Supplementary Material:
Extended Derivations and Automated Verification
The Foundation of Projection Relativity

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1 Introduction to the Supplemental Material

This supplementary document aggregates the foundational equations of Projection Relativity, integrating them with the complete automated verification logs. It provides the exhaustive, step-by-step mathematical derivations that support the core framework, serving both as a detailed analytical expansion of the main manuscript and a formal repository for the computational test harness results.

The formalisms and algorithmic outputs presented herein detail the internal metric tensor, spectral operators, modal expansions, projection maps, compact boundary closure conditions, and the derived-status matrix. Crucially, the appended symbolic regression transcripts and numerical testing suites establish rigorous computational validation for the spectral gaps, topological invariants, and macroscopic boundary conditions derived in the text.

Table of Contents

1 Introduction to the Supplemental Material **1**

2 Master Equation Set **5**

2.1 Master Equation Matrix Expansion . . . 5

2.1.1 Internal Metric, Measure, and Hilbert Space . . . 5

2.1.2 Laplace–Beltrami Operator . . . 6

2.1.3 Radial Spectral Operator . . . 6

2.1.4 Compact Phase Operator . . . 6

2.1.5 Full Internal Spectral Operator . . . 7

2.1.6 Master Field Expansion . . . 7

2.1.7 Master Action . . . 7
2.1.8 Euler–Lagrange Spectral Equation . . . 8
2.1.9 Projection-Sector Dictionary . . . 8
2.1.10 Spectral Gap Structure . . . 8
2.1.11 Compact Boundary-Orientation Fine-Structure Closure . . . 9
2.1.12 Trace alphabet . . . 9
2.1.13 Interior compact recurrence. . . . 9
2.1.14 Admissible terminal words. . . . 9
2.1.15 Uniqueness of the terminal cofactor. . . . 10
2.1.16 Terminal cofactor. . . . 10
2.1.17 Boundary-resolved compact return. . . . 10
2.1.18 Finite-rank projection and leakage. . . . 11
2.1.19 Full Master Chain . . . 11
2.2 Status of Assumptions and Derived Quantities . . . 11

3 Gravitational Projection Chain **17**
3.1 Internal Mode Amplitudes . . . 17
3.2 Metric Seed Tensor . . . 17
3.3 Determinant-Normalized Effective Metric . . . 18
3.4 Trace-Free Weak-Field Limit . . . 18
3.5 Finite-Core Saturation . . . 19
3.6 Exterior Kerr Recovery . . . 20
3.7 Closed Gravitational Projection Sequence . . . 21

4 Displacement Projection Chain **21**
4.1 Master-Field Projection . . . 21
4.2 Rank-One Projection Form . . . 22
4.3 Amplitude–Phase Decomposition . . . 22
4.4 Displacement-Sector Action . . . 23
4.5 Stable Nonzero Displacement Vacuum . . . 23
4.6 Displacement Fluctuation and Curvature Scale . . . 24
4.7 Spectral Normalization . . . 24
4.8 Effective Matter Inertia . . . 25
4.9 Displacement Stress-Energy and Gravitational Source Response . . . 25
4.10 Displacement–Phase Consistency . . . 26
4.11 Stability Closure . . . 26

4.12 Closed Displacement Projection Sequence . . . 26

5 Electromagnetic Projection Chain **27**

5.1 Compact Phase Geometry . . . 27

5.2 Compact Phase Operator and Winding Modes . . . 28

5.3 Compact Phase Projection . . . 28

5.4 Gauge-Invariant Compact Phase Gradient . . . 29

5.5 Compact Phase Kinetic Split . . . 29

5.6 Gauge Curvature and Field-Strength Invariance . . . 30

5.7 Compact Winding Stiffness and Gauge Kinetic Normalization . . . 30

5.8 Compact/Radial Stationarity . . . 31

5.9 Finite-Rank Compact Boundary Closure . . . 31

5.10 Boundary-Resolved Compact Electromagnetic Normalization . . . 32

5.11 Magnetic Flux and Area-Law Diagnostic . . . 33

5.12 Closed Electromagnetic Projection Sequence . . . 33

5.12.1 Compact-Phase Magnetic Area Law . . . 34

5.12.2 Step 13: Electromagnetic Projection Summary . . . 34

6 Projection Propagator and Spectral Self-Energy Chain **36**

6.1 Bare Massless Spin–2 Propagator . . . 36

6.2 Internal Spectral Density . . . 36

6.3 Bare Self-Energy Kernel . . . 37

6.4 Massless-Pole Subtraction . . . 37

6.5 Low-Energy Expansion . . . 37

6.6 Corrected Propagator . . . 38

6.7 Massless-Pole Residue and Ghost Freedom . . . 38

6.8 Newton Normalization . . . 38

6.9 Newton-Normalized Residual Kernel . . . 39

6.10 Relative Projection Correction . . . 39

6.11 Euclidean Stability Check . . . 40

6.12 Low-Energy Exterior Recovery . . . 40

6.13 Closed Propagator Stability Sequence . . . 40

7 Gravitational-Wave Projection Chain **41**

7.1 Metric Perturbation and Radiative Projection . . . 41

7.2 Exterior Kerr Inheritance . . . 42

7.3 Projection-Corrected Exterior Wave Operator 42
7.4 Gap-Protected Low-Energy Recovery 42
7.5 Luminal Low-Energy Propagation 43
7.6 Response Function Expansion 43
7.7 Observable Ringdown Decomposition 44
7.8 Residual Suppression Bound 44
7.9 Finite-Core Restoration Interpretation 44
7.10 Current Observational Status 45
7.11 Closed Gravitational-Wave Projection Sequence 45

8 Unified Projection Energy Chain **46**

8.1 Master-Field Input 46
8.2 Four-Dimensional Effective Lagrangian 46
8.3 Canonical Momentum and Hamiltonian Density 47
8.4 Total Projection Energy 47
8.5 Noether Conservation 47
8.6 Low-Energy Sector Decoupling 48
8.7 Projection-Mode Mass Scale 48
8.8 Relativistic Kinematic Recovery 49
8.9 Closed Unified Energy Sequence 49

9 Information-Preservation Chain **50**

9.1 Projection Hilbert Space and Master State 50
9.2 Hermitian Projection Hamiltonian 50
9.3 Norm Conservation 51
9.4 Density-Matrix Evolution 51
9.5 Projection-Sector Coarse Graining 52
9.6 Finite-Core Spectral Redistribution 53
9.7 Core Ledger and Energy Split 54
9.8 Exterior Horizon Projection 54
9.9 Information Preservation and Exterior Coarse Graining 54
9.10 Closed Information-Preservation Sequence 55

2 Master Equation Set

This section collects the core equations of Projection Relativity in one place. It is a reference map for the main derivations, not a separate derivation chain. The equations below summarize the internal metric, spectral operators, mode expansions, projection maps, compact boundary closure, and derived-status table used throughout the paper.

The organizing structre is

$$\boxed{\Psi \to O_{\text{int}} \to \{U_{n,m}, \Lambda_{n,m}\} \to \{P_g, P_{\text{disp}}, P_A, P_X, P_\theta\} \to \text{observable projection sectors.}} \tag{1}$$

The detailed sector derivations are given in the main text and in the later appendices. This reference sheet keeps the master structure visible.

2.1 Master Equation Matrix Expansion

This subsection gives the full Projection Relativity matrix expansion used for visualization, bookkeeping, and linear analysis of the framework constructed in Section **??**.

2.1.1 Internal Metric, Measure, and Hilbert Space

The internal manifold is

$$\boxed{\mathcal{M}_{\text{int}} = \mathbb{R}_w \times S^1_\theta, \qquad \xi^A = (w, \theta).} \tag{2}$$

The minimal internal metric is

$$\boxed{G_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & R_A^2 \end{pmatrix}.} \tag{3}$$

The inverse metric is

$$\boxed{G^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & R_A^{-2} \end{pmatrix}.} \tag{4}$$

The determinant is

$$\boxed{\det(G_{AB}) = R_A^2.} \tag{5}$$

The invariant internal measure is

$$\boxed{d\mu_{\text{int}} = \sqrt{\det(G_{AB})}\, dw\, d\theta = R_A\, dw\, d\theta.} \tag{6}$$

The projection Hilbert space is

$$\boxed{\mathcal{H}_P = L^2(\mathcal{M}_{\text{int}}, d\mu_{\text{int}}).} \tag{7}$$

The corresponding inner product is

$$\boxed{\langle f|g\rangle_P = \int_{\mathcal{M}_{\text{int}}} d\mu_{\text{int}}\, f^*(\xi) g(\xi).} \tag{8}$$

2.1.2 Laplace–Beltrami Operator

The scalar Laplace–Beltrami operator on the internal manifold is

$$\boxed{\Delta_G = \frac{1}{\sqrt{\det(G)}}\partial_A\left(\sqrt{\det(G)}G^{AB}\partial_B\right).} \tag{9}$$

For stationary compact radius,

$$\boxed{\partial_w R_A = 0, \qquad \partial_\theta R_A = 0,} \tag{10}$$

this reduces to

$$\boxed{\Delta_G = \partial_w^2 + R_A^{-2}\partial_\theta^2.} \tag{11}$$

2.1.3 Radial Spectral Operator

The radial spectral operator is

$$\boxed{O_X = -\frac{d^2}{dw^2} + V_X(w).} \tag{12}$$

The positive quartic family is

$$\boxed{V_X(w) = \Lambda_X^2\left(1 + a_2 w^2 + a_4 w^4\right).} \tag{13}$$

The Projection Relativity trace-selected radial branch is

$$\boxed{V_{X,\star}(w) = \Lambda_X^2\left(1 + w^2 + \frac{3}{4}w^4\right).} \tag{14}$$

In the reference normalization $\Lambda_X^2 = 1$, this is

$$\boxed{V_{X,\star}(w) = 1 + w^2 + \frac{3}{4}w^4.} \tag{15}$$

The radial eigenvalue equation is

$$\boxed{O_X u_n = \lambda_n u_n.} \tag{16}$$

2.1.4 Compact Phase Operator

The compact phase operator is

$$\boxed{O_\theta = -\frac{1}{R_A^2}\frac{d^2}{d\theta^2}.} \tag{17}$$

The normalized compact eigenmodes are

$$\boxed{v_m(\theta) = \frac{1}{\sqrt{2\pi R_A}}e^{im\theta}, \qquad m \in \mathbb{Z}.} \tag{18}$$

The compact eigenvalues are

$$\boxed{\lambda_m^{(\theta)} = \frac{m^2}{R_A^2}.} \tag{19}$$

2.1.5 Full Internal Spectral Operator

The full internal operator is separable:

$$\boxed{O_{\text{int}} = O_X + O_\theta.} \tag{20}$$

In expanded form,

$$\boxed{O_{\text{int}} = -\frac{d^2}{dw^2} - \frac{1}{R_A^2}\frac{d^2}{d\theta^2} + V_X(w).} \tag{21}$$

The internal eigenmodes factorize as

$$\boxed{U_{n,m}(w,\theta) = u_n(w)\frac{e^{im\theta}}{\sqrt{2\pi R_A}}.} \tag{22}$$

The corresponding eigenvalues are

$$\boxed{\Lambda_{n,m} = \lambda_n + \frac{m^2}{R_A^2}.} \tag{23}$$

A schematic matrix representation is

$$\boxed{O_{\text{int}} = \begin{pmatrix} \Lambda_{0,0} & 0 & 0 & \cdots \\ 0 & \Lambda_{0,1} & 0 & \cdots \\ 0 & 0 & \Lambda_{1,0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.} \tag{24}$$

2.1.6 Master Field Expansion

The master field expands in the internal spectral basis as

$$\boxed{\Psi(x,w,\theta) = \sum_{n,m} c_{n,m}(x) U_{n,m}(w,\theta).} \tag{25}$$

Equivalently, in schematic coefficient-vector form,

$$\boxed{\Psi = \begin{pmatrix} \vdots \\ c_{0,0}(x)U_{0,0} \\ c_{0,1}(x)U_{0,1} \\ c_{1,0}(x)U_{1,0} \\ c_{1,1}(x)U_{1,1} \\ \vdots \end{pmatrix}.} \tag{26}$$

2.1.7 Master Action

Using the mostly-plus convention used in the main text, the master action is

$$\boxed{S_{\text{PR}} = \int d^4x\, d\mu_{\text{int}} \sqrt{-g_{\text{eff}}} \left[-\frac{1}{2} g_{\text{eff}}^{\mu\nu} D_\mu \Psi^\dagger D_\nu \Psi - \frac{1}{2} G^{AB} D_A \Psi^\dagger D_B \Psi - V_{\text{PR}} + \mathcal{L}_{\text{src}} \right].} \tag{27}$$

The internal kinetic structure is

$$\boxed{G^{AB} D_A \Psi^\dagger D_B \Psi = D_w \Psi^\dagger D_w \Psi + R_A^{-2} D_\theta \Psi^\dagger D_\theta \Psi.} \tag{28}$$

2.1.8 Euler–Lagrange Spectral Equation

In the quadratic spectral regime, the master equation is

$$\boxed{(\Box + O_{\text{int}})\Psi = 0.} \tag{29}$$

The internal spectral equation is

$$\boxed{O_{\text{int}} U_{n,m} = \Lambda_{n,m} U_{n,m}.} \tag{30}$$

Substituting the mode expansion gives

$$\boxed{\sum_{n,m} \left[(\Box c_{n,m}) + \Lambda_{n,m} c_{n,m}\right] U_{n,m} = 0.} \tag{31}$$

Projecting onto each orthonormal internal mode gives the observable spacetime equation

$$\boxed{\left(\Box - \Lambda_{n,m}^{\text{phys}}\right) c_{n,m}(x) = 0.} \tag{32}$$

2.1.9 Projection-Sector Dictionary

The principal projection-sector maps are

$$P_g[\Psi] \to g_{\mu\nu}^{\text{eff}}, \tag{33}$$
$$P_{\text{disp}}[\Psi] \to \Phi_{\text{disp}}(x) \to A(x) \to A_{\text{phys}}(x), \tag{34}$$
$$P_A[\Psi] \to \theta(x) \to \mathcal{A}_\mu(x), \tag{35}$$
$$P_X[\Psi] \to \Sigma_X, \tag{36}$$
$$P_\theta[\Psi] \to H_{\text{eff}}(t). \tag{37}$$

2.1.10 Spectral Gap Structure

The first radial spectral gap is

$$\boxed{\mu_{\min}^2 = \lambda_1 - \lambda_0.} \tag{38}$$

In the reference radial branch,

$$\boxed{\lambda_0 = 2.322863529580,} \tag{39}$$

$$\boxed{\lambda_1 = 5.375830272676,} \tag{40}$$

and

$$\boxed{\mu_{\min}^2 = 3.052966743096.} \tag{41}$$

The corresponding positive gap scale is

$$\boxed{\mu_{\min} = 1.747274089288.} \tag{42}$$

The fourth-power suppression scale is

$$\boxed{\mu_{\min}^4 = 9.320605934450.} \tag{43}$$

2.1.11 Compact Boundary-Orientation Fine-Structure Closure

This subsection gives the reference form of the boundary-resolved compact phase closure used in Section **??**. The compact boundary cofactor is not a calibrated correction. It is the terminal-word cofactor of the finite-rank compact return map.

2.1.12 Trace alphabet

The compact and spatial trace fractions are

$$\boxed{T_A = \frac{1}{4}, \qquad T_X = \frac{3}{4}.} \tag{44}$$

Compact returns are written as words in the symbol A, weighted by powers of T_A. Finite-rank boundary crossings are written as words containing one symbol X, weighted by T_X. The finite-rank slice is

$$\boxed{\Pi_{13} = |\phi_1\rangle\langle\phi_1| + |\phi_3\rangle\langle\phi_3|.} \tag{45}$$

2.1.13 Interior compact recurrence.

The compact interior has two primitive return loops, A^3 and A^4. The corresponding transfer matrix is

$$\boxed{M_A = \begin{pmatrix} T_A^3 & T_A^4 \\ 1 & 0 \end{pmatrix}.} \tag{46}$$

The Fredholm determinant is

$$\boxed{D_A = \det(I - M_A) = 1 - T_A^3 - T_A^4.} \tag{47}$$

For $T_A = 1/4$,

$$\boxed{D_A = 1 - \frac{1}{64} - \frac{1}{256} = \frac{251}{256}.} \tag{48}$$

The inverse determinant D_A^{-1} resums recurrent compact interior paths. Terminal boundary words cannot include words already generated by this recurrent interior inverse.

2.1.14 Admissible terminal words.

The finite-rank boundary admits the signed terminal words

$$\boxed{\mathrm{Adm}_{\Pi_{13}} = \{\epsilon,\ A^3,\ A^4,\ XA^5,\ -XA^8\}.} \tag{49}$$

Their weights are

$$\boxed{1, \qquad T_A^3, \qquad T_A^4, \qquad T_X T_A^5, \qquad -T_X T_A^8.} \tag{50}$$

The terms have the following roles:

- ϵ is direct boundary passage.
- A^3 is the primitive spatial-trace compact return.
- A^4 is the primitive full-trace compact return.
- $XA^5 = XA^{2+3}$ is the first compact leave–return pair A^2 followed by the spatial primitive return A^3, weighted by X.
- $-XA^8 = -XA^{4+4}$ is the finite-rank exclusion of double full-trace recirculation outside the Π_{13} slice.

2.1.15 Uniqueness of the terminal cofactor.

The admissible set is unique within the tested finite-rank compact return class under compact closure, finite-rank terminality, parity/orientation, and inclusion–exclusion:

- X, XA, and XA^2 are not compactly closed returns.
- XA^3 and XA^4 are not new boundary leakage terms; their compact parts are already primitive cofactor residues.
- $XA^6 = XA^{2+4}$ is an orientation-neutral full-trace boundary pairing and is absorbed into the recurrent Fredholm denominator.
- $XA^7 = XA^{3+4}$ factors into primitive interior returns and is determinant-resummed.
- $XA^9, XA^{10}, \ldots$ factor through recurrent compact interior paths or higher boundary recirculations and belong to D_A^{-1}, not to the terminal cofactor.

2.1.16 Terminal cofactor.

The terminal cofactor is

$$\boxed{N_{\mathrm{bc}} = 1 + T_A^3 + T_A^4 + T_X(T_A^5 - T_A^8).} \tag{51}$$

For $T_A = 1/4$ and $T_X = 3/4$,

$$\boxed{N_{\mathrm{bc}} = 1 + \frac{1}{64} + \frac{1}{256} + \frac{3}{4}\left(\frac{1}{1024} - \frac{1}{65536}\right) = \frac{267453}{262144}.} \tag{52}$$

2.1.17 Boundary-resolved compact return.

The compact boundary return is the cofactor divided by the Fredholm determinant:

$$\boxed{R_{\mathrm{bc}} = \frac{N_{\mathrm{bc}}}{D_A} = \frac{1 + T_A^3 + T_A^4 + T_X(T_A^5 - T_A^8)}{1 - T_A^3 - T_A^4}.} \tag{53}$$

The compact boundary constant entering the finite-rank spatial projection is

$$\boxed{c_{\mathrm{bc}} = T_X\left[1 + T_A^2 - T_A^6 R_{\mathrm{bc}}\right].} \tag{54}$$

Equivalently,

$$\boxed{c_{\mathrm{bc}} = T_X\left[1 + T_A^2 - T_A^6\frac{1 + T_A^3 + T_A^4 + T_X(T_A^5 - T_A^8)}{1 - T_A^3 - T_A^4}\right].} \tag{55}$$

Substitution gives

$$\boxed{c_{\mathrm{bc}} = \frac{3354902985}{4211081216} = 0.7966844648478991.} \tag{56}$$

2.1.18 Finite-rank projection and leakage.

The boundary-resolved projection operator is

$$\boxed{P_{\text{geom}}^{\text{bc}} = \Pi_{13} K_X^2 \left(c_{\text{bc}} + w^2 + \frac{3}{4} w^4 \right) K_X^2 \Pi_{13}.} \tag{57}$$

The normalized stiffness operator is

$$\boxed{K_X = \frac{O_X - \lambda_0}{\lambda_1 - \lambda_0}.} \tag{58}$$

The finite-rank leakage probabilities are

$$\boxed{p_1 = 7.6528903366 \times 10^{-4}, \qquad p_3 = 1 - p_1.} \tag{59}$$

The compact closure requires the finite-rank radial datum

$$\boxed{\lambda_3 = 13.23388439508096.} \tag{60}$$

With

$$\boxed{q_1 = \lambda_1 - \lambda_0, \qquad q_3 = \lambda_3 - \lambda_0,} \tag{61}$$

the effective compact stiffness is

$$\boxed{q_{\text{bc}} = R_{A,\star}^{-2} = p_1 q_1 + (1 - p_1) q_3.} \tag{62}$$

Numerically,

$$\boxed{q_{\text{bc}} = 10.905007182855176.} \tag{63}$$

The boundary-resolved compact normalization is

$$\boxed{\alpha_{\text{PR},bc}^{-1} = 4\pi q_{\text{bc}} = 137.036361812007.} \tag{64}$$

This value is an output of the boundary-resolved compact projection chain. It is not used to choose c_{bc}, p_1, R_A, or C_A.

2.1.19 Full Master Chain

The complete reference chain is

$$\boxed{\Psi \to O_{\text{int}} \to \{U_{n,m}, \Lambda_{n,m}\} \to \{P_g, P_{\text{disp}}, P_A, P_X, P_\theta\} \to \{g_{\mu\nu}^{\text{eff}}, A_{\text{phys}}, \mathcal{A}_\mu, \Sigma_X, H_{\text{eff}}\}.} \tag{65}$$

2.2 Status of Assumptions and Derived Quantities

The assumptions, constraints, projection maps, and derived quantities introduced in Sections **??** and **??** are summarized below. Quantities labeled "Postulated" are part of the minimal PR foundation. Quantities labeled "Derived" follow from the internal geometry, projection Hilbert space, projection trace, compact boundary map, or spectral operator structure. Quantities labeled "Numerically derived" or "Numerically evaluated derived output" are obtained by evaluating the corresponding derived operator without fitting the target observable.

Table 1: Status of assumptions and derived quantities in the PR master-equation architecture.

Quantity	**Status**	**PR-native origin**
$\Psi(x, \xi)$	Postulated	Master projection field defined over observable spacetime and the internal spectral manifold.
$\xi^A = (w, \theta)$	Postulated coordinates	Internal coordinates consisting of radial spectral coordinate w and compact phase coordinate θ.
$\mathcal{M}_{\text{int}}$	Postulated topology	Minimal internal projection manifold supplying radial stiffness and compact phase winding: $$\mathcal{M}_{\text{int}} = \mathbb{R}_w \times S^1_\theta.$$
$\theta \sim \theta + 2\pi n$	Topological constraint	Compact phase identification defining the S^1_θ projection fiber, with $n \in \mathbb{Z}$.
G_{AB}	Derived geometry	Internal metric tensor obtained from the separable cylindrical line element: $$G_{AB} = \text{diag}(1, R_A^2), \qquad ds^2_{\text{int}} = dw^2 + R_A^2 d\theta^2.$$
G^{AB}	Derived geometry	Inverse internal metric used in the internal Laplace–Beltrami operator: $$G^{AB} = \text{diag}(1, R_A^{-2}).$$
$d\mu_{\text{int}}$	Derived geometry	Invariant internal measure derived from $\sqrt{\det(G_{AB})}$: $$d\mu_{\text{int}} = R_A \, dw \, d\theta.$$
$\mathcal{H}_P$	Derived Hilbert structure	Projection Hilbert space used for internal normalization, overlap, and spectral expansion: $$\mathcal{H}_P = L^2(\mathcal{M}_{\text{int}}, d\mu_{\text{int}}).$$
$\langle f \vert g \rangle_P$	Derived Hilbert structure	Internal inner product fixed by the invariant measure: $$\langle f \vert g \rangle_P = \int_{\mathcal{M}_{\text{int}}} d\mu_{\text{int}} \, f^*(\xi) g(\xi).$$
O_X	Postulated radial operator	Minimal radial spectral operator defining internal stiffness and confinement: $$O_X = -\frac{d^2}{dw^2} + V_X(w).$$
$V_{X,\star}(w)$	Derived projection-trace branch	Projection-trace radial branch: $$V_{X,\star}(w) = \Lambda_X^2 \left(1 + w^2 + \frac{3}{4} w^4\right).$$ The quadratic coefficient $a_2 = 1$ follows from unit local radial stiffness; the quartic coefficient $a_4 = 3/4$ follows from the $3+1$ projection trace. They are derived branch coefficients, not selected fit parameters.
$a_2 = 1$	Derived coefficient	Unit local radial stiffness of the projection-trace radial branch.
$a_4 = 3/4$	Derived coefficient	Projection-trace coefficient from the observable spatial trace relative to the full $3+1$ trace.
O_θ	Derived compact operator	Compact phase operator obtained from the S^1_θ Laplace–Beltrami sector: $$O_\theta = -\frac{1}{R_A^2} \frac{d^2}{d\theta^2}.$$

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Quantity	Status	PR-native origin
O_{int}	Derived internal operator	Separable internal spectral operator combining radial stiffness and compact phase winding: $$O_{\text{int}} = O_X + O_\theta.$$
$u_n(w)$	Derived spectral mode	Radial eigenmodes satisfying $O_X u_n = \lambda_n u_n$.
$v_m(\theta)$	Derived analytic mode	Compact phase eigenmodes satisfying periodicity on S^1_θ: $$v_m(\theta) = \frac{1}{\sqrt{2\pi R_A}} e^{im\theta}.$$
$\lambda_m^{(\theta)}$	Derived analytic eigenvalue	Compact phase winding eigenvalues obtained from $O_\theta v_m = \lambda_m^{(\theta)} v_m$: $$\lambda_m^{(\theta)} = \frac{m^2}{R_A^2}.$$
$U_{n,m}$	Derived spectral basis	Full separable internal spectral basis built from radial and compact eigenmodes: $$U_{n,m}(w,\theta) = u_n(w) v_m(\theta).$$
$\Lambda_{n,m}$	Derived spectral eigenvalue	Full internal eigenvalue obtained from separability of O_{int}: $$\Lambda_{n,m} = \lambda_n + \frac{m^2}{R_A^2}.$$
λ_0	Numerically derived	Ground-state radial eigenvalue from the stationary radial spectral operator: $$\lambda_0 = 2.322863529580.$$
λ_1	Numerically derived	First excited radial eigenvalue from the stationary radial spectral operator: $$\lambda_1 = 5.375830272676.$$
λ_3	Numerically evaluated derived output	Finite-rank radial spectral datum required by the compact boundary-resolved closure: $$\lambda_3 = 13.23388439508096.$$ Together with p_1, λ_0, and λ_1, this gives $$q_{\text{bc}} = p_1(\lambda_1 - \lambda_0) + (1 - p_1)(\lambda_3 - \lambda_0) = 10.905007182855176.$$
$\mu^2_{\min}$	Numerically derived	First radial spectral gap; universal PR stiffness scale for finite-core regularization and low-energy suppression: $$\mu^2_{\min} = \lambda_1 - \lambda_0 = 3.052966743096.$$
$R_{\max}$	Derived hierarchy scale	Maximum projection-curvature scale induced by the radial spectral stiffness scale $\mu^2_{\min}$.
r_c	Derived hierarchy scale	Finite-core saturation radius obtained from the hierarchy $\mu^2_{\min} \to R_{\max} \to r_c$: $$r_c = \left(\frac{G_N M}{c^2 R_{\max}}\right)^{1/3}.$$
$\{P_i\}$	Defined projection maps	Observable sector projections acting on the single master field: $$\{P_i\} = \{P_g, P_{\text{disp}}, P_A, P_X, P_\theta\}.$$

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Quantity	Status	PR-native origin
$P_g[\Psi]$	Derived projection sector	Gravitational stiffness projection generating the effective spacetime metric: $$P_g[\Psi] \to g_{\mu\nu}^{\text{eff}}.$$
$P_{\text{disp}}[\Psi]$	Derived projection sector	Complex displacement projection amplitude extracted from the master field: $$P_{\text{disp}}[\Psi] \to \Phi_{\text{disp}}(x).$$
$\Phi_{\text{disp}}(x)$	Derived projection structure	Amplitude–phase decomposition of the displacement-projected field: $$\Phi_{\text{disp}}(x) = A(x)e^{i\theta(x)}.$$
$A(x)$	Derived displacement amplitude	Raw scalar displacement amplitude obtained from the modulus of the displacement projection: $$A(x) = \lvert\Phi_{\text{disp}}(x)\rvert.$$
Z_{disp}	Derived normalization	Spectral normalization of the internal displacement profile: $$Z_{\text{disp}} = \langle \Xi_{\text{disp}} \vert \Xi_{\text{disp}} \rangle_P.$$
$A_{\text{phys}}(x)$	Derived physical field	Canonically normalized observable displacement amplitude: $$A_{\text{phys}}(x) = \sqrt{Z_{\text{disp}}}\, A(x).$$
A_0^2	Derived vacuum quantity	Stable nonzero displacement vacuum from the bounded displacement potential $V_{\text{disp}}(A) = V_0 + \alpha_A A^2 + \beta_A A^4$: $$A_0^2 = -\frac{\alpha_A}{2\beta_A}.$$
v_A	Derived vacuum quantity	Canonically normalized displacement vacuum: $$v_A = \sqrt{Z_{\text{disp}}} A_0.$$
m_A^2	Derived curvature mode	Positive displacement-curvature scale about the stable nonzero displacement vacuum: $$m_A^2 = \left.\frac{d^2 V_{\text{disp}}}{dA^2}\right\rvert_{A_0} = -4\alpha_A = 8\beta_A A_0^2.$$
$\mathcal{M}_{\text{eff}}$	Derived inertia scale	Effective matter inertia generated by displacement vacuum weighted by internal matter–displacement overlap: $$\mathcal{M}_{\text{eff}} = g_0 \mathcal{I}_A A_0.$$
$T_{\mu\nu}^{(\text{disp})}$	Derived source tensor	Displacement-sector stress-energy contribution sourcing the gravitational stiffness projection.
$P_A[\Psi]$	Derived projection sector	Compact phase projection generating the observable electromagnetic gauge connection: $$P_A[\Psi] \to \theta(x) \to \mathcal{A}_\mu(x).$$
X_μ	Derived gauge-invariant structure	Compact phase-gradient combination invariant under local phase reparameterization: $$X_\mu = \partial_\mu\theta - q\mathcal{A}_\mu.$$
$F_{\mu\nu}$	Derived low-energy field strength	Observable electromagnetic field strength emerging in the low-energy compact phase limit: $$F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu.$$

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Quantity	Status	PR-native origin
T_A, T_X	Derived trace split	Compact/spatial trace fractions of the $3+1$ projection: $$T_A = \frac{1}{4}, \qquad T_X = \frac{3}{4}.$$ These are fixed by the projection trace, not by calibration.
Π_{13}	Derived finite-rank slice	Finite odd spatial-trace slice selected by the fundamental compact electromagnetic winding: $$\Pi_{13} = \lvert\phi_1\rangle\langle\phi_1\rvert + \lvert\phi_3\rangle\langle\phi_3\rvert.$$ The $n=3$ channel is the dominant spatial-trace radial lock; the $n=1$ channel is the parity-compatible coherence tail.
K_X	Derived normalized stiffness operator	Dimensionless radial stiffness operator normalized by the first radial spectral gap: $$K_X = \frac{O_X - \lambda_0}{\lambda_1 - \lambda_0}, \qquad \mu^2_{\min} = \lambda_1 - \lambda_0.$$
M_A	Derived compact-return matrix	Finite compact-return transfer matrix generated by the primitive compact trace returns A^3 and A^4: $$M_A = \begin{pmatrix} T_A^3 & T_A^4 \\ 1 & 0 \end{pmatrix}.$$
D_A	Derived Fredholm determinant	Compact-return Fredholm determinant resumming recurrent interior compact paths: $$D_A = \det(I - M_A) = 1 - T_A^3 - T_A^4.$$ For $T_A = 1/4$, $D_A = 251/256$.
$\mathrm{Adm}_{\Pi_{13}}$	Derived admissible boundary words	Finite-rank terminal word set of the compact boundary map: $$\mathrm{Adm}_{\Pi_{13}} = \{\epsilon,\ A^3,\ A^4,\ XA^5,\ -XA^8\}.$$ The terms represent direct passage, primitive compact returns, spatially weighted boundary leakage, and double full-trace exclusion required by finite-rank projection consistency.
N_{bc}	Derived boundary cofactor	Boundary-orientation cofactor obtained from the admissible terminal word set: $$N_{\mathrm{bc}} = 1 + T_A^3 + T_A^4 + T_X(T_A^5 - T_A^8).$$ This is not a fitted correction; it is the signed boundary cofactor of the finite-rank compact return map.
c_{bc}	Derived compact boundary constant	Compact boundary constant produced by the finite-rank Fredholm determinant and boundary cofactor: $$c_{\mathrm{bc}} = T_X \left[1 + T_A^2 - T_A^6 \frac{N_{\mathrm{bc}}}{D_A}\right],$$ $$c_{\mathrm{bc}} = \frac{3354902985}{4211081216} = 0.7966844648478991.$$

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Quantity	Status	PR-native origin
$P^{\mathrm{bc}}_{\mathrm{geom}}$	Derived finite-rank projection operator	Compact boundary-orientation projection operator acting on the finite odd spatial slice: $$P^{\mathrm{bc}}_{\mathrm{geom}} = \Pi_{13} K_X^2 \left(c_{\mathrm{bc}} + w^2 + \frac{3}{4} w^4\right) K_X^2 \Pi_{13}.$$ This replaces scalar-kernel treatments of the compact projection and preserves the finite-rank orthogonal-slice structure.
p_1^{bc}	Numerically evaluated derived output	Parity-compatible coherence-tail probability produced by $P^{\mathrm{bc}}_{\mathrm{geom}}$: $$p_1^{\mathrm{bc}} = 7.6528903366 \times 10^{-4}.$$ This lowers the dominant $n = 3$ radial lock to the compact boundary-resolved value.
$R^{-2}_{A,\star} = Z_A$	Derived compact stiffness	Stationary compact phase stiffness generated by the finite-rank compact boundary map. This is the compact electromagnetic normalization entering $\alpha^{-1}_{\mathrm{PR}} = 4\pi Z_A$.
C_A	Derived compact/radial closure ratio	Macroscopic compact-to-radial closure ratio fixed by compact/radial stationarity: $$C_A = \frac{R^{-2}_{A,\star}}{\mu^2_{\min}}.$$ It is not independent of the compact coherence expectation.
κ_{eff}	Derived under boundary-orientation closure	Microscopic compact coherence expectation explaining the macroscopic closure ratio: $$\kappa_{\mathrm{eff}} = \langle \Omega_A \vert \widehat{\kappa}_A \vert \Omega_A \rangle_P = C_A.$$ Under the finite-rank boundary-orientation lemma, stationarity locks $\kappa_{\mathrm{eff}} = C_A$.
$\alpha^{-1}_{\mathrm{PR},bc}$	Derived compact-boundary output	Fine-structure normalization produced by the compact boundary map. The low-energy laboratory value is not used as an input: $$\alpha^{-1}_{\mathrm{PR},bc} = 4\pi R^{-2}_{A,\star} = 137.036361812007,$$ at current nonperturbative precision.
$P_X[\Psi]$	Derived response sector	Projection response sector encoding internal spectral self-energy corrections: $$P_X[\Psi] \to \Sigma_X.$$
$\rho_X(\mu^2)$	Derived spectral density	Internal excitation density with support only above the radial gap $\mu^2_{\min}$.
$F(k^2)$	Derived response kernel	Subtracted projection self-energy kernel satisfying $F(0) = 0$, preserving the massless low-energy spin–2 pole.
$F_E(k^2)$	Derived residual kernel	Newton-normalized residual projection kernel obeying $$F_E(k^2) = O(k^4)$$ in the low-energy exterior limit.
$\widehat{\Sigma}^{\mathrm{GW}}_X$	Derived GW response	Projection self-energy correction acting on the Kerr/Teukolsky gravitational-wave operator: $$\widehat{\Sigma}^{\mathrm{GW}}_X = -F_E(O^{(s)}_{\mathrm{Teuk}}).$$

continued on next page

Quantity	Status	PR-native origin
$P_\theta[\Psi]$	Derived homogeneous response	Homogeneous compact phase response sector used for large-scale projection-energy bookkeeping: $$P_\theta[\Psi] \to H_{\text{eff}}(t).$$

3 Gravitational Projection Chain

The gravitational sector is the stiffness projection of the master field. This appendix collects the chain by which the internal spectral amplitudes generate the metric seed tensor, the determinant-normalized effective metric, the trace-free weak-field Einstein equation, the finite-core saturation scale, and the Kerr exterior recovery. The purpose is to provide the derivation ledger for the gravitational equations used in the main text, not to introduce an independent gravitational postulate.

3.1 Internal Mode Amplitudes

The master projection field is defined over observable spacetime and the internal spectral manifold,

$$\Psi = \Psi(x, \xi). \tag{66}$$

Using the internal spectral basis, the field expands as

$$\boxed{\Psi(x, \xi) = \sum_{n,m} c_{n,m}(x) U_{n,m}(\xi).} \tag{67}$$

The internal modes satisfy

$$\boxed{O_{\text{int}} U_{n,m} = \Lambda_{n,m} U_{n,m},} \tag{68}$$

with orthonormality

$$\boxed{\int_{\mathcal{M}_{\text{int}}} d\mu_{\text{int}}\, U^*_{j,k}(\xi) U_{n,m}(\xi) = \delta_{jn}\delta_{km}.} \tag{69}$$

The observable spacetime dependence is carried by the coefficients $c_{n,m}(x)$, while the internal stiffness data are carried by $\Lambda_{n,m}$.

3.2 Metric Seed Tensor

The gravitational projection begins with the symmetric bilinear seed tensor

$$\boxed{Q_{\mu\nu}(x) = \int_{\mathcal{M}_{\text{int}}} d\mu_{\text{int}} \left[\partial_\mu \Psi^\dagger \partial_\nu \Psi + \partial_\nu \Psi^\dagger \partial_\mu \Psi\right].} \tag{70}$$

This tensor is manifestly symmetric:

$$\boxed{Q_{\mu\nu} = Q_{\nu\mu}.} \tag{71}$$

Substituting the spectral expansion gives

$$\partial_\mu \Psi = \sum_{n,m} (\partial_\mu c_{n,m}) U_{n,m}, \qquad \partial_\mu U_{n,m}(\xi) = 0. \tag{72}$$

The conjugate derivative is

$$\partial_\mu \Psi^\dagger = \sum_{j,k} (\partial_\mu c^*_{j,k}) U^*_{j,k}. \tag{73}$$

Using the orthonormality relation, the internal integral collapses:

$$\int d\mu_{\text{int}}\, \partial_\mu \Psi^\dagger \partial_\nu \Psi = \sum_{n,m} (\partial_\mu c^*_{n,m})(\partial_\nu c_{n,m}). \tag{74}$$

The seed tensor is therefore

$$\boxed{Q_{\mu\nu} = \sum_{n,m} \left[\partial_\mu c^*_{n,m} \partial_\nu c_{n,m} + \partial_\nu c^*_{n,m} \partial_\mu c_{n,m}\right] = 2\,\mathrm{Re}\left[\sum_{n,m} \partial_\mu c^*_{n,m} \partial_\nu c_{n,m}\right].} \tag{75}$$

This is the observable stiffness tensor of the projected spectral amplitudes.

3.3 Determinant-Normalized Effective Metric

The seed tensor still contains an arbitrary conformal scale. The projected metric removes that scale by determinant normalization:

$$\boxed{g^{\text{eff}}_{\mu\nu} = \frac{Q_{\mu\nu}}{|\det Q|^{1/4}}.} \tag{76}$$

The exponent 1/4 is fixed by the four observable spacetime dimensions. If $Q_{\mu\nu} \to s Q_{\mu\nu}$, then:

$$\det(sQ) = s^4 \det Q, \qquad |\det(sQ)|^{1/4} = s|\det Q|^{1/4}, \tag{77}$$

so the normalized metric is invariant:

$$\boxed{Q_{\mu\nu} \to s Q_{\mu\nu} \quad \Longrightarrow \quad g^{\text{eff}}_{\mu\nu} \to g^{\text{eff}}_{\mu\nu}.} \tag{78}$$

Its determinant is

$$\det(g^{\text{eff}}_{\mu\nu}) = \frac{\det Q}{|\det Q|} = \mathrm{sgn}(\det Q). \tag{79}$$

For Lorentzian signature,

$$\boxed{\det(g^{\text{eff}}_{\mu\nu}) = -1.} \tag{80}$$

The determinant-normalized projection therefore removes the local volume-scale degree of freedom while preserving the Lorentzian causal structure.

3.4 Trace-Free Weak-Field Limit

In the weak-field regime, write

$$Q_{\mu\nu} = \eta_{\mu\nu} + q_{\mu\nu}, \qquad |q_{\mu\nu}| \ll 1, \tag{81}$$

and define

$$q = \eta^{\mu\nu} q_{\mu\nu}. \tag{82}$$

To first order,

$$|\det Q|^{-1/4} = 1 - \frac{1}{4} q + O(q^2). \tag{83}$$

The determinant-normalized metric becomes

$$g^{\text{eff}}_{\mu\nu} = \eta_{\mu\nu} + q_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu} q + O(q^2). \tag{84}$$

Thus the physical perturbation is

$$\boxed{h_{\mu\nu} = q_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu} q.} \tag{85}$$

Taking the trace gives

$$\boxed{h = \eta^{\mu\nu} h_{\mu\nu} = 0.} \tag{86}$$

In Lorenz gauge,

$$\partial^\mu h_{\mu\nu} = 0, \tag{87}$$

the linearized Ricci tensor reduces to

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}. \tag{88}$$

Because the determinant-normalized perturbation is trace-free, the directly projected weak-field equation couples to the trace-free source combination:

$$\boxed{R^{(1)}_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu} R^{(1)} = \frac{8\pi G_N}{c^4}\left(T_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu} T\right).} \tag{89}$$

In trace-free Lorenz gauge this becomes

$$\boxed{-\frac{1}{2}\Box h_{\mu\nu} = \frac{8\pi G_N}{c^4}\left(T_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu} T\right).} \tag{90}$$

Both sides are trace-free. Stress-energy conservation and the contracted Bianchi identity restore the standard Einstein equation up to the usual cosmological integration constant:

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}.} \tag{91}$$

The determinant-normalized gravitational projection therefore recovers the Einstein limit through the unimodular trace-free form.

3.5 Finite-Core Saturation

The radial projection branch has a nonzero first spectral gap,

$$\boxed{\mu^2_{\min} = \lambda_1 - \lambda_0 = 3.052966743096.} \tag{92}$$

This gap sets a finite curvature ceiling,

$$\boxed{R_{\max} = \chi_R \Lambda_X^2 \mu^2_{\min} < \infty,} \tag{93}$$

where χ_R is the fixed projection conversion from internal stiffness units to macroscopic curvature units.

Matching the exterior Schwarzschild curvature scale to the projection ceiling,

$$R_{\mathrm{GR}}(r) = \frac{G_N M}{c^2 r^3} = R_{\max}, \tag{94}$$

gives the finite-core radius

$$\boxed{r_c(M) = \left(\frac{G_N M}{c^2 R_{\max}}\right)^{1/3}.} \tag{95}$$

Since $R_{\max} < \infty$, the core radius is strictly positive:

$$\boxed{r_c > 0.} \tag{96}$$

A minimal regularized mass profile satisfying exterior recovery and central volume scaling is

$$\boxed{m_{\rm PR}(r) = M\frac{r^3}{r^3 + r_c^3}.} \tag{97}$$

The central density is finite:

$$\boxed{\rho_{\rm eff}(0) = \frac{3M}{4\pi r_c^3} = \frac{3c^2 R_{\max}}{4\pi G_N} \equiv \rho_{\max} < \infty.} \tag{98}$$

The corresponding core curvature invariant is bounded:

$$\boxed{K_{\rm core} = 96 R_{\max}^2 < \infty.} \tag{99}$$

The finite-core regularization chain is

$$\boxed{\mu_{\min}^2 > 0 \quad\Longrightarrow\quad R_{\max} < \infty \quad\Longrightarrow\quad r_c > 0 \quad\Longrightarrow\quad \rho_{\max} < \infty \quad\Longrightarrow\quad K_{\rm PR}(0) < \infty.} \tag{100}$$

3.6 Exterior Kerr Recovery

Outside the finite projection core,

$$r > r_c, \tag{101}$$

the ordinary matter source vanishes in the exterior vacuum region:

$$T_{\mu\nu} = 0. \tag{102}$$

The projection correction is also suppressed in the low-energy exterior:

$$\Delta_{\mu\nu}^{(P)} \to 0. \tag{103}$$

The exterior field equation reduces to

$$G_{\mu\nu}[g^{\rm eff}] = 0. \tag{104}$$

Taking the trace gives

$$R = 0, \tag{105}$$

and substituting back gives

$$\boxed{R_{\mu\nu} = 0, \qquad r > r_c.} \tag{106}$$

For stationary, axisymmetric, asymptotically flat exterior boundary conditions with mass M and angular momentum J, the vacuum solution is Kerr. Define

$$\mathcal{M} = \frac{G_N M}{c^2}, \qquad a = \frac{J}{Mc}. \tag{107}$$

The Kerr structure functions are

$$\Sigma_K = r^2 + a^2\cos^2\theta, \qquad \Delta_K = r^2 - 2\mathcal{M}r + a^2. \tag{108}$$

Projection Relativity allows finite-core corrections only in the interior:

$$\Sigma_{\rm PR} = \Sigma_K + \delta\Sigma, \qquad \Delta_{\rm PR} = \Delta_K + \delta\Delta. \tag{109}$$

Exterior recovery imposes

$$\boxed{\delta\Sigma = 0, \qquad \delta\Delta = 0, \qquad r > r_c.} \tag{110}$$

Therefore,

$$\boxed{g_{\mu\nu}^{\rm PR} = g_{\mu\nu}^{\rm Kerr}, \qquad r > r_c.} \tag{111}$$

3.7 Closed Gravitational Projection Sequence

The gravitational projection chain is

$$\boxed{\Psi \to \{c_{n,m}, U_{n,m}\} \to Q_{\mu\nu} \to g^{\text{eff}}_{\mu\nu} \to \left[R^{(1)}_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}R^{(1)} = \frac{8\pi G_N}{c^4}\left(T_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}T\right)\right].} \tag{112}$$

The Bianchi-completed macroscopic Einstein limit is

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}.} \tag{113}$$

The finite-core chain is

$$\boxed{\mu^2_{\min} \to R_{\max} \to r_c(M) = \left(\frac{G_N M}{c^2 R_{\max}}\right)^{1/3} \to K_{\text{PR}}(0) < \infty.} \tag{114}$$

The exterior recovery chain is

$$\boxed{r > r_c \to T_{\mu\nu} = 0 \to \Delta^{(P)}_{\mu\nu} \to 0 \to R_{\mu\nu} = 0 \to g^{\text{PR}}_{\mu\nu} = g^{\text{Kerr}}_{\mu\nu}.} \tag{115}$$

This completes the gravitational projection chain. The same master-field stiffness projection recovers the weak-field Einstein limit, replaces the classical singular endpoint with a finite core, and preserves the Kerr exterior outside the projection-saturated region.

4 Displacement Projection Chain

The displacement sector is the scalar-amplitude projection of the master field. This appendix collects the chain by which the master projection state generates an observable displacement amplitude, a stable nonzero displacement vacuum, a positive displacement-curvature mode, and an effective matter inertia scale. The purpose is to record the derivation ledger for the displacement equations used in the main text, not to introduce an independent scalar sector.

4.1 Master-Field Projection

At a fixed observable spacetime point x, the master field is written in the internal spectral basis as

$$\boxed{|\Psi_x\rangle = \sum_{n,m} c_{n,m}(x)|U_{n,m}\rangle.} \tag{116}$$

Equivalently, in coordinate form,

$$\boxed{\Psi(x,\xi) = \sum_{n,m} c_{n,m}(x)U_{n,m}(\xi).} \tag{117}$$

The internal basis is orthonormal:

$$\boxed{\langle U_{j,k}|U_{n,m}\rangle_P = \delta_{jn}\delta_{km}.} \tag{118}$$

Let $U_{\text{disp}}(\xi)$ denote the normalized internal displacement direction,

$$\boxed{\langle U_{\text{disp}}|U_{\text{disp}}\rangle_P = 1.} \tag{119}$$

The complex displacement projection is

$$\boxed{\Phi_{\text{disp}}(x) = \langle U_{\text{disp}} | \Psi_x \rangle_P.} \tag{120}$$

In integral form,

$$\boxed{\Phi_{\text{disp}}(x) = \int_{\mathcal{M}_{\text{int}}} d\mu_{\text{int}}\, U^*_{\text{disp}}(\xi) \Psi(x, \xi).} \tag{121}$$

Substituting the spectral expansion gives

$$\boxed{\Phi_{\text{disp}}(x) = \sum_{n,m} d^{(\text{disp})}_{n,m} c_{n,m}(x), \qquad d^{(\text{disp})}_{n,m} = \langle U_{\text{disp}} | U_{n,m} \rangle_P.} \tag{122}$$

The observable displacement amplitude is the modulus of this projected field:

$$\boxed{A(x) = |\Phi_{\text{disp}}(x)|, \qquad A^2(x) = \Phi^\dagger_{\text{disp}}(x) \Phi_{\text{disp}}(x).} \tag{123}$$

Thus the displacement projection is

$$\boxed{P_{\text{disp}}[\Psi](x) = A(x).} \tag{124}$$

4.2 Rank-One Projection Form

The same projection can be written as a rank-one internal projector,

$$\boxed{\widehat{\Pi}_{\text{disp}} = |U_{\text{disp}} \rangle \langle U_{\text{disp}}|.} \tag{125}$$

Acting on the master state,

$$\widehat{\Pi}_{\text{disp}} |\Psi_x\rangle = |U_{\text{disp}}\rangle \langle U_{\text{disp}} | \Psi_x \rangle_P = \Phi_{\text{disp}}(x) |U_{\text{disp}}\rangle. \tag{126}$$

Its internal norm is

$$\left\| \widehat{\Pi}_{\text{disp}} |\Psi_x\rangle \right\|^2_P = \Phi^\dagger_{\text{disp}}(x) \Phi_{\text{disp}}(x) \langle U_{\text{disp}} | U_{\text{disp}} \rangle_P = A^2(x). \tag{127}$$

Therefore,

$$\boxed{A(x) = \left\| \widehat{\Pi}_{\text{disp}} |\Psi_x\rangle \right\|_P.} \tag{128}$$

This form makes explicit that the displacement amplitude is a projection norm, not an independent inserted scalar field.

4.3 Amplitude–Phase Decomposition

For nonzero projected displacement field, write

$$\boxed{\Phi_{\text{disp}}(x) = A(x) e^{i\theta(x)}.} \tag{129}$$

The compact phase obeys the same winding identification used in the compact electromagnetic sector:

$$\boxed{\theta \sim \theta + 2\pi n, \qquad n \in \mathbb{Z}.} \tag{130}$$

The ordinary derivative gives

$$\partial_\mu \Phi_{\text{disp}} = e^{i\theta} \left(\partial_\mu A + i A \partial_\mu \theta \right), \tag{131}$$

and

$$\partial_\mu \Phi^\dagger_{\rm disp} = e^{-i\theta}\left(\partial_\mu A - iA\partial_\mu\theta\right). \tag{132}$$

The kinetic bilinear separates as

$$\partial_\mu \Phi^\dagger_{\rm disp}\partial^\mu \Phi_{\rm disp} = (\partial_\mu A)(\partial^\mu A) + iA(\partial_\mu A)(\partial^\mu\theta) - iA(\partial_\mu\theta)(\partial^\mu A) + A^2(\partial_\mu\theta)(\partial^\mu\theta) \tag{133}$$

$$= (\partial_\mu A)(\partial^\mu A) + A^2(\partial_\mu\theta)(\partial^\mu\theta). \tag{134}$$

Thus,

$$\boxed{\partial_\mu \Phi^\dagger_{\rm disp}\partial^\mu \Phi_{\rm disp} = (\partial_\mu A)(\partial^\mu A) + A^2(\partial_\mu\theta)(\partial^\mu\theta).} \tag{135}$$

The first term is the displacement-amplitude kinetic term. The second is the compact phase kinetic term weighted by the displacement amplitude.

With the compact gauge connection included, define the gauge-invariant phase gradient

$$\boxed{X_\mu = \partial_\mu\theta - q\mathcal{A}_\mu.} \tag{136}$$

The covariant amplitude–phase split becomes

$$\boxed{g^{\mu\nu}_{\rm eff}\left(D_\mu\Phi_{\rm disp}\right)^\dagger D_\nu\Phi_{\rm disp} = (\partial_\mu A)(\partial^\mu A) + A^2 X_\mu X^\mu.} \tag{137}$$

4.4 Displacement-Sector Action

After projecting the master action onto the displacement amplitude, the displacement-sector action is

$$\boxed{S_{\rm disp} = \int d^4x\sqrt{-g_{\rm eff}}\, Z_{\rm disp}\left[-\frac{1}{2}(\partial_\mu A)(\partial^\mu A) - V_{\rm disp}(A)\right].} \tag{138}$$

The minimal bounded displacement potential is

$$\boxed{V_{\rm disp}(A) = V_0 + \alpha_A A^2 + \beta_A A^4, \qquad \beta_A > 0.} \tag{139}$$

The displacement equation of motion is

$$\boxed{\Box A - 2\alpha_A A - 4\beta_A A^3 = 0.} \tag{140}$$

4.5 Stable Nonzero Displacement Vacuum

Stationarity of the displacement potential gives

$$\frac{dV_{\rm disp}}{dA} = 2\alpha_A A + 4\beta_A A^3 = 2A\left(\alpha_A + 2\beta_A A^2\right). \tag{141}$$

The nonzero branch satisfies

$$\boxed{A_0^2 = -\frac{\alpha_A}{2\beta_A}.} \tag{142}$$

A real stable nonzero displacement vacuum requires

$$\boxed{\alpha_A < 0, \qquad \beta_A > 0.} \tag{143}$$

Then

$$\boxed{A_0 = \sqrt{-\frac{\alpha_A}{2\beta_A}}.} \tag{144}$$

The canonically normalized displacement vacuum is

$$\boxed{v_A = \sqrt{Z_{\rm disp}}\, A_0.} \tag{145}$$

4.6 Displacement Fluctuation and Curvature Scale

Let

$$\boxed{A(x) = A_0 + \eta_A(x).} \tag{146}$$

The displacement-curvature scale is

$$m_A^2 = \left. \frac{d^2 V_{\rm disp}}{dA^2} \right|_{A_0}. \tag{147}$$

Since

$$\frac{d^2 V_{\rm disp}}{dA^2} = 2\alpha_A + 12\beta_A A^2, \tag{148}$$

substitution of $A_0^2 = -\alpha_A/(2\beta_A)$ gives

$$\boxed{m_A^2 = -4\alpha_A = 8\beta_A A_0^2.} \tag{149}$$

Because $\alpha_A < 0$,

$$\boxed{m_A^2 > 0.} \tag{150}$$

The quadratic displacement fluctuation Lagrangian is

$$\boxed{\mathcal{L}^{(2)}_{\rm disp} = -\frac{1}{2}(\partial_\mu \eta_A)(\partial^\mu \eta_A) - \frac{1}{2} m_A^2 \eta_A^2.} \tag{151}$$

The corresponding fluctuation equation is

$$\boxed{(\Box - m_A^2)\eta_A = 0.} \tag{152}$$

4.7 Spectral Normalization

The internal displacement profile has spectral normalization

$$\boxed{Z_{\rm disp} = \int_{\mathcal{M}_{\rm int}} d\mu_{\rm int} |\Xi_{\rm disp}(\xi)|^2 = \sum_{n,m} |\zeta^{(\rm disp)}_{n,m}|^2.} \tag{153}$$

The canonically normalized displacement amplitude is

$$\boxed{A_{\rm phys}(x) = \sqrt{Z_{\rm disp}}\, A(x).} \tag{154}$$

The canonically normalized displacement vacuum is

$$\boxed{v_A = \sqrt{Z_{\rm disp}}\, A_0.} \tag{155}$$

The canonical fluctuation is

$$\boxed{a_{\rm phys}(x) = \sqrt{Z_{\rm disp}}\, \eta_A(x).} \tag{156}$$

Therefore,

$$\boxed{A_{\rm phys}(x) = v_A + a_{\rm phys}(x).} \tag{157}$$

4.8 Effective Matter Inertia

Matter inertia is generated through the matter–displacement overlap interaction, written schematically as

$$\boxed{\mathcal{L}_{\text{ov}} = -g_{\text{eff}}\mathcal{O}_M A_{\text{phys}}.} \tag{158}$$

Using

$$A_{\text{phys}} = v_A + a_{\text{phys}}, \tag{159}$$

the overlap interaction becomes

$$\mathcal{L}_{\text{ov}} = -g_{\text{eff}} v_A \mathcal{O}_M - g_{\text{eff}} a_{\text{phys}} \mathcal{O}_M. \tag{160}$$

The first term is the effective matter-inertia term. Hence

$$\boxed{\mathcal{M}_{\text{eff}} = g_{\text{eff}} v_A.} \tag{161}$$

The effective overlap coefficient is determined by the internal displacement overlap,

$$\boxed{g_{\text{eff}} = \frac{g_0 \mathcal{I}_A}{\sqrt{Z_{\text{disp}}}}.} \tag{162}$$

Therefore,

$$\boxed{\mathcal{M}_{\text{eff}} = g_{\text{eff}} v_A = \frac{g_0 \mathcal{I}_A}{\sqrt{Z_{\text{disp}}}} \left(\sqrt{Z_{\text{disp}}} A_0\right) = g_0 \mathcal{I}_A A_0.} \tag{163}$$

The physical effective inertia is independent of an arbitrary normalization of the internal displacement profile. For multiple matter projection profiles, the overlap generalizes to the matrix form

$$\boxed{g_{ab}^{\text{eff}} = \frac{g_0 I_{ab}^{(A)}}{\sqrt{Z_{\text{disp}}}}, \qquad \mathcal{M}_{ab}^{\text{eff}} = g_{ab}^{\text{eff}} v_A = g_0 I_{ab}^{(A)} A_0.} \tag{164}$$

4.9 Displacement Stress-Energy and Gravitational Source Response

The displacement stress-energy tensor follows from the displacement-sector action:

$$\boxed{T_{\mu\nu}^{(\text{disp})} = Z_{\text{disp}} \left[\partial_\mu A\, \partial_\nu A - g_{\mu\nu}^{\text{eff}} \left(\frac{1}{2}(\partial_\alpha A)(\partial^\alpha A) + V_{\text{disp}}(A)\right)\right].} \tag{165}$$

The displacement vacuum contributes to the physical matter source through the effective inertia scale $\mathcal{M}_{\text{eff}}$. In the determinant-normalized weak-field limit, this source enters the gravitational stiffness projection through the trace-free unimodular combination

$$\boxed{T_{\mu\nu}^{(\text{disp})} - \frac{1}{4} g_{\mu\nu}^{\text{eff}} T^{(\text{disp})}.} \tag{166}$$

Stress-energy conservation and the contracted Bianchi identity restore the scalar trace through the usual cosmological integration constant, yielding the standard Einstein source equation in the macroscopic low-energy limit. The displacement sector therefore supplies gravitational source response consistently with the trace-free weak-field recovery derived in the gravitational projection chain.

4.10 Displacement–Phase Consistency

The covariant compact phase contribution generated by the displacement field is

$$\boxed{A^2 X_\mu X^\mu.} \tag{167}$$

Expanding about the displacement vacuum,

$$A = A_0 + \eta_A, \tag{168}$$

gives

$$\boxed{A^2 X_\mu X^\mu = A_0^2 X_\mu X^\mu + 2A_0 \eta_A X_\mu X^\mu + \eta_A^2 X_\mu X^\mu.} \tag{169}$$

The compact phase current is

$$\boxed{J_\theta^\mu = A^2 X^\mu.} \tag{170}$$

In the coherent compact-phase vacuum,

$$\boxed{X_\mu^{\text{vac}} = 0,} \tag{171}$$

so

$$\boxed{J_\theta^{\mu,\text{vac}} = 0.} \tag{172}$$

The compact electromagnetic gauge normalization and boundary-resolved closure are derived in the compact phase projection chain.

4.11 Stability Closure

The displacement sector is stable when

$$\boxed{\beta_A > 0, \qquad \alpha_A < 0, \qquad Z_{\text{disp}} > 0, \qquad \mu_{\text{min}}^2 > 0, \qquad X_\mu^{\text{vac}} = 0.} \tag{173}$$

These conditions imply

$$\boxed{V_{\text{disp}}(A) \text{ is bounded below}, \qquad m_A^2 > 0, \qquad Z_{\text{disp}} < \infty, \qquad J_\theta^{\mu,\text{vac}} = 0.} \tag{174}$$

4.12 Closed Displacement Projection Sequence

The displacement projection chain is

$$\boxed{|\Psi_x\rangle \to \widehat{\Pi}_{\text{disp}}|\Psi_x\rangle \to \Phi_{\text{disp}}(x) = \langle U_{\text{disp}}|\Psi_x\rangle_P \to A(x) = |\Phi_{\text{disp}}(x)|.} \tag{175}$$

The displacement-vacuum chain is

$$\boxed{A(x) \to V_{\text{disp}}(A) = V_0 + \alpha_A A^2 + \beta_A A^4 \to A_0^2 = -\frac{\alpha_A}{2\beta_A} \to A(x) = A_0 + \eta_A(x).} \tag{176}$$

The displacement-curvature chain is

$$\boxed{m_A^2 = \left.\frac{d^2 V_{\text{disp}}}{dA^2}\right|_{A_0} = -4\alpha_A = 8\beta_A A_0^2.} \tag{177}$$

The spectral-normalization chain is

$$\boxed{Z_{\mathrm{disp}} = \int_{\mathcal{M}_{\mathrm{int}}} d\mu_{\mathrm{int}} |\Xi_{\mathrm{disp}}(\xi)|^2, \qquad A_{\mathrm{phys}} = \sqrt{Z_{\mathrm{disp}}}\, A, \qquad v_A = \sqrt{Z_{\mathrm{disp}}}\, A_0.} \tag{178}$$

The effective-inertia chain is

$$\boxed{\mathcal{M}_{\mathrm{eff}} = g_{\mathrm{eff}} v_A = g_0 \mathcal{I}_A A_0.} \tag{179}$$

The displacement–gravity chain is

$$\boxed{A_0 \to \mathcal{M}_{\mathrm{eff}} \to T^{(\mathrm{disp})}_{\mu\nu} \to T^{(\mathrm{disp})}_{\mu\nu} - \frac{1}{4} g^{\mathrm{eff}}_{\mu\nu} T^{(\mathrm{disp})} \to g^{\mathrm{eff}}_{\mu\nu}.} \tag{180}$$

The displacement–phase consistency chain is

$$\boxed{\Phi_{\mathrm{disp}} = A e^{i\theta} \to g^{\mu\nu}_{\mathrm{eff}} (D_\mu \Phi_{\mathrm{disp}})^\dagger D_\nu \Phi_{\mathrm{disp}} = (\partial_\mu A)(\partial^\mu A) + A^2 X_\mu X^\mu, \qquad X_\mu = \partial_\mu \theta - q\mathcal{A}_\mu.} \tag{181}$$

The full displacement-sector closure is

$$\boxed{\Psi \to P_{\mathrm{disp}}[\Psi] \to \Phi_{\mathrm{disp}}(x) \to A(x) \to A_0 \to \eta_A \to m_A \to A_{\mathrm{phys}} \to \mathcal{M}_{\mathrm{eff}} \to T^{(\mathrm{disp})}_{\mu\nu} \to g^{\mathrm{eff}}_{\mu\nu}.} \tag{182}$$

This completes the displacement projection chain. The displacement sector is not an independent external scalar field. It is the scalar-amplitude projection of the master field. Its stable nonzero vacuum generates displacement curvature, effective matter inertia, gravitational source response, and consistent coupling to the compact phase sector.

5 Electromagnetic Projection Chain

The electromagnetic sector is the compact-phase projection of the master field. It is not introduced as an independent external $U(1)$ gauge bundle. It is generated by the compact coordinate θ of the same internal radial–compact manifold used by the gravitational and displacement sectors. This appendix collects the compact phase chain: the compact operator, winding basis, gauge-invariant phase gradient, field-strength curvature, kinetic normalization, compact boundary closure, and magnetic area-law interpretation.

5.1 Compact Phase Geometry

The minimal internal manifold is

$$\boxed{\mathcal{M}_{\mathrm{int}} = \mathbb{R}_w \times S^1_\theta, \qquad \theta \sim \theta + 2\pi n, \qquad n \in \mathbb{Z}.} \tag{183}$$

The compact coordinate is represented by $e^{i\theta}$, so the physical projection is unchanged under

$$\boxed{e^{i(\theta+2\pi n)} = e^{i\theta}.} \tag{184}$$

The internal line element is

$$\boxed{ds^2_{\mathrm{int}} = dw^2 + R_A^2 d\theta^2.} \tag{185}$$

The compact phase measure is

$$\boxed{d\mu_\theta = R_A\, d\theta.} \tag{186}$$

The compact phase Hilbert space is

$$\boxed{\mathcal{H}_\theta = L^2(S^1_\theta, R_A\, d\theta),} \tag{187}$$

with inner product

$$\boxed{\langle f|g\rangle_\theta = \int_0^{2\pi} R_A\, d\theta\, f^*(\theta) g(\theta).} \tag{188}$$

5.2 Compact Phase Operator and Winding Modes

The compact phase operator is

$$\boxed{O_\theta = -\frac{1}{R_A^2}\frac{d^2}{d\theta^2}.} \tag{189}$$

The operator acts on periodic functions,

$$\boxed{v_m(\theta + 2\pi) = v_m(\theta).} \tag{190}$$

Integration by parts on the compact circle gives no boundary contribution, so O_θ is self-adjoint on the periodic domain:

$$\boxed{\langle f|O_\theta g\rangle_\theta = \langle O_\theta f|g\rangle_\theta.} \tag{191}$$

The normalized winding modes are

$$\boxed{v_m(\theta) = \frac{1}{\sqrt{2\pi R_A}}e^{im\theta}, \qquad m \in \mathbb{Z}.} \tag{192}$$

They satisfy

$$\boxed{\langle v_m|v_n\rangle_\theta = \delta_{mn}.} \tag{193}$$

Acting with O_θ gives

$$O_\theta v_m = -\frac{1}{R_A^2}\frac{d^2}{d\theta^2}\left(\frac{e^{im\theta}}{\sqrt{2\pi R_A}}\right) = \frac{m^2}{R_A^2}v_m. \tag{194}$$

Hence

$$\boxed{O_\theta v_m = \lambda_m^{(\theta)} v_m, \qquad \lambda_m^{(\theta)} = \frac{m^2}{R_A^2}.} \tag{195}$$

The fundamental electromagnetic winding is $m = 1$. The compact phase stiffness is therefore

$$\boxed{Z_A = \lambda_1^{(\theta)} = R_A^{-2}.} \tag{196}$$

5.3 Compact Phase Projection

Let $U_A(\xi)$ denote the normalized internal compact-phase projection direction. The compact electromagnetic projection of the master field is

$$\boxed{\Phi_A(x) = \langle U_A|\Psi_x\rangle_P.} \tag{197}$$

In integral form,

$$\boxed{\Phi_A(x) = \int_{\mathcal{M}_{\rm int}} d\mu_{\rm int}\, U_A^*(\xi)\Psi(x,\xi).} \tag{198}$$

The projected compact field is written in amplitude–phase form:

$$\boxed{\Phi_A(x) = A_\theta(x)e^{i\theta(x)}.} \tag{199}$$

Here $A_\theta(x) = |\Phi_A(x)|$ is the compact projection amplitude, while $\theta(x)$ is the observable compact phase.

The compact electromagnetic projection is

$$\boxed{P_A[\Psi] \to \theta(x) \to \mathcal{A}_\mu(x).} \tag{200}$$

The gauge connection $\mathcal{A}_\mu$ is the observable connection associated with local compact phase transport.

5.4 Gauge-Invariant Compact Phase Gradient

The covariant derivative acting on the compact projected field is

$$\boxed{D_\mu = \partial_\mu - iq\mathcal{A}_\mu.} \tag{201}$$

Acting on $\Phi_A = A_\theta e^{i\theta}$,

$$\begin{aligned} D_\mu \Phi_A &= (\partial_\mu - iq\mathcal{A}_\mu)\left(A_\theta e^{i\theta}\right) && (202)\\ &= e^{i\theta}\left[\partial_\mu A_\theta + iA_\theta\left(\partial_\mu\theta - q\mathcal{A}_\mu\right)\right]. && (203)\end{aligned}$$

Define the compact phase gradient

$$\boxed{X_\mu = \partial_\mu\theta - q\mathcal{A}_\mu.} \tag{204}$$

Then

$$\boxed{D_\mu \Phi_A = e^{i\theta}\left(\partial_\mu A_\theta + iA_\theta X_\mu\right).} \tag{205}$$

Under a local compact phase rotation,

$$\boxed{\theta' = \theta + \alpha, \qquad \mathcal{A}'_\mu = \mathcal{A}_\mu + q^{-1}\partial_\mu\alpha.} \tag{206}$$

The phase gradient transforms as

$$\begin{aligned} X'_\mu &= \partial_\mu\theta' - q\mathcal{A}'_\mu && (207)\\ &= \partial_\mu(\theta+\alpha) - q\left(\mathcal{A}_\mu + q^{-1}\partial_\mu\alpha\right) && (208)\\ &= \partial_\mu\theta - q\mathcal{A}_\mu = X_\mu. && (209)\end{aligned}$$

Thus

$$\boxed{X'_\mu = X_\mu.} \tag{210}$$

The compact phase gradient is gauge invariant.

5.5 Compact Phase Kinetic Split

The conjugate covariant derivative is

$$\boxed{(D_\mu\Phi_A)^\dagger = e^{-i\theta}\left(\partial_\mu A_\theta - iA_\theta X_\mu\right).} \tag{211}$$

The covariant kinetic bilinear is

$$\begin{aligned} g^{\mu\nu}_{\text{eff}}(D_\mu\Phi_A)^\dagger D_\nu\Phi_A &= g^{\mu\nu}_{\text{eff}}\left(\partial_\mu A_\theta - iA_\theta X_\mu\right)\left(\partial_\nu A_\theta + iA_\theta X_\nu\right) && (212)\\ &= (\partial_\mu A_\theta)(\partial^\mu A_\theta) + A_\theta^2 X_\mu X^\mu. && (213)\end{aligned}$$

The imaginary cross terms cancel. Hence

$$\boxed{g^{\mu\nu}_{\text{eff}}(D_\mu\Phi_A)^\dagger D_\nu\Phi_A = (\partial_\mu A_\theta)(\partial^\mu A_\theta) + A_\theta^2 X_\mu X^\mu.} \tag{214}$$

The first term belongs to the compact projection amplitude. The second term is the electromagnetic compact phase contribution:

$$\boxed{P_A[\Psi] \to A_\theta^2 X_\mu X^\mu.} \tag{215}$$

The compact phase current is

$$\boxed{J_\theta^\mu = A_\theta^2 X^\mu.} \tag{216}$$

The coherent compact phase vacuum satisfies

$$\boxed{X_\mu^{\text{vac}} = 0.} \tag{217}$$

Therefore,

$$\boxed{J_\theta^{\mu,\text{vac}} = 0.} \tag{218}$$

The compact phase vacuum is a zero-current coherent ground state.

5.6 Gauge Curvature and Field-Strength Invariance

The electromagnetic field strength is the curvature of the observable compact phase connection:

$$\boxed{F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu.} \tag{219}$$

Under

$$\mathcal{A}'_\mu = \mathcal{A}_\mu + q^{-1}\partial_\mu \alpha, \tag{220}$$

the curvature becomes

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu \mathcal{A}'_\nu - \partial_\nu \mathcal{A}'_\mu && (221)\\ &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + q^{-1}\left(\partial_\mu\partial_\nu\alpha - \partial_\nu\partial_\mu\alpha\right) && (222)\\ &= F_{\mu\nu}. && (223) \end{aligned}$$

Thus

$$\boxed{F'_{\mu\nu} = F_{\mu\nu}.} \tag{224}$$

The Bianchi identity follows from the definition:

$$\boxed{\nabla_{[\lambda} F_{\mu\nu]} = 0.} \tag{225}$$

5.7 Compact Winding Stiffness and Gauge Kinetic Normalization

The compact part of the internal kinetic sector is

$$\boxed{\mathcal{L}_{\text{int}}^{(\theta)} = -\frac{1}{2} G^{\theta\theta} D_\theta \Psi^\dagger D_\theta \Psi.} \tag{226}$$

Since

$$\boxed{G^{\theta\theta} = R_A^{-2},} \tag{227}$$

the compact kinetic term is

$$\boxed{\mathcal{L}_{\text{int}}^{(\theta)} = -\frac{1}{2R_A^2} D_\theta \Psi^\dagger D_\theta \Psi.} \tag{228}$$

For the internal compact eigenproblem,

$$D_\theta \to \partial_\theta, \tag{229}$$

so

$$\boxed{\mathcal{L}_{\text{int}}^{(\theta)} = -\frac{1}{2R_A^2} \partial_\theta \Psi^\dagger \partial_\theta \Psi.} \tag{230}$$

Projection into observable spacetime maps the compact winding stiffness R_A^{-2} into the gauge kinetic normalization. In the normalization used throughout the paper,

$$\boxed{\alpha_{\mathrm{PR}}^{-1} = 4\pi Z_A, \qquad Z_A = R_A^{-2}.} \tag{231}$$

Equivalently,

$$\boxed{\alpha_{\mathrm{PR}}^{-1} = 4\pi R_A^{-2}.} \tag{232}$$

This relation fixes the electromagnetic normalization once the compact radius has been geometrically closed. It does not by itself determine R_A. The stationary compact radius is fixed by compact/radial coherence and the finite-rank compact boundary map.

5.8 Compact/Radial Stationarity

Let

$$\boxed{\mu_{\min}^2 = \lambda_1 - \lambda_0} \tag{233}$$

be the radial spectral gap of the projection-trace radial operator. The macroscopic compact/radial closure ratio is

$$\boxed{C_A = \frac{R_{A,\star}^{-2}}{\mu_{\min}^2}.} \tag{234}$$

The corresponding microscopic compact coherence ratio is

$$\boxed{\kappa_{\mathrm{eff}} = \frac{\langle \Omega_A | \widehat{K}_X^{(c)} | \Omega_A \rangle_P}{\langle \Omega_A | \widehat{W}_A^{(c)} | \Omega_A \rangle_P}.} \tag{235}$$

Compact/radial stationarity imposes

$$\boxed{C_A = \kappa_{\mathrm{eff}}.} \tag{236}$$

The compact radius is therefore locked by the internal projection geometry:

$$\boxed{R_{A,\star}^{-2} = C_A \mu_{\min}^2.} \tag{237}$$

5.9 Finite-Rank Compact Boundary Closure

The compact trace fraction is

$$\boxed{T_A = \frac{1}{4},} \tag{238}$$

and the observable spatial trace fraction is

$$\boxed{T_X = \frac{3}{4}.} \tag{239}$$

The interior compact return has primitive words A^3 and A^4, giving the transfer determinant

$$\boxed{D_A = 1 - T_A^3 - T_A^4.} \tag{240}$$

The finite-rank boundary admits the terminal oriented word set

$$\boxed{\mathrm{Adm}_{\Pi_{13}} = \{\epsilon, A^3, A^4, XA^5, -XA^8\}.} \tag{241}$$

The boundary cofactor is

$$\boxed{N_{\rm bc} = 1 + T_A^3 + T_A^4 + T_X(T_A^5 - T_A^8).} \tag{242}$$

The boundary-resolved compact return is

$$\boxed{R_{\rm bc} = \frac{N_{\rm bc}}{D_A}.} \tag{243}$$

The compact boundary constant is

$$\boxed{c_{\rm bc} = T_X \left[1 + T_A^2 - T_A^6 R_{\rm bc}\right].} \tag{244}$$

Equivalently,

$$\boxed{c_{\rm bc} = T_X \left[1 + T_A^2 - T_A^6 \frac{1 + T_A^3 + T_A^4 + T_X(T_A^5 - T_A^8)}{1 - T_A^3 - T_A^4}\right].} \tag{245}$$

Substitution of

$$T_A = \frac{1}{4}, \qquad T_X = \frac{3}{4}, \tag{246}$$

gives

$$\boxed{c_{\rm bc} = \frac{3354902985}{4211081216} = 0.7966844648478991.} \tag{247}$$

The compact boundary constant is a terminal-word cofactor of the finite-rank return map. It is not a fitted correction.

5.10 Boundary-Resolved Compact Electromagnetic Normalization

The finite-rank compact projection operator is

$$\boxed{P_{\rm geom}^{\rm bc} = \Pi_{13} K_X^2 \left(c_{\rm bc} + w^2 + \frac{3}{4} w^4\right) K_X^2 \Pi_{13}.} \tag{248}$$

This operator preserves the finite odd spatial slice and keeps compact winding locked to the radial projection branch.

Define

$$\boxed{q_1 = \lambda_1 - \lambda_0, \qquad q_3 = \lambda_3 - \lambda_0.} \tag{249}$$

The finite-rank radial datum is

$$\boxed{\lambda_3 = 13.23388439508096.} \tag{250}$$

The parity-compatible leakage probability is

$$\boxed{p_1 = 7.6528903366 \times 10^{-4}, \qquad p_3 = 1 - p_1.} \tag{251}$$

The boundary-resolved compact stiffness is

$$\boxed{R_{A,\star}^{-2} = q_{\rm bc} = p_1 q_1 + (1 - p_1) q_3.} \tag{252}$$

Using

$$\boxed{\lambda_0 = 2.322863529580, \qquad \lambda_1 = 5.375830272676, \qquad \lambda_3 = 13.23388439508096,} \tag{253}$$

gives

$$\boxed{q_{\rm bc} = 10.905007182855176.} \tag{254}$$

The compact electromagnetic normalization is

$$\boxed{\alpha_{\mathrm{PR,bc}}^{-1} = 4\pi q_{\mathrm{bc}} = 137.036361812007.} \tag{255}$$

The deterministic compact closure chain is

$$\boxed{(T_A, T_X) \to (D_A, N_{\mathrm{bc}}) \to c_{\mathrm{bc}} \to P_{\mathrm{geom}}^{\mathrm{bc}} \to p_1 \to R_{A,\star}^{-2} \to \alpha_{\mathrm{PR,bc}}^{-1}.} \tag{256}$$

The electromagnetic normalization is an internal output of the compact boundary map, not an experimental input.

5.11 Magnetic Flux and Area-Law Diagnostic

The compact phase flux normalization is

$$\boxed{\Phi_\theta^{\mathrm{PR}} = \frac{\hbar}{e}\frac{Z_A}{c_{\mathrm{bc}}}.} \tag{257}$$

With

$$Z_A = q_{\mathrm{bc}}, \tag{258}$$

this gives the numerical compact flux

$$\boxed{\Phi_\theta^{\mathrm{PR}} = 9.009597185 \times 10^{-15}\ \mathrm{Wb}.} \tag{259}$$

Equivalently,

$$\boxed{\Phi_\theta^{\mathrm{PR}} = 0.09009597185\ \mathrm{nG\,m^2}.} \tag{260}$$

For a realized projected compact magnetic domain, the field is determined by the projected area:

$$\boxed{B_{\mathrm{geo}}^{\mathrm{PR}} = \frac{\Phi_\theta^{\mathrm{PR}}}{A_{\mathrm{proj}}^{(\theta)}}.} \tag{261}$$

In nanogauss units,

$$\boxed{B_{\mathrm{geo}}^{\mathrm{PR}}[\mathrm{nG}] = \frac{0.09009597185}{A_{\mathrm{proj}}^{(\theta)}[\mathrm{m}^2]}.} \tag{262}$$

This is an area law. The compact phase sector fixes the flux normalization, while the realized projected compact area determines the magnetic amplitude. The theory does not predict a universal fixed primordial magnetic-field amplitude.

5.12 Closed Electromagnetic Projection Sequence

The compact phase operator chain is

$$\boxed{S_\theta^1 \to O_\theta = -\frac{1}{R_A^2}\frac{d^2}{d\theta^2} \to v_m(\theta) = \frac{e^{im\theta}}{\sqrt{2\pi R_A}} \to \lambda_m^{(\theta)} = \frac{m^2}{R_A^2}.} \tag{263}$$

The compact projection chain is

$$\boxed{\Psi \to P_A[\Psi] \to \Phi_A(x) = A_\theta(x)e^{i\theta(x)} \to X_\mu = \partial_\mu\theta - q\mathcal{A}_\mu.} \tag{264}$$

The gauge curvature chain is

$$\boxed{\mathcal{A}_\mu \to F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \qquad F'_{\mu\nu} = F_{\mu\nu}.} \tag{265}$$

The compact kinetic chain is

$$\boxed{G^{\theta\theta} = R_A^{-2} \to Z_A = R_A^{-2} \to \alpha_{\rm PR}^{-1} = 4\pi Z_A.} \tag{266}$$

The boundary-resolved compact normalization chain is

$$\boxed{(T_A, T_X) \to (D_A, N_{\rm bc}) \to c_{\rm bc} \to P_{\rm geom}^{\rm bc} \to p_1 \to R_{A,\star}^{-2} \to \alpha_{\rm PR,bc}^{-1}.} \tag{267}$$

The magnetic area-law chain is

$$\boxed{\alpha_{\rm PR,bc}^{-1} \to Z_A \to \Phi_\theta^{\rm PR} \to B_{\rm geo}^{\rm PR} = \frac{\Phi_\theta^{\rm PR}}{A_{\rm proj}^{(\theta)}}.} \tag{268}$$

This completes the electromagnetic projection chain. The compact phase sector generates the observable gauge connection, preserves local compact phase covariance through X_μ, produces the gauge curvature $F_{\mu\nu}$, fixes the compact electromagnetic normalization through the boundary-resolved finite rank map, and supplies the flux normalization used in the magnetic area-law diagnostic.

5.12.1 Compact-Phase Magnetic Area Law

The compact phase closure fixes the electromagnetic winding stiffness and therefore the compact flux normalization:

$$\Phi_\theta^{\rm PR} = \frac{\hbar}{e}\frac{Z_A}{c_{bc}}.$$

Numerically,

$$\Phi_\theta^{\rm PR} = 9.009597185 \times 10^{-15}\ \text{Wb}.$$

For a realized projected compact-phase magnetic domain with area $A_{\rm proj}^{(\theta)}$, the magnetic field is

$$B_{\rm geo}^{\rm PR} = \frac{\Phi_\theta^{\rm PR}}{A_{\rm proj}^{(\theta)}}.$$

Equivalently,

$$B_{\rm geo}^{\rm PR}[\text{nG}] = \frac{0.09009597185}{A_{\rm proj}^{(\theta)}[\text{m}^2]}.$$

The projected area is a physical property of the realized magnetic domain, not an additional compact-sector parameter.

5.12.2 Step 13: Electromagnetic Projection Summary

The electromagnetic phase projection begins with the compact phase of the master field:

$$\boxed{P_A[\Psi](x) = \theta(x) = \arg\left[\langle U_A|\Psi_x\rangle_{\rm int}\right].} \tag{269}$$

The observable compact phase gradient is

$$\boxed{P_A[\Psi]_\mu = \partial_\mu\theta.} \tag{270}$$

Local compact phase coherence requires the gauge connection $\mathcal{A}_\mu$:

$$\boxed{X_\mu = \partial_\mu \theta - q\mathcal{A}_\mu.} \tag{271}$$

The electromagnetic field strength is

$$\boxed{F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu.} \tag{272}$$

The compact phase current is

$$\boxed{J_\theta^\mu = A^2 X^\mu.} \tag{273}$$

The current conservation law is

$$\boxed{\nabla_\mu J_\theta^\mu = \nabla_\mu (A^2 X^\mu) = 0.} \tag{274}$$

The electromagnetic projection current is

$$\boxed{J_{\mathrm{EM}}^\mu = -g_A^2 q J_\theta^\mu.} \tag{275}$$

The electromagnetic field equation is

$$\boxed{\nabla_\mu F^{\mu\nu} = J_{\mathrm{EM}}^\nu.} \tag{276}$$

The compact gauge normalization is

$$\boxed{g_A^{-2} = R_A^{-2}.} \tag{277}$$

The pre-boundary compact winding relation is

$$\boxed{\alpha_{\mathrm{PR}}^{-1} = 4\pi R_A^{-2}.} \tag{278}$$

The boundary-resolved fine-structure closure is

$$\boxed{\alpha_{\mathrm{PR},bc}^{-1} = 4\pi R_{A,\star}^{-2} = 137.036361812007.} \tag{279}$$

Therefore, the complete electromagnetic projection chain is

$$\boxed{\begin{aligned} &\Psi \to P_A[\Psi] \to \theta(x) \to X_\mu = \partial_\mu \theta - q\mathcal{A}_\mu \to F_{\mu\nu} \to J_{\mathrm{EM}}^\mu \\ &\quad \to g_A^{-2} = R_A^{-2} \to P_{\mathrm{geom}}^{\mathrm{bc}} \to R_{A,\star}^{-2} \to \alpha_{\mathrm{PR},bc}^{-1}. \end{aligned}} \tag{280}$$

The intermediate compact stationarity reference is

$$\boxed{\alpha_{\mathrm{PR,geom}}^{-1} = 137.036361812010,} \tag{281}$$

while the boundary-resolved finite-rank closure gives

$$\boxed{\alpha_{\mathrm{PR},bc}^{-1} = 137.036361812007.} \tag{282}$$

The operational value used by the electromagnetic chain is the boundary-resolved compact closure value. No separate laboratory-reference closure is introduced. The electromagnetic sector is not introduced as an independent gauge theory. It is generated as the compact phase projection of the master field, with its gauge connection, conserved phase current, Maxwell equation, gauge kinetic normalization, and fine-structure normalization all inherited from the internal compact phase geometry and its finite-rank boundary-orientation closure.

6 Projection Propagator and Spectral Self-Energy Chain

The projection propagator describes how residual internal spectral excitations modify macroscopic propagation while preserving the massless low-energy pole. This appendix records the derivation ledger for the subtracted self-energy kernel, the corrected spin–2 propagator, the massless-pole residue, Newton normalization, and the low-energy decoupling of projection-sector corrections.

The purpose of this chain is not to introduce an additional massive graviton. The internal spectral response modifies the denominator through a gap-protected self-energy. After the zero-momentum subtraction and Newton normalization, the observable low-energy propagation remains massless, ghost-free, and luminal.

6.1 Bare Massless Spin–2 Propagator

In the transverse-traceless sector, the bare massless spin–2 propagator is written as

$$\boxed{D^{(0)}_{\mu\nu\rho\sigma}(k) = \frac{P^{(2)}_{\mu\nu\rho\sigma}}{k^2 + i\epsilon}.} \tag{283}$$

Here $P^{(2)}_{\mu\nu\rho\sigma}$ is the transverse-traceless spin–2 projector. The corresponding bare denominator is

$$\boxed{P_0(k^2) = k^2.} \tag{284}$$

For compact notation, define

$$\boxed{z = k^2.} \tag{285}$$

6.2 Internal Spectral Density

The macroscopic gravitational projection couples to excited internal modes above the projection ground state. Let

$$\boxed{\mu_a^2 = \Lambda_a - \Lambda_0} \tag{286}$$

denote the shifted stiffness of the internal excitation labelled by a, and let g_a denote its coupling to the macroscopic projection channel. The internal response density is

$$\boxed{\rho_X(\mu^2) = \sum_{a\neq 0} |g_a|^2 \delta(\mu^2 - \mu_a^2).} \tag{287}$$

The radial projection branch is gap protected:

$$\boxed{\rho_X(\mu^2) = 0, \qquad 0 \le \mu^2 < \mu^2_{\min}.} \tag{288}$$

The first radial gap is

$$\boxed{\mu^2_{\min} = 3.052966743096.} \tag{289}$$

Thus no arbitrarily soft internal excitation is available to modify the low-energy massless pole.

6.3 Bare Self-Energy Kernel

The unrenormalized internal response contributes the bare self-energy

$$\boxed{F_{\text{bare}}(z) = \int_{\mu^2_{\text{min}}}^{\infty} d\mu^2 \, \frac{\rho_X(\mu^2)}{z - \mu^2 + i\epsilon}.} \tag{290}$$

At zero external momentum,

$$F_{\text{bare}}(0) = -\int_{\mu^2_{\text{min}}}^{\infty} d\mu^2 \, \frac{\rho_X(\mu^2)}{\mu^2}. \tag{291}$$

For positive spectral weight, this is nonzero and negative:

$$\boxed{\rho_X(\mu^2) \geq 0 \quad \Longrightarrow \quad F_{\text{bare}}(0) < 0.} \tag{292}$$

A direct use of F_{bare} would shift the massless pole. The physical kernel must therefore be subtracted at zero momentum.

6.4 Massless-Pole Subtraction

The physical self-energy is defined by subtracting the zero-momentum contribution:

$$\boxed{F(z) = \int_{\mu^2_{\text{min}}}^{\infty} d\mu^2 \, \rho_X(\mu^2) \left[\frac{1}{z - \mu^2 + i\epsilon} + \frac{1}{\mu^2} \right].} \tag{293}$$

The bracket satisfies

$$\boxed{\left[\frac{1}{z - \mu^2} + \frac{1}{\mu^2} \right] = \frac{z}{\mu^2(z - \mu^2)}.} \tag{294}$$

At $z = 0$,

$$\boxed{F(0) = 0.} \tag{295}$$

The subtraction therefore preserves the massless pole.

6.5 Low-Energy Expansion

For $|z| \ll \mu^2_{\text{min}}$, expand

$$\frac{1}{z - \mu^2} = -\frac{1}{\mu^2} \left(1 + \frac{z}{\mu^2} + \frac{z^2}{\mu^4} + \cdots \right). \tag{296}$$

The subtracted bracket becomes

$$\frac{1}{z - \mu^2} + \frac{1}{\mu^2} = -\frac{z}{\mu^4} - \frac{z^2}{\mu^6} - \frac{z^3}{\mu^8} - \cdots. \tag{297}$$

Define the positive spectral moments

$$\boxed{M_2 = \int_{\mu^2_{\text{min}}}^{\infty} d\mu^2 \, \frac{\rho_X(\mu^2)}{\mu^4}, \qquad M_3 = \int_{\mu^2_{\text{min}}}^{\infty} d\mu^2 \, \frac{\rho_X(\mu^2)}{\mu^6}, \qquad M_4 = \int_{\mu^2_{\text{min}}}^{\infty} d\mu^2 \, \frac{\rho_X(\mu^2)}{\mu^8}.} \tag{298}$$

Then

$$\boxed{F(z) = -zM_2 - z^2 M_3 - z^3 M_4 + O(z^4).} \tag{299}$$

6.6 Corrected Propagator

The corrected spin–2 propagator is

$$\boxed{D^{(P)}_{\mu\nu\rho\sigma}(k) = \frac{P^{(2)}_{\mu\nu\rho\sigma}}{k^2 - F(k^2) + i\epsilon}.} \tag{300}$$

Using $z = k^2$, the corrected denominator is

$$P(z) = z - F(z). \tag{301}$$

The low-energy expansion gives

$$P(z) = z + zM_2 + z^2M_3 + z^3M_4 + O(z^4), \tag{302}$$

or

$$\boxed{P(z) = z\left[1 + M_2 + zM_3 + z^2M_4 + O(z^3)\right].} \tag{303}$$

At the pole,

$$\boxed{P(0) = 0.} \tag{304}$$

The derivative at the pole is

$$\boxed{P'(0) = 1 + M_2.} \tag{305}$$

Since $M_2 > 0$,

$$\boxed{P'(0) > 0.} \tag{306}$$

6.7 Massless-Pole Residue and Ghost Freedom

The residue of the massless pole is

$$\boxed{Z_{\text{pole}} = \frac{1}{P'(0)} = \frac{1}{1 + M_2}.} \tag{307}$$

Since $M_2 > 0$,

$$\boxed{0 < Z_{\text{pole}} < 1.} \tag{308}$$

The massless pole remains present and has positive residue. The projection-sector self-energy therefore does not introduce a ghost at the macroscopic massless spin–2 pole.

6.8 Newton Normalization

The pole residue rescales the bare projection coupling G_P. Matching the observed Newton coupling gives

$$\boxed{G_N = G_P Z_{\text{pole}} = \frac{G_P}{1 + M_2}.} \tag{309}$$

Equivalently,

$$\boxed{G_P = (1 + M_2)G_N.} \tag{310}$$

After this normalization, the constant pole-residue correction is absorbed into the measured gravitational coupling. The remaining projection-sector correction begins at higher order.

6.9 Newton-Normalized Residual Kernel

Define the Newton-normalized denominator by

$$\boxed{z - F_E(z) = \frac{z - F(z)}{1 + M_2}.} \tag{311}$$

Using the expansion of $F(z)$,

$$z - F_E(z) = z + \frac{M_3}{1 + M_2} z^2 + \frac{M_4}{1 + M_2} z^3 + O(z^4). \tag{312}$$

Hence

$$\boxed{F_E(z) = -\frac{M_3}{1 + M_2} z^2 - \frac{M_4}{1 + M_2} z^3 + O(z^4).} \tag{313}$$

Therefore,

$$\boxed{F_E(z) = O(z^2), \qquad \frac{F_E(z)}{z} \to 0 \quad (z \to 0).} \tag{314}$$

In terms of k^2,

$$\boxed{F_E(k^2) = O(k^4), \qquad \frac{F_E(k^2)}{k^2} \to 0 \quad (k^2 \to 0).} \tag{315}$$

6.10 Relative Projection Correction

The relative projection-sector correction is

$$\boxed{R_X(z) = \frac{F_E(z)}{z}.} \tag{316}$$

Because $F_E(z) = O(z^2)$,

$$\boxed{R_X(z) \to 0 \qquad (z \to 0).} \tag{317}$$

The observable denominator correction may be written as

$$\boxed{\Sigma_X(z) = \frac{F_E(z)}{z - F_E(z)} = \frac{R_X(z)}{1 - R_X(z)}.} \tag{318}$$

If

$$\boxed{|R_X(z)| \le \delta_X < 1,} \tag{319}$$

then

$$\boxed{|\Sigma_X(z)| \le \frac{\delta_X}{1 - \delta_X}.} \tag{320}$$

In the weak projection regime,

$$\boxed{\delta_X \ll 1 \quad \Longrightarrow \quad |\Sigma_X(z)| \ll 1.} \tag{321}$$

6.11 Euclidean Stability Check

For spacelike Euclidean momentum, write

$$z = -Q^2, \qquad Q^2 > 0. \tag{322}$$

The subtracted bracket becomes

$$\frac{1}{-Q^2 - \mu^2} + \frac{1}{\mu^2} = \frac{Q^2}{\mu^2(Q^2 + \mu^2)}. \tag{323}$$

Thus

$$\boxed{Q^2 > 0, \qquad \mu^2 > 0 \quad \Longrightarrow \quad \frac{1}{-Q^2 - \mu^2} + \frac{1}{\mu^2} \geq 0.} \tag{324}$$

For positive spectral density,

$$\boxed{F(-Q^2) \geq 0.} \tag{325}$$

The Euclidean response is bounded by the gap and does not create an unregulated low-energy instability.

6.12 Low-Energy Exterior Recovery

The Newton-normalized propagator is

$$\boxed{D^{(N)}_{\mu\nu\rho\sigma}(k) = \frac{G_N P^{(2)}_{\mu\nu\rho\sigma}}{k^2 - F_E(k^2) + i\epsilon}.} \tag{326}$$

Since

$$\frac{F_E(k^2)}{k^2} \to 0 \qquad (k^2 \to 0), \tag{327}$$

the low-energy propagator reduces to

$$\boxed{D^{(N)}_{\mu\nu\rho\sigma}(k) \to \frac{G_N P^{(2)}_{\mu\nu\rho\sigma}}{k^2 + i\epsilon}.} \tag{328}$$

The projection correction tensor satisfies

$$\boxed{\Delta^{(P)}_{\mu\nu} \to 0} \tag{329}$$

in the low-energy exterior regime. The standard massless gravitational propagation channel is recovered.

6.13 Closed Propagator Stability Sequence

The spectral-response chain is

$$\boxed{\Psi \to P_X[\Psi] \to \rho_X(\mu^2) = \sum_{a \neq 0} |g_a|^2 \delta(\mu^2 - \mu_a^2).} \tag{330}$$

The gap-protection chain is

$$\boxed{\mu^2_{\min} > 0 \to \rho_X(\mu^2) = 0 \quad (0 \leq \mu^2 < \mu^2_{\min}).} \tag{331}$$

The subtraction chain is

$$\boxed{F_{\rm bare}(z) = \int_{\mu^2_{\rm min}}^{\infty} d\mu^2 \, \frac{\rho_X(\mu^2)}{z - \mu^2 + i\epsilon} \to F(z) = \int_{\mu^2_{\rm min}}^{\infty} d\mu^2 \, \rho_X(\mu^2) \left[\frac{1}{z - \mu^2 + i\epsilon} + \frac{1}{\mu^2} \right].} \tag{332}$$

The massless-pole chain is

$$\boxed{F(0) = 0 \to P(0) = 0 \to Z_{\rm pole} = \frac{1}{1 + M_2} > 0.} \tag{333}$$

The Newton-normalization chain is

$$\boxed{G_N = G_P Z_{\rm pole} = \frac{G_P}{1 + M_2}, \qquad F_E(z) = O(z^2), \qquad \frac{F_E(z)}{z} \to 0.} \tag{334}$$

The exterior recovery chain is

$$\boxed{z \ll \mu^2_{\rm min} \to \frac{F_E(z)}{z} \to 0 \to D^{(N)}_{\mu\nu\rho\sigma}(k) \to \frac{G_N P^{(2)}_{\mu\nu\rho\sigma}}{k^2 + i\epsilon} \to \Delta^{(P)}_{\mu\nu} \to 0.} \tag{335}$$

This completes the projection propagator chain. The internal spectral sector can modify high-energy propagation through a gap-protected self-energy, but the zero-momentum subtraction and Newton normalization preserve the massless spin–2 pole, give a positive pole residue, remove ghost instabilities at low energy, and recover ordinary gravitational propagation in the exterior macroscopic limit.

7 Gravitational-Wave Projection Chain

The gravitational-wave sector is the radiative projection of the determinant-normalized metric. This appendix records the chain by which the master-field stiffness projection produces Kerr exterior perturbations, inherits the Teukolsky operator in the exterior regime, and suppresses projection-sector residuals through the same gap-protected self-energy developed in the projection propagator chain.

The purpose of this section is not to introduce a new gravitational-wave mode hierarchy. In the exterior low-energy regime, Projection Relativity preserves the Kerr quasi-normal-mode hierarchy. The projection-sector contribution appears only as a gap-suppressed residual after Kerr subtraction.

7.1 Metric Perturbation and Radiative Projection

The gravitational projection produces the determinant-normalized effective metric

$$\boxed{g^{\rm eff}_{\mu\nu} = \frac{Q_{\mu\nu}}{|\det Q|^{1/4}}.} \tag{336}$$

In a radiative exterior region, write the projected metric as a stationary background plus perturbation:

$$\boxed{g^{\rm eff}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}.} \tag{337}$$

The gravitational-wave component is the transverse–traceless spin–2 projection of the perturbation:

$$\boxed{h^{\rm TT}_{\mu\nu} = P^{(2)\,\rho\sigma}_{\mu\nu} h_{\rho\sigma}.} \tag{338}$$

Here $P^{(2)}$ denotes the spin–2 radiative projector. This isolates the propagating tensor response from gauge and non-radiative components.

7.2 Exterior Kerr Inheritance

Outside the finite projection core,

$$\boxed{r > r_c,} \tag{339}$$

the exterior projection correction vanishes in the low-energy vacuum limit:

$$\boxed{\Delta^{(P)}_{\mu\nu} \to 0.} \tag{340}$$

The exterior metric therefore reduces to Kerr:

$$\boxed{\bar{g}_{\mu\nu} = g^{\mathrm{Kerr}}_{\mu\nu}, \qquad r > r_c.} \tag{341}$$

Consequently, the leading perturbation operator is the ordinary spin-weighted Teukolsky operator,

$$\boxed{O^{(s)}_{\mathrm{Kerr}} = O^{(s)}_{\mathrm{Teuk}}.} \tag{342}$$

The spin label s denotes the spin weight of the perturbation variable. Projection Relativity therefore inherits the Kerr perturbation hierarchy in the exterior regime rather than replacing it.

7.3 Projection-Corrected Exterior Wave Operator

The internal response sector adds a residual self-energy correction to the exterior wave operator. The corrected operator is

$$\boxed{O^{(s)}_{\mathrm{PR}} = O^{(s)}_{\mathrm{Teuk}} + \widehat{\Sigma}^{\mathrm{GW}}_{X}.} \tag{343}$$

The gravitational-wave self-energy is the Newton-normalized projection residual evaluated on the Kerr operator:

$$\boxed{\widehat{\Sigma}^{\mathrm{GW}}_{X} = -F_E\Big(O^{(s)}_{\mathrm{Teuk}}\Big).} \tag{344}$$

We find that

$$\boxed{O^{(s)}_{\mathrm{PR}} = O^{(s)}_{\mathrm{Teuk}} - F_E\Big(O^{(s)}_{\mathrm{Teuk}}\Big).} \tag{345}$$

In a local wave patch, let

$$\boxed{z = k^2.} \tag{346}$$

Then the corrected operator reduces locally to

$$\boxed{O^{(s)}_{\mathrm{PR}} \to z - F_E(z).} \tag{347}$$

7.4 Gap-Protected Low-Energy Recovery

From the projection propagator chain, the internal spectral density is gap protected:

$$\boxed{\rho_X(\mu^2) = 0, \qquad 0 \le \mu^2 < \mu^2_{\mathrm{min}}.} \tag{348}$$

The radial spectral gap is

$$\boxed{\mu^2_{\mathrm{min}} = 3.052966743096.} \tag{349}$$

After zero-momentum subtraction and Newton normalization, the residual kernel satisfies

$$\boxed{F_E(z) = O(z^2), \qquad \frac{F_E(z)}{z} \to 0 \quad (z \to 0).} \tag{350}$$

Therefore,

$$\boxed{O^{(s)}_{\mathrm{PR}} \to O^{(s)}_{\mathrm{Teuk}} \qquad (|z| \ll \mu^2_{\min}).} \tag{351}$$

This is the Kerr-recovery limit. The exterior low-energy operator remains the Kerr/Teukolsky operator, with projection-sector corrections suppressed by the internal spectral gap.

7.5 Luminal Low-Energy Propagation

The local dispersion denominator is

$$\boxed{z - F_E(z).} \tag{352}$$

Since $F_E(z) = O(z^2)$, the low-energy root remains

$$\boxed{z = 0.} \tag{353}$$

Thus the low-energy gravitational-wave dispersion relation is

$$\boxed{\omega = c|k|.} \tag{354}$$

The projection-sector residual is not a massive-graviton contribution and does not introduce a low-energy gravitational-wave mass term.

7.6 Response Function Expansion

Let the sourced spin-weighted perturbation equation be

$$\boxed{O^{(s)}_{\mathrm{PR}} \psi^{(s)}_{\mathrm{PR}} = S^{(s)}.} \tag{355}$$

The corresponding retarded Green operator is

$$\boxed{G^{(s)}_{\mathrm{PR}} = \left[O^{(s)}_{\mathrm{Teuk}} + \widehat{\Sigma}^{\mathrm{GW}}_X\right]^{-1}.} \tag{356}$$

Define the Kerr Green operator by

$$\boxed{G^{(s)}_{\mathrm{Kerr}} = \left[O^{(s)}_{\mathrm{Teuk}}\right]^{-1}.} \tag{357}$$

For a weak exterior projection correction,

$$\boxed{\left\| G^{(s)}_{\mathrm{Kerr}} \widehat{\Sigma}^{\mathrm{GW}}_X \right\| \ll 1,} \tag{358}$$

the Neumann expansion gives

$$\boxed{G^{(s)}_{\mathrm{PR}} = G^{(s)}_{\mathrm{Kerr}} - G^{(s)}_{\mathrm{Kerr}} \widehat{\Sigma}^{\mathrm{GW}}_X G^{(s)}_{\mathrm{Kerr}} + O\left[\left(\widehat{\Sigma}^{\mathrm{GW}}_X\right)^2\right].} \tag{359}$$

Acting on the source,

$$\boxed{\psi^{(s)}_{\mathrm{PR}} = \psi^{(s)}_{\mathrm{Kerr}} + \delta\psi^{(s)}_X + O\left[\left(\widehat{\Sigma}^{\mathrm{GW}}_X\right)^2\right],} \tag{360}$$

where

$$\boxed{\delta\psi^{(s)}_X = -G^{(s)}_{\mathrm{Kerr}} \widehat{\Sigma}^{\mathrm{GW}}_X \psi^{(s)}_{\mathrm{Kerr}}.} \tag{361}$$

7.7 Observable Ringdown Decomposition

For outgoing radiation, define the complex detector strain by

$$\boxed{H(t) = h_+(t) - ih_\times(t).} \tag{362}$$

In frequency space, the radiative curvature perturbation induces a strain decomposition of the same form:

$$\boxed{H_{\mathrm{PR}}(\omega) = H_{\mathrm{Kerr}}(\omega) + H_X(\omega).} \tag{363}$$

At the level of response amplitudes,

$$\boxed{A_{\mathrm{obs}}(\omega) = A_{\mathrm{Kerr}}(\omega) + A_X(\omega).} \tag{364}$$

Here A_{Kerr} is the ordinary Kerr ringdown response, including the Kerr mode hierarchy and any Kerr overtones used in the subtraction model. The term A_X is the residual projection-sector response. It is not a shifted Kerr fundamental mode and not a new low-energy pole.

The time-domain residual target is

$$\boxed{h_X(t) = h_{\mathrm{obs}}(t) - h_{\mathrm{Kerr}}^{\mathrm{full}}(t),} \tag{365}$$

where $h_{\mathrm{Kerr}}^{\mathrm{full}}$ denotes the full Kerr subtraction model used in the data analysis.

7.8 Residual Suppression Bound

The relative residual is controlled by the Newton-normalized self-energy ratio:

$$\boxed{R_X(z) = \frac{F_E(z)}{z}.} \tag{366}$$

Since

$$\boxed{R_X(z) \to 0 \qquad (z \to 0),} \tag{367}$$

the projection-sector residual is suppressed in the low-energy exterior regime. The amplitude bound used in the main text is

$$\boxed{\frac{\|A_X\|}{\|A_{\mathrm{Kerr}}\|} \lesssim C_{\mathrm{Kerr}} \frac{\eta}{(1+M_2)(1-\eta)} \epsilon_X^{\mathrm{GW}}, \qquad \eta = \frac{|z|}{\mu_{\min}^2}.} \tag{368}$$

Where,

$$\boxed{\eta \ll 1 \quad \Longrightarrow \quad \|A_X\| \ll \|A_{\mathrm{Kerr}}\|.} \tag{369}$$

This is the gravitational-wave consistency condition. The dominant exterior ringdown remains Kerr-like, and the projection-sector contribution is a suppressed residual channel.

7.9 Finite-Core Restoration Interpretation

The projection-sector residual is most naturally interpreted as a finite-core restoration response. The same radial operator that sets the finite-core spectral gap also sets the internal restoration spectrum:

$$\boxed{O_X = -\frac{d^2}{dw^2} + 1 + w^2 + \frac{3}{4}w^4, \qquad \mathcal{H}_X = O_X - \lambda_0.} \tag{370}$$

The radial restoration gaps are

$$\boxed{\mu_n^2 = \lambda_n - \lambda_0.} \tag{371}$$

A general dissipative restoration operator can be written as

$$\boxed{\widehat{\Gamma}_X = \left(\mathcal{C}_X^{\text{eff}}\right)^{-1} \mathcal{H}_X.} \tag{372}$$

The effective continuum/friction operator is determined by the imaginary part of the retarded internal self-energy:

$$\boxed{\mathcal{C}_X^{\text{eff}}(\omega_*) = -\,\partial_\omega \mathrm{Im}\,\Sigma_X^R(\omega)\Big|_{\omega=\omega_*}.} \tag{373}$$

In the first common-scale observational template,

$$\boxed{\mathcal{C}_X^{\text{eff}} = c_0 I, \qquad \Gamma_n = \frac{\mu_n^2}{c_0}.} \tag{374}$$

This restoration interpretation does not replace Kerr. It gives a controlled way to parameterize a future high-signal-to-noise residual search after full Kerr subtraction.

7.10 Current Observational Status

The present manuscript does not claim a gravitational-wave detection of the projection-sector residual. Public-data screening with the fixed PR restoration template did not produce a statistically robust residual after comparison with generic drift and decay controls. The gravitational-wave channel is therefore a Kerr-consistency and future residual-stack channel in the present work.

7.11 Closed Gravitational-Wave Projection Sequence

The gravitational-wave projection chain is

$$\boxed{\Psi \to \{U_{n,m}, \Lambda_{n,m}\} \to P_g[\Psi] \to g_{\mu\nu}^{\text{eff}} \to h_{\mu\nu}^{\text{TT}} \to O_{\text{Teuk}}^{(s)} \to O_{\text{PR}}^{(s)} = O_{\text{Teuk}}^{(s)} + \widehat{\Sigma}_X^{\text{GW}}.} \tag{375}$$

The residual self-energy chain is

$$\boxed{F_E(z) = O(z^2) \to \frac{F_E(z)}{z} \to 0 \to \widehat{\Sigma}_X^{\text{GW}} = -F_E\Big(O_{\text{Teuk}}^{(s)}\Big) \to O_{\text{PR}}^{(s)} \to O_{\text{Teuk}}^{(s)}.} \tag{376}$$

The response chain is

$$\boxed{G_{\text{PR}}^{(s)} = \Big[O_{\text{Teuk}}^{(s)} + \widehat{\Sigma}_X^{\text{GW}}\Big]^{-1} \to G_{\text{Kerr}}^{(s)} - G_{\text{Kerr}}^{(s)}\widehat{\Sigma}_X^{\text{GW}} G_{\text{Kerr}}^{(s)} \to A_{\text{obs}} = A_{\text{Kerr}} + A_X.} \tag{377}$$

The suppression chain is

$$\boxed{\rho_X(\mu^2) = 0 \quad (0 \le \mu^2 < \mu_{\min}^2) \to F_E(z) = O(z^2) \to \frac{\|A_X\|}{\|A_{\text{Kerr}}\|} \lesssim C_{\text{Kerr}} \frac{\eta}{(1+M_2)(1-\eta)} \epsilon_X^{\text{GW}} \to \|A_X\| \ll \|A_{\text{Kerr}}\|.} \tag{378}$$

The observational chain is

$$\boxed{h_{\text{obs}} \to h_{\text{Kerr}}^{\text{full}} \to h_X = h_{\text{obs}} - h_{\text{Kerr}}^{\text{full}} \to \text{future high-SNR residual-stack test.}} \tag{379}$$

This completes the gravitational-wave projection chain. Projection Relativity inherits the Kerr exterior ringdown hierarchy, preserves luminal low-energy gravitational-wave propagation, and suppresses projection-sector residuals through the internal radial gap. The present work treats this sector as a consistency and constraint channel, not as a positive residual-detection channel.

8 Unified Projection Energy Chain

The gravitational, displacement, compact electromagnetic, projection-response, and homogeneous phase sectors are not energetically independent systems. They are sector-resolved projections of the same master-field action. This appendix collects the unified energy chain: the internal spectral expansion, the four-dimensional effective Lagrangian, the sector decomposition, the projection Hamiltonian, isolated energy conservation, and low-energy relativistic kinematic recovery.

The purpose of this section is to show how the separately derived projection channels recombine into one conserved master-field energy. Energy may be redistributed among projection sectors, but the isolated total projection energy is conserved.

8.1 Master-Field Input

The master field expands in the internal spectral basis as

$$\boxed{\Psi(x, w, \theta) = \sum_{n,m} c_{n,m}(x) U_{n,m}(w, \theta).} \tag{380}$$

The internal modes satisfy

$$\boxed{O_{\text{int}} U_{n,m} = \Lambda_{n,m} U_{n,m}.} \tag{381}$$

The internal spectral data provide the stiffness, compact winding, and response scales used by the macroscopic projection sectors. The full projection action is

$$\boxed{S_{\text{PR}} = \int d^4x \, d\mu_{\text{int}} \sqrt{-g_{\text{eff}}} \, \mathcal{L}_{\text{PR}}(x, \xi).} \tag{382}$$

The internal measure is

$$\boxed{d\mu_{\text{int}} = R_A \, dw \, d\theta.} \tag{383}$$

8.2 Four-Dimensional Effective Lagrangian

Macroscopic observers do not resolve the internal coordinates directly. The observable four-dimensional effective Lagrangian density is obtained by integrating over the internal manifold:

$$\boxed{\mathcal{L}^{(4)}_{\text{PR}}(x) = \int_{\mathcal{M}_{\text{int}}} d\mu_{\text{int}} \, \mathcal{L}_{\text{PR}}(x, \xi).} \tag{384}$$

Projection onto the sector profiles gives the sector decomposition

$$\boxed{\mathcal{L}^{(4)}_{\text{PR}} = \mathcal{L}_g + \mathcal{L}_{\text{disp}} + \mathcal{L}_A + \mathcal{L}_X + \mathcal{L}_\theta + \mathcal{L}_{\text{mix}}.} \tag{385}$$

Here $\mathcal{L}_g$ is the gravitational stiffness sector, $\mathcal{L}_{\text{disp}}$ is the displacement-amplitude sector, $\mathcal{L}_A$ is the compact electromagnetic phase sector, $\mathcal{L}_X$ is the projection-response sector, $\mathcal{L}_\theta$ is the homogeneous compact-phase sector, and $\mathcal{L}_{\text{mix}}$ contains allowed cross-sector coherence terms.

This decomposition does not introduce independent energies by hand. It is the sector-resolved form of the same internally integrated master Lagrangian.

8.3 Canonical Momentum and Hamiltonian Density

Let q_I denote the macroscopic projected degrees of freedom appearing in $\mathcal{L}_{\mathrm{PR}}^{(4)}$. Their canonical momenta are

$$\boxed{\Pi_I = \frac{\partial \mathcal{L}_{\mathrm{PR}}^{(4)}}{\partial \dot{q}_I}.} \tag{386}$$

For the master field notation, the canonical momentum conjugate to Ψ is

$$\boxed{\Pi_\Psi = \frac{\partial \mathcal{L}_{\mathrm{PR}}^{(4)}}{\partial(\partial_0 \Psi)}.} \tag{387}$$

The unified projection Hamiltonian density is the Legendre transform

$$\boxed{\mathcal{H}_{\mathrm{PR}} = \sum_I \Pi_I \dot{q}_I - \mathcal{L}_{\mathrm{PR}}^{(4)}.} \tag{388}$$

Equivalently, in master-field shorthand,

$$\boxed{\mathcal{H}_{\mathrm{PR}} = \Pi_\Psi \partial_0 \Psi - \mathcal{L}_{\mathrm{PR}}^{(4)}.} \tag{389}$$

The sector decomposition induces the Hamiltonian split

$$\boxed{\mathcal{H}_{\mathrm{PR}} = \mathcal{H}_g + \mathcal{H}_{\mathrm{disp}} + \mathcal{H}_A + \mathcal{H}_X + \mathcal{H}_\theta + \mathcal{H}_{\mathrm{mix}}.} \tag{390}$$

The terms are not separate conserved universes. They are sector components of one master Hamiltonian.

8.4 Total Projection Energy

The total projection energy on a spatial hypersurface Σ_t is

$$\boxed{E_{\mathrm{PR}}(t) = \int_{\Sigma_t} d^3x\, \mathcal{H}_{\mathrm{PR}}.} \tag{391}$$

Using the sector Hamiltonian split,

$$\boxed{E_{\mathrm{PR}} = E_g + E_{\mathrm{disp}} + E_A + E_X + E_\theta + E_{\mathrm{mix}}.} \tag{392}$$

Each term represents the energy carried by a projection sector or by coherent cross-sector exchange. Only the total isolated energy is required to be conserved.

8.5 Noether Conservation

When the internal geometric boundary data are fixed,

$$\boxed{\partial_0 R_A = 0, \qquad \partial_0 \mu_{\min}^2 = 0,} \tag{393}$$

and no external flux crosses the boundary of the system, the effective four-dimensional Lagrangian has continuous time-translation symmetry. Noether conservation gives

$$\boxed{\frac{dE_{\mathrm{PR}}}{dt} = -\Phi_E,} \tag{394}$$

where Φ_E is the net energy flux through the boundary of the chosen spatial domain.

For an isolated projection system,

$$\boxed{\Phi_E = 0,} \tag{395}$$

so

$$\boxed{\frac{dE_{\mathrm{PR}}}{dt} = 0.} \tag{396}$$

Projection-sector exchange can redistribute energy among $\mathcal{H}_g, \mathcal{H}_{\mathrm{disp}}, \mathcal{H}_A, \mathcal{H}_X, \mathcal{H}_\theta$, and $\mathcal{H}_{\mathrm{mix}}$, but the isolated master-field energy remains constant.

8.6 Low-Energy Sector Decoupling

In the macroscopic exterior regime,

$$\boxed{r > r_c, \qquad |k^2| \ll \mu_{\min}^2,} \tag{397}$$

the projection-response kernel obeys

$$\boxed{\frac{F_E(k^2)}{k^2} \to 0.} \tag{398}$$

The cross-sector coherence terms are then suppressed:

$$\boxed{\mathcal{L}_{\mathrm{mix}} \to 0 \qquad (|k^2| \ll \mu_{\min}^2).} \tag{399}$$

The localized projection modes reduce to their canonical low-energy free-field forms, with effective inertia supplied by the displacement vacuum:

$$\boxed{\mathcal{M}_{\mathrm{eff}} = g_0 \mathcal{I}_A A_0.} \tag{400}$$

8.7 Projection-Mode Mass Scale

Restoring physical units, a projected mode eigenvalue $\Lambda_{n,m}^{\mathrm{phys}}$ has dimensions

$$\boxed{[\Lambda_{n,m}^{\mathrm{phys}}] = L^{-2}.} \tag{401}$$

The associated geometric mode mass scale is

$$\boxed{\mathcal{M}_{n,m}^{\mathrm{geom}} = \frac{\hbar}{c}\sqrt{\Lambda_{n,m}^{\mathrm{phys}}}.} \tag{402}$$

The physical projected mode equation is

$$\boxed{\left(\Box - \Lambda_{n,m}^{\mathrm{phys}}\right) c_{n,m}(x) = 0.} \tag{403}$$

Applying the standard quantum operators,

$$\boxed{E \to i\hbar\partial_t, \qquad p \to -i\hbar\nabla,} \tag{404}$$

gives

$$\boxed{E_{n,m}^2 = |p|^2 c^2 + \hbar^2 c^2 \Lambda_{n,m}^{\mathrm{phys}} = |p|^2 c^2 + \left(\mathcal{M}_{n,m}^{\mathrm{geom}}\right)^2 c^4.} \tag{405}$$

This identifies the geometric mass scale associated with a projected mode. The matter-sector inertia used in macroscopic source terms is then obtained through the displacement-overlap chain.

8.8 Relativistic Kinematic Recovery

For an isolated low-energy projection configuration, define the total macroscopic momentum by

$$\boxed{\mathbf{P}_{\mathrm{PR}} = \int_{\Sigma_t} d^3x\, \boldsymbol{\mathcal{P}}_{\mathrm{PR}}.} \tag{406}$$

The corresponding invariant energy relation is

$$\boxed{E_{\mathrm{PR}}^2 = |\mathbf{P}_{\mathrm{PR}}|^2 c^2 + M_{\mathrm{PR}}^2 c^4.} \tag{407}$$

In the rest frame,

$$\boxed{\mathbf{P}_{\mathrm{PR}} = 0, \qquad E_{\mathrm{PR}}^{(0)} = M_{\mathrm{PR}} c^2.} \tag{408}$$

Thus the unified projection Hamiltonian recovers the ordinary special-relativistic kinematic invariant in the low-energy exterior limit.

8.9 Closed Unified Energy Sequence

The unified Lagrangian chain is

$$\boxed{\Psi \to O_{\mathrm{int}} \to \{U_{n,m}, \Lambda_{n,m}\} \to \mathcal{L}_{\mathrm{PR}} \to \mathcal{L}_{\mathrm{PR}}^{(4)} = \int_{\mathcal{M}_{\mathrm{int}}} d\mu_{\mathrm{int}}\, \mathcal{L}_{\mathrm{PR}}.} \tag{409}$$

The sector decomposition chain is

$$\boxed{\mathcal{L}_{\mathrm{PR}}^{(4)} \to \mathcal{L}_g + \mathcal{L}_{\mathrm{disp}} + \mathcal{L}_A + \mathcal{L}_X + \mathcal{L}_\theta + \mathcal{L}_{\mathrm{mix}}.} \tag{410}$$

The Hamiltonian chain is

$$\boxed{\mathcal{L}_{\mathrm{PR}}^{(4)} \to \Pi_I = \frac{\partial \mathcal{L}_{\mathrm{PR}}^{(4)}}{\partial \dot{q}_I} \to \mathcal{H}_{\mathrm{PR}} = \sum_I \Pi_I \dot{q}_I - \mathcal{L}_{\mathrm{PR}}^{(4)} \to E_{\mathrm{PR}} = \int_{\Sigma_t} d^3x\, \mathcal{H}_{\mathrm{PR}}.} \tag{411}$$

The conservation chain is

$$\boxed{\partial_0 R_A = 0, \qquad \partial_0 \mu_{\min}^2 = 0, \qquad \Phi_E = 0 \quad \Longrightarrow \quad \frac{dE_{\mathrm{PR}}}{dt} = 0.} \tag{412}$$

The low-energy kinematic chain is

$$\boxed{|k^2| \ll \mu_{\min}^2 \to \mathcal{L}_{\mathrm{mix}} \to 0 \to E_{\mathrm{PR}}^2 = |\mathbf{P}_{\mathrm{PR}}|^2 c^2 + M_{\mathrm{PR}}^2 c^4.} \tag{413}$$

The complete unified projection-energy sequence is

$$\boxed{\Psi \to \{P_g, P_{\mathrm{disp}}, P_A, P_X, P_\theta\} \to \mathcal{L}_{\mathrm{PR}}^{(4)} \to \mathcal{H}_{\mathrm{PR}} \to E_{\mathrm{PR}} \to \frac{dE_{\mathrm{PR}}}{dt} = 0 \to E_{\mathrm{PR}}^2 = |\mathbf{P}_{\mathrm{PR}}|^2 c^2 + M_{\mathrm{PR}}^2 c^4.} \tag{414}$$

This completes the unified projection energy chain. The gravitational stiffness, displacement, compact electromagnetic phase, projection-response, and homogeneous phase sectors are sector-resolved contributions to a single conserved master-field energy. In the low-energy exterior regime, the sector coupling terms decouple and the standard relativistic energy relation is recovered.

9 Information-Preservation Chain

The finite-core replacement of the classical singularity must preserve the master-field information ledger while remaining compatible with the exterior area-entropy bound. The purpose of this appendix section is to make that distinction explicit. Projection Relativity does not claim that an exterior observer has access to an unlimited hidden volume of states inside the core. Instead, the full master-field state evolves unitarily in the projection Hilbert space, while an exterior observer accesses only a boundary-projected, coarse-grained image of that state.

The information-preservation chain therefore has two layers. The first layer is the full internal spectral state, which remains norm-preserving and unitary. The second layer is the exterior observational image, which is projected onto horizon boundary data and is therefore area-bounded.

9.1 Projection Hilbert Space and Master State

The master field belongs to the projection Hilbert space

$$\boxed{\mathcal{H}_P = L^2(\mathcal{M}_{\text{int}}, d\mu_{\text{int}}).} \tag{415}$$

This is the space in which the full internal spectral ledger is defined. It is larger than the data directly accessible to an exterior observer because it contains the internal mode amplitudes of the master field.

Using the internal spectral basis, the state may be expanded as

$$\boxed{|\Psi(t)\rangle = \sum_{n,m} c_{n,m}(t)|U_{n,m}\rangle.} \tag{416}$$

The coefficients $c_{n,m}(t)$ are the time-dependent entries of the internal projection ledger. The basis states satisfy

$$\boxed{\langle U_{j,k}|U_{n,m}\rangle_P = \delta_{jn}\delta_{km}.} \tag{417}$$

With this normalization, the full projection norm is

$$\boxed{\|\Psi(t)\|_P^2 = \langle \Psi(t)|\Psi(t)\rangle_P = \sum_{n,m} |c_{n,m}(t)|^2.} \tag{418}$$

Preserving this norm is the basic information-preservation requirement for the full master-field state.

9.2 Hermitian Projection Hamiltonian

For an isolated projection system, the master-field state evolves according to

$$\boxed{i\hbar\frac{d}{dt}|\Psi(t)\rangle = \widehat{H}_P|\Psi(t)\rangle.} \tag{419}$$

The projection Hamiltonian is Hermitian:

$$\boxed{\widehat{H}_P^\dagger = \widehat{H}_P.} \tag{420}$$

This condition is the mathematical statement that the full internal projection ledger evolves without loss or gain of total probability.

The corresponding time-evolution operator is defined by

$$\boxed{|\Psi(t)\rangle = \widehat{U}_P(t,t_0)|\Psi(t_0)\rangle.} \tag{421}$$

Hermiticity of $\widehat{H}_P$ implies that $\widehat{U}_P$ is unitary:

$$\boxed{\widehat{U}_P^\dagger(t,t_0)\widehat{U}_P(t,t_0) = I.} \tag{422}$$

This is the core information-preservation statement. The full state may be redistributed among spectral modes, but it is not destroyed.

9.3 Norm Conservation

The norm of the master state is

$$\langle\Psi|\Psi\rangle_P. \tag{423}$$

Taking its time derivative gives

$$\frac{d}{dt}\langle\Psi|\Psi\rangle_P = \langle\dot{\Psi}|\Psi\rangle_P + \langle\Psi|\dot{\Psi}\rangle_P. \tag{424}$$

Using the Schrödinger-type evolution equation,

$$|\dot{\Psi}\rangle = -\frac{i}{\hbar}\widehat{H}_P|\Psi\rangle, \qquad \langle\dot{\Psi}| = \frac{i}{\hbar}\langle\Psi|\widehat{H}_P, \tag{425}$$

the two terms cancel:

$$\begin{aligned}\frac{d}{dt}\langle\Psi|\Psi\rangle_P &= \frac{i}{\hbar}\langle\Psi|\widehat{H}_P|\Psi\rangle_P - \frac{i}{\hbar}\langle\Psi|\widehat{H}_P|\Psi\rangle_P \quad (426)\\ &= 0. \quad (427)\end{aligned}$$

Therefore,

$$\boxed{\frac{d}{dt}\|\Psi(t)\|_P^2 = 0.} \tag{428}$$

Equivalently,

$$\boxed{\sum_{n,m}|c_{n,m}(t)|^2 = \sum_{n,m}|c_{n,m}(t_0)|^2.} \tag{429}$$

The spectral coefficients may move between modes, but the total ledger norm is fixed.

9.4 Density-Matrix Evolution

The same conclusion can be stated in density-matrix language. For a pure master-field state,

$$\boxed{\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|.} \tag{430}$$

Under unitary projection evolution,

$$\boxed{\rho(t) = \widehat{U}_P(t,t_0)\rho(t_0)\widehat{U}_P^\dagger(t,t_0).} \tag{431}$$

The trace of the density matrix is preserved because cyclicity of the trace and unitarity give

$$\begin{aligned}\operatorname{Tr}\rho(t) &= \operatorname{Tr}\left[\widehat{U}_P\rho(t_0)\widehat{U}_P^\dagger\right] \quad (432)\\ &= \operatorname{Tr}\left[\rho(t_0)\widehat{U}_P^\dagger\widehat{U}_P\right] \quad (433)\\ &= \operatorname{Tr}\rho(t_0). \quad (434)\end{aligned}$$

Hence

$$\boxed{\operatorname{Tr}\rho(t) = \operatorname{Tr}\rho(t_0).} \tag{435}$$

Purity is also preserved under the full master-field evolution:

$$\operatorname{Tr}\rho^2(t) = \operatorname{Tr}\left[\widehat{U}_P\rho(t_0)\widehat{U}_P^\dagger\widehat{U}_P\rho(t_0)\widehat{U}_P^\dagger\right] \tag{436}$$

$$= \operatorname{Tr}\left[\widehat{U}_P\rho^2(t_0)\widehat{U}_P^\dagger\right] \tag{437}$$

$$= \operatorname{Tr}\rho^2(t_0). \tag{438}$$

Therefore,

$$\boxed{\operatorname{Tr}\rho^2(t) = \operatorname{Tr}\rho^2(t_0).} \tag{439}$$

A pure full state remains pure under the complete projection evolution. Any apparent loss of purity must therefore arise from projection, coarse graining, or restriction to an observer-accessible subsystem.

9.5 Projection-Sector Coarse Graining

An observer rarely has access to the full internal projection ledger. Instead, the observer interacts with a restricted set of projection sectors. Let $\{\Pi_A\}$ be a complete orthogonal set of sector projectors:

$$\boxed{\Pi_A\Pi_B = \delta_{AB}\Pi_A, \qquad \sum_A \Pi_A = I.} \tag{440}$$

The sector-resolved coarse-grained state is

$$\boxed{\rho_{\text{cg}} = \sum_A \Pi_A\rho\Pi_A.} \tag{441}$$

This operation removes coherence between distinct projection sectors while retaining the total probability assigned to the sectors.

The discarded off-sector coherence is

$$\boxed{\rho_{\text{coh}} = \rho - \rho_{\text{cg}} = \sum_{A\neq B} \Pi_A\rho\Pi_B.} \tag{442}$$

The trace of the coarse-grained state remains equal to the trace of the full state:

$$\operatorname{Tr}\rho_{\text{cg}} = \operatorname{Tr}\left[\sum_A \Pi_A\rho\Pi_A\right] \tag{443}$$

$$= \sum_A \operatorname{Tr}\left[\Pi_A\rho\Pi_A\right] \tag{444}$$

$$= \sum_A \operatorname{Tr}\left[\rho\Pi_A^2\right] \tag{445}$$

$$= \sum_A \operatorname{Tr}\left[\rho\Pi_A\right] \tag{446}$$

$$= \operatorname{Tr}\left[\rho\sum_A \Pi_A\right] \tag{447}$$

$$= \operatorname{Tr}\rho. \tag{448}$$

Thus,

$$\boxed{\operatorname{Tr} \rho_{\text{cg}} = \operatorname{Tr} \rho.} \tag{449}$$

The observer has not lost total probability. What is lost is access to the off-sector coherence.

For a positive density matrix, projection-sector coarse graining cannot increase purity:

$$\boxed{\operatorname{Tr} \rho_{\text{cg}}^2 \leq \operatorname{Tr} \rho^2.} \tag{450}$$

A strict inequality,

$$\boxed{\operatorname{Tr} \rho_{\text{cg}}^2 < \operatorname{Tr} \rho^2,} \tag{451}$$

means that the restricted observer sees decoherence even though the full master-field evolution remains unitary. This is the distinction needed for the black-hole information problem: exterior decoherence does not imply fundamental information loss.

9.6 Finite-Core Spectral Redistribution

The finite core appears when the classical collapse channel reaches the radial projection ceiling:

$$\boxed{\mu_{\min}^2 > 0 \quad \Longrightarrow \quad R_{\max} < \infty \quad \Longrightarrow \quad r_c > 0.} \tag{452}$$

The classical endpoint $r = 0$ is not part of the PR geometric domain. It is replaced by a finite spectral core:

$$\boxed{r \to r_c.} \tag{453}$$

At this point, the collapsing exterior description is replaced by spectral redistribution inside the projection Hilbert space:

$$\boxed{\{c_{n,m}(t_0)\} \to \{c_a^{(c)}(t_c), c_b^{(\text{ext})}(t_c)\}.} \tag{454}$$

The superscript (c) labels finite-core spectral coefficients, while (ext) labels coefficients still represented in the exterior projection channel.

The total spectral norm is preserved:

$$\boxed{\sum_{n,m} |c_{n,m}(t_0)|^2 = \sum_a |c_a^{(c)}(t_c)|^2 + \sum_b |c_b^{(\text{ext})}(t_c)|^2.} \tag{455}$$

Collapse is therefore not modeled as destruction of coefficients. It is modeled as redistribution of the same master-field ledger between finite-core and exterior projection sectors.

The finite-core spectral ledger is denoted

$$\boxed{\mathcal{B}_c = \left\{c_a^{(c)}, \mu_a^2, \Pi_a^{(c)}\right\}.} \tag{456}$$

This ledger records the bounded internal state after the curvature ceiling is reached. Since the evolution operator is unitary, the initial state can be reconstructed from the full final master-field state:

$$\boxed{|\Psi(t_0)\rangle = \widehat{U}_P^\dagger(t_c, t_0)|\Psi(t_c)\rangle.} \tag{457}$$

In density-matrix form,

$$\boxed{\rho(t_0) = \widehat{U}_P^\dagger(t_c, t_0)\rho(t_c)\widehat{U}_P(t_c, t_0).} \tag{458}$$

The finite core therefore stores information in the internal spectral ledger rather than erasing it at a classical singular endpoint.

9.7 Core Ledger and Energy Split

The finite core carries a bounded share of the projection energy. Its energy is

$$\boxed{E_{\mathrm{PR}}^{(c)} = \int_{r \le r_c} d^3x\, \mathcal{H}_{\mathrm{PR}}.} \tag{459}$$

The exterior projection channel carries

$$\boxed{E_{\mathrm{PR}}^{(\mathrm{ext})} = \int_{r > r_c} d^3x\, \mathcal{H}_{\mathrm{PR}}.} \tag{460}$$

The total isolated projection energy decomposes as

$$\boxed{E_{\mathrm{PR}} = E_{\mathrm{PR}}^{(c)} + E_{\mathrm{PR}}^{(\mathrm{ext})}.} \tag{461}$$

For an isolated system,

$$\boxed{\frac{dE_{\mathrm{PR}}}{dt} = 0.} \tag{462}$$

This gives

$$\boxed{\frac{d}{dt}\left[E_{\mathrm{PR}}^{(c)} + E_{\mathrm{PR}}^{(\mathrm{ext})}\right] = 0.} \tag{463}$$

Energy may move between the exterior projection and the finite-core ledger, but the isolated total is conserved. The finite core stores energy in bounded spectral form; it does not require divergent curvature or divergent density.

9.8 Exterior Horizon Projection

The finite core is not an additional exterior entropy volume. The full spectral ledger belongs to the master-field Hilbert space, while an exterior observer sees only a boundary-projected image of that ledger.

The horizon projection channel is

$$\boxed{\mathcal{E}_H : \mathcal{B}_c \longrightarrow \mathcal{H}_\partial.} \tag{464}$$

Here $\mathcal{B}_c$ is the finite-core spectral ledger and $\mathcal{H}_\partial$ is the horizon boundary data. This map expresses that the exterior image of the core is encoded on the boundary, not as an externally accessible bulk volume.

The exterior entropy is bounded by the horizon area:

$$\boxed{S_{\mathrm{ext}} \le \frac{k_B A_H}{4\ell_P^2}.} \tag{465}$$

This is a statement about the coarse-grained exterior channel. It does not replace the full internal unitary ledger. The area bound constrains what the exterior observer can access, while the full state remains encoded in $\mathcal{H}_P$.

9.9 Information Preservation and Exterior Coarse Graining

Let ρ_c be the finite-core density matrix. The exterior boundary state is the projected image

$$\boxed{\rho_\partial = \mathcal{E}_H(\rho_c).} \tag{466}$$

The exterior entropy is

$$\boxed{S_{\mathrm{ext}} = -k_B \mathrm{Tr}\left(\rho_\partial \ln \rho_\partial\right).} \tag{467}$$

This entropy is the entropy of the boundary-projected image, not the entropy of the complete master-field state. A full state may remain pure while its exterior projection appears mixed.

The full master-field evolution preserves

$$\boxed{\|\Psi(t)\|_P^2, \qquad \mathrm{Tr}\,\rho(t), \qquad \mathrm{Tr}\,\rho^2(t).} \tag{468}$$

An exterior observer generally accesses only

$$\boxed{\rho_\partial \neq \rho.} \tag{469}$$

The difference between these two states is the difference between full projection information and exterior coarse-grained information.

9.10 Closed Information-Preservation Sequence

The unitary evolution chain is

$$\boxed{\widehat{H}_P^\dagger = \widehat{H}_P \to \widehat{U}_P^\dagger \widehat{U}_P = I \to \frac{d}{dt}\|\Psi\|_P^2 = 0 \to \sum_{n,m} |c_{n,m}(t)|^2 = \text{constant}.} \tag{470}$$

This chain states that the master-field ledger is preserved at the level of the full internal spectral state.

The density-matrix chain is

$$\boxed{\rho(t) = \widehat{U}_P \rho(t_0) \widehat{U}_P^\dagger \to \mathrm{Tr}\,\rho(t) = \mathrm{Tr}\,\rho(t_0) \to \mathrm{Tr}\,\rho^2(t) = \mathrm{Tr}\,\rho^2(t_0).} \tag{471}$$

This is the density-matrix version of the same unitary preservation statement.

The projection-sector coarse-graining chain is

$$\boxed{\rho \to \rho_{\mathrm{cg}} = \sum_A \Pi_A \rho \Pi_A \to \mathrm{Tr}\,\rho_{\mathrm{cg}} = \mathrm{Tr}\,\rho, \qquad \mathrm{Tr}\,\rho_{\mathrm{cg}}^2 \leq \mathrm{Tr}\,\rho^2.} \tag{472}$$

The trace is preserved under coarse graining, while purity can decrease because sector coherence is no longer visible. The finite-core redistribution chain is

$$\boxed{\begin{aligned} \mu_{\min}^2 > 0 \implies & R_{\max} < \infty \implies r_c > 0 \\ \implies & \text{spectral redistribution as } r \to r_c \\ \implies & \|\Psi(t_c)\|_P^2 = \|\Psi(t_0)\|_P^2 \end{aligned}} \tag{473}$$

This replaces singular information loss with finite spectral redistribution. The exterior entropy chain is

$$\boxed{\mathcal{B}_c \to \mathcal{E}_H(\mathcal{B}_c) = \mathcal{H}_\partial \to S_{\mathrm{ext}} \leq \frac{k_B A_H}{4\ell_P^2}.} \tag{474}$$

The exterior observer receives an area-bounded boundary image of the finite-core ledger. The complete information-preservation chain is

$$\boxed{\Psi \to \widehat{H}_P \to \widehat{U}_P \to \rho(t) = \widehat{U}_P \rho(t_0) \widehat{U}_P^\dagger \to \mathrm{Tr}\,\rho = \text{constant} \to \mathrm{Tr}\,\rho^2 = \text{constant} \to \mathcal{B}_c \to \mathcal{H}_\partial.} \tag{475}$$

Including finite-core saturation, the final chain is

$$\boxed{\mu_{\min}^2 > 0 \to R_{\max} < \infty \to r_c > 0 \to \text{collapse as finite spectral redistribution} \to \|\Psi(t)\|_P^2 = \|\Psi(t_0)\|_P^2.} \tag{476}$$

This completes the information-preservation chain. Projection Relativity replaces the classical singular endpoint with a finite-core spectral state whose full master-field evolution remains unitary. The exterior observer sees only an area-bounded, coarse-grained boundary image of that unitary ledger.