omega_1},\ q_{\omega_1+1} \cdots q_{\omega_2},\ q_{\omega_2+1} \cdots q_n,$$

but so that this sign changes in passing from one series to another, it is easily seen, by the same reasoning as in the preceding case, that the two positive and two negative products on the right-hand side of the equation all give the same sign to the coefficient of the highest power of x, and consequently that if L and Λ be superior and inferior limits to

$$[q_1 \cdots q_{\omega_1}],\ [q_{\omega_1+1} \cdots q_{\omega_2}],\ [q_{\omega_2+1} \cdots q_n],$$

and consequently by Lemma A, to

$$[q_1 q_2 \cdots q_{\omega_1-1}],\ [q_{\omega_1+2} \cdots q_{\omega_2}],\ [q_{\omega_1+1} \cdots q_{\omega_2-1}],\ [q_{\omega_1+2} \cdots q_{\omega_2-1}],$$

and to $$[q_{\omega_2+2} \cdots q_n],$$

L or Λ substituted for x will cause $[q_1 q_2 \cdots q_n]$ to retain always the same sign, and will consequently be superior and inferior limits thereto; and so in general; whence it follows, returning to the theorem to be demonstrated, that the greatest and least roots of

$$[q_1 q_2 \cdots q_i] \times [q_{i+1} q_{i+2} \cdots q_{i'}] \times \ldots \times [q_{(i)+1} \cdots q_n],$$

will be superior and inferior limits to the cumulant $[q_1 q_2 \cdots q_n]$, that is to Cfx*, and therefore to fx, as was to be proved.

Art. (γ). The second theorem is the following: if $q_1, q_2 \cdots q_n$ be linear functions of x, say $a_1 x + b_1,\ a_2 x + b_2 \ldots a_n x + b_n$, in which the coefficients of x

* If $\dfrac{\phi x}{fx}$ expanded as a continued fraction by means of the common measure process gives rise to the quotients $q_1, q_2 \ldots q_n$, and if $L_1, L_2 \ldots L_{n-1}, L_n$ be the leading coefficients of the successive simplified residues, (L_n being, in fact, the final simplified residue, that is, the resultant to $\phi x, fx$), we must have $\phi x = C[q_2, q_3 \cdots q_n]$, $fx = C[q_1, q_2 \cdots q_n]$, where (supposing ϕx to be of $n-1$, and fx of n dimensions in x),

$$C = \frac{1}{L_n} \left\{ \frac{L_n{}^2 L^2{}_{n-2} L^2{}_{n-4} \,\&\text{c.}}{L^2{}_{n-1} L^2{}_{n-3} L^2{}_{n-5} \,\&\text{c.}} \right\}.$$

have all the same sign, and if we take the quantities $\mu_1, \mu_2 \ldots \mu_{n-1}$, all having the same sign as $a_1, a_2 \ldots a_n$, but otherwise arbitrary, and make

$$k_1 = \mu_1, \quad k_2 = \mu_2 + \frac{1}{\mu_1}, \quad k_3 = \mu_3 + \frac{1}{\mu_2} \ldots k_{n-1} = \mu_{n-1} + \frac{1}{\mu_{n-2}}, \quad k_n = \frac{1}{\mu_{n-1}},$$

then the greatest of the quantities

$$\frac{k_1 - b_1}{a_1}, \quad \frac{k_2 - b_2}{a_2} \ldots \frac{k_n - b_n}{a_n},$$

say L, is a superior limit, and the least of the quantities

$$\frac{-k_1 - b_1}{a_1}, \quad \frac{-k_2 - b_2}{a_2} \ldots \frac{-k_n - b_n}{a_n},$$

say Λ, is an inferior limit to the roots of fx.

L and any value greater than L substituted for x will evidently make $q_1 - k_1, q_2 - k_2 \ldots q_n - k_n$, all of them positive.

Hence, when $x =$ or $> L$, q_1 is positive and $> \mu_1'$, and

$$q_2 - \frac{1}{q_1} > k_2 - \frac{1}{\mu_1} > \mu_2 + \frac{1}{\mu_1} - \frac{1}{\mu_1}, \text{ that is, is positive, and } > \mu_2,$$

$$q_3 - \frac{1}{q_2 -} \frac{1}{q_1} > k_3 - \frac{1}{\mu_2} > \mu_3 + \frac{1}{\mu_2} - \frac{1}{\mu_2}, \text{ that is, is positive, and } > \mu_3,$$

....

....

and $$q_n - \frac{1}{q_{n-1} -} \frac{1}{q_{n-2}} \ldots \frac{1}{q_1} > \frac{1}{\mu_{n-1}} - \frac{1}{\mu_{n-1}}, \text{ that is, is positive,}$$

and consequently the cumulant $[q_1 q_2 q_3 \ldots q_n]$, which

$$= q_1 \times \left(q_2 - \frac{1}{q_1}\right) \times \left(q_3 - \frac{1}{q_2 -} \frac{1}{q_1}\right) \times \text{\&c.},$$

remains of a constant sign when L and any quantity greater than L is substituted for x. Hence L is a superior limit. In like manner Λ and any quantity less than Λ will evidently make $q_1 + k_1, q_2 + k_2 \ldots q_n + k_n$ all of them negative, so that, when $x =$ or $< \Lambda$, q_1 is negative, and $< -\mu_1$,

$$q_2 - \frac{1}{q_1} < k_2 - \frac{1}{\mu_1} \text{ is negative, and } < -\mu_2,$$

$$q_3 - \frac{1}{q_2} < k_3 - \frac{1}{\mu_2} \text{ is negative, and } < -\mu_3,$$

....

....

and $$q_n - \frac{1}{q_{n-1} -} \frac{1}{q_{n-2}} \ldots \frac{1}{q_1} < \frac{1}{\mu_{n-1}} - \frac{1}{\mu_{n-1}} \text{ is negative.}$$

So that $[q_1, q_2 \dots q_n]$ for all values of x less than Λ will preserve an invariable sign, and consequently Λ is an inferior limit to fx.

Art. (δ). It may be remarked that the quantities

$$\mu_1,\ \mu_2+\frac{1}{\mu_1},\ \mu_3+\frac{1}{\mu_2},\ \dots \mu_{n-2}+\frac{1}{\mu_{n-3}},\ \mu_{n-1}+\frac{1}{\mu_{n-2}},\ \frac{1}{\mu_{n-1}}$$

may be derived successively from one another, according to the same law, from whichever end of the series we begin.

If we take any two consecutive terms as

$$\mu_i+\frac{1}{\mu_{i-1}},\ \mu_{i+1}+\frac{1}{\mu_i},$$

the effect of diminishing μ_i is to decrease the first of these two terms, and *pro tanto*, to tend to reduce the limit; but on the other hand, $\frac{1}{\mu_i}$ being increased, there is brought into play an opposite tendency, which operates *pro tanto* to increase the value of the limit.

Art. (ϵ). It is of importance to remark, that by a right selection of the system of quantities $\mu_1, \mu_2 \dots \mu_{n-1}$, which enter into the composition of $k_1, k_2 \dots k_n$, L may be made to coincide with the greatest root of $[q_1, q_2 \dots q_n]$; and so in like manner by a right selection of another system of these quantities, whereby to form $k_1, k_2 \dots k_n$, Λ may be made to coincide with the least root of the same. Thus let $\mu_1, \mu_2 \dots \mu_{n-1}$ be so chosen, that

$$q_1-k_1=0,\ \ q_2-k_2=0 \ \dots \ q_n-k_n=0,$$

are all satisfied by the same value of x.

Then $$q_1=\mu_1,\ \ q_2=\mu_2+\frac{1}{\mu_1},\ \ q_3=\mu_3+\frac{1}{\mu_2}\ \dots q_n=\frac{1}{\mu_{n-1}},$$
exist simultaneously.

Hence $$\mu_2=q_2-\frac{1}{q_1},\ \ \mu_3=q_3-\frac{1}{\mu_2}=q_3-\frac{1}{q_2-}\frac{1}{q_1},$$

$$\mu_{n-1}=q_{n-1}-\frac{1}{q_{n-2}-}\frac{1}{q_{n-3}}-\dots\frac{1}{q_1},$$

$$\mu_n=\frac{1}{q_{n-1}-}\frac{1}{q_{n-2}-}\dots\frac{1}{q_1},$$

which is satisfied by making

$$[q_n, q_{n-1}, q_{n-2} \dots q_1]=0.$$

It remains then only to show that the greatest root of x in this equation substituted for x in $q_1, q_2 \dots q_n$ will make $\mu_1, \mu_2 \dots \mu_{n-1}$ all of one sign, and that the least root of x similarly substituted, will also make them all of one, but a contrary sign, which may be proved as follows.

We have

$$\mu_1 = q_1, \ \mu_2 = [q_1, q_2] \div q_1, \ \mu_3 = [q_1 q_2 q_3] \div [q_1, q_2] \text{ \&c.}$$
$$\mu_{n-1} = [q_1 q_2 \ldots q_{n-1}] \div [q_1 q_2 \ldots q_{n-2}];$$

and by Lemma B the superior limit to $[q_1 q_2 \ldots q_n]$ will be a superior limit also to $[q_1 q_2 \ldots q_{n-2}]$, and to

$$[q_1 q_2], \ [q_1 q_2 q_3] \ldots, \ [q_1 q_2 \ldots q_{n-1}].$$

Consequently this superior limit will make $\mu_1, \mu_2 \ldots \mu_{n-1}$ have all the same sign as that of the coefficients of x in $q_1, q_2 \ldots q_n$. And in like manner, the inferior limit to $[q_1 q_2 \ldots q_n]$ will cause $\mu_1, \mu_2 \ldots \mu_{n-1}$ to have all the contrary sign to that of these coefficients.

Thus then we see that when the coefficients of x in the partial quotients to $\dfrac{\phi x}{fx}$ expressed as an improper continued fraction form a single series of continuations of signs, by a right choice of the arbitrary constants $\mu_1, \mu_2 \ldots \mu_{n-1}$ the superior or inferior limit given by this new method may severally and separately be made to coincide with the greatest and least real root, or each in turn with the sole real root of fx, if there be but one.

Art. (ζ). The general method of enclosing the roots of fx within limits is founded upon the combination of the two theorems above demonstrated. An arbitrary function ϕx, one degree in x below fx, being assumed, and by aid of the auxiliary function ϕx, fx being thrown under the form

$$C[q_1 q_2 \ldots q_i, \ q_1' q_2' \ldots q'_{i'}, \ q_1'' \ldots (q)_1 (q)_2 \ldots, \ (q)_{(i)}],$$

in which the coefficient of x is supposed to change sign in the passage from q_i to q_1', from $q'_{i'}$ to q_1'', &c., a superior limit is found to each of the cumulants

$$[q_1 q_2 \ldots q_i], \ [q_1' q_2' \ldots q'_{i'}] \ldots [(q)_1 (q)_2 \ldots (q)_{(i)}],$$

taken separately, by means of the second theorem, and then by virtue of the first theorem the greatest of these superior limits is a superior limit to the cumulant

$$[q_1 q_2 \ldots q_i \ldots (q)_1 \ldots (q)_{(i)}],$$

and consequently to fx, and so *mutatis mutandis* the least of the inferior limits of the same partial cumulants is an inferior limit to the total cumulant

$$[q_1 q_2 \ldots q_i \ldots (q)_1 (q)_2 \ldots (q)_{(i)}].$$

Art. (η). When all the roots of fx are real, if ϕx be so assumed that all its roots are intercalated between those of fx, the partial quotients to $\dfrac{\phi x}{fx}$ will form but one single series. In order that ϕx may fulfil this condition, it is necessary that the coefficients of ϕx shall be subject to certain conditions of inequality, not necessary to be investigated here; but no conditions of equality, that is, no equations between the coefficients of ϕx, are introduced by this condition; or in other words, the coefficients* of ϕx, the auxiliary function, are independent and arbitrary within limits; and we have shown that in this case the auxiliary constants $\mu_1, \mu_2 \ldots \mu_{n-1}$ may be so determined that the limits may be made to come separately and respectively into contact with the two extreme roots. When all the roots of fx are not real, the quotients (however ϕx is chosen) can no longer be made to form a single series. It still however remains true, that, by a due choice of the auxiliary function followed by a due choice of the auxiliary constants, this coincidence may be brought about, so long as there is a single real root in fx.

It is rather important to demonstrate this universal possibility of effecting a coincidence of the limits to the roots with the extreme roots themselves, because it is the most striking feature which distinguishes the method of limitation here developed from all others previously brought to light.

Art. (θ). Before entering upon this demonstration I may make the passing remark, that every method of root-limitation is implicitly a method of root-approximation.

For instance, let e be any given quantity between which and $+\infty$ it is known that a root of fx lies. Then if we write $x = e + \frac{1}{y}$, and form the equation $y^n f\left(e + \frac{1}{y}\right) = 0$, and find L a superior limit to y, it is clear that $e + \frac{1}{L}$ will lie between e and the root of fx say E, next superior to e. Again, making $x = e + \frac{1}{L} + \frac{1}{y'}$, and finding a superior limit L' to y', we shall have $e + \frac{1}{L} + \frac{1}{L'}$ still nearer to E than $e + \frac{1}{L}$ was; and so we may proceed advancing nearer and nearer, and always from the same side towards E at each step, and finally obtain E under the form $e + \frac{1}{L} + \frac{1}{L'} + \frac{1}{L''} + \&c.$ And in like manner calling $E_{\prime}$ the root next below e, we may find

$$E_{\prime} = e - \frac{1}{\Lambda} - \frac{1}{\Lambda'} - \frac{1}{\Lambda''}, \&c.$$

Art. (ι). In establishing the theorem of coincidence above adverted to, the following notation will be found very advantageous. Let Ω denote a Type of any number of Elements, as $q_1, q_2 \ldots q_{i-1}, q_i$, and let Ω' denote this

* It need scarcely be stated that $f'x$ is the simplest form of ϕx, which satisfies the condition in question.

same type when the last element, and $'\Omega$ the same type when the first element is cut off, and $'\Omega'$ the same type when both extremes are cut off, so that the apocopated type Ω' will mean $q_1, q_2 \ldots q_{i-1}$; the apocopated type $'\Omega$ will mean $q_2 q_3 \ldots q_i$, and the doubly apocopated type $'\Omega'$ will mean $q_2, q_3 \ldots q_{i-1}$.

If now a type Ω be made up of the types $\Omega_1, \Omega_2 \ldots \Omega_i$ put in apposition, and if we use in general $[\Omega]$ to denote the cumulant corresponding to the type Ω, there will be a very simple law* connecting $[\Omega]$ with

$$[\Omega_1], [\Omega_2], [\Omega_3] \ldots [\Omega_{i-2}], [\Omega_{i-1}], [\Omega_i],$$
$$[\Omega'_1], [\Omega'_2], [\Omega'_3] \ldots [\Omega'_{i-2}], [\Omega'_{i-1}],$$
$$['\Omega_2], ['\Omega_3] \ldots ['\Omega_{i-2}], ['\Omega_{i-1}], ['\Omega_i],$$
$$['\Omega'_2], ['\Omega'_3] \ldots ['\Omega'_{i-2}], ['\Omega'_{i-1}].$$

This law will be seen to be obviously deducible by successive steps of expansion from the fundamental theorem given in Lemma B, Art. (α), for the case of $\Omega = \Omega_1 \Omega_2$, and will be best understood by showing its operation in a few simple cases.

Thus let $\Omega = \Omega_1 \Omega_2$†. Then

$$[\Omega] = [\Omega_1] \times [\Omega_2] - [\Omega'_1] \times ['\Omega_2].$$

Let $\Omega = \Omega_1 \Omega_2 \Omega_3$. Then

$$[\Omega] = [\Omega_1] \times [\Omega_2] \times [\Omega_3] - [\Omega'_1] \times ['\Omega_2] \times [\Omega_3] - [\Omega_1] \times [\Omega'_2] \times ['\Omega_3] + [\Omega'_1] \times ['\Omega'_2] \times ['\Omega_3].$$

Let $\Omega = \Omega_1 \Omega_2 \Omega_3 \Omega_4$. Then

$$\begin{aligned}[\Omega] = {} & [\Omega_1] \times [\Omega_2] \times [\Omega_3] \times [\Omega_4] \\ & - [\Omega'_1] \times ['\Omega_2] \times [\Omega_3] \times [\Omega_4] - [\Omega_1] \times [\Omega'_2] \times ['\Omega_3] \times [\Omega_4] \\ & - [\Omega_1] \times [\Omega_2] \times [\Omega'_3] \times ['\Omega_4]\ddagger + [\Omega'_1] \times ['\Omega'_2] \times ['\Omega_3] \times [\Omega_4] \\ & + [\Omega'_1] \times ['\Omega_2] \times [\Omega'_3] \times ['\Omega_4] + [\Omega_1] \times [\Omega'_2] \times ['\Omega'_3] \times ['\Omega_4] \\ & - [\Omega'_1] \times ['\Omega'_2] \times ['\Omega'_3] \times ['\Omega_4],\end{aligned}$$

* The cumulant corresponding to any portion or fragment of a type may be said to be a partial cumulant to the entire type, and a type whose elements are constituted out of the elements of two or more types placed in juxtaposition may be said to be the aggregate of these types; the law given in the text above may then be said to have for its object the expansion of the complete cumulant to any type in terms of complete and partial cumulants to the types of which the given type is the aggregate.

† The sign of equality is employed here to denote the relation between a concrete whole and the aggregate of its parts.

‡ The number of distinct factors entering into these products, taken collectively, is evidently $i + 2(i-1) + (i-2)$, that is $4(i-1)$.

and so in general if $\Omega = \Omega_1 \Omega_2 \dots \Omega_i$, $[\Omega]$ may be expanded under the form of the sum of 2^{i-1} products separable into i alternately positive and negative groups containing respectively 1, $(i-1)$, $\frac{1}{2}(i-1)(i-2)$, ... $(i-1)$, 1 products.

Art. (κ). In every one of the above groups forming a product the accents enter in pairs and between contiguous factors, it being a condition that if any Ω have an accent on the right the next Ω must have one on the left, and if it have one on the left the preceding Ω must have an accent on the right, and the number of pairs of accents goes on increasing in each group from 0 to $i-1$. This rule serves completely to define the development in question*.

For greater brevity let $[\Omega_e]$, $[\Omega'_e]$, $['\Omega_e]$, $['\Omega'_e]$ be denoted respectively by ω_e, ω'_e, $'\omega_e$, $'\omega'_e$, then when the type Ω_e consists of a single element,

$$\omega'_e = 1, \quad '\omega_e = 1, \quad '\omega'_e = 0.$$

It should be observed that the two equations $\omega_e = 0$, $\omega'_e = 0$ cannot exist simultaneously, for if Ω_e represent $q_1, q_2 \dots q_i$,

$$\omega_e = q_i \omega'_e - \omega''_e, \quad \omega'_e = q_{i-1}\omega''_e - \omega'''_e, \text{ \&c.},$$

so that if $\omega_e = 0$ and $\omega'_e = 0$, we have $\omega''_e = 0$, $\omega'''_e = 0$, &c., and thus, finally, $-1 = 0$, which is absurd.

Now, if we suppose $\Omega_1, \Omega_2 \dots \Omega_e$ to be types every element in each of which is a linear function of x, the coefficients of x in these elements being positive in Ω_1, negative in Ω_2, and so on alternately, and Ω is the aggregate of $\Omega_1, \Omega_2 \dots \Omega_e$, it may easily be made out that each term in the development of ω in terms of $\omega_1, \omega'_1, '\omega_1, '\omega'_1$; $\omega_2, \omega'_2, '\omega_2, '\omega'_2$, &c. will have the same sign when we give to x a value which is a superior limit, or an inferior limit to

* When each partial type Ω consists of a single element, every doubly accented Ω will vanish, and every singly accented Ω will become unity; hence we may derive the rule for the expansion of the cumulant $[a_1 a_2 a_3 \dots a_i]$ in terms of $a_1, a_2 \dots a_i$, which will accordingly consist of

$$a_1 a_2 a_3 \dots a_i - \Sigma \frac{1}{a_e a_{e+1}} (a_1 a_2 \dots a_i) + \Sigma \frac{1}{a_e a_{e+1} \times a_f a_{f+1}} (a_1 a_2 \dots a_i) \mp \text{\&c.},$$

the indices e and f, $e+1$ and f, &c. being understood to be all distinct integers (which agrees with the known rule for the expression of the denominator of a continued fraction in terms of the quotients). The number of terms in this expansion, in consequence of the vanishing of the quantities affected with a double accent, reduces from 2^{i-1} down to the ith term in the series commencing with 1, 2, 3, &c. defined by the equation $u_{i+1} = u_i + u_{i-1}$, that is

$$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{i+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{i+1};$$

the number, therefore, of products in which double accents occur in the general expansion of $[\omega_1 \omega_2 \dots \omega_i]$ is

$$2^{i-1} - \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{i+1} + \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{i+1}.$$

the roots of each of the cumulants $\omega_1, \omega_2 \ldots \omega_e$, and consequently to those of the cumulants $\omega'_1, \omega'_2 \ldots \omega'_e$; $'\omega_1, '\omega_2 \ldots '\omega_e$; $'\omega'_1, '\omega'_2 \ldots '\omega'_e$; the products affected with positive signs being all positive or negative in themselves, and those affected with negative signs being reversely all negative, or all positive.

Thus, for example, if

$$\Omega = \Omega_1 \Omega_2,$$

$$\omega = \omega_1 \omega_2 - \omega'_1 {}'\omega_2,$$

and the sign of the leading coefficient in $'\omega_2$ will be the contrary of that in ω_2, but ω_1 and ω'_1 have both the same positive sign; so again if $\Omega = \Omega_1 \Omega_2 \Omega_3$,

$$\omega = \omega_1 \omega_2 \omega_3 - \omega'_1 {}'\omega_2 \omega_3 - \omega_1 \omega'_2 {}'\omega_3 + \omega'_1 {}'\omega'_2 {}'\omega_3,$$

where the leading coefficients in ω_2 and $'\omega_2$ have contrary signs, as have also those in ω_2 and ω'_2 while ω_2 and $'\omega'_2$ have the same sign; and of course the leading coefficients in $\omega_1, \omega_3, \omega'_1, '\omega_3$ have all the same sign, they being all positive, and so in general. But the superior limit to the roots of any integral algebraical function of x substituted in place of x causes the signs of the resulting values of the functions to coincide with the signs of the leading coefficients, so that in the example last above given, L a superior limit to all the factors in the several products in the equation substituted for x will make $\omega_1\omega_2\omega_3$, $-\omega'_1 {}'\omega_2\omega_3$, $-\omega_1\omega'_2 {}'\omega_3$, $\omega_1' {}'\omega'_2 {}'\omega_3$ to have all the same sign. The like will be true of Λ the inferior limit; for if $\Omega_1, \Omega_2, \Omega_3$ contain respectively n_1, n_2, n_3 elements, the values of the four products last above written, when $x = -\infty$, will be to the values of the same when $x = +\infty$ in the respective ratios of

$$(-)^{m_1+m_2+m_3} : 1, \quad (-)^{m_1+m_2+m_3-2} : 1, \quad (-)^{m_1+m_2+m_3-2} : 1, \quad (-)^{m_1+m_2+m_3-4} : 1,$$

and so in general. Hence we deduce the theorem, that if the total type Ω represent the aggregate in apposition of the partial orders $\Omega_1, \Omega_2 \ldots \Omega_e$ (the elements being understood to be linear functions of x, which are subject to the law of alternation in the signs of the coefficients of x in passing from one partial type to another), no superior limit to $\omega_1, \omega_2 \ldots \omega_e$ can make ω vanish unless each separate product in the expansion of ω in terms of $\omega_1, \omega_2 \ldots \omega_e$ and the appurtenant apocopated cumulants vanish separately.

Art. (λ). From the above theorem we may deduce the following law, namely, that if the roots of $\omega_1, \omega_2 \ldots \omega_e$ be supposed to be arranged in order of magnitude, and λ to be that one of them which is nearest to $+\infty$ or to $-\infty$, then if e is even it is impossible for λ to be a root of ω. Thus suppose $e = 2$, and consequently $\omega = \omega_1 \omega_2 - \omega'_1 {}'\omega_2$; if λ be a root of ω_1 and one of the two extremes of the roots of ω_1, ω_2 put in order of magnitude, λ cannot be a root of $'\omega_2$, for the roots of $'\omega_2$ are confined between the roots of ω_2; but if λ make ω and ω_1 each vanish, we must have $\omega'_1{}'\omega_2 = 0$, hence $\omega_1{}' = 0$ as well as $\omega_1 = 0$, which is impossible. In like manner if a root of ω_2 were the extreme root, the same impossibility could be in like manner established.

Again, suppose $e = 4$, so that

$$\omega = \omega_1\omega_2\omega_3\omega_4\left\{1 - \frac{\omega'_1{}'\omega_2}{\omega_1\omega_2} - \frac{\omega'_2{}'\omega_3}{\omega_2\omega_3} - \frac{\omega'_3{}'\omega_4}{\omega_3\omega_4} + \frac{\omega'_1{}'\omega'_2{}'\omega_3}{\omega_1\omega_2\omega_3} + \frac{\omega'_1{}'\omega_2\omega'_3{}'\omega_4}{\omega_1\omega_2\omega_3\omega_4} + \frac{\omega'_2{}'\omega'_3{}'\omega_4}{\omega_2\omega_3\omega_4} - \frac{\omega'_1{}'\omega'_2{}'\omega'_3{}'\omega_4}{\omega_1\omega_2\omega_3\omega_4}\right\}.$$

Let λ continue to denote one or the other extreme of the roots of $\omega_1\omega_2\omega_3\omega_4$. If λ makes $\omega = 0$ we have

$$\omega_1\omega_2\omega_3\omega_4 = 0, \quad \omega'_1{}'\omega_2\omega_3\omega_4 = 0, \quad \omega_1\omega'_2{}'\omega_3\omega_4 = 0, \quad \omega_1\omega_2\omega'_3{}'\omega_4 = 0,$$
$$\omega'_1{}'\omega'_2{}'\omega_3\omega_4 = 0, \quad \omega'_1{}'\omega_2\omega'_3{}'\omega_4 = 0, \quad \omega_1\omega'_2{}'\omega'_3{}'\omega_4 = 0, \quad \omega'_1{}'\omega'_2{}'\omega'_3{}'\omega_4 = 0.$$

Now suppose that λ is a root of ω_1, then the equations remaining to be satisfied are

$$\omega'_1{}'\omega_2\omega_3\omega_4 = 0, \quad \omega'_1{}'\omega'_2{}'\omega_3\omega_4 = 0, \quad \omega'_1{}'\omega_2\omega'_3{}'\omega_4 = 0, \quad \omega'_1{}'\omega'_2{}'\omega'_3{}'\omega_4 = 0.$$

Since ω_1 and ω'_1 cannot both be zero together, λ cannot make ω'_1 or ${}'\omega_1$ zero; and because λ is an extreme to the roots of ω_2, ω_3, ω_4, λ cannot make ω'_2 or ${}'\omega_2$ or ω_3 or ${}'\omega_3$ or ω'_4 or ${}'\omega_4$ zero, so that in fact when $x = \lambda$ none of the singly accented quantities ω can be zero. As regards the doubly accented quantities ω, the same thing cannot be affirmed, because if any Ω contains only one element the corresponding value of ω with a double accent vanishes spontaneously. Again, any of the unaccented quantities ω may vanish, because we may suppose any of these to have an extreme root λ. Consequently the first, second and fourth of the equations remaining to be satisfied, might be satisfied on making the necessary suppositions as to the form of the quantities ω and the values of the extreme roots; but the third remaining equation $\omega'_1{}'\omega_2\omega'_3{}'\omega_4 = 0$, in which only singly accented quantities ω occur, remains incapable of being satisfied on any supposition whatever. And the same thing would be true if we suppose λ to be a root of any other ω instead of ω_1. Hence λ cannot make $\omega = 0$ when $e = 4$.

In like manner, if e be any even number 2ϵ, there will be an equation

$$\omega'_1{}'\omega_2\omega'_3{}'\omega_4\omega'_5{}'\omega_6 \ldots \omega'_{2\epsilon-1}{}'\omega_{2\epsilon} = 0,$$

to be satisfied by that value (if it exist) of x which, besides being an extreme (on either side) of the roots of $\omega_1, \omega_2 \ldots \omega_{2\epsilon}$ arranged in order of magnitude, also makes $\omega = 0$. But as such equation cannot be satisfied, neither extreme root of the roots of $\omega_1, \omega_2 \ldots \omega_{2\epsilon}$ can be a root of ω, as was to be proved. Consequently, unless ϕx is so assumed that the number of changes of sign in the coefficients of x in the quotients resulting from $\frac{\phi x}{fx}$ expanded as an improper continued fraction is even (for if the *changes* from sequence to sequence are odd the number of *sequences* themselves is even), the method of limitation in the text cannot give the means of drawing either limit indefinitely near to one or the other extreme roots of fx.

Art. (μ). It now remains to prove the converse, and to show, first, that when the number of changes is even, that is, the number of sequences odd, this coincidence can always be effected; and secondly, that it is always possible when fx has one or more real roots, so to assume ϕx that the number of sequences shall be odd.

The first part of the proposition is easily proved. Thus suppose $e=3$, so that

$$\omega = \omega_1\omega_2\omega_3 - \omega'_1{}'\omega_2\omega_3 - \omega_1\omega'_2{}'\omega_3 + \omega'_1{}'\omega'_2{}'\omega_3.$$

If we suppose λ, either extreme of the scale formed by writing in order of magnitude the roots of ω_1, ω_2, ω_3, to be a root common to ω_1 and to ω_3, and if $'\omega'_2=0$, which last equation may be satisfied by supposing the type Ω_2 to consist of a single element, the separate equations

$$\omega_1\omega_2\omega_3=0, \quad \omega'_1{}'\omega_2\omega_3=0, \quad \omega_1\omega'_2{}'\omega_3=0, \quad \omega'_1{}'\omega'_2{}'\omega_3=0$$

will all be satisfied; and so in general it may be shown without difficulty that if $e=2\epsilon+1$, and if λ be a root common to

$$\omega_1=0, \quad \omega_3=0, \quad \omega_5=0 \ldots \omega_{2\epsilon+1}=0,$$

and if ω_2, $\omega_4 \ldots \omega_{2\epsilon}$ be all *simple linear functions* of x, so that consequently $'\omega'_2=0$, $'\omega'_4=0 \ldots {}'\omega'_{2\epsilon}=0$, each separate term in the development of ω will vanish singly and separately, and consequently λ will be a root of ω: for since λ makes $\omega_1=0$, $\omega_3=0 \ldots \omega_{2\epsilon+1}=0$, every product in the developed form ω, in which ω_1, $\omega_3 \ldots \omega_{2\epsilon+1}$ do not each bear at least one accent, will vanish; and if we consider any product in which ω_1, $\omega_3 \ldots \omega_{2\epsilon+1}$ are all accented, if in any two of these immediately following one after the other as ω_{2k-1}, ω_{2k+1}, an accent falls to the right of the first, and to the left of the second, the intervening term ω_{2k} will bear a double accent, and will therefore vanish, since ω_{2k} is supposed to be a linear function of x; but it is impossible when every ω is accented to prevent two accents of contiguous odd terms in any such product, from falling to the right of the left, and to the left of the right, term of the two, since the contrary would imply that all the accents would fall to the right, or all to the left, which, as above remarked, is impossible, on account of the two extreme terms being only simply accentable, that is, ω_1 only to the right, and $\omega_{2\epsilon+1}$ only to the left. Hence, when x substituted for λ makes ω_1, $\omega_3 \ldots \omega_{2\epsilon+1}$ all vanish, and when ω_2, $\omega_4 \ldots \omega_{2k}$ are all linear functions of x, $x=\lambda$ will be a root of ω.

Art. (ν). I believe that the remaining part of the proposition may be rigorously demonstrated, namely that when any of the roots of fx are real, and the number of odd integers not exceeding the index of the degree of fx is m, and the number of imaginary pairs of roots in fx is μ, ϕx may be so assumed that the quotients to $\frac{\phi x}{fx}$ expanded under the form of an improper continued fraction, may be made to take the form $\Omega_1, \Omega_2, \Omega_3, \Omega_4 \ldots \Omega_{2i+1}$, where $\Omega_2, \Omega_4 \ldots \Omega_{2i}$ are linear functions of x, and i is any number assumed at will, not less than μ, and of course not greater than m; and where $\omega_1, \omega_3 \ldots \omega_{2i+1}$ will have in common a root λ, which may be made at will the greatest or the least root of $\omega_1\omega_2\omega_3 \ldots \omega_{2i+1}$; the investigation, however, according to the present light which I possess on the subject, appears complicated and tedious, and therefore, in order that the press, which is waiting for the completion of these supplemental articles, may not be kept standing, must be adjourned to some future occasion. For the present I content myself with showing the truth of the law for the simple case where fx is a cubic function of x.

Firstly. If $\frac{\phi x}{fx}$ gives rise to a single sequence of quotients Ω, we know, from the theory of intercalations, that it is necessary that all the roots of fx shall be real, and in order that when this is the case the quotients may form a single sequence Ω, it is only necessary so to assume ϕx, that its roots may be intermediate between those of fx.

Secondly. If the roots of fx are not all real, or if they are all real, but do not comprise the roots of ϕx intercalated between them, and if for greater brevity of ratiocination we stipulate that ϕx shall have its leading coefficients of the same sign as that of the leading coefficient of fx, the leading coefficients of the three quotients will either bear the respective signs $++-$, or the respective signs $+-+$, or the respective signs $+--$; in the first and last of these cases there would be two sequences, and therefore, by what has been shown above, the method of limitation of the text could not give a limit coincident with a root. Let us then look to the remaining case, and inquire whether, and how, ϕx may be assumed so that fx shall become representable to a constant factor *près* by the cumulant $[p(x-a),\ -q(x-\beta),\ r(x-a)]$, where p, q, r are all positive, and a is a root of fx.

Let this cumulant be called hfx.

Nothing in point of generality will be lost if we suppose the leading coefficient of hfx to be -1. We then have

$$hfx = [p(x-a),\ -q(x-\beta),\ r(x-a)]$$
$$= -pqr(x-a)^2(x-b) - (p+r)(x-a)$$

and writing $-\dfrac{hfx}{x-a}=x^2+Bx+C$ and making $x=a$, we find from the above identity that

$$p+r=a^2+Ba+C, \text{ that is, } p=a^2+Ba+C-r,$$

and
$$pqr\,(x-\beta)=x+a+B,$$

hence
$$\beta+a+B=0, \text{ that is, } \beta=-B-a,$$

and
$$pqr=1, \text{ and therefore } qr=\frac{1}{p}=\frac{1}{a^2+Ba+C-r}.$$

Hence if ϕx be so assumed that the quotients to $\dfrac{\phi x}{fx}$ are $p\,(x-a)$, $-q\,(x-\beta)$, $r\,(x-a)$, we have

$$\begin{aligned} h\phi x &= [-q\,(x-\beta),\ r\,(x-a)] = -qr\,(x+B+a)\,(x-a)-1 \\ &= -qr\,(x^2+Bx-a^2-aB)-1=-\frac{1}{p}\{x^2+Bx-a^2-aB+p\}. \end{aligned}$$

Hence $\phi\,(x)$ is of the form

$$m\,\{x^2+Bx-a^2-aB+(a^2+aB+C-r)\}=m\,(x^2+Bx+C-r).$$

If we call the three roots of fx, a, b, c respectively, we have

$$q=\frac{1}{r\,(a^2+Ba+C-r)}=\frac{1}{r\,\{(a-b)\,(a-c)-r\}};$$

and since q and r are both to be positive, we see that a must be taken the greatest or least of the three roots if they are all real, so that a^2+Ba+C may be positive, which it will of course necessarily be if b and c are imaginary; we must also have $a^2+Ba+C-r$ positive, so that the form of ϕx is $m\,\{(x^2-a^2)+B\,(x-a)-t\}$, t being necessarily positive, but otherwise arbitrary, a form containing two arbitrary constants, one of which is subject to satisfy a certain condition of inequality; whereas when fx is of such a form as to *admit*, and ϕx is supposed to be so assumed as to *cause it to come to pass* that the quotients to $\dfrac{\phi x}{fx}$ form a single sequence, then the three coefficients in ϕx remain exempt from all conditions of equality but are subject to two conditions of inequality. And so in general when the degree of fx is x and the number of sequences $2i+1$, it is to be inferred that the n coefficients of ϕx will be subject to satisfy $n-i-1$ conditions of inequality and i conditions of equality.

Art. (ξ). The theory of the determination of the minimum interval between either limit determinable by this method and the nearest root, or between the two limits so determinable when ϕx is so assumed that $\dfrac{\phi x}{fx}$ gives rise to a defined even number of sequences (which will include the theory of the case where all the roots of fx are imaginary), must be deferred to an opportunity more favourable for leisurely contemplation. As regards the application of the theory to the very interesting case of all the roots being imaginary, the principal point remaining to be cleared up is the determination of the least value that can be assigned to the greatest, and the greatest value that can be assigned to the least root of the algebraical product $X_1X_2X_3 \dots X_{2n}$, where $X_1, X_2 \dots X_{2n}$ are all of them real linear functions of x, subject to the condition that the cumulant $[X_1, X_2, X_3 \dots X_{2n}]$ shall (to a numerical factor *près*) be equal to a given function of the degree $2n$ in x incapable of changing its sign, which condition implies, as a necessary consequence, that the coefficients of x in each of the terms $X_1, X_2 \dots X_{2n}$ must be affected with the same algebraical sign.

Art. (o). It should be observed that in the application of the above method, the division of the series of quotients into distinct sequences governed by the signs of the coefficients of x is introduced for the purpose of drawing the limits closer to the roots, but is not necessary for the mere object of assigning limits.

Thus, for instance, if there be two sequences so that

$$[q_1q_2 \cdots q_i, \quad q_{i+1}q_{i+2} \cdots q_{i+i'}],$$

$$q_1^2 = \mu_1^2,\ q_2^2 = \left(\mu_2 + \frac{1}{\mu_1}\right)^2,\ q_3^2 = \left(\mu_3 + \frac{1}{\mu_2}\right)^2 \cdots q_i^2 = \left(\frac{1}{\mu_{i-1}}\right)^2,$$

and $$q^2_{i+1} = \nu_1^2,\ q^2_{i+2} = \left(\nu_2 + \frac{1}{\nu_1}\right)^2 \cdots q_{i+i'} = \left(\frac{1}{\nu_{i'-1}}\right)^2,$$

the greatest and least roots of x deduced from these equations will be superior and inferior limits respectively to the roots of fx; from which it is clear that if leaving all the other equations unaltered, except those which contain respectively q_i^2 and q^2_{i+1}, we write in place of these

$$q_i^2 = \left(\rho + \frac{1}{\mu_{i-1}}\right)^2,$$

$$q^2_{i+1} = \left(\frac{1}{\rho} + \nu_1\right)^2$$

the roots of the system of $i + i'$ equations thus modified will *à fortiori* be limits to the roots of fx, but then the quantities

$$\mu_1,\ \mu_2 + \frac{1}{\mu_1} \cdots \mu_{i-1} + \frac{1}{\mu_{i-2}},\ \rho + \frac{1}{\mu_{i-1}},\ \nu_1 + \frac{1}{\rho},\ \nu_2 + \frac{1}{\nu_1} \cdots \frac{1}{\nu_{i'-1}},$$

form the same single series as would correspond to the two sequences

$$q_1\, q_2 \cdots q_i\, q_{i+1} \cdots q_{i+i'},$$

treated as a single sequence, and the same is obviously the case for any number of sequences*.

Art. (π). If we consider a single sequence as $q_1, q_2 \ldots q_n$, and write

$$q_1 = a_1(x - c_1), \quad q_2 = a_2(x - c_2) \ldots q_n = a_n(x - c_n),$$

where $a_1, a_2 \ldots a_n$ are supposed to have all the same sign, and write

$$a_1^2(x - c_1)^2 = \mu_1^2, \quad a_2^2(x - c_2)^2 = \left(\mu_2 + \frac{1}{\mu_1}\right)^2 \ldots a_n^2(x - c_n)^2 = \left(\frac{1}{\mu_{n-1}}\right)^2,$$

it seems not unlikely that the interval between the greatest and least of the roots of the above equations will be a minimum when the interval between any pair is the same for each pair, that is, when

$$\frac{\mu_1}{a_1} = \frac{\mu_2 + \frac{1}{\mu_1}}{a_2} = \frac{\mu_3 + \frac{1}{\mu_2}}{a_3} = \ldots = \frac{\frac{1}{\mu_{n-1}}}{a_n}.$$

If we assume these equations, and write $\mu_1 = a_1\xi$, the equation for determining ξ will be

$$[a_1\xi, \quad a_2\xi, \quad a_3\xi \ldots a_n\xi] = 0.$$

If $n = 2$ this equation becomes $a_1a_2\xi^2 - 1 = 0$.

If $n = 3$, rejecting the factor ξ, it becomes

$$a_1a_2a_3\xi^2 - (a_1 + a_3) = 0.$$

If $n = 4$ it becomes

$$a_1a_2a_3a_4\xi^4 - (a_1a_2 + a_3a_4 + a_1a_4)\xi^2 + 1 = 0.$$

If $n = 5$, rejecting the factor ξ, it becomes

$$a_1a_2a_3a_4a_5\xi^4 - (a_1a_2a_3 + a_1a_2a_5 + a_1a_4a_5 + a_3a_4a_5)\xi^2 + (a_1 + a_3 + a_5) = 0,$$

* It follows from this, that if $q_1, q_2 \ldots q_n$ be all linear functions of x, and if

$$Q = (q_1^2 - \mu_1^2)\left\{q_2^2 - \left(\mu_2 + \frac{1}{\mu_1}\right)^2\right\} \left\{q_3^2 - \left(\mu_3 + \frac{1}{\mu_2}\right)^2\right\} \ldots\ldots \left(q_n^2 - \frac{1}{\mu^2_{n-1}}\right),$$

no root of Q can lie between the extreme roots of the function K, used to denote the cumulant

$$[\sqrt{q_1^2}, \quad -\sqrt{q_2^2}, \quad \sqrt{q_3^2}, \ldots\ldots, \pm\sqrt{q_n^2}],$$

the square roots being understood to be taken so as to make the sign of the coefficients of x all of them positive; and from a preceding article we know that either extreme root of Q can be made to coincide with a corresponding extreme root of K. Hence we have an *à priori* solution of the following question, namely, "To determine the $(n-1)$ positive quantities $\mu_1, \mu_2 \ldots \mu_{n-1}$, so as to make the greatest root of Q a minimum and its least root a maximum;" for the greatest root of K will be the minimum greatest root of Q, and the least root of K the maximum least root of Q. Calling these respectively l and λ, the two systems of values of $\mu_1, \mu_2 \ldots \mu_{n-1}$ required will be obtained by substituting respectively l and λ for x in the equations

$$\mu_1 = \sqrt{q_1^2}, \quad \mu_2 = -\sqrt{q_2^2} - \frac{1}{\mu_1}, \quad \mu_3 = +\sqrt{q_3^2} - \frac{1}{\mu_2} \ldots\ldots \mu_{n-1} = \pm\sqrt{q^2_{n-1}} - \frac{1}{\mu_{n-2}}.$$

and so in general, the equation in ξ^2 being always of a degree measured by the integer nearest to and not exceeding $\frac{n}{2}$; and it is easy to be seen that for all values of n, the second coefficient divided by the first will be an inferior limit to ξ^2 (of course actually coinciding with it for the cases of $n=2$ and $n=3$). Hence we have the following valuable practical rule for finding a superior and inferior limit to the cumulant

$$[a_1(x-c_1),\quad a_2(x-c_2)\dots a_n(x-c_n)],$$

where $a_1, a_2\dots a_n$ have the same sign, namely if C be the greatest, and K be the least of the quantities $c_1, c_2\dots c_n$, $C+\Delta$ will be a superior, and $K-\Delta$ an inferior limit, Δ being taken equal to the positive value of

$$\sqrt{\left(\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\frac{1}{a_3a_4}+\dots+\frac{1}{a_{n-1}a_n}\right)};$$

and it may be noticed that C and K are the quantities which would themselves be the superior and inferior limits to the given cumulant if the series of terms $a_1, a_2\dots a_n$, instead of presenting only a sequence of continuations or permanencies, presented only a sequence of changes or variations of sign.

Section V.

On the Theory of Intercalations as applicable to two functions of the same degree, and on the formal properties of the Bezoutiant with reference to the method of Invariants.

Art. 56. If fx and ϕx be any two given functions of x of the same degree m, we may form a system of m Bezoutics to f and ϕ (as shown in the first section), the coefficients of the powers of $x^{m-1}, x^{m-2}\dots x^1, x^0$ in which will compose a square matrix of m lines of m terms each, which will be symmetrical in respect to the diagonal which passes through the first coefficient of the first Bezoutic and the last coefficient of the last Bezoutic; and we may construct a quadratic homogeneous function of m new variables, such that its determinantive matrix shall coincide with the Bezoutic square so formed. This quadratic form may be considered in the light of a generating function. All its coefficients will be formed of quantities obtained by taking any two coefficients in one of the given functions, and two corresponding coefficients in the other given function, multiplying them in cross order, and taking the difference: each coefficient of the generating function in question will consist of one or more such differences, and will thus be of two dimensions altogether, being linear in respect to the coefficients of f, and also linear in respect to the coefficients of ϕ. This generating function I term the

Bezoutiant, and it may be denoted by the symbol $B(f, \phi)$: the determinant of B is of course the resultant to f, ϕ, and the matrix to B is the Bezoutic square to f, ϕ. Now we have seen that the decrease in the number of continuations of sign in the series 1, $B_1(x)$, $B_2(x) \ldots B_m(x)$ (where $B_1(x)$, $B_2(x) \ldots B_m(x)$ are the m Bezoutics to f, ϕ), as x changes from a to b, measures the number of roots of fx retained in the effective scale of intercalations taken between the limits a and b. If we take the entire scale between $+\infty$ and $-\infty$ the total number of effective intercalations will be the same, whether reckoned by the number of roots of f or of ϕ remaining; for these two numbers can never differ except by a unit, since no two of either can ever come together; but the number of each remaining in the effective scale will be $m - 2i$ and $m - 2i'$ respectively, i being the number of pairs of imaginary roots and pairs of unseparated real roots of f, and i' being the similar number for ϕ; so that we must have $i = i'$.

Now obviously this number becomes measured by the number of continuations of sign in the *signaletic* series 1, (B_1), $(B_2) \ldots (B_m)$, where in general (B_i) denotes the principal coefficient in $B_i(x)$.

But (B_1), $(B_2) \ldots (B_m)$ are the successive ascending coaxal minor determinants about the axis of symmetry to the Bezoutic square; and accordingly the number of continuations just spoken of, measures the number of positive terms in the Bezoutiant when linearly transformed, so as to contain only positive and negative squares, or in other words, measures the *inertia* of the Bezoutiant, the constant integer which adheres to it under all its real linear transformations.

Art. 57. This inertia is the same number as, in the case of a homogeneous quadratic function of three variables used to express a conic referred to trilinear coordinates, serves to determine whether such conic belongs to the impossible class or to the possible class of conics, being 3 or 0 in the former case, and 1 or 2 in the latter; or as in the case of a homogeneous quadratic function of four variables used to denote a surface referred to quadriplanar or tetrahedral coordinates, serves to determine whether such surface belongs to the impossible class or to the class consisting of the ellipsoid and the hyperboloid of two sheets (which are *descriptively* indistinguishable), or to the hyperboloid of one sheet, being 0 or 4 in the first case, 1 or 3 in the second, and 2 in the third. The most *symmetrical* (but least expeditious) method of finding the *inertia* of any quadratic form is that which corresponds to the method of orthogonal transformations, and is, in fact, the usual method employed in geometrical treatises on lines and surfaces of the second degree. If we apply this method to the Bezoutiant B considered as a homogeneous quadratic function of the m arbitrarily named variables u_1, u_2, $u_3 \ldots u_m$ in order to measure its inertia, that is to say, the number of effective interpositions between the two systems of roots, we must construct the determinant

$$D(\lambda) = \begin{vmatrix} \frac{d^2B}{du_1^2}+\lambda, & \frac{d^2B}{du_1du_2}, & \frac{d^2B}{du_1du_3} \cdots & \frac{d^2B}{du_1du_m} \\ \frac{d^2B}{du_2du_1}, & \frac{d^2B}{du_2^2}+\lambda, & \frac{d^2B}{du_2du_3} \cdots & \frac{d^2B}{du_2du_m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^2B}{du_mdu_1}, & \frac{d^2B}{du_mdu_2}, & \frac{d^2B}{du_mdu_3} \cdots & \frac{d^2B}{du_m^2}+\lambda \end{vmatrix}.$$

All the roots of $D(\lambda)=0$, as is well known, are real; the inertia of B, being measured by the number of positive roots of $D(-\lambda)$, will be equal to the number of continuations of sign in $D(\lambda)$ expressed as a function of λ of the mth degree.

If in fx and ϕx we reverse the order of the coefficients, and fx and ϕx so transformed become f_1x and ϕ_1x, it is obvious that the roots of f_1 and ϕ_1 being the reciprocals of the roots of f and ϕ respectively, the number of effective intercalations to f_1 and ϕ_1 must be the same as for f and ϕ. Accordingly we find that the form of the Bezoutiant to f and ϕ is the same as that of the Bezoutiant of f_1 and ϕ_1, the sole difference (one only of *names*) being that $B(u_1, u_2 \ldots u_{m-1}, u_m)$ for the one becomes $B(u_m, u_{m-1} \ldots u_2, u_1)$ for the other. The equation $D(\lambda)$, which determines the *inertia* of B, remains precisely the same, as it ought to do, for either of the two systems f and ϕ or f_1 and ϕ_1.

Art. 58. The theory in the preceding articles of this section may be made to embrace the case involved in Sturm's theorem; for if

$$fx = a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x^{m-1} + a_mx^m,$$

$$f'x = ma_0x^{m-1} + (m-1)\,a_1x^{m-2} + \ldots + a_{m-1},$$

and

$$f_1x = mfx - f'x$$

$$= a_1x^{m-1} + 2a_2x^{m-2} + \ldots + ma_m,$$

the Bezoutian secondaries, or which is the same thing, the simplified Sturmian residues to fx and $f'x$, will evidently be the same as those to f_1x and $f'x$. Accordingly, if we form the signaletic series

$$f_1x,\ f'x,\ B_1,\ B_2 \ldots B_{m-1},$$

where B_1, $B_2 \ldots B_{m-1}$ are the Bezoutian secondaries to f_1x and $f'x$, the number of *variations* of sign between consecutive terms in this series, when

x is made $+\infty$, will measure the number of pairs of imaginary roots in fx; and fx and $f'x$ forming always a continuation, and the highest coefficient of $f'x$ being supposed positive, we see that the terms of the rhizoristic series will be 1, (B_1), $(B_2) \dots (B_{m-1})$, consisting of positive unity and the successive ascending coaxal determinants of the Bezoutian matrix to $f'x$ and f_1x. Hence then the form of the Bezoutiant to $f'x$ and f_1x will serve to determine the number of pairs of imaginary, and consequently also the number of real roots to fx. It should be remarked that the form of the Bezoutiant to $f'x$ and f_1x, considered as a quadratic function of $u_1, u_2 \dots u_{m-1}$ and of the coefficients in fx, will remain unaltered when for fx we write f_1x, for this will change the signs throughout of fx and f_1x; and consequently the coefficients in the Bezoutiant, which contains in every term one coefficient from $f'x$, and one from f_1x, will remain unaltered in sign.

Art. 59. It appears then from the preceding article, that for every function of x of the degree m, there exists a homogeneous quadratic function of $(m-1)$ variables, the *inertia* of which augmented by unity will represent the number of real roots in the given function. Now this *inertia* itself may be measured by the number of positive roots of a certain equation in λ formed from the quadratic function (in fact the well-known equation for the secular inequalities of the planets), all whose roots will be real. Hence then we are led to the following remarkable statement. "*An algebraical equation of any degree being given, an equation whose degree is one unit lower may be formed, all the roots of which shall be real, and of which the number of positive roots shall be one less than the total number of real roots of the given equation.*"

Let us suppose fx written in its most general form, the first and last as well as all the intermediate coefficients being anything whatever: by reversing the order of the coefficients $f'x$ will become f_1x and f_1x will become $f'x$; the Bezoutiant to f_1x and $f'x$ (which we may term the Bezoutoid to fx) will remain unaltered except in sign, and the equation of the $(m-1)$th degree in λ formed from the Bezoutoid remain unchanged; consequently the equation in λ enables us to substitute, for the purpose of calculating the total number of real roots in fx, in lieu of Sturm's auxiliary functions to fx, another set of functions which remain unaltered when the order of the coefficients is completely reversed, that is in effect, when we consider the number of real roots of $f\left(\frac{1}{x}\right)$ in lieu of those of $f(x)$. And of course more generally the equation of the mth degree in λ formed from the Bezoutiant to any two functions fx and ϕx of the mth degree each in x, supplies a set of functions for determining the total number of effective intercalations between the roots of fx and ϕx, which do not alter when we consider in lieu of these the roots of $f\left(\frac{1}{x}\right)$ and $\phi\left(\frac{1}{x}\right)$. This substitution of functions symmetrically formed in respect to the two ends of an equation for the purpose of assigning the total number of real roots in lieu of the unsymmetrical ones furnished by the ordinary method of M. Sturm, had been long felt by me to be a desideratum, and as an object the accomplishment of which was indispensable to the ulterior development of the theory, and it is certain that I did not in anticipation exaggerate the importance of the result to be attained.

Art. 60. It may happen that the Bezoutiant to f and ϕ (each of the mth degree) may become a quadratic function of less than m independent variables, or the Bezoutoid to f (a function in x of the mth degree) of less than $(m-1)$ independent variables. This will take place whenever f and ϕ have roots in common, or whenever f has equal roots. The number of independent relations of equality between the roots of f and ϕ, and the amount of multiplicity, however distributed, among the roots of f, will be indicated by the number of *orders* thus disappearing out of the general form of the Bezoutiant and Bezoutoid in the respective cases*. In what particular mode the form of each would be affected according to the manner of the distribution of the equalities and the multiplicity requires a specific discussion, which I must reserve for some future occasion.

Art. 61. I shall devote the remainder of this memoir to a consideration of the properties and affinities of Bezoutiants or Bezoutoids, regarded from the point of view of the Calculus of Invariants. For this purpose it will be more convenient hereafter to convert all the functions which we are concerned with into homogeneous forms, and I shall accordingly for the future use f and ϕ to denote functions each of x and y, which I shall write under the form

$$f = a_0x^m + ma_1x^{m-1}y + \tfrac{1}{2}m(m-1)\,a_2x^{m-2}y^2 + \ldots + a_my^m,$$

$$\phi = b_0x^m + mb_1x^{m-1}y + \tfrac{1}{2}m(m-1)\,b_2x^{m-2}y^2 + \ldots + b_my^m.$$

In what follows a knowledge of the general principles of the Method of Invariants is presupposed, but a perusal of my two papers on the Calculus of Forms† in the *Cambridge and Dublin Mathematical Journal*, February and May, 1852, will furnish nearly all the information that is strictly necessary for the present purpose. The first point to be established is, that B, the

* I have elsewhere defined how this word order, as here employed, is to be understood. If F, a homogeneous function of $x_1, x_2 \ldots x_n$, can be expressed as a function of $u_1, u_2 \ldots u_{n-i}$ (all linear functions of $x_1, x_2 \ldots x_n$), F is said to be a function of $n-i$ orders, or to have lost i of the orders belonging to the complete form.

[† See pp. 284, 328, 411 above.]

Bezoutiant of fx and ϕx, is a Covariant to the system f, ϕ; the variables in B being in compound relation of cogredience with the combinations of powers of x and y,

$$x^{m-1},\ x^{m-2}y,\ x^{m-3}y^2 \dots y^{m-1}.$$

That is to say, I propose to show that if f, g, h, k be any four quantities, taken for greater simplicity subject to the relation $fk - gh = 1$, and if on substituting $fx + gy$ for x and $hx + ky$ for y, $f(x, y)$ becomes

$$A_0x^m + mA_1x^{m-1}y + \tfrac{1}{2}m(m-1)A_2x^{m-2}y^2 + A_my^m, \text{ say } G(x,\ y),$$

and $\phi(x, y)$ becomes

$$B_0x^m + mB_1x^{m-1}y + \tfrac{1}{2}m(m-1)B_2x^{m-2}y^2 + B_my^m, \text{ say } T(x,\ y),$$

and if $B'(u_1', u_2' \dots u_m')$ be the Bezoutiant to G and T, $B(u_1, u_2 \dots u_m)$ being that to f and ϕ, then, on making u_1, $u_2 \dots u_m$, the same *linear* functions of u_1', $u_2' \dots u_m'$ as

$$(fx+gy)^{m-1},\quad (fx+gy)^{m-2}(hx+ky) \dots (fx+gy)(hx+ky)^{m-2},\quad (hx+ky)^{m-1},$$

are respectively of

$$x^{m-1},\ x^{m-2}y \dots xy^{m-2},\ y^{m-1},$$

B will become identical with B'. I was led to suspect the high probability of the truth of this proposition concerning the invariance of the Bezoutiant from the following considerations: Firstly, that for the particular case where f and ϕ are the differential derivatives in respect to x and y respectively of the same function $F(x, y)$, the Bezoutiant of f and ϕ, which then becomes the Bezoutoid of F, determines the number of real factors in F, which obviously remains the same for all linear transformations of F. Secondly, that taking f and ϕ in their most general form, the invariant to their Bezoutiant, that is the determinant of their Bezoutiant, is an invariant of f and ϕ, being in fact the resultant of these two functions; now as every concomitant (an invariantive form of the most general kind) to a concomitant is itself a concomitant to the primitive, so it appeared to me, and is I believe true (although awaiting strict proof), that any form satisfying certain necessary and tolerably obvious conditions of homogeneity and isobarism, a concomitant to which is also a concomitant to a given form, will be itself a concomitant to such form; this principle, if admitted, would be of course at once conclusive as to the Bezoutiant being an invariantive concomitant to the functions from which it is derived.

Art. 61*. Since the publication of the two papers above referred to on the Calculus of Forms, I have made the important observation that every species of concomitant, however complex, to a given system of functions, may be treated as a simple invariant of a system including the given system together with an appropriate superadded system of absolute functions; thus an ordinary covariant involving only one system of variables, as $u, v, w \ldots$ cogredient with $x, y, z \ldots$ the variables of a system S, is in fact an invariant of the system S combined with the system $uy - vx$, $vz - wy$, $wx - uz$, &c., $u, v, w \ldots$ being treated as constants; so again a simple contravariant of S is an invariant of S combined with the form $ux + vy + wz + \&c.$; so again, to meet the case before us, a covariant to the binary system f and ϕ expressed as a function of $u_1, u_2 \ldots u_m$, where $u_1, u_2 \ldots u_m$ are cogredient with x^{m-1}, $x^{m-2}y \ldots y^{m-1}$, may be regarded as an invariant of the ternary system f, ϕ, Ω, where

$$\Omega = u_1 y^{m-1} - m u_2 y^{m-2} x + \tfrac{1}{2} m (m-1) u_3 y^{m-3} x^2 \ldots + (-)^{m-1} u_m x^{m-1},$$

($u_1, u_2 \ldots u_m$ being here to be treated as constants); and accordingly the differential equations which serve to define in the most general and absolute manner such covariant of f, ϕ, or invariant to f, ϕ, Ω, say I, will take the form

$$\left\{\begin{aligned} &\left(a_0 \frac{d}{da_1} + b_0 \frac{d}{db_1}\right) + 2\left(a_1 \frac{d}{da_2} + b_1 \frac{d}{db_2}\right) + 3\left(a_2 \frac{d}{da_3} + b_2 \frac{d}{db_3}\right) + \ldots \\ &\qquad + m\left(a_{m-1} \frac{d}{da_m} + b_{m-1} \frac{d}{db_m}\right) \\ &\qquad - \left(u_1 \frac{d}{du_2} + 2u_2 \frac{d}{du_3} + 3u_3 \frac{d}{du_4} + \ldots + (m-1) u_{m-1} \frac{d}{du_m}\right) \end{aligned}\right\} I = 0,$$

$$\left\{\begin{aligned} &\left(a_m \frac{d}{da_{m-1}} + b_m \frac{d}{db_{m-1}}\right) + 2\left(a_{m-1} \frac{d}{da_{m-2}} + b_{m-1} \frac{d}{db_{m-2}}\right) \\ &\qquad + 3\left(a_{m-2} \frac{d}{da_{m-3}} + b_{m-2} \frac{d}{db_{m-3}}\right) + \ldots + m\left(a_1 \frac{d}{da_0} + b_1 \frac{d}{db_0}\right) \\ &- \left(u_m \frac{d}{du_{m-1}} + 2u_{m-1} \frac{d}{du_{m-2}} + 3u_{m-2} \frac{d}{du_{m-3}} + \ldots + (m-1) u_2 \frac{d}{du_1}\right) \end{aligned}\right\} I = 0.$$

These equations may be proved to be satisfied when I is taken $= B$, the Bezoutiant to f, ϕ, and thus B may be proved to be a covariant to f, ϕ, but the demonstration is long and tedious. An admirable suggestion, well worthy of its keen-witted author, for which I am indebted to Mr Cayley, will enable us to prove the invariantive character of B by a much more expeditious method.

Art. 62. For greater simplicity begin with considering functions of a single variable x; and in order to fix the ideas, suppose m to be taken 5, and write

$$fx = ax^5 + bx^4 + cx^3 + dx^2 + ex + l,$$

$$\phi x = \alpha x^5 + \beta x^4 + \gamma x^3 + \delta x^2 + \epsilon x + \lambda,$$

and let $\vartheta = \dfrac{fx\phi x' - fx'\phi x}{x - x'}$; this is of course an integral function of x and x', since the numerator vanishes when $x = x'$; and we have by performing the actual operations,

$$\begin{aligned}\vartheta = &(a\beta - b\alpha)\, x^4 x'^4 + (a\gamma + c\alpha)\, x^3 x'^3 (x + x') + (a\delta - d\alpha)\, x^2 x'^2 (x^2 + xx' + x'^2)\\ &+ (a\epsilon - e\alpha)\, xx' (x^3 + x^2x' + xx'^2 + x'^3) + (a\lambda - l\alpha)(x^4 + x^3x' + x^2x'^2 + xx'^3 + x'^4)\\ &+ (b\gamma - c\beta)\, x^3x'^3 + (b\delta - d\beta)\, x^2x'^2 (x + x') + (b\epsilon - e\beta)\, xx' (x^2 + xx' + x'^2)\\ &+ (b\lambda - l\beta)(x^3 + x^2x' + xx'^2 + x'^3)\\ &+ (c\delta - d\gamma)\, x^2x'^2 + (c\epsilon - e\gamma)\, xx' (x + x') + (c\lambda - l\gamma)(x^2 + xx' + x'^2)\\ &+ (d\epsilon - e\delta)\, xx' + (d\lambda - l\delta)(x + x')\\ &+ (e\lambda - l\epsilon);\end{aligned}$$

and if we arrange ϑ under the form

$$\begin{aligned}&A_{4,4}\, x^4x'^4 + A_{4,3}\, x^4x'^3 + A_{4,2}\, x^4x'^2 + A_{4,1}\, x^4x' + A_{4,0}\, x^4\\ +\, &A_{3,4}\, x^3x'^4 + A_{3,3}\, x^3x'^3 + A_{3,2}\, x^3x'^2 + A_{3,1}\, x^3x' + A_{3,0}\, x^3\\ +\, &A_{2,4}\, x^2x'^4 + A_{2,3}\, x^2x'^3 + A_{2,2}\, x^2x'^2 + A_{2,1}\, x^2x' + A_{2,0}\, x^2\\ +\, &A_{1,4}\, xx'^4 + A_{1,3}\, xx'^3 + A_{1,2}\, xx'^2 + A_{1,1}\, xx' + A_{1,0}\, x\\ +\, &A_{0,4}\, x'^4 + A_{0,3}\, x'^3 + A_{0,2}\, x'^2 + A_{0,1}\, x' + A_{0,0}\end{aligned}$$

it will readily be perceived that the matrix formed by the twenty-five coefficients, namely

$$\begin{matrix} A_{4,4}, & A_{4,3}, & A_{4,2}, & A_{4,1}, & A_{4,0},\\ A_{3,4}, & A_{3,3}, & A_{3,2}, & A_{3,1}, & A_{3,0},\\ A_{2,4}, & A_{2,3}, & A_{2,2}, & A_{2,1}, & A_{2,0},\\ A_{1,4}, & A_{1,3}, & A_{1,2}, & A_{1,1}, & A_{1,0},\\ A_{0,4}, & A_{0,3}, & A_{0,2}, & A_{0,1}, & A_{0,0}, \end{matrix}$$

will be symmetrical about its dexter diagonal (that one, namely, which passes through $A_{4,4}$ and $A_{0,0}$), and will be identical with the Bezoutian square corresponding to the system f, ϕ; in fact, using the notation previously employed in the first section, it becomes

$$\begin{matrix} (0,\ 1) & (0,\ 2) & (0,\ 3) & (0,\ 4) & (0,\ 5)\\ (0,\ 2) & \begin{Bmatrix}(0,\ 3)\\ +\\ (1,\ 2)\end{Bmatrix} & \begin{Bmatrix}(0,\ 4)\\ +\\ (1,\ 3)\end{Bmatrix} & \begin{Bmatrix}(0,\ 5)\\ +\\ (1,\ 4)\end{Bmatrix} & (1,\ 5)\\ (0,\ 3) & \begin{Bmatrix}(0,\ 4)\\ +\\ (1,\ 3)\end{Bmatrix} & \begin{Bmatrix}(0,\ 5)\\ +\\ (1,\ 4)\\ +\\ (2,\ 3)\end{Bmatrix} & \begin{Bmatrix}(1,\ 5)\\ +\\ (2,\ 3)\end{Bmatrix} & (2,\ 5)\\ (0,\ 4) & \begin{Bmatrix}(0,\ 5)\\ +\\ (1,\ 4)\end{Bmatrix} & \begin{Bmatrix}(1,\ 5)\\ +\\ (2,\ 3)\end{Bmatrix} & \begin{Bmatrix}(2,\ 5)\\ +\\ (3,\ 4)\end{Bmatrix} & (3,\ 5)\\ (0,\ 5) & (1,\ 5) & (2,\ 5) & (3,\ 5) & (4,\ 5), \end{matrix} \qquad (\alpha)$$

(r, s) being used in general to denote the difference between the cross products of the coefficients of x^{5-r} and x^{5-s} in f and ϕ. Restoring now to m its general value, and taking f and ϕ homogeneous functions of x and y, and making

$$\vartheta = \frac{f(x, y)\,\phi(x', y') - f(x', y')\,\phi(x, y)}{xy' - x'y},$$

we see without difficulty that

$$\vartheta = \Sigma A_{r,s}\,\{x^r y^{m-1-r} x'^s y'^{m-1-s}\},$$

where $A_{r,s}$ is the term in the rth line and sth column of the Bezoutiant matrix to f and ϕ. This is the identification, the idea of which, as before observed, is due to Mr Cayley.

Art. 63. If, now, we consider the system of functions

$$f(x, y) = a_0 x^m + m a_1 x^{m-1} y + \ldots + a_m y^m,$$

$$\phi(x, y) = b_0 x^m + m b_1 x^{m-1} y + \ldots + b_m y^m,$$

$$\Omega(x, y) = u_m y^{m-1} - (m-1)\,u_{m-1} y^{m-2} x \pm \ldots + (-)^{m-1} u_1 x^{m-1},$$

evidently $f(x, y)\,\phi(x', y') - f(x', y')\,\phi(x, y)$ is a covariant with f and ϕ, and therefore (which is a mere truism) with the entire system f, ϕ, Ω. So also is $xy' - x'y$, and therefore ϑ, the quotient of these two, is a covariant to the system. Hence, therefore, by virtue of a general theorem given in my Calculus of Forms,

$$\Omega\left(\frac{d}{dy}, \; -\frac{d}{dx}\right)\vartheta$$

is a covariant to the system; and, again, therefore,

$$\Omega\left(\frac{d}{dy'}, \; -\frac{d}{dx'}\right)\Omega\left(\frac{d}{dy}, \; -\frac{d}{dx}\right)\vartheta$$

is a covariant thereto. Now ϑ is of $(m-1)$ dimensions in x, y and also of the same in x', y'. Consequently this latter form will contain only the quantities $u_1, u_2 \ldots u_{m-1}$, and the coefficients of f and ϕ, so that the powers of x, y; x', y' will not appear in it.

Now $$\vartheta = \sum_{m-1}^{0} \sum_{m-1}^{0} A_{r,s}\,\{x^r y^{m-1-r} x'^s y'^{m-1-s}\},$$

$$(-)^{m-1}\,\Omega\left(\frac{d}{dy}, -\frac{d}{dx}\right) = u_m \left(\frac{d}{dx}\right)^{m-1} + (m-1)\,u_{m-1}\left(\frac{d}{dx}\right)^{m-2}\frac{d}{dy} + \ldots$$
$$\ldots + u_1\left(\frac{d}{dy}\right)^{m-1},$$

$$(-)^{m-1}\,\Omega\left(\frac{d}{dy'}, -\frac{d}{dx'}\right) = u_m \left(\frac{d}{dx'}\right)^{m-1} + (m-1)\,u_{m-1}\left(\frac{d}{dx'}\right)^{m-2}\frac{d}{dy'} + \ldots$$
$$\ldots + u_1\left(\frac{d}{dy'}\right)^{m-1},$$

therefore
$$\frac{1}{\{1\,.\,2\,.\,3\,\ldots\,(m-1)\}^2}\,\Omega\left(\frac{d}{dy'},\,-\frac{d}{dx'}\right)\Omega\left(\frac{d}{dy},\,-\frac{d}{dx}\right)\vartheta$$
$$=\sum_{m-1}^{0}(A_{r,r}u^2_{r+1})+2\sum_{m-1}^{0}\sum_{m-1}^{0}(A_{rs}u_{r+1}u_{s+1}),$$

r and s being excluded in the latter sum from being made equal; but this latter expression is the Bezoutiant to f, ϕ. Hence the Bezoutiant of f, ϕ is an invariant to f, ϕ, Ω, that is a covariant to the system f, ϕ, as was to be proved. The mode of obtaining the covariant ϑ, used in this and the preceding article, is very remarkable. I believe that the true *suggestive* view of the process for finding it, is to consider

$$f(x,\,y)\,\phi\,(x',\,y')-f(x',\,y')\,\phi\,(x,\,y)$$

as a concomitant capable of being expressed under the form of a function of ϑ and ω, ω standing for the universal covariant $xy'-x'y$; ϑ is then to be considered, not properly as a quotient, but rather as an invariant of the form $\vartheta\omega$, a function of ω of the first degree, where ϑ is treated as constant.

Art. 64. B is not an ordinary covariant of f and ϕ, it belongs to that special and most important family of invariants to a system to which I have given the name of Combinants*, namely Invariants, which, besides the ordinary character of invariance when linear substitutions are impressed upon the variables, possess the same character of invariance when linear substitutions are impressed upon the functions themselves containing the variables; combinants being, as it were, invariants to a system of functions in their corporate combined capacity *quâ* system. That the Bezoutiant possesses this property is evident; for if instead of f and ϕ we write $kf+i\phi$ and $k'f+i'\phi$, any such quantity as $a_rb_s-a_sb_r$ (a_r, b_r being coefficients in f, and a_s, b_s the corresponding ones in ϕ) becomes

$$(ka_r+ib_r)(k'a_s+i'b_s)-(ka_s+ib_s)(k'a_r+i'b_r),$$

that is
$$(ki'-k'i)(a_rb_s-a_sb_r),$$

so that B, the Bezoutiant, becomes increased in the ratio of $(ki'-k'i)^m$, that is remains always unaltered in point of form and absolutely immutable, provided that $ki'-k'i$ be taken, as we may always suppose to be the case, equal to 1.

We derive immediately from this observation, the somewhat remarkable geometrical proposition, that the intersections with the axis of x made by any two curves of the family of curves $u=\lambda f(x)+\mu\phi(x)$, ($f$ and ϕ being functions of x of the same degree) give rise to a constant number of effective intercalations, whatever values be given to λ or μ for the two curves so selected.

* For some remarks on the Classification of Combinants, see *Cambridge and Dublin Mathematical Journal*, November, 1853 [p. 411 above].

Art. 65. $B(u_1, u_2 \dots u_m)$ being a covariant of the system f and ϕ, and $u_1, u_2 \dots u_m$ cogredient with $x^{m-1}, x^{m-2}y \dots y^{m-1}$, it follows from a general principle in the theory of invariants, that on making $u_1, u_2 \dots u_m$ respectively equal to the quantities with which they are cogredient, B will become an ordinary covariant to f and ϕ. By this transformation B becomes a function of x and y of the degree $2(m-1)$ in x and y conjointly, and linear in respect to the coefficients of f, and also in respect to those of ϕ. The only covariant capable of answering this description is what I am in the habit of calling the Jacobian (after the name of the late but ever-illustrious Jacobi), a term capable of application to any number of homogeneous functions of as many variables. In the case before us, where we have two functions of two variables, the Jacobian

$$J(f, \phi) = \begin{vmatrix} \dfrac{df}{dx}, & \dfrac{d\phi}{dx} \\[2ex] \dfrac{df}{dy}, & \dfrac{d\phi}{dy} \end{vmatrix} = \frac{df}{dx}\frac{d\phi}{dy} - \frac{df}{dy}\frac{d\phi}{dx}.$$

We have then the interesting proposition*, that the Bezoutiant to two functions, when the variables in the former are replaced by the combinations of the variables in the latter, with which they are cogredient, becomes the Jacobian†. So in the case of a single function F of the degree m, the Bezoutoid, that is the Bezoutiant to $\dfrac{dF}{dx}, \dfrac{dF}{dy}$, on making the $(m-1)$ variables which it contains identical with $x^{m-2}, x^{m-3}y \dots y^{m-2}$ respectively, becomes identical with the Jacobian to $\dfrac{dF}{dx}, \dfrac{dF}{dy}$, that is the Hessian of F, namely

$$\begin{vmatrix} \dfrac{d^2F}{dx^2}, & \dfrac{d^2F}{dxdy} \\[2ex] \dfrac{d^2F}{dxdy}, & \dfrac{d^2F}{dy^2} \end{vmatrix}.$$

As an example of this property of the Bezoutiant, suppose

$$f = ax^3 + bx^2y + cxy^2 + dy^3,$$
$$\phi = \alpha x^3 + \beta x^2y + \gamma xy^2 + \delta y^3.$$

The Bezoutiant matrix becomes

$$\begin{matrix} a\beta - b\alpha, & a\gamma - c\alpha, & a\delta - d\alpha, \\[1ex] a\gamma - c\alpha, & \begin{pmatrix} a\delta - d\alpha \\ + \\ b\gamma - c\beta \end{pmatrix}, & b\gamma - c\beta, \\[1ex] a\delta - d\alpha, & b\gamma - c\beta, & c\delta - d\gamma. \end{matrix}$$

* I have subsequently found that this proposition is contained under another mode of statement, at the end of Section 2 of the memoir of Jacobi, "De Eliminatione," above referred to.

† For a strict proof of this proposition see Supplement to Third Section of this memoir.

The Bezoutiant accordingly will be the quadratic function

$$(a\beta - b\alpha)\, u_1^2 + \{(a\delta - d\alpha) + (b\gamma - c\beta)\}\, u_2^2 + (c\delta - d\gamma)\, u_3^2$$
$$+ 2\,(a\gamma - c\alpha)\, u_1 u_2 + 2\,(a\delta - d\alpha)\, u_3 u_1 + 2\,(b\gamma - c\beta)\, u_2 u_3,$$

which on making

$$u_1 = x^2, \quad u_2 = xy, \quad u_3 = y^2,$$

becomes

$$Lx^4 + Mx^3y + Nx^2y^2 + Pxy^3 + Qy^4, \qquad (\beta)$$

where L, M, N, P, Q respectively will be the sum of the terms lying in the successive bands drawn parallel to the sinister diagonal of the Bezoutiant matrix, that is

$$L = a\beta - b\alpha,$$
$$M = 2\,(a\gamma - c\alpha),$$
$$N = 3\,(a\delta - d\alpha) + (b\gamma - c\beta),$$
$$P = 2\,(b\gamma - c\beta),$$
$$Q = c\delta - d\gamma.$$

The biquadratic function in x and y, (β), above written, will be found on computation to be identical in point of form with the Jacobian to f, ϕ, namely

$$(3ax^2 + 2bxy + cy^2)\,(\beta x^2 + 2\gamma xy + 3\delta y^2) - (3\alpha x^2 + 2\beta xy + \gamma y^2)\,(bx^2 + 2cxy + dy^2),$$

this latter being in fact

$$3Lx^4 + 3Mx^3y + 3Nx^2y^2 + 3Pxy^3 + 3Qy^4.$$

The remark is not without some interest, that in fact the Bezoutiant, which is capable (as has been shown already) of being mechanically constructed, gives the best and readiest means of calculating the Jacobian; for in summing the sinister bands transverse to the axis of symmetry the only numerical operation to be performed is that of addition of positive integers, whereas the direct method involves the necessity of numerical subtractions as well as additions, inasmuch as the same terms will be repeated with different signs. Thus if

$$f = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + ly^5,$$
$$\phi = \alpha x^5 + \beta x^4y + \gamma x^3y^2 + \delta x^2y^3 + \epsilon xy^4 + \lambda y^5,$$

using (r, s) in the ordinary sense that has been considered throughout, we obtain by taking the sum of the sinister bands in (α)* for the value of B when we write $x^4, x^3y, x^2y^2, xy^3, y^4$ in place of u_1, u_2, u_3, u_4, u_5,

$$(0,\ 1)\, x^8 + 2\,(0,\ 2)\, x^7y + \{3\,(0,\ 3) + (1,\ 2)\}\, x^6y^2 + \{4\,(0,\ 4) + 2\,(1,\ 3)\}\, x^5y^3$$
$$+ \{5\,(0,\ 5) + 3\,(1,\ 4) + (2,\ 3)\}\, x^4y^4 + \{4\,(1,\ 5) + 2\,(2,\ 4)\}\, x^3y^5$$
$$+ \{3\,(2,\ 5) + (3,\ 4)\}\, x^2y^6 + 2\,(3,\ 5)\, xy^7 + (4,\ 5)\, y^8.$$

* *Vide* Art. 62 [p. 552 above].

The direct process requires the calculation of

$$(5ax^4 + 4bx^3y + 3cx^2y^2 + 2dxy^3 + ey^4)(\beta x^4 + 2\gamma x^3y + 3\delta x^2y^2 + 4\epsilon xy^3 + 5\lambda y^4)$$
$$-(5ax^4 + 4\beta x^3y + 3\gamma x^2y^2 + 2\delta xy^3 + \epsilon y^4)(bx^4 + 2cx^3y + 3dx^2y^2 + 4exy^3 + 5ly^4),$$

each coefficient of which will contain the numerical factor 5; so that to reduce the Jacobian to its simplest form each coefficient will necessitate the employment of additions, subtractions, and a division, instead of additions merely, as when the Bezoutic square is employed. For instance, to find the coefficient of x^4y from the above expression (α) we have to calculate

$$\tfrac{1}{5}\{25(0,\ 5) + 16(1,\ 4) + 9(2,\ 3) + 4(3,\ 2) + (4,\ 1)\},$$

that is

$$\tfrac{1}{5}\{25(0,\ 5) + (16-1)(1,\ 4) + (9-4)(2,\ 3)\},$$

which is $5(0, 5) + 3(1, 4) + (2, 3)$, agreeing with what has been found above for the value of such coefficient, by a simple process of counting. The same remark will, of course, also apply to the computation of the Hessian of F by means of its Bezoutoid.

Art. 66. This relation between the Bezoutiant and the Jacobian led me to inquire whether, as would at first sight appear probable, the Bezoutiant were the only lineo-linear quadratic function of m variables covariantive to f and ϕ (the word lineo-linear being used to denote the form of coefficients, such as those in the Bezoutiant, linear in respect of the coefficients in f and the coefficients of ϕ). If so, then there would have existed a method of performing the inverse process of recovering the Bezoutiant from the Jacobian, almost as simple as that of deriving the Jacobian from the Bezoutiant. On investigating the matter, however, I found that such is by no means the case*, but that there exists a whole family of independent

* This might have been concluded immediately from the following observation. Let J, the Jacobian of f and ϕ, be expressed under the form

$$A_0x^{2m-2} + (2m-2)A_1x^{2m-1}y + \tfrac{1}{2}(2m-2)(2m-3)A_2x^{2m-2}y^2 + \ldots + A_{2m-2}y^{2m-2},$$

then we know [p. 282 above] from the Calculus of Forms, that, D being taken to represent the persymmetrical Determinant

$$\begin{vmatrix} A_0, & A_1, & A_2, & \ldots\ldots, & A_{m-1} \\ A_1, & A_2, & A_3, & \ldots\ldots, & A_m \\ A_2, & A_3, & A_4, & \ldots\ldots, & A_{m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{m-1}, & A_m, & A_{m+1}, & \ldots\ldots, & A_{2m-2} \end{vmatrix},$$

$D=0$ is the condition to be satisfied in order that J may be representable under the form of the sum of powers of $(m-1)$ linear functions of x and y, and D itself is an invariant to J, and consequently an invariant and (as is obvious from its form) a combinantive invariant to f and ϕ. Moreover, which is more immediately to the point, we know that the quadratic form Q

$$A_0u_1^2 + 2A_1\{u_1(m-1)u_2\} + A_2\left\{\{(m-1)u_2\}^2 + 2u_1\left(\frac{(m-1)(m-2)}{2}\right)u_3\right\} + \&c. + A_{2m-2}u_m^2,$$

lineo-linear quadratic covariants of m variables to every two homogeneous functions of x and y of the mth degree. I have, moreover, I believe, succeeded in determining the number of such lineo-linear quadratic forms for any value of m, of which all the rest, in whatever manner obtained, may be expressed as linear functions, the coefficients of the linear relations moreover being abstract numbers; in other words, I have succeeded in forming the fundamental or constituent scale of lineo-linear quadratic forms of m variables covariantive to f and ϕ; a result of too great interest, as exhibiting the affinities of the Bezoutiant to its cognate forms, to be altogether passed over in silence. Supposing the number of linearly independent forms of the kind to be ν, then speaking *à priori* any of the forms taken at random might seem to be equally eligible to form one of the ν included in the fundamental scale, combined with any $(\nu - 1)$ others independent *inter se*, and of which the selected one is also independent. In fact, however, this is not so; for it will always be more satisfactory to contemplate the fundamental scale of forms as generated successively or simultaneously by a uniform process; and in the case before us, the process which I have hit upon, and which I believe is the simplest that can be employed for generating the fundamental scale, will be found not to include directly the Bezoutiant among the number. There will thus arise two subjects of inquiry; firstly, the mode of forming the fundamental scale, and proving its fundamental character; secondly, determining the numerical relations

will be an invariant to f, ϕ and Ω (this last quantity Ω being defined as in p. [551]), and a combinantive covariant to f and ϕ in the same sense precisely as the Bezoutiant is a covariant to the same, and like the Bezoutiant is lineo-linear in respect of the coefficients of f and ϕ. If we operate with the symbol E, where E represents

$$v_1^2 \frac{d}{dA_0} + 2v_1v_2 \frac{d}{dA_1} + (v_2^2 + 2v_1v_3) \frac{d}{dA_2} + \&c. + v_m^2 \frac{d}{dA_{2m-2}},$$

upon K any invariant of f and ϕ, we shall obtain EK, a quadratic function of $v_1, v_2 \dots v_m$, which by the rules of the Calculus of Forms we know will be a contravariant to f and ϕ, and the matrix corresponding to which must evidently be persymmetrical. It is an interesting subject of inquiry, which I reserve for some future occasion, to determine the Co-bezoutiant, the Discriminant of which must be employed for K, so that when this discriminant is operated upon by E, the matrix corresponding to EK may become identical (term for term) with the matrix which is the inverse to the Bezoutiant matrix, which inverse, as Jacobi has so simply and beautifully demonstrated, possesses this persymmetrical character. *Vide* the "De Eliminatione," Section 5. The investigation of the arithmetical connexion between the Q of this note and the fundamental Co-bezoutiants must be also similarly reserved. I believe it to be generally true, and have verified the fact for the case of two cubic functions, that EQ gives a quadratic form such that the corresponding matrix is the inverse to the matrix of Q. The calculations necessary for extending the verification of this remarkable proposition for functions of x, y exceeding the third degree (notwithstanding that they are much abbreviated by the application of the rules of the calculus) still remain excessively laborious. The abbreviation alluded to consists in confining the verification in question to the comparison of either one of the two unreiterated terms at opposite corners of the matrix to EQ with the corresponding term in the inverse matrix of Q; if these coincide, it is easy to prove that every other pair of corresponding terms in the two matrices must also coincide respectively with one another.

which connect that very important form, perhaps of all its kind the most important, with the forms comprised in the fundamental or constituent scale. These questions I propose to consider more fully at a future period. For the present I shall content myself with giving a method of forming the constituent scale (without, however, seeking the proof of all the forms *extra* to such assumed scale being linear functions of those comprised within it), and with determining the numerical relations between the forms in this scale and the Bezoutiant for a limited number of values of m. All the forms which we are seeking, besides being lineo-linear quadratics, must also be combinantive invariants to f and ϕ, remaining (as forms) unaltered for any linear substitutions impressed either upon the variables or upon the functions containing the variables.

Art. 67. I must here premise that if there be any two forms of the same degree (and that degree odd) in x and y, a combinant may be formed from them, which will be linear in respect to each set of coefficients*. Thus calling the two functions

$$a_0 x^{2n+1} + (2n+1)\, a_1 x^{2n} y + \tfrac{1}{2}(2n+1)\, 2n a_2 x^{2n-1} y^2 + \ldots + a_{2n+1} y^{2n+1}$$

$$\alpha_0 x^{2n+1} + (2n+1)\, \alpha_1 x^{2n} y + \tfrac{1}{2}(2n+1)\, 2n \alpha_2 x^{2n-1} y^2 + \ldots + \alpha_{2n+1} y^{2n+1},$$

the lineo-linear combinant in question will be

$$T = a_0 \alpha_{2n+1} - (2n+1)\, a_1 \alpha_{2n} + \tfrac{1}{2}(2n+1)\, 2n\, a_2 \alpha_{2n-1} + \frac{(2n+1)(2n)(2n-1)}{1\,.\,2\,.\,3}\, a_3 \alpha_{2n-2}\ \&\text{c.} - \&\text{c.},$$

which, using our customary notation, will be of the form

$$(0,\ 2n+1) - (2n+1)(1,\ 2n) + \frac{(2n+1)\, 2n}{1\,.\,2}(2,\ 2n-1) \pm \&\text{c.} + (-)^n \frac{(2n+1)(2n)(2n-1)\ldots(n+2)}{1\,.\,2\,.\,3\ldots n}(n,\ n+1).$$

As a corollary to this proposition (which, as well as the proposition itself, will be needed for the purposes of the ensuing determination), taking any function of an even degree in x, y, $F(x, y)$, there will exist a combinant to $\frac{dF}{dx}$ and $\frac{dF}{dy}$, by virtue of what has been stated above, which will be

* I may add here incidentally (although not wanted for our present purposes) that as a combinant in which each set of coefficients enters linearly can always be formed to a system of functions two in number of as many variables and of any odd degree, so reciprocally can a combinant in which each set of coefficients enters linearly be always formed to a system of functions each of the degree 2, of which and of the variables contained in them, the number is any odd integer [cf. p. 606 below].

Mr Cayley's well-known quadrinvariant to F; namely, if

$$F = a_0 x^{2n} + 2na_1 x^{2n-1} y + \ldots + a_{2n} y^{2n},$$

this will be

$$a_0 a_{2n} - 2na_1 a_{2n-1} + \frac{2n(2n-1)}{2} a_2 a_{2n-2} + \ldots + \frac{1}{2}(-)^n \frac{2n(2n-1)\ldots(n+1)}{1.2\ldots n} a_n^2.$$

The proposition itself is easily proved; first, the expression T being expressed entirely in terms of quantities of the form (r, s) remains unaltered for linear substitutions impressed upon the forms f and ϕ; it remains then only to show that T satisfies the differential equations to T treated as a mere invariant, namely

$$\left\{\begin{array}{l} a_0 \dfrac{d}{da_1} + 2a_1 \dfrac{d}{da_2} + 3a_2 \dfrac{d}{da_3} + \ldots + (2n+1) a_{2n} \dfrac{d}{da_{2n+1}} \\ + \alpha_0 \dfrac{d}{d\alpha_1} + 2\alpha_1 \dfrac{d}{d\alpha_2} + 3\alpha_2 \dfrac{d}{d\alpha_3} + \ldots + (2n+1) \alpha_{2n} \dfrac{d}{d\alpha_{2n+1}} \end{array}\right\} T = 0,$$

and

$$\left\{\begin{array}{l} a_{2n+1} \dfrac{d}{da_{2n}} + 2a_{2n} \dfrac{d}{da_{2n-1}} + \ldots + (2n+1) a_1 \dfrac{d}{da_0} \\ + \alpha_{2n+1} \dfrac{d}{d\alpha_{2n}} + 2\alpha_{2n} \dfrac{d}{d\alpha_{2n-1}} + \ldots + (2n+1) \alpha_1 \dfrac{d}{d\alpha_0} \end{array}\right\} T = 0.$$

From the hemihedral symmetry of T, which only changes its sign when the order of the coefficients in f and ϕ is simultaneously reversed, it is obvious that one of these equations cannot be satisfied without the other being so too. Looking then exclusively at the first of them, we see that this is satisfied by virtue of the equations

$$\left\{ a_0 \frac{d}{da_1} + (2n+1) \alpha_{2n} \frac{d}{d\alpha_{2n+1}} \right\} T = 0,$$

$$\left\{ 2a_1 \frac{d}{da_2} + 2n \alpha_{2n-1} \frac{d}{d\alpha_{2n}} \right\} T = 0,$$

....

....

$$\left\{ (2n+1) a_{2n} \frac{d}{da_{2n+1}} + \alpha_0 \frac{d}{d\alpha_1} \right\} T = 0.$$

Hence then the differential equations to T being satisfied proves that it is an invariant, and, as above observed, its form shows upon its face that it is a combinant.

Precisely in the same way it may be demonstrated, that to two functions each of the same even degree $2m$ as

$$a_0 x^{2m} + 2ma_1 x^{2m-1} y + \frac{2m(2m-1)}{2} a_2 x^{2m-2} y^2 + \ldots + a_{2m} y^{2m},$$

and $$\alpha_0 x^{2m} + 2m\alpha_1 x^{2m-1}y + \frac{2m(2m-1)}{2}\alpha_2 x^{2m-2}y^2 + \ldots + \alpha_{2m}y^{2m},$$

there will be a quantity

$$G = a_0\alpha_{2m} - 2ma_1\alpha_{2m-1} + \frac{2m(2m-1)}{2}a_2\alpha_{2m-2} \pm \&\text{c.} - 2m\alpha_1 a_{2m-1} + \alpha_0 a_{2m},$$

which, although not a combinant, will satisfy the differential equations necessary to prove it to be an ordinary invariant to the two given functions.

Art. 68. Now let us consider the three forms, f, ϕ and the subsidiary form Ω, where

$$f = a_0x^m + ma_1x^{m-1}y + \ldots + a_my^m,$$

$$\phi = b_0x^m + mb_1x^{m-1}y + \ldots + b_my^m,$$

$$\Omega = u_1y^{m-1} - (m-1)u_2y^{m-2}x \pm \&\text{c.} + (-)^{m-1}u_mx^{m-1},$$

where $u_1, u_2 \ldots u_m$ are to be treated as constants.

Make $$E_{2i+1}f = \frac{1}{m(m-1)\ldots(m-2i)}\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^{2i+1} f,$$

$$E_{2i+1}\phi = \frac{1}{m(m-1)\ldots(m-2i)}\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^{2i+1} \phi,$$

i being any integer such that $2i+1$ does not exceed m, and now consider $E_{2i+1}f$, $E_{2i+1}\phi$ as two functions of the degree $2i+1$ in ξ, η (x and y being regarded as constants); and by virtue of the formula in the last article, form T_i, the lineo-linear combinant of $E_{2i+1}f$ and $E_{2i+1}\phi$; T_i will then be lineo-linear in respect to the coefficients in f and ϕ, and of the degree $2\{m-(2i+1)\}$ in respect to x and y. Again, let

$$E_{2i}\Omega = \frac{1}{m(m-1)\ldots(m-2i+1)}\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^{2i} \Omega.$$

$E_{2i}\Omega$ treated as a function of ξ and η of the degree $2i$ will furnish a quadrinvariant Q_i of the degree $2(m-1-2i)$ in respect of x and y, and quadratic in respect of the system $u_1, u_2 \ldots u_m$. We have thus two forms, T_i and Q_i, each of the same even degree $2\{m-(2i+1)\}$ in respect of x, y. Forming between these the lineo-linear invariant G_i, G_i will be a function lineo-linear in respect of the coefficients of f and ϕ, and quadratic in respect of the system $u_1, u_2 \ldots u_m$. Moreover, G_i will (by the general principle of successive concomitance) be an invariant in respect to the system f, ϕ, Ω, and combinantive in respect to f and ϕ. Thus then G_i for all admissible values of i will belong to the family of forms to which the Bezoutiant is to be referred.

It requires to be noticed, that when i is taken zero, so that T_i and G_i are of the degree $2(m-1)$, E_{2i} for this case must be taken equal to Ω^2, which evidently fulfils the required conditions of being of the degree $2(m-1)$ in (x, y), and quadratic in respect of the coefficients of Ω. If, now, m be even, we may take for $2i+1$ successively all the odd numbers from 1 to $(m-1)$ inclusively, and there will be $\frac{1}{2}m$ forms G_i; when m is odd we may take for $2i+1$ successively all the odd numbers from 1 to m, and the number of forms of G_i will be $\frac{1}{2}(m+1)$. It should be observed, that when m is odd and $2i+1=m$, T_i will become identical with the lineo-linear combinant to f and ϕ, and Q_i with the quadrinvariant to Ω; and no power of x or y will enter into either, so that G_m will become simply $T_m \times Q_m$. I am now able to enunciate the proposition, that G_0, $G_1 \ldots G_{\frac{m}{2}-1}$, when m is even, and G_0, $G_1 \ldots G_{\frac{m-1}{2}}$, when m is odd, form the constituent scale of forms, of which the Bezoutiant and all other lineo-linear quadratic functions of m variables, which are combinants of the system f, ϕ, will be numerically-linear functions. I propose to term the members of this scale Co-bezoutiants.

As regards the present memoir, I shall content myself with exhibiting a partial verification of this law as regards the connection of the Bezoutiant with the G scale of Co-bezoutiants, and a complete determination of the numerical multipliers which express this connection for the cases comprised between $m=2$ and $m=6$ taken inclusively. It is impossible to predict for what ulterior purposes in the development of the Calculus of Invariants these numbers may or may not be required, and it seems to me desirable that a commencement of a table containing them should be made and placed on record. The remaining pages of this memoir will accordingly be devoted to the ascertainment of them.

The theory of the Bezoutoid being included within that of the Bezoutiant, need not hereafter call for any special attention; I may merely notice that the Bezoutoid to a function of the degree m will be a numerico-linear function of $\frac{1}{2}(m-3)$ of the G's if m be odd, and $\frac{1}{2}(m-4)$ of the G's if m be even.

It will be more convenient hereafter to denote the G's as G_1, G_3, G_5 respectively, in lieu of G_0, G_1, G_2, &c., and to continue at the same time to give to the T's and Q's the same subscripts as the corresponding G's.

Art. 69. Firstly. Suppose $m=2$,

$$f = ax^2 + 2bxy + cy^2,$$

$$\phi = \alpha x^2 + 2\beta xy + \gamma y^2,$$

$$\Omega = u_1 y - u_2 x.$$

Then

$$E_1 f = (ax + by)\,\xi + (bx + cy)\,\eta,$$

$$E_1 \phi = (\alpha x + \beta y)\,\xi + (\beta x + \gamma y)\,\eta,$$

$$\begin{aligned} T_1 &= (ax + by)(\beta x + \gamma y) - (bx + cy)(\alpha x + \beta y) \\ &= (a\beta - b\alpha)\,x^2 + (a\gamma - c\alpha)\,xy + (b\gamma - c\beta)\,y^2, \end{aligned}$$

$$Q_1 = \Omega^2 = u_1^2 y^2 - 2u_1 u_2 xy + u_2^2 x^2,$$

and therefore

$$G_1 = (a\beta - b\alpha)\,u_1^2 + (a\gamma - c\alpha)\,u_1 u_2 + (b\gamma - c\beta)\,u_2^2.$$

Let us now form in the usual manner the Bezoutiant to f, ϕ; this is the quadratic function which corresponds to the matrix

$$\left.\begin{matrix} (2a\beta - 2b\alpha), & (a\gamma - c\alpha) \\ (a\gamma - c\alpha), & (2b\gamma - 2c\beta) \end{matrix}\right\},$$

that is

$$\tfrac{1}{2}B = (a\beta - b\alpha)\,u_1^2 + (a\gamma - c\alpha)\,u_1 u_2 + (b\gamma - c\beta)\,u_2^2 = G_1 \text{ or } B = 2G_1.$$

Secondly. Suppose $m = 3$.

$$f = ax^3 + 3bx^2 y + 3cxy^2 + dy^3,$$

$$\phi = \alpha x^3 + 3\beta x^2 y + 3\gamma xy^2 + \delta y^3,$$

$$\Omega = u_1 y^2 - 2u_2 yx + u_3 x^2.$$

We have then

$$E_1 f = (ax^2 + 2bxy + cy^2)\,\xi + (bx^2 + 2cxy + dy^2)\,\eta,$$

$$E_1 \phi = (\alpha x^2 + 2\beta xy + \gamma y^2)\,\xi + (\beta x^2 + 2\gamma xy + \delta y^2)\,\eta,$$

$$\begin{aligned} T_1 &= (ax^2 + 2bxy + cy^2)(\beta x^2 + 2\gamma xy + \delta y^2) - (bx^2 + 2cxy + dy^2)(\alpha x^2 + 2\beta xy + \gamma y^2) \\ &= (a\beta - b\alpha)\,x^4 + 2\,(a\gamma - c\alpha)\,x^3 y + \{3\,(b\gamma - c\beta) + (a\delta - d\alpha)\}\,x^2 y^2 \\ &\qquad + 2\,(b\delta - d\beta)\,xy^3 + (c\delta - d\gamma)\,y^4, \end{aligned}$$

$$Q_1 = \Omega^2 = u_1^2 y^4 - 4u_1 u_2 y^3 x + (4u_2^2 + 2u_1 u_3)\,y^2 x^2 - 4u_2 u_3 y x^3 + u_3^2 x^4.$$

Supplying for facility of computation the reciprocals of the binomial coefficients to the index 4, namely

$$1, \quad -\tfrac{1}{4}, \quad \tfrac{1}{6}, \quad -\tfrac{1}{4}, \quad 1,$$

we obtain

$$\begin{aligned} G_1 &= (a\beta - b\alpha)\,u_1^2 + 2\,(a\gamma - c\alpha)\,u_1 u_2 + \{2\,(b\gamma - c\beta) + \tfrac{2}{3}\,(a\delta - d\alpha)\}\,u_2^2 \\ &\quad + \{(b\gamma - c\beta) + \tfrac{1}{3}\,(a\delta - d\alpha)\}\,u_1 u_3 + 2\,(b\delta - d\beta)\,u_2 u_3 + (c\delta - d\gamma)\,u_3^2. \end{aligned}$$

It will here and henceforth be more useful to employ $[r, s]$ to denote, not the difference of the cross products of the $(r+1)$th and $(s+1)$th *entire* coefficients in f and ϕ, but the difference of the cross products of these coefficients divided each by its appropriate binomial coefficient. We may then write

$$G_1 = [0, 1]\, u_1^2 + 2\,[0, 2]\, u_1 u_2 + ([1, 2] + \tfrac{1}{3}\,[0, 3])\, u_1 u_3 + (2\,[1, 2] + \tfrac{2}{3}\,[0, 3])\, u_2^2$$
$$+ 2\,[1, 3]\, u_2 u_3 + [2, 3]\, u_3^2.$$

Again,

$$G_3 = \{(a\delta - d\alpha) - 3\,(b\gamma - c\beta)\}\,(u_1 u_3 - u_2^2) = ([0, 3] - 3\,[1, 2])\, u_1 u_3$$
$$- ([0, 3] - 3\,[1, 2])\, u_2^2.$$

Hence

$$G_1 - \tfrac{1}{3}\, G_3 = [0, 1]\, u_1^2 + 2\,[0, 2]\, u_1 u_2 + 2\,[1, 2]\, u_1 u_3 + ([0, 3] + [1, 2])\, u_2^2$$
$$+ 2\,[1, 3]\, u_2 u_3 + [2, 3]\, u_2^2.$$

But, again, the Bezoutiant of f, ϕ corresponds to the matrix

$$\begin{array}{ccc} 3\,[0, 1], & 3\,[0, 2], & [0, 3], \\ 3\,[0, 2], & [0, 3] + 9\,[1, 2], & 3\,[1, 3], \\ [0, 3], & 3\,[1, 3], & [3, 4]. \end{array}$$

Hence summing the sinister bands to form the coefficients, we have

$$B = 3\,[0, 1]\, u_1^2 + 6\,[0, 2]\, u_1^2 u_2 + (3\,[0, 3] + 9\,[1, 2])\, u_2^2 + 6\,[1, 3]\, u_2 u_3$$
$$+ [2, 3]\, u_3^2 = 3G_1 - G_3.$$

Thirdly. Suppose $m = 4$,

$$f = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$
$$\phi = \alpha x^4 + 4\beta x^3 y + 6\gamma x^2 y^2 + 4\delta x y^3 + \epsilon y^4,$$
$$\Omega = u_1 y^3 - 3u_2 y^2 x + 3u_3 y x^2 - u_4 x^3.$$

Then

$$E_3 f = (ax + by)\,\xi^3 + 3\,(bx + cy)\,\xi^2\eta + 3\,(cx + dy)\,\xi\eta^2 + (dx + ey)\,\eta^3,$$

therefore

$$T_3 = \begin{Bmatrix} (ax + by)\,(\delta x + \epsilon y) \\ -\,(\alpha x + \beta y)\,(dx + ey) \end{Bmatrix} - 3 \begin{Bmatrix} (bx + cy)\,(\gamma x + \delta y) \\ -\,(\beta x + \gamma y)\,(cx + dy) \end{Bmatrix}$$
$$= ([0, 3] - 3\,[1, 2])\, x^2 + ([0, 4] - 2\,[1, 3])\, xy + ([1, 4] - 3\,[2, 3])\, y^2$$

and

$$Q_3 = (u_1 y - u_2 x)\,(u_3 y - u_4 x) - (u_2 y - u_3 x)^2$$
$$= (u_1 u_3 - u_2^2)\, y^2 - (u_1 u_4 - u_2 u_3)\, xy + (u_2 u_4 - u_3^2)\, x^2.$$

Hence supplying the binomial reciprocals

$$1, \quad -\tfrac{1}{2}, \quad 1,$$

we have

$$G_3 = ([0, 3] - 3\,[1, 2])\,(u_1 u_3 - u_2^2) + \tfrac{1}{2}\,([0, 4] - 2\,[1, 3])\,(u_1 u_4 - u_2 u_3)$$
$$+ ([1, 4] - 3\,[2, 3])\,(u_2 u_4 - u_3^2).$$

Again,

$$T_1 = (ax^3 + 3bx^2y + 3cxy^2 + dy^3)(\beta x^3 + 3\gamma x^2y + 3\delta xy^2 + \epsilon y^3)$$
$$- (\alpha x^3 + 3\beta x^2y + 3\gamma xy^2 + \delta y^3)(bx^3 + 3cx^2y + 3dxy^2 + ey^3)$$
$$= [0, 1]\, x^6 + 3\,[0, 2]\, x^5y + (3\,[0, 3] + 6\,[1, 2])\, x^4y^2 + ([0, 4] + 8\,[1, 3])\, x^3y^3$$
$$+ (3\,[1, 4] + 6\,[2, 3])\, x^2y^4 + 3\,[2, 4]\, xy^5 + [3, 4]\, y^6,$$

and

$$Q_1 = \Omega^2$$
$$= u_1^2y^6 - 6u_1u_2y^5x + (9u_2^2 + 6u_1u_3)\, y^4x^2 - (2u_1u_4 + 18u_2u_3)\, x^3y^3$$
$$+ (9u_3^2 + 6u_2u_4)\, y^2x^4 - 6u_3u_4yx^5 + u_4^2x^6.$$

Hence, supplying the reciprocal binomial coefficients,

$$1,\quad -\tfrac{1}{6},\quad +\tfrac{1}{15},\quad -\tfrac{1}{20},\quad \tfrac{1}{15},\quad -\tfrac{1}{6},\quad 1,$$

we find

$$G_1 = [0, 1]\, u_1^2 + 3\,[0, 2]\, u_1u_2 + (\tfrac{1}{5}\,[0, 3] + \tfrac{2}{5}\,[1, 2])(9u_2^2 + 6u_1u_3)$$
$$+ (\tfrac{1}{10}\,[0, 4] + \tfrac{8}{10}\,[1, 3])(u_1u_4 + 9u_2u_3)$$
$$+ (\tfrac{1}{5}\,[1, 4] + \tfrac{2}{5}\,[2, 3])(9u_3^2 + 6u_2u_4) + 3\,[2, 4]\, u_3u_4 + [3, 4]\, u_4^2.$$

Now the Bezoutic square, taking account of the binomial factors in f and ϕ, may be written under the form

$$\begin{array}{cccc}
4\,[0, 1], & 6\,[0, 2], & 4\,[0, 3], & [0, 4], \\
6\,[0, 2], & \begin{bmatrix} 4\,[0, 3] \\ +\,24\,[1, 2] \end{bmatrix}, & \begin{bmatrix} [0, 4] \\ +\,16\,[1, 3] \end{bmatrix}, & 4\,[1, 4], \\
4\,[0, 3], & \begin{bmatrix} [0, 4] \\ +\,16\,[1, 3] \end{bmatrix}, & \begin{bmatrix} [1, 4] \\ +\,24\,[2, 3] \end{bmatrix}, & 6\,[2, 4], \\
[0, 4], & 4\,[1, 4], & 6\,[2, 4], & [3, 4].
\end{array}$$

Hence the Bezoutiant B becomes

$$4\,[0, 1]\, u_1^2 + 12\,[0, 2]\, u_1u_2 + (4\,[0, 3] + 24\,[1, 2])\, u_2^2 + 2\,[0, 4]\, u_1u_4$$
$$+ (2\,[0, 4] + 32\,[1, 3])\, u_2u_3 + 8\,[1, 4]\, u_2u_4 + ([1, 4] + 24\,[2, 3])\, u_3^2$$
$$+ 12\,[2, 4]\, u_3u_4 + [3, 4]\, u_4^2.$$

And we ought to have $B = cG_1 + eG_3$, to satisfy which equation we must manifestly have $c = 4$; to find e, compare the coefficients of u_2^2, this gives

$$4\,[0, 3] + 24\,[1, 2] = \tfrac{36}{5}\,[0, 3] + \tfrac{72}{5}\,[1, 2] + e\,(3\,[1, 2] - [0, 3]);$$

accordingly we ought to be able to satisfy the two equations

$$\tfrac{36}{5} - e = 4, \quad \tfrac{72}{5} + 3e = 24,$$

each of which accordingly we find is satisfied by the equality $e = \tfrac{16}{5}$.

Substituting in the equation for B above written, we thus obtain

$$B = 4G_1 + \tfrac{16}{5}\, G_3,$$

which will be found to be identically true.

Art. 70. We may now see our way to a more concise mode of obtaining the numerical coefficients, by which they may in fact be computed and verified with comparatively little labour, connecting the Bezoutiant with the Co-bezoutiant forms of the constituent scale. It will not fail to have been remarked, that throughout the preceding determinations I have presumed the truth of the formula, which admits of an immediate verification, that for all values of m and ω we have the identical equation

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{\omega} \left\{c_0 x^m + mc_1 x^{m-1}y + \tfrac{1}{2}m(m-1)c_2 x^{m-2}y^2 + \ldots + mc_{m-1}xy^{m-1} + c_m y^m\right\}$$

$$= m(m-1)\ldots(m-\omega+1)\left\{L_0\xi^{\omega} + \omega L_1 \xi^{\omega-1}\eta + \tfrac{1}{2}\omega(\omega-1)L_2\xi^{\omega-2}\eta^2 + \ldots + L_{\omega}\eta^{\omega}\right\},$$

where

$$\left.\begin{array}{l} L_0 = c_0 x^{m-\omega} + (m-\omega)c_1 x^{m-\omega-1}y + (m-\omega)\dfrac{m-\omega-1}{2}c_2 x^{m-\omega-2}y^2 + \ldots + c_{m-\omega}y^m \\ L_1 = c_1 x^{m-\omega} + (m-\omega)c_2 x^{m-\omega-1}y + (m-\omega)\dfrac{m-\omega-1}{2}c_2 x^{m-\omega-2}y^2 + \ldots + c_{m-\omega+1}y^m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_{\omega} = c_{\omega}x^{m-\omega} + (m-\omega)c_{\omega+1}x^{m-\omega-1}y + (m-\omega)\dfrac{m-\omega-1}{2}c_2 x^{m-\omega-2}y^2 + \ldots + c_m y^m. \end{array}\right\}$$

Let us now proceed to determine by an abridged method the linear relations corresponding to the cases of $m=5$, $m=6$, and first for $m=5$.

Let

$$f = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + hy^5,$$

$$\phi = \alpha x^5 + 5\beta x^4 y + 10\gamma x^3y^2 + 10\delta x^2y^3 + 5\epsilon xy^4 + \eta y^5,$$

$$\Omega = u_1y^4 - 4u_2y^3x + 6u_3y^2x^2 - 4u_4yx^3 + u_5x^4.$$

In forming G_5, G_3, G_1, let us confine our attention to the terms u_1^2, u_1u_3, u_1u_4.

A comparison of the coefficients of these with those in the Bezoutiant (B) will be sufficient for assigning the three numerical quantities which connect B with G_1, G_3, G_5. I omit u_1u_2, because G_1 is the only one of the G's for any value of m which contains u_1^2 or u_1u_2, and in G_1 the terms containing u_1^2 and u_1u_2 are

$$[0, 1]\,u_1^2 + (m-1)\,[0, 2]\,u_1u_2,$$

and the corresponding part of the Bezoutiant is

$$m\,[0, 1]\,u_1^2 + m(m-1)\,[0, 2]\,u_1u_2;$$

so that if we write

$$B = c_1G_1 + c_3G_3 + c_5G_5 + \&c.,$$

the two terms u_1^2 and u_1u_2 will only enable us to form one equation with the c's, namely, $c_1 = m$. Again, instead of considering the entire coefficients of u_1u_3 and u_1u_4, it will be sufficient to take a single argument of either of these coefficients (in the forms to be compared), as for instance [0, 3] and [1, 3]. Then c_1 being known, c_3, c_5 will be determined; but for the purposes of verification I shall furthermore compute the whole of the coefficient of u_1u_5.

Accordingly, calculating the G system in reverse order, we have

$$G_5 = \{[0, 5] - 5\,[1, 4] + 10\,[2, 3]\}\,\{u_1u_5 - 4u_2u_4 + 3u_3^2\}$$
$$= \{[0, 5] - 5\,[1, 4] + 10\,[2, 3]\}\,u_1u_5 + \ldots,$$

$$E_3 f = (ax^2 + 2bxy + cy^2)\,\xi^3 + 3\,(bx^2 + 2cxy + dy^2)\,\xi^2\eta + 3\,(cx^2 + 2dxy + ey^2)\,\xi\eta^2 + (dx^2 + 2exy + fy^2)\,\eta^3$$

$E_3\phi =$ &c. &c.;

therefore

$$T_3 = \{(ax^2 + 2bxy + cy^2)(\delta x^2 + 2\epsilon xy + \eta y^2) - (\alpha x^2 + 2\beta xy + \gamma y^2)(dx^2 + 2exy + hy^2)\}$$
$$- 3\,\{(bx^2 + 2cxy + dy^2)(\gamma x^2 + 2\delta xy + \epsilon y^2) - (\beta x^2 + 2\gamma xy + \delta y^2)(cx^2 + 2dxy + ey^2)\}$$
$$= ([0, 3] - 3\,[1, 2])\,x^4 + (2\,[0, 4] + \ldots)\,x^3y + \{[0, 5] + [1, 4] - 8\,[2, 3]\}\,x^2y^2 + \&c.$$

The number -8 results from the calculation $1 - 3\,(4 - 1) = -8$.

Again,

$$E_2\Omega = (u_1y^2 - 2u_2yx + u_3x^2)\,\xi^2 - 2\,(u_2y^2 - 2u_3yx + u_4x^2)\,\xi\eta + (u_3y^2 - 2u_4yx + u_5x^2)\,\eta^2,$$

therefore

$$Q_3 = (u_1y^2 - 2u_2yx + u_3x^2)(u_3y^2 - 2u_4yx + u_5x^2) - (u_2y^2 - 2u_3yx + u_4x^2)^2$$
$$= u_1u_3y^4 - 2u_1u_4y^3x + u_1u_5y^2x^2 + \&c.,$$

all the terms and parts of terms unexpressed being free of u_1, and therefore not necessary for our purpose. Hence supplying the reciprocal factors

$$1,\ -\tfrac{1}{4},\ \tfrac{1}{6},\ \ldots,$$

we have

$$G_3 = [0, 3]\,u_1u_3 + ([0, 4] +)\,u_1u_4 + \tfrac{1}{6}\,\{[0, 5] + [1, 4] + [2, 3]\}\,u_1u_5 + \&c.$$

Again, expressing $E_1 f$ and $E_1\phi$ in the usual way, we obtain

$$T_1 = (ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4)(\beta x^4 + 4\gamma x^3y + 6\delta x^2y^2 + 4\epsilon xy^3 + \eta y^4)$$
$$- (\alpha x^4 + 4\beta x^3y + 6\gamma x^2y^2 + 4\delta xy^3 + \epsilon y^4)(bx^4 + 4cx^3y + 6dx^2y^2 + 4exy^3 + hy^4)$$
$$= [0, 1]\,x^8 + 4\,[0, 2]\,x^7y + (6\,[0, 3] +)\,x^6y^2 + (4\,[0, 4] +)\,x^5y^3$$
$$+ ([0, 5] + 15\,[1, 4] + 20\,[2, 3])\,x^4y^4 + \&c.$$

(where it may be observed that the numbers 15 and 20 in the coefficient of x^4y^4 arise from the quantities $4^2 - 1$, $6^2 - 4^2$).

Again,

$$Q_1 = \Omega^2 = u_1^2 x^8 + 8u_1u_2x^7y + 12u_1u_3x^6y^2 - 8u_1u_4x^5y^3 + 2u_1u_5x^4y^4 + \&c.$$

Hence supplying the multipliers

$$1,\ \frac{-1}{8},\ \frac{1}{28},\ \frac{-1}{56},\ +\frac{1}{70},\ \&c.$$

we have

$$G_1 = [0,\ 1]\, u_1^2 + 4\,[0,\ 2]\, u_1u_2 + \tfrac{18}{7}\,[0,\ 3]\, u_1u_3 + \tfrac{4}{7}\,[0,\ 4]\, u_1u_4$$
$$+ \tfrac{1}{35}\,([0,\ 5] + 15\,[1,\ 4] + 20\,[2,\ 3])\, u_1u_5.$$

Again, the Bezoutiant

$$B = 5\,[0,\ 1]\, u_1^2 + 2\,.\,10\,[0,\ 2]\, u_1u_2 + 2\,.\,10\,[0,\ 3]\, u_1u_3$$
$$+ 2\,.\,5\,[0,\ 4]\, u_1u_4 + 2\,[0,\ 5]\, u_1u_5 + \&c.$$

Accordingly, if we write $B = c_1G_1 + c_3G_3 + c_5G_5$, we have, as above remarked, $c_1 = 5$; and to determine c_3, c_5, we have, by comparing the coefficients of u_1u_3, u_1u_4 in B, G_1, G_3, G_5,

$$20 = \tfrac{90}{7} + c_3$$
$$10 = \tfrac{20}{7} + c_3.$$

These two equations, then, as it turns out, are not independent, but are satisfied simultaneously by

$$c_3 = \tfrac{50}{7}.$$

Finally, equating the coefficients of the several arguments in u_1u_5, we have

$$0 = 5 \times \tfrac{1}{35} + \tfrac{50}{7} \times \tfrac{1}{6} + c_5 \quad \text{from the argument } [0,\ 5],$$
$$0 = 5 \times \tfrac{15}{35} + \tfrac{50}{7} \times \tfrac{1}{6} + 5c_5 \quad \text{from the argument } [1,\ 4],$$
$$0 = 5 \times \tfrac{20}{35} + \tfrac{50}{7} \times \tfrac{8}{6} + 10c_5 \quad \text{from the argument } [2,\ 3].$$

The first of which equations gives

$$c_5 = 2 - \tfrac{1}{7} - \tfrac{25}{21} = \tfrac{14}{21} = \tfrac{2}{3};$$

the second gives

$$c_5 = \tfrac{3}{7} + \tfrac{5}{21} = \tfrac{2}{3},$$

and the third gives

$$c_5 = \tfrac{20}{21} + \tfrac{2}{7} = \tfrac{2}{3}.$$

We have thus abundantly verified the accuracy of the calculation, and there results the relation

$$B = 5G_1 + \tfrac{50}{7}G_3 + \tfrac{2}{3}G_5.$$

Lastly, let $m = 6$,

$$f = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6hxy^5 + ly^6,$$
$$\phi = \alpha x^6 + 6\beta x^5y + 15\gamma x^4y^2 + 20\delta x^3y^3 + 15\epsilon x^2y^4 + 6\eta xy^5 + \lambda y^6,$$
$$\Omega = u_1y^5 - 5u_2y^4x + 10u_3y^3x^2 - 10u_4y^2x^3 + 5u_5yx^4 - u_6x^5.$$

I shall here confine myself to the determination of a single argument in each of the terms u_1^2, u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_1u_6; this will be ample for the purpose of verification, as the equation to be assigned is of the form

$$B = c_1G_1 + c_3G_3 + c_5G_5.$$

The arguments which I select as the most simple, will be those expressed by the symbols (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6) respectively; then we have

$$T_5 = (ax + by)(\eta x + \lambda y) \mp \text{\&c.} - (hx + ly)(\alpha x + \beta y)$$

$$= ([0, 5] + \ldots) x^2 + ([0, 6] + \ldots) xy + (\ldots) y^2,$$

$$Q_5 = (u_1y - u_2x)(u_5y - u_6x) \mp \text{\&c.}$$

$$= (u_1u_5 + \ldots) y^2 - (u_1u_6 + \ldots) yx + (\ldots) x^2.$$

Hence supplying the binomial reciprocals

$$1, \ -\tfrac{1}{2}, \ 1,$$

$$G_5 = ([0, 5] + \ldots) u_1u_5 + \tfrac{1}{2}([0, 6] + \ldots) u_1u_6 + \text{\&c.}$$

Again,

$$T_3 = (ax^3 + \ldots)(\delta x^3 + 3\epsilon x^2y + 3\eta xy^2 + \lambda y^3) \mp \text{\&c.}$$
$$- (dx^3 + 3ex^2y + 3hxy^2 + ly^3)(\alpha x^3 + \ldots)$$

$$= ([0, 3] + \ldots) x^6 + (3[0, 4] + \ldots) x^5y + (3[0, 5] + \ldots) x^4y^2$$
$$+ ([0, 6] + \ldots) x^3y^3 + \text{\&c.}$$

$$Q_3 = (u_1y^3 \mp \text{\&c.})(u_3y^3 \mp 3u_4y^2 + 3u_57x^2 - u_6x^3) - \text{\&c.}$$

$$= (u_1u_3 + \ldots) y^6 - (3u_1u_4 + \ldots) y^5x + (3u_1u_5 + \ldots) y^4x^2 - (u_1u_6 + \ldots) y^3x^3 + \text{\&c.},$$

and the reciprocal binomial multipliers will be

$$1, \ \frac{-1}{6}, \ \frac{+1}{15}, \ \frac{-1}{20}, \ \text{\&c.}$$

Hence

$$G_3 = [0, 3] u_1u_3 + \tfrac{3}{2}[0, 4] u_1u_4 + \tfrac{3}{5}[0, 5] u_1u_5 + \tfrac{1}{20}[0, 6] u_1u_6 \ \text{\&c. \&c.}$$

Finally,

$$T_1 = (ax^5 + \text{\&c.})(\beta x^5 + 5\gamma x^4y + 10\delta x^3y^2 + 10\epsilon x^2y^3 + 5\eta xy^4 + \lambda y^5) - \text{\&c.}$$

$$= ([0, 1] + \ldots) x^{10} + 5([0, 2] + \ldots) x^9y + (10[0, 3] + \ldots) x^8y^2$$
$$+ (10[0, 4] + \ldots) x^7y^3 + (5[0, 5] + \ldots) x^6y^4 + ([0, 6] + \ldots) x^5y^5 + \text{\&c.}$$

$$Q_1 = \Omega^2 = u_1^2y^{10} + (10u_1u_2 + \ldots) y^9x + (20u_1u_3 + \ldots) y^8x^2 + (20u_1u_4 + \ldots) y^7x^3$$
$$+ (10u_1u_5 + \ldots) y^6x^4 + (2u_1u_6 + \ldots) y^5x^5 + \text{\&c.};$$

and supplying the numerical series

$$1, \quad -\frac{1}{10}, \quad \frac{1}{45}, \quad \frac{-1}{120}, \quad \frac{1}{210}, \quad \frac{-1}{252}, \text{ \&c.,}$$

we have

$$G_1 = [0,\ 1]\,u_1^2 + 5\,[0,\ 2]\,u_1u_2 + \tfrac{40}{9}\,[0,\ 3]\,u_1u_3 + \tfrac{5}{3}\,[0,\ 4]\,u_1u_4$$
$$+ \tfrac{5}{21}\,[0,\ 5]\,u_1u_5 + \tfrac{1}{126}\,[0,\ 6]\,u_1u_6 + \text{\&c.}$$

Again, the Bezoutiant

$$= 6\,[0,\ 1]\,u_1^2 + 30\,[0,\ 2]\,u_1u_2 + 40\,[0,\ 3]\,u_1u_3 + 30\,[0,\ 4]\,u_1u_4$$
$$+ 12\,[0,\ 5]\,u_1u_5 + 2\,[0,\ 6]\,u_1u_6 + \text{\&c. \&c.} = B.$$

Hence making

$$B = c_1\,G_1 + c_3\,G_3 + c_5\,G_5,$$

from u_1^2 and u_1u_2 we obtain respectively

$$c_1 = 6,$$

$$5c_1 = 30;$$

hence from u_1u_3 and u_1u_4 we obtain respectively

$$\left.\begin{aligned} \tfrac{240}{9} + \ \ c_3 &= 40 \\ \tfrac{30}{3} + \tfrac{3}{2}c_3 &= 30 \end{aligned}\right\} \text{ or } c_3 = \tfrac{40}{3};$$

hence from u_1u_5 and u_1u_6 we obtain respectively

$$6 \times \tfrac{5}{21} + \tfrac{40}{3}\tfrac{3}{5} + c_5 = 12, \text{ that is } c_5 = 12 - 8 - \tfrac{10}{7} = \tfrac{18}{7},$$

$$6 \times \tfrac{1}{126} + \tfrac{40}{3}\tfrac{1}{20} + \tfrac{1}{2}\,c_5 = 2, \text{ that is } \tfrac{1}{2}c_5 = 2 - \tfrac{2}{3} - \tfrac{1}{21} = \tfrac{9}{7};$$

hence

$$c_5 = \tfrac{18}{7},$$

and the equation sought for is

$$B = 6G_1 + \tfrac{40}{3}G_3 + \tfrac{18}{7}G_5.$$

Art. 71. The following table exhibits the relations between the Bezoutiant and the correspondent system of Co-bezoutiants for all values of m between 1 and 6 under a synoptical form.

$$m = 1, \quad B = G_1,$$

$$m = 2, \quad B = 2G_1,$$

$$m = 3, \quad B = 3G_1 - G_3,$$

$$m = 4, \quad B = 4G_1 + \tfrac{16}{5}G_3,$$

$$m = 5, \quad B = 5G_1 + \tfrac{50}{7}G_3 + \tfrac{2}{3}G_5,$$

$$m = 6, \quad B = 6G_1 + \tfrac{40}{3}G_2 + \tfrac{18}{7}G_5.$$

These series could if wanted be easily extended, and the calculation of the coefficients reduced to a mere mechanical procedure.

If we suppose m to be $2i$ or $2i-1$, we have the equation

$$B = c_1 G_1 + c_3 G_3 + \dots + c_{2i-1} G_{2i-1},$$

and it appears from the foregoing instances that the comparison of the coefficients, either of u_1^2, or of $u_1 u_2$ on the two sides of the equation, will serve to give c_1 (m being known), c_3 may be found by a comparison of the coefficients either of $u_1 u_3$, or of $u_1 u_4$, and so on for $c_5 \dots c_{2i-1}$; all the coefficients in the equation for B above given, thus admitting of being found separately and successively and in two modes, so that there is a check at each step upon the correctness of the computations: the only exception to this last remark is (when m is odd) for the last coefficient of which the above condensed method affords only a single determination. I need hardly add the remark, that in substituting $x^{m-1}, x^{m-2}y, \dots xy^{m-2}, y^{m-1}$ in place of $u_1, u_2 \dots u_{m-1}, u_m$ respectively, all the G's become (to a numerical factor *près*) identical with one another and with the Jacobian to the system (f, ϕ).

Art. 72. The foregoing theory took its origin (as will have been readily imagined) in meditations growing out of the celebrated theorem of M. Sturm. There appear to be several directions in which a development or extension of the subject matter of that theorem may be sought for. Thus a theory may be constructed relative to a single function of one or more variables, viewed in all cases as representing a geometrical locus. In the limiting case, when this locus becomes a system of points in a right line, we have the theorem of Sturm; generally the theory will be that of contours. Or, again, a theory may be formed in which the number of functions is always kept equal to that of the variables. We have then a theory of discrete points corresponding to roots, the number of real ones of which comprised within given limits it is the object of such theory to determine. M. Hermite, in a memoir recently presented to the French Institute, appears to have made a valuable addition to the Sturmian theory extended in this direction, to which the beautiful researches of M. Cauchy and the joint labours of MM. Liouville and Sturm, with reference to the disposition of the imaginary roots of equations appear to have led the way. Finally, the number of variables may be supposed to be arbitrarily increased, but made always inferior by a unit to the number of the functions in which they are contained, or which comes to the same thing, we may construct the theory of a system of homogeneous functions equal in number to the variables in them, which in its simplest case becomes the theory of Intercalations which has been here partially considered, and which (as has been shown) embraces (not as a particular case, but as an implied consequence and easily extricated result) the theorem of M. Sturm.

General and Concluding Supplement.

Art. (א). The expressions given in Art. (n) [p. 507 above] for the partial quotients of the continued fraction represented by $\frac{\phi x}{fx}$, are restricted to the supposition of all these partial quotients (except the first) being linear in x; when the first partial quotient is linear the formula (B) of that article continues applicable on replacing $(D_i h_\theta)$ by 1. I was forcibly struck by the peculiarity of these formulæ not ceasing to be true in consequence of the first partial quotient being supposed non-linear; and reflecting upon this, I was soon led to perceive that all the partial quotients might be supposed to be arbitrary integral functions of x, and the formulæ would still continue to apply to any such of them as might happen to be linear, although, as it were, imbedded among a group of other non-linear partial quotients. From this it was but an easy step to perceive that the formulæ (A) and (B) must admit of extension to the representation of partial quotients of any form, and that the dimorphism of the representation of the linear partial quotients could only be a consequence of the equation in integers $u + \nu = 1$ having two solutions $u = 0$, $\nu = 1$ and $u = 1$, $\nu = 0$. I now proceed to enunciate the very remarkable general theorem (or as it may perhaps not inappropriately be termed Algebraical Porism), by virtue of which any partial quotient of a given degree in x belonging to an infinite continued fraction, all of whose partial quotients are algebraical functions of x, may be expressed to a constant factor *près*, by means of the numerator and denominator (or if we please either one of these) of the convergent *immediately* antecedent to and of the numerator and denominator of *any* convergent *not* antecedent to the partial quotient which is to be determined.

Art. (ב). *Theorem.* Let $Q_1, Q_2 \ldots Q_i, Q_{i+1} \ldots Q_n$, &c. each of an arbitrary degree in x, be the n first partial quotients of an algebraical continued fraction; let Q_{i+1} be the partial quotient to be determined and of the given degree ω_{i+1}; let

$$\frac{1}{Q_1 -}\,\frac{1}{Q_2 -}\,\frac{1}{Q_3 -}\cdots\frac{1}{Q_i} = \frac{\phi_i(x)}{f_i(x)},$$

and

$$\frac{1}{Q_1 -}\,\frac{1}{Q_2 -}\,\frac{1}{Q_3 -}\,\frac{1}{Q_i -}\,\frac{1}{Q_{i+1}}\cdots\frac{1}{Q_n} = \frac{\Phi(x)}{F(x)};$$

let u and ν be any couple of integers of the $\omega_{i+1} + 1$ couples which satisfy the equation $\nu + u = \omega_{i+1}$; then, as usual, denoting the product of the differences of each of one set of terms from each of another set, by writing the former under the latter, and calling $\eta_1, \eta_2 \ldots \eta_\mu$ the μ roots of $\Phi(x)$, and $h_1, h_2 \ldots h_m$

the m roots of $F(x)$, (Φ and F being supposed respectively of μ and m dimensions in x), and forming the disjunctive equations

$$\theta_1,\ \theta_2,\ \theta_3 \ldots \theta_\mu = 1,\ 2,\ 3 \ldots \mu,$$

$$t_1,\ t_2,\ t_3 \ldots t_m = 1,\ 2,\ 3 \ldots m,$$

we have the following equation, wherein ϕ and f are written for ϕ_i and f_i,

$$Q_{i+1} = K_{u,\nu} \times \Sigma \left\{ (\phi\eta_{\theta_1}\phi\eta_{\theta_2} \ldots \phi\eta_{\theta_\nu})^2 \times (fh_{t_1}fh_{t_2} \ldots fh_{t_u})^2 \right.$$

$$\times \frac{\begin{bmatrix} \eta_{\theta_1}, & \eta_{\theta_2} & \ldots \eta_{\theta_\nu} \\ h_{t_{u+1}}, & h_{t_{u+2}} & \ldots h_{t_m} \end{bmatrix} \times \begin{bmatrix} h_{t_1}, & h_{t_2} & \ldots h_{t_u} \\ \eta_{\theta_{\nu+1}}, & \eta_{\theta_{\nu+2}} & \ldots \eta_{\theta_\mu} \end{bmatrix}}{\begin{bmatrix} \eta_{\theta_1}, & \eta_{\theta_2} & \ldots \eta_{\theta_\nu} \\ \eta_{\theta_{\nu+1}}, & \eta_{\theta_{\nu+2}} & \ldots \eta_{\theta_\mu} \end{bmatrix} \times \begin{bmatrix} h_{t_1}, & h_{t_2} & \ldots h_{t_u} \\ h_{t_{u+1}}, & h_{t_{u+2}} & \ldots h_{t_m} \end{bmatrix}}$$

$$\left. \times \{(x - \eta_{\theta_1})(x - \eta_{\theta_2}) \ldots (x - \eta_{\theta_\nu})\} \{(x - h_{t_1})(x - h_{t_2}) \ldots (x - h_{t_u})\} \right\},$$

and moreover the different values of $K_{u,\nu}$ depending upon the different modes of breaking up ω_{i+1} into two parts u and ν are all (to a numerical factor *près*) equal to one another. Thus then the theorem pointed at in Art. (p) is discovered, and the way laid open (by an unexpected channel) for a complete discussion of the theory of the singular cases which may occur in the expansion of any rational algebraical fraction under the form of a continued fraction.

Art. (ג). In the above expression, if we suppose $\omega_{i+1} = 1$, we have $u = 1$ and $\nu = 0$, or $u = 0$ and $\nu = 1$, and remembering that

$$\begin{bmatrix} h & \\ \eta_1, & \eta_2 \ldots \eta_\mu \end{bmatrix} = \Phi h \text{ and } \begin{bmatrix} \eta & \\ h_1, & h_2 \ldots h_m \end{bmatrix} = F\eta,$$

$$\begin{bmatrix} h_{t_1} & \\ h_{t_2}, & h_{t_3} \ldots h_{t_m} \end{bmatrix} = F'h_{t_1} \text{ and } \begin{bmatrix} \eta_{\theta_1} & \\ \eta_{\theta_2}, & \eta_{\theta_3} \ldots \eta_{\theta_\mu} \end{bmatrix} = \Phi'\eta_{\theta_1},$$

Q_{i+1} becomes by virtue of the general formula representable under either of the equivalent forms

$$K_{0,1}\Sigma^\theta \left\{ (\phi\eta_\theta)^2 \frac{F\eta_\theta}{\Phi'\eta_\theta}(x - \eta_\theta) \right\} \text{ and } K_{1,0}\Sigma^t \left\{ (fh_t)^2 \frac{\Phi h_t}{F'h_t}(x - h_t) \right\},$$

$K_{0,1}$ and $K_{1,0}$ being either equal, or differing only in the sign, agreeably to the formulæ (A) and (B) [p. 508 above].

Art. (ד). It may be worth while to notice, that, although (of course) these formulæ and the general formulæ of Art. (ב), when supposed converted into functions of x and of the coefficients of F and of Φ by the reduction, integration and summation of the symmetrical functions of the roots which enter into them remain universally valid, and subject to no cases of exception, yet antecedently to these processes being performed the formulæ as they stand may become illusory when any relations of equality exist between the roots of Φ *inter se*, or between the roots of F *inter se.* Thus in the case before us, if Φ have equal roots the formula commencing with $K_{0,1}$ is illusory, and if F have equal roots the other of the two formulæ becomes illusory.

Let us take the second of these and suppose that $F(x)$ has

$$k_1 \text{ roots } c_1,\ k_2 \text{ roots } c_2 \ldots k_p \text{ roots } c_p,$$

we may pass to the actual case from any case where the roots are infinitesimally near to the actual roots of $F(x)$, and all infinitesimally different from one another. Moreover the choice of the infinitesimal variations being arbitrary, let the k_1 roots c_1 be replaced by a group of roots

$$c_1+\delta,\quad c_1+\delta\rho_1,\quad c_1+\delta\rho_1^2 \ldots c_1+\delta\rho_1^{k_1-1},$$

where ρ_1 is a prime root of the equation $\rho_1^{k_1}=0$, and δ is an infinitesimal quantity, and suppose each of the other groups to be varied in an analogous manner. Then it may easily be shown from this that the second of the formulæ in question will become

$$K_{10} \sum_{t=1}^{p} k_t \frac{\left(\dfrac{d}{dc}\right)^{k-1} \{(fc_t)^2 (\Phi c_t)(x-c_t)\}}{\left(\dfrac{d}{dc_t}\right)^{k} Fc_t},$$

and similarly, the twin formula becomes

$$K_{01} \sum_{\theta=1}^{\pi} \kappa_\theta \frac{\left(\dfrac{d}{d\gamma}\right)^{\kappa-1} \{(\phi\gamma_\theta)^2 (F\gamma_\theta)(x-\gamma_\theta)\}}{\left(\dfrac{d}{d\gamma_\theta}\right)^{\kappa} \Phi\gamma_\theta} *.$$

Corresponding modifications will admit of being made by aid of a like method in the general formulæ of Art. (ב) upon a similar supposition as to equalities springing up between the roots of fx *per se* and of $\phi(x)$ *per se*, or between the roots of fx and ϕx *inter se.*

* For in general if ρ is a prime root of the equation $\rho^\omega=1$, and if fx have ω roots all equal to c and ψx is any other function of x and if δ is an infinitesimal quantity, then rejecting all powers of δ higher than the $(\omega-1)$th degree,

$$\frac{\psi(c+\delta)}{f'(c+\delta)}+\frac{\psi(c+\rho\delta)}{f'(c+\rho\delta)}+\frac{\psi(c+\rho^2\delta)}{f'(c+\rho^2\delta)}+\ldots+\frac{\psi(c+\rho^{\omega-1}\delta)}{f'(c+\rho^{\omega-1}\delta)}$$

$$=\frac{1}{\left(\dfrac{d}{dc}\right)^{\omega} fc\delta^{\omega-1}}\{\psi(c+\delta)+\rho\psi(c+\rho\delta)+\rho^2\psi(c+\rho^2\delta)+\ldots+\rho^{\omega-1}\psi(c+\rho^{\omega-1}\delta)\}$$

$$=\frac{\left(\dfrac{d}{dc}\right)^{\omega-1}\psi c\omega\delta^{\omega-1}}{\left(\dfrac{d}{dc}\right)^{\omega} fc\delta^{\omega-1}}=\omega\,\frac{\left(\dfrac{d}{dc}\right)^{\omega-1}\psi c}{\left(\dfrac{d}{dc}\right)^{\omega} fc}.$$

Art. (ה). If in Art. (ב) we take $i=0$, the formula for Q_{i+1} will become

$$Q_1 = K_{u,\nu} \frac{\begin{bmatrix} \eta_{\theta_1}, & \eta_{\theta_2} & \dots \eta_{\theta_\nu} \\ h_{t_{u+1}}, & h_{t_{u+2}} & \dots h_{t_m} \end{bmatrix} \times \begin{bmatrix} h_{t_1}, & h_{t_2} & \dots h_{t_u} \\ \eta_{\theta_{\nu+1}}, & \eta_{\theta_{\nu+2}} & \dots \eta_{\theta_\mu} \end{bmatrix}}{\begin{bmatrix} \eta_{\theta_1}, & \eta_{\theta_2} & \dots \eta_{\theta_\nu} \\ \eta_{\theta_{\nu+1}}, & \eta_{\theta_{\nu+2}} & \dots \eta_{\theta_\mu} \end{bmatrix} \times \begin{bmatrix} h_{t_1}, & h_{t_2} & \dots h_{t_u} \\ h_{t_{u+1}}, & h_{t_{u+2}} & \dots h_{t_m} \end{bmatrix}}$$
$$\times \{(x-\eta_{\theta_1}) \dots (x-\eta_{\theta_\nu})\} \{(x-h_{t_1}) \dots (x-h_{t_u})\},$$

u and ν being any two integers whose sum is ω_1, which is identical (as it ought to be) with the expression virtually contained in the formulæ of Section II. for the syzygetic multiplier of $\Phi(x)$ in the syzygetic equation connecting Fx and Φx with their first residue when Φx is supposed to be ω_1 dimensions in x lower than Fx identical, *videlicet,* in other words, with the integer part of the algebraical fraction $\frac{F(x)}{\Phi(x)}$.

Art. (ו). When $\Phi(x) = F'(x)$,

$$\frac{\Phi(h_1),\ \Phi(h_2) \dots \Phi(h_{\omega_{i+1}})}{\begin{bmatrix} h_1, & h_2 & \dots h_{\omega_{i+1}} \\ h_{1+\omega_{i+1}}, & h_{2+\omega_{i+1}} & \dots h_m \end{bmatrix}} \text{ becomes identical with } (-)^{\frac{1}{2}(\omega_{i+1}-1)\omega_{i+1}} \zeta(h_1, h_2 \dots h_{\omega_{i+1}}),$$

and we may consequently (using an extreme term in the forms in the polymorphic scale of forms representing Q_{i+1}), write

$$Q_{i+1} = (-)^{\frac{1}{2}(\omega_{i+1}-1)\omega_{i+1}} K_{0,\omega_{i+1}} \Sigma\zeta(h_1, h_2 \dots h_{\omega_{i+1}}) (f_i h_1)^2 (f_i h_2)^2 \dots$$
$$(f_i h_{\omega_{i+1}})^2 (x-h_1)(x-h_2) \dots (x-h_{\omega_{i+1}}).$$

Art. (ז). The following observations will serve to complete the theory of the singular cases in the expansion of an algebraical continued fraction.

Preserving the notation of Art. (ב), let

$$\sigma_i = m - (\omega_1 + \omega_2 + \dots + \omega_{i-1} + 1).$$

Then (calling the roots of Fx, $h_1, h_2 \dots h_m$) the (i)th simplified residue to $\frac{\Phi x}{Fx}$, in accordance with the general formulæ for the residues in the second section (for greater simplicity selecting an extreme term of the polymorphic scale), will be represented by

$$\Sigma \frac{\Phi h_1,\ \Phi h_2,\ \Phi h_3 \dots \Phi h_{\sigma_i}}{\begin{bmatrix} h_1, & h_2, & h_3 & \dots h_{\sigma_i} \\ h_{1+\sigma_i}, & h_{2+\sigma_i}, & h_{3+\sigma_i} & \dots h_m \end{bmatrix}} (x-h_1)(x-h_2)(x-h_3) \dots (x-h_{\sigma_i}),$$

which will be of the form $L_i x^{\sigma_i - \omega_i + 1} + \&c.$, all the terms containing higher powers of x vanishing by the coefficients becoming zero. If in the above expression we should use σ_i' in lieu of σ_i, where σ_i' is σ_i diminished by any integer inferior to ω_i, we should get other forms of the same residue, but these will all be of higher dimensions in the roots or coefficients than the one just given, and in fact the forms thus obtained corresponding to the values σ_i, $\sigma_i - 1$, $\sigma_i - 2 \ldots \sigma_i - \omega_i + 1$ substituted for σ_i in succession, would, by aid of the relations of condition between the coefficients of Φx and Fx implied in the value of ω_i, admit of being exhibited as a scale in which each form would be an exact algebraical product of the form which precedes it, multiplied by a function of the coefficients, and did space permit thereof it would be perfectly easy to give the forms of these multiplicators. But I pass on to the representation of what is more material, namely, the form of the complete residue in the case supposed, merely observing (as an *obiter dictum*) that the existence of each singular partial quotient (meaning thereby a quotient non-linear in x) only affects the form of the single simplified residue in immediate connexion with itself, and not at all the form of the other residues antecedent or subsequent to that one.

Art. (ח). Let the ith simplified residue be called R_i and the corresponding complete residue $[R_i]$, then applying a method similar to the method given in Section I., we shall find that

$$(-)^{?}[R_i] = \frac{L_{i-2}^{\omega_{i-1}+1}\, L_{i-4}^{\omega_{i-4}+1} \text{ \&c.}}{L_{i-1}^{\omega_{i-1}+1}\, L_{i-3}^{\omega_{i-3}+1} \text{ \&c.}} R_i,$$

L_i representing the leading coefficient in the ith simplified residue, and the sign of interrogation (?) denoting some function of $\omega_1, \omega_2 \ldots \omega_i$ (possibly a constant) remaining to be determined. And reverting to Art. (ב), the quantity that would be called $K_{0,\,\omega_i}$ according to the notation employed in the formulæ expressing Q_{i+1} in that article, will (abstraction being made of the algebraical sign and using for greater brevity (ι), $(\iota-1)$, &c. to express $1+\omega_i$, $1+\omega_{i-1}$, &c.) come to be represented by

$$\frac{L_{i-1}^{(\iota-1)}}{L_i^{(\iota)}}\;\frac{L_{i-3}^{4(\iota-3)}\, L_{i-5}^{4(\iota-5)} \text{ \&c.}}{L_{i-2}^{4(\iota-2)}\, L_{i-4}^{4(\iota-4)} \text{ \&c.}},$$

a similar convention being supposed to be made respecting the numerator and denominator of each convergent as was made respecting them in the particular case treated of in Art. (f), page [502].

Art. (ט). I will merely add a very few words in generalization of the method of limiting the roots of fx given in the Supplement to the fourth Section [p. 528 above]. As an inferior limit to fx is identical with a superior limit to $f(-x)$, we may confine our attention to superior limits alone. Suppose then that

$$\frac{\phi x}{fx} = \frac{1}{Q_1 -}\frac{1}{Q_2 -}\cdots\frac{1}{Q_i -}\frac{1}{Q_1' -}\frac{1}{Q_2' -}\cdots\frac{1}{Q'_{i'}}\cdots\frac{1}{(Q)_1 -}\frac{1}{(Q)_2 -}\cdots\frac{1}{(Q)_{(i)}},$$

where the partial quotients Q are each of any arbitrary degree in x, and have all one algebraical sign in the coefficients of the highest powers of x from Q_1 to Q_i, and all the same sign (contrary to the former), in the coefficients of the highest powers of x from Q_i' to $Q'_{i'}$, and so on alternately, then firstly a superior limit to the superior limits of the cumulants $[Q_1, Q_2 \dots Q_i]$, $[Q_1', Q_2' \dots Q'_{i'}]$, ... $[(Q)_1, (Q)_2 \dots (Q)_{(i)}]$ will be a superior limit to fx, so that it remains only to give a rule for finding a superior limit to a cumulant $[Q_1, Q_2, Q_3 \dots Q_i]$, which, secondly, is to be found by making

$$Q_1 - M_1 = 0,\ Q_2 - M_2 = 0,\ Q_3 - M_3 = 0 \dots Q_i - M_i = 0,$$

where $$M_1 = \mu_1,\ M_2 = \mu_2 + \frac{1}{\mu_1},\ M_3 = \mu_3 + \frac{1}{\mu_2} \dots M_i = \frac{1}{\mu_{i-1}},$$

$\mu_1, \mu_2 \dots \mu_{i-1}$ being any quantities entirely independent and arbitrary except in regard to their being all of the same sign as the leading coefficients in the elements $Q_1, Q_2 \dots Q_i$.

We may then find $L_1, L_2 \dots L_i$ any superior limits to the roots of x in these i equations respectively; L, the greatest of these, will be a superior limit to the proposed cumulant $[Q_1, Q_2 \dots Q_i]$; and it may be observed that $M_1, M_2 \dots M_i$ are the general values which satisfy the equation

$$M_1 - \frac{1}{M_2 -}\,\frac{1}{M_3 -} \dots \frac{1}{M_i} = 0,$$

subject to the condition that for all values of e

$$\frac{1}{M_e -}\,\frac{1}{M_{e-1} -}\,\frac{1}{M_{e-2} -} \dots \frac{1}{M_1}$$

shall have a given invariable sign. The first part of the process, as just shown, consists in separating the type of the total cumulant which represents fx into partial types, the point for each fracture of the total type being marked by a change of sign in the elements of the type for the value $x = +\infty$; it is easily seen therefore from this, that if $\frac{\Phi x}{Fx}$ is the generatrix of the cumulant in question, the number of such fractures (that is, the number one less than the number of partial cumulants) will be the number of changes of algebraical sign in the signaletic series, consisting of the leading coefficients in Fx and in each of the odd-placed complete residues respectively, together with the number of changes of sign in the signaletic series, consisting of the leading coefficients in Φx and in each of the even-placed complete residues respectively.

The syzygetic theory of two algebraical functions, and the allied theory of algebraical continued fractions with their principal applications, may, I think, now be said to be completely made out, as well for the singular cases as for the general hypothesis.

Art. (ʼ). I will conclude with observing that the theory within developed gives the means of transforming (explicitly and without the aid of symmetrical functions) into an algebraical continued fraction, any given sum of algebraical fractions of the form

$$\frac{c_1}{x-h_1}+\frac{c_2}{x-h_2}+\frac{c_3}{x-h_3}+\ldots+\frac{c_n}{x-h_n},$$

where each c and h is supposed known. For let the above sum be called $\frac{\Phi x}{Fx}$, then if h_θ, c_θ be used to denote any pair of corresponding terms of the h series and the c series, we have $\frac{\Phi h_\theta}{F'h_\theta}=c_\theta$, as is well known and easily proved. Again, if $D_i x$ represent the simplified denominator of the ith convergent to the continued fraction equal to $\frac{\Phi x}{Fx}$ which is to be found, say

$$\frac{1}{(A_1x+B_1)-}\,\frac{1}{(A_2x+B_2)-}\cdots\frac{1}{(A_nx+B_n)},$$

we have [p. 476 above]

$$D_ix=\Sigma\frac{\Phi h_1,\ \Phi h_2\ldots\Phi h_i}{\begin{vmatrix} h_1, & h_2 & \ldots h_i \\ h_{i+1}, & h_{i+2} & \ldots\ h_n\end{vmatrix}}(x-h_1)(x-h_2)\ldots(x-h_i)$$

$$=\Sigma(-)^{i\frac{(i-1)}{2}}\frac{\zeta(h_1,h_2\ldots h_i)\,\Phi h_1,\ \Phi h_2\ldots\Phi h_i}{F'h_1F'h_2\ldots F'h_i}(x-h_1)(x-h_2)\ldots(x-h_i)$$

$$=(-)^{i\frac{i-1}{2}}\Sigma\{c_1c_2\ldots c_i\,\zeta(h_1,h_2\ldots h_i)(x-h_1)(x-h_2)\ldots(x-h_i)\}.$$

Therefore

$$(D_ih_1)^2=\{\Sigma(c_2c_3\ldots c_{i+1})\,\zeta(h_2,h_3\ldots h_{i+1})(h_1-h_2)(h_1-h_3)\ldots(h_1-h_{i+1})\}^2$$
$$=\{\Sigma(c_2c_3\ldots c_{i+1})\,\zeta^{\frac{1}{2}}(h_1,h_2\ldots h_{i+1})\,\zeta^{\frac{1}{2}}(h_2,h_3\ldots h_{i+1})\}^2;$$

and the simplified $(i+1)$th quotient, that is, the value of $A_{i+1}x+B_{i+1}$, when divested of the allotrious factor, has been proved [cf. p. 508 above] to be equal to

$$\Sigma(D_ih_1)^2\frac{\Phi h_1}{F'h_1}(x-h_1);$$

it is therefore now known as a rational and *integral* function of x; $h_1, h_2\ldots h_n$; $c_1, c_2\ldots c_n$. The allotrious factor itself is made up of the product of squares of quantities all of the same form as the leading coefficient in D_ix, which, from what has been shown above, is seen to be equal to

$$(-)^{\frac{i-1}{2}}\Sigma\{(c_1c_2\ldots c_i)\,\zeta(h_1,h_2\ldots h_i)\}.$$

Hence each term in the continued fraction

$$\frac{1}{(A_1x+B_1)-}\,\frac{1}{(A_2x+B_2)-}\cdots\frac{1}{(A_nx+B_n)},$$

which is to be made equal to

$$\frac{c_1}{x-h_1}+\frac{c_2}{x-h_2}+\ldots+\frac{c_n}{x-h_n},$$

is completely assigned in terms of x and the given quantities c and h.

Art. (כ). The number of effective intercalations between the roots of Φx, Fx is easily seen to be equal to the excess of the number of positive real numerators over the number of negative real numerators in the partial fractions of which $\frac{\Phi x}{Fx}$ is the sum, and hence we see *à priori*, as an obvious consequence of a simple extension of the reasoning in Art. 47 [p. 515 above], that the inertia of the quadratic function

$$\Sigma\left\{c_\theta\,(u_1+h_\theta u_2+h_\theta^2u_3+\ldots+h_\theta^{n-1}u_n)^2\,\frac{1}{x-h_\theta}\right\},$$

where $c_\theta=\frac{\Phi h_\theta}{F'h_\theta}$, will represent the value of the index in question. So too we may see that the formulæ given for the residues to fx, $f'x$ in Art. 46 continue to apply to the residues Fx, Φx. That is to say, these residues when divided out by Fx will be respectively represented by the successive principal coaxal determinants to the matrix

$$\begin{array}{llll} S_0, & S_1, & S_2 & \ldots S_{m-1}, \\ S_1, & S_2, & S_3 & \ldots S_m, \\ S_2, & S_3, & S_4 & \ldots S_{m+1}, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ S_{m-1}, & S_m, & S_{m+1} & \ldots S_{2m-2}, \end{array}$$

where in general

$$S_r=\frac{c_1}{x-h_1}h_1^r+\frac{c_2}{x-h_2}h_2^r+\ldots+\frac{c_n}{x-h_n}h_n^r,$$

and using the same matrix as above written with S' substituted for S, where in general

$$S_r'=c_1(x-h_1)\,h_1^r+c_2(x-h_2)\,h_2^r+\ldots+c_n(x-h_n)\,h_n^r,$$

the successive principal coaxal determinants of the new matrix represent the successive denominators to the convergents of the continued fraction which expresses $\frac{\Phi x}{Fx}$.

The expression for the numerators to the convergents may also, there is no doubt, be obtained by some simple modification (dependent on introducing the quantities $c_1, c_2 \ldots c_n$) of the formula in Art. 41, p. [492].

I annex, more with the hope of suggesting than (in all instances) of conveying a full conception of the force of the definitions, a Glossary, or rather a Repertory of the principal terms of art employed in the preceding pages, which might otherwise be apt to occasion some difficulty to persons unfamiliar with the subject.

Glossary of new or unusual Terms, or of Terms used in a new or unusual sense, in the preceding Memoir.

Allotrious.—The allotrious factor to a residue or quotient in the process of common measure applied to two algebraical functions is the constant factor of which such residue or quotient must be divested in order to become an integral and irreducible function.

Apocopated.—Applied to a type in the Theory of Cumulants, denotes a type the final or initial element of which has been taken away. If both are taken away, the type is said to be doubly apocopated.

Bezoutic.—For definition of Primary and Secondary Bezoutics see first Section. *Bezoutiant* to two functions, each of degree n, is a homogeneous quadratic invariantive function of n variables, the form of which serves to assign the index of the scale of the effective intercalations of the real roots of the two given functions.

Bezoutoid.—The Bezoutiant to two homogeneous functions obtained by differentiation from one homogeneous function of two variables. The Bezoutoid to a given function of m dimensions in the variables is accordingly a quadratic function of $(m-1)$ variables, the form of which is sufficient for determining the number of real roots in the given function.

Characteristic.—The employment of this word has been avoided in the preceding memoir; but as it contains an idea of capital importance in analysis, and especially in all inquiries of the kind here treated of, I subjoin the definition of its meaning. The characteristic of a simple condition of any kind is the rational integral function (in its lowest terms) whose evanescence necessarily and universally implies and is implied by the satisfaction of such condition. A simple condition has always a single characteristic, abstraction being made of the algebraical sign, which remains indeterminate. In like manner, a multiple condition, or a system of conditions, will have for its characteristic a plexus of rational integral functions, whose evanescence necessarily and universally implies and is implied by the satisfaction of such multiple condition or system of conditions. The number of functions in the characteristic plexus will however in general greatly exceed the index of the multiplicity of the conditions, and need not always be a unique system. There are however exceptions to this: thus the duplex condition, that a biquadratic function of x shall contain a cubic factor, or that a curve of the third degree shall have a cusp, will each be definitely characterized by a plexus of two functions, and no more.

The spirit of the higher analysis resides, and is to be sought for, in the *logic* of characteristics.

Co-bezoutiant.—Any homogeneous quadratic function similar in form and in its property of invariance to the Bezoutiant.

Cogredient and Contragredient.—A system of variables is cogredient to another system when it is subject to undergo simultaneously therewith linear substitutions of a like kind, and contragredient when it is subject to undergo linear substitutions simultaneously therewith but of a contrary kind.

Combinant.—A function of the quantities appearing in a given set of functions which remains unaltered as well for linear substitutions impressed upon the variables as for linear combinations of the functions themselves.

Concomitant.—*Nomen generalissimum* for a form invariantively connected with a given form or system of forms.

Conjunctive.—A syzygetic function of a given set of functions. Any function which universally, and *subject to no cases of exception*, vanishes when a certain number of other functions all vanish together must be a conjunctive (that is a syzygetic function), or a root of a conjunctive of such functions. But if its vanishing is subject to cases of exception, then all that can be predicated of it is that it is *syzygetically related* to such functions, but it may, and usually does happen, that it will be syzygetically related to them in more than one way.

Contravariant.—A function which stands in the same relation to the primitive function from which it is derived as any of its linear transforms to an inversely derived transform of its primitive.

Covariant.—A function which stands in the same relation to the primitive function from which it is derived as any of its linear transforms to a similarly derived transform of its primitive.

Cumulant.—The denominator of the simple algebraical fraction which expresses the value of an improper continued fraction. See *Type*, infra.

Determinant.—This word is used throughout in the single sense, after which it denotes the alternate or hemihedral function the vanishing of which is the condition of the possibility of the coexistence of a system of a certain number of homogeneous linear equations of as many variables.

Dialytic.—If there be a system of functions containing in each term different combinations of the powers of the variables in number equal to the number of the functions, a resultant may be formed from these functions by, as it were, *dissolving* the relations which connect together the different combinations of the powers of the variables, and treating them as simple independent quantities linearly involved in the functions. The resultant so formed is called the Dialytic Resultant of the functions supposed; and any method by which the elimination between two or more equations can be made to depend on the formation of such a resultant is called a dialytic method of elimination. In such method accordingly the process of elimination between equations of a higher degree than the first is always reduced to a question of elimination between equations which are of the first degree only.

Discriminant.—The resultant of the n differential coefficients of a homogeneous function of n variables. See *Resultant*, infra.

Disjunctive.—A disjunctive equation is a relation between two sets of quantities such that each one of either set is equal according to some unspecified order of connexion with some one of the other set.

Effective scale of intercalations is the series of the real roots of two functions of x written in order of magnitude after repeated processes of removing pairs of roots belonging to either the same function (when not separated by roots of the other function): the roots of the two functions follow each other alternately.

Effluent.—From every homogeneous function of any number i of variables of the degree mm', where m, m' are any two integers, may be formed (as shown in the Calculus of Forms, Section II.) a covariantive function of the degree m and of μ variables, where μ is the number of permutations that can be obtained by dividing m' into i parts (zeros admissible), in which all the coefficients are numerical multiples of the given coefficients; covariants so formed may be termed effluents of their primitive. An example of this occurs in the footnote to Section V., [p. 557], where the quantity there called Q is a quadratic effluent of the Jacobian.

Element.—A simple component of the type to a cumulant. See *Cumulant*, supra.

Emanant.—The result of operating any number of times (suppose i times) upon a given homogeneous function of any number of variables x, y, z ... t with the operative symbol

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}+\ldots+t'\frac{d}{dt}\right),$$

is called the ith emanant of the function operated upon. Every emanant is a covariant to its primitive, the new variables x', y', z' ... t' being cogredient with the variables x, y, z ... t with which they are respectively associated. $E_{2i+i}f$, $E_{2i+i}\phi$, page [561], are emanants of f and ϕ. The process of emanation is one of incessant occurrence in the theory of invariants. When the order of the emanant is the same as the degree of the function (supposed to be rational and integral) from which the emanant proceeds, the form of the original function is reproduced in the final emanant, the names only of the variables being changed.

Endoscopic, *Exoscopic.*—When the coefficients of the functions concerned in any investigation are regarded as integral indecomposable monads, the method is called exoscopic, and endoscopic when the coefficients are treated with reference to their internal constitution as composed of roots or other elements.

In addition to the examples in the footnote to Section I.*, these words have a marked and most important application in the theory of Invariants, especially of two variables.

Form.—Any function may be regarded as an *opus operatum*; the matter operated upon being the variables, and the substance of the operations being the form, which resides in the function as the soul in the body. A form is always common to an infinity of functions, but for greater brevity may be and frequently is called by the name of some specified function in which it is contained.

[* p. 431 above.]

Fundamental.—The fundamental scale of a system of Invariants or Concomitants is a set of the same, whereof every other is a Rational Integral Function.

Hessian or *Hessean*, named after Dr Otto Hesse, of Königsberg (the worthy pupil of his illustrious master, Jacobi, but who, to the scandal of the mathematical world, remains still without a Chair in the University which he adorns with his presence and his name), is the Jacobian to the differential coefficients of a homogeneous function of any number of variables. It is to a Jacobian what a Bezoutoid is to a Bezoutiant, or a Discriminant to a Resultant.

Hyperdeterminants.—See Memoir of Mr Cayley, *Cambridge and Dublin Mathematical Journal*, May 1845, and *Crelle's Journal* of about the same date.

Improper continued fraction is a continued fraction differing only from an ordinary one in the circumstance of negative signs being substituted for positive signs to connect the terms.

Inertia.—The unchangeable number of integers in the excess of positive over negative signs which adheres to a quadratic form expressed as the sum of positive and negative squares, notwithstanding any real linear transformations impressed upon such form.

Intercalations.—The theory of intercalations is the theory of the relative distribution of the real roots, or point-roots, of two or more equations, but in this theory the number of roots mutually interposed is to be taken only with reference to the number 2 as a modulus.

Invariance.—The property (under prescribed or implied conditions) of remaining invariable.

Invariant.—A function of the coefficients of one or more forms which remains unaltered when these undergo suitable linear transformations.

Inverse.—The inverse to a given square matrix is formed by selecting in its turn each component of the given matrix, substituting unity in its place, making all the other components in the same line and column therewith zero, and finally writing the value of the determinant corresponding to the matrix thus modified in lieu of the selected component. If the determinant to the matrix be equal to unity, its second inverse, that is the inverse to its inverse, will be identical, term for term, with the original matrix.

Jacobian.—The Jacobian to n homogeneous functions of n variables is the determinant represented by the symmetrical collocation in a square of the n differential coefficients of each of the n functions.

Kenotheme.—A finite system of discrete points defined by one or more homogeneous equations in number one less than the number of variables contained therein.

Limiting Series.—One set of quantities whose extreme values are exterior to the extreme values of a second set is set to limit the latter.

Matrix.—A square or rectangular arrangement of terms in lines and columns.

Minor Determinant.—Any determinant retained represented by a square group of terms arbitrarily chosen out of a matrix is a minor determinant thereto. The simple terms of the matrix are the last minors, and of course if the matrix is a square, it will itself in its totality represent a single complete determinant.

Monotheme.—A line, or finite system of lines, defined by one or more homogeneous equations two less in number than the number of the variables contained therein.

Order.—The orders of a homogeneous function are the linear functions of the variables the least in number by aid of which the function admits of being expressed.

Persymmetrical.—A symmetrical matrix, in which all the terms in the diagonal bands transverse to the axis of symmetry are identical, is said to be persymmetrical. *Example.* An addition table.

Quadrinvariant.—An invariant of which the terms are quadratic functions of the coefficients of the primitive.

Relation (simple and compound). Vide *Substitution*, infra.

Resultant.—The resultant of n homogeneous general functions of n variables is that function of their coefficients which, equated to zero, expresses in the simplest terms the condition of the possibility of their coexistence.

Rhizoristic.—A rhizoristic series is a series of disconnected functions which serve to fix the number of real roots of a given function lying between any assigned limits.

Signaletic.—A signaletic or *Semaphoretic* series is a sequence of disjunctive terms, considered solely with reference to the algebraical signs of *plus* and *minus* which they respectively carry.

Singular.—A proper algebraical function of a given degree, n, in one variable in its most general form, will, in respect to that variable, be of the nth degree in the denominator and the $(n-1)$th degree in the numerator, and will admit of being represented by a continued algebraical fraction of n terms, all of them linear.

But for particular values of, or relations among, the coefficients entering into the given fraction this mode of representation fails, and the continued fraction, instead of consisting of linear terms n in number, will consist of terms, some of them at least, non-linear, and fewer than n in number. These then are the *singular* cases (or cases of singularity) in the theory of the development of an algebraical fraction under the continued fraction form; and it will be seen that according to this definition the case of the development of any proper algebraical fraction in which the degree of the numerator is more than one unit below that of the denominator, belongs (strictly speaking) to the class of singular cases; and this view of the case supposed is perfectly correct and conformable to the analogies of the subject.

Substitution (linear, *similar* or *contrary*).—A linear substitution is said to be impressed upon a system of variables when each variable is replaced by a linear conjunctive of all the variables. The matrix formed by the coefficients of substitution arranged in regular order is called the Matrix of Substitution, and is of course a square. When two substitutions (impressed on two systems of variables) have the same matrix, they are said to be *similar*, and *contrary* when their matrices are contrary, that is mutually inverse to each other. When two systems of variables are supposed to be subject to the condition that their substitutions are always similar or always contrary, they are said to be related or in simple relation, the relation being of cogredience in the one case and of contragredience in the other.

When a linear substitution is impressed upon a system of independent variables, a corresponding linear substitution is necessarily impressed at the same time upon every complete system of homogeneous combinations (that is, products and powers and products of powers) of these variables, the matrix to which latter substitution will consist of terms which will be functions (depending upon the degree of the homogeneous combinations) of the terms of the matrix to the primitive substitution. This matrix may be termed a compound matrix, having the primitive matrix for its base.

If, now, two systems of independent variables are subject to be synchronously impressed with substitutions, the matrices to which (not being both of them simple matrices) have for their bases matrices which are either similar or contrary, these two systems will be said to be in *compound relation* of cogredience in the one case, and of contragredience in the other.

Syrrhizoristic.—A syrrhizoristic series is a series of disconnected functions which serve to determine the effective intercalations of the real roots of two functions lying between any assigned limits.

Syzygetic.—A syzygetic function or conjunctive of a number of given rational integral functions is the sum of these affected respectively with arbitrary functional multipliers, which are termed the syzygetic multipliers. When a syzygetic function of a given set of functions can be made to vanish, they are said to be syzygetically related.

Transform.—Equivalent to the French noun substantive "*transformée.*"

Type.—The type of a cumulant is the series of the simple elements (or quotients), arranged in a fixed order, of which the cumulant is composed.

Umbral.—The umbral notation is a notation according to which simple quantities are denoted by syllables, instead of by single letters (the composition of these syllables being governed by the mode in which the quantities which they express are obtained); and the single letters of such syllables are termed umbral quantities or *umbræ.*

Weight.—In this memoir (throughout the earlier sections) the weight of any quantity composed of the product of the coefficients of any given function or functions of x is used to denote the number of roots of x appertaining to the given function or functions which must be employed to express such quantity. More generally, when dealing with a system of homogeneous functions, the *weight* of a quantity may be defined with respect to *any selected variable* therein as the sum of the weights in respect to such variable of the several coefficients of which the quantity is composed (the weight of each several coefficient meaning the index of the power of the selected variable in that term of the given function or functions which is affected with such coefficient). These two definitions of *weight* may be perfectly well reconciled with each other by understanding the weight of a quantity formed from the coefficients of a function or system of functions of x to mean the weight, in respect *to unity*, of such quantity when the given functions are treated as homogeneous functions of x and 1.

Zeta.—The symbol ζ (preceding a row of bracketed terms) is used to denote the product of the squared differences of the terms which it affects.

[ ]. A bracket of this form, when enclosing a superior and an inferior row of terms m and n in number respectively, indicates the mn products of the differences obtained by subtracting each term in the second row from each term in the first row; when enclosing an arrangement of terms in a single line, it is used to denote the cumulant of which such an arrangement is the type.

58.

ON THE CONDITIONS NECESSARY AND SUFFICIENT TO BE SATISFIED IN ORDER THAT A FUNCTION OF ANY NUMBER OF VARIABLES MAY BE LINEARLY EQUIVALENT TO A FUNCTION OF ANY LESS NUMBER OF VARIABLES.

[*Philosophical Magazine*, v. (1853), pp. 119—126.]

In the *Cambridge and Dublin Mathematical Journal* for November 1850*, I defined an order as signifying any linear function of a given set of variables, and spoke of a general function of n variables as losing r orders when the relation between its coefficients is such that it is capable of being expressed as a function of $(n-r)$ orders only. It will be highly convenient to preserve the same nomenclature for the purposes of the present investigation.

Dr Otto Hesse, in a long memoir in *Crelle's Journal*, the contents of which have been described to me†, but which I have not yet been able to procure, has given a rule for determining the analytical conditions for the loss of one order. I propose to give a more simple and comprehensive scheme of conditions than Professor Hesse appears to have discovered, applicable not to this case only, but to that of the loss of any number whatever of orders, and shall moreover show in what relation the substituted orders stand to the given variables.

Dr Hesse's rule had been previously stated by me in the 4th section of my Calculus of Forms (*Cambridge and Dublin Mathematical Journal*, May 1852‡) as applicable to the case of a general function of the 3rd degree

[* p. 171 above.]

† A distinguished mathematical friend in Paris communicated to me with great admiration Professor Hesse's result overnight. I ventured to affirm that, to one conversant with the calculus of forms, the problem could offer no manner of difficulty. An hour's quiet reflection in bed the following morning, or morning after, sufficed to disclose to me the true principle of the solution. [Cf. Noether, *Math. Annal.* L. (1898) p. 138. Ed.]

‡ *Vide* Vol. VII. p. 187 [p. 335 above]. "When U represents a pencil of three rays meeting in a point, $\frac{dS}{da}=0$, $\frac{dS}{db}=0$, &c., and also therefore $T=0$" (S and T being the two Aronholdian invariants of U, and a, b, c, &c. the coefficients of U); "also in place of this system may be substituted the system obtained *by taking all the coefficients of the Hessian zero.*"

of three variables becoming the representative of three right lines diverging from the same point, which is the case of a cubic function of three variables becoming a function of two linear functions of these variables, that is to say, losing one order: this, perhaps, might have been noticed in the Professor's memoir. I gave also another rule for the same case; but the true fundamental scheme of conditions about to be set forth will be seen to embrace as mere corollaries all such and such-like rules, which in fact supply more or less arbitrary combinations of the conditions, rather than the naked conditions themselves in their simple form and absolute totality.

I shall call the function to be dealt with U, and shall consider U to be a *homogeneous** rational function of m dimensions in respect of $x_1, x_2 \ldots x_n$, and shall inquire what are the conditions which must obtain when U is capable of being expressed as a function of only $(n-r)$ orders, say $l_1, l_2 \ldots l_{n-r}$, each of which is of course a homogeneous linear function of the given n variables.

Let the term derivative of U be understood to mean any result obtained by differentiating U any number of times with respect to one or more of the variables $x_1, x_2 \ldots x_n$. The first derivatives will be of $(m-1)$ dimensions, the second derivatives of $(m-2)$ dimensions, and so on; and finally, the $(m-1)$th derivatives will be homogeneous linear functions of $x_1, x_2 \ldots x_n$. Suppose U to be expressible as a function of $l_1, l_2 \ldots l_{n-r}$. It is immediately obvious that the derivatives from the 1st to the $(m-1)$th inclusive will be all expressible as homogeneous functions of $l_1, l_2 \ldots l_{n-r}$, and vanish when these vanish. But this statement is in substance pleonastic; for by means of Euler's well-known law, any derivative of U, say K, may be expressed (to a numerical factor *près*) under the form of

$$x_1 \frac{dK}{dx_1} + x_2 \frac{dK}{dx_2} + \ldots + x_n \frac{dK}{dx_n},$$

and consequently, whenever the linear derivatives of U vanish, all the upper derivatives of U, including U itself, must vanish at the same time. The number of these linear derivatives, say ν, will be the number of terms in a homogeneous function of n variables of $(m-1)$ dimensions, that is to say,

$$\frac{n(n-1)\ldots(n-m+2)}{1.2\ldots(m-1)}.$$

Again, if all the ν linear derivatives vanish when the $(n-r)$ equations $l_1 = 0, l_2 = 0 \ldots l_{n-r} = 0$ are satisfied, r being greater than zero, this can only happen by virtue of these ν derivatives being linear functions of $(n-r)$

* It is a common error to regard homogeneity of expression as merely a means for satisfying the desire for symmetry; the ground of its application and utility in analysis lies, in fact, much deeper; it is essentially a *method* and a *power*.

of them. Now, conversely, I shall prove, that if it be true that all the linear derivatives of U are linear functions $(n-r)$ of them, then U may be expressed as a function of these $(n-r)$; and this rule, as will be immediately made apparent, will give the necessary and sufficient conditions for the loss of r orders in the most simple and complete form by which they admit of being expressed. For the proof of the rule, only one additional remark has to be made in addition to that already made, of the vanishing of the linear derivatives necessarily implying the simultaneous evanescence of all the other derivatives; this additional remark being, that if the derivatives of any class, linear or otherwise, *quâ* one set of variables, become all zero, the derivatives of the same class, *quâ* any other set of variables linear functions of the first set and the same in number, will also become zero, for they are evidently expressible as linear functions of the first set.

Now let $d_1, d_2 \ldots d_{n-r}$ be any $(n-r)$ linear derivatives of U, of which all the other of the ν derivatives of this class are linear functions, so that they vanish when these $(n-r)$ vanish, and let U be expressed as a function of $(d_1, d_2 \ldots d_{n-r}; x_1, x_2 \ldots x_r)$. Then we may write

$$U = \phi_{m,0} + \phi_{m-1,1} + \phi_{m-2,2} + \ldots + \phi_{1,m-1} + \phi_{0,m},$$

where in general $\phi_{m-\epsilon,\epsilon}$ denotes a function homogeneous and of $m-\epsilon$ dimensions in respect to $d_1, d_2 \ldots d_{n-r}$, and homogeneous and of ϵ dimensions in respect to $x_1, x_2 \ldots x_r$. Now the linear derivatives of U all vanish when $d_1 = 0$, $d_2 = 0 \ldots d_{n-r} = 0$ for all values of $x_1, x_2 \ldots x_r$. Hence $U = 0$ on the same supposition, and hence $\phi_{0,m}$ is similarly zero. Also the first derivatives of U, *quâ* $d_1, d_2 \ldots d_{n-r}$, must vanish on the same supposition. Hence $\phi_{1,m-1}$ is identically zero; and so by taking the 2nd, 3rd ... up to the $(m-1)$th or linear derivatives of U in respect to $d_1, d_2 \ldots d_{n-r}$, we find successively $\phi_{2,m-2}, \phi_{3,m-3} \cdots \phi_{m-1,1}$ each identically zero, and consequently

$$U = \phi_{m,0} = \phi(d_1, d_2 \ldots d_{n-r}),$$

as was to be proved. To express the fact of the ν derivatives being linear functions of $(n-r)$ of them, form a rectangular matrix with the coefficients of the ν linear derivatives. This matrix will be n terms in breadth and ν terms in depth. Let $r = 1$: it is a direct consequence of the rule which has been established, that every full determinant consisting of a square n terms by n terms that can be formed out of this rectangular matrix must be zero: again, let $r = 2$; all the first minors, that is to say, all the determinants composed of squares $(n-1)$ terms by $(n-1)$ terms, must be zero, and so in general a loss of r orders will require that the $(r-1)$th minors shall all vanish; if $r = n$, the $(n-1)$th minors, that is the simple terms of the matrix which are all coefficients of U, must vanish, or in other words, when the function is of zero order all the coefficients vanish (an obvious truism).

Thus, then, we see that the true rule for the loss of one order in a polynomial of any degree is precisely the same as the well-known rule for the loss of one order in a quadratic function; the speciality in the latter case consisting merely in the fact that ν being equal to n, the rectangular matrix becomes a square, and there is only one full determinant. Moreover, for any other value of r the above rule coincides with that given by me some time back in the *Philosophical Magazine* for the case of quadratic functions.

Professor Hesse's rule for finding conditions applicable to the loss of one order is, as I have already stated, a consequence of the more simple scheme of conditions above given. It consists in forming the determinant

$$\left|\begin{array}{l} \dfrac{d^2U}{dx_1^2},\ \dfrac{d^2U}{dx_1dx_2} \cdots \dfrac{d^2U}{dx_1dx_n} \\ \dfrac{d^2U}{dx_2dx_1},\ \dfrac{d^2U}{dx_2dx_2} \cdots \dfrac{d^2U}{dx_2dx_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{d^2U}{dx_ndx_1},\ \dfrac{d^2U}{dx_ndx_2} \cdots \dfrac{d^2U}{dx_ndx_n} \end{array}\right|,$$

and equating the coefficients of this determinant fully developed separately to zero*. The attachment of the Professor to this particular form of covariant (I use the language of the calculus of forms) is readily intelligible, seeing the admirable application which he has made of it to the canonization of the cubic function of three variables, but it is really foreign to the nature of the present question; the coefficients of this covariant may easily be shown to be merely the full determinants of the $n \times \nu$ rectangular matrix above described, or linear functions of these said determinants with numerical coefficients. Hence the ground of its applicability.

Returning to the rule of the matrix, if we suppose the number of variables to be two, and call the coefficients of U

$$a_0,\ na_1,\ \tfrac{1}{2}n(n-1)a_2 \ldots a_n,$$

our rectangle becomes

$$\left|\begin{array}{ll} a_0, & a_1 \\ a_1, & a_2 \\ a_2, & a_3 \\ \cdots & \cdots \\ \cdots & \cdots \\ a_{n-1}, & a_n \end{array}\right|$$

* A form capable of being so derived I have elsewhere termed (in compliment to M. Hesse) the Hessian of the function to which it appertains. This is the trivial name which is much needed on account of the frequent occurrence of the form, and has been adopted by Mr Salmon

and the conditions become

$$a_0a_2 - a_1^2 = 0, \qquad a_0\,a_3 - a_1\,a_2 = 0,$$
$$a_1a_3 - a_2^2 = 0, \qquad \cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots$$
$$a_{n-2}a_n - a^2_{n-1} = 0, \quad a_{n-3}a_n - a_{n-2}a_{n-1} = 0, \text{ \&c.,}$$

all of which equations are obviously true (when the function loses an order, that is to say, becomes a perfect power) and are satisfied (special cases excepted) when any $(n-1)$ independent equations out of the entire number obtain; so that the number of conditions implied in the property to be represented is in exact conformity with the number of independent equations derived from the matrix, that is equations which, when satisfied, will in general cause all the rest to be satisfied. This conformity manifests itself also in the case of a quadratic function of n variables. But except in these two limiting (and, in an occult sense, reciprocal*) cases of a function of two variables of the nth degree, or of the degree 2 and n variables, this conformity in measure as the degree or number of variables rises, although it must substantially continue to exist, becomes, and in an accelerated degree, less and less apparent.

Thus, take the simple case of a cubic function of three variables, and let us confine ourselves to the consideration of the conditions which must be satisfied when it loses a single order. Let U be written out at length,

$$ax^3 + by^3 + cz^3 + 3hyz^2 + 3izx^2 + 3jxy^2 + 3h'y^2z + 3i'z^2x + 3j'x^2y + 6mxyz.$$

in his admirable treatise on the higher plane curves. In systematic nomenclature it would be termed the discriminant of the quadratic emanant, or more briefly, the quadremanative discriminant. I have discovered quite recently that the long sought for symmetrical, and by far the most easy practical process for discovering the number of the real roots of an equation, is contained in, and may be deduced immediately from, a certain transformation of its Hessian!

* There are frequent cases occurring in the calculus of forms of interchange between the degree of a function and the number of variables which it contains. Thus, to select a striking example (although one where the interchange is not exact), the theory of the real and imaginary roots or factors of a homogeneous function of two variables and of the nth degree may be shown to be immediately dependent upon the determination of the specific nature of a concomitant homogeneous function of the 2nd degree and of $(n-1)$ variables. For instance, if any ordinary algebraical equation of the 5th degree be given, a homogeneous quadratic function of four variables may be constructed, representing, consequently, a surface of the 2nd degree [the coefficients of which (as indeed is true whatever be the degree of the equation) will be quadratic functions of the coefficients of the given equation]; and such that, according as the surface so represented belongs to the class of (1), impossible surfaces; (2), the ellipsoid or hyperboloid of two sheets; (3), the hyperboloid of one sheet; the given equation will have 5, 3, or only 1 real root! Moreover, an equality between two of the roots of the equation will be denoted by the loss of one order in the associated quadratic function; and so many orders altogether will be lost as there are independent equalities existing between the roots. An entirely new light is thus thrown on M. Sturm's theorem; and the number of real and imaginary roots in an equation is for the first time made to depend upon the signs of functions symmetrically constructed in respect to the two ends of the equation, which has long been felt as a desideratum.

The matrix formed out of the coefficients of the linear derivatives becomes

$$\left|\begin{matrix} a, & j', & i \\ j, & b, & h' \\ i', & h, & c \\ m, & h', & h \\ i, & m, & i' \\ j', & j, & m \end{matrix}\right|.$$

Now by the homaloidal law, if the terms in this rectangle were all unlike, the number of full determinants (3 terms by 3 terms) whose evanescence (except for special values) determines the evanescence of all the rest, should be $(6-3+1)(3-3+1)$, that is 4; but in the actual case, since the evanescence of all the full determinants is a necessary consequence of the function becoming a cubic function of two orders (that is, breaking up into the product of three linear functions of x, y, z), and as this decomposability, as is well known, implies only the existence of three affirmative conditions, the four full determinants

$$\left|\begin{matrix} a, & j', & i \\ j, & b, & h' \\ i', & h, & c \end{matrix}\right| \quad \left|\begin{matrix} a, & j', & i \\ j, & b, & h' \\ m, & h', & h \end{matrix}\right| \quad \left|\begin{matrix} a, & j', & i \\ j, & b, & h' \\ i, & m, & i' \end{matrix}\right| \quad \left|\begin{matrix} a, & j', & i \\ j, & b, & h' \\ j', & j, & m \end{matrix}\right| *$$

* That is to say, a syzygetic relation must connect these four determinants. I may as well here repeat, that when the vanishing of a set of i rational integral functions necessarily, and without cases of exception, implies the vanishing of another rational integral function, then this function is termed a syzygetic function of the others; and some power of it must be expressible under the form of a sum of i binary products of rational integral functions, one factor of each of which products must be one of the i given functions. When the vanishing of all but one of a set of functions in general necessarily implies the vanishing of that one, but subject to cases of exception for specific values of the variables, then it can only be affirmed that the functions of the set are in syzygy; that is to say, that the sum of the products of each of them respectively by some rational integral function will be zero: the equation expressing this relation is termed a syzygetic equation.

Thus, if we take the three full determinants that can be formed out of the matrix

$$\begin{matrix} a, & \alpha, \\ b, & \beta, \\ c, & \gamma, \end{matrix}$$

that is $$a\beta - b\alpha, \quad b\gamma - c\beta, \quad c\alpha - a\gamma,$$

these are in syzygy, for we can form the equation

$$c(a\beta - b\alpha) + a(b\gamma - c\beta) + b(c\alpha - a\gamma) = 0.$$

This, however, is not the only equation of the kind that can be formed, for

$$\gamma(a\beta - b\alpha) + \alpha(b\gamma - c\beta) + \beta(c\alpha - a\gamma) = 0$$

is also identically true. We see in this case that the evanescence of any two of the three functions which in the general case would be entirely independent, in this case cease to be so; and the vanishing of three of them must draw along with it by necessary implication (except for special values) the evanescence of the 4th, for thus only can the necessary conformity between the number of affirmative conditions and the number of unimplicated equations come to take effect. The clear and direct putting in evidence of this peculiar species of implication demands and deserves to be minutely considered; and as it must in part borrow its explanation from the very little yet known of syzygetic relations, so it must also throw new light on that great and important, but as yet unformed and scarcely more than nascent theory.

In conclusion, it is apparent from the demonstration above given, that when U, a function of n variables, becomes expressible as a function of $(n-r)$ orders, these orders may be taken respectively any independent linear functions of the linear derivatives of U, which remark completes the theory of functions subject to the loss of one or more orders. It is obvious (and I am indebted to my esteemed friend Mr Cayley for the remark), that the conditions furnished as above by the $(m-1)$th, that is linear derivatives, are identical with and may be more elegantly replaced by those involved in the assertion of the existence of linear relations between the 1st or $(m-1)$th degreed derivatives, and we have then this very simple rule; *if ϕ, a function of x_1, $x_2 \dots x_n$, is expressible as a function of $n-r$ linear functions of x_1, $x_2 \dots x_n$, it is necessary and sufficient that r independent linear relations shall exist between*

$$\frac{d\phi}{dx_1}, \ \frac{d\phi}{dx_2} \dots \frac{d\phi}{dx_n}.$$

$a\beta - b\alpha$; $b\gamma - c\beta$; $c\alpha - a\gamma$ will in general imply the third, subject, however, to special cases of exception. Thus, if the 1st and 2nd vanish, the 3rd must vanish unless b and β *both* vanish; if the 2nd and 3rd vanish, the 1st must vanish unless c and γ both vanish; if the 3rd and 1st vanish, the second will vanish unless a and α both vanish. It will thus be seen that a peculiar species of *astricted* syzygy obtains between the three proposed functions, which enables us to affirm that *in general*, and except under *extra special* conditions, all three must vanish simultaneously. If two out of the three vanish, and the 3rd does not vanish, it is not merely (as might at the first blush of the theory of syzygy be conjectured) because some one other function vanishes in its place, but necessarily because a *plurality* of entirely independent functions (two simple letters as it happens here) each separately vanish. Thus we see how all but one of a set of functions $\chi_1, \chi_2 \dots \chi_n$ may *in general*, and yet *not universally*, necessarily vanish when all the rest vanish: to say that *one* syzygetic equation such as

$$\chi_1 \chi_1' + \chi_2 \chi_2' + \dots + \chi_n \chi_n' = 0$$

obtains, is not enough to explain the circumstances of the case; the fact is, that several distinct systems of values of χ_1', $\chi_2' \dots \chi_n'$ will be found capable of satisfying the equation, so that each of the functions $\chi_1, \chi_2 \dots \chi_n$ will have a *system* of syzygetic factors attached to it, and these unrelated, in the wide sense that, if we take χ_n', χ_n'', any two of the syzygetic factors attached to χ_n, they will *not be in syzygy* with $\chi_1, \chi_2 \dots \chi_{n-1}$; so that when these $(n-1)$ functions vanish, the vanishing of χ_n' and χ_n'' represents two distinct and completely independent conditions. Thus, in fine, the mutual implication of functions will in general denote the possibility of forming a *series* of syzygetic equations between them,—a remark, this, of no minor importance.

This rule itself also, it is evident, is capable of an independent and immediate demonstration by means of integrating the partial differential equation or equations by which it admits of being expressed. The above theory may readily be extended to functions of several systems of variables. Thus, for instance, the determinant

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

vanishing will be indicative of the function

$$\left\{\begin{array}{r} a\ xu + b\ xv + c\ xw \\ + a'\,yu + b'\,yv + c'\,yw \\ + a''zu + b''zv + c''zw \end{array}\right\},$$

being linearly equivalent to a function of the form

$$\left\{\begin{array}{r} Ax'u' + Bx'v' \\ + Cy'u' + Dy'v' \end{array}\right\},$$

that is losing an order in respect of each of the two systems x, y, z; u, v, w; and so in general.