ON THE REPRESENTATION OF CERTAIN ASYMPTOTIC SERIES AS CONVERGENT CONTINUED FRACTIONS

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1. It has been proved by Prof. T. Muir (*Edinburgh Trans.*, Vol. XXVII., 667–671) that, in general, a power series in x, $a_0+a_1x+a_2x^2+\ldots$, can be represented as a continued fraction of the form

$$\frac{e_0}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-}\,\ldots,$$

and, for convenience of reference, it will be useful to indicate the proof of the determinant expressions which give the e's in terms of the a's.

If P_n and Q_n are respectively numerator and denominator of the n-th convergent left unreduced, so that

$$P_1 = e_0, \quad P_2 = e_0, \quad P_n = P_{n-1}-e_{n-1}xP_{n-2};$$

and $$Q_1 = 1, \quad Q_2 = 1-e_1x, \quad Q_n = Q_{n-1}-e_{n-1}xQ_{n-2},$$

it will be readily seen that P_{2n-1}, P_{2n}, Q_{2n-1} are of degree $n-1$ in x, and that Q_n is of degree n.

Moveover, if E_n denote $\frac{e_n}{1-}\,\frac{e_{n+1}x}{1-}\,\ldots$, then

$$E_0Q_n-P_n = E_0E_1\ldots E_nx^n. \tag{1}$$

If $n = 2m$, we get, by equating coefficients of x^m, x^{m+1}, ..., x^{2m}, the relation

$$e_0e_1\,\ldots\,e_{2m} = \alpha_{2m}/\alpha_{2m-2}, \tag{2}$$

where $$\alpha_{2m} = \begin{vmatrix} a_{2m}, & a_{2m-1}, & \ldots, & a_m \\ a_{2m-1}, & a_{2m-2}, & \ldots, & a_{m-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_m, & a_{m-1}, & \ldots, & a_0 \end{vmatrix},$$

which is a determinant of a type called persymmetric.

Also, by considering the coefficients of x^{2m+1}, we get

$$e_0e_1\,\ldots\,e_{2m}(e_1+e_2+\ldots+e_{2m+1}) = \alpha_{2m,\,1}/\alpha_{2m-2}, \tag{3}$$

where $\alpha_{2m,1}$ denotes what α_{2m} becomes when the suffixes of the first row are all increased by unity.

Similarly, by making $n = 2m+1$, we get

$$e_0 e_1 \dots e_{2m+1} = a_{2m+1}/a_{2m-1} \tag{4}$$

and
$$e_0 e_1 \dots e_{2m+1}(e_1+e_2+\dots e_{2m+2}) = a_{2m+1,\,1}/a_{2m-1}, \tag{5}$$

where
$$a_{2m+1} = \begin{vmatrix} a_{2m+1}, & a_{2m}, & \dots, & a_{m+1} \\ a_{2m}, & a_{2m-1}, & \dots, & a_m \\ \vdots & \vdots & \vdots & \vdots \\ a_{m+1}, & a_m, & \dots, & a_1 \end{vmatrix},$$

and $a_{2m+1,\,1}$ is what a_{2m+1} becomes when the suffixes of the first row are all increased by unity.

The converse relations, viz., the expressions giving a_0, a_1, ... in terms of the e's, can be determined successively, but apparently not generally. For, if δ denote an operation which changes a_n into a_{n+1}, we get $\delta a_n = a_{n,\,1}$, both when n is even and when n is odd.

Hence, if $n > 1$,

$$\delta e_0 e_1 \dots e_n = \frac{a_n}{a_{n-2}}\left(\frac{a_{n,\,1}}{a_n} - \frac{a_{n-2,\,1}}{a_{n-2}}\right), \quad \text{by (2) or (4),}$$
$$= e_0 e_1 \dots e_n (e_n + e_{n+1}), \quad \text{by (3) or (5),}$$

therefore
$$\delta e_n = e_n (e_{n+1} - e_{n-1})$$

while
$$\delta e_0 = e_0 e_1, \quad \text{and} \quad \delta e_1 = e_1 e_2.$$

Thus
$$a_0 = e_0,$$
$$a_1 = e_0 e_1,$$
$$a_2 = \delta e_0 e_1 = e_0 e_1 (e_1 + e_2),$$
$$a_3 = \delta e_0 e_1 (e_1+e_2) = e_0 e_1 (e_1+e_2)^2 + e_0 e_1 e_2 e_3,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots .$$

The $2n$-th convergent denotes that rational algebraic fraction in x, of degree in numerator $n-1$ and of degree in denominator n, which is identical with the given series up to $2n$ terms, *i.e.*, as far as x^{2n-1}; while the $(2n+1)$-th convergent is identical with the given series up to $2n+1$ terms, and has numerator and denominator both of degree n.

The alternate convergents thus form a series of fractions which may in certain cases approach a finite limit, even when the given series does not absolutely converge.

Such will be found to be the case when the series is of the asymptotic type derived from certain definite integrals by continued process of integration by parts.

It will be necessary first to establish certain lemmata which will hereafter be found important.

LEMMA I.—The $2m$-th convergent of the fraction

$$\frac{e_0}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-}\,\dots$$

is identical with the m-th convergent of

$$\frac{e_0}{1-e_1x-}\,\frac{e_1e_2x^2}{1-(e_2+e_3)x-}\,\frac{e_3e_4x^2}{1-(e_4+e_5)x-}\,\dots.$$

For, from the relation

$$P_{2m}=P_{2m-1}-e_{2m-1}xP_{2m-2},\quad P_{2m-1}=P_{2m-2}-e_{2m-2}xP_{2m-3},\quad \dots,$$

we may, by eliminating P_{2m-1}, P_{2m-3}, obtain

$$P_{2m}=\{1-(e_{2m-2}+e_{2m-1})x\}P_{2m-2}-e_{2m-3}e_{2m-2}x^2P_{2m-4},$$

together with a similar relation connecting alternate Q's; which two equations, since the necessary initial conditions are satisfied, give the required result.

LEMMA II.—If $\quad \dfrac{e_0}{1-}\,\dfrac{e_1x}{1-}\dots=e_0+\dfrac{f_1x}{1-}\,\dfrac{f_2x}{1-}\dots,$

then $\quad f_1=e_0e_1,\quad f_2=e_1+e_2,\quad f_2f_3=e_2e_3,\quad f_3+f_4=e_3+e_4,\quad \dots.$

For
$$E_0=\frac{e_0}{1-xE_1}=e_0+\frac{e_0e_1x}{\dfrac{e_1}{E_1}-e_1x},$$

while
$$E_1=\frac{e_1}{1-e_2x-}\,\frac{e_2e_3x^2}{1-(e_3+e_4)x-}\dots\quad \text{(by Lemma I.)};$$

therefore
$$E_0=e_0+\frac{e_0e_1x}{1-(e_1+e_2)x-}\,\frac{e_2e_3x^2}{1-(e_3+e_4)x-}\,\dots.$$

But
$$E_0=e_0+\frac{f_1x}{1-f_2x-}\,\frac{f_2f_3x^2}{1-(f_3+f_4)x-}\dots\quad \text{(by Lemma I.)},$$

whence, by comparing the two expressions for E_0, we obtain the required relations between the f's and the e's.

LEMMA III.—If in a continued fraction of the second class

$$\frac{e_0}{1-}\,\frac{e_1}{1-}\,\frac{e_2}{1-}\,\dots,$$

where the e's are all positive, the convergent denominators are all positive and the fraction is known to be convergent, then also will the fraction

$$\frac{e_0}{1-}\,\frac{f_1}{1-}\,\frac{f_2}{1-}\,\dots$$

be convergent, provided $f_n<e_n$.

Let p_n/q_n be the n-th convergent of the first fraction, and P_n/Q_n that of the second.

Then, since $p_{n+1}=p_n-e_n p_{n-1}, \ldots$, we have

$$\frac{p_{n+1}}{q_{n+1}}-\frac{p_n}{q_n}=\frac{e_n q_{n-1}}{q_{n+1}}\left(\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}\right).$$

Now it is evident that, if $\dfrac{f_n Q_{n-1}}{Q_{n+1}}<\dfrac{e_n q_{n-1}}{q_{n+1}}$ for all values of n above some fixed integer, the second continued fraction converges more rapidly than the first.

Now

$$\frac{e_n q_{n-1}}{q_{n+1}}=\frac{q_n}{q_{n+1}}-1 \quad \text{and} \quad \frac{q_n}{q_{n+1}}=\frac{1}{1-}\,\frac{e_n}{1-}\,\frac{e_{n-1}}{1-}\cdots\frac{e_2}{1-e_1}.$$

Since, however, $f_1<e_1$, $f_2<e_2$,

$$\frac{f_2}{1-f_1}<\frac{e_2}{1-e_1},$$

and, since $f_3<e_3$,
$$\frac{f_3}{1-}\,\frac{f_2}{1-f_1}<\frac{e_3}{1-}\,\frac{e_2}{1-e_1},$$

and generally
$$\frac{Q_n}{Q_{n+1}}<\frac{q_n}{q_{n+1}},$$

which is the condition required in order that the second continued fraction should converge more rapidly than the first.

COROLLARY.—If $m_0, m_1, m_2, \ldots>1$, then $\dfrac{e_0}{m_0-}\,\dfrac{e_1}{m_1-}\,\dfrac{e_2}{m_2-}\cdots$ converges more rapidly than $\dfrac{e_0}{1-}\,\dfrac{e_1}{1-}\cdots$.

As Lemma IV., I introduce for reference the well known condition for fractions of the first class

$$\frac{e_0}{1+}\,\frac{e_1}{1+}\,\frac{e_2}{1+}\cdots,$$

where all the e's are positive; viz., that, if the fraction be reduced to the form

$$\frac{1}{d_0+}\,\frac{1}{d_1+}\,\frac{1}{d_2+}\cdots,$$

it will converge, provided at least one of the series $d_0+d_2+\ldots$ or $d_1+d_3+\ldots$ is divergent.

It may be remarked that in this case the fraction

$$\frac{e_0}{1+}\,\frac{e_1 x}{1+}\,\frac{e_2 x}{1+}\cdots$$

is also convergent for all positive values of x.

2. Let u_n denote $\int_0^\infty \operatorname{sn}^n t\, e^{-t/x}\, dt$. Then, if $n > 2$,

$$u_n = \Big[-xe^{-t/x}\operatorname{sn}^n t\Big]_0^\infty + x\int_0^\infty \frac{d}{dt}\operatorname{sn}^n t\, e^{-t/x}\, dt = x^2\int_0^\infty \frac{d^2}{dt^2}\operatorname{sn}^n t\, e^{-t/x}\, dt$$

$$= x^2 n(n-1)\, u_{n-2} - n^2 x^2 (1+k^2)\, u_n + n^2(n+1)\, x^2 u_{n+2}.$$

If $n = 1$, we have

$$u_1 = x\int_0^\infty \operatorname{cn} t \operatorname{dn} t\, e^{-t/x}\, dt$$

$$= \Big[-x^2 \operatorname{cn} t \operatorname{dn} t\, e^{-t/x}\Big]_0^\infty + x^2\int_0^\infty e^{-t/x}\{-(1+k^2)\operatorname{sn} t - 2k^2 \operatorname{sn}^3 t\}\, dt$$

$$= x^2 - x^2(1+k^2)\, u_1 - 2k^2 x^2 u_3,$$

whence $$u_1 = \frac{x^2}{1+(1+k^2)x^2 - 2k^2x^2 u_3/u_1}$$

$$= \frac{x^2}{1+(1+k^2)x^2-}\;\frac{1.2^2.3k^2x^4}{1+3^2(1+k^2)x^2-}\;\frac{3.4^2.5k^2x^4}{1+5^2(1+k^2)x^2-}. \qquad (1)$$

Let $x = 1/u$; then

$$\int_0^\infty \operatorname{sn} t\, e^{-ut}\, dt = \frac{1}{u^2+1+k^2-}\;\frac{1.2^2.3k^2}{u^2+3^2(1+k^2)-}\;\frac{3.4^2.5k^2}{u^2+5^2(1+k^2)-}\;\cdots.$$

To prove that this fraction is convergent for all real values of u and k, where $k < 1$, let us write

$$u^2 = (1+k^2)v^2 \quad \text{and} \quad \frac{2k}{1+k^2} = m.$$

$$(1+k^2)\int_0^\infty \operatorname{sn} t\, e^{-ut}\, dt = \frac{1+k^2}{(1+k^2)(v^2+1)-}\;\frac{2^2.1.3.k^2}{(1+k^2)(v^2+3^2)-}\;\cdots$$

$$= \frac{1}{v^2+1-}\;\frac{m^2.1.3}{v^2+3^2-}\;\frac{2^2m^2.3.5}{v^2+5^2-}$$

$$= \frac{1}{v^2+1-}\;\frac{m^2}{\frac{1}{3}(v^2+3^2)-}\;\frac{2^2m^2}{\frac{1}{5}(v^2+5^2)-}\;\frac{3^2m^2}{\frac{1}{7}(v^2+7^2)-}. \qquad (2)$$

If $v = 0$, this becomes

$$\frac{1}{1-}\;\frac{m^2}{3-}\;\frac{2^2m^2}{5-}\;\frac{3^2m^2}{7-} \qquad (3)$$

where $m < 1$.

This fraction is known to be convergent and equal to $m^{-1}\tanh^{-1} m$; so that, by Lemma III., the right-hand side of (2) is also convergent.

If $k=1$, the identity (1) becomes

$$\int_0^\infty \tanh t\, e^{-t/x}\,dt = \frac{x^2}{1+2x^2-}\,\frac{1.2^2.3x^4}{1+2.3^2x^2-}\,\frac{3.4^2.5x^4}{1+2.5^2x^2-},$$

which, by Lemma I., $$= \frac{x^2}{1+}\,\frac{1.2x^2}{1+}\,\frac{2.3x^2}{1+}\,\frac{3.4x^2}{1+}\,\dots. \tag{4}$$

The convergence of this fraction may be established also from Lemma IV.

From considerations similar to the foregoing, we have also

$$\int_0^\infty \operatorname{sn}^2 t\, e^{-t/x}\,dt = \frac{2x^3}{1+2^2(1+k^2)x^2-}\,\frac{2.3^2.4k^2x^4}{1+4^2(1+k^2)x^2-}, \tag{5}$$

of which the convergency is, by Lemma III., more rapid than that of $\frac{m^2}{2-}\,\frac{3^2m^2}{4-}\,\frac{5^2m^2}{6-}\,\dots$, which, again, is more rapid than that of (3).

3. If v_n denote $\int_0^\infty \operatorname{cn}^n t\, e^{-t/x}\,dt$, we shall, as in the last section, obtain a general linear relation connecting v_{n-2}, v_n, and v_{n+2}.

By Lemma I., it will be found that the integral may be expressed in the form

$$\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{2^2k^2x^2}{1+}\,\frac{3^2x^2}{1+}\,\frac{4^2k^2x^2}{1+}\,\dots, \tag{1}$$

the convergency of which can be established by Lemma IV.

Similarly, $$\int_0^\infty \operatorname{dn} t\, e^{-t/x}\,dt = \frac{x}{1+}\,\frac{k^2x^2}{1+}\,\frac{2^2x^2}{1+}\,\frac{3^2k^2x^2}{1+}\,\dots,$$

and, by putting $k=1$,

$$\int_0^\infty \operatorname{sech} t\, e^{-t/x}\,dt = \frac{x}{1+}\,\frac{x^2}{1+}\,\frac{2^2x^2}{1+}\,\frac{3^2x^2}{1+}\,\dots.$$

4. If Landen's theorem be applied to the results of § 2, (1) and (5), we get from (1), after some reductions,

$$\int_0^\infty \frac{\operatorname{sn} t \operatorname{cn} t}{\operatorname{dn} t} e^{-t/x}\,dt$$

$$= \frac{x^2}{1+2(1+k'^2)x^2-}\,\frac{1.2^2.3k^4x^4}{1+2.3^2(1+k'^2)x^2-}\,\frac{3.4^2.5k^4x^4}{1+2.5^2(1+k'^2)x^2-\dots}$$

$$= \frac{x^2}{1+2^2x^2-2k^2x^2-}\,\frac{1.2^2.3k^4x^4}{1+2^2.3^2x^2-2.3^2k^2x^2-}\,\frac{3.4^2.5k^4x^4}{1+2^2.5^2x^2-2.5^2k^2x^2-\dots},$$

which also $$= \frac{x^2}{1+2^2x^2-}\,\frac{1.2k^2x^2}{1-}\,\frac{2.3k^2x^2}{1+6^2x^2-}\,\frac{3.4k^2x^2}{1-}\,\frac{4.5k^2x^2}{1+10^2x^2-\dots}$$

by Lemma I.; the x of the lemma being the k^2 of this section. So, too,

$$\int_0^\infty \frac{k^2 \operatorname{sn}^2 t \operatorname{cn}^2 t}{\operatorname{dn}^2 t} e^{-t/x} dt = \frac{k^2x^3}{1+2.2^2(1+k'^2)x^2-} \frac{2.3^2.4k^4x^4}{1+2.4^2(1+k'^2)x^2-\dots},$$

whence $$\int_0^\infty \frac{1-k^2 \operatorname{sn}^4 t}{\operatorname{dn}^2 t} e^{-t/x} dt = x + \frac{k^2x^3}{1+2.2^2(1+k'^2)x^2-\dots}.$$

The right-hand side of this equation may be treated by Lemma II., where k^2 is the x of the lemma; and, transforming the elliptic function at the same time, we have

$$\int_0^\infty \frac{2e^{-t/x}}{1+\operatorname{dn} 2t} dt = \frac{x}{1-} \frac{2k^2x^2}{1+4^2x^2-} \frac{2.3k^2x^2}{1-} \frac{3.4k^2x^2}{1+8^2x^2-\dots}.$$

5. A very general theorem relating to the conversion of a definite integral of the form $\int_0^\infty f(t)\, e^{-t/x} dt$ may be obtained from the following considerations.

Suppose $f(x)$ to be represented by a power series in x with positive integral indices, and that the number of terms is indefinite. Then it is not difficult to see that we have sufficient arbitrary constants at our disposal to assume that

$$f(x+y) = A_0 f(x) f(y) + A_1 f_1(x) f_1(y) + A_2 f_2(x) f_2(y) + \dots \tag{1}$$

where A_0, A_1, A_2 are independent of x and y. Also the leading power of x in $f_n(x)$ may be assumed to be x^n, the leading coefficient being at present arbitrary. If $f(x)$ consists of a finite number of terms, the functions f_1, f_2, ... are, of course, finite in number; otherwise the number of these functions is indefinite, but, as we are only concerned with the manner of deducing successively the constants A_0, A_1, A_2, ... and the coefficients in f_1, f_2, ... from those in f, we need not consider the question of convergency of any of the series.

We shall, moreover, assume that the constants A_0, A_1, A_2, ... do not vanish identically, which assumption includes the supposition that $f(0)$ is not equal to zero.

Differentiating (1) n times for y, and putting $y = 0$, we have

$$\frac{d^n}{dx^n} f(x) = \text{linear function of } f, f_1, \dots, f_n.$$

Using all such deductions for values of n from 1 to n, we see that f_n is a linear function of f and its first n derivatives, and that the condition that the leading power of x shall be x^n makes this representation of f_n unique.

Hence $Lf_{n-1}+M\frac{d}{dx}f_{n-1}+Nf_{n-2}$, where L, M, N are constants, is a linear function of f and its first n derivatives, and in general its leading power of x is x^{n-2}.

But we may so choose L, M, and N that the coefficients of x^{n-2} and x^{n-1} are zero, and in this case the function can be none other than a constant multiple of f_n. We may write then

$$f_n = Lf_{n-1}+M\frac{d}{dx}f_{n-1}+Nf_{n-2}. \tag{2}$$

Now, let
$$\int_0^\infty f_n(t)\,e^{-t/x}\,dt = x^{n+1}\phi_n(x).$$

Then
$$\int_0^\infty \frac{d}{dt}f_{n-1}(t)\,e^{-t/x}\,dt = \Big[f_{n-1}(t)e^{-t/x}\Big]_0^\infty + \frac{1}{x}\int_0^\infty f_{n-1}(t)\,e^{-t/x}\,dt.$$

Assuming that the integrated part vanishes at both limits, we see that (2) reduces to
$$x^2\phi_n = Lx\phi_{n-1}+M\phi_{n-1}+N\phi_{n-2}.$$

Since the leading coefficients in the ϕ's are still arbitrary, we may write this relation in the form

$$\phi_{n-2} = (1-a_n x)\,\phi_{n-1}+\beta_n x^2\phi_n. \tag{3}$$

The integrated part in the above equation will not, however, vanish at the lower limit when $n=1$; so that this case will require special consideration.

We have
$$\frac{d}{dt}f(t) = Af'(0)f(t)+Bf_1'(0)f_1(t)\,;$$

therefore

$$\Big[f(t)e^{-t/x}\Big]_0^\infty + \frac{1}{x}\int_0^\infty f(t)\,e^{-t/x}\,dt = Af'(0)\int_0^\infty f(t)\,e^{-t/x}dt+Bf_1'(0)\int_0^\infty f_1(t)\,e^{-t/x}dt.$$

If
$$f(t) = a_0+a_1t+\frac{a_2}{2!}t^2+\frac{a_3}{3!}t^3+\ldots,$$

then, still assuming that $t=\infty$ gives a zero value for $f(t)e^{-t/x}$, we have

$$-a_0+\phi_0 = xAa_1\phi_0+Bf_1'(0)x^2\phi_1$$

or
$$(1-xa_1)\,\phi_0 = a_0+\beta_1x^2\phi_1,$$

while, by (3),
$$\phi_0 = (1-a_2x)\,\phi_1+\beta_2x^2\phi_2,$$
$$\phi_1 = (1-a_3x)\,\phi_2+\beta_3x^2\phi_3,$$
$$\ldots \qquad \ldots \qquad \ldots .$$

Hence
$$\phi_0 = a_0 + a_1 x + a_2 x^2 + \dots$$
$$= \frac{a_0}{1 - a_1 x - \beta_1 x^2 \phi_1/\phi_0}$$
$$= \frac{a_0}{1 - a_1 x -} \, \frac{\beta_1 x^2}{1 - a_2 x -} \, \frac{\beta_2 x^2}{1 - a_3 x - \dots}. \tag{4}$$

It may be noticed that the terms independent of x in all the functions ϕ are the same and equal to a_0. Moreover, it should be mentioned that (4) fails if any of the β's are zero, *i.e.*, if any of the persymmetrical determinants called a_{2n} in § 1 vanish.

We may now easily deduce the value of A_n. For $\frac{d^n}{dx^n} f(x) =$ a linear function in $f, f_1, \dots, f_n$, where the coefficient in f_n is $A_n \frac{d^n}{dy^n} f_n(y)$, when $y = 0$. But
$$f_n(y) = \frac{a_0 y^n}{n!} + \dots;$$
therefore this coefficient is $A_n a_0$. Thus
$$\frac{d^n}{dx^n} f(x) = A_n a_0 f_n(x) + \text{terms in } f_{n-1}(x) \dots .$$

Similarly, $$\frac{d^{n+1}}{dx^{n+1}} f(x) = A_{n+1} a_0 f_{n+1}(x) + \dots .$$

But $$\frac{d^{n+1}}{dx^{n+1}} f(x) = A_n a_0 \frac{d}{dx} f_n(x) + \dots$$
$$= A_n a_0 \{f_{n-1}(x) + a_n f_n(x) + \beta_n f_{n+1}(x)\} + \dots .$$

Hence, equating coefficients of $f_{n+1}(x)$, we have
$$A_{n+1} = \beta_n A_n.$$

We see, then, that, if $\phi_0(x)$ is known as a continued fraction
$$\frac{e_0}{1-} \, \frac{e_1 x}{1 - \dots},$$
or, by Lemma I., as $$\frac{a_0}{1 - a_1 x -} \, \frac{\beta_1 x^2}{1 - a_2 x - \dots},$$
we may obtain a series of functions $\phi_1, \phi_2, \dots$; so that an identity
$$f(x+y) = A_0 f(x) f(y) + A_1 f_1(x) f_1(y) + \dots$$
may be obtained, where the leading power of $f_n(x)$ is x^n.

Conversely, if the relation (1) is known, we may convert $\phi_0(x)$ into a continued fraction.

The connection is simpler when $f(x)$ is an even function of x; for then

$$a_1 = a_2 \dots = 0.$$

For instance, the "addition" theorem in Bessel's function

$$J_0(x+y) = J_0(x)\,J_0(y) + 2J_1(x)\,J_1(y) + \dots$$

is connected with the chain-fraction form for $(1+x^2)^{-\frac{1}{2}}$, which is obtained from Gauss's formula for converting the quotient

$$F\{\alpha,\ \beta+1,\ \gamma+1,\ x\} / F\{\alpha,\ \beta,\ \gamma,\ x\}$$

into a continued fraction. Again, the relation

$$J_0\sqrt{(x^2+y^2)} = J_0(x)\,J_0(y) - 2J_2(x)\,J_2(y) + \dots$$

is connected with the chain-fraction form for e^x, also obtainable from Gauss's formula.

A very general relation of the type of the Bessel formula may be got from Gauss's by putting $\beta = 0$, so that $f(x)$ is of the form

$$1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \dots;$$

but is not of sufficient importance to work out here. The functions $f_1, f_2, \dots$ have all factorial coefficients and are absolutely convergent; but the relation (1) would not be valid unless the series of functions on the right-hand side were also convergent.

It is perhaps interesting to note that, if $f(x)$ is not algebraic, the series in (1) cannot be finite unless the fraction in (4) is algebraic. It then can be expressed as a series of partial fractions of the form $\dfrac{A}{1+ax}$, and the corresponding $f(x)$ would consist of a finite series of exponential functions of x, the simplest cases of (1) being

$$e^{x+y} = e^x . e^y \quad \text{and} \quad \cos(x+y) = \cos x \cos y - \sin x \sin y.$$

The elliptic functions $\operatorname{cn} x$ and $\operatorname{dn} x$ lend themselves to the formula of this section in a very elegant way. We have

$$\begin{aligned}\operatorname{cn}(x+y) &= \frac{\operatorname{cn} x \operatorname{cn} y - \operatorname{sn} x \operatorname{dn} x \operatorname{sn} y \operatorname{dn} y}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y} \\ &= \operatorname{cn} x \operatorname{cn} y - \operatorname{sn} x \operatorname{dn} x \operatorname{sn} y \operatorname{dn} y + k^2 \operatorname{sn}^2 x \operatorname{cn} x \operatorname{sn}^2 y \operatorname{dn} y - \dots;\end{aligned}$$

so that the series $f_1, f_2, \dots$ are

$$-\operatorname{sn} x \operatorname{dn} x, \qquad \frac{1}{2}\operatorname{sn}^2 x \operatorname{dn} x, \qquad -\frac{1}{3!}\operatorname{sn}^3 x \operatorname{cn} x, \qquad \frac{1}{4!}\operatorname{sn}^3 x \operatorname{dn} x, \quad \dots;$$

so that $A_1 = -1, \quad A_2 = 2^2k^2, \quad A_3 = -(3\,!)^2 k^2, \quad A_4 = -(4\,!)^2 k^4, \quad \ldots,$

and $\quad \beta_1 = -1, \quad \beta_2 = -2^2k^2, \quad \beta_3 = -3^2, \quad \beta_4 = -4^2k^2, \quad \ldots,$

while the a's of (4) are all zero, since $\operatorname{cn} x$ is even. Hence we obtain the formula (1) of § 3.

Similarly, $\quad \sec^n(x+y) = (\cos x \cos y - \sin x \sin y)^{-n}$;

so that the functions $f_1, f_2, \ldots$ are

$$\sec^{n+1} x \sin x, \quad \frac{1}{2}\sec^{n+2} x \sin^2 x, \quad \frac{1}{3\,!}\sec^{n+3} x \sin^3 x, \quad \ldots,$$

while $\quad A_1 = n, \quad A_2 = 2\,!\, n(n+1), \quad A_0 = 3\,!\,(n+1)(n+2), \quad \ldots,$

and $\quad \beta_1 = n, \quad \beta_2 = 2(n+1), \quad \beta_3 = 3(n+2), \quad \ldots,$

whence $$\int_0^\infty \sec^n t\, e^{-t/x}\, dt = \frac{x}{1-}\,\frac{nx^2}{1-}\,\frac{2(n+1)x^2}{1-\ldots}.$$

This formula is, however, obviously absurd, since the integrand will pass through an infinity of infinite values. We may correct the formula by taking the hyperbolic secant, and derive the relation

$$\int_0^\infty \operatorname{sech}^n t\, e^{-t/x}\, dt = \frac{x}{1+}\,\frac{nx^2}{1+}\,\frac{2(n+1)x^2}{1+}\,\frac{3(n+2)x^2}{1+\ldots}.$$

The convergency of the fraction is readily established by Lemma IV.

The identity $\left\{\dfrac{1+x+y}{(1+x)(1+y)}\right\}^{-n} = \left\{1 - \dfrac{xy}{(1+x)(1+y)}\right\}^{-n}$

gives a formula of the type § 4 (1), but the corresponding integral

$$\int_0^\infty (1+t)^{-n} e^{-t/x}\, dt,$$

i.e., the asymptotic series

$$x - nx^2 + n(n+1)x^3 - \ldots,$$

may be represented as a continued fraction more simply by Gauss's formula

$$\frac{1}{\gamma} F\{a, \beta+1, \gamma+1, x\} / F\{a, \beta, \gamma, x\}$$
$$= \frac{1}{\gamma-}\,\frac{a(\gamma-\beta)x}{\gamma+1-}\,\frac{(\beta+1)(\gamma-a+1)x}{\gamma+2-}\,\frac{(a+1)(\gamma-\beta+1)x}{\gamma-3-\ldots},$$

by making x and γ infinite, while x/γ remains finite. If $\beta = 0$, we have

$$1 + ax + a(a+1)x^2 + \ldots = \frac{1}{1-}\,\frac{ax}{1-}\,\frac{x}{1-}\,\frac{(a+1)x}{1-}\,\frac{2x}{1-}\,\frac{(a+2)x}{1-}\,\frac{3x}{1-\ldots};$$

so that

$$\int_0^\infty (1+t)^{-n} e^{-t/x} dt = \frac{x}{1+}\,\frac{nx}{1+}\,\frac{x}{1+}\,\frac{(n+1)x}{1+}\,\frac{2x}{1+}$$

$$= \frac{x}{1+nx-}\,\frac{nx^2}{1+(n+2)x-}\,\frac{2(n+1)x^2}{1+(n+4)x-}\,\frac{3(n+2)x^2}{1+\dots}.$$

If we put m for $\frac{1}{x}$ and $-n$ for n, we have

$$\int_0^\infty (1+t)^n e^{-mt} dt = \frac{1}{m-n+}\,\frac{n}{m-n+2+}\,\frac{2(n+1)}{m-n+4+}\,\frac{3(n+2)}{m-n+6+\dots}.$$

$$\text{This integral} = \int_{-1}^\infty (1+t)^n e^{-mt} dt - \int_{-1}^0 (1+t)^n e^{-mt} dt$$

$$= \int_0^\infty u^n e^{-mu} e^m du - \int_0^1 u^n e^{-mu} e^m du$$

$$= \frac{e^m}{m^{n+1}} \Gamma(n+1) - e^m \left\{ \frac{1}{n+1} - \frac{m}{1(n+2)} + \frac{m^2}{2!\,(n+3)} - \dots \right\}$$

$$= \frac{e^m}{m^{n+1}} \Gamma(n+1) - \frac{1}{\gamma} F\{\alpha, \beta+1, \gamma+1, x\} / F\{\alpha, \beta, \gamma, x\},$$

where $\gamma = n+1, \quad \alpha = \gamma, \quad \beta = \infty, \quad x\beta = -m.$

This ratio of hypergeometric series is therefore

$$\frac{1}{n+1-}\,\frac{(n+1)m}{n+2+}\,\frac{m}{n+3-}\,\frac{(n+2)m}{n+4+\dots};$$

so that we have $\frac{e^m}{m^{n+1}} \Gamma(n+1)$ expressed as the sum of two continued fractions.

An alternative form of the identity is

$$\frac{e^m \Gamma(n+1)}{m^{n+1}} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{m^2}{(n+1)(n+2)(n+3)} + \dots$$

$$+ \frac{1}{m-n+}\,\frac{n}{m-n+2+}\,\frac{2(n+1)}{m-n+4+}\,\frac{3(n+2)}{m-n+6+\dots}.$$

6. The integral $\int_0^\infty \frac{t}{\sinh t} e^{-t/x} dt$ may be expressed as an asymptotic series in odd powers of x; so that we may assume it equivalent to a continued fraction of the form

$$\frac{x}{1-}\,\frac{e_1 e_2 x^2}{1-}\,\frac{e_3 e_4 x^2}{1-}\,\dots.$$

Calling this ϕx, we may notice that

$$\phi\frac{x}{1-x}-\phi\frac{x}{1+x}=\int_0^\infty\frac{t}{\sinh t}\left\{e^{-t/x(1-x)}-e^{-t/x(1+x)}\right\}dt = 2\int_0^\infty t\,e^{-t/x}dt=2x^2$$

Hence $$\frac{x}{1-x-}\,\frac{e_1e_2x^2}{1-x-}\,\frac{e_3e_4x^2}{1-x-}\dots = 2x^2+\frac{x}{1+x-}\,\frac{e_1e_2x^2}{1+x-}\dots. \qquad (1)$$

Dividing by x, and using Lemmas I. and II.,

$$\frac{1}{1-x-}\,\frac{e_1e_2x^2}{1-x-}\dots = 1+\frac{f_1x}{1-}\,\frac{f_2x}{1-}\dots,$$

where $$e_1=1,\quad e_2+e_3=e_4+e_5=\dots$$

and $$f_1=e_1,\quad f_2=e_1+e_2,\quad f_2f_3=e_2e_3,\quad f_3+f_4=e_3+e_4\dots.$$

But now $$\frac{f_1}{1-}\,\frac{f_2x}{1-}\dots = 2-\frac{f_1}{1+}\,\frac{f_2x}{1+}\dots,$$

from (1); so that, by Lemma II.,

$$f_2+f_3=0=f_4+f_5=f_6+f_7=\dots:$$

Hence $$e_1=1,\quad f_2=e_1+e_2,\quad f_2^2=e_2(e_2-1),$$

and therefore $e_2(e_2-1)=(1+e_2)^2$ or $3e_2=-1$; so that $e_2=-\frac{1}{3}$, $e_3=\frac{4}{3}$, while $f_2=-f_3=e_1+e_2=\frac{2}{3}$. Again, $f_4f_5=e_4e_5$; therefore

$$-f_4^2=e_4(1-e_4).$$

But $$f_4=e_3+e_4-f_3=e_4+\tfrac{4}{3}+\tfrac{2}{3}=e_4+2;$$

therefore $(e_4+2)^2=e_4(e_4-1)$ or $5e_4=-4$, and $e_5=\frac{9}{4}$.

In this way all the successive e's can be found, and, by induction, we have

$$e_{2n}=-\frac{n^2}{2n+1},\qquad e_{2n+1}=\frac{(n+1)^2}{2n+1},$$

$$f_{2n}=-f_{2n+1}=\frac{n(n+1)}{2n+1}.$$

Hence $$\int_0^\infty\frac{t}{\sinh t}\,e^{-t/x}dt=\frac{x}{1+}\,\frac{1^4x^2}{3+}\,\frac{2^4x^2}{5+}\,\frac{3^4x^2}{7+}\dots.$$

Changing x into $\dfrac{x}{x+2}$, and t into $\frac{1}{2}t$, we have

$$\int_0^\infty\frac{t}{e^t-1}\,e^{-t/x}dt=\frac{2x}{a+2+}\,\frac{1^4x^2}{3(x+2)+}\,\frac{2^4x^2}{5(x+2)+}\dots.$$

From this, by application of Lemmas I. and II., we get

$$B_2-B_4x^2+B_6x^4-\dots=\frac{1}{6+}\,\frac{1.2^2.3x^2}{10+}\,\frac{2.3^2.4x^2}{14+}\,\frac{3.4^2.5x^2}{18+}\dots.$$

7. The fraction $\dfrac{1}{1-x+}\,\dfrac{x^2}{2(1-x)+}\,\dfrac{3^2x^2}{2(1-x)+}\,\dots$, which we shall write $f(x)$, has been discussed by Prof. Muir (*Phil. Trans.*, 1877). It may be reduced to a definite integral form in the following manner:—

By Lemma I., we may write the fraction in the form

$$\frac{1}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-}\,\dots,$$

where

$$e_1=1,\;\; e_2=-\frac{1}{2},\;\; e_2+e_3=1,\;\; e_3e_4=-\frac{3^2}{4},\;\; e_4+e_5=1,\;\; e_5e_6=-\frac{5^2}{3},\;\dots,$$

i.e., $$e_3=\tfrac{3}{2},\quad e_4=-\tfrac{3}{2},\quad e_5=\tfrac{5}{2},\quad e_6=-\tfrac{5}{2}.$$

Moreover, by Lemma I.,

$$f(x)=\frac{1}{1-}\,\frac{e_1x}{1-e_2x-}\,\frac{e_2e_3x^2}{1-(e_3+e_4)x-}\,\dots.$$

But evidently $e_{2n-1}+e_{2n}=0$; therefore

$$f(x)=\frac{1}{1-}\,\frac{x}{1+\frac{1}{2}x+u},$$

where u is an even function of x.

Thus $$f(x)=\frac{1+\frac{1}{2}x+u}{1-\frac{1}{2}x+u},$$

and therefore $f(x)f(-x)=1$, as is shewn by Prof. Muir in *Phil. Trans.*, 1877.

If we put $\log f(x)=\phi(x)$, we have $\phi(x)+\phi(-x)=0$; so that $\phi(x)$ is an odd function, while, since

$$f\left(\frac{x}{1+x}\right)=\frac{1+x}{1+}\,\frac{x^2}{2+}\,\frac{3^2x^2}{2+}\,\dots,$$

we see that $\dfrac{1}{1+x}f\left(\dfrac{x}{1+x}\right)$ is an even function and $=\dfrac{x}{1-x}f\left(-\dfrac{x}{1-x}\right)$.

Hence $$\phi\left(\frac{x}{1+x}\right)-\log(1+x)=\phi\left(-\frac{x}{1-x}\right)-\log(1-x),$$

and therefore $$\phi\left(\frac{x}{1+x}\right)+\phi\left(\frac{x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right).$$

Assume $\phi(x)=\int^{\infty}\psi(t)e^{-t/x}\,dt$, where evidently $\psi(t)$ is even. Then

$$\int_0^{\infty}\psi(t)(e^t+e^{-t})e^{-t/x}\,dt=2\left(x+\frac{x^3}{3}+\dots\right),$$

which series $$=2\int_0^{\infty}\left(t+\frac{t^2}{3!}+\frac{t^4}{5!}+\dots\right)e^{-t/x}\,dt.$$

Hence $$\psi(t)\cosh t = \frac{1}{t}\sinh t,$$

and $$\phi(x) = \int_0^\infty \frac{\tanh t}{t} e^{-t/x}\,dt.$$

8. A fraction $$\frac{e_0}{1-}\,\frac{e_1 x}{1-}\,\frac{e_2 x}{1-\dots} \tag{1}$$

cannot in general be reduced to the form

$$\frac{e_0}{1-e_1x\mp\beta_1x^2-}\,\frac{\gamma_1 x^4}{1-\alpha_2x\mp\beta_2x^2-}\,\frac{\gamma_2 x^4}{1-\alpha_3x\mp\beta_3x^2-\dots}, \tag{2}$$

but it may be shewn that this reduction is possible if

$$e_2+e_3 = 0 = e_6+e_7 = \text{in general } e_{4n+2}+e_{4n+3}.$$

Now, it can easily be seen that the first convergent of (2) implies neglect of x^4 and higher powers in equating (2) to the equivalent power series; in other words, this first convergent of (2) is the fourth convergent of (1). The second convergent of (2) implies the neglect of x^8, and is therefore the eighth convergent of (1).

Again, by Lemma I., if

$$1-e_1x = D_1,\quad 1-(e_2+e_3)x = D_3,\quad 1-(e_4+e_5)x = D_5,\quad \dots,$$

we have

$$4n\text{-th convergent of (1)} = 2n\text{-th convergent of } \frac{e_0}{D_1-}\,\frac{e_1e_2x^2}{D_3-}\,\dots,$$

which $$= \frac{\dfrac{e_0}{D_1}}{1-}\,\frac{\dfrac{e_1e_2x^2}{D_1D_3}}{1-}\,\frac{\dfrac{e_3e_4x^2}{D_3D_5}}{1-};$$

therefore the $4n$-th convergent of (1) = n-th convergent of

$$\frac{\dfrac{e_0}{D_1}}{1-\dfrac{e_1e_2x^2}{D_1D_3}-}\,\frac{\dfrac{e_1e_2e_3e_4x^4}{D_1D_3^2D_5}}{1-\dfrac{e_3e_4x^2}{D_3D_5}-\dfrac{e_5e_6x^2}{D_5D_7}-}\,\frac{\dfrac{e_5e_6e_7e_8x^4}{D_5D_7^2D_9}}{1-\dfrac{e_7e_8x^2}{D_7D_9}-\dfrac{e_9e_{10}x^2}{D_9D_{11}}-}\dots. \tag{3}$$

This fraction should be identical, convergent by convergent, with (2).

Hence $D_3 = 1$, *i.e.*, $e_2+e_3=0$, and (3) becomes

$$\frac{e_0}{D_1-e_1e_2x^2-}\,\frac{\dfrac{e_1e_2e_3e_4x^4}{D_5}}{1-\dfrac{e_3e_4x^2}{D_5}-\dfrac{e_5e_6x^2}{D_5D_7}-}.$$

This second constituent must be simplified by multiplying numerator and denominator by $D_5 D_7$, but, in order to be equivalent to the second constituent of (2), we must have $D_7 = 1$, *i.e.*, $e_6+e_7 = 0$.

Finally, $D_{11} = D_{15} = D_{4n+3} = 1$, and (3) becomes

$$\frac{e_0}{1-e_1x-e_1e_2x^2-}\;\frac{e_1e_2e_3e_4x^4}{1-(e_4+e_5)x-(e_3e_4+e_5e_6)x^2-}\;\frac{e_5e_6e_7e_8x^4}{1-(e_8+e_9)x-(e_7e_8+e_9e_{10})x^2-}. \tag{4}$$

If $\phi(x) = \int_0^\infty \frac{t\,e^{-t/x}}{\cosh t}\,dt = \frac{f_0x^2}{1-}\,\frac{f_1x^2}{1-}\,\frac{f_2x^2}{1-\dots} = \frac{f_0x^2}{1-f_1x^2-}\,\frac{f_1f_2x^4}{1-(f_2+f_3)x^2-\dots}$,

then $$\phi\left(\frac{x}{1-x}\right) = \frac{f_0x^2}{(1-x)^2-f_1x^2-}\,\frac{f_1f_2x^4}{(1-x)^2-(f_2+f_3)x^2-\dots},$$

which is a fraction of the type just discussed.

Hence, if $$\phi\left(\frac{x}{1-x}\right) = \frac{e_0x^2}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-\dots},$$

we have $$e_2+e_3 = 0 = e_6+e_7 = e_{10}+e_{11} = \dots,$$

while $$e_1 = e_4+e_5 = e_8+e_9 = \dots = 2.$$

But $$\phi\left(\frac{x}{1-x}\right)+\phi\left(\frac{x}{1+x}\right) = 2\int_0^\infty t\,e^{-t/x}\,dt = 2x^2;$$

therefore $$\frac{e_0}{1-}\,\frac{e_1x}{1-}\dots+\frac{e_0}{1+}\,\frac{e_1x}{1+}\dots = 2.$$

By Lemma II., this implies $e_1+e_2 = 0 = e_3+e_4 = e_5+e_6 = \dots$. Thus

$$\begin{aligned} e_1 &= -e_2 = e_3 = -e_4 = 2,\\ e_5 &= 4 = -e_6 = e_7 = -e_8,\\ e_9 &= 6 = -e_{10} = e_{11} = -e_{12},\\ &\dots\quad \dots\quad \dots\quad \dots \end{aligned}$$

and $$\phi\left(\frac{x}{1-x}\right) = \frac{x^2}{1-2x+2^2x^2-}\,\frac{2^4x^4}{1-2x+(2^2+4^2)x^2-}\,\frac{4^4x^4}{1-2x+(4^2+6^2)x^2-};$$

so that $$\phi(x) = \frac{x^2}{1+(2^2-1)x^2-}\,\frac{2^4x^4}{1+(2^2+4^2-1)x^2-}.$$

Let $x^2/(1-x^2) = y$; then

$$\begin{aligned}\phi(x) &= \frac{y}{1+2^2y-}\,\frac{2^4y^2}{1+(2^2+4^2)y-}\,\frac{4^4y^2}{1+(4^2+6^2)y-}\\ &= \frac{y}{1+}\,\frac{2^2y}{1+}\,\frac{2^2y}{1+}\,\frac{4^2y}{1+}\,\frac{4^2y}{1+}\,\frac{6^2y}{1+}\,\frac{6^2y}{1+}.\end{aligned}$$

If $x < 1$, y is positive, and, by Lemma IV., the condition for convergency is satisfied.

9. If $\psi(x)$ denote $$2x^3+\int_0^{\infty}\frac{t^2}{e^t-1}e^{-t/x}\,dt,$$

which may be represented by the asymptotic series

$$x^2+x^3+3B_2x^4-5B_4x^5+\ldots,$$

then $$\psi\left(\frac{x}{1+x}\right)=\psi(-x). \tag{1}$$

If, then, $$\psi(x)=\frac{x^2}{1-}\,\frac{e_1x}{1-}\,\frac{e_2x}{1-\ldots}=\frac{x^2}{1-ax-\beta x^2-\gamma x^3-\ldots},$$

we have

$$\frac{x^2}{(1+x)^2-ax(1+x)-\beta x^2-\gamma x^3-\ldots}=\frac{x^2}{1+ax-\beta x^2+\gamma x^3-\ldots},$$

and evidently $$\gamma=0,\quad a=1.$$

Let then $$\psi(x)=\frac{x^2}{1-e_1x-e_1e_2x^2-x^2\psi_1x},$$

so that, from (1),

$$(1+x)^2-e_1x(1+x)-e_1e_2x^2-x^2\gamma_1\left(\frac{x}{1+x}\right)=1+e_1x-e_1e_2x^2-x^2\psi_1(-x)$$

and $$\psi_1\left(\frac{x}{1+x}\right)=\psi_1(-x).$$

Again, if $$\psi_1(x)=\frac{e_1e_2e_3e_4x^2}{1-a_1x-\beta_1x^2-x^2\psi_2x},$$

we shall have $$\psi_2\left(\frac{x}{1+x}\right)=\psi_2(-x)\quad\text{and}\quad x_1=1.$$

Proceeding in this way, we see that $\psi(x)$ takes the form (4) in § 8, where

$$e_1=e_4+e_5=e_8+e_9=\ldots=1$$

and $$e_2+e_3=e_6+e_7=\ldots=0.$$

The actual calculation of the e's may be effected by considering the relation $$\psi(x)-\psi(-x)=2x^3; \tag{2}$$

so that $$\frac{c_0}{1-}\,\frac{e_1x}{1-}\ldots=2x+\frac{e_0}{1+}\,\frac{e_1x}{1+}\ldots.$$

By using Lemma II., and putting

$$\frac{e_0}{1-}\,\frac{e_1x}{1-}\ldots=1+\frac{f_1x}{1-}\,\frac{f_2x}{1-\ldots},$$

we have $$\frac{f_1}{1-}\,\frac{f_2x}{1-}\ldots+\frac{f_1}{1+}\,\frac{f_2x}{1+\ldots}=1,$$

whence $$f_1 = 1, \quad f_2+f_3 = 0, \quad f_4+f_5 = 0, \quad \ldots,$$

where $f_1 = e_0 e_1$, $f_2 = e_1+e_2$, $f_2 f_3 = e_2 e_3$, $f_3+f_4 = e_3+e_4$, $\ldots$.

The general values of the e's may be deduced inductively, viz.,

$$e_{4n+1} = \frac{(n+1)^2}{2n+1}, \quad e_{4n+2} = -\frac{n+1}{2} = -e_{4n+3}, \quad e_{4n+4} = -\frac{(n+1)^2}{2n+3}.$$

Since $$\psi(x) = \frac{x^2}{1-x-e_1 e_2 x^2-}\,\frac{e_1 e_2 e_3 e_4 x^4}{1-x-(e_3 e_4+e_5 e_6)\,x^2-\ldots},$$

we get, by writing $x^2/(1-x) = y$,

$$\psi(x) = \frac{y}{1-e_1 e_2 y-}\,\frac{e_1 e_2 e_3 e_4 y^2}{1-(e_3 e_4+e_5 e_6)y-\ldots}$$

$$= \frac{y}{1-}\,\frac{e_1 e_2 y}{1-}\,\frac{e_3 e_4 y}{1-}\,\frac{e_5 e_6 y}{1-\ldots} \quad \text{(by Lemma I.)}$$

$$= \frac{y}{1+}\,\frac{y}{2+}\,\frac{1^3 y}{3+}\,\frac{1^3 y}{2+}\,\frac{2^3 y}{5+}\,\frac{2^3 y}{2+}\,\frac{3^3 y}{7+}\,\frac{3^3 y}{2+\ldots}.$$

Moreover I find that

$$3B_2 x^2 - 5B_4 x^4 + \ldots = \frac{x^2}{2+}\,\frac{1^2.2x^2}{3+}\,\frac{1.2^2x^2}{2+}\,\frac{2^2.3x^2}{5+}\,\frac{2.3^2x^2}{2+\ldots}.$$

The types of definite integral to which the methods of this memoir are applicable are evidently very limited in number. Those treated of in § 2 to § 5 depend upon the fact that each integral is the leader of a series of functions connected by a difference equation of the second order. Such integrals as $\int_0^\infty (1+t^2)^n e^{-t/x}\,dt$ depend on difference equations of the third order, and would not be readily adaptable to the above methods, but it is not impossible that some form of convergent approximation may exist, depending on algebraic fractions in x.

The integrals treated of in the later sections depend upon relations of the type § 9, (1) and (2), and may have an analogue of the general form $\int_0^\infty t^n e^{-t/x}(e^t-1)\,dt$, but I have not succeeded in finding a general law for the form of continued fraction when n is greater than 2.