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ON THE LAW OF HYSTERESIS (PART III.), AND THE THEORY OF FERRIC INDUCTANCES.

BY CHARLES PROTEUS STEINMETZ.

CHAPTER I.—COEFFICIENT OF MOLECULAR MAGNETIC FRICTION.

In two former papers, of January 19 and September 27, 1892, I have shown that the loss of energy by magnetic hysteresis, due to molecular friction, can, with sufficient exactness, be expressed by the empirical formula—

$$H = \eta\, B^{1.6}$$

where H = loss of energy per cm³. and per cycle, in ergs,
B = amplitude of magnetic variation,
η = coefficient of molecular friction,
the loss of energy by eddy currents can be expressed by

$$h = \varepsilon\, N\, B^2,$$

where h = loss of energy per cm³. and per cycle, in ergs,
ε = coefficient of eddy currents.

Since then it has been shown by Mr. R. Arno, of Turin, that the loss of energy by static dielectric hysteresis, *i.e.*, the loss of energy in a dielectric in an electro-static field can be expressed by the same formula :

$$H = \delta\, F^{\mathcal{H}},$$

where H = loss of energy per cycle,
F = electro-static field intensity or intensity of dielectric stress in the material,
δ = coefficient of dielectric hysteresis.

Here the exponent $\mathcal{H}$ was found approximately to = 1.6 at the low electro-static field intensities used.

At the frequencies and electro-static field strengths met in condensers used in alternate current circuits, I found the loss of energy by dielectric hysteresis proportional to the square of the field strength.

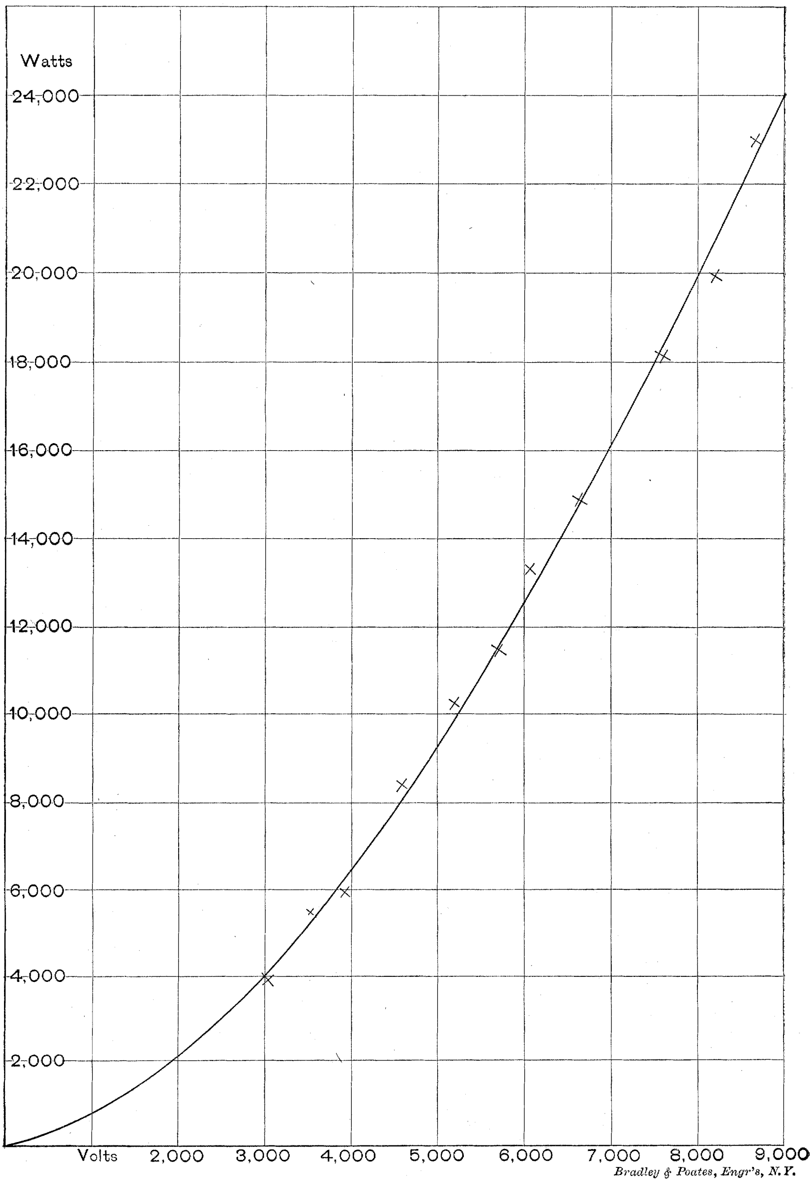

FIG. 1.

Other observations made afterwards agreed with this result. With regard to magnetic hysteresis, essentially new discoveries have not been made since, and the explanation of this exponent 1.6 is still unknown.

In the calculation of the core losses in dynamo electrical machinery and in transformers, the law of hysteresis has found its application, and so far as it is not obscured by the superposition of eddy currents has been fully confirmed by practical experience.

As an instance is shown in Fig. 1, the observed core loss of a high voltage 500 K. W. alternate current generator for power transmission. The curve is plotted with the core loss as abscissæ and the terminal volts as ordinates. The observed values are marked by crosses, while the curve of 1.6 power is shown by the drawn line.

The core loss is a very large and in alternators like the present machine, even the largest part of the total loss of energy in the machine.

With regard to the numerical values of the coefficient of hysteresis, the observations up to the time of my last paper cover the range,

$$\eta \times 10^3 =$$

Materials	From	To	Average.
Wrought iron; Sheet iron and sheet steel	2.00	5.48	3.0 to 3.3
Cast iron	11.3	16.2	13.0
Soft cast steel and mitis metal	3.18	12 0	6.0
Hard cast steel	. . .	27.9	
Welded steel	14.5	74.8	
Magnetite	20.4	23.5	
Nickel	12.2	38.5	
Cobalt	11.9		

While no new materials have been investigated in the meantime, for some, especially sheet iron and sheet steel, the range of observed value of η has been greatly extended, and, I am glad to state, mostly towards lower value of η, that is, better iron.

While at the time of my former paper, the value of hysteresis $\eta \times 10^3 = 2.0$, taken from Ewing's tests, was unequaled, and the best material I could secure, a very soft Norway iron, gave $\eta \times 10^3 = 2.275$, now quite frequently values considerably better than Ewing's soft iron wire are found, as the following table shows, which gives the lowest and the highest values of hysteretic loss observed in sheet iron and sheet steel, intended for electrical machinery.

The values are taken at random from the factory records of the General Electric Company.

Values of $\eta \times 10^3$.

Lowest.	Highest.
1.24	5.30
1.33	5.15
1.35	5.12
1.58	4.78
1.59	4.77
1.59	4.72
1.66	4.58
1.66	4.55
1.68	4.27
1.70	
1.71	
1.76	
1.80	
1.82	
1.88	
1.90	
1.93	
1.94	
1.94	

As seen, all the values of the first column refer to iron superior in its quality even to the sample of Ewing $\eta \times 10^3 = 2.0$, unequaled before.

The lowest value is $\eta \times 10^3 = 1.24$, that is, 38 per cent. better than Ewing's iron. A sample of this iron I have here. As you see, it is very soft material. Its chemical analysis does not show anything special. The chemical constitution of the next best sample $\eta \times 10^3 = 1.33$ is almost exactly the same as the constitution of samples $\eta \times 10^3 = 4.77$ and $\eta \times 10^3 = 3.22$, showing quite conclusively that the chemical constitution has no direct influence upon the hysteretic loss.

In consequence of this extension of η towards lower values, the total range of η yet known in iron and steel is from $\eta \times 10^3 = 1.24$ in best sheet iron to $\eta \times 10^3 = 74.8$ in glass-hard steel, and $\eta \times 10^3 = 81.8$ in manganese steel, giving a ratio of 1 to 66.

With regard to the exponent $\mathcal{H}$ in

$$H = \eta \, B^{\mathcal{H}},$$

which I found to be approximately $= 1.6$ over the whole range of magnetization, Ewing has investigated its variation, and found that it varies somewhat at different magnetizations, and that its variation corresponds to the shape of the magnetization curve, showing its three stages.[1]

1. J. A. Ewing, *Philosophical Transactions of the Royal Society*, London, June 15, 1893.

Tests of the variation of the hysteretic loss per cycle as function of the temperature have been published by Dr. W. Kunz[1], for temperatures from 20° and 800° Cent. They show that with rising temperature, the hysteretic loss decreases very greatly, and this decrease consists of two parts, one part, which disappears again with the decrease of temperature and is directly proportional to the increase of temperature, thus making the hysteretic loss a linear function of the temperature, and another part, which has become permanent, and seems to be due to a permanent change of the molecular structure produced by heating. This latter part is in soft iron, proportional to the temperature also, but irregular in steel.

Chapter II.—Molecular Friction and Magnetic Hysteresis.

In an alternating magnetic circuit in iron and other magnetic material, energy is converted into heat by molecular magnetic friction. The area of the hysteretic loop, with the M. M. F. as abscissæ and the magnetization as ordinates, represents the energy expended by the M. M. F. during the cyclic change of magnetization.

If energy is neither consumed nor applied outside of the magnetic circuit by any other source, the area of the hysteretic loop, *i. e.*, the energy consumed by hysteresis, measures and represents the energy wasted by molecular magnetic friction.

In general, however, the energy expended by the M. M. F.—the area of the hysteretic loop—needs not to be equal to the molecular friction. In the armature of the dynamo machine, it probably is not, but, while the hysteretic loop more or less collapses under the influence of mechanical vibration, the loss of energy by molecular friction remains the same, hence is no longer measured by the area of the hysteretic loop.

Thus a sharp distinction is to be drawn between the phenomenon of magnetic hysteresis, which represents the expenditure of energy by the M. M. F., and the molecular friction.

In stationary alternating current apparatus, as ferric inductances, hysteretic loss and molecular magnetic friction are generally identical.

In revolving machinery, the discrepancy between molecular friction and magnetic hysteresis may become very large, and the magnetic loop may even be overturned and represent, not expen-

1. *Elektrotechnische Zeitschrift,* April 5th, 1894.

diture, but production of electrical energy from mechanical energy; or inversely, the magnetic loop may represent not only the electrical energy converted into heat by molecular friction, but also electrical energy converted into mechanical motion.

Two such cases are shown in Figs. 2 and 3 and in Figs. 4 and 5. In these cases the magnetic reluctance and thus the inductance of the circuit was variable. That is, the magnetic circuit was opened and closed by the revolution of a shuttle-shaped armature.

The curve *s* represents the inductances of the magnetic circuit

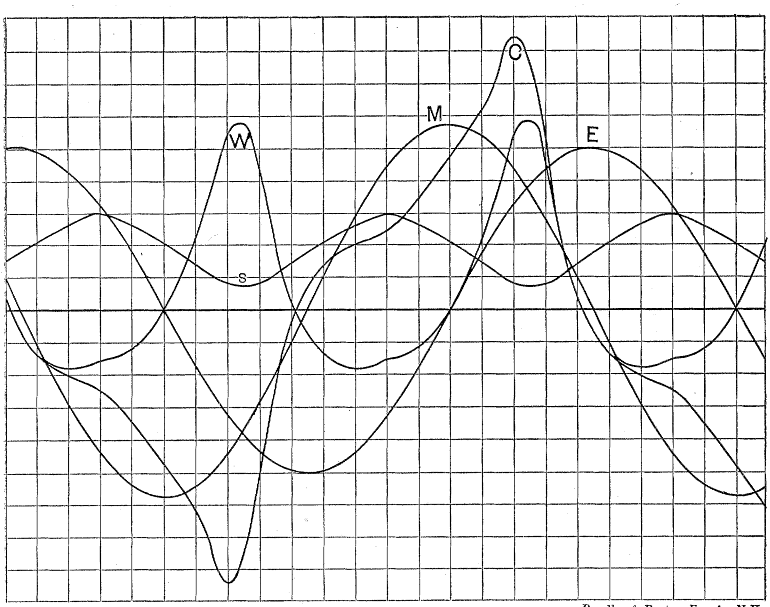

FIG. 2.

as function of the position. The curve E = counter E. M. F. or, since the internal resistance is negligible, the impressed E. M. F. and curve M = magnetism. If the impressed E. M. F., E is a sine wave, the current C assumes a distorted wave shape, and the product of current and E. M. F., $W = C\,E$ represents the energy. As seen, in this case the total energy is not equal to zero, *i. e.*, the E. M. F. or self-induction E not wattless as usually supposed, but represents production of electrical energy in the first, consumption in the second case. Thus, if the apparatus is driven by exterior power, it assumes the phase relation shown in

Fig. 2, and yields electrical energy as a self-exciting alternate current generator; if now the driving power is withdrawn it drops into the phase relation shown in Fig. 4, and then continues to revolve and to yield mechanical energy as a synchronous motor.

The magnetic cycles or H-B curves, or rather for convenience, the C-M curves, are shown in Figs. 3 and 5.

As seen in Fig. 5, the magnetic loop is greatly increased in area and represents not only the energy consumed by molecular magnetic friction, but also the energy converted into mechanical power, while the loop in Fig. 3 is overturned or negative, thus representing the electrical energy produced, minus loss by molecular friction.

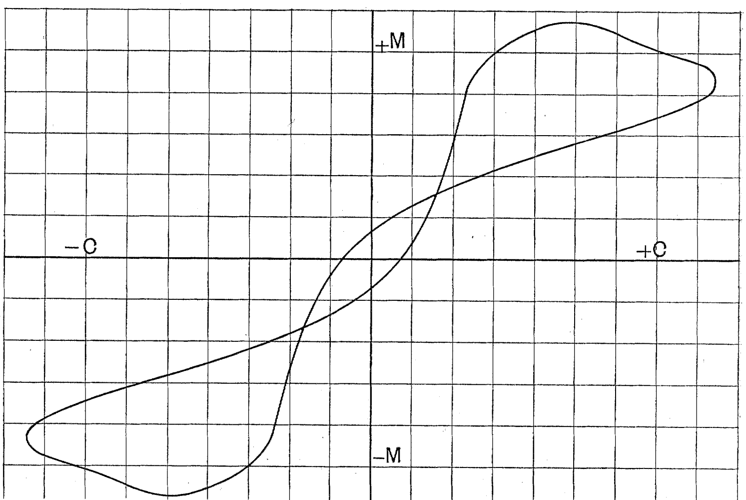

FIG. 3.

This is the same apparatus, of which two hysteretic loops were shown in my last paper, an indicator-alternator of the "humming bird" type.

Thus magnetic hysteresis is not identical with molecular magnetic friction, but is one of the phenomena caused by it.

CHAPTER III.—THEORY AND CALCULATION OF FERRIC INDUCTANCES.

In the discussion of inductive circuits, generally the assumption is made, that the circuit contains no iron. Such non-ferric inductances are, however, of little interest, since inductances are almost always ironclad or ferric inductances.

With our present knowledge of the alternating magnetic circuit, the ferric inductances can now be treated analytically with the same exactness and almost the same simplicity as non-ferric inductances.

Before entering into the discussion of ferric inductances, some terms will be introduced, which are of great value in simplifying the treatment.

Referring back to the continuous current circuit, it is known that, if in a continuous current circuit a number of resistances,

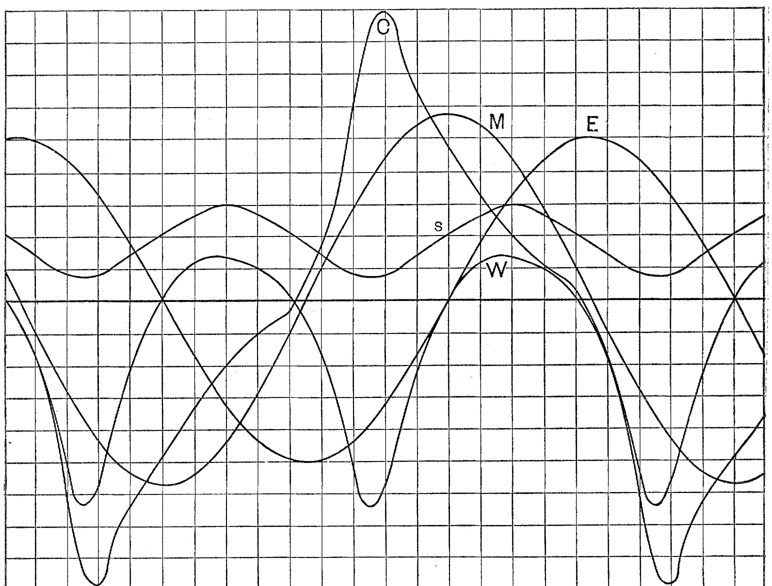

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FIG. 4.

$r_1, r_2, r_3 \ldots.$ are connected in series, their joint resistance, R, is the sum of the individual resistances:

$$R = r_1 + r_2 + r_3 + \ldots.$$

If, however, a number of resistances, $r_1, r_2, r_3 \ldots$, are connected in parallel, or in multiple, their joint resistance, R, cannot be expressed in a simple form, but is:

$$R = \frac{1}{\frac{1}{r^1} + \frac{1}{r_2} + \frac{1}{r_3} + \ldots}$$

Hence, in the latter case, it is preferable, instead of the term "resistance," to introduce its reciprocal, or inverse value, the term "*conductance*" $\rho = \frac{1}{r}$. Then we get:

"If a number of conductances, $\rho_1, \rho_2, \rho_3 \ldots$, are connected in parallel, their joined conductance is the sum of the individual conductances:

$$P = \rho_1 + \rho_2 + \rho_3 + \ldots.$$

When using the term conductance, the joined conductance of

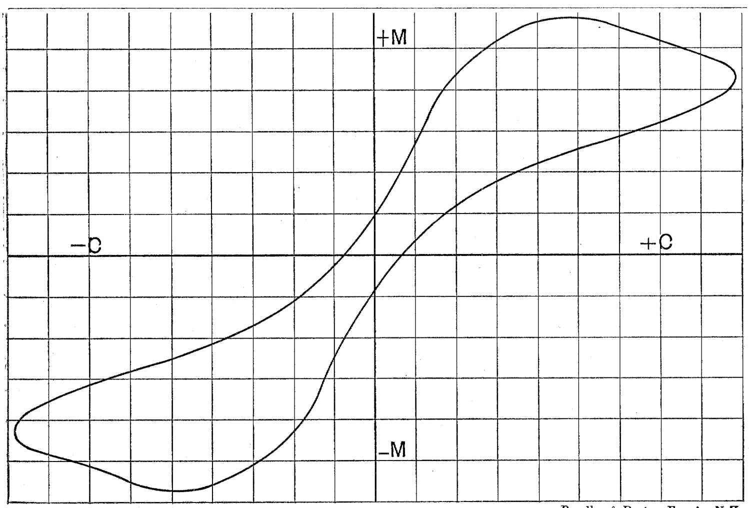

FIG. 5.

a number of series connected conductances, $\rho_1, \rho_2, \rho_3 \ldots$ becomes a complicated expression:

$$P = \frac{1}{\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \ldots}$$

Hence the use of the term "resistance" is preferable in the case of series connection, the use of the reciprocal term "conductance," in parallel connection, and we have thus:

"*The joined resistance of a number of series connected resistances is equal to the sum of the individual resistances, the joined conductance of a number of parallel connected conductances is equal to the sum of the individual conductances.*"

In alternating current circuits, in place of the term "resistance" we have the term "*impedance*," expressed in complex quantities by the symbol:

$$U = r - j\,s,$$

with its two components, the "*resistance*" r and the "*reactance*" s, in the formula of Ohm's law:

$$E = C\,U.$$ [1]

The resistance, r, gives the coefficient of the E. M. F. in phase with the current, or the energy component of E. M. F., $C\,r$; the reactance, s, gives the coefficient of the E. M. F. in quadrature with the current, or the wattless component of E. M. F., $C\,s$, both combined give the total E. M. F.

$$C\,u' = C\,\sqrt{r^2 + s^2}.$$

This *reactance*, s, is positive as inductive reactance:

$$s = 2\,\pi\,N\,L,$$

or negative as capacity reactance:

$$s = -\frac{1}{2\,\pi\,N\,K},$$

where,

N = frequency,
L = coefficient of self-induction, in henrys,
K = capacity, in farads.

Since E. M. F.'s are combined by adding their complex expressions, we have:

"The joined impedance of a number of series connected impedances, is the sum of the individual impedances, when expressed in complex quantities."

In graphical representation, impedances have not to be added, but combined in their proper phase, by the law of parallelogram, like the E. M. F.'s consumed by them.

The term "impedance" becomes inconvenient, however, when dealing with parallel connected circuits, or, in other words, when several currents are produced by the same E. M. F., in cases where Ohm's law is expressed in the form:

$$C = \frac{E}{U}.$$

It is preferable then, to introduce the reciprocal of "impe-

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1. "Complex Quantities and their use in Electrical Engineering," a paper read before Section A of the International Electrical Congress at Chicago, 1893.

dance," which may be called the "*admittance*" of the circuit:

$$Y = \frac{1}{U}.$$

As the reciprocal of the complex quantity

$$U = r - j\,s,$$

the admittance is a complex quantity also:

$$Y = \rho + j\,\sigma,$$

consisting of the component, ρ, which represents the coefficient of current in phase with the E. M. F., or energy current, $\rho\,E$, in the equation of Ohm's law:

$$C = Y\,E = (\rho + j\,\sigma)\,E,$$

and the component, σ, which represents the coefficient of current in quadrature with the E. M. F., or wattless component of current, $\sigma\,E$.

ρ may be called the "*conductance*," σ the "*susceptance*" of the circuit. Hence the conductance, ρ, is the energy component, the susceptance, σ, the wattless component of the admittance

$$Y = \rho + j\,\sigma,$$

and the numerical value of admittance is:

$$v = \sqrt{\rho^2 + \sigma^2};$$

the resistance, r, is the energy component, the reactance, s, the wattless component of the impedance

$$U = r - j\,s,$$

and the numerical value of impedance is

$$u = \sqrt{r^2 + s^2}.$$

As seen, the term "admittance" means dissolving the current into two components, in phase and in quadrature with the E. M. F., or the energy current and the wattless current; while the term "impedance" means dissolving the E. M. F. into two components, in phase and in quadrature with the current, or the energy E. M. F. and the wattless E. M. F.

It must be understood, however, that the "conductance" is not the reciprocal of the resistance, but depends upon the resistance as well as upon the reactance. Only when the reactance $s = 0$, or in continuous current circuits, is the conductance the reciprocal of resistance.

Again, only in circuits with zero resistance $r = 0$, is the susceptance the reciprocal of reactance; otherwise the susceptance depends upon reactance and upon resistance.

From the definition of the admittance:

$$Y = \rho + j\,\sigma$$

as the reciprocal of the impedance:

$$U = r - j\,s$$

we get

$$Y = \frac{1}{U},$$

or

$$\rho + j\,\sigma = \frac{1}{r - j\,s}$$

or, multiplying on the right side numerator and denominator by $(r + j\,s)$:

$$\rho + j\,\sigma = \frac{r + j\,s}{(r - j\,s)\,(r + j\,s)},$$

hence, since

$$(r - j\,s)\,(r + j\,s) = r^2 + s^2 = u^2:$$

$$\rho + j\,\sigma = \frac{r}{r^2 + s^2} + j\,\frac{s}{r^2 + s^2} = \frac{r}{u^2} + j\,\frac{s}{u^2},$$

or,

$$\rho = \frac{r}{r^2 + s^2} = \frac{r}{u^2}$$

$$\sigma = \frac{s}{r^2 + s^2} = \frac{s}{u^2},$$

and inversely:

$$r = \frac{\rho}{\rho^2 + \sigma^2} = \frac{\rho}{v^2}$$

$$s = \frac{\sigma}{\rho^2 + \sigma^2} = \frac{\sigma}{v^2}.$$

By these equations, from resistance and reactance, the conductance and susceptance can be calculated, and inversely.

Multiplying the equations for ρ and r, we get:

$$\rho\,r = \frac{r\,\rho}{u^2\,v^2},$$

hence,

$$u^2\,v^2 = (r^2 + s^2)\,(\rho^2 + \sigma^2) = 1,$$

and

$$u = \frac{1}{v} = \frac{1}{\sqrt{\rho^2 + \sigma^2}}$$

the absolute value of impedance,

$$v = \frac{1}{u} = \frac{1}{\sqrt{r^2 + s^2}}$$

the absolute value of admittance.

The sign of "admittance" is always opposite to that of "impedance," that means, if the current lags behind the E. M. F., the E. M. F. leads the current, and inversely, as obvious.

Thus we can express Ohm's law in the two forms:

$$E = C\ U.$$

$$C = E\ Y,$$

and have

"*The joined impedance of a number of series connected impedances is equal to the sum of the individual impedances; the joined admittance of a number of parallel connected admittances is equal to the sum of the individual admittances, if expressed in complex quantities; in diagrammatic representation, combination by the parallelogram law takes the place of addition of the complex quantities.*"

The resistance of an electric circuit is determined:

1. By direct comparison with a known resistance (Wheatstone bridge method, etc.). This method gives what may be called the true ohmic resistance of the circuit.

2. By the ratio:

$$\frac{\text{Volts consumed in circuit}}{\text{Amperes in circuit}}.$$

In an alternating current circuit, this method gives not the resistance, but the impedance

$$u = \sqrt{r^2 + s^2}$$

of the circuit.

3. By the ratio:

$$r = \frac{\text{Power consumed}}{(\text{current})^2} = \frac{(\text{E. M. F.})^2}{\text{Power consumed}},$$

where, however, the "power" and the "E. M. F." do not include the work done by the circuit, and the counter E. M. F.'s representing it, as for instance, the counter E. M. F. of a motor.

In alternating current circuits, this value of resistance is the energy coefficient of the E. M. F., and is:

$$r = \frac{\text{Energy component of E. M. F.}}{\text{Total current}}.$$

It is called the "*equivalent resistance*" of the circuit, and the energy coefficient of current:

$$\rho = \frac{\text{Energy component of current}}{\text{Total E. M. F.}},$$

is called the "*equivalent conductance*" of the circuit.

In the same way the value:

$$s = \frac{\text{Wattless component of E. M. F.}}{\text{Total current}}$$

is the "*equivalent reactance*," and

$$\sigma = \frac{\text{Wattless component of current}}{\text{Total E. M. F.}}$$

is the "*equivalent susceptance*" of the circuit.

While the true ohmic resistance represents the expenditure of energy as heat, inside of the electric conductor, by a current of uniform density, the "equivalent resistance" represents the total expenditure of energy.

Since in an alternating current circuit in general, energy is expended not only in the conductor, but also outside thereof, by hysteresis, secondary currents, etc., the equivalent resistance frequently differs from the true ohmic resistance, in such way as to represent a larger expenditure of energy.

In dealing with alternating current circuits, it is necessary, therefore, to substitute everywhere the values "equivalent resistance," "equivalent reactance," "equivalent conductance," "equivalent susceptance," to make the calculation applicable to general alternating current circuits, as ferric inductance, etc.

While the true ohmic resistance is a constant of the circuit, depending upon the temperature only, but not upon the E. M. F., etc., the "equivalent resistance" and "equivalent reactance" is in general not a constant, but depends upon the E. M. F., current, etc.

This dependence is the cause of most of the difficulties met in dealing analytically with alternating current circuits containing iron.

The foremost sources of energy loss in alternating current circuits, outside of the true ohmic resistance loss, are:

1. Molecular friction, as:

   (*a*) magnetic hysteresis;

   (*b*) dielectric hysteresis.

2. Primary electric currents, as:
   (*a*) leakage or escape of current through the insulation, brush discharge;
   (*b*) eddy-currents in the conductor, or unequal current distribution.
3. Secondary or induced currents, as:
   (*a*) eddy or Foucault currents in surrounding magnetic materials;
   (*b*) eddy or Foucault currents in surrounding conducting materials;
   (*c*) secondary currents of mutual inductance in neighboring circuits.
4. Induced electric charges, electro-static influence.

While all these losses can be included in the terms "equivalent resistance," etc., only the magnetic hysteresis and the eddy-currents in the iron will form the object of the present paper.

I.—*Magnetic Hysteresis.*

To examine this phenomenon, first a circuit of very high inductance, but negligible true ohmic resistance may be considered, that is, a circuit entirely surrounded by iron; for instance, the primary circuit of an alternating current transformer with open secondary circuit.

The wave of current produces in the iron an alternating magnetic flux, which induces in the electric circuit an E. M. F., the counter E. M. F. of self-induction. If the ohmic resistance is negligible, the counter E. M. F. equals the impressed E. M. F., hence, if the impressed E. M. F. is a sine-wave, the counter E. M. F., and therefore the magnetism which induces the counter E. M. F. must be sine-waves also. The alternating wave of current is not a sine-wave in this case, but is distorted by hysteresis. It is possible, however, to plot the current wave in this case from the hysteretic cycle of magnetization.

From the number of turns n of the electric circuit, the effective counter E. M. F. E, and the frequency N of the current, the maximum magnetic flux M is found by the formula:

$$E = \sqrt{2}\, \pi\, n\, N\, M\, 10^{-8};$$

hence:

$$M = \frac{E\, 10^8}{\sqrt{2}\, \pi\, n\, N}.$$

Maximum flux M and magnetic cross-section S give the maximum magnetic induction $B = \frac{M}{S}$.

If the magnetic induction varies periodically between $+B$ and $-B$, the M. M. F. varies between the corresponding values $+F$ and $-F$, and describes a looped curve, the cycle of hysteresis.

If the ordinates are given in lines of magnetic force, the ab-

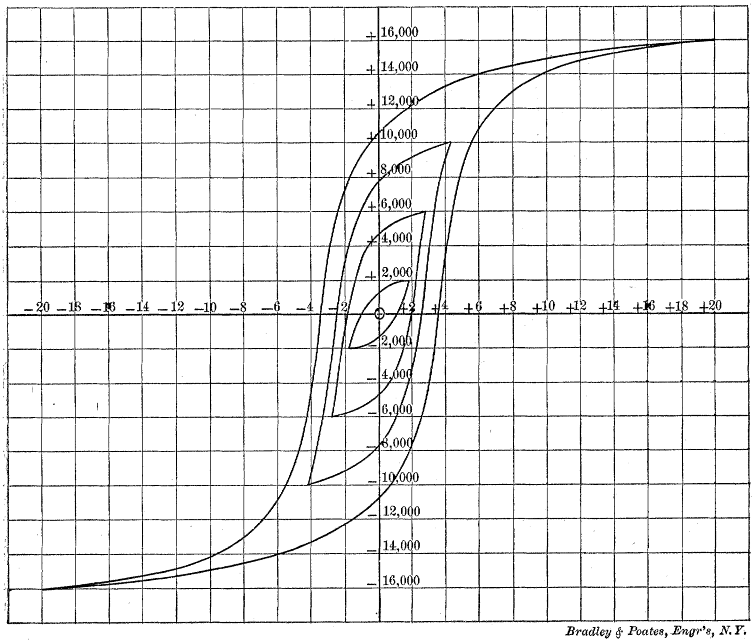


FIG. 6.

scissæ in tens of ampere-turns, the area of the loop equals the energy consumed by hysteresis, in ergs per cycle.

From the hysteretic loop is found the instantaneous value of M. M. F. corresponding to an instantaneous value of magnetic flux, that is of induced E. M. F., and from the M. M. F., F, in ampere-turns per unit length of magnetic circuit, the length l of the magnetic circuit, and the number of turns n of the electric circuit, are found the instantaneous values of current c corresponding to a M. M. F. F, that is a magnetic induction B and thus induced E. M. F. e, as:

$$c = \frac{Fl}{n}.$$

In Fig. 6 four magnetic cycles are plotted, with the maximum values of magnetic inductions: $B = 2{,}000$, $6{,}000$, $10{,}000$ and $16{,}000$, and the corresponding maximum M. M. F.'s: $F = 1.8$, 2.8, 4.3, 20.0. They show the well-known hysteretic loop, which becomes pointed when magnetic saturation is approached.

These magnetic cycles correspond to average good sheet iron or sheet steel of hysteretic coefficient: $\eta = .0033$, and are given

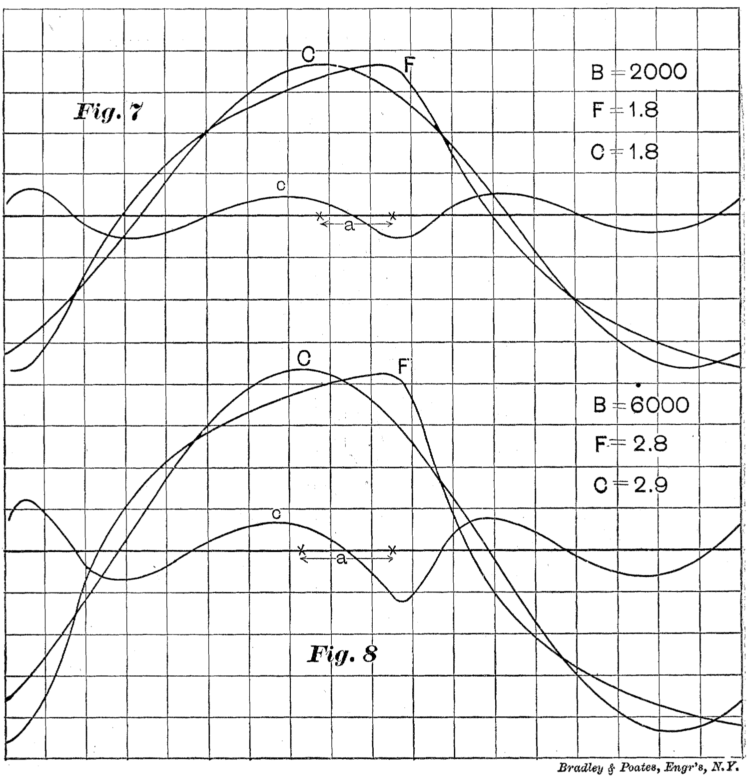


Fig. 7

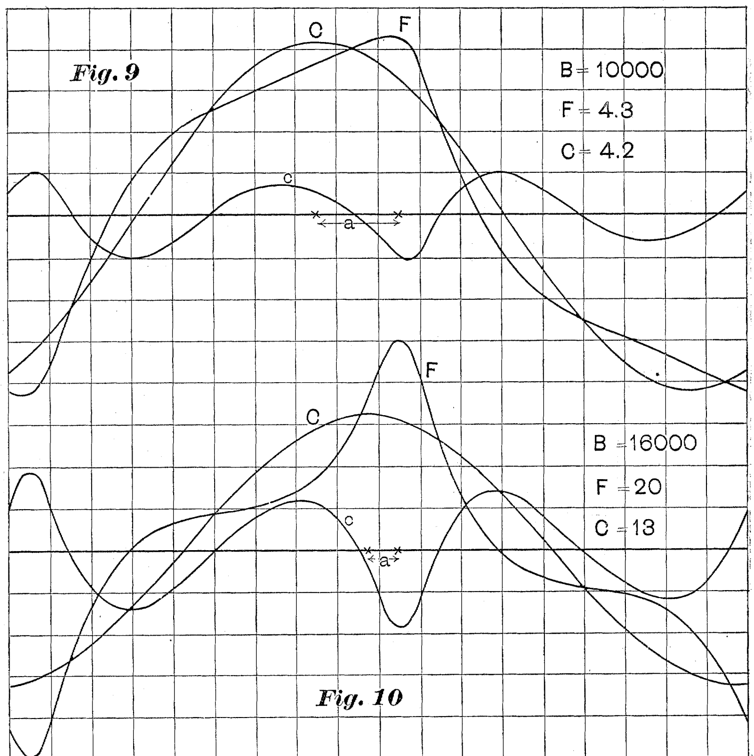
Fig. 8

with ampere-turns per cm. as abscissæ, and kilolines of magnetic force as ordinates.

In Figs. 7, 8, 9 and 10 the magnetism, or rather the magnetic induction, as derived from the induced E. M. F., is assumed as sine-curve. For the different values of magnetic induction of this sine-curve, the corresponding values of M. M. F., hence of current, are taken from Fig. 6, and plotted, giving thus the exciting current required to produce the sine-wave of magnetism;

that is, the wave of current, which a sine-wave of impressed E. M. F. will send through the circuit.

As seen from Figs. 4 to 10, these waves of alternating current F are not sine-waves, but are distorted by the superposition of higher harmonics, that is, are complex harmonic waves. They reach their maximum value at the same time with the maximum of magnetism, that is, 90° ahead of the maximum induced E. M. F.,



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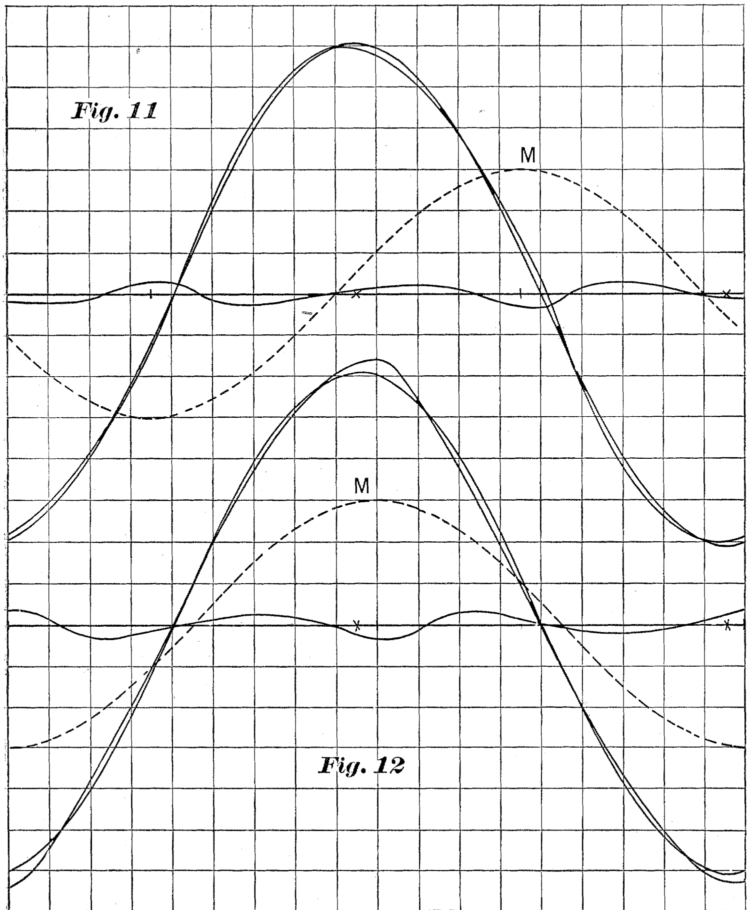
hence about 90° behind the maximum impressed E. M. F., but pass the zero line considerably ahead of the zero value of magnetism: 42, 52, 50 and 41 degrees respectively.

The general character of these current waves is, that the maximum point of the wave coincides in time with the maximum point of the sine-wave of magnetism, but the current wave is bulged out greatly at the rising; hollowed in at the decreasing side. With increasing magnetization, the maximum of the current wave becomes more pointed, as the curve of Fig. 9, for $B = 10{,}000$ shows, and at still higher saturation a peak is formed at the maximum point, as in the curve of Fig. 10, for $B = 16{,}000$. This is the case, when the curve of magnetization reaches within the range of magnetic saturation, since in the proximity of saturation



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the current near the maximum point of magnetization has to rise abnormally, to cause a small increase of magnetization only.

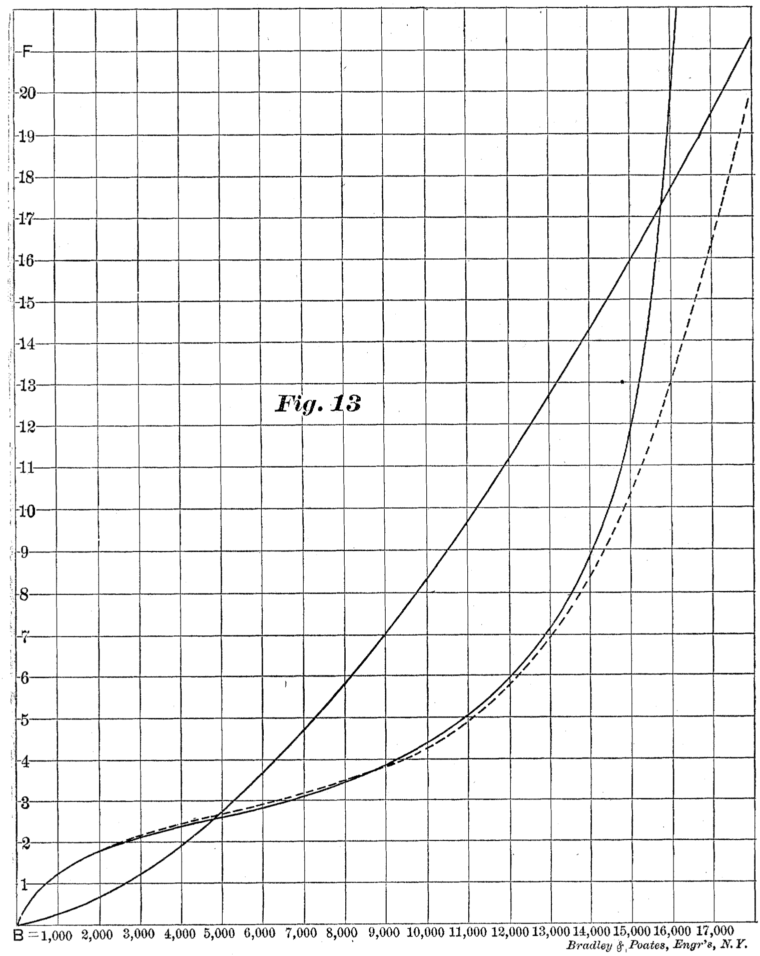
The distortion of the wave of magnetizing current is so large as shown here, only in an iron closed magnetic circuit expending energy by hysteresis only, as in the ironclad transformer at open secondary circuit. As soon as the circuit expends energy in any other way, as in resistance, or by mutual inductance, or if an air-gap is introduced in the magnetic circuit, the distortion of the current wave rapidly decreases and practically disappears, and the current becomes more sinusoidal. That is, while the distorting component remains the same, the sinusoidal component of current greatly increases, and obscures the distortion. For instance, in Figs. 11 and 12 two waves are shown, corresponding in magnetization to the curve of Fig. 8, as the worst distorted. The curve in Fig. 11 is the current wave of a transformer at $\frac{1}{10}$ load. At higher load the distortion is still correspondingly less. The curve of Fig. 12 is the exciting current of a magnetic circuit, containing an air-gap, whose length equals $\frac{1}{400}$ the length of the magnetic circuit. These two curves are drawn in $\frac{1}{3}$ the size of the curve in Fig. 8. As seen, both curves are practically sine-waves.

The distorted wave of current can be dissolved in two components: *a true sine-wave of equal effective intensity and equal power with the distorted wave*, called the "*equivalent sine-wave*," and a *wattless higher harmonic*, consisting chiefly of a term of triple frequency.

In Figs. 7 to 12 are shown, in drawn lines, the equivalent sine-waves, and the wattless complex higher harmonics, which together form the distorted current wave. The equivalent sine-wave of M. M. F., or of current, in Figs. 7 to 10, leads the magnetism by 34, 44, 38 and 15.5 degrees respectively. In Figs. 11 and 12 the equivalent sine-wave almost coincides with the distorted curve, and leads the magnetism by only 9°.

It is interesting to note, that even in the greatly distorted curves of Figs. 7 to 9 the maximum value of the equivalent sine-wave is nearly the same as the maximum value of the original distorted wave of M. M. F., as long as magnetic saturation is not approached, being 1.8, 2.9 and 4.2 respectively, against 1.8, 2.8 and 4.3 as maximum values of the distorted curve. Since by the definition the effective value of the equivalent sine-wave is the same as that of the distorted wave, this means, that the distorted wave of exciting current shares with the sine-wave the feature, that the maximum value and the effective value have the ratio: $\sqrt{2} \div 1$. Hence, below saturation, the maximum value of the distorted curve can be calculated from the effective value—which is given by the reading of an electro-dynamometer—by the same ratio as with a true sine-wave, and the magnetic characteristic can thus be determined by means of alternating currents, by the electro-dynamometer method, with sufficient exactness.

In Fig. 13 is shown the true magnetic characteristic of a sample of average good sheet iron, as found by the method of slow



reversals by the magnetometer, and for comparison in dotted lines the same characteristic, as determined by alternating currents, by the electro-dynamometer, with ampere-turns per cm. as ordinates, and magnetic inductions as abscissæ. As seen, the two curves practically coincide up to $B = 10{,}000 \sim 14{,}000$.

For higher saturations, the curves rapidly diverge, and the electro-dynamometer curve shows comparatively small M. M. F.'s producing apparently very high magnetizations.

The same Fig. 13 gives the curve of hysteretic loss, in ergs per cm.^{3} and cycle, as ordinates, and magnetic inductions as abscissæ.

So far as current strength and energy consumption is concerned, the distorted wave can be replaced by the equivalent sine-wave, and the higher harmonics neglected.

All the measurements of alternating currents, with the only exception of instantaneous readings, yield the equivalent sine-wave only, but suppress the higher harmonic, since all measuring instruments give either the mean square of the current wave, or the mean product of instantaneous values of current and E. M. F., which are by definition the same in the equivalent sine-wave as in the distorted wave.

Hence, in all practical applications, it is permissible to neglect the higher harmonic altogether, and replace the distorted wave by its equivalent sine-wave, keeping in mind, however, the existence of a higher harmonic as a possible disturbing factor, which may become noticeable in those very infrequent cases, where the frequency of the higher harmonic is near the frequency of resonance of the circuit.

The equivalent sine-wave of exciting current leads the sine-wave of magnetism by an angle α, which is called the "*angle of hysteretic advance of phase.*" Hence the current lags behind the E. M. F. by $90° - \alpha$, and the power is, therefore:

$$P = C\,E \cos (90° - \alpha) = C\,E \sin \alpha.$$

Thus the *exciting current* C consists of an energy component: $C \sin \alpha$, which is called the "*hysteretic energy current,*" and a wattless component: $C \cos \alpha$, which is called the "*magnetizing current.*" Or inversely, the E. M. F. consists of an energy component: $E \sin \alpha$, the "*hysteretic energy* E. M. F.", and a wattless component: $E \cos \alpha$, the "E. M. F. *of self-induction.*"

Denoting the absolute value of the impedance of the circuit $\frac{E}{C}$, by u—where u is determined by the magnetic characteristic of the iron, and the shape of the magnetic and electric circuit—the impedance is represented, in phase and intensity, by the symbolic expression:

$$U = r - j\,s = u \sin \alpha - j\,u \cos \alpha,$$

and the admittance by:

$$Y = \rho + j\,\sigma = \frac{1}{u}\sin\alpha + j\,\frac{1}{u}\cos\alpha = v\sin\alpha + j\,v\cos\alpha.$$

The quantities: u, r, s and v, ρ, σ are not constants, however, in this case, as in the circuit without iron, but depend upon the intensity of magnetization, B, that is, upon the E. M. F.

This dependence complicates the investigation of circuits containing iron.

In a circuit entirely enclosed by iron, α is quite considerable, from 30 to 50 degrees for values below saturation. Hence even with negligible true ohmic resistance *no great lag can be produced in ironclad alternating current circuits.*

As I have proved, the loss of energy by hysteresis due to molecular friction is with sufficient exactness proportional to the 1.6th power of magnetic induction, B. Hence, it can be expressed by the formula:

$$H = \eta\, B^{1.6},$$

where

H = loss of energy per cycle, in ergs or (C. G. S.) units (= 10^{-7} Joules) per cm.^{3},

B = maximum magnetic induction, in lines of force per cm.^{2}, and,

η = the "coefficient of hysteresis."

At the frequency, N, in the volume, V, the loss of power is by this formula:

$$P = \eta\, N\, V\, B^{1.6}\, 10^{-7} \text{ watts,}$$

$$= \eta\, N\, V \left(\frac{M}{S}\right)^{1.6} 10^{-7} \text{ watts,}$$

where S is the cross-section of the total magnetic flux, M.

The maximum magnetic flux, M, depends upon the counter E. M. F. of self-induction, E, by the equation:

$$E = \sqrt{2}\,\pi\, N\, n\, M\, 10^{-8},$$

or,

$$M = \frac{E\, 10^{8}}{\sqrt{2}\,\pi\, N\, n},$$

where n = number of turns of the electric circuit.

Substituting this in the value of the power, P, and cancelling, we get:

$$P = \eta\,\frac{E^{1.6}}{N^{.6}}\,\frac{V\,10^{5.8}}{2^{.8}\,\pi^{1.6}\,S^{1.6}\,n^{1.6}} = 58\,\eta\,\frac{E^{1.6}}{N^{1.6}}\,\frac{V\,10^{3}}{S^{1.6}\,n^{1.6}}$$

or

$$P = a\,\frac{E^{1.6}}{N^{.6}},\ \text{where}: a = \eta\,\frac{V\,10^{5.8}}{2^{8}\,\pi^{1.6}\,S^{1.6}\,n^{1.6}} = 58\,\eta\,\frac{V\,10^{3}}{S^{1.6}\,n^{1.6}},$$

or, substituting

$$\eta = .0033:$$

$$a = 191.4\,\frac{V}{S^{1.6}\,n^{1.6}},$$

or, substituting

$$V = S\,L,\ \text{where } L = \text{length of magnetic circuit}:$$

$$a = \eta\,\frac{L\,10^{5.8}}{2^{.8}\,\pi^{1.6}\,S^{\,6}\,n^{1.6}} = \frac{58\,\eta\,L\,10^{3}}{S^{.6}\,n^{1\,6}} = 191.4\,\frac{L}{S^{.6}\,n^{1.6}}$$

and

$$P = \frac{58\,\eta\,E^{1.6}\,L\,10^{3}}{N^{.6}\,S^{.6}\,n^{1.6}} = \frac{191.4\,E^{1.6}\,L}{N^{.6}\,S^{.6}\,n^{1.6}}.$$

As seen, the hysteretic loss is proportional to the 1.6th power of the E. M. F., inverse proportional to the 1.6th power of the number of turns, and inverse proportional to the .6th power of frequency, and of cross-section.

If ρ = equivalent conductance, the energy component of current is $C = E\,\rho$, and the energy consumed in conductance ρ is:

$$P = C\,E = E^{2}\,\rho.$$

Since, however,

$$P = a\,\frac{E^{1.6}}{N^{.6}},$$

it is:

$$a\,\frac{E^{1\,6}}{N^{.6}}\,E^{2}\,\rho,$$

or,

$$\rho = \frac{a}{N^{.6}\,E^{.4}} = \frac{58\,\eta\,L\,10^{3}}{E^{.4}\,N^{.6}\,S^{.6}\,n^{1.6}} = 191.4\,\frac{L}{E^{.4}\,N^{.6}\,S^{.6}\,n^{1.6}}$$

That is:

"*The equivalent conductance due to magnetic hysteresis, is proportional to the coefficient of hysteresis, η, and to the length of the magnetic circuit, L, and inverse proportional to the .4th power of the E. M. F., E, to the .6th power of the frequency, N, and of the cross-section of the magnetic circuit, S, and to the 1.6th power of the number of turns, n.*"

Hence, the equivalent hysteretic conductance increases with decreasing E. M. F., and decreases with increasing E. M. F.; it varies, however, much slower than the E. M. F., so that, if the hysteretic conductance represents only a part of the total energy consumption, it can within a limited range of variation, as for instance, in constant potential transformers, without serious error be assumed as constant.

If:

P = magnetic reluctance of a circuit,

F = maximum M. M. F.,

C = effective current, hence

$C\sqrt{2}$ = maximum current, it is the magnetic flux:

$$M = \frac{F}{P} = \frac{n\,C\,\sqrt{2}}{P}.$$

Substituting this in the equation of the counter E. M. F. of self-induction:

$$E = \sqrt{2}\,\pi\,N\,n\,M\,10^{-8},$$

it is:

$$E = \frac{2\,\pi\,n^2\,N\,C\,10^{-8}}{P},$$

hence, the absolute admittance of the circuit:

$$v = \sqrt{\rho^2 + \sigma^2} = \frac{C}{E} = \frac{P\,10^8}{2\,\pi\,n^2\,N} = \frac{b\,P}{N},$$

where

$$b = \frac{10^8}{2\,\pi\,n^2}\text{ is a constant.}$$

Thus:

"*The absolute admittance, v, of a circuit of negligible resistance is proportional to the magnetic reluctance, P, and inverse proportional to the frequency, N, and to the square of the number of turns, n.*"

In a circuit containing iron, the reluctance, P, varies with the magnetization, that is, with the E. M. F. Hence, the admittance of such a circuit is not a constant, but is variable also.

In an ironclad electric circuit, that is, a circuit whose magnetic field exists entirely within iron, as the magnetic circuit of a well-designed alternating current transformer, P, is the reluctance of the iron circuit. Hence, if μ = permeability, since,

$$P = \frac{\mathfrak{F}}{M},$$

and

$$\mathfrak{F} = L\,F = \frac{10}{4\,\pi}\,L\,H = \text{M. M. F.},$$

$$M = S\,B = \mu\,S\,H = \text{magnetism},$$

it is:

$$P = \frac{10\ L}{4\ \pi\ \mu\ S},$$

and, substituting this value in the equation of the admittance:

$$v = \frac{P\ 10^8}{2\ \pi\ n^2\ N} = \frac{L\ 10^9}{8\ \pi^2\ n^2\ \mu\ S\ N} = \frac{d}{N\ \mu},$$

where:

$$d = \frac{L\ 10^9}{8\ \pi^2\ n^2\ S} = \frac{127\ L\ 10^6}{n^2\ S}.$$

Thus:

"*In an ironclad circuit, the absolute admittance, v, is inverse proportional to the frequency, N, to the permeablity, μ, the cross-section, S, and square of the number of turns, n, and directly proportional to the length of the magnetic circuit, L.*"

The conductance is:

$$\rho = \frac{a}{N^{.6}\ E^{.4}}$$

the admittance:

$$v = \frac{d}{N\ \mu};$$

hence, the angle of hysteretic advance:

$$\sin\ a = \frac{\rho}{v} = \frac{a\ \mu\ N^{.4}}{d\ E^{.4}};$$

or, substituting for a and d:

$$\sin\ a = \mu\ \frac{N^{.4}}{E^{.4}}\ \frac{\eta\ L\ 10^{5.8}}{2^{.8}\ \pi^{1.6}\ S^{.6}\ n^{1.6}}\ \frac{8\ \pi^2\ n^2\ S}{L\ 10^9}$$

$$= \frac{\mu\ \eta\ N^{.4}\ n^{.4}\ S^{.4}\ \pi^{.4}\ 2^{2.2}}{E^{.4}\ 10^{3.2}};$$

or, substituting:

$$E = 2^{.5}\ \pi\ N\ n\ S\ B\ 10^{-8}:$$

$$\sin\ a = \frac{4\ \mu\ \eta}{B^{.4}};$$

hence, independent of frequency, number of turns, shape and size of magnetic and electric circuit.

Thus:

"*In an ironclad inductance, the angle of hysteretic advance, a, depends upon the magnetic constants: permeability and coefficient of hysteresis, and upon the maximum magnetic induction, but is entirely independent of the frequency, of the shape and other conditions of the magnetic and electric circuit, and,* *therefore, all the ironclad magnetic circuits constructed of the same quality of iron, and using the same magnetic density, give the same angle of hysteretic advance.*"

"The angle of hysteretic advance, α, in a closed circuit transformer, depends upon the quality of the iron, and the magnetic density only."

"The sine of the angle of hysteretic advance equals four times the product of permeability and coefficient of hysteresis, divided by the .4th power of the magnetic density:

$$\sin \alpha = \frac{4\,\mu\,\eta}{B^{.4}}."$$

If the magnetic circuit is not entirely ironclad, but the magnetic structure contains air-gaps, the total reluctance is the sum of the iron reluctance and the air reluctance:

$$P = P_i + P_a;$$

hence, the admittance is:

$$v = \sqrt{\rho^2 + \sigma^2} = \frac{b}{N}(P_i + P_a),$$

or:

"In a circuit containing iron, the admittance is the sum of the admittance due to the iron part of the circuit:

$$v_i = \frac{b}{N}\,P_i,$$

and the admittance due to the air part of the circuit:

$$v_a = \frac{b}{N}\,P_a,$$

if the iron and the air are in series in the magnetic circuit."

The conductance, ρ, represents the loss of energy in the iron, and, since air has no magnetic hysteresis, is not changed by the introduction of an air-gap.

Hence, the angle of hysteretic advance of phase is:

$$\sin \alpha = \frac{\rho}{v} = \frac{\rho}{v_i + v_a} = \frac{\rho}{v_i}\,\frac{P_i}{P_i + P_a},$$

and is a maximum $= \frac{\rho}{v_i}$, for the ironclad circuit, but decreases with increasing width of the air-gap. The introduction of the air-gap of reluctance, P_a, decreases $\sin \alpha$ in the ratio $\frac{P_i}{P_i + P_a}$.

In the range of practical application, from $B = 2{,}000$ to

$B = 12{,}000$, the permeability of the iron varies between 900 and 2,000 approximately, while sin α in an ironclad circuit varies in this range from .51 to .69. In air, $u = 1$.

If, consequently, one per cent. of the length of the iron is replaced by an air-gap, the total reluctance varies only in the proportion of $1\frac{1}{9}$ to $1\frac{1}{20}$, or by about six per cent.; that is, is practically constant, while the angle of hysteretic advance varies from sin $\alpha = .035$ to sin $\alpha = .064$. Thus ρ is already negligible compared with σ, and σ practically equal to v.

Hence:

"In an electric circuit containing iron, but forming an open magnetic circuit whose air-gap is not less than $\frac{1}{100}$ the length of the iron, the susceptance is practically constant and equal to the admittance, as long as saturation is not yet approached, and it is:

$$\sigma = \frac{P\ b}{N}, \text{ or : } s = \frac{N}{P\ b}.$$

The angle of hysteretic advance is small, below 4°, and the hysteretic conductance is

$$\rho = \frac{a}{E^{.4}\ N^{.6}}.$$

At a sine-wave of impressed E. M. F., the current wave is practically a sine-wave."

To determine the electric constants of a circuit containing iron, we shall proceed in the following way:

Let E = counter E. M. F. of self-induction;
then from the equation:

$$E = \sqrt{2}\ \pi\ n\ N\ M\ 10^{-8},$$

where:

$$N = \text{frequency},$$
$$n = \text{number of turns},$$

we get the magnetism, M, and by means of the magnetic cross-section, S, the maximum magnetic induction:

$$B = \frac{M}{S}.$$

From B we get, by means of the magnetic characteristic of the iron, the M. M. F., F, in ampere-turns per cm. length, where

$$F = \frac{10}{4\ \pi}\ H,$$

H = M. M. F. in (C. G. S.) units.

Hence, if

L_l = length of iron circuit, $F_l = L_l F$ = ampere-turns required in the iron,

L_a = length of air circuit, $F_a \dfrac{10\, L_a\, B}{4\, \pi}$ = ampere-turns required in the air,

hence,

$F = F_l + F_a$ = total ampere-turns, maximum value, and

$$\frac{F}{\sqrt{2}} = \text{effective value.}$$

The exciting current is:

$$C = \frac{F}{n\ \sqrt{2}},$$

and the absolute admittance:

$$v = \sqrt{\rho^2 + \sigma^2} = \frac{C}{E}.$$

If F_l is not negligible against F_a, this admittance, v, is variable with the E. M. F., E.

If:

V = volume of iron,

η = coefficient of hysteresis,

the loss of energy by hysteresis due to molecular magnetic friction is:

$$W = \eta\ N\ V\ B^{1.6},$$

hence the hysteretic conductance:

$$\rho = \frac{W}{E^2},$$

and is variable with the E. M. F., E.

The angle of hysteretic advance is:

$$\sin \alpha = \frac{\rho}{v},$$

the susceptance:

$$\sigma = \sqrt{v^2 - \rho^2},$$

the equivalent resistance:

$$r = \frac{\rho}{v^2},$$

the reactance:

$$s = \frac{\sigma}{v^2}.$$

As conclusions we derive from this chapter:

1. In an alternating current circuit surrounded by iron, the current produced by a sine-wave of E. M. F. is not a true sine-wave, but is distorted by hysteresis.

2. This distortion is excessive only with a closed magnetic circuit transferring no energy into a secondary circuit by mutual inductance.

3. The distorted wave of current can be replaced by the equivalent sine-wave, that is, a sine-wave of equal effective intensity and equal power, and the superposed higher harmonic, consisting mainly of a term of triple frequency, can be neglected except in resonating circuits.

4. Below saturation, the distorted curve of current and its equivalent sine-wave have approximately the same maximum value.

5. The angle of hysteretic advance, that is, the phase difference between magnetism and equivalent sine-wave of M. M. F., is a maximum for the closed magnetic circuit, and depends then only upon the magnetic constants of the iron: the permeability μ and the coefficient of hysteresis η, and upon the maximum magnetic induction, by the equation:

$$\sin \alpha = \frac{4 \mu \eta}{B^{.4}}.$$

6. The effect of hysteresis can be represented by an admittance: $Y = \rho + j\,\sigma$, or an impedance: $U = r - j\,s$.

7. The hysteretic admittance, or impedance, varies with the magnetic induction, that is, with the E. M. F., etc.

8. The hysteretic conductance ρ is proportional to the coefficient of hysteresis η and to the length of the magnetic circuit L, inverse proportional to the .4th power of the E. M. F., E, to the .6th power of frequency N and of cross-section of the magnetic circuit S, and to the 1.6th power of the number of turns of the electric circuit n, thus expressed by the equation:

$$S = \frac{58\, \eta\, L\, 10^3}{E^4\, N^{.6}\, S^6\, n^{1\,6}}.$$

9. The absolute value of hysteretic admittance $v = \sqrt{\rho^2 + \sigma^2}$ is proportional to the magnetic reluctance: $P = P_i + P_a$, and inverse proportional to the frequency N and to the square of the number of turns n, hence expressed by the equation:

$$v = \frac{(P_i + P_a)\, 10^8}{2\, \pi\, N\, n^2}.$$

10. In an ironclad circuit, the absolute value of admittance is proportional to the length of the magnetic circuit, and inverse proportional to cross-section S, frequency N, permeability μ, and square of the number of turns n:

$$v_1 = \frac{127\, L\, 10^6}{n^2\, S\, N\, \mu}.$$

11. In an open magnetic circuit, the conductance ρ is the same as in a closed magnetic circuit of the same iron part.

12. In an open magnetic circuit, the admittance v is practically constant, if the length of the air-gap is at least $\frac{1}{100}$ of the length of the magnetic circuit, and saturation is not approached.

13. In a closed magnetic circuit, conductance, susceptance and admittance can be assumed as constant in a limited range only.

14. From the shape and the dimensions of the circuits, and the magnetic constants of the iron, all the electric constants: ρ, σ, v; r, s, u, can be calculated.

II.—*Foucault or Eddy-Currents.*

While magnetic hysteresis or molecular friction is a magnetic phenomenon, eddy-currents are rather an electrical phenomenon. When passing through the iron, the magnetic field causes a loss of energy by hysteresis, which, however, does not react magnetically upon the field. When impinging upon an electric conductor, the magnetic field induces a current therein. The M. M. F. of this current reacts upon and affects the magnetic field more or less, and thus an alternating magnetic field cannot penetrate deeply into a solid conductor, but a kind of screening effect is produced, which makes solid masses of iron unsuitable for alternating fields, and necessitates the use of laminated iron, or iron wire, as the carrier of magnetism.

The eddy-currents are true electric currents, though flowing in minute circuits, and follow all the laws of electric circuits.

Their E. M. F. is proportional to the intensity of magnetization B, and to the frequency N.

Thus the eddy-currents are proportional to the magnetization B, the frequency N, and the electric conductivity γ of the iron, hence can be expressed by:

$$c = \beta\, \gamma\, B\, N.$$

The power consumed by the eddy-currents is proportional to their square, and inversely proportional to the electric conductivity, hence can be expressed by:

$$W = \rho\, \gamma\, B^2\, N^2,$$

or, since $B\,N$ is proportional to the induced E. M. F., E, by the equation:

$$E = \sqrt{2}\, \pi\, S\, n\, N\, B\, 10^{-8}.$$

"*The loss of power by eddy-currents is proportional to the square of the* E. M. F., *and proportional to the electric conductivity of the iron:*

$$W = a\, E^2\, \gamma."$$

Hence that component of the effective conductance, which is due to eddy-currents, is:

$$\rho = \frac{W}{E^2} = a\, \gamma;$$

that is:

"*The equivalent conductance due to eddy-currents in the iron is a constant of the magnetic circuit, independent of* E. M. F., *frequency, etc., but proportional to the electric conductivity of the iron* γ."

Eddy-currents cause an advance of phase of the current also, like magnetic hysteresis, by an *angle of advance*, β, but unlike hysteresis, eddy-currents in general do not distort the current wave.

The angle of advance of phase due to eddy-currents is:

$$\sin \beta = \frac{\rho}{v},$$

where v = absolute admittance of the circuit, ρ = eddy-current conductance.

While the equivalent conductance, ρ due to eddy-currents, is a constant of the circuit, independent of E. M. F., frequency, etc., the loss of power by eddy-currents is proportional to the square of the E. M. F., of self-induction, hence proportional to the square of frequency and the square of magnetization.

Of eddy-currents, only the energy component, $\rho\, E$, is of interest, since the wattless component is identical with the wattless component of hysteresis, discussed before.

The calculation of the losses of power by eddy-currents is the following:

Let V = volume of iron,

B = maximum magnetic induction,

$N =$ frequency,
$\gamma =$ electric conductivity of iron,
$\varepsilon =$ coefficient of eddy-currents.

The loss of energy per cm.^{3}, in ergs per cycle, is :

$$h = a \gamma N B^2,$$

hence, the total loss of power by eddy-currents is :

$$W = \varepsilon \gamma \; V N^2 B^2 \, 10^{-7} \text{ watts},$$

and the equivalent conductance due to eddy-currents :

$$\rho = \frac{W}{E^2} = \frac{10 \, \varepsilon \gamma L}{2 \pi^2 S n^2} = \frac{.507 \, \varepsilon \gamma L}{S n^2},$$

where :

$L =$ length of magnetic circuit,
$S =$ section of magnetic circuit,
$n =$ number of turns of electric circuit.

The coefficient of eddy currents, ε, depends merely upon the shape of the constituent parts of the magnetic circuit, that is, whether iron plates or wire, and thickness of plates or diameter of wire, etc.

The two most important cases are :

(*a*) laminated iron,
(*b*) iron wire.

a. Laminated Iron.

Let, in Fig. 14,

$d =$ thickness of the iron plates,
$B =$ maximum magnetic induction,
$N =$ frequency,
$\gamma =$ electric conductivity of the iron.

Then, if x is the distance of a zone, $d\,x$, from the center of the sheet, the conductance of a zone of thickness, $d\,x$, and one cm. length and width is, $\gamma \; d\,x$; and the magnetic flux cut by this zone is, $B\,x$. Hence, the E. M. F. induced in this zone is :

$$\delta E = \sqrt{2} \, \pi N B x \text{ (C. G. S.) units.}$$

This E. M. F. produces the current :

$$d\,C = \delta E \gamma \, d\,x = \sqrt{2} \, \pi N B \gamma x \, d\,x \text{ (C. G. S.) units,}$$

if the thickness of the plate is negligible compared with the length, so that the current can be assumed as flowing parallel to the sheet, in the one direction at the one, in the other direction at the other side.

The power consumed by the induced current in this zone, $d\, x$, is:

$d\, W = \delta\, E\, d\, C = 2\, \pi^2\, N^2\, B^2\, \gamma\, x^2\, d\, x$ (C. G. S.) units or erg seconds, and, consequently, the total power consumed in one cm.^{2} of the sheet of thickness, d:

$$\delta\, W = \int_{-\frac{d}{2}}^{+\frac{d}{2}} d\, W = 2\, \pi^2\, N^2\, B^2\, \gamma \int_{-\frac{d}{2}}^{+\frac{d}{2}} x^2\, d\, k$$

$$= \frac{\pi^2\, N^2\, B^2\, \gamma\, d^3}{6} \text{ (C. G. S.) units,}$$

hence, the power consumed per cm.^{3} of iron:

$$w = \frac{\delta\, W}{d} = \frac{\pi^2\, N^2\, B^2\, \gamma\, d^2}{6} \text{ (C. G. S.) units or erg seconds,}$$

and the energy consumed per cycle and per cm.^{3} of iron;

$$h = \frac{w}{N} = \frac{\pi^2\, \gamma\, d^2\, N\, B^2}{6} \text{ ergs.}$$

Thus, the coefficient of eddy-currents for laminated iron is:

$$\varepsilon = \frac{\pi^2\, d^2}{6} = 1.645\, d^2,$$

where γ is expressed in (C. G. S.) units. Hence, if γ is expressed in practical units, or mho-centimetres, it is:

$$\varepsilon = \frac{\pi^2\, d^2\, 10^{-9}}{6} = 1.645\, d^2\, 10^{-9}.$$

Substituting for the conductivity of sheet iron the approximate value:

$$\gamma = 10^5,$$

we get:

Coefficient of eddy-currents for laminated iron:

$$\varepsilon = \frac{\pi^2}{6}\, d^2\, 10^{-9} = 1.645\, d^2\, 10^{-9}.$$

Loss of energy per cm.^{3} and cycle:

$$h = \varepsilon\, \gamma\, N\, B^2 = \frac{\pi^2}{6}\, d^2\, \gamma\, N\, B^2\, 10^{-9} = 1.645\, d^2\, \gamma\, N\, B^2\, 10^{-9} \text{ ergs}$$

$$= 1.645\, d^2\, N\, B^2\, 10^{-4} \text{ ergs;}$$

or,

$$h = \varepsilon\, \gamma\, N\, B^2\, 10^{-7} = 1.645\, d^2\, N\, B^2\, 10^{-11} \text{ joules.}$$

Loss of power per cm.^{3} at frequency N:

$$w = N\,h = \varepsilon\,\gamma\,N^2\,B^2\,10^{-7} = 1.645\,d^2\,N^2\,B^2\,10^{-11}\text{ watts,}$$

and, total loss of power, in volume V:

$$W = V\,w = 1.645\;V\,d^2\;N^2\,B^2\,10^{-11}\text{ watts.}$$

Instance:

$d = 1$ mm. $= .1$ cm. $N = 100$. $B = 5{,}000$. $V = 1{,}000$ cm.^{3}

$$\varepsilon = 1{,}645 \times 10^{-11},$$
$$h = 4110\text{ ergs} = .000411\text{ joules,}$$
$$w = .0411\text{ watts,}$$
$$W = 41.1\text{ watts.}$$

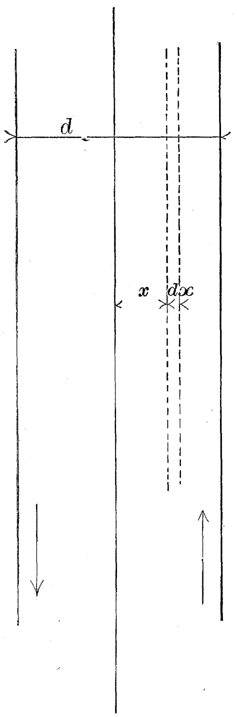


FIG. 14.

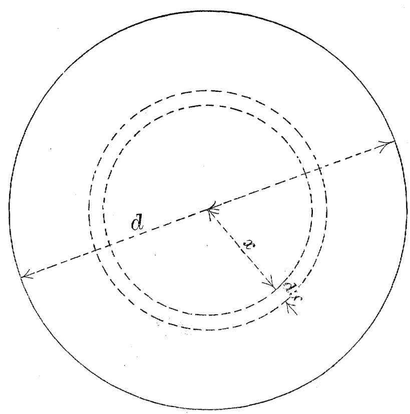


FIG. 15.

b. Iron Wire.—Let, in Fig. 15, d = diameter of wire; then, if x is the radius of a circular zone of thickness, dx, and one cm. length, the conductance of this zone is, $\frac{\gamma\,d\,x}{2\,\pi\,x}$, and the magnetic flux enclosed by the zone is $B\,x^2\,\pi$.

Hence, the E. M. F. induced in this zone is:

$$\delta\,E = \sqrt{2}\,\pi^2\,N\,B\,x^2\text{ (C. G. S.) units,}$$

and the current produced thereby:

$$d\,C = \frac{\gamma\,d\,x}{2\,\pi\,x} \times \sqrt{2}\,\pi^2\,N\,B\,x^2$$

$$= \frac{\sqrt{2}\,\pi}{2}\,\gamma\,N\,B\,x\,d\,x \text{ (c. g. s.) units,}$$

hence, the power consumed in this zone:

$$d\,W = \delta\,E\,d\,C = \pi^3\,\gamma\,N^2\,B^2\,x^3\,d\,x \text{ (c. g. s.) units,}$$

and, consequently, the total power consumed in one cm. length of wire:

$$\delta\,W = \int_0^{\frac{d}{2}} d\,W = \pi^3\,\gamma\,N^2\,B^2 \int_0^{\frac{d}{2}} x^3\,d\,x$$

$$= \frac{\pi^3}{6^4}\,\gamma\,N^2\,B^2\,d^4 \text{ (c. g. s.) units.}$$

Since the volume of one cm. length of wire is:

$$v = \frac{d^2\,\pi}{4},$$

power consumed in one cm.^{3} of iron is:

$$w = \frac{\delta\,W}{v} = \frac{\pi^2}{16}\,\gamma\,N^2\,B^2\,d^2 \text{ (c. g. s.) units or erg seconds,}$$

and the energy consumed per cycle and cm.^{3} of iron:

$$h = \frac{w}{N} = \frac{\pi^2}{16}\,\gamma\,N\,B^2 \text{ ergs.}$$

Thus, the coefficient of eddy-currents for iron wire is:

$$\varepsilon = \frac{\pi^2}{16}\,d^2 = .617\,d^2,$$

or, if γ is expressed in practical units or mho centimetres $= 10^{-9}$ absolute units:

$$\varepsilon = \frac{\pi^2}{16}\,d^2\,10^{-9} = .617\,d^2\,10^{-9}.$$

Substituting:

$$\gamma = 10^5,$$

we get:

Coefficient of eddy-currents for iron wire:

$$\varepsilon = \frac{\pi^2}{16}\,d^2\,10^{-9} = .617\,d^2\,10^{-9}.$$

Loss of energy per cm.^{3} of iron, and per cycle:

$$h = \varepsilon\,\gamma\,N\,B^2 = \frac{\pi^2}{16}\,d^2\,\gamma\,N\,B^2\,10^9 = .617\,d^2\,\gamma\,N\,B^2\,10^{-9}$$

$$= .617\ d^2\ N\ B^2\ 10^{-4} \text{ ergs},$$

$$= \varepsilon\ \gamma\ N\ B^2\ 10^{-7} = .617\ d^2\ N\ B^2\ 10^{-11} \text{ joules}.$$

Loss of power per cm.^{3}, at frequency N:

$$w = N\ h = \varepsilon\ \gamma\ N^2\ B^2\ 10^{-7} = .617\ d^2\ N^2\ B^2\ 10^{-11} \text{ watts},$$

and, total loss of power, in volume V:

$$W = V\ w = .617\ V\ d^2\ N^2\ B^2\ 10^{-11} \text{ watts}.$$

Instance:

$d = 1$ mm. $= 1$ cm. $N = 100$. $B = 5000$. $V = 1000$ cm.^{3}

$\varepsilon = .617 \times 10^{-11}$,

$h = 1540$ ergs $= .000154$ joules,

$w = .0154$ watts,

$W = 15.4$ watts, hence very much less than in sheet iron of equal thickness.

Comparison of sheet iron and iron wire.

If

$d_1 =$ thickness of lamination of sheet iron, and

$d_2 =$ diameter of iron wire, it is:

coefficient of eddies in sheet iron:

$$\varepsilon_1 = \frac{\pi^2}{6}\ d_1^2\ 10^{-9};$$

coefficient of eddies in iron wire;

$$\varepsilon_2 = \frac{\pi^2}{16}\ d_2^2\ 10^{-9}.$$

The loss of power is equal in both—other things being equal—if $\varepsilon_1 = \varepsilon_2$, that is:

$$d_2^2 = \frac{8}{3}\ d_1^2,$$

or,

$$d_2 = 1.63\ d_1.$$

That is:

The diameter of iron wire can be 1.63 times, or roughly $1\frac{2}{3}$ as large as the thickness of laminated iron, to give the same loss of energy by eddy-currents.

Alternating Current Transformer.

The relative proportions of wire and lamina are shown in Fig. 16.

The same formulas obviously apply to the eddy-currents in masses of any other material, substituting for γ the proper value.

As an instance of the calculation of ferric inductances, the general equations of the alternate current transformer may be given.

Let:

$Y_0 = \rho_0 + j\,\sigma_0 =$ hysteretic admittance of primary coil,

$U_0 = r_0 - j\,s_0 =$ impedance of primary coil,

$U_1 = r_1 - j\,s_1 =$ impedance of secondary coil,

where the inductances, s_0 and s_1, refer to the flow of true self-induction, that is, that magnetism, which surrounds one of the transformer coils only, but not the other.

Let $a = \frac{n_0}{n_1} =$ ratio of turns of primary and of secondary coil.

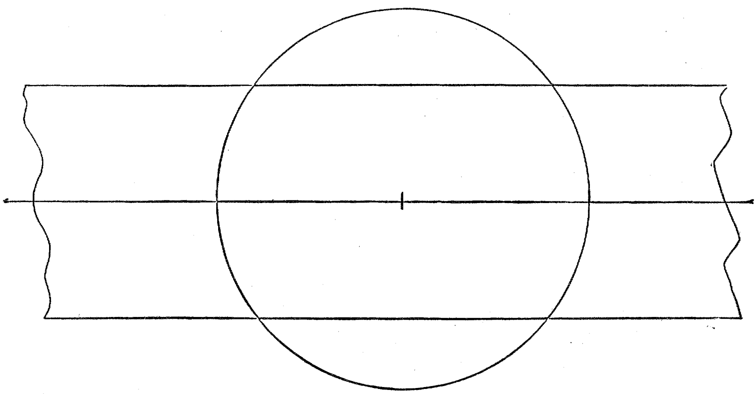


Fig. 16.

Then, denoting the terminal voltage of primary and of secondary coil by E_0 and E_1, and the E. M. F.'s induced in these coils by the magnetic flux surrounding them by E_0^1 and E_1^1, it is:

$$E_0^1 = a\,E_1^1.$$

Denoting the total admittance of the secondary circuit—including the internal impedance of the secondary coil—by:

$$Y_1 = \rho_1 + j\,\sigma_1$$

the secondary current is:

$$C_1 = Y_1\,E_1^1,$$

consisting of the energy component, $\rho_1\,E_1^1$, and the wattless component, $\sigma_1\,E_1^1$.

Hereto corresponds the component of primary current, by the ratio of turns:

$$C = \frac{C_1}{a} = \frac{Y_1\,E_1^1}{a}.$$

The primary exciting current (current at open secondary circuit) is:

$$
\begin{aligned}
C_{00} &= Y_0 \, E_0^1 \\
&= Y_0 \, a \, E_1^1,
\end{aligned}
$$

hence, the total primary current:

$$C_0 = C + C_{00} = \frac{E_1^1}{a} \left(Y_1 + a^2 \, Y_0 \right),$$

and, the ratio of primary and of secondary current:

$$\frac{C_0}{C_1} = \frac{1}{a} \left(1 + \frac{a^2 \, Y_0}{Y_1} \right).$$

The terminal voltage of the secondary coil is:

$$
\begin{aligned}
E_1 &= E_1 - U_1 \, C_1 \\
&= E_1^1 \left(1 - U_1 \, Y_1 \right).
\end{aligned}
$$

The terminal voltage of the primary coil is:

$$
\begin{aligned}
E_0 &= E_0 + U_0 \, C_0 \\
&= a \, E_1^1 + \frac{E_1^1 \, U_0}{a} \left(Y_1 + a^2 \, Y_0 \right) \\
&= a \, E_1^1 \left(1 + U_0 \, Y_0 = \frac{U_0 \, Y_0}{a^2} \right),
\end{aligned}
$$

hence the ratio of primary and of secondary terminal voltage:

$$\frac{E_0}{E_1} = a \, \frac{1 + U_0 \, Y_0 + \dfrac{U_0 \, Y_1}{a^2}}{1 - U_1 \, Y_1}.$$

That is, if, at the primary impressed E. M. F., E_0, the secondary circuit is closed by the admittance Y_1, it is:

Ratio of transformation of E. M. F.'s:

$$\frac{E_0}{E_1} = a \, \frac{1 + U_0 \, Y_0 + \dfrac{U_0 \, Y_1}{a^2}}{1 - U_1 \, Y_1}.$$

Ratio of transformation of currents:

$$\frac{C_0}{C_1} = \frac{1}{a} \left(1 + \frac{a^2 \, Y_0}{Y_1} \right)$$

where these ratios are complex quantities of the form:

$$p \, (\cos \tilde{\omega} + j \sin \tilde{\omega}),$$

thus denoting the numerical value of the ratio of transformation by the vector p, and the phase difference between primary and secondary circuit by angle $\tilde{\omega}$.

Discussion.

Dr. Bedell:—Mr. President, I would like to comment on the remarks of Mr. Steinmetz on the idea of the equivalent sine-wave.[1]

The distorted nature of actual current waves has been particularly emphasized in the valuable paper to which we have listened this morning. Although we know that this distortion exists, we still find it convenient to make what we call the "sine assumption." Now this sine assumption does not mean as commonly supposed, that we consider that the current is actually harmonic. When we assume a harmonic current we simply assume a harmonic current to which the actual current is equivalent. This has, I think, been already pointed out by Mr. Steinmetz as well as by Dr. Crehore and myself.[2] The sine assumption with this meaning has proved very useful in combining experimental and theoretical results, and is not open to the criticism which is often given, that we do not have perfect sine currents under ordinary circumstances.

I would like to question Mr. Steinmetz in regard to one other point; that is in regard to the hysteresis loss in the revolving armature as compared to the hysteresis loss in the transformer, and I would like to ask how he applies his law to the two cases.

Mr. Steinmetz:—With regard to the loss of energy by magnetic friction in a rotary magnetic field, as for instance in the revolving armature of a bipolar smooth core dynamo, I found no essential difference with the loss in an alternating field. But I found that occasionally the observed core loss in the armature of a machine is not the molecular magnetic friction only, but superimposed upon it are eddy-current losses in the iron, the shields, etc., and in the conductors, which losses are proportional to the square of the magnetization. Thus, the observed core loss sometimes rises with a power higher than 1.6, sometimes nearly approaching the square. But by laminating the iron very carefully, designing the mechanical construction so as to expose no solid metal to the alternating field, and shaping the conductors so as to exclude eddy currents, I always got curves very nearly proportional to the 1.6 power, like the one I show here for a variation of voltage up to 9,000 volts, that is, up to very high magnetic densities (about $B = 19{,}000$). There you see the curve of 1.6 power in drawn line, very closely representing the observed core losses. The points marked by crosses are the observed values of the power consumed by the generator less the friction of the belt. So I think the law holds for generators just the same, and therefore I believe the law applies not to the hysteresis loss, but to the loss by molecular magnetic friction, since in the generators we probably have no hysteresis. I took

1. Transactions, vol. xi, p. 46.
2. Geometrical Proof of the Three-ammeter Method of Measuring Power. *Physical Review*, vol. 1, No. 1, p. 61.

pains once to find out if there is a lag of the magnetism behind the resultant magnetizing force in a generator, which would distort the wave of electromotive force, but I did not find anything of the kind. I found no hysteretic lag. Thus the total loss of energy, which as you see here in this case is many kilowatts, is supplied directly by the mechanical power, in which way I am not able to say, but it is not in the form of a hysteretic loop, at least not a hysteretic loop of noticeable size.

PROF. ANTHONY:—I would like to ask one question simply to see whether I have properly understood Mr. Steinmetz. I understand him to mean when he speaks of equivalent sine-curves the several component sine-curves into which the distorted curve could be resolved.

MR. STEINMETZ:—No, I meant a true sine-wave of current of the same frequency as the fundamental, the same effective intensity as the total distorted wave, and shifted against the equivalent sine-wave of electromotive force by such an angle that its power in watts equals that of the distorted wave. I can say that the eqivalent sine wave is not identical with the fundamental sine-wave, except in the case where the sum total of higher harmonics is wattless, because the equivalent sine-wave includes the energy of the higher harmonics also, and thus the remainder, or the difference between distorted wave and equivalent sine-wave, generally includes a component of the same frequency as the fundamental.

MR. KENNELLY:—This paper seems to me to be valuable, first, for its bearing upon the subject of hysteresis and its nature, and, secondly, upon the practical determination of inductances or of equivalent inductances in coils containing iron, such as transformers. The main point, it seems to me, can be stated in a very few words. When the current is no longer a sinusoidal wave, if it becomes distorted by the action of iron in the circuit, it is a complicated wave such as shown at F in the Figs. 7 and 8, etc. But the ammeter or dynamometer which is used to measure that distorted current will show some effective current strength which might be attributable to a pure sinusoidal current. It would show a current strength in amperes which would be represented by the curve C, so that the real current F, whose shape can only be determined by a long series of experiments, has an equivalent representation in the dynamometer such as would be produced by a current of the pure sine shape of C. But if you do not carry the magnetization too high, the amplitude of the pure sine-wave C, such as the dynamometer, would lead you to suppose exists, and the amplitude of the actual distorted wave F are equal. This, if true, is an important and valuable proposition, because it gives you the maximum number of ampere-turns on the magnetic circuit, the maximum cyclic magneto-motive force. But it is pointed out that when you get beyond 10 kilogausses in your iron, you will no longer have this relation maintained. That is in agreement with the observations in Dr. Pupin's valuable paper read this morning, where it is shown that the harmonics of his primary currents remained proportional to the current strength if he did not go up too far in flux intensity, and that is bearing directly on this paper. If you do not go beyond 10 kilogausses you will probably have those two wave crests on the same line.

Dr. Bedell:—There is one point to which a little further attention might be given, and that is in regard to the lag of the current behind the electromotive force when the current and electromotive force are not harmonic. Those who have had occasion to make a study of currents which are not strictly harmonic and desire to find the phase relations, have doubtless met this question. The phase difference between the maximum values and zero values or any other values of the current and electromotive force are not the same. The use of the equivalent sine function is the solution of this question. We assume an equivalent electromotive force which is harmonic and has the same mean square value as the electromotive force which is not harmonic, and we do the same with the current. We then set these two with such an angle of lag between them that the power is the same. Now we can get our power from other measurements and by these measurements of the power, the current and the electromotive force, we thus have a measure of the angle of lag in degrees, which cannot be otherwise obtained when the currents are far from being harmonic. In other words, we say the power is $W = E\,I \cos\theta$. By measuring W, E and I, *we may find a value for the angle* θ, *whether the current is harmonic or not.*

Mr. Steinmetz:—I would like to point out one thing here, not to allow a misconception to arise. This dissolving of the distorted wave into an equivalent sine-wave, and a wattless remainder is not identical with the dissolving of it by Fourier's theorem into a series of sine-waves, because the equivalent sine-wave c is not the fundamental component of the total wave, but the wattless remainder of apparently triple frequency, shown here, may contain a term of simple frequency.

To fix a definition of this equivalent sine-wave, it is "a sine-wave of equal effective intensity and equal power with the true wave." If you take a wave of electromotive force, for instance, and a wave of current, then the higher harmonics may, but need not, be powerless. This is especially the case if you have the current distorted by hysteresis.

Dr. Pupin:—I might say a word or two on this paper of Mr. Steinmetz, a very interesting paper indeed. In the first place in studying these harmonics in the course of last year I had, especially, Prof. Rowland's paper of 1892 to guide me, in which a radically different view was taken from that of Prof. Fleming. Comparing these two views with my own work, it seemed to me that they could be reconciled to a certain extent in this way:

The hysteresis loop reminds us of two things: In the first place, of the loss of energy, and, in the second place, of the variation of permeability. Now Dr. Fleming ascribed the generation of harmonics to the action of hysteresis in general, not saying exactly what he meant by it. Hysteresis is a very broad term and may be made to mean a great many things. Prof. Rowland specified his view and ascribed the presence of harmonics to the variation of permeability. Both views, therefore, refer to the hysteresis loop for an explanation of the distortion of the current wave. In the course of a discussion[1] at a meeting of this INSTITUTE, I suggested that the distortion of alternating current waves could be very well studied by studying, with the aid of the hysteresis loop, the process of magnetization and demagnetization during each cycle. Mr. Steinmetz's method is exactly the method to which I referred at that time. I am sorry that Mr. Steinmetz has not explained the details of the method of his investigation and the data obtained by it, which enabled him to plot the harmonics of various frequencies from the hysteretic loop.

Another point that I would like to mention refers to what Mr. Steinmetz calls "molecular friction." The distinction between molecular friction and hysteresis does not seem quite clear from Mr. Steinmetz's paper. I have expressed my opinion on several occasions in the course of this and last year, that there are certain phenomena going on during each complete cycle of magnetization of iron which cannot very well be explained by Foucault current and hysteresis as commonly understood, but which phenomena seem to point out clearly the existence of additional passive resistances. Possibly Mr. Steinmetz means the same thing when he speaks of molecular friction. There is certainly a very marked difference between the action of iron when it forms a closed magnetic circuit and when it does not form such a circuit, especially in its damping action upon a resonating current. Again, certain kinds of iron may have a large hysteretic constant, but only a small damping constant, etc. These differences appear at all magnetizations, even at magnetizations due to telephonic currents, and are especially marked at higher frequencies. There is a certain magnetic sluggishness in every piece of iron, and it is my opinion that this sluggishness is not measured by the hysteretic action as ordinarily understood, nor by Foucault current losses. Now what this sluggishness is, it is difficult to tell. The invention of a new name like "molecular friction" certainly does not advance our knowledge one bit. It may retard it if the new name should lead us to believe that further inquiry into the matter will lead to nothing more than mere commonplace molecular friction.

MR. STEINMETZ:—I think Dr. Pupin is mistaken in his state-

1. See discussion of Dr. Bell's paper, "Practical Properties of Polyphase Apparatus." TRANSACTIONS, vol. xi, p. 46.

ment with regard to the name hysteresis. The word has a well-defined meaning. It was introduced merely to denote the lag of the magnetism behind the magnetomotive force, as the derivation of the word signifies, which lag causes the magnetism as function of an alternating M. M. F. to describe a closed curve, the "loop of hysteresis."

Afterward it was shown by Warburg and Ewing that the area of the hysteretic loop represents energy, and represents the energy expended by the magnetomotive force during the cycle of magnetism, and from this, the erroneous conclusion has been drawn that this hysteretic energy is the energy lost in the iron by molecular magnetic friction, that is, by changing the magnetic state of the iron. That is what I want to make clear—that this conclusion is wrong; that this energy expended by the magnetomotive force is not necessarily the energy wasted in the iron. The energy represented by the hysteretic loop or a part of it may be converted into mechanical motion, or the energy lost in molecular magnetic friction may be supplied by mechanical energy, and the hysteretic loop may collapse, or may expand considerably, so that between the area of the hysteretic loop and the loss of energy in the iron there is no direct relation. I have explained this quite fully and shown by tests in my second paper on hysteresis.[1] Since, however, it seems to have escaped attention, probably due to the length of aforesaid paper, I thought it advisable to discuss it again more fully in my present paper.

Now with regard to the changes of permeability and to hysteresis as producers of higher harmonics, the statement that hysteresis produces higher harmonics, is quite correct. It produces higher harmonics, but change of permeability does the same, or rather, hysteresis is nothing but a change of permeability. Take this case I show here on pages 575-7, Figs. 2 and 4. There you have the loop of hysteresis produced by the variable permeability. What Prof. Pupin means in his statement that hysteresis does not produce higher harmonics is probably that molecular magnetic friction does not necessarily cause higher harmonics, and with that I agree; higher harmonics of current appear only when the molecular magnetic friction causes a variation of permeability in the form of hysteresis. But beside this, there are undoubtedly still other causes, which produce higher harmonics, which are neither change of permeability nor hysteresis.

Of any sluggishness displayed by the iron in changing its magnetic state, I have never found any trace which could not be explained as the effect of the hysteretic loop, and thus do not believe that any such sluggishness or viscous hysteresis exists at ordinary frequencies of a few hundred cycles.

The difference in the action of a closed circuit transformer and an open circuit transformer is fully explained by the fact that the open circuit transformer is at open secondary circuit highly in-

1. TRANSACTIONS, 1892, vol. ix, chapter v, p. 711.

ductive; that is, the current passing through it is almost all idle or wattless current, having a small energy component only. In the closed circuit transformer the magnetizing current is so small that the exciting current is largely energy current—hysteretic energy current—the angle of lag being even at open secondary circuit only from 40 to 60 degrees. This explains that no resonance can be produced by a closed circuit transformer, since resonance presupposes a highly inductive circuit, which the transformer is not.

Can anyone inform me when the relation between the distortion of the alternating current wave and the hysteretic loop was first stated by Fleming?

DR. PUPIN:—It is in the second volume of his book.

MR. STEINMETZ:—If you go back, for instance, in our TRANSACTIONS to Prof. Ryan's paper[1], I think it came out in 1889, he plotted the hysteretic loop from the wave shape of the current, thereby making use of the feature, that the distortion of the current wave is due to the hysteresis, and that the hysteretic loop can be reproduced from the distortion. What I did here was merely to reverse the process. But this has probably also been done before that. Thus I did not need to give a very explicit description. But I think the credit of having first shown this relation between distortion and hysteresis is due to Prof. Ryan.

DR. PUPIN:—I do not think that Prof. Ryan employed the hysteretic loop for plotting the various harmonics. If I remember correctly, the curves of current and electromotive force were plotted by sliding contact, and then the harmonics were determined by the ordinary method of harmonic analysis.

MR. STEINMETZ:—I think he did it directly from the shape of the wave of the current, not from the watt curve, if I am not mistaken. I really do not remember exactly.

DR. PUPIN:—Perhaps Dr. Bedell can tell us?

DR. BEDELL:—I think that the relation between hysteresis and the shape of the current curve was first brought out by Professor Ryan and described by him in his paper[2] on transformers before this INSTITUTE in 1889. In conjunction with Professor Merritt, he constructed a hysteresis loop from the curves of current and electromotive force taken by the method of instantaneous contact. From these curves for current and electromotive force, they did construct a watt curve, as Dr. Pupin states, but they made no use of this in determining the hysteresis loop, obtaining the latter directly from the instantaneous curves. That this relation between the current curve and the hysteresis loop existed had been pointed out a little before this time by Dr. Hopkinson,[3] who showed the relation by means of a graphical construction

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1. TRANSACTIONS, vol. vii. p. 1.

2. *Ibid.*

3. Hopkinson: "Induction Coils or Transformers." *Proceedings of the Royal Society*, Feb. 17, 1887. Also given on p. 184 of his re-printed papers.

involving three dimensions which was based upon some results obtained analytically from fundamental differential equations. As far as I am aware, however, it has not been until recently that Dr. Hopkinson has made any investigations in this direction. In a paper[1] published a year ago or so, he described an extended investigation in which hysteresis loops were obtained for different frequencies from curves taken by the method of instantaneous contact. This is, I think, the most complete investigation upon this line of work which has thus far been published; but it differs from the work of Ryan and Merritt only in its greater completeness.

In a paper[2] published about a year before the work done by Professors Ryan and Merritt, Dr. Sumpner showed a very pretty graphical construction for obtaining the current curve when we are given the electromotive force and a curve showing the relation between the current and the time-constant of the circuit. This is at least of considerable theoretical interest; but he could have carried it further. Furthermore, if I remember rightly, he did not take a different time-constant curve for his ascending and descending values. He did not accomplish by his method, however, that which was done by Ryan and Merritt, viz., the construction of a hysteresis loop from the current curve.

Dielectric hysteresis, as well as magnetic, affects the shape of the current curve. I have already had the pleasure of calling the attention of the INSTITUTE to this relation, and of describing a method for determining the hysteresis loop for a condenser. Such a loop is given in the TRANSACTIONS[3] for last year.

Each one of the papers I have referred to has contributed something of value to the question at hand, and due credit should be given to each of the several writers; but I think that to Professors Ryan and Merritt must be given the credit for the practical development of the subject. The harmonic analysis of these curves according to Fourier's theorem was worked out by them and is given by Dr. Fleming in the second volume[4] of his work on transformers. The fundamental together with the third and fifth harmonics were found to closely represent the actual distorted wave.

In conclusion I would say that I consider all this work of particular significance, combining, as it does, observed phenomena and mathematical analysis. Theoretical deductions are always based upon certain premises, and in many cases these premises have consisted of artificial conditions. The conclusions are rigorously true under the assumed conditions, but the conditions are unobtainable. We are acquiring greater ability in making our

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1. Drs. J. and B. Hopkinson: London *Electrician*, Sept. 9, 1892. Also: "Gray's Absolute Measurements in Electricity and Magnetism," vol. ii, p. 752.

2. Sumpner: *Philosophical Magazine*, June, 1888, p. 468.

3. TRANSACTIONS, vol. x, p. 525.

4. "Alternate Current Transformer," vol. ii, p. 452.

conditions accord with facts. It has often happened that our conclusions are only true in case hysteresis be absent and the current is a true sine-wave. But this need not be: we may make quantitative assumptions as to the hysteresis present, and may assume the presence of such harmonics in addition to the fundamental wave as occasion demands; predetermination becomes possible, and our work becomes definite and exact.