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Supplemental Material: Have Optical Wigner's Friend Experiments Been Blind to a Geometric Degree of Freedom?

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S1 Literature Search and Algebraic Proof

S1.1 Full Proof of Equatorial Cancellation Theorem

Theorem. Let Friend F measure in z-basis and Superobserver W at Bloch angles (θ, ϕ). With $f_\perp(b, d) = 1 - |\langle b|d\rangle|^2$:

$$f_\perp(+1, H) - f_\perp(-1, H) = -\cos\theta \tag{S1}$$

Vanishes iff $\theta = \pi/2$.

Step 1: W's measurement basis

$$|b = +1\rangle = \cos(\theta/2)|H\rangle + e^{i\phi}\sin(\theta/2)|V\rangle, \tag{S2}$$
$$|b = -1\rangle = \sin(\theta/2)|H\rangle - e^{i\phi}\cos(\theta/2)|V\rangle. \tag{S3}$$

Step 2: Overlaps with F's outcomes

$$|\langle b = +1|H\rangle|^2 = \cos^2(\theta/2), \quad |\langle b = +1|V\rangle|^2 = \sin^2(\theta/2), \tag{S4}$$
$$|\langle b = -1|H\rangle|^2 = \sin^2(\theta/2), \quad |\langle b = -1|V\rangle|^2 = \cos^2(\theta/2). \tag{S5}$$

ϕ drops out: $|e^{i\phi}|^2 = 1$. Overlaps depend *only* on θ.

Step 3: $f_\perp$ values

$$f_\perp(+1, H) = 1 - \cos^2(\theta/2) = \sin^2(\theta/2), \tag{S6}$$
$$f_\perp(-1, H) = 1 - \sin^2(\theta/2) = \cos^2(\theta/2), \tag{S7}$$
$$f_\perp(+1, V) = 1 - \sin^2(\theta/2) = \cos^2(\theta/2), \tag{S8}$$
$$f_\perp(-1, V) = 1 - \cos^2(\theta/2) = \sin^2(\theta/2). \tag{S9}$$

Step 4: Outcome-dependence parameter

$$f_\perp(+1, H) - f_\perp(-1, H) = \sin^2(\theta/2) - \cos^2(\theta/2) \tag{S10}$$
$$= -(\cos^2(\theta/2) - \sin^2(\theta/2)) \tag{S11}$$
$$= -\cos\theta. \tag{S12}$$

Vanishes iff $\theta = \pi/2$. At this angle, all four $f_\perp = 1/2$, constant across all outcome pairs. □

Step 5: $K9\_E$ reduction

When $f_\perp$ is outcome-independent:

$$P(o|K) = \mathrm{Tr}(E_o\rho) \cdot \frac{1 - \beta \cdot \text{constant}}{1 - \beta \cdot \text{constant}} = \mathrm{Tr}(E_o\rho) \tag{S13}$$

$K9\_E = 0$ for all equatorial measurements. □

Sympy verification

```
import sympy as sp
theta = sp.Symbol('theta', real=True)
assert sp.simplify(sp.sin(theta/2)**2 - sp.cos(theta/2)**2 + sp.cos(theta)) == 0
```

Generality

At $\theta = \pi/2$, $|\langle b|d\rangle|^2 = 1/2$ for all b, d. Any $g(|\langle b|d\rangle|^2)$ therefore takes the same value for all outcome pairs. The equatorial plane is a fixed point for the entire class — the cancellation is not specific to $f_\perp$.

S1.2 BSM Equivalence for Proietti et al. (2019)

Proietti et al. use Bell-state measurement (BSM) rather than projective measurement. BSM projects onto four Bell states $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$. For the singlet-state source, each Bell outcome occurs with equal probability 1/4, and conditioned on any Bell outcome b, the Friend's recorded outcome d is equally likely to be H or V: $P(d|b) = 1/2$ for all b, d. Because b is a two-photon Bell state and d a single-photon record, the relevant object is this conditional probability rather than a single-photon overlap $|\langle b|d\rangle|^2$; the value 1/2 plays the same role as the equatorial overlap $|\langle b|d\rangle|^2 = 1/2$ derived in the main text, so the BSM configuration is effectively equatorial ($\theta = \pi/2$).

S1.3 Literature Search Methodology

Search date: 2026-05-15 to 2026-05-24.
Databases: Google Scholar, arXiv (quant-ph), Web of Science, InspireHEP.
Search strings (Boolean combinations):

- **Group A (subject):** "Wigner's friend" OR "extended Wigner" OR "Frauchiger-Renner" OR "Local Friendliness"
- **Group B (parameter):** "equatorial measurement" OR "Bloch sphere polar angle" OR "outcome dependence" OR "geometric constraint" OR "measurement basis angle"

Inclusion criteria:

1. Optical EWF implementation with Friend+Superobserver structure
2. Published in peer-reviewed venue or arXiv
3. Reports measurement settings from which polar angle θ can be determined

Results: ∼200 titles screened → 47 full-text examined → 2 published optical EWF experiments identified. Both operate at equatorial geometry ($\theta = \pi/2$ or equivalent via BSM). No published experiment varies θ from $\pi/2$ for any purpose.
Additional sources examined: Supplemental Material of Bong et al. (2020); Methods and Supplementary Information of Proietti et al. (2019); LF derivations in Frauchiger-Renner (2018) and Wiseman-Cavalcanti-Rieffel (2023); Bell/LF review by Brunner et al. (2014); multipartite, sequential, and possibilistic LF extensions.
Limitation: We cannot rule out unpublished results or implementations outside our database scope. The survey establishes that within the indexed, peer-reviewed literature, θ has not been varied.

S2 Numerical Methods and Statistical Robustness

S2.1 Exact Numerical Integration Method

All numerical predictions use the density matrix for the singlet state with finite visibility under the SPDC noise model (where noise is confined to the $\{|HV\rangle, |VH\rangle\}$ subspace):

$$\rho_\mu = \mu|\Psi^-\rangle\langle\Psi^-| + \frac{1-\mu}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH|). \tag{S14}$$

For the overlap-dependent model Eq. (2–3) of the main text:

$$P(a, b \mid x, y) = P_{\mathrm{QM}}(a, b \mid x, y) \cdot [1 - \beta f_\perp(b, d)]/Z, \tag{S15}$$

with full renormalization $Z = \sum_{a,b} P_{\mathrm{QM}}(a, b \mid x, y) \cdot [1 - \beta f_\perp(b, d)]$.
All computations performed via `K9S12_proposal.py`, `statistical_significance.py`.
The algebraic cancellation theorem is independently verified via symbolic computation (`universal_theorem_lf_check.py`).

S2.2 Derivation of the Sensitivity Formula and First-Order Overestimate Warning

S2.2.1 f_perp values at polar angle θ

The Superobserver measurement basis at (θ, ϕ) is:

$$|b = +1\rangle = \cos(\theta/2)|H\rangle + e^{i\phi}\sin(\theta/2)|V\rangle, \tag{S16}$$
$$|b = -1\rangle = \sin(\theta/2)|H\rangle - e^{i\phi}\cos(\theta/2)|V\rangle. \tag{S17}$$

The squared overlaps with the Friend's z-basis outcomes (where ϕ drops out since $|e^{i\phi}|^2 = 1$) are:

$$|\langle b = +1|H\rangle|^2 = \cos^2(\theta/2), \quad |\langle b = +1|V\rangle|^2 = \sin^2(\theta/2), \tag{S18}$$
$$|\langle b = -1|H\rangle|^2 = \sin^2(\theta/2), \quad |\langle b = -1|V\rangle|^2 = \cos^2(\theta/2). \tag{S19}$$

Using the incompatibility function $f_\perp(b, d) = 1 - |\langle b|d\rangle|^2$, we obtain:

$$f_\perp(+1, H) = \sin^2(\theta/2), \quad f_\perp(+1, V) = \cos^2(\theta/2), \tag{S20}$$
$$f_\perp(-1, H) = \cos^2(\theta/2), \quad f_\perp(-1, V) = \sin^2(\theta/2). \tag{S21}$$

The outcome-dependence difference parameter is:

$$|f_\perp(+1, H) - f_\perp(-1, H)|/2 = |\cos\theta|/2. \tag{S22}$$

At the tilted angle $\theta = 31°$, this yields:

$$|\cos 31°|/2 = 0.8572/2 = 0.4286. \tag{S23}$$

S2.2.2 QM correlator for mixed setting $(x = 1, y = 2)$

For the SPDC noise model (where noise is confined to the $\{|HV\rangle, |VH\rangle\}$ subspace), both the pure singlet and the noise term yield identical z-vs-tilted correlators $(-\cos\theta)$, so the mixed-setting correlator is independent of visibility μ. At the reference angle $\theta = 31°$:

$$\langle A_1 B_2\rangle_{\mathrm{QM}} = -\cos(31°) = -0.8572. \tag{S24}$$

S2.2.3 K9_E deformed correlator

For the multiplicative model, the K9_E probability is:

$$P_{\mathrm{K9E}}(a, b|x, y) \propto P_{\mathrm{QM}}(a, b|x, y) \cdot [1 - \beta f_\perp(b, d)], \tag{S25}$$

with the full normalization $Z = \sum_{a,b} P_{\mathrm{QM}} \cdot [1 - \beta f_\perp(b, d)]$ ensuring probability conservation. Because the normalization factor Z depends on both β and θ, the shift in the correlator is coupled across all outcome pairs. Numerical results at $\theta = 31°, \mu = 0.95$ yield:

β	$\langle A_1 B_2\rangle_{\mathrm{QM}}$	$\langle A_1 B_2\rangle_{\mathrm{K9E}}$	δ
0.10	-0.8572	-0.8687	-0.0115
0.30	-0.8572	-0.8927	-0.0355
0.50	-0.8572	-0.9180	-0.0609

The deformation shifts the correlator to more negative values (enhancing anti-correlation) because the renormalization shifts probability weight toward outcome pairs with larger $f_\perp$ values, amplifying the geometric asymmetry.

S2.2.4 First-order expansion and overestimate warning

Expanding the renormalized probability to first order in β (before full renormalization by Z):

$$\delta\langle A_1 B_2\rangle \approx -\beta \cdot |\cos\theta| \cdot \langle A_1 B_2\rangle_{\mathrm{QM}}^2 + O(\beta^2). \tag{S26}$$

At $\theta = 31^\circ$, this unrenormalized expansion yields:

$$\delta \approx -\beta \cdot 0.8572 \cdot 0.7347 = -0.6298\beta. \tag{S27}$$

Warning: This unrenormalized first-order expansion overestimates $|\delta|$ by approximately a factor of 5.5. The ratio of the numerical value to the leading-order expansion is nearly constant across all β:

β	δ (LO)	δ (numerical)	ratio
0.01	-0.0063	-0.0011	0.181
0.07	-0.0441	-0.0080	0.182
0.10	-0.0630	-0.0115	0.183
0.30	-0.1889	-0.0355	0.188
0.50	-0.3149	-0.0609	0.193

The discrepancy arises because the normalization factor $Z = \sum P_{\mathrm{QM}}[1 - \beta f_\perp]$ is also $O(\beta)$, leading to a cancellation between the numerator and denominator corrections. Therefore, the unrenormalized formula captures the correct qualitative structural dependence (vanishing iff $\theta = \pi/2$) but severely overestimates its magnitude. All manuscript values use exact numerical computation.

S2.3 Complete Correlator Table ($\theta = 31^\circ$, $\mu = 0.95$)

S2.4 Grid Search Optimization

A grid search over $(\theta, \phi_2, \phi_3, \beta_{\mathrm{Bob}})$ maximizes:

$$\mathrm{FOM}(\theta) = \min\left(n_\sigma^{\mathrm{LF}}(\theta), n_\sigma^{\mathrm{signal}}(\theta, \beta)\right), \tag{S28}$$

yielding $\theta = 31^\circ$, $\phi_2 = 112^\circ$, $\phi_3 = 217^\circ$, $\beta_{\mathrm{Bob}} = 20^\circ$. FOM remains above 5σ for $\theta \in [20^\circ, 45^\circ]$ at $\beta = 0.30$ and for $\theta \in [35^\circ, 46^\circ]$ at $\beta = 0.07$, $N = 91{,}000$.

Table S1: All nine $\langle A_x B_y \rangle$ correlators with uncertainties ($N = 91{,}000$) and $K9\_E$ deformed predictions.

(x, y)	$\langle AB \rangle_{\mathrm{QM}}$	σ	$\langle AB \rangle_{\mathrm{K9E}}$ ($\beta = 0.1$)	$\langle AB \rangle_{\mathrm{K9E}}$ ($\beta = 0.3$)	$\langle AB \rangle_{\mathrm{K9E}}$ ($\beta = 0.5$)
$(1, 1)$	-1.0000	0.0000	-1.0000	-1.0000	-1.0000
$(1, 2)$	-0.8572	0.0017	-0.8687	-0.8927	-0.9181
$(1, 3)$	-0.8572	0.0017	-0.8687	-0.8927	-0.9181
$(2, 1)$	-0.8572	0.0017	-0.8687	-0.8927	-0.9181
$(2, 2)$	-0.5045	0.0029	-0.5045	-0.5045	-0.5045
$(2, 3)$	-0.8933	0.0015	-0.8933	-0.8933	-0.8933
$(3, 1)$	-0.8572	0.0017	-0.8687	-0.8927	-0.9181
$(3, 2)$	-0.8933	0.0015	-0.8933	-0.8933	-0.8933
$(3, 3)$	-0.8829	0.0016	-0.8829	-0.8829	-0.8829

S2.5 Monte Carlo Validation

10,000 runs: Gen LF 1 $\geq 5\sigma$ in 99.97%. $\beta = 0.07$: >99% power (four combined settings, $n_\sigma = 9.4$; ∼38% single-setting, $n_\sigma = 4.7$); $\beta = 0.05$: ∼90% (combined). Conservative Bayesian analysis (20% inflation): $\beta_{\min} \approx 0.046$ (combined).

S2.6 Angular Stability

LF significance varies <3% across $\Delta\theta = \pm 5^\circ$. $\delta\langle AB \rangle$ (exact numerical θ-dependence): <1.5% variation at $\pm 1^\circ$ (Bong's reported precision).

S2.7 ϕ-Scramble Control Protocol

Rotate HWP to $N_\phi \geq 10$ random azimuthal angles $\phi \in [0^\circ, 360^\circ)$, fit $\delta\langle AB \rangle(\phi) = A + B\cos(2\phi) + C\sin(2\phi)$. Test: $A \neq 0$ with $B, C \approx 0$ indicates geometric θ-dependent signal; B or $C \neq 0$ indicates birefringence/alignment artifacts.