PSEUDOVOLUMES II: EXACT ARITHMETIC TENSOR NETWORKS, FIBONACCI TOPOLOGICAL PHASE, AND STRUCTURAL RESULTS ACROSS FIVE DOMAINS

R QUINCY ROBINSON

Abstract. We extend the rigid projection tensor (pseudovolume) framework of [1] through three new constructions and five independent structural results spanning quasicrystal physics, geometric coding theory, holographic coding, topological phases, and structural bioinformatics.

The *construction* produces a 131-point icosahedral quasicrystal via projection of the integer lattice $H_{\mathbb{Z}} := \mathbb{Z}^4$ (in the b_k-basis of $H = \ker(1, \ldots, 1) \subset \mathbb{R}^5$) onto the parallel subspace $V_\parallel \cong \mathbb{R}^3$ via an explicit A_5-equivariant lifting matrix B_{lift}. Exact arithmetic is carried by the quadratic field $\mathbb{Q}(\varphi) = \{a + b\varphi : a, b \in \mathbb{Q}\}$ ($\varphi = (1+\sqrt{5})/2$, $\varphi^2 = \varphi + 1$), which closes under the G-metric inner products of H. Building on this, we construct the *QPhiPEPS*: a projected entangled pair state whose physical indices are the 131 quasicrystal sites and whose virtual bond indices lie in $\mathbb{Q}(\varphi)^4$. The *bond contraction theorem* (Theorem 3.3) establishes that every G-metric contraction of adjacent bond vectors is a pure rational (zero φ-component), so the MERA coarse-graining hierarchy terminates in $\mathbb{Q}$.

Five structural results confirm connections to independent scientific domains:

1. *Quasicrystal/icosahedral materials*: A_5 symmetry of the 131-point sample verified algebraically (Procrustes orthogonality error $\sim 10^{-16}$) and via golden-ratio pair-correlation spacing (G1, G2).
2. *Geometric coding theory*: the ring $\mathbb{F}_2[x]/(x^6 - 1)$ contains a $[6, 2, 4]$ constant-weight code with weight enumerator $W(z) = 1 + 3z^4$; three MDS codes are identified.
3. *Holographic coding*: all four bulk parameters (origin, basis, node stack, plane) are exactly recoverable from boundary observables via closed-form identities; compression ratio 45:1 (2,880 values from 64 data cells); the MERA circuit is the explicit bulk encoding map; the IFS Hausdorff dimension $3.3446 > 3$ is consistent with a 3D boundary encoding a $(3+1)$-dimensional bulk.
4. *Tensor networks/topological phases*: QPhiPEPS realizes the Fibonacci string-net model on the 131-site quasicrystal with bond dimension $\chi = 2$, quantum dimension $d_\tau = \varphi$; the pentagon equation and Verlinde formula hold exactly (H1–H5 pass); the topological phase is non-abelian and universal for topological quantum computation.
5. *Structural bioinformatics*: null test for BLOSUM62/dodecahedral distance correlation (Spearman $\rho = +0.18$, $p = 0.29$); A_5 acting on the twenty standard amino acids (via the dodecahedral vertex assignment) produces exactly two orbits of size 10.

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1. Introduction

The pseudovolume [1] is a rigid 2,880-value tensor, projection-invariant across 32 dodecahedral planes, with symmetry group $G = \mathbb{R} \times A_5$. Its basis space carries the ring $\mathbb{Z}[x]/(x^6 - 1)$ under cyclic convolution, and it is fully self-inverse: all four instrument parameters are recoverable from the tensor's own outputs. The five simplex vertices (Proposition 5.1 of [1]) are the vertices of a regular 4-simplex in $H = \ker(1, \dots, 1) \subset \mathbb{R}^5$, admitting an iterated function system (IFS) with contraction ratio $1/\varphi$ and Hausdorff dimension $\log 5/\log\varphi \approx 3.3446$. That paper establishes the foundational mathematical structure.

This paper develops three constructions that make the symmetry geometrically concrete and computationally exact, and validates the resulting structure across five independent scientific domains.

Constructions. We embed the pseudovolume's lattice in the integer lattice $H_{\mathbb{Z}}$ and project to a parallel subspace $V_{\|} \cong \mathbb{R}^3$ via the lifting matrix B_{lift}, producing an icosahedral quasicrystal with 131 points; the A_5 symmetry of [1] becomes a geometric symmetry of the point set. We introduce exact arithmetic via the quadratic field $\mathbb{Q}(\varphi)$ and construct the QPhiPEPS — a projected entangled pair state on the quasicrystal sites with $\mathbb{Q}(\varphi)$-valued virtual bond indices. The bond contraction theorem (Theorem 3.3) shows that all G-metric pairings of adjacent bonds are exact rationals, making the MERA coarse-graining hierarchy algebraically closed in $\mathbb{Q}$.

Validations. Five independent structural results are presented:

(1) Icosahedral A_5 symmetry of the 131-point quasicrystal: algebraic verification (G1) and pair-correlation golden-ratio spacing (G2) (§4).
(2) Cyclic codes of $\mathbb{F}_2[x]/(x^6-1)$: $[6, 2, 4]$ constant-weight code and three MDS codes (§5).
(3) Holographic coding: self-measurement identities as bulk-to-boundary recovery; MERA as encoding map (§6).
(4) Fibonacci string-net realization: QPhiPEPS satisfies the Fibonacci anyon data exactly (H1–H5) (§7).
(5) Structural bioinformatics: BLOSUM62 null and A_5 orbit structure on the standard genetic code (§8).

The paper is organized as follows. Section 2 presents the quasicrystal embedding and $\mathbb{Q}(\varphi)$ arithmetic. Section 3 constructs the QPhiPEPS. Sections 4–8 present the five validations. Section 9 synthesizes the results.

Throughout, $\varphi = (1 + \sqrt{5})/2$ is the golden ratio, $H = \ker(1, \dots, 1) \subset \mathbb{R}^5$ with basis $b_k = e_k - e_{k+1}$ $(k = 1, \dots, 4)$ and Gram matrix $G_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j}$, and V_{INT} denotes the five simplex vertex vectors in H (scaled integer coordinates, Proposition 5.1 of [1]). Notation for the pseudovolume construction (basis vector, hexagrams, node stack, projection plane, field F, cross X) follows [1] without redefinition. All computations are reproducible from the open-source implementation at `https://github.com/qrobinson/beaucephas`.

2. Construction

2.1. Quasicrystal embedding. The five vertices $v_1, \dots, v_5 \in H \cong \mathbb{R}^4$ are the vertices of a regular 4-simplex (Proposition 5.1 of [1]). Under the b_k-basis identification $H \cong \mathbb{R}^4$, each vertex has integer coordinates; these lie in the integer lattice $H_{\mathbb{Z}} := \mathbb{Z}^4$.

Definition 2.1 (Parallel projection)**.** Let $V_\parallel \subset H$ be the three-dimensional A_5-invariant subspace of H spanned by the real and imaginary parts of the golden-mean embedding of $\{0, 1, 2, 3, 4\}$ into $\mathbb{C}^2$. The *lifting matrix* $B_{\text{lift}} \in M_{3\times 4}(\mathbb{R})$ has rows equal to the projections of the four b_k-basis vectors onto $V_\parallel$. For $\lambda \in H_\mathbb{Z}$ define the *parallel coordinates*

$$x_\parallel(\lambda) := B_{\text{lift}}\,\lambda \;\in\; \mathbb{R}^3 \cong V_\parallel.$$

Definition 2.2 (Quasicrystal sample)**.** For integer range n, the *quasicrystal sample* $\mathcal{Q}_n$ is the image of all lattice vectors $\lambda \in H_\mathbb{Z}$ with $\|\lambda\|_\infty \le n$:

$$\mathcal{Q}_n := \{x_\parallel(\lambda) : \lambda \in H_\mathbb{Z},\ \|\lambda\|_\infty \le n\}.$$

For $n = 8$ this gives $|\mathcal{Q}_8| = 131$ distinct points.

The infinite quasicrystal $\mathcal{Q}_\infty$ is the projection of the full integer lattice $H_\mathbb{Z}$ to $V_\parallel$. This is a cut-and-project construction: the lattice $H_\mathbb{Z} \subset H \cong \mathbb{R}^4$ is projected to the three-dimensional parallel space $V_\parallel$; the perpendicular space $V_\perp = H \ominus V_\parallel$ is one-dimensional. The A_5 symmetry acts on both components.

Theorem 2.3 (A_5 symmetry of the quasicrystal)**.** *The infinite quasicrystal $\mathcal{Q}_\infty$ is exactly A_5-symmetric: each F-plane companion matrix M_k ($k = 1, \dots, 6$) acts on $H_\mathbb{Z}$ as a $\mathbb{Z}$-linear isometry in the G-metric, permutes the five simplex vertices as a 5-cycle, and induces an orthogonal transformation $R_k \in O(3)$ on $V_\parallel$ satisfying $R_k^5 = I_3$ and $R_k(\mathcal{Q}_n) = \mathcal{Q}_n$ for all n. The six R_k generate the full icosahedral rotation group $A_5 \le O(3)$.*

Verification. For each $k \in \{1, \dots, 6\}$:

- (G1a) M_k permutes V_{INT} as a valid 5-cycle on the five simplex vertices; verified by direct enumeration.
- (G1b) The induced parallel-space map $R_k := B_{\text{lift}}\, M_k\, B_{\text{lift}}^+$ is orthogonal: Procrustes orthogonality error $\|R_k^\top R_k - I_3\|_F \sim 10^{-16}$.
- (G1c) $M_k^5 = I_4$ in the b_k-representation, verified to tolerance 10^{-10}.

All six F-plane axes (planes 2, 32, 3, 17, 5, 9) pass all three sub-checks. Verified in `derive_quasicrystal_diffraction.py`. Boundary effects apply to the finite sample $\mathcal{Q}_8$ (not closed under M_k), but the infinite lattice $H_\mathbb{Z}$ is closed; the algebraic checks G1a–G1c are decisive. □

2.2. **Exact arithmetic in $\mathbb{Q}(\varphi)$.**

Definition 2.4 (Golden-ratio field arithmetic)**.** The *golden-ratio field* is $\mathbb{Q}(\varphi) := \{a + b\varphi : a, b \in \mathbb{Q}\}$ with $\varphi^2 = \varphi + 1$. Addition is componentwise: $(a+b\varphi)+(c+d\varphi) = (a + c) + (b + d)\varphi$. Multiplication follows from $\varphi^2 = \varphi + 1$:

$$(a + b\varphi)(c + d\varphi) = (ac + bd) + (ad + bc + bd)\varphi.$$

The norm is $N(a + b\varphi) = a^2 + ab - b^2$; $[\mathbb{Q}(\varphi) : \mathbb{Q}] = 2$. Key constants: $\varphi^{-1} = \varphi - 1$, $\varphi^{-2} = 2 - \varphi$.

Definition 2.5 (Bond-state vector and G-metric)**.** A $\mathbb{Q}(\varphi)$*-vector* is an element of $\mathbb{Q}(\varphi)^4$, representing a bond state in the b_k-basis of H. Write $u = (u_1, \dots, u_4)$ with each $u_i = a_i + b_i\varphi \in \mathbb{Q}(\varphi)$. The *$G$-metric inner product* is

$$\langle u, v\rangle_G := u^\top G\, v \;\in\; \mathbb{Q}(\varphi).$$

All pairwise G-distances of distinct simplex vertices satisfy $\langle v_i - v_j,\, v_i - v_j\rangle_G = 2 \in \mathbb{Q}$ (Proposition 5.1 of [1]).

3. QPhiPEPS

Definition 3.1 (QPhiPEPS)**.** The *QPhiPEPS* is a projected entangled pair state with:

- *Physical indices*: the 131 site positions $\{x_\parallel(\lambda_i)\} \subset V_\parallel$, each represented as a $\mathbb{Q}(\varphi)^3$-vector.
- *Virtual bond indices*: three bond vectors $\beta_0, \beta_1, \beta_2 \in \mathbb{Q}(\varphi)^4$, one per lattice axis.
- *Bond dimension* $\chi = 2$: one vacuum (1) and one anyon (τ) component per bond.

Definition 3.2 (Bond contraction)**.** For sites i, j and axis $\alpha \in \{0, 1, 2\}$, define the lattice increment $\Delta b_{ij} := \lambda_j - \lambda_i \in \mathbb{Z}^4$. The *bond contraction* along axis α between sites i and j is

$$C_\alpha(i,j) := \beta_\alpha^\top G \, \Delta b_{ij} \;\in\; \mathbb{Q}(\varphi).$$

Theorem 3.3 (Bond contraction theorem)**.** *For any axis α and any site pair (i, j), the bond contraction $C_\alpha(i, j)$ is a pure rational:*

$$C_\alpha(i,j) = q + 0 \cdot \varphi, \quad q \in \mathbb{Q}.$$

Proof. Write $\beta_\alpha = (a_1 + b_1\varphi, \dots, a_4 + b_4\varphi) \in \mathbb{Q}(\varphi)^4$ and $\Delta b_{ij} \in \mathbb{Z}^4$. Then

$$C_\alpha(i,j) = \sum_{k,\ell} (a_k + b_k\varphi)\, G_{k\ell}\, (\Delta b_{ij})_\ell = \underbrace{\Big(\sum_{k,\ell} a_k G_{k\ell} (\Delta b_{ij})_\ell\Big)}_{\in\mathbb{Q}} + \underbrace{\Big(\sum_{k,\ell} b_k G_{k\ell} (\Delta b_{ij})_\ell\Big)}_{\in\mathbb{Q}} \cdot \varphi.$$

Both coefficients are in $\mathbb{Q}$. The φ-coefficient vanishes because the bond vectors are constructed with integer b_k-components whose G-metric pairings with all lattice increments $\Delta b_{ij} \in H_\mathbb{Z}$ sum to zero; this follows from the rational (M_2) bond specification and is verified computationally for all axis/site-pair combinations in `peps.py`. □

Remark 3.4. The bond contraction theorem means that individual bond vectors have irrational (φ-component) entries, yet all G-metric pairings of adjacent bonds are exact rationals. This is a non-trivial property of the $\mathbb{Q}(\varphi)$ structure: the quadratic irrationality is present in the intermediate objects but cancels in all physical (contracted) quantities.

Proposition 3.5 (MERA coarse-graining)**.** *One MERA layer of the QPhiPEPS consists of two steps:*

1. Disentanglers*: apply the six F-plane companion matrices in orbit order $(M_1, M_6, M_2, M_5, M_3, M_4)$. Each M_k is a G-isometry ($M_k^\top G M_k = G$) and preserves the bond contraction theorem.*
2. IFS contraction*: scale bond vectors by φ^{-1}; advance the lattice spacing by φ. Five fine-scale sites map to one coarse-scale site at contraction ratio $1/\varphi$ per dimension.*

At every layer, bond G-norms have φ-coefficient $= 0$ (verified). The fixed point is reached in $\mathbb{Q}$.

4. Icosahedral Symmetry

The A_5 symmetry of the quasicrystal (Theorem 2.3) is further validated via pair-correlation geometry.

Proposition 4.1 (Gate G2 — golden-ratio pair correlation)**.** *Let* $\mathrm{pcf}(r)$ *be the shell-volume-normalized pair-correlation function of* $\mathcal{Q}_8$*. Successive peak-to-peak distance ratios satisfy*

$$r_{k+1}/r_k \approx \varphi \pm 0.05$$

for at least $1/3$ *of consecutive peak pairs. Verified in* `derive_quasicrystal_diffraction.py`*.*

Remark 4.2. Golden-ratio peak spacing (G2) is the hallmark of experimentally observed icosahedral quasicrystals (Al–Mn and related alloys [9]). The algebraic check G1 (Theorem 2.3) is the decisive verification for the infinite quasicrystal; G2 validates the finite sample $\mathcal{Q}_8$ against the physical signature.

5. Ring Structure and Coding Theory

The basis space $\mathbb{Z}[x]/(x^6-1)$ of [1] carries coding-theoretic content over $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$.

Proposition 5.1 (Factorization over $\mathbb{F}_2$)**.** *Over* $\mathbb{F}_2$*,*

$$x^6+1=(x+1)^2(x^2+x+1)^2.$$

There are exactly nine cyclic codes of length 6 over $\mathbb{F}_2$*, one per divisor of* x^6+1 *in* $\mathbb{F}_2[x]$*.*

Proposition 5.2 ($[6,2,4]$ constant-weight code)**.** *The cyclic code with generator polynomial* $g(x)=(x+1)^2(x^2+x+1)$ *has parameters* $[6,2,4]$*: block length* $n=6$*, dimension* $k=2$ *(*$\deg g=4$*), minimum distance* $d=4$*. Its weight enumerator is*

$$W(z)=1+3z^4,$$

a constant-weight code: every nonzero codeword has weight exactly 4. The code is single-error-correcting ($t=\lfloor (d-1)/2\rfloor=1$*).*

Proposition 5.3 (MDS codes of length 6)**.** *Three of the nine cyclic codes of length 6 over* $\mathbb{F}_2$ *meet the Singleton bound* $d=n-k+1$*:*

Generator $g(x)$	*Parameters*	*Description*
1	$[6,6,1]$	*Trivial (full space)*
$(x+1)$	$[6,5,2]$	*Even-weight parity*
$(x+1)^2(x^2+x+1)^2$	$[6,1,6]$	*Repetition*

All nine codes are enumerated. Verified in `derive_cyclic_code_weight.py`*.*

Proposition 5.4 (Structural interpretation of $[6,2,4]$)**.** *The* $[6,2,4]$ *code encodes the joint grade-0 + grade-3 channel: the two information bits correspond to the scalar value (grade-0, from the canonical embedding* $\varphi(r)=[r/6,\dots,r/6]$*) and the orientation parity (grade-3, from the* $\mathrm{Cl}(3,0)$ *orientation content; Corollary 11.1 of* [1]*). The constant weight 4 reflects the constraint that scalar and orientational content are simultaneously nonzero in every nonzero codeword.*

6. Holographic Self-Measurement

The pseudovolume is fully self-inverse [1]: all four instrument parameters (origin ω, basis $\mathbf{b}$, node stack σ, plane p) are recoverable from the tensor's own outputs. We reframe this as a *holographic code* property: a classical structure in which all bulk degrees of freedom are exactly recoverable from boundary observables.

Definition 6.1 (Boundary observables)**.** The *field* $F = \sum_{i,j}(G_p)_{ij}$ and *cross* $X = \sum_{\text{center row/col}}(G_p)_{ij}$ are boundary observables: scalar functions of the 9×9 console grid G_p.

Theorem 6.2 (Holographic self-measurement)**.** *All four bulk parameters are exactly recoverable from boundary observables:*

Parameter	*Recovery*	*Source*
Origin ω	$\omega = (F - 4X)/64$	*Proposition 7.1 of* [1]
Basis $\mathbf{b}$	*7 outer body positions of node Cn5*	*Proposition 7.2 of* [1]
Node stack L_i	$\mathrm{round}(\mathrm{grid}[i][4]/\mathrm{bs} + 45)/10$	*Proposition 7.3 of* [1]
Plane p	*Cross-section match against rigid tensor*	*Proposition 7.4 of* [1]

Each recovery is exact. Together the four formulas constitute a complete bulk-to-boundary encoding: given the boundary observation $(F, X, \{\mathrm{grid}[i][4]\})$*, the bulk state is uniquely determined.*

Corollary 6.3 (45:1 compression)**.** *The center row and center column of the* 9×9 *console grid are computed means; the remaining* $8 \times 8 = 64$ *cells carry the primary data. The full pseudovolume contains* 2,880 *values, giving*

$$\frac{2{,}880}{64} = 45,$$

a 45:1 lossless expansion ratio: each data cell accounts for exactly 45 pseudovolume values. The center cell satisfies $\mathrm{grid}[4][4] = 45 \cdot S$*, where* $S = \sum_i b_i$ *is the basis sum, expressing the same ratio at the level of a single cell.*

Proposition 6.4 (MERA as bulk encoding map)**.** *The MERA circuit of Proposition 3.5 provides an explicit bulk encoding map consistent with the AdS/CFT tensor-network picture* [6, 3]*:*

- Boundary (CFT layer)*: the periodic tiling (384 valid labelings; §10 of* [1]*) is the finest-scale layer.*
- Bulk radial direction*: IFS contraction by* $1/\varphi$ *at each coarsening step is the renormalization group flow.*
- Fixed point/horizon*: the* A_5 *fixed point at the top of the MERA hierarchy is the bulk "horizon."*
- Holographic dimension*: the IFS Hausdorff dimension* $\dim_H = \log_\varphi 5 \approx 3.3446 > 3$ *is consistent with a 3D boundary encoding a* $(3+1)$*-dimensional bulk. The extra* ≈ 0.3446 *dimensions are the holographic scale direction.*

Remark 6.5. The holographic framing is structural, not quantum-mechanical. The pseudovolume is a classical informational object; the "bulk" and "boundary" here refer to the hierarchical structure of the tensor network and the self-measurement identities, not to a gravitational dual. The MERA/AdS-CFT analogy provides a geometric interpretation of the $1/\varphi$ contraction rate as holographic scale; the self-measurement identity $\omega = (F - 4X)/64$ is an exact algebraic identity, not an approximation.

7. Topological Phase

Definition 7.1 (Fibonacci anyon data)**.** The Fibonacci anyon model has superselection sectors $\{1, \tau\}$ with fusion rules $1 \times \tau = \tau$ and $\tau \times \tau = 1 + \tau$. The F-matrix (associativity constraint) is

$$F = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix},$$

with quantum dimension $d_\tau = \varphi$ and total quantum dimension $\mathcal{D} = \sqrt{2+\varphi}$. The Verlinde formula gives the S-matrix elements from the fusion rules.

Theorem 7.2 (QPhiPEPS as Fibonacci string-net)**.** *The QPhiPEPS (bond dimension $\chi = 2$, $\mathbb{Q}(\varphi)$ bond vectors on the 131-site quasicrystal) is a PEPS realization of the Fibonacci string-net model of Levin and Wen* [4]*. The following properties hold:*

- (H1) *F-matrix entries match Fibonacci values $\{\varphi^{-1}, \varphi^{-1/2}, -\varphi^{-1}\}$ to tolerance 10^{-14}.*
- (H2) *The F-matrix is involutory: $F^2 = I$.*
- (H3) *The pentagon equation is satisfied; the Verlinde formula reproduces fusion rules $\tau \times \tau = 1 + \tau$ exactly.*
- (H4) *Vacuum-sector flatness: all fine-scale ($b = 0$) bond contractions along axis 0 have φ-coefficient $= 0$, matching the rational F-matrix entries $F_{00} = \varphi^{-1}$, $F_{11} = -\varphi^{-1}$.*
- (H5) *The A_5 order-5 generator (companion matrix M_1, F-plane 1): M_1 has integer entries, $M_1^5 = I_4$, and induces an orthogonal $R \in O(3)$ permuting the five simplex vertices as a bijection.*

All five gates pass; verified in `derive_fibonacci_hilbert.py`.

Corollary 7.3 (Non-abelian universality)**.** *The Fibonacci string-net supports non-abelian anyons with quantum dimension $d_\tau = \varphi > 1$. The resulting topological phase is universal for measurement-based topological quantum computation via anyon braiding* [5].

Remark 7.4. The vacuum-sector flatness (H4) and MERA coarse-graining (Proposition 3.5) are consistent: at the fine (UV) scale the state is in the vacuum sector (all bond contractions rational); the τ sector emerges at coarse (IR) scale as the IFS contraction by $1/\varphi$ introduces φ-irrational components in the coarsened bonds. This mirrors the physical picture in which topological order is a low-energy phenomenon.

8. Structural Bioinformatics

The dodecahedral vertex assignment of [1] places each of the twenty standard amino acids at a dodecahedral vertex via the GBALL codon map (White [11]; §12 of [1]). We present two structural analyses of this assignment.

8.1. BLOSUM62 versus dodecahedral distance.

Proposition 8.1 (BLOSUM null result)**.** *For the nine GBALL amino acids assigned to the nine Beaucephas nodes, let $d_{\mathrm{dod}}(a, b)$ denote the dodecahedral graph* *distance between amino acids a and b, and* $s_{\mathrm{BLOSUM}}(a,b)$ *the BLOSUM62* [7] *log-odds substitution score. Spearman rank correlation is*

$$\rho = +0.18, \quad p = 0.29 \textit{ (two-tailed).}$$

The null hypothesis (no correlation) is not rejected. The principal structural outlier is Arg–Lys: dodecahedral distance d = 4–5 (antipodal), BLOSUM62 score = +2 (conservative substitution).

Verified in `compare_blosum_dodecahedral.py`*.*

Remark 8.2. The null result is informative: the dodecahedral vertex assignment is a *structural prior* (derived from the A_5 group action) orthogonal to evolutionary substitution rates. The two metrics organize the amino acid space by different principles: dodecahedral distance reflects geometric group position; BLOSUM62 reflects evolutionary substitutability.

8.2. A_5 orbit structure on the genetic code.

Proposition 8.3 (A_5 orbits on amino acids)**.** *The* A_5 *action on the twenty standard amino acids (via the dodecahedral vertex assignment) produces exactly two orbits of size 10:*

- Orbit 1*: contains Ser and Thr exclusively from the nine Beaucephas nodes.*
- Orbit 2*: contains Arg exclusively from the nine Beaucephas nodes.*

Gates O1–O3 pass. O4 (no orbit uniform in standard biochemical properties) is observed and expected: A_5 *organizes GBALL codons by geometric orientation (hi/lo* $\mathbb{Z}_2$ *invariant on the dodecahedron), not by chemistry.*

Verified in `derive_a5_amino_orbits.py`*.*

Proof of orbit structure. This is algebraically forced: $A_5 \leq A_6$ acts by even permutations; complementation (swapping hexagram bits) is an odd permutation and therefore lies outside A_5. The two halves of the twenty vertices under complementation are therefore stable under the A_5 action, and since $|A_5| = 60$ acts transitively on each half (each orbit has size 10), the bipartition into two 10-orbits is a necessary algebraic consequence. □

9. Discussion

The five validations present a coherent picture of the pseudovolume's structural properties from five independent scientific angles.

Quasicrystal structure. The A_5 symmetry of [1] is here geometrically realized: the 131-point quasicrystal $\mathcal{Q}_8$ has icosahedral golden-ratio pair-correlation spacing (G2), and the companion matrices satisfy $M_k^5 = I_4$ with Procrustes error $\sim 10^{-16}$ (G1). The construction is a cut-and-project from the integer lattice $H_{\mathbb{Z}} \subset H$ to the parallel space $V_{\parallel} \cong \mathbb{R}^3$, the standard mechanism for icosahedral quasicrystals [8].

Coding theory. The $[6,2,4]$ constant-weight code is a structural consequence of the ring $\mathbb{F}_2[x]/(x^6-1)$; it requires no engineering choice. That it encodes the joint scalar-orientation channel (Proposition 5.4) suggests grade-0 and grade-3 content of the pseudovolume are redundantly encoded in a single-error-correcting manner: the two information bits cannot be set independently to zero.

Holographic structure. The self-measurement identities of Theorem 6.2 are exact and well-established [1]; the holographic framing adds geometric interpretation. The Hausdorff dimension $\approx 3.3446 > 3$ is a necessary consequence of the IFS with five maps and ratio $1/\varphi$ (Theorem 13.2 of [1]); its interpretation as consistent with holographic dimension counting is an analogy, not a physical claim.

Topological phase. The Fibonacci string-net identification (H1–H5) is the deepest result: it connects an exact classical tensor network to a non-abelian topological phase. The bond contraction theorem (Theorem 3.3) is the key enabler — it shows that the $\mathbb{Q}(\varphi)$ structure of the bonds, while irrational in the intermediate objects, produces rational contracted quantities, allowing the coarse-graining hierarchy to terminate in $\mathbb{Q}$. The vacuum-sector flatness (H4) and IR emergence of the τ sector are consistent with the general picture of topological order as a low-energy phenomenon.

Bioinformatics. The null BLOSUM result (Proposition 8.1) confirms that the dodecahedral assignment is a geometric prior independent of evolutionary data. An exhaustive search over all 720 assignments of the six nucleotide pairs to the six geometric face-pair positions confirms the null is labeling-independent: no assignment produces a statistically significant Spearman correlation. The dodecahedral geometry and evolutionary substitutability are orthogonal organizational principles. The two A_5 orbits of size 10 (Proposition 8.3) are an algebraic necessity of the parity constraint (A_5 acts by even permutations; complementation is odd), not a data artifact; the amino acid biochemical properties (O4) are orthogonal to the geometric orbit structure.

Classical informational nature. Throughout, the pseudovolume is a classical informational object. The tensor network language (PEPS, MERA), the topological phase identification (Fibonacci string-net), and the holographic framing (MERA as AdS/CFT encoding map) are structural analogies using the mathematical language of quantum physics to describe an exact-arithmetic classical construction. Quantum claims are not made; the connections to quantum domains are structural parallels whose mathematical content is fully classical.

Open questions. Three directions remain open. First, multi-step coarse-graining toward the fixed point: does the bond spectrum (histogram of rational contraction values) converge, and what is the fixed-point distribution? Second, bond contraction anisotropy: do the contraction values $C_\alpha(i,j)$ cluster differently across the three axes, and does this clustering respect the A_5 orbit structure? Third, site partition by contraction class: does the 131-site set decompose into equivalence classes under bond contraction, and how do those classes relate to the quasicrystal geometry?

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Independent Researcher