Deterministic Full Clifford Closure, Coefficient Spaces, and Grade Projections for Variable-Probability Signature Clifford Fields

VPSCF Paper 2

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Abstract

This paper develops the deterministic carrier, closure, and coefficient-semantics layer underlying Variable-Probability Signature Clifford Fields. The variable-probability structure was introduced in VPSCF1 through signature laws, local kernels, and averaged multiplication tensors; the present paper does not introduce a new stochastic dynamics. Instead, it proves the deterministic carrier, closure, and coefficient-semantics firewall that every pathwise or averaged signature law must respect before probability-dependent signed forms, projected dynamics, or analytic field equations can be stated without ambiguity. VPSCF1 [1] fixed a common real blade carrier

$$\mathcal{A} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}$$

and placed all signature dependence in frozen multiplication tensors

$$m_\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \qquad \sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \{\pm 1\}^3.$$

The present paper studies what VPSCF1 deliberately left open: full Clifford closure generated by the scalar-vector sector, coefficient-valued products in spaces of the form $V \otimes \mathcal{A}$, and grade projections back to truncated sectors. The first structural firewall is not a new Clifford-generation fact, but its consequence for VPSCF semantics: the scalar-vector sector

$$\mathcal{A}_{\leq 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$$

is not a closed algebraic carrier: repeated homogeneous grade-one generator products collapse to scalar signature signs, whereas exact frozen products of distinct homogeneous grade-one generators produce ordered bivector labels. The second point is that $V \otimes \mathcal{A}$ is only a coefficient carrier until one specifies coefficient semantics. VPSCF2 isolates three baseline coefficient regimes—pure metric or topological data, associative coefficient multiplication, and free tensor-hierarchy bookkeeping—without claiming that they exhaust all possible coefficient semantics. The third point is that projected products

$$U \star_{\sigma,\leq r} W = \Pi_{\leq r}(U \cdot_\sigma W)$$

create closed effective truncated operations only by discarding higher grades, and those discarded grades generate associator defects. Thus exact multiplication remains the frozen full-grade product in $\mathcal{A}$, while projected and averaged products are effective operations.

Keywords. Variable-probability signature; Clifford algebra; frozen signature; Clifford blade carrier; full Clifford closure; full-carrier closure certificate; coefficient semantics; coefficient-valued Clifford fields; grade projection; projected product; tensor hierarchy; associator defect; non-associativity; scalar-vector sector; homogeneous grade-one subspace; effective multiplication.

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Series predecessor. This paper continues VPSCF Paper 1, *Axioms, Explicit Structure Constants, and Signature-Field State Spaces for Variable-Probability Signature Clifford Fields*, DOI: 10.5281/zenodo.20523472.

Contents

1 Introduction

1.1 Starting point and relation to VPSCF1

The starting expression of the Variable-Probability Signature Clifford Field program is

$$U = a + bi + cj + dk. \tag{1}$$

At first sight, (1) resembles the usual scalar-plus-vector part of a small Clifford algebra; fixed-sign Clifford background may be found in standard references such as [2, 3, 5]. In the VPSCF framework, however, the signs of the squares of the formal generators are not fixed once and for all. A frozen signature is a triple

$$\sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \Sigma := \{\pm 1\}^3, \tag{2}$$

with square relations

$$i^2 = \varepsilon_i, \qquad j^2 = \varepsilon_j, \qquad k^2 = \varepsilon_k, \tag{3}$$

and anticommutation relations

$$ij = -ji, \qquad ik = -ki, \qquad jk = -kj. \tag{4}$$

For each fixed σ, these relations determine a frozen Clifford-type product. The key move of VPSCF1 was not to let the underlying vector space vary with σ. Instead, VPSCF1 fixed the common real blade carrier

$$\mathcal{A} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}, \tag{5}$$

and placed the signature dependence entirely in the multiplication tensor

$$m_\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}. \tag{6}$$

Thus the frozen algebra is

$$\mathrm{Cl}_\sigma = (\mathcal{A}, m_\sigma), \tag{7}$$

where the vector space is fixed and the product varies with the signature.

Remark 1.1 (Relation with the standard $\mathrm{Cl}_{p,q}(\mathbb{R})$ notation)**.** For a frozen signature

$$\sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \{\pm 1\}^3,$$

set

$$p(\sigma) := \#\{\ell \in \{i, j, k\} : \varepsilon_\ell = +1\}, \qquad q(\sigma) := \#\{\ell \in \{i, j, k\} : \varepsilon_\ell = -1\}.$$

With the convention that positive generators square to $+1$ and negative generators square to -1, the frozen algebra $(\mathcal{A}, m_\sigma)$ is the rank-three real Clifford algebra of type $\mathrm{Cl}_{p(\sigma),q(\sigma)}(\mathbb{R})$. The notation Cl_σ is used for a VPSCF-specific reason: the ordered blade carrier

$$\mathcal{A} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}$$

is kept fixed, and the signature dependence is recorded in the labelled multiplication tensor m_σ. Consequently, two signatures with the same pair (p, q) may be isomorphic as ordinary Clifford algebras, while still being distinct labelled multiplication tensors on the common VPSCF carrier when the signs are attached to different generators.

The first paper established the finite algebraic and measurable probability layer of this construction [1]: explicit blade multiplication, structure constants, frozen-signature associativity, measurable random multiplication on $\mathcal{A}$-valued fields, local one-point kernels, signature-field laws, and averaged multiplication tensors. It also emphasized the doctrine

$$\boxed{\text{Exact algebra is pathwise; averaged multiplication is effective.}} \tag{8}$$

The present paper keeps that doctrine and develops the next algebraic layer: closure, coefficient semantics, and grade projection.

Remark 1.2 (Scope of VPSCF2)**.** Although the series is named Variable-Probability Signature Clifford Fields, the present paper is not a new stochastic-dynamics paper. Its role is the deterministic probability-interface layer: it fixes the common full-grade carrier, identifies the closure forced by the scalar-vector sector, separates coefficient carriers from coefficient algebras, and analyzes the defect caused by grade projection. The probability layer remains pathwise/effective in the sense of VPSCF1; VPSCF2 supplies the carrier and semantic firewall that every later probability-dependent construction must use.

1.2 The scalar-vector sector is not a closed algebraic universe

The expression (1) belongs to the scalar-vector sector

$$\mathcal{A}_{\leq 1} := \operatorname{span}_{\mathbb{R}}\{1, i, j, k\}. \tag{9}$$

This sector is the natural home of the motivating object $a + bi + cj + dk$. It is not, however, a closed algebra under the frozen product m_σ. The obstruction is immediate:

$$i, j \in \mathcal{A}_{\leq 1}, \qquad i \cdot_\sigma j = ij, \qquad ij \notin \mathcal{A}_{\leq 1}. \tag{10}$$

Thus products of distinct homogeneous grade-one generators leave the scalar-vector sector, while repeated homogeneous grade-one generator products return scalar signature signs. This is not a notational inconvenience. It is the first structural closure problem of the VPSCF series.

Consequently, (1) must not be interpreted as an element of a closed four-dimensional algebra unless an additional effective operation has been specified. The exact frozen product of two elements of $\mathcal{A}_{\leq 1}$ lives in the full carrier $\mathcal{A}$. In scalar coefficients, if

$$U = a + bi + cj + dk, \qquad W = \alpha + \beta i + \gamma j + \delta k, \tag{11}$$

then the product $U \cdot_\sigma W$ generally contains a bivector component. The exact formula is derived in Section 3; for the purposes of the introduction, the relevant point is simply

$$\mathcal{A}_{\leq 1} \cdot_\sigma \mathcal{A}_{\leq 1} \subseteq \mathcal{A}_{\leq 2}, \qquad \mathcal{A}_{\leq 1} \cdot_\sigma \mathcal{A}_{\leq 1} \not\subseteq \mathcal{A}_{\leq 1}. \tag{12}$$

The full carrier $\mathcal{A}$, not $\mathcal{A}_{\leq 1}$, is therefore the exact multiplicative carrier.

Remark 1.3 (The role of full closure)**.** The fact that $m_\sigma(\mathcal{A}, \mathcal{A}) \subseteq \mathcal{A}$ was already part of the frozen algebraic foundation constructed in VPSCF1 [1]. The new role of VPSCF2 is different: it analyzes the closure generated by the scalar-vector sector and shows that the full blade carrier is forced by the motivating scalar-vector expression. Full closure is recalled; the full-carrier closure certificate inside the fixed carrier is extracted and used.

1.3 The coefficient-space problem

A second ambiguity appears when the coefficients in (1) are no longer scalars. One may want to write

$$U = a \otimes 1 + b \otimes i + c \otimes j + d \otimes k, \qquad a, b, c, d \in V, \tag{13}$$

so that

$$U \in V \otimes \mathcal{A}_{\leq 1}. \tag{14}$$

But the symbol $V \otimes \mathcal{A}$ is not automatically an algebra. If V is merely a Hilbert, Banach, normed, or topological vector space, then it may carry norms, inner products, convergence structures, or pairings, but those data alone do not define a coefficient multiplication

$$\mu_V : V \times V \to V. \tag{15}$$

Without such a multiplication, products such as

$$b\gamma, \qquad c\beta, \qquad d\delta \tag{16}$$

are not intrinsically meaningful.

This paper separates three baseline coefficient regimes used in VPSCF2.

(i) **Pure metric or topological coefficient spaces.** The space V provides linear, metric, topological, or pairing structure, but no canonical multiplication. The object $V \otimes \mathcal{A}$ is then a coefficient carrier, not automatically an algebra.

(ii) **Associative coefficient algebras.** If V is equipped with an associative product $\cdot_V$, then one may define an ordinary ungraded tensor product algebra by

$$(v \otimes A) \cdot_\sigma (w \otimes B) = (v \cdot_V w) \otimes m_\sigma(A, B). \tag{17}$$

No commutativity of V is required.

(iii) **Free tensor-hierarchy semantics.** If no multiplication on V is chosen, products can still be recorded freely by increasing tensor order:

$$(v \otimes A)(w \otimes B) = (v \otimes w) \otimes m_\sigma(A, B), \tag{18}$$

which lands in $V^{\otimes 2} \otimes \mathcal{A}$. Repeated products live in $V^{\otimes n} \otimes \mathcal{A}$.

These baseline coefficient regimes are needed before any signed quadratic form, analytic field space, PDE operator, or interface condition can be stated without ambiguity. They are not an exhaustive classification of all possible coefficient semantics. Pairings, contractions, quotient maps, module actions, bilinear maps into external targets, and distribution-valued products are also possible, but each is additional semantics and must be declared explicitly. In particular, a Hilbert pairing may later support signed quadratic forms, but it does not by itself define coefficient multiplication. Conversely, an associative algebra product gives multiplication but may not encode the metric or positivity structure needed for later energy-like constructions.

1.4 Projection is effective, not exact

One may create an effective closed operation on the scalar-vector sector by projecting products back to a truncated grade space. For $0 \leq r \leq 3$, let

$$\Pi_{\leq r} : \mathcal{A} \to \mathcal{A}_{\leq r} \tag{19}$$

be the vector-space projection onto grades at most r. The corresponding projected product is

$$U \star_{\sigma,\le r} W := \Pi_{\le r}(U \cdot_\sigma W). \tag{20}$$

This product is closed by construction on $\mathcal{A}_{\le r}$, but closure is achieved by discarding higher-grade components. Therefore $\star_{\sigma,\le r}$ is not generally the exact Clifford product.

The main case for the motivating expression (1) is

$$U \star_\sigma W := \Pi_{\le 1}(U \cdot_\sigma W). \tag{21}$$

For the motivating scalar-vector projection $r = 1$, this returns a scalar-vector object, but it is non-associative for every frozen signature. The mechanism is not mysterious. Put

$$P_r = \Pi_{\le r}, \qquad Q_r = I - P_r. \tag{22}$$

Then the associator of the projected product satisfies the defect formula

$$\mathrm{Assoc}_{\sigma,\le r}(U, V, W) = P_r\left(-Q_r(U \cdot_\sigma V) \cdot_\sigma W + U \cdot_\sigma Q_r(V \cdot_\sigma W)\right). \tag{23}$$

The products appearing on the right are exact frozen products in $(\mathcal{A}, m_\sigma)$. Thus discarded grades control the associator defect.

The classification is simple but important:

$$\star_{\sigma,\le 0} \text{ and } \star_{\sigma,\le 3} \text{ are associative,} \tag{24}$$

whereas

$$\star_{\sigma,\le 1} \text{ and } \star_{\sigma,\le 2} \text{ are non-associative for every frozen signature on their full truncated domains.} \tag{25}$$

The first associative case is ordinary scalar multiplication. The second is the full frozen Clifford product. The intermediate truncations are effective projected operations.

1.5 Exact, projected, and averaged products

The word "effective" is used in this paper in the same structural sense as in VPSCF1. An operation is effective when it is derived from exact pathwise products by an additional operation such as averaging or projection. The exact algebraic product for a fixed signature remains

$$m_\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}. \tag{26}$$

If a signature field is realized as

$$\sigma : X \times \Omega \to \Sigma, \tag{27}$$

then the pathwise product is

$$(U \cdot_\sigma W)(x, \omega) = m_{\sigma(x,\omega)}(U(x, \omega), W(x, \omega)). \tag{28}$$

Averaged multiplication, as studied in VPSCF1, has the schematic form

$$\bar{m}_x = \sum_{\tau \in \Sigma} \pi_x(\tau) m_\tau, \tag{29}$$

where π_x is a local one-point signature law. It is useful, but it is not the exact frozen product. Similarly, projected products such as $\star_{\sigma,\le 1}$ are useful, but they are not exact Clifford products unless the projection is the identity.

The paper therefore keeps three product types separated:

Product	Domain	Associativity	Status
m_σ	$\mathcal{A} \times \mathcal{A}$	yes	exact frozen product
$\star_{\sigma,\le r}$	$\mathcal{A}_{\le r} \times \mathcal{A}_{\le r}$	depends on r	projected effective product
$\bar{m}_x$	$\mathcal{A} \times \mathcal{A}$	law-dependent; not automatic	averaged effective product

This taxonomy is a safeguard against a common error: using a closed or averaged effective operation as though it were the exact pathwise algebra.

1.6 Main results of this paper

The paper proves the following results.

(R1) Scalar-vector non-closure. The sector

$$\mathcal{A}_{\le 1} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k\}$$

is not closed under m_σ, since $i \cdot_\sigma j = ij \notin \mathcal{A}_{\le 1}$.

(R2) Scalar-vector product bound. Products of scalar-vector elements satisfy

$$\mathcal{A}_{\le 1} \cdot_\sigma \mathcal{A}_{\le 1} \subseteq \mathcal{A}_{\le 2},$$

and the bivector component is generally nonzero.

(R3) Full-carrier closure certificate. Inside the fixed carrier $\mathcal{A}$, every real linear subspace containing $\mathcal{A}_{\le 1}$ and closed under m_σ is the full carrier $\mathcal{A}$. Hence the blade-generated closure is also $\mathcal{A}$, and the generated blade set is independent of the signs $\varepsilon_i, \varepsilon_j, \varepsilon_k$.

(R4) Baseline coefficient-regime taxonomy. Pure metric or topological coefficient data do not define multiplication on $V \otimes \mathcal{A}$. Associative coefficient algebras do. Product-free coefficient spaces admit the tensor-algebra hierarchy $\mathcal{H}_\bullet = T(V) \otimes \mathcal{A}$, with scalar level $\mathcal{H}_0 \cong \mathcal{A}$.

(R5) General projection-defect formula. For $P_r = \Pi_{\le r}$ and $Q_r = I - P_r$, the projected associator is

$$\operatorname{Assoc}_{\sigma,\le r}(U, V, W) = P_r\left(-Q_r(U \cdot_\sigma V) \cdot_\sigma W + U \cdot_\sigma Q_r(V \cdot_\sigma W)\right).$$

(R6) Associativity classification of projected products. For every frozen signature, $\star_{\sigma,\le 0}$ and $\star_{\sigma,\le 3}$ are associative, whereas $\star_{\sigma,\le 1}$ and $\star_{\sigma,\le 2}$ are non-associative as products on their full truncated domains.

These results settle the algebraic status of $a + bi + cj + dk$ before the series moves to signed quadratic forms, quotient reductions, PDE operators, interface spaces, probability dynamics, and full-grade obstruction theory.

1.7 Scope and claims not made

The present paper is intentionally algebraic. It does not construct a PDE theory, a variational principle, a smooth Clifford bundle, a physical model, or an interface transmission calculus. It also does not develop signed quadratic forms. Those forms require the carrier and coefficient semantics fixed here, but their own positivity, degeneracy, and realizability issues belong to VPSCF3.

The paper also does not claim that projected products are replacements for frozen Clifford multiplication. A projected product is a useful effective operation when one insists on remaining inside a truncated sector, but it is not the exact product unless the projection is the identity. Similarly, the tensor hierarchy is not claimed to be the only possible semantics for product-free coefficient spaces. It is the tensor-algebra bookkeeping construction $T(V) \otimes \mathcal{A}$, free on the coefficient side before one chooses a contraction, quotient, pairing, or multiplication on V.

Remark 1.4 (Why this boundary matters)**.** If closure, coefficient multiplication, and projection are not separated, later formulas become ambiguous. For example, an expression that looks like a signed quadratic form may secretly require a Hilbert pairing on V, whereas an expression that looks like a product may require an associative multiplication on V. VPSCF2 prevents this ambiguity by fixing the algebraic and coefficient-space layer before the analytic and energetic layers are introduced.

1.8 Organization of the paper

Section 2 recalls the frozen carrier, the signature space, and the grade decomposition. Section 3 proves scalar-vector non-closure and derives the scalar product expansion. Section 4 recalls full closure from VPSCF1 and proves the full-carrier closure certificate forced by the scalar-vector sector inside the fixed carrier. Section 5 introduces coefficient-valued carriers $V \otimes \mathcal{A}$. Section 6 develops the three coefficient regimes: pure metric or topological spaces, associative coefficient algebras, and free tensor hierarchies, including measurable coefficient-valued Regime-B fields and the full tensor-hierarchy mapping property. Section 7 defines truncated grade projections and projected products. Section 8 proves the general associator-defect formula and the associativity classification. Section 9 separates exact, projected, and averaged products and gives finite associativity criteria for averaged and projected averaged products. Section 10 records the exact status of $a + bi + cj + dk$. Section 11 explains the transition to signed quadratic forms in VPSCF3.

2 Frozen Carrier and Grade Decomposition

2.1 Purpose of this section

This section fixes the common notation used throughout the rest of the paper. VPSCF1 already constructed the frozen products on a common blade carrier [1]. Here we recall precisely the part of that construction needed for the closure and projection analysis of VPSCF2: the finite signature space, the ordered blade basis, the frozen multiplication tensors, the grade decomposition, and the grade projections.

The essential separation is the following:

$$\boxed{\text{the vector space } \mathcal{A} \text{ is fixed, while the product } m_\sigma \text{ depends on } \sigma.} \tag{30}$$

This separation is the reason one can compare products, projections, and coefficient regimes across different signatures without changing the underlying coordinate carrier.

Convention 2.1 (Algebra terminology)**.** A real algebra means a real vector space equipped with a bilinear multiplication. It is called associative if its multiplication is associative. It is called unital only when a multiplicative identity is explicitly specified. Unless explicitly stated, algebras in this paper are not assumed to be unital.

2.2 Signature space and square signs

The finite signature space is

$$\Sigma = \{\pm 1\}^3. \tag{31}$$

A frozen signature is written

$$\sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \Sigma. \tag{32}$$

For such a fixed σ, the intended square signs are

$$i^2 = \varepsilon_i, \qquad j^2 = \varepsilon_j, \qquad k^2 = \varepsilon_k. \tag{33}$$

The generators anticommute:

$$ij = -ji, \qquad ik = -ki, \qquad jk = -kj. \tag{34}$$

These relations are not imposed by changing the vector space. They are encoded in the frozen multiplication tensor m_σ on the fixed carrier.

For any subset $C \subseteq \{i, j, k\}$, define the sign monomial

$$\varepsilon_C(\sigma) = \prod_{\ell \in C} \varepsilon_\ell, \qquad \varepsilon_\varnothing(\sigma) = 1. \tag{35}$$

When no confusion is possible, we write simply ε_C. For example,

$$\varepsilon_{\{i,k\}} = \varepsilon_i \varepsilon_k, \qquad \varepsilon_{\{i,j,k\}} = \varepsilon_i \varepsilon_j \varepsilon_k. \tag{36}$$

2.3 Ordered blade basis

Fix the ordered generator set

$$I = \{i, j, k\}, \qquad i < j < k. \tag{37}$$

For every subset $S \subseteq I$, let e_S denote the corresponding ordered blade. Thus

$$e_\varnothing = 1, \qquad e_{\{i\}} = i, \qquad e_{\{j\}} = j, \qquad e_{\{k\}} = k, \tag{38}$$

while

$$e_{\{i,j\}} = ij, \qquad e_{\{i,k\}} = ik, \qquad e_{\{j,k\}} = jk, \qquad e_{\{i,j,k\}} = ijk. \tag{39}$$

The notation ij, ik, jk, and ijk denotes ordered basis symbols in the fixed carrier. These symbols become products only after the frozen multiplication tensor has been chosen.

Definition 2.2 (Fixed blade carrier)**.** The common real blade carrier is

$$\mathcal{A} = \operatorname{span}_{\mathbb{R}}\{e_S : S \subseteq I\} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}. \tag{40}$$

It is an eight-dimensional real vector space with ordered basis

$$\mathcal{B} = (1, i, j, k, ij, ik, jk, ijk). \tag{41}$$

Every element $U \in \mathcal{A}$ has a unique coordinate expansion

$$\begin{aligned} U &= u_\varnothing + u_i i + u_j j + u_k k \\ &\quad + u_{ij} ij + u_{ik} ik + u_{jk} jk + u_{ijk} ijk, \qquad u_S \in \mathbb{R}. \end{aligned} \tag{42}$$

Here $S \subseteq I$ denotes the subset corresponding to the displayed blade coordinate. This coordinate uniqueness is independent of σ. The signature affects how elements multiply, not how they are expanded in the common basis.

Remark 2.3 (Formal blade notation)**.** In the fixed-carrier model, ij is first a basis vector. The equality

$$i \cdot_\sigma j = ij$$

is then a statement about the frozen product m_σ. This distinction prevents one from confusing formal blade labels with products before the product has been specified.

Convention 2.4 (Blade labels and product notation)**.** The symbols

$$ij, \qquad ik, \qquad jk, \qquad ijk$$

are ordered basis labels in the fixed carrier $\mathcal{A}$. After a frozen signature σ has been fixed, identities such as

$$i \cdot_\sigma j = ij, \qquad i \cdot_\sigma k = ik, \qquad j \cdot_\sigma k = jk$$

state that the exact frozen product sends pairs of distinct homogeneous grade-one basis elements to the corresponding ordered blade labels.

Throughout the paper the product notation is semantic rather than decorative:

(i) $\cdot_\sigma$ denotes the exact frozen Clifford product;

(ii) $\star_{\sigma,\leq r}$ denotes the projected product $P_r(U \cdot_\sigma W)$;

(iii) $\bar{m}_x$ denotes an averaged effective product over signatures;

(iv) $\star^V_{\sigma,\leq r}$ denotes a coefficient-valued projected product after Regime-B coefficient multiplication has been declared.

Juxtaposition UW is avoided in theorem statements and defect formulas. When used locally inside elementary computations after σ has been fixed, it means only the exact frozen product $U \cdot_\sigma W$, never a projected, averaged, or coefficient-valued product.

Convention 2.5 (Index subsets versus algebra elements)**.** Index subsets of

$$I = \{i, j, k\}$$

are denoted by $S, T, R \subseteq I$. Their ordered blades are denoted by e_S, e_T, e_R. General elements of the carrier $\mathcal{A}$ are denoted by U, V, W. Thus expressions such as

$$N(S,T), \qquad S \cap T, \qquad S \triangle T$$

refer to set operations on index subsets, whereas products such as

$$U \cdot_\sigma V$$

refer to algebraic multiplication in the frozen carrier. This convention prevents subset labels from being confused with arbitrary elements of $\mathcal{A}$.

2.4 Frozen multiplication tensors

The frozen product is recalled from VPSCF1. For subsets $S, T \subseteq I$, let

$$N(S,T) = \#\{(a,b) \in S \times T : b < a\}, \tag{43}$$

where the order is $i < j < k$. This is the number of swaps needed to move the ordered word associated with S and the ordered word associated with T into increasing order after repeated symbols have been paired.

The frozen blade product is

$$e_S \cdot_\sigma e_T = (-1)^{N(S,T)} \varepsilon_{S\cap T} e_{S\triangle T}, \tag{44}$$

where $S\triangle T$ is the symmetric difference. Bilinear extension defines

$$m_\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}. \tag{45}$$

For each frozen signature, the corresponding fixed-carrier algebra is denoted

$$\mathcal{A}_\sigma := (\mathcal{A}, m_\sigma, 1), \qquad \mathrm{Cl}_\sigma := \mathcal{A}_\sigma. \tag{46}$$

Thus $\mathcal{A}$ names the common vector carrier, while $\mathcal{A}_\sigma$ or Cl_σ names the signature-dependent algebra structure on that carrier.

Convention 2.6 (Common vector carrier, signature-dependent algebra structure)**.** The symbol $\mathcal{A}$ denotes the fixed eight-dimensional real vector space with ordered blade basis

$$1,\ i,\ j,\ k,\ ij,\ ik,\ jk,\ ijk.$$

For each frozen signature $\sigma \in \Sigma$, the notation

$$\mathcal{A}_\sigma = (\mathcal{A}, m_\sigma, 1)$$

denotes the algebra structure obtained by using the multiplication tensor m_σ. Thus the vector space is common, but the algebra product is signature-dependent. No canonical algebra identification between different $\mathcal{A}_\sigma$'s is assumed by the carrier convention.

The rule (44) immediately gives

$$i \cdot_\sigma i = \varepsilon_i, \qquad j \cdot_\sigma j = \varepsilon_j, \qquad k \cdot_\sigma k = \varepsilon_k, \tag{47}$$

and

$$i \cdot_\sigma j = ij, \qquad j \cdot_\sigma i = -ij, \tag{48}$$

with analogous formulas for the pairs (i,k) and (j,k). Thus the usual square and anticommutation rules are recovered inside the fixed-carrier product.

Proposition 2.7 (The identity carrier map is not generally an algebra map)**.** *Let $\sigma, \tau \in \Sigma$. The identity linear map*

$$\mathrm{id}_{\mathcal{A}} : \mathcal{A}_\sigma \to \mathcal{A}_\tau$$

is an algebra homomorphism preserving the ordered generators i, j, k if and only if

$$\sigma = \tau.$$

Proof. If $\mathrm{id}_{\mathcal{A}}$ is an algebra homomorphism, then for each generator $a \in \{i, j, k\}$,

$$\mathrm{id}_{\mathcal{A}}(m_\sigma(a,a)) = m_\tau(\mathrm{id}_{\mathcal{A}}(a), \mathrm{id}_{\mathcal{A}}(a)).$$

Thus

$$\varepsilon_a(\sigma)1 = \varepsilon_a(\tau)1.$$

Hence $\varepsilon_a(\sigma) = \varepsilon_a(\tau)$ for $a = i, j, k$, so $\sigma = \tau$. Conversely, if $\sigma = \tau$, the identity map trivially preserves the product. $\square$

Remark 2.8 (No abstract isomorphism classification)**.** Proposition 2.7 concerns only the fixed carrier identification used in this paper. It does not classify the abstract real Clifford algebras associated with different signatures, nor does it rule out nontrivial algebra isomorphisms in special cases. The point is only that the VPSCF carrier convention identifies the underlying vector space, not the multiplication tensor.

Proposition 2.9 (Structure-constant cocycle identity)**.** *Define*

$$\kappa_\sigma(S,T) := (-1)^{N(S,T)}\varepsilon_{S\cap T}, \qquad S,T \subseteq I.$$

Then, for all $S,T,R \subseteq I$,

$$\kappa_\sigma(S,T)\kappa_\sigma(S\triangle T,R) = \kappa_\sigma(T,R)\kappa_\sigma(S,T\triangle R). \tag{49}$$

Consequently, the blade rule

$$e_S \cdot_\sigma e_T = \kappa_\sigma(S,T)e_{S\triangle T}$$

is associative on basis blades.

Proof. Let $\mathbf{1}_S(x)$ denote the indicator of membership $x \in S$. All exponents below are computed modulo 2. The sign exponent on the left-hand side of (49) is

$$N(S,T) + N(S\triangle T,R) = \sum_{x>y}\Big(\mathbf{1}_S(x)\mathbf{1}_T(y) + (\mathbf{1}_S(x)+\mathbf{1}_T(x))\mathbf{1}_R(y)\Big),$$

which expands to

$$\sum_{x>y}\Big(\mathbf{1}_S(x)\mathbf{1}_T(y) + \mathbf{1}_S(x)\mathbf{1}_R(y) + \mathbf{1}_T(x)\mathbf{1}_R(y)\Big).$$

The sign exponent on the right-hand side,

$$N(T,R) + N(S,T\triangle R),$$

has the same expansion. Hence the (-1)-factors agree.

It remains to compare the square-sign factors. For each generator $x \in I$, the exponent of ε_x on the left-hand side is

$$\mathbf{1}_S(x)\mathbf{1}_T(x) + (\mathbf{1}_S(x)+\mathbf{1}_T(x))\mathbf{1}_R(x),$$

whereas the exponent on the right-hand side is

$$\mathbf{1}_T(x)\mathbf{1}_R(x) + \mathbf{1}_S(x)(\mathbf{1}_T(x)+\mathbf{1}_R(x)).$$

Both are equal modulo 2 to

$$\mathbf{1}_S(x)\mathbf{1}_T(x) + \mathbf{1}_S(x)\mathbf{1}_R(x) + \mathbf{1}_T(x)\mathbf{1}_R(x).$$

Since $\varepsilon_x^2 = 1$, the square-sign factors agree. This proves (49).

Finally,

$$(e_S \cdot_\sigma e_T)\cdot_\sigma e_R = \kappa_\sigma(S,T)\kappa_\sigma(S\triangle T,R)e_{S\triangle T\triangle R},$$

whereas

$$e_S \cdot_\sigma (e_T \cdot_\sigma e_R) = \kappa_\sigma(T,R)\kappa_\sigma(S,T\triangle R)e_{S\triangle T\triangle R}.$$

The cocycle identity gives equality. ∎

Lemma 2.10 (Normal forms and PBW basis). *For every frozen signature $\sigma \in \Sigma$, the quotient*

$$C_\sigma = \mathbb{R}\langle e_i, e_j, e_k\rangle / I_\sigma,$$

where I_σ is generated by

$$e_i^2 - \varepsilon_i, \qquad e_j^2 - \varepsilon_j, \qquad e_k^2 - \varepsilon_k,$$

and by the anticommutation relations

$$e_i e_j + e_j e_i, \qquad e_i e_k + e_k e_i, \qquad e_j e_k + e_k e_j,$$

has the eight ordered square-free monomials

$$1, \quad e_i, \quad e_j, \quad e_k, \quad e_i e_j, \quad e_i e_k, \quad e_j e_k, \quad e_i e_j e_k$$

as a real basis. Hence $\dim_{\mathbb{R}} C_\sigma = 8$, and every element has a unique ordered-blade normal form.

Proof. Use the rewriting rules

$$e_j e_i \mapsto -e_i e_j, \qquad e_k e_i \mapsto -e_i e_k, \qquad e_k e_j \mapsto -e_j e_k,$$

and

$$e_i^2 \mapsto \varepsilon_i, \qquad e_j^2 \mapsto \varepsilon_j, \qquad e_k^2 \mapsto \varepsilon_k.$$

The lexicographic measure given by word length and inversion number strictly decreases under these reductions, so the system terminates. The irreducible words are precisely the displayed ordered square-free monomials.

The only nontrivial critical ambiguities are square–swap ambiguities and the three-generator ordering ambiguity. For example,

$$e_k e_j e_i \mapsto -e_j e_k e_i \mapsto e_j e_i e_k \mapsto -e_i e_j e_k,$$

while

$$e_k e_j e_i \mapsto -e_k e_i e_j \mapsto e_i e_k e_j \mapsto -e_i e_j e_k.$$

Both reduction paths give the same normal form. A representative square–swap ambiguity is

$$e_j e_i^2 \mapsto \varepsilon_i e_j, \qquad e_j e_i^2 \mapsto -e_i e_j e_i \mapsto e_i^2 e_j \mapsto \varepsilon_i e_j.$$

The remaining cases are identical after relabeling i, j, k. Thus the terminating rewriting system is locally confluent and hence confluent. Therefore every word has a unique ordered square-free normal form, and the eight irreducible monomials form a basis of C_σ. This is the finite-rank PBW normal-form mechanism for Clifford algebras; see [2, 8, 9]. □

Lemma 2.11 (Frozen Clifford associativity). *For every fixed signature*

$$\sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \Sigma,$$

the product m_σ on

$$\mathcal{A} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}$$

is associative:

$$(U \cdot_\sigma V) \cdot_\sigma W = U \cdot_\sigma (V \cdot_\sigma W), \qquad U, V, W \in \mathcal{A}.$$

Proof. By Proposition 2.9, associativity holds on the ordered blade basis. Since m_σ is defined by bilinear extension from basis blades, associativity holds for arbitrary $U, V, W \in \mathcal{A}$. Lemma 2.10 additionally identifies the finite structure-constant rule (44) with the usual rank-three Clifford quotient normal form, so no hidden lower-dimensional quotient relation is being imposed. □

Remark 2.12 (Associativity is frozen, not averaged). For every fixed $\sigma \in \Sigma$, Lemma 2.11 shows that m_σ is associative. This is the frozen associativity used throughout the paper. No associativity is inferred from this lemma for projected or averaged products.

2.5 Probability interface of the deterministic carrier

The variable-probability layer of VPSCF1 varies the signature law, not the deterministic blade carrier. The following theorem records the precise interface needed later in this paper: pathwise randomization and local averaging both remain operations on the same finite carrier $\mathcal{A}$.

Theorem 2.13 (Probability-interface theorem for the common carrier)**.** *Let* $(X, \mathcal{X})$ *and* $(\Omega, \mathcal{F})$ *be measurable spaces, and let*

$$\sigma : X \times \Omega \to \Sigma$$

be a measurable frozen-signature field. Let

$$U, W : X \times \Omega \to \mathcal{A}$$

be measurable $\mathcal{A}$*-valued fields, where* $\mathcal{A}$ *is identified with* $\mathbb{R}^8$ *through the fixed ordered blade basis. Then the pathwise product*

$$(x, \omega) \longmapsto m_{\sigma(x,\omega)}(U(x, \omega), W(x, \omega))$$

is again a measurable $\mathcal{A}$*-valued field.*

If $\pi_x \in \mathcal{P}(\Sigma)$ *is a measurable local signature kernel, meaning that each coordinate function* $x \mapsto \pi_x(\tau)$ *is measurable, then the averaged product*

$$\bar{m}_x(U, W) := \sum_{\tau \in \Sigma} \pi_x(\tau) m_\tau(U, W)$$

is a bilinear map

$$\bar{m}_x : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

on the same deterministic carrier. Hence neither pathwise randomization nor finite local averaging changes the underlying blade carrier.

Proof. The signature space Σ is finite, and each multiplication tensor m_τ is a fixed bilinear map on the finite-dimensional carrier $\mathcal{A}$. In blade coordinates, the pathwise product is a finite sum of measurable coordinate functions multiplied by structure constants determined by the measurable finite-valued map σ. Therefore the pathwise product is measurable.

For the averaged product, the expression

$$\bar{m}_x(U, W) = \sum_{\tau \in \Sigma} \pi_x(\tau) m_\tau(U, W)$$

is a finite linear combination of bilinear maps $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, with measurable coefficient functions in x. Hence, for each fixed x, it is again a bilinear map on $\mathcal{A}$, and as x varies its structure constants are measurable. The carrier remains $\mathcal{A}$ in both the pathwise and averaged interpretations. □

Corollary 2.14 (Law-independent scalar-vector closure)**.** *The exact multiplicative closure forced by the scalar-vector sector*

$$\mathcal{A}_{\leq 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$$

is independent of the signature law, local kernel, or measurable signature field. For every frozen signature σ*, the minimal real linear subspace of* $\mathcal{A}$ *containing* $\mathcal{A}_{\leq 1}$ *and closed under* m_σ *is the full carrier* $\mathcal{A}$*.*

Proof. The probability-interface theorem shows that pathwise and averaged signature semantics do not change the carrier. The closure statement is the frozen closure result proved below in Theorem 4.4; that theorem holds for every fixed $\sigma \in \Sigma$. Therefore changing the law or kernel changes the multiplication tensor used pathwise or effectively, but it does not change the full carrier forced by exact scalar-vector closure. □

2.6 Grade decomposition

The ordered blade basis carries a natural grade decomposition by subset cardinality. Define

$$\mathcal{A}^{(q)} = \mathrm{span}_{\mathbb{R}}\{e_S : S \subseteq I,\ |S| = q\}, \qquad q = 0, 1, 2, 3. \tag{50}$$

Explicitly,

$$\mathcal{A}^{(0)} = \mathrm{span}_{\mathbb{R}}\{1\}, \tag{51}$$

$$\mathcal{A}^{(1)} = \mathrm{span}_{\mathbb{R}}\{i, j, k\}, \tag{52}$$

$$\mathcal{A}^{(2)} = \mathrm{span}_{\mathbb{R}}\{ij, ik, jk\}, \tag{53}$$

and

$$\mathcal{A}^{(3)} = \mathrm{span}_{\mathbb{R}}\{ijk\}. \tag{54}$$

Thus

$$\mathcal{A} = \mathcal{A}^{(0)} \oplus \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \mathcal{A}^{(3)}. \tag{55}$$

For $0 \leq r \leq 3$, define the truncated grade sector

$$\mathcal{A}_{\leq r} = \bigoplus_{q=0}^{r} \mathcal{A}^{(q)}. \tag{56}$$

The sector central to the motivating expression $a + bi + cj + dk$ is

$$\mathcal{A}_{\leq 1} = \mathcal{A}^{(0)} \oplus \mathcal{A}^{(1)} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}. \tag{57}$$

This paper calls $\mathcal{A}_{\leq 1}$ the *scalar-vector sector* or the *level-one truncated sector*. The phrase *grade-one sector* is reserved for the homogeneous vector subspace $\mathcal{A}^{(1)} = \mathrm{span}_{\mathbb{R}}\{i, j, k\}$. Thus scalar-vector expressions contain grades zero and one, whereas homogeneous grade-one expressions contain only vector blades.

The dimensions of the truncated sectors are

$$\dim \mathcal{A}_{\leq 0} = 1, \qquad \dim \mathcal{A}_{\leq 1} = 4, \qquad \dim \mathcal{A}_{\leq 2} = 7, \qquad \dim \mathcal{A}_{\leq 3} = 8. \tag{58}$$

The full carrier is exactly $\mathcal{A}_{\leq 3} = \mathcal{A}$.

Remark 2.15 (Grading is vector-space data)**.** The decomposition (55) is a decomposition of the fixed vector space $\mathcal{A}$. It does not depend on σ. What depends on σ is the multiplication rule that sends products of homogeneous blades into other blade sectors.

2.7 Grade projections

Let

$$\Pi_q : \mathcal{A} \to \mathcal{A}^{(q)} \tag{59}$$

be the projection associated with the direct sum (55). For $0 \leq r \leq 3$, define

$$\Pi_{\leq r} = \sum_{q=0}^{r} \Pi_q : \mathcal{A} \to \mathcal{A}_{\leq r}. \tag{60}$$

For the full coordinate expansion (42), the level-one projection is

$$\begin{aligned} \Pi_{\leq 1} U &= u_{\varnothing} + u_i i + u_j j + u_k k, \\ (I - \Pi_{\leq 1}) U &= u_{ij} ij + u_{ik} ik + u_{jk} jk + u_{ijk} ijk. \end{aligned} \tag{61}$$

Similarly, $\Pi_{\leq 2}$ retains the scalar, vector, and bivector parts and discards only the trivector part.

Convention 2.16 (Projection versus multiplication)**.** Throughout this paper, projections Π_q and $\Pi_{\le r}$ are fixed vector-space maps on $\mathcal{A}$. They are independent of σ. Products are written using $\cdot_\sigma$ or m_σ and depend on the frozen signature. Projected products, introduced later, are therefore composite operations: exact frozen multiplication followed by a signature-independent projection.

2.8 Why this setup is needed for the rest of the paper

The later arguments depend on three distinctions fixed in this section.

First, the scalar-vector sector

$$\mathcal{A}_{\le 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$$

is a vector subspace, not automatically a subalgebra. Section 3 proves that it is not closed under m_σ.

Second, the full carrier

$$\mathcal{A} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}$$

is the exact frozen multiplicative carrier inherited from VPSCF1. Section 4 explains why this full carrier is forced as the exact closure certificate generated by $\mathcal{A}_{\le 1}$.

Third, projections such as $\Pi_{\le 1}$ and $\Pi_{\le 2}$ are not products. They become part of an effective product only after being composed with frozen multiplication:

$$U \star_{\sigma,\le r} W := \Pi_{\le r}(U \cdot_\sigma W).$$

The algebraic defect of such projected products is the main subject of Section 8.

3 Scalar Grade-One Non-Closure

3.1 Purpose of this section

The preceding section fixed the common carrier, the frozen products, and the grade decomposition. This section proves the first structural claim of the paper: the scalar-vector sector

$$\mathcal{A}_{\le 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\} \tag{62}$$

is not closed under frozen Clifford multiplication. The point is elementary but foundational. Expressions of the form

$$a + bi + cj + dk \tag{63}$$

are legitimate elements of $\mathcal{A}_{\le 1}$, but the exact product of two such elements generally contains bivector terms. Hence the exact product does not remain in $\mathcal{A}_{\le 1}$.

This section is deliberately scalar. The coefficients appearing below lie in $\mathbb{R}$. Coefficient-valued objects of the form $V \otimes \mathcal{A}$ require additional coefficient semantics and are treated later, beginning in Section 5. Keeping the scalar case separate prevents one from silently multiplying coefficients in a vector space V where no multiplication has been chosen.

3.2 Scalar-vector elements

Throughout this section, assume

$$a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}. \tag{64}$$

Let

$$U = a + bi + cj + dk, \qquad W = \alpha + \beta i + \gamma j + \delta k. \tag{65}$$

Then

$$U, W \in \mathcal{A}_{\leq 1}. \tag{66}$$

The space $\mathcal{A}_{\leq 1}$ is a four-dimensional vector subspace of the fixed carrier $\mathcal{A}$. It contains the scalar unit and the three grade-one generators, but it does not contain the bivectors

$$ij, \qquad ik, \qquad jk. \tag{67}$$

Thus closure of $\mathcal{A}_{\leq 1}$ under m_σ would require that all bivector terms cancel for every product of scalar-vector elements. The next theorem shows that this is false in the strongest possible way: even a single product of basis generators leaves the sector.

3.3 Non-closure theorem

Theorem 3.1 (Scalar-vector non-closure)**.** *For every frozen signature*

$$\sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \Sigma,$$

the scalar-vector sector

$$\mathcal{A}_{\leq 1} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k\}$$

is not closed under the frozen product m_σ.

Proof. By definition,

$$i, j \in \mathcal{A}_{\leq 1}.$$

Using the frozen blade product recalled in (44), or equivalently the anticommutation convention with ordered blades, one has

$$i \cdot_\sigma j = ij.$$

But

$$ij \in \mathcal{A}^{(2)}, \qquad ij \notin \mathcal{A}_{\leq 1}.$$

Therefore $\mathcal{A}_{\leq 1}$ is not closed under m_σ. The argument is independent of the values of $\varepsilon_i, \varepsilon_j, \varepsilon_k$, because no generator is squared in the product $i \cdot_\sigma j$. □

Remark 3.2 (Non-closure is not a probabilistic effect)**.** The obstruction in Theorem 3.1 occurs for every fixed frozen signature. It is not caused by averaging, randomization, or variation of σ. It is already present in each frozen Clifford algebra $(\mathcal{A}, m_\sigma)$. Random or averaged signature layers may add further effective defects, but the failure of $\mathcal{A}_{\leq 1}$ to be a subalgebra is a purely frozen algebraic fact.

3.4 Product expansion for scalar-vector elements

The non-closure theorem can be sharpened by writing the full product of two scalar-vector elements. The expansion separates into scalar, vector, and bivector parts.

Definition 3.3 (Exterior coefficient bivector map)**.** For coefficient triples

$$u = (b, c, d), \qquad w = (\beta, \gamma, \delta) \in \mathbb{R}^3,$$

define the ordered exterior coefficient bivector map

$$u \wedge_{\mathcal{B}} w := (b\gamma - c\beta)ij + (b\delta - d\beta)ik + (c\delta - d\gamma)jk \in \mathcal{A}^{(2)}.$$

Equivalently, this is the ordinary exterior product $u \wedge w \in \Lambda^2\mathbb{R}^3$ transported to the ordered bivector basis

$$e_i \wedge e_j \mapsto ij, \qquad e_i \wedge e_k \mapsto ik, \qquad e_j \wedge e_k \mapsto jk.$$

The subscript $\mathcal{B}$ records that the output is expressed in the fixed ordered blade basis of $\mathcal{A}$, not in an abstract exterior algebra.

Proposition 3.4 (Scalar-vector product expansion)**.** *Let $U, W \in \mathcal{A}_{\leq 1}$ be given by (65). Then*

$$U \cdot_\sigma W = S_\sigma(U, W) + V_\sigma(U, W) + B_\sigma(U, W), \tag{68}$$

where the scalar part is

$$S_\sigma(U, W) = a\alpha + \varepsilon_i b\beta + \varepsilon_j c\gamma + \varepsilon_k d\delta, \tag{69}$$

the vector part is

$$V_\sigma(U, W) = (a\beta + b\alpha)i + (a\gamma + c\alpha)j + (a\delta + d\alpha)k, \tag{70}$$

and the bivector part is

$$B_\sigma(U, W) = (b\gamma - c\beta)ij + (b\delta - d\beta)ik + (c\delta - d\gamma)jk = (b, c, d) \wedge_{\mathcal{B}} (\beta, \gamma, \delta). \tag{71}$$

Consequently,

$$U \cdot_\sigma W \in \mathcal{A}_{\leq 2}. \tag{72}$$

Proof. Expand bilinearly:

$$\begin{aligned} U \cdot_\sigma W &= (a + bi + cj + dk)(\alpha + \beta i + \gamma j + \delta k) \\ &= a\alpha + a\beta i + a\gamma j + a\delta k \\ &\quad + b\alpha i + b\beta i^2 + b\gamma ij + b\delta ik \\ &\quad + c\alpha j + c\beta ji + c\gamma j^2 + c\delta jk \\ &\quad + d\alpha k + d\beta ki + d\gamma kj + d\delta k^2. \end{aligned}$$

Using

$$i^2 = \varepsilon_i, \qquad j^2 = \varepsilon_j, \qquad k^2 = \varepsilon_k,$$

and

$$ji = -ij, \qquad ki = -ik, \qquad kj = -jk,$$

we collect the scalar, vector, and bivector terms. The scalar terms are

$$a\alpha + \varepsilon_i b\beta + \varepsilon_j c\gamma + \varepsilon_k d\delta.$$

The vector terms are

$$(a\beta + b\alpha)i + (a\gamma + c\alpha)j + (a\delta + d\alpha)k.$$

The bivector terms are

$$(b\gamma - c\beta)ij + (b\delta - d\beta)ik + (c\delta - d\gamma)jk.$$

There is no trivector term, because the product of two scalar-vector elements contains at most two generators in each blade. This proves the formula and the inclusion in $\mathcal{A}_{\leq 2}$. □

Remark 3.5 (Dependence on σ)**.** In the scalar product expansion, the square signs $\varepsilon_i, \varepsilon_j, \varepsilon_k$ appear only in the scalar part. The vector part and the bivector wedge-type part are independent of the square signs. This reflects the fact that changing σ changes contractions produced by repeated generators, while the ordered bivector labels generated from distinct generators are the same in the fixed carrier.

3.5 Generic escape from the scalar-vector sector

The expansion in Proposition 3.4 gives an exact criterion for a scalar-vector product to remain scalar-vector.

Corollary 3.6 (Bivector cancellation criterion)**.** *For scalar-vector elements*

$$U = a + bi + cj + dk, \qquad W = \alpha + \beta i + \gamma j + \delta k,$$

one has

$$U \cdot_\sigma W \in \mathcal{A}_{\leq 1}$$

if and only if

$$(b, c, d) \wedge_{\mathcal{B}} (\beta, \gamma, \delta) = 0.$$

Equivalently,

$$b\gamma - c\beta = 0, \qquad b\delta - d\beta = 0, \qquad c\delta - d\gamma = 0.$$

Equivalently, the vector coefficient triples (b, c, d) and (β, γ, δ) are linearly dependent in $\mathbb{R}^3$, with the zero cases included.

Proof. By Proposition 3.4, the only possible component of $U \cdot_\sigma W$ outside $\mathcal{A}_{\leq 1}$ is

$$B_\sigma(U, W) = (b, c, d) \wedge_{\mathcal{B}} (\beta, \gamma, \delta) \in \mathcal{A}^{(2)}.$$

Since ij, ik, jk are linearly independent ordered blade labels, this component vanishes exactly when the three displayed coefficients vanish. This is precisely the usual rank-one condition for two vectors in $\mathbb{R}^3$, equivalently linear dependence, with zero vectors included. $\square$

3.6 Grade bookkeeping

The preceding expansion is a concrete instance of a simple grade bookkeeping principle.

Lemma 3.7 (Scalar-vector product bound)**.** *For every frozen signature $\sigma \in \Sigma$,*

$$\mathcal{A}_{\leq 1} \cdot_\sigma \mathcal{A}_{\leq 1} \subseteq \mathcal{A}_{\leq 2}. \tag{73}$$

Moreover,

$$\mathcal{A}_{\leq 1} \cdot_\sigma \mathcal{A}_{\leq 1} \not\subseteq \mathcal{A}_{\leq 1}. \tag{74}$$

Proof. Every element of $\mathcal{A}_{\leq 1}$ is a scalar plus a linear combination of i, j, k. A scalar times a scalar has grade zero. A scalar times a vector, or a vector times a scalar, has grade one. A product of two grade-one generators either squares to a scalar, if the two generators are equal, or gives an ordered bivector up to sign, if they are distinct. Therefore a product of two elements of $\mathcal{A}_{\leq 1}$ can have only grades $0, 1, 2$. This proves (73). The strict failure of containment in $\mathcal{A}_{\leq 1}$ follows from $i \cdot_\sigma j = ij \notin \mathcal{A}_{\leq 1}$. $\square$

Remark 3.8 (No trivector from two homogeneous grade-one factors)**.** A trivector term requires three distinct generators. Since the product of two scalar-vector factors contains at most one generator from each factor, it cannot create an ijk term. Trivectors first appear when one multiplies a vector by a bivector, or when three homogeneous grade-one factors are multiplied without truncation.

3.7 Why non-closure forces full carrier analysis

The failure of closure has two consequences.

First, the exact product of scalar-vector inputs must be computed in a larger carrier. For two scalar-vector inputs, $\mathcal{A}_{\leq 2}$ is enough. But once one multiplies again, bivectors interact with vectors and generate the trivector. For example,

$$i \cdot_\sigma jk = ijk. \tag{75}$$

Thus repeated exact multiplication starting from $1, i, j, k$ forces the full carrier

$$\mathcal{A} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}. \tag{76}$$

This is developed formally in Section 4.

Second, if one insists on staying inside $\mathcal{A}_{\leq 1}$, one must introduce an additional operation. The natural operation is projected multiplication,

$$U \star_\sigma W = \Pi_{\leq 1}(U \cdot_\sigma W). \tag{77}$$

This operation is closed on $\mathcal{A}_{\leq 1}$, but it is no longer the exact frozen Clifford product. Its associator defect is caused by the bivector terms discarded by $\Pi_{\leq 1}$. This projected-product mechanism is developed in Sections 7 and 8.

3.8 Coefficient warning and transition

All formulas in this section were scalar formulas. The products

$$b\gamma, \qquad c\beta, \qquad d\delta \tag{78}$$

that appear in Proposition 3.4 are ordinary real products. If instead

$$a, b, c, d, \alpha, \beta, \gamma, \delta \in V \tag{79}$$

for a general vector space V, then these products are not defined until one specifies how coefficients multiply.

In the associative coefficient-algebra regime developed later, the same scalar expansion has an ordered analogue obtained by replacing scalar multiplication with the chosen product $\cdot_V$. If V is noncommutative, order matters: for example, the ij-coefficient becomes

$$b \cdot_V \gamma - c \cdot_V \beta, \tag{80}$$

not an unordered product. In the tensor-hierarchy regime, the corresponding terms are represented by tensor concatenations such as $b \otimes \gamma$ and $c \otimes \beta$, not by products in V. If V is merely a Hilbert or Banach space with no specified multiplication, neither expression is automatically meaningful as a coefficient in V.

This is why the paper next recalls full closure and then separates coefficient-valued carriers from coefficient-valued algebras. The non-closure of $\mathcal{A}_{\leq 1}$ is a blade-level issue; the meaning of coefficient multiplication is a separate coefficient-level issue. VPSCF2 keeps them separate.

4 Full Clifford Closure and Minimality

4.1 Purpose of this section

Section 3 proved that the scalar-vector sector

$$\mathcal{A}_{\le 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$$

is not closed under frozen multiplication. The present section identifies the exact blade carrier forced by this failure. The point is not novelty at the level of classical Clifford generation. The point is semantic minimality inside the VPSCF carrier convention. The full-carrier closure was already built into the frozen product construction recalled from VPSCF1. The sharper statement needed here is:

$$\boxed{\text{once one starts from } \mathcal{A}_{\le 1} \text{ and insists on exact multiplication, all blades are forced.}}$$

In other words, the full carrier is not an optional enlargement. It is the exact closure certificate inside the fixed carrier generated by the scalar-vector input sector.

This distinction is important for the rest of VPSCF2. Later sections will introduce coefficient-valued objects and projected products. Those constructions must be understood against the exact full-grade background developed here. Projection may return a product to a smaller sector, but exact multiplication does not.

4.2 Recalled full-carrier closure from VPSCF1

We first recall the closure property inherited from VPSCF1 [1]. The common blade carrier is

$$\mathcal{A} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}. \tag{81}$$

For each frozen signature

$$\sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \Sigma,$$

VPSCF1 defines a multiplication tensor

$$m_\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}. \tag{82}$$

Equivalently, if

$$A, B \in \mathcal{A},$$

then

$$A \cdot_\sigma B \in \mathcal{A}. \tag{83}$$

Proposition 4.1 (Recalled full-carrier closure)**.** *For every frozen signature $\sigma \in \Sigma$, the full carrier $\mathcal{A}$ is closed under m_σ:*

$$m_\sigma(\mathcal{A}, \mathcal{A}) \subseteq \mathcal{A}. \tag{84}$$

Consequently,

$$(\mathcal{A}, m_\sigma) \tag{85}$$

is a closed frozen-signature Clifford algebra.

Proof. This is the full-carrier closure property already encoded by the VPSCF1 blade multiplication law [1]. In the ordered blade basis

$$1, i, j, k, ij, ik, jk, ijk,$$

the product of any two basis blades is again a signed scalar multiple of a basis blade. By bilinearity, products of arbitrary elements of $\mathcal{A}$ remain in $\mathcal{A}$. $\square$

The role of Proposition 4.1 in VPSCF2 is foundational rather than novel. The contribution below is the semantic closure certificate: starting only from the scalar-vector sector, exact closure regenerates the whole carrier.

Remark 4.2 (Novelty is semantic, not classical generation)**.** The minimality result below is not intended as a new theorem about ordinary real Clifford algebras. In a rank-three Clifford algebra, degree-one generators classically generate the full algebra. Its role in VPSCF2 is different: it certifies that the motivating scalar-vector expression $a + bi + cj + dk$ cannot be treated as a closed exact algebra unless one either passes to the full carrier $\mathcal{A}$ or explicitly replaces exact multiplication by a projected or otherwise effective operation.

4.3 Generated subspaces inside the fixed carrier

We now make precise the closure operation used in this section. A *linear subspace inside the fixed carrier* means a real linear subspace $W \subseteq \mathcal{A}$. It is m_σ-closed if

$$m_\sigma(W, W) \subseteq W. \tag{86}$$

A *blade subspace* is a special case: a linear subspace spanned by a subset of the ordered blade basis

$$\mathcal{B} = \{1, i, j, k, ij, ik, jk, ijk\}. \tag{87}$$

Since the present section concerns internal closure, not arbitrary quotient algebras or external completions, all closure statements are made inside the fixed carrier $\mathcal{A}$.

Definition 4.3 (Blade algebra generated inside the fixed carrier)**.** Let $S \subseteq \mathcal{A}$ be a blade-spanned subspace. We write

$$\mathrm{BladeAlg}_{m_\sigma}\langle S\rangle \tag{88}$$

for the smallest blade subspace of the fixed carrier $\mathcal{A}$ that contains S and is closed under the frozen product m_σ.

The stronger minimality theorem below is not restricted to blade subspaces: any real linear subspace of $\mathcal{A}$ containing $\mathcal{A}_{\le 1}$ and closed under m_σ is already the full carrier. The blade-generated statement is then an immediate special case.

4.4 Minimal full linear closure

The scalar-vector sector contains the unit and the three grade-one generators:

$$\mathcal{A}_{\le 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}. \tag{89}$$

The products among the grade-one generators force the bivectors:

$$i \cdot_\sigma j = ij, \qquad i \cdot_\sigma k = ik, \qquad j \cdot_\sigma k = jk. \tag{90}$$

Once the bivectors are present, products of a vector and a bivector force the trivector. For instance,

$$i \cdot_\sigma jk = ijk. \tag{91}$$

Thus closure under exact multiplication does not stop at scalar-vector or scalar-vector-bivector level.

Theorem 4.4 (Full-carrier closure forced by scalar-vector data)**.** *For every frozen signature $\sigma \in \Sigma$, let $W \subseteq \mathcal{A}$ be a real linear subspace satisfying*

$$\mathcal{A}_{\le 1} \subseteq W \tag{92}$$

and

$$m_\sigma(W, W) \subseteq W. \tag{93}$$

Then

$$W = \mathcal{A}. \tag{94}$$

Consequently, the smallest real linear subspace of the fixed carrier $\mathcal{A}$ containing $\mathcal{A}_{\le 1}$ and closed under m_σ is the full carrier $\mathcal{A}$. In particular,

$$\mathrm{BladeAlg}_{m_\sigma}\langle \mathcal{A}_{\le 1} \rangle = \mathcal{A}. \tag{95}$$

Proof. Since $\mathcal{A}_{\le 1} \subseteq W$, we have

$$1, i, j, k \in W.$$

Because W is closed under m_σ,

$$ij = i \cdot_\sigma j \in W, \qquad ik = i \cdot_\sigma k \in W, \qquad jk = j \cdot_\sigma k \in W. \tag{96}$$

Now $i \in W$ and $jk \in W$. Hence closure gives

$$ijk = i \cdot_\sigma jk \in W. \tag{97}$$

Therefore W contains every ordered blade:

$$1, i, j, k, ij, ik, jk, ijk \in W. \tag{98}$$

Since W is a real linear subspace, it contains the real span of these blades, namely $\mathcal{A}$. Thus $\mathcal{A} \subseteq W$. The reverse inclusion $W \subseteq \mathcal{A}$ holds by assumption, so $W = \mathcal{A}$.

Applying this conclusion to blade subspaces gives the displayed blade-generated identity. The theorem is therefore a closure certificate for the VPSCF carrier convention: scalar-vector truncation may be used as an effective projected model, but it is not an exact multiplicative carrier. □

Corollary 4.5 (Exact closure versus effective scalar-vector truncation)**.** *The scalar-vector sector $\mathcal{A}_{\le 1}$ can be used as a projected or effective state space only after the product has been modified, for example by*

$$U \star_{\sigma, \le 1} V = P_1 m_\sigma(U, V).$$

It is not an exact frozen Clifford algebra under m_σ. Exact scalar-vector multiplication forces the full carrier $\mathcal{A}$ by Theorem 4.4.

4.5 Signature-independence of the generated blade set

The signs $\varepsilon_i, \varepsilon_j, \varepsilon_k$ influence contractions and squares, such as

$$i^2 = \varepsilon_i, \qquad j^2 = \varepsilon_j, \qquad k^2 = \varepsilon_k.$$

They do not change which blades are forced by closure of the scalar-vector sector. The generators ij, ik, jk, ijk arise from multiplying distinct generators, so no square sign is needed to generate the blade set itself.

Corollary 4.6 (Signature-independent blade generation)**.** *For every $\sigma \in \Sigma$,*

$$\mathrm{BladeAlg}_{m_\sigma}\langle \mathcal{A}_{\le 1}\rangle = \mathcal{A}. \tag{99}$$

Moreover, the generated blade set

$$\{1, i, j, k, ij, ik, jk, ijk\} \tag{100}$$

is independent of the signs $\varepsilon_i, \varepsilon_j, \varepsilon_k$.

Proof. The proof of Theorem 4.4 uses the products

$$i \cdot_\sigma j = ij, \qquad i \cdot_\sigma k = ik, \qquad j \cdot_\sigma k = jk, \qquad i \cdot_\sigma jk = ijk.$$

These blade-generation steps do not require replacing any repeated generator by a square sign. Therefore the set of generated blades is the same for every $\sigma \in \Sigma$. □

This corollary is not saying that all frozen algebras $(\mathcal{A}, m_\sigma)$ are the same algebra. Their multiplication tensors differ through the signs in contractions. It says only that the common blade carrier forced by exact closure is the same.

4.6 Why intermediate truncated sectors do not suffice

Theorem 4.4 can also be read as a negative statement about intermediate truncations. The scalar sector

$$\mathcal{A}_{\le 0} = \mathrm{span}_{\mathbb{R}}\{1\}$$

is closed, but it does not contain the motivating scalar-vector data. The scalar-vector sector $\mathcal{A}_{\le 1}$ contains the motivating data, but Theorem 3.1 shows that it is not closed. The next possible truncation is

$$\mathcal{A}_{\le 2} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk\}. \tag{101}$$

This sector contains all products of two scalar-vector elements, but it is still not closed under exact multiplication.

Proposition 4.7 (The scalar-vector-bivector sector is not exact-closed)**.** *For every $\sigma \in \Sigma$,*

$$m_\sigma(\mathcal{A}_{\le 2}, \mathcal{A}_{\le 2}) \not\subseteq \mathcal{A}_{\le 2}. \tag{102}$$

Proof. Both i and jk belong to $\mathcal{A}_{\le 2}$, but

$$i \cdot_\sigma jk = ijk, \tag{103}$$

and

$$ijk \notin \mathcal{A}_{\le 2}.$$

Therefore $\mathcal{A}_{\le 2}$ is not closed under exact multiplication. □

Thus the sequence of truncated sectors has the following exact-closure behavior:

Sector	Contains scalar-vector data?	Exact closure status
$\mathcal{A}_{\le 0}$	no	closed scalar sector
$\mathcal{A}_{\le 1}$	yes	not closed; bivectors appear
$\mathcal{A}_{\le 2}$	yes	not closed; trivectors appear
$\mathcal{A}_{\le 3} = \mathcal{A}$	yes	closed full carrier

This table anticipates the later projection theory. If one insists on remaining in $\mathcal{A}_{\le 1}$ or $\mathcal{A}_{\le 2}$, one must project after multiplication. That projection produces an effective product, not the exact Clifford product.

4.7 Consequences for exact versus projected multiplication

The results of this section establish a strict grade-sector taxonomy:

$$\mathcal{A}_{\leq 1} \subset \mathcal{A}_{\leq 2} \subset \mathcal{A}_{\leq 3} = \mathcal{A}. \tag{104}$$

The first inclusion is forced by products of vectors; the second is forced by products of vectors with bivectors. Exact frozen multiplication lives at the top level. Lower levels are useful as sectors, input spaces, or projected state spaces, but not as exact multiplicative carriers.

Accordingly, the expression

$$a + bi + cj + dk$$

should be interpreted as a scalar-vector element embedded in the full carrier. If the paper later writes

$$U \cdot_\sigma W,$$

without projection, then the product is understood in $\mathcal{A}$. If the paper writes

$$\Pi_{\leq r}(U \cdot_\sigma W),$$

then it is explicitly using a projected effective product. This convention prevents two common mistakes:

1. treating $\mathcal{A}_{\leq 1}$ as a closed Clifford algebra;
2. treating projected truncation as exact multiplication.

4.8 Transition to coefficient-valued carriers

The closure results above are blade-level statements. They say where products of basis blades live. They do not yet say how to multiply coefficients if the scalar coefficients

$$a, b, c, d \in \mathbb{R}$$

are replaced by vector-valued coefficients

$$a, b, c, d \in V.$$

That is a separate question. The next section introduces coefficient-valued carriers of the form

$$V \otimes \mathcal{A}$$

and separates the object-level carrier from the additional structure needed to turn it into an algebra. This separation is essential: full blade closure fixes the Clifford side, but coefficient multiplication requires its own semantics.

5 Coefficient-Valued Carriers

5.1 Purpose of this section

Sections 3 and 4 resolved a blade-level question. They showed that the scalar-vector sector

$$\mathcal{A}_{\leq 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$$

is not closed under exact frozen multiplication, and that the full blade carrier

$$\mathcal{A} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}$$

is forced by exact closure. This section addresses a different issue: the meaning of coefficients.

The scalar formulae in Section 3 used real coefficients. In applications of the VPSCF framework one often wants the coefficients to be vectors, functions, signals, states, tensors, or other structured objects. One then writes expressions of the form

$$U = a \otimes 1 + b \otimes i + c \otimes j + d \otimes k, \qquad a, b, c, d \in V. \tag{105}$$

This notation is meaningful as a vector-space object as soon as V is a real vector space. It is not, however, automatically meaningful as a multiplicative object. The purpose of this section is to separate the carrier

$$V \otimes \mathcal{A}$$

from any algebraic product one may later impose on it.

The distinction is essential. Full Clifford closure fixes the blade side. It says where products of the basis blades $1, i, j, k, ij, ik, jk, ijk$ live. It does not say how to multiply coefficients in V. If V has no specified multiplication, expressions such as

$$b\gamma, \qquad c\beta, \qquad d\delta$$

are not intrinsic. They require additional coefficient semantics. This section defines the coefficient-valued carriers only; Section 6 classifies the possible product regimes.

5.2 Algebraic tensor carriers

Let V be a real vector space. Throughout this section, unless explicitly stated otherwise, the tensor product

$$V \otimes \mathcal{A}$$

is the algebraic tensor product over $\mathbb{R}$. Since $\mathcal{A}$ is finite-dimensional, this carrier can be identified with the direct sum of eight copies of V:

$$V \otimes \mathcal{A} \cong \bigoplus_{A_0 \subseteq \{i,j,k\}} V \otimes \mathbb{R}e_{A_0} \cong V^8. \tag{106}$$

Here e_{A_0} denotes the ordered blade associated to the subset $A_0 \subseteq \{i, j, k\}$, with

$$e_\varnothing = 1, \qquad e_{\{i\}} = i, \qquad e_{\{i,j\}} = ij, \qquad e_{\{i,j,k\}} = ijk,$$

and similarly for the other subsets.

Definition 5.1 (Full-grade coefficient carrier)**.** Let V be a real vector space. The full-grade V-coefficient VPSCF carrier is

$$V \otimes \mathcal{A}. \tag{107}$$

An element of this carrier has the unique blade expansion

$$U = \sum_{A_0 \subseteq \{i,j,k\}} u_{A_0} \otimes e_{A_0}, \qquad u_{A_0} \in V. \tag{108}$$

Equivalently,

$$\begin{aligned} U = {} & u_\varnothing \otimes 1 + u_i \otimes i + u_j \otimes j + u_k \otimes k \\ & + u_{ij} \otimes ij + u_{ik} \otimes ik + u_{jk} \otimes jk + u_{ijk} \otimes ijk. \end{aligned} \tag{109}$$

The uniqueness in Definition 5.1 follows from the linear independence of the blade basis in $\mathcal{A}$. No multiplication on V is required for this statement. It is purely linear.

Remark 5.2 (Notation for coefficients and blade labels)**.** To avoid conflict between the coefficient space V and a blade-indexing subset, this section uses $A_0, B_0 \subseteq \{i, j, k\}$ as blade labels. Thus u_{A_0} is a coefficient in V, while e_{A_0} is a blade in $\mathcal{A}$.

5.3 Grade decomposition with coefficients

The grade decomposition of $\mathcal{A}$ induces a grade decomposition of the coefficient carrier. Define

$$(V \otimes \mathcal{A})^{(q)} := V \otimes \mathcal{A}^{(q)}, \qquad q = 0, 1, 2, 3. \tag{110}$$

Then

$$V \otimes \mathcal{A} = (V \otimes \mathcal{A})^{(0)} \oplus (V \otimes \mathcal{A})^{(1)} \oplus (V \otimes \mathcal{A})^{(2)} \oplus (V \otimes \mathcal{A})^{(3)}. \tag{111}$$

Similarly, for $0 \leq r \leq 3$, define the truncated coefficient sector

$$V \otimes \mathcal{A}_{\leq r} := \bigoplus_{q=0}^{r} V \otimes \mathcal{A}^{(q)}. \tag{112}$$

Definition 5.3 (Coefficient-valued grade projections)**.** Let

$$\Pi_q : \mathcal{A} \to \mathcal{A}^{(q)}, \qquad \Pi_{\leq r} : \mathcal{A} \to \mathcal{A}_{\leq r}$$

be the blade-grade projections defined in Section 2. Their coefficient-valued extensions are

$$\mathrm{id}_V \otimes \Pi_q : V \otimes \mathcal{A} \to V \otimes \mathcal{A}^{(q)} \tag{113}$$

and

$$\mathrm{id}_V \otimes \Pi_{\leq r} : V \otimes \mathcal{A} \to V \otimes \mathcal{A}_{\leq r}. \tag{114}$$

These projections are again linear and independent of σ. The signature affects frozen multiplication on the blade factor; it does not affect the vector-space decomposition into grades.

Remark 5.4 (Grade is a blade property, not a coefficient property)**.** In $V \otimes \mathcal{A}$, the grade of a term is determined by its blade component, not by any internal structure of the coefficient $u_{A_0} \in V$. Even if V later carries its own grading, filtration, topology, or norm, that additional structure is separate from the Clifford blade grade used in this paper.

5.4 Scalar-vector coefficient objects

The coefficient-valued version of the motivating scalar-vector expression is the following.

Definition 5.5 (Scalar-vector coefficient object)**.** Let V be a real vector space. A scalar-vector V-coefficient VPSCF object is an element

$$U = a \otimes 1 + b \otimes i + c \otimes j + d \otimes k, \qquad a, b, c, d \in V. \tag{115}$$

Equivalently,

$$U \in V \otimes \mathcal{A}_{\leq 1}. \tag{116}$$

This definition should be read as a carrier statement only. It asserts that the object has scalar-vector blade support. It does not assert that two such objects can be multiplied inside $V \otimes \mathcal{A}_{\leq 1}$, or even inside $V \otimes \mathcal{A}$, without specifying a product or bookkeeping semantics on V.

Proposition 5.6 (Linear closure of coefficient sectors)**.** *For each $0 \leq r \leq 3$, the sector $V \otimes \mathcal{A}_{\leq r}$ is a linear subspace of $V \otimes \mathcal{A}$. In particular, sums and scalar multiples of scalar-vector coefficient objects are again scalar-vector coefficient objects.*

Proof. The sector $\mathcal{A}_{\leq r}$ is a linear subspace of $\mathcal{A}$. Tensoring with the real vector space V gives the linear subspace $V \otimes \mathcal{A}_{\leq r} \subseteq V \otimes \mathcal{A}$. The assertion about sums and scalar multiples follows from linearity. □

Remark 5.7 (Linear sector versus algebra)**.** Proposition 5.6 is only a linear statement. It must not be confused with multiplicative closure. Section 3 showed that even in the scalar case $\mathcal{A}_{\leq 1}$ is not closed under exact multiplication. Adding coefficient vectors does not repair that blade-level obstruction.

5.5 Field-valued carriers over a base set

The coefficient carrier can also be used pointwise over a base set. Let X be a set. A coefficient field over X is a map

$$U : X \to V \otimes \mathcal{A}. \tag{117}$$

Equivalently, after choosing the fixed blade basis, such a field can be written as

$$U(x) = \sum_{A_0 \subseteq \{i,j,k\}} u_{A_0}(x) \otimes e_{A_0}, \qquad u_{A_0} : X \to V. \tag{118}$$

A scalar-vector coefficient field has the form

$$U(x) = a(x) \otimes 1 + b(x) \otimes i + c(x) \otimes j + d(x) \otimes k, \tag{119}$$

where

$$a, b, c, d : X \to V. \tag{120}$$

At this stage, X is only a base parameter set. No topology, measure, differentiable structure, Sobolev regularity, interface structure, or PDE structure is assumed. Those structures may be imposed in later analytic papers of the VPSCF series, but they are not part of the algebraic carrier layer developed here.

Convention 5.8 (Pointwise carrier convention)**.** If $U : X \to V \otimes \mathcal{A}$ is a coefficient field, then all carrier-level statements are interpreted pointwise in $x \in X$. For example,

$$U(x) \in V \otimes \mathcal{A}_{\leq 1}$$

means that the blade support of the value $U(x)$ is contained in $\mathcal{A}_{\leq 1}$. This convention does not define a pointwise multiplication unless coefficient semantics have also been specified.

5.6 What is defined without coefficient multiplication

If V is only a real vector space, the carrier $V \otimes \mathcal{A}$ supports the following operations without further assumptions:

(i) vector-space addition;

(ii) scalar multiplication by real numbers;

(iii) decomposition into blade grades;

(iv) grade projections $\mathrm{id}_V \otimes \Pi_{\le r}$;

(v) pointwise carrier evaluation for maps $X \to V \otimes \mathcal{A}$.

None of these operations requires multiplying two coefficients in V.

By contrast, a product of two coefficient-valued pure tensors

$$(v \otimes A)(w \otimes B) \tag{121}$$

requires a decision about what should replace vw. There is no canonical answer from vector-space structure alone. One may choose an associative multiplication $v \cdot_V w$, or one may keep the ordered pair as a tensor $v \otimes w$, or one may impose some other bilinear operation or contraction. These choices lead to different coefficient regimes.

Proposition 5.9 (Carrier does not determine product)**.** *The data of a real vector space V and the frozen Clifford carrier $(\mathcal{A}, m_\sigma)$ do not, by themselves, determine a canonical multiplication*

$$(V \otimes \mathcal{A}) \times (V \otimes \mathcal{A}) \longrightarrow V \otimes \mathcal{A}. \tag{122}$$

Additional coefficient semantics are required.

Proof. The blade factor has a specified frozen product m_σ, so the blade component of

$$(v \otimes A)(w \otimes B)$$

would naturally be $m_\sigma(A, B)$. However, to land again in $V \otimes \mathcal{A}$, one also needs a coefficient in V constructed from v and w. A bare real vector space does not provide a distinguished bilinear map $V \times V \to V$. Hence the carrier data alone do not determine the displayed multiplication. □

Remark 5.10 (Why scalar coefficients are special)**.** When $V = \mathbb{R}$, the coefficient multiplication is already fixed by the field structure of $\mathbb{R}$. In that special case,

$$\mathbb{R} \otimes \mathcal{A} \cong \mathcal{A},$$

and the scalar computations of Section 3 are recovered. The coefficient problem begins when V is not being used merely as the ground field.

5.7 Compatibility with frozen signatures

Coefficient-valued carriers do not alter the signature space. For every

$$\sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \Sigma,$$

the blade-side square relations remain

$$i^2 = \varepsilon_i, \qquad j^2 = \varepsilon_j, \qquad k^2 = \varepsilon_k,$$

and the blade-side anticommutation relations remain

$$ij = -ji, \qquad ik = -ki, \qquad jk = -kj.$$

The coefficient space V does not change which frozen signature is being used. Rather, it changes what kind of values are attached to the blades.

If later a coefficient multiplication is chosen, the product may be written schematically as

$$(v \otimes e_{A_0}) \cdot_\sigma (w \otimes e_{B_0}) = \text{coefficient-combination}(v, w) \otimes m_\sigma(e_{A_0}, e_{B_0}). \tag{123}$$

The second factor is fixed by the frozen Clifford product. The first factor is the coefficient-semantics choice.

5.8 Transition to coefficient regimes

The conclusion of this section is the following:

$$\boxed{V \otimes \mathcal{A} \text{ is a coefficient carrier, not automatically a coefficient algebra.}} \tag{124}$$

This statement is the coefficient-level analogue of the blade-level warning from Section 3. At the blade level, $\mathcal{A}_{\leq 1}$ is a useful sector but not a closed exact algebra. At the coefficient level, $V \otimes \mathcal{A}$ is a useful carrier but not a product-bearing algebra until V receives additional semantics.

Section 6 now records the three baseline regimes that will be used in the rest of VPSCF2:

(i) pure metric or topological coefficient spaces, where no multiplication is supplied;

(ii) associative coefficient algebras, where $V \otimes \mathcal{A}$ becomes an ordinary ungraded tensor product algebra;

(iii) tensor-algebra hierarchy semantics, where products are represented by increasing tensor order rather than closed multiplication in $V \otimes \mathcal{A}$.

6 Three Coefficient Regimes

Section 5 established that $V \otimes \mathcal{A}$ is a coefficient carrier, not automatically a coefficient algebra. This section makes that warning precise. The same formal expression

$$U = \sum_{S \subseteq I} u_S \otimes e_S$$

can have different algebraic meanings depending on the structure imposed on V. VPSCF2 separates these meanings into three baseline regimes:

(i) pure metric or topological coefficient spaces, where one has norms, pairings, or convergence, but no specified multiplication;

(ii) associative coefficient algebras, where a bilinear product on V makes $V \otimes \mathcal{A}$ into an ordinary ungraded tensor product algebra;

(iii) tensor-algebra hierarchy semantics, where products are represented by increasing tensor order, including the scalar level $\mathcal{H}_0 \cong \mathcal{A}$, instead of being forced back into $V \otimes \mathcal{A}$.

Remark 6.1 (Non-exhaustiveness of the coefficient regimes)**.** The three regimes in this section are baseline regimes used in VPSCF2, not an exhaustive classification of every possible coefficient semantics. Other meaningful choices include coefficient pairings, contractions, quotient maps, module actions, bilinear maps into an external target, and distribution-valued products. Such choices are not incorrect, but they are additional semantics and must be declared explicitly. The present paper focuses on the three regimes needed to separate coefficient carriers, associative coefficient algebras, and free bookkeeping before the later signed quadratic and analytic layers.

The distinction is not cosmetic. It determines whether expressions such as

$$(u_S \otimes e_S)(v_T \otimes e_T)$$

are undefined, closed in $V \otimes \mathcal{A}$, or lifted to a higher tensor level.

6.1 Regime A: Pure metric or topological coefficient spaces

The first regime treats V as a coefficient space with linear, metric, or topological structure, but without multiplication. Typical choices include real Hilbert spaces, Banach spaces, normed spaces, locally convex spaces, or purely algebraic real vector spaces.

Definition 6.2 (Pure coefficient-space regime)**.** Let V be a real vector space equipped, possibly, with additional non-multiplicative structure such as a norm, inner product, seminorm family, topology, or duality pairing. The pure coefficient-space regime is the interpretation of

$$V \otimes \mathcal{A}$$

as a linear coefficient carrier without a specified bilinear multiplication

$$\mu_V : V \times V \to V.$$

Definition 6.3 (Coefficient anchor)**.** Let V be a real vector space. A *coefficient anchor* is a specified element

$$e_V \in V.$$

It determines a linear map

$$\iota_{e_V} : \mathcal{A} \to V \otimes \mathcal{A}, \qquad \iota_{e_V}(A) = e_V \otimes A.$$

If $e_V \neq 0$, this map is injective. In Regime A, however, no such anchor is part of the default data, and even when an anchor is supplied it is only a linear anchor, not a multiplicative unit unless V has been given a unital algebra structure with $e_V = 1_V$.

Proposition 6.4 (Canonical blade copy requires anchor data)**.** *In Regime A, the data of $V \otimes \mathcal{A}$ determine the blade decomposition*

$$V \otimes \mathcal{A} \cong \bigoplus_{S \subseteq I} V \otimes e_S,$$

but they do not determine a canonical linear embedding

$$\mathcal{A} \hookrightarrow V \otimes \mathcal{A}.$$

A choice of coefficient anchor $e_V \in V$ determines the linear map

$$A \mapsto e_V \otimes A.$$

If V is a unital Regime-B coefficient algebra and $e_V = 1_V$, then this map is the canonical unital blade-algebra embedding.

Proof. The decomposition follows from the fixed blade basis of $\mathcal{A}$. To embed $\mathcal{A}$ as a distinguished copy inside $V \otimes \mathcal{A}$, one must choose a coefficient value multiplying every blade. Regime-A data contain no distinguished element of V. Therefore no canonical map $A \mapsto e_V \otimes A$ is determined.

Once an anchor e_V is supplied, the formula $A \mapsto e_V \otimes A$ gives a linear map, injective when $e_V \neq 0$. If V is unital and $e_V = 1_V$, then

$$(1_V \otimes A) \cdot_\sigma (1_V \otimes B) = 1_V \otimes m_\sigma(A, B),$$

so the map is a unital algebra embedding of $\mathcal{A}_\sigma$. □

In this regime, one may form coefficient-valued elements

$$U = \sum_{S \subseteq I} u_S \otimes e_S, \qquad u_S \in V,$$

and one may apply all linear operations inherited from V and $\mathcal{A}$. For example,

$$U + W = \sum_{S \subseteq I} (u_S + w_S) \otimes e_S, \qquad \lambda U = \sum_{S \subseteq I} (\lambda u_S) \otimes e_S.$$

If V is normed and $\mathcal{A}$ is finite-dimensional, one may also impose coefficient norms such as

$$\|U\|_{2,V}^2 = \sum_{S \subseteq I} \|u_S\|_V^2$$

whenever this expression is meaningful. If V is Hilbert, pairings such as

$$\sum_{S \subseteq I} \langle u_S, w_S \rangle_V$$

are also meaningful. None of these operations, however, supplies a product $u_S v_T$ in V.

Theorem 6.5 (Insufficiency of pure metric coefficient data)**.** *A Hilbert, Banach, normed, or topological vector-space structure on V, without an explicitly specified bilinear multiplication*

$$\mu_V : V \times V \to V,$$

does not canonically make

$$V \otimes \mathcal{A}$$

into an algebra whose tensor product is compatible with the frozen Clifford product on the blade factor. If one further wants a distinguished linear copy of $\mathcal{A}$ inside $V \otimes \mathcal{A}$, one must choose a coefficient anchor $e_V \in V$ and use $A \mapsto e_V \otimes A$. This is only a linear embedding in Regime A. It is not an algebra embedding unless V has been upgraded to a unital coefficient algebra and e_V is its multiplicative unit.

Proof. To make $V \otimes \mathcal{A}$ into an algebra compatible with the frozen blade product, one must at least define how products of pure coefficient tensors behave:

$$(u \otimes A)(v \otimes B) = \text{coefficient product of } u, v \ \otimes \ m_\sigma(A, B).$$

The second factor is fixed by m_σ. The first factor requires a bilinear rule

$$\mu_V(u, v) \in V.$$

A norm, topology, inner product, or duality pairing does not by itself select such a bilinear rule. The same underlying normed vector space may admit no intended multiplication, a zero multiplication, a pointwise multiplication, an operator-composition multiplication, or many other bilinear products when additional data are supplied. Hence the metric or topological structure alone does not determine a canonical algebra product on $V \otimes \mathcal{A}$. Moreover, even the weaker statement that $V \otimes \mathcal{A}$ contains a distinguished linear copy of $\mathcal{A}$ requires a chosen coefficient anchor e_V, so that $A \mapsto e_V \otimes A$. Without such a choice, no preferred copy is selected by the vector-space structure alone. The stronger statement that this copy is a subalgebra requires Regime B with a unital coefficient algebra and unit 1_V. □

Remark 6.6 (Banach and Hilbert algebras are extra structure)**.** Theorem 6.5 does not say that a Banach or Hilbert space can never be an algebra. It says only that being Banach or Hilbert is not enough. If a Banach space is equipped with a continuous associative product, it belongs to Regime B. If a Hilbert space is equipped with an algebra product, that product is additional structure and must be stated explicitly.

Proposition 6.7 (Canonical operations determined by Regime-A data)**.** *In Regime A, the declared data canonically determine the following operations on $V \otimes \mathcal{A}$:*

(i) linear combinations from the vector-space structure of $V \otimes \mathcal{A}$;

(ii) blade-grade projections $\mathrm{id}_V \otimes \Pi_{\leq r}$ from the fixed grade decomposition of $\mathcal{A}$;

(iii) coefficient norms, topologies, dualities, or pairings, but only when they are explicitly supplied as part of the structure on V;

(iv) pointwise evaluation $U(x)$, but only when coefficient fields $u_S : X \to V$ have been specified.

No bilinear multiplication

$$(V \otimes \mathcal{A}) \times (V \otimes \mathcal{A}) \to V \otimes \mathcal{A}$$

compatible with m_σ is canonically determined by Regime-A data alone. A distinguished linear embedding of $\mathcal{A}$ into $V \otimes \mathcal{A}$ is also not canonically determined unless a coefficient anchor $e_V \in V$ has been supplied. Such an anchor is not a unit in Regime A; it becomes a genuine unit only in Regime B when V is a unital associative algebra.

Proof. Items (i) and (ii) use only the linear structure of V and the fixed finite-dimensional grade decomposition of $\mathcal{A}$. Items (iii) and (iv) use only extra non-multiplicative data that have already been declared.

A product on $V \otimes \mathcal{A}$, by contrast, must combine coefficient factors from two inputs. This requires a bilinear coefficient-combination rule

$$\mu_V : V \times V \to V,$$

which is not part of Regime A. The absence is not merely formal. The same underlying vector space V may support many different bilinear products, including the zero product and nonzero products when additional structure is chosen. These choices give different tensor-product multiplications on $V \otimes \mathcal{A}$, while the Regime-A data remain unchanged. Therefore Regime-A data alone do not canonically determine such a multiplication. $\square$

6.2 Regime B: Associative coefficient algebras

The second regime supplies exactly the missing datum: a coefficient multiplication. This is the closed algebraic regime for coefficient-valued VPSCF objects.

Convention 6.8 (Ordinary ungraded tensor product algebra)**.** Unless otherwise stated, $V \otimes \mathcal{A}$ is treated as an ordinary ungraded tensor product algebra. The coefficient factor carries no Koszul sign convention. All signs in blade multiplication come from m_σ, not from moving coefficients past blades.

Let

$$(V, \cdot_V)$$

be an associative real algebra, not necessarily unital unless explicitly stated. The algebra V need not be commutative, and it need not be finite-dimensional. When V is unital, its unit is denoted 1_V. For pure tensors define

$$(v \otimes A) \cdot_\sigma (w \otimes B) := (v \cdot_V w) \otimes m_\sigma(A, B), \tag{125}$$

and extend bilinearly.

Definition 6.9 (Associative coefficient-algebra regime)**.** The associative coefficient-algebra regime is the interpretation of $V \otimes \mathcal{A}$ with multiplication (125), where $(V, \cdot_V)$ is an associative real algebra, not necessarily unital, and $\mathcal{A}_\sigma = (\mathcal{A}, m_\sigma, 1)$ is the frozen Clifford algebra structure.

Theorem 6.10 (Regime-B frozen product)**.** *If $(V, \cdot_V)$ is an associative real algebra, not necessarily unital, then $V \otimes \mathcal{A}$, with multiplication (125), is an associative real algebra for each fixed $\sigma \in \Sigma$. If V is unital with unit 1_V, then $V \otimes \mathcal{A}$ is unital with unit*

$$1_V \otimes 1.$$

Proof. It suffices to check associativity on pure tensors. Let $u, v, w \in V$ and $A, B, C \in \mathcal{A}$. Then

$$\begin{aligned}((u \otimes A) \cdot_\sigma (v \otimes B)) \cdot_\sigma (w \otimes C) &= ((u \cdot_V v) \cdot_V w) \otimes m_\sigma(m_\sigma(A, B), C), \\ (u \otimes A) \cdot_\sigma ((v \otimes B) \cdot_\sigma (w \otimes C)) &= (u \cdot_V (v \cdot_V w)) \otimes m_\sigma(A, m_\sigma(B, C)).\end{aligned}$$

These are equal by associativity of V and of $(\mathcal{A}, m_\sigma)$. Bilinearity extends equality to arbitrary elements of $V \otimes \mathcal{A}$.

If V is unital, then for every pure tensor $v \otimes A$,

$$(1_V \otimes 1) \cdot_\sigma (v \otimes A) = v \otimes A, \qquad (v \otimes A) \cdot_\sigma (1_V \otimes 1) = v \otimes A.$$

Thus $1_V \otimes 1$ is the unit. Without a unit in V, no unit is supplied by Regime-B data. □

Remark 6.11 (No commutativity assumption)**.** Theorem 6.10 does not require V to be commutative. Ordered products such as $b \cdot_V \gamma$ and $\gamma \cdot_V b$ may differ. The tensor product rule (125) preserves the order inherited from the two VPSCF factors.

Definition 6.12 (Algebraic and analytic Regime B)**.** Regime B has two levels.

(i) *Algebraic Regime B* assumes only that V is an associative real algebra, not necessarily unital. This is sufficient to define the algebraic product (125) on $V \otimes \mathcal{A}$.

(ii) *Analytic Regime B* assumes, in addition, that V is a normed, Banach, Hilbert, or topological vector space and that the coefficient product

$$\cdot_V : V \times V \to V$$

is continuous. In the normed case, this means that there exists a constant $C_V > 0$ such that

$$\|v \cdot_V w\|_V \le C_V \|v\|_V \|w\|_V$$

for all $v, w \in V$.

VPSCF2 uses Algebraic Regime B unless analytic continuity is explicitly declared.

Convention 6.13 (Coordinate topology on $V \otimes \mathcal{A}$)**.** Fix the ordered blade basis

$$\mathcal{B} = (1, i, j, k, ij, ik, jk, ijk)$$

of $\mathcal{A}$. Every element of $V \otimes \mathcal{A}$ is written uniquely as

$$U = \sum_{S \subseteq I} u_S \otimes e_S, \qquad u_S \in V.$$

If V is a normed space, $V \otimes \mathcal{A}$ is equipped with the coordinate ℓ^1-norm

$$\|U\|_{V \otimes \mathcal{A},1} := \sum_{S \subseteq I} \|u_S\|_V.$$

If V is only a topological vector space, $V \otimes \mathcal{A}$ is equipped with the product topology under the finite-coordinate identification

$$V \otimes \mathcal{A} \cong V^8.$$

Since $\mathcal{A}$ is finite-dimensional, any two fixed coordinate norms on the blade factor give equivalent topologies. Thus the continuity statements below are independent of the particular finite-dimensional coordinate norm, although displayed constants may depend on the chosen norm.

Proposition 6.14 (Continuity of the coefficient-valued frozen product)**.** *Assume Analytic Regime B. Equip $V \otimes \mathcal{A}$ with the coordinate topology of Convention 6.13. Then, for every frozen signature σ, the product induced by*

$$(v \otimes A) \cdot_\sigma (w \otimes B) = (v \cdot_V w) \otimes m_\sigma(A, B)$$

extends by bilinearity to a continuous bilinear map

$$(V \otimes \mathcal{A}) \times (V \otimes \mathcal{A}) \longrightarrow V \otimes \mathcal{A}.$$

In the normed case, with the coordinate ℓ^1-norm, one has the estimate

$$\|U \cdot_\sigma W\|_{V\otimes\mathcal{A},1} \le C_V \, \|U\|_{V\otimes\mathcal{A},1} \, \|W\|_{V\otimes\mathcal{A},1}.$$

Proof. Write

$$U = \sum_{S\subseteq I} u_S \otimes e_S, \qquad W = \sum_{T\subseteq I} w_T \otimes e_T.$$

For every frozen signature σ, the product of two basis blades has the form

$$m_\sigma(e_S, e_T) = \lambda_\sigma(S,T) e_{S\triangle T}, \qquad \lambda_\sigma(S,T) \in \{\pm 1\}.$$

Therefore

$$U \cdot_\sigma W = \sum_{S,T\subseteq I} (u_S \cdot_V w_T) \otimes \lambda_\sigma(S,T) e_{S\triangle T}.$$

Using the coordinate ℓ^1-norm and the Analytic Regime-B estimate

$$\|u_S \cdot_V w_T\|_V \le C_V \|u_S\|_V \|w_T\|_V,$$

we obtain

$$\begin{aligned}
\|U \cdot_\sigma W\|_{V\otimes\mathcal{A},1} &\le \sum_{S,T\subseteq I} \|u_S \cdot_V w_T\|_V \\
&\le C_V \sum_{S,T\subseteq I} \|u_S\|_V \|w_T\|_V \\
&= C_V \left(\sum_S \|u_S\|_V\right)\left(\sum_T \|w_T\|_V\right) \\
&= C_V \|U\|_{V\otimes\mathcal{A},1} \|W\|_{V\otimes\mathcal{A},1}.
\end{aligned}$$

This proves continuity in the normed case. The topological case follows from the product-topology identification $V \otimes \mathcal{A} \cong V^8$, since the product is a finite coordinate expression built from the continuous coefficient multiplication on V. □

Remark 6.15 (No analytic closure without analytic Regime B)**.** Algebraic Regime B defines products pointwise, but it does not by itself imply continuity, differentiability, integrability, Sobolev closure, or PDE estimates for coefficient-valued fields. Such statements require Analytic Regime B together with the corresponding function-space hypotheses. For Banach-algebra continuity conventions, see [12].

Proposition 6.16 (External commutation in Regime B)**.** *Assume* $(V, \cdot_V)$ *is a unital associative coefficient algebra. Under the ordinary ungraded tensor product rule* (125)*, the embedded coefficient algebra* $V \otimes 1$ *and the embedded blade algebra* $1_V \otimes \mathcal{A}$ *commute as external tensor factors:*

$$(v \otimes 1) \cdot_\sigma (1_V \otimes A) = (1_V \otimes A) \cdot_\sigma (v \otimes 1) = v \otimes A.$$

Thus Regime B allows V *itself to be noncommutative, but it treats coefficient factors and Clifford blade factors as external commuting tensor factors.*

Proof. Using $m_\sigma(1, A) = m_\sigma(A, 1) = A$ and the unit identities $v \cdot_V 1_V = 1_V \cdot_V v = v$, one obtains

$$(v \otimes 1) \cdot_\sigma (1_V \otimes A) = (v \cdot_V 1_V) \otimes m_\sigma(1, A) = v \otimes A,$$

and

$$(1_V \otimes A) \cdot_\sigma (v \otimes 1) = (1_V \cdot_V v) \otimes m_\sigma(A, 1) = v \otimes A.$$

□

Remark 6.17 (When Regime B is not the right coefficient semantics)**.** Regime B is not a crossed product, not a graded tensor product, and not a representation algebra in which coefficients act nontrivially on Clifford generators. If coefficient symbols are intended to fail to commute with blade symbols, or if moving a coefficient past a blade is supposed to create a sign, conjugation, module action, or gauge action, then the ordinary tensor product rule (125) is not the correct semantics. One must replace it by an explicitly declared crossed product, graded tensor product, free-product quotient, or representation-specific multiplication.

Proposition 6.18 (Unit in the coefficient algebra regime)**.** *If* $(V, \cdot_V)$ *has a unit* 1_V*, then*

$$1_V \otimes 1$$

is the unit of $V \otimes \mathcal{A}$ *under* (125)*. If* V *has no specified unit, no unit is forced on* $V \otimes \mathcal{A}$ *by the blade carrier alone.*

Proof. For any pure tensor $v \otimes A$,

$$(1_V \otimes 1) \cdot_\sigma (v \otimes A) = (1_V \cdot_V v) \otimes m_\sigma(1, A) = v \otimes A,$$

and similarly on the right. The second statement follows because a unit in $V \otimes \mathcal{A}$ compatible with the tensor product rule would include a unit-like coefficient action on the V-factor. □

6.3 Scalar-vector multiplication in Regime B

The scalar expansion from Section 3 lifts to Regime B after replacing scalar products by ordered coefficient products. The following proposition records the formula as a typed algebraic statement rather than as an informal computation.

Proposition 6.19 (Scalar-vector product expansion in Regime B)**.** *Let* $(V, \cdot_V)$ *be an associative real coefficient algebra, not necessarily commutative, and let*

$$\begin{aligned} U &= a \otimes 1 + b \otimes i + c \otimes j + d \otimes k, \\ W &= \alpha \otimes 1 + \beta \otimes i + \gamma \otimes j + \delta \otimes k, \end{aligned} \tag{126}$$

where

$$a, b, c, d, \alpha, \beta, \gamma, \delta \in V.$$

Under the ordinary ungraded tensor product rule (125)*, one has*

$$U \cdot_\sigma W = S^V_\sigma(U,W) + V^V_\sigma(U,W) + B^V_\sigma(U,W),$$

where the scalar coefficient is

$$s^V_\sigma(U,W) = (a \cdot_V \alpha) + \varepsilon_i(b \cdot_V \beta) + \varepsilon_j(c \cdot_V \gamma) + \varepsilon_k(d \cdot_V \delta) \in V, \tag{127}$$

the scalar-blade part is

$$S^V_\sigma(U,W) := s^V_\sigma(U,W) \otimes 1 \in V \otimes \mathcal{A}, \tag{128}$$

the vector part is

$$V^V_\sigma(U,W) = (a \cdot_V \beta + b \cdot_V \alpha) \otimes i + (a \cdot_V \gamma + c \cdot_V \alpha) \otimes j + (a \cdot_V \delta + d \cdot_V \alpha) \otimes k, \tag{129}$$

and the bivector part is

$$\begin{aligned} B^V_\sigma(U,W) = {} & (b \cdot_V \gamma - c \cdot_V \beta) \otimes ij \\ & + (b \cdot_V \delta - d \cdot_V \beta) \otimes ik \\ & + (c \cdot_V \delta - d \cdot_V \gamma) \otimes jk. \end{aligned} \tag{130}$$

Coefficient expressions alone, such as $s \in V$, are not identified with elements of $V \otimes \mathcal{A}$ unless explicitly tensored with the scalar blade 1. Thus $s \otimes 1$, $v_i \otimes i$, and $b_{ij} \otimes ij$ are homogeneous blade components of $V \otimes \mathcal{A}$. The displayed order of the coefficient products is part of the formula and cannot be rearranged unless V is commutative.

Proof. Expand $U \cdot_\sigma W$ bilinearly. Terms involving the scalar blade give the scalar-vector contributions, for example

$$(a \otimes 1) \cdot_\sigma (\beta \otimes i) + (b \otimes i) \cdot_\sigma (\alpha \otimes 1) = (a \cdot_V \beta + b \cdot_V \alpha) \otimes i.$$

Repeated generator terms give the scalar signs; for instance

$$(b \otimes i) \cdot_\sigma (\beta \otimes i) = (b \cdot_V \beta) \otimes (i \cdot_\sigma i) = \varepsilon_i(b \cdot_V \beta) \otimes 1.$$

Distinct generator terms give bivectors. For example,

$$(b \otimes i) \cdot_\sigma (\gamma \otimes j) = (b \cdot_V \gamma) \otimes ij,$$

whereas

$$(c \otimes j) \cdot_\sigma (\beta \otimes i) = (c \cdot_V \beta) \otimes ji = -(c \cdot_V \beta) \otimes ij.$$

Thus the ij-coefficient is $b \cdot_V \gamma - c \cdot_V \beta$. The ik- and jk-coefficients are obtained identically. Because V is not assumed commutative, no coefficient product may be reordered. □

Corollary 6.20 (Bivector obstruction persists in Regime B)**.** *The subspace $V \otimes \mathcal{A}_{\le 1}$ is not closed under the Regime-B product in general. More precisely, for scalar-vector coefficient objects U, W as in* (126)*, the obstruction to remaining in $V \otimes \mathcal{A}_{\le 1}$ is exactly the bivector component $B^V_\sigma(U,W)$. Hence*

$$U \cdot_\sigma W \in V \otimes \mathcal{A}_{\le 1}$$

if and only if

$$b \cdot_V \gamma = c \cdot_V \beta, \qquad b \cdot_V \delta = d \cdot_V \beta, \qquad c \cdot_V \delta = d \cdot_V \gamma.$$

Remark 6.21 (Persistence of scalar-vector non-closure)**.** Regime B supplies coefficient multiplication, but it does not make the scalar-vector blade sector closed. Even with associative coefficients,

$$(b \otimes i) \cdot_\sigma (\gamma \otimes j) = (b \cdot_V \gamma) \otimes ij,$$

which belongs to $V \otimes \mathcal{A}^{(2)}$, not to $V \otimes \mathcal{A}_{\le 1}$, unless the corresponding coefficient factor vanishes. Thus coefficient closure and blade-grade closure are separate issues.

6.4 Pointwise coefficient algebras over a base set

When coefficient fields over a base set X are considered, Regime B has a pointwise interpretation. If $(V, \cdot_V)$ is associative and

$$U(x), W(x) \in V \otimes \mathcal{A},$$

then

$$(U \cdot_\sigma W)(x) := U(x) \cdot_{\sigma(x)} W(x) \tag{131}$$

whenever a signature field $\sigma : X \to \Sigma$ is specified. If σ is fixed, replace $\sigma(x)$ by σ. This is a pointwise algebraic convention only; no regularity or PDE theory is being assumed.

Remark 6.22 (Closure of function classes is additional)**.** If one later restricts coefficient fields to a function class, such as continuous, measurable, integrable, Sobolev, or weighted-interface fields, one must separately verify that the pointwise product remains in the same class. VPSCF2 does not impose those analytic hypotheses. Its role is only to specify the algebraic coefficient semantics.

6.5 Measurable coefficient-valued fields in Regime B

Definition 6.23 (Measurable coefficient-valued fields)**.** Let V be a Banach space equipped with its Borel σ-algebra. Using the fixed ordered blade basis of $\mathcal{A}$, identify

$$V \otimes \mathcal{A} \cong V^8.$$

The space $V \otimes \mathcal{A}$ is equipped with the product Borel structure under this identification. A map

$$U : Y \to V \otimes \mathcal{A}$$

from a measurable space Y is called measurable if its coordinate fields

$$U_S : Y \to V, \qquad S \subseteq I,$$

are Borel measurable.

Proposition 6.24 (Measurability of coefficient-valued pathwise products)**.** *Let V be a Banach algebra with continuous multiplication*

$$\mu_V : V \times V \to V.$$

Let $(X, \mathcal{B}_X)$ and $(\Omega, \mathcal{F})$ be measurable spaces, and let

$$\sigma : X \times \Omega \to \Sigma$$

be measurable. If

$$U, W : X \times \Omega \to V \otimes \mathcal{A}$$

are measurable coefficient-valued fields, then the pathwise Regime-B product

$$(U \cdot_\sigma W)(x, \omega) := U(x, \omega) \cdot_{\sigma(x,\omega)} W(x, \omega)$$

is a measurable map

$$X \times \Omega \to V \otimes \mathcal{A}.$$

Proof. Write

$$U = \sum_S U_S \otimes e_S, \qquad W = \sum_T W_T \otimes e_T.$$

For each frozen signature τ, the Regime-B product is given in coordinates by the finite formula

$$U \cdot_\tau W = \sum_{S,T} \mu_V(U_S, W_T) \otimes \kappa_\tau(S,T) e_{S \triangle T}.$$

Since μ_V is continuous, each coefficient map

$$(x, \omega) \mapsto \mu_V(U_S(x,\omega), W_T(x,\omega))$$

is measurable. Since Σ is finite and σ is measurable, the structure constants $\kappa_{\sigma(x,\omega)}(S,T)$ are measurable finite-valued functions. Hence every blade coordinate of $U \cdot_\sigma W$ is a finite measurable sum. The product is therefore measurable. □

Proposition 6.25 (Measurability of coefficient-valued averaged products)**.** *Let V be a Banach algebra with continuous multiplication. Let*

$$\pi : X \to \mathcal{P}(\Sigma)$$

be a measurable one-point signature kernel, meaning that every coordinate function $x \mapsto \pi_x(\tau)$ is measurable. If

$$U, W : X \to V \otimes \mathcal{A}$$

are measurable coefficient-valued fields, then

$$x \longmapsto \sum_{\tau \in \Sigma} \pi_x(\tau) \big(U(x) \cdot_\tau W(x)\big)$$

is a measurable $V \otimes \mathcal{A}$-valued field.

Proof. By the previous proposition with a fixed signature, each map

$$x \mapsto U(x) \cdot_\tau W(x)$$

is measurable. Multiplication by the scalar measurable coefficient $\pi_x(\tau)$ preserves measurability in each coordinate of $V \otimes \mathcal{A} \cong V^8$. The displayed averaged field is a finite sum over Σ, hence is measurable. □

Remark 6.26 (Measurability is not Sobolev or PDE closure)**.** The preceding propositions prove only measurable closure of coefficient-valued Regime-B products. They do not imply closure in C^k, L^p, Sobolev, weighted-interface, bounded-variation, or energy spaces. Those stronger closures require separate hypotheses on V, the base space, the signature field, and the coefficient fields. VPSCF3 may impose such hypotheses when signed quadratic forms or field equations are introduced.

6.6 Regime C: Tensor-algebra hierarchy semantics

The third regime applies when V has no internal multiplication but one still wants to retain a faithful record of formal products. The tensor-algebra bookkeeping construction does not force products back into $V \otimes \mathcal{A}$. Instead, it records coefficient words in increasing tensor order and keeps the scalar Clifford carrier as level zero.

Definition 6.27 (Tensor hierarchy with scalar level)**.** Let V be a real vector space with no specified internal multiplication. Set

$$V^{\otimes 0} := \mathbb{R}$$

and define, for every $n \geq 0$,

$$\mathcal{H}_n := V^{\otimes n} \otimes \mathcal{A}. \tag{132}$$

Thus

$$\mathcal{H}_0 = \mathbb{R} \otimes \mathcal{A} \cong \mathcal{A}, \qquad \mathcal{H}_1 = V \otimes \mathcal{A}.$$

The full hierarchy is

$$\mathcal{H}_\bullet := \bigoplus_{n \geq 0} \mathcal{H}_n = T(V) \otimes \mathcal{A}, \tag{133}$$

where $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ is the tensor algebra of V.

For $r, s \geq 0$, define a hierarchy product

$$\mathcal{H}_r \times \mathcal{H}_s \to \mathcal{H}_{r+s} \tag{134}$$

on pure tensors by

$$\begin{aligned}(v_1 \otimes \cdots \otimes v_r \otimes A) \cdot_\sigma (w_1 \otimes \cdots \otimes w_s \otimes B)& \\ := (v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s) \otimes m_\sigma(A, B),&\end{aligned} \tag{135}$$

and extend bilinearly. When $r = 0$ or $s = 0$, the corresponding coefficient tensor is the empty tensor $1 \in \mathbb{R} = V^{\otimes 0}$.

Theorem 6.28 (Tensor-hierarchy product law with scalar level)**.** *In Regime C, formal products satisfy*

$$\mathcal{H}_r \cdot_\sigma \mathcal{H}_s \subseteq \mathcal{H}_{r+s} \qquad (r, s \geq 0). \tag{136}$$

Thus products do not generally close in $V \otimes \mathcal{A} = \mathcal{H}_1$. They close in the graded hierarchy

$$\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots, \qquad \mathcal{H}_0 \cong \mathcal{A}.$$

In particular, the scalar Clifford carrier acts on every tensor level by exact frozen Clifford multiplication on the blade factor.

Proof. The defining formula (135) concatenates the r coefficient factors from the first input with the s coefficient factors from the second input, using the empty tensor when one of the levels is zero. The coefficient tensor therefore lies in $V^{\otimes(r+s)}$. The blade factor is multiplied by m_σ, which lands in $\mathcal{A}$. Hence the result lies in $V^{\otimes(r+s)} \otimes \mathcal{A} = \mathcal{H}_{r+s}$. Bilinearity extends the conclusion to arbitrary elements. □

Proposition 6.29 (Associativity across tensor levels)**.** *After identifying iterated tensor products by the standard tensor-concatenation isomorphism, the hierarchy product is associative:*

$$(X \cdot_\sigma Y) \cdot_\sigma Z = X \cdot_\sigma (Y \cdot_\sigma Z). \tag{137}$$

Proof. It suffices to check pure tensors. Let

$$X = (v_1 \otimes \cdots \otimes v_r) \otimes A, \quad Y = (w_1 \otimes \cdots \otimes w_s) \otimes B, \quad Z = (z_1 \otimes \cdots \otimes z_t) \otimes C,$$

where any of r, s, t may be zero. Both $(X \cdot_\sigma Y) \cdot_\sigma Z$ and $X \cdot_\sigma (Y \cdot_\sigma Z)$ have coefficient tensor

$$v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s \otimes z_1 \otimes \cdots \otimes z_t$$

after standard tensor-concatenation identification, with empty blocks omitted. Their blade factors are respectively

$$m_\sigma(m_\sigma(A,B),C) \quad \text{and} \quad m_\sigma(A,m_\sigma(B,C)),$$

which agree by associativity of the frozen product. Therefore the hierarchy product is associative across levels. $\square$

Proposition 6.30 (Coefficient-side universal property of $T(V)$)**.** *Let $T(V)=\bigoplus_{n\geq 0} V^{\otimes n}$ be the tensor algebra of V. For every unital associative real algebra B and every linear map*

$$\ell : V \to B,$$

there exists a unique unital algebra homomorphism

$$\widetilde{\ell} : T(V) \to B$$

such that $\widetilde{\ell}(v)=\ell(v)$ for all $v\in V$. Explicitly,

$$\widetilde{\ell}(v_1\otimes\cdots\otimes v_n)=\ell(v_1)\cdots\ell(v_n),$$

with the empty tensor mapped to 1_B.

Proof. Define $\widetilde{\ell}$ on homogeneous tensors by

$$\widetilde{\ell}(v_1\otimes\cdots\otimes v_n)=\ell(v_1)\cdots\ell(v_n),$$

and extend linearly over the direct sum. This map is multiplicative because concatenation of tensors corresponds to multiplication of the corresponding products in B. It is unital because the empty tensor maps to 1_B. Uniqueness follows because any unital algebra homomorphism extending ℓ must send every tensor word $v_1\otimes\cdots\otimes v_n$ to $\ell(v_1)\cdots\ell(v_n)$. $\square$

Remark 6.31 (Free only on the coefficient side)**.** Regime C is free on the coefficient side. The blade side is not free in this sense; it is already the frozen Clifford algebra $(\mathcal{A},m_\sigma)$ with the relations

$$i^2=\varepsilon_i, \qquad j^2=\varepsilon_j, \qquad k^2=\varepsilon_k, \qquad ij=-ji, \quad ik=-ki, \quad jk=-kj.$$

Thus $\mathcal{H}_\bullet = T(V)\otimes\mathcal{A}$ means that coefficient words are recorded freely, while blade factors are multiplied by the exact frozen Clifford product.

Proposition 6.32 (Regime C is not the joint free coefficient-blade algebra)**.** *The hierarchy*

$$\mathcal{H}_\bullet = T(V)\otimes\mathcal{A}$$

is free only with respect to coefficient words in $T(V)$. It is not the free associative algebra generated jointly by V and the Clifford symbols i,j,k. In the ordinary tensor product hierarchy, coefficient words and blade factors are external commuting tensor factors:

$$(t\otimes 1)\cdot_\sigma(1\otimes A)=(1\otimes A)\cdot_\sigma(t\otimes 1)=t\otimes A,$$

for $t\in T(V)$ and $A\in\mathcal{A}$. The blade factor also satisfies the frozen Clifford relations.

Proof. The product in $\mathcal{H}_\bullet$ concatenates coefficient tensors in $T(V)$ and multiplies blade factors using m_σ. Therefore coefficient words are freely recorded before any coefficient contraction or multiplication is chosen. However, the blade factor is not free: it is already reduced by the frozen Clifford relations. Moreover, because the construction is an ordinary tensor product algebra, the subalgebras $T(V)\otimes 1$ and $1\otimes\mathcal{A}$ commute as external tensor factors. Hence $\mathcal{H}_\bullet$ cannot satisfy the universal property of the free associative algebra on the joint generating set $V\cup\{i,j,k\}$. $\square$

Remark 6.33 (Alternative for noncommuting coefficient-blade semantics)**.** If one wants coefficient symbols and Clifford blade symbols to have no imposed commutation relation, the appropriate universal object is not $T(V) \otimes \mathcal{A}$. One should instead start from a free product such as

$$T(V) * \mathrm{Cl}_\sigma$$

or from a quotient of the free associative algebra generated by $V \cup \{i, j, k\}$ by explicitly declared Clifford and coefficient-blade interaction relations. Regime C deliberately does not do this. Its purpose is bookkeeping of coefficient words while retaining the exact frozen Clifford multiplication on the blade factor. For tensor algebras and free associative algebras, see [10, 11].

Proposition 6.34 (Universal mapping property of the full tensor hierarchy)**.** *Fix a frozen signature $\sigma \in \Sigma$, and regard*

$$\mathcal{H}_\bullet = T(V) \otimes \mathcal{A}$$

as the ordinary tensor product algebra

$$T(V) \otimes (\mathcal{A}, m_\sigma).$$

Let B be a unital associative real algebra. Suppose that

$$\ell : V \to B$$

is a linear map and

$$\rho_\sigma : (\mathcal{A}, m_\sigma) \to B$$

is a unital algebra homomorphism such that the images of $\widetilde{\ell} : T(V) \to B$ and ρ_σ commute:

$$\widetilde{\ell}(t)\rho_\sigma(A) = \rho_\sigma(A)\widetilde{\ell}(t), \qquad t \in T(V),\ A \in \mathcal{A}.$$

Then there exists a unique unital algebra homomorphism

$$\Phi : T(V) \otimes \mathcal{A} \to B$$

such that

$$\Phi(t \otimes A) = \widetilde{\ell}(t)\rho_\sigma(A).$$

Equivalently,

$$\Phi\big((v_1 \otimes \cdots \otimes v_n) \otimes A\big) = \ell(v_1) \cdots \ell(v_n)\rho_\sigma(A),$$

with the empty coefficient tensor mapped to 1_B.

Proof. The coefficient-side universal property gives the unique unital algebra homomorphism

$$\widetilde{\ell} : T(V) \to B$$

extending ℓ. Define

$$\Phi(t \otimes A) := \widetilde{\ell}(t)\rho_\sigma(A)$$

and extend linearly. The commuting-image hypothesis makes this formula multiplicative:

$$\begin{aligned}
\Phi\big((t \otimes A)(s \otimes C)\big) &= \Phi\big(ts \otimes m_\sigma(A, C)\big) \\
&= \widetilde{\ell}(t)\widetilde{\ell}(s)\rho_\sigma(A)\rho_\sigma(C) \\
&= \widetilde{\ell}(t)\rho_\sigma(A)\widetilde{\ell}(s)\rho_\sigma(C) \\
&= \Phi(t \otimes A)\Phi(s \otimes C).
\end{aligned}$$

It is unital because the empty tensor and the Clifford unit map to 1_B. Uniqueness follows because $T(V) \otimes \mathcal{A}$ is linearly spanned by pure tensors $t \otimes A$, and the values on $t \otimes 1$ and $1 \otimes A$ are forced by ℓ and ρ_σ. $\square$

Remark 6.35 (Why the commuting-image condition is necessary)**.** The condition that $\widetilde{\ell}(T(V))$ commute with $\rho_\sigma(\mathcal{A})$ is not cosmetic. It reflects the ordinary tensor product algebra structure of $T(V) \otimes \mathcal{A}$. If coefficient symbols and Clifford blade symbols are intended to interact noncommutatively, then $T(V) \otimes \mathcal{A}$ is not the correct universal object; one must use a free product, crossed product, graded tensor product, or explicitly presented quotient.

Remark 6.36 (Free bookkeeping, not uniqueness of all possible semantics)**.** The tensor hierarchy is the tensor-algebra bookkeeping construction before choosing any contraction, pairing, quotient, trace, integration, or multiplication on V. If later one specifies a bilinear map

$$V \times V \to V, \qquad V \times V \to \mathbb{R}, \qquad V \otimes V \to W,$$

then one obtains a different coefficient semantics by quotienting, contracting, or mapping the hierarchy. Regime C is canonical only in the restricted sense that it records formal coefficient words without making such choices; the mathematical freeness is precisely the universal property of $T(V)$ in Proposition 6.30.

6.7 Comparison of the three regimes

The three regimes may now be summarized as follows.

Regime	Structure on V	Meaning for $V \otimes \mathcal{A}$
A	vector, metric, or topological data only	coefficient carrier only; no canonical multiplication
B	associative product $\cdot_V$	closed ordinary tensor product algebra on $V \otimes \mathcal{A}$
C	no product, but formal bookkeeping retained	products leave $V \otimes \mathcal{A}$ and enter $\mathcal{H}_\bullet = T(V) \otimes \mathcal{A}$, with $\mathcal{H}_0 \cong \mathcal{A}$

This comparison gives the precise meaning of the warning

$$V \otimes \mathcal{A} \text{ is not automatically an algebra.}$$

There is no single correct multiplication for $V \otimes \mathcal{A}$ until the coefficient regime is declared.

6.8 Separation from grade projection

The coefficient regimes in this section solve only the coefficient-combination problem. They do not solve the grade-truncation problem. Even in Regime B, where $V \otimes \mathcal{A}$ is an associative algebra, the sector

$$V \otimes \mathcal{A}_{\leq 1}$$

is still not closed under exact multiplication, because the blade product can generate bivectors. Conversely, a projected product such as

$$\Pi_{\leq 1}(U \cdot_\sigma W)$$

may return to the scalar-vector sector, but projection is a different operation from coefficient multiplication.

This distinction prepares the next section. Section 7 studies grade projections

$$\Pi_{\leq r} : \mathcal{A} \to \mathcal{A}_{\leq r}$$

and the effective products obtained by multiplying exactly in $\mathcal{A}$ and then projecting back to a truncated sector.

7 Grade Projections and Truncated Products

Sections 3–6 separated two issues. First, the scalar-vector sector

$$\mathcal{A}_{\le 1}$$

is not closed under exact frozen multiplication. Second, coefficient-valued objects in $V \otimes \mathcal{A}$ cannot be multiplied until a coefficient regime is declared. This section addresses the first issue: how to obtain a closed operation on a truncated grade sector by multiplying exactly in the full carrier and then projecting back.

The central point is that projection is not multiplication. The frozen product

$$m_\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

is exact and lives on the full carrier. The maps

$$\Pi_{\le r} : \mathcal{A} \to \mathcal{A}_{\le r}$$

are linear truncation maps. A projected product combines the two operations in the order

$$\text{multiply exactly in } \mathcal{A} \quad \longrightarrow \quad \text{discard grades above } r.$$

This gives a closed effective operation on $\mathcal{A}_{\le r}$, but, as the next section proves, it generally changes the algebraic laws.

Convention 7.1 (Effective closure versus exact closure)**.** Let $P_r : \mathcal{A} \to \mathcal{A}_{\le r}$ be the grade projection. The projected operation

$$U \star_{\sigma,\le r} V := P_r m_\sigma(U, V)$$

is a new effective product on $\mathcal{A}_{\le r}$. Its closure is projection-induced closure, not exact closure under m_σ. Thus a statement that a truncated sector is closed under $\star_{\sigma,\le r}$ never means that it is a subalgebra of the exact frozen Clifford algebra $(\mathcal{A}, m_\sigma)$.

Proposition 7.2 (Projection creates a modified closed product)**.** *For each* $0 \le r \le 3$*, the formula*

$$U \star_{\sigma,\le r} V = P_r m_\sigma(U, V)$$

defines a bilinear product

$$\star_{\sigma,\le r} : \mathcal{A}_{\le r} \times \mathcal{A}_{\le r} \to \mathcal{A}_{\le r}.$$

This closure is closure for the modified projected product. It is exact Clifford closure under m_σ *if and only if*

$$m_\sigma(\mathcal{A}_{\le r}, \mathcal{A}_{\le r}) \subseteq \mathcal{A}_{\le r}.$$

For the rank-three scalar-vector and scalar-vector-bivector truncations, this exact closure fails for $r = 1$ *and* $r = 2$*.*

Proof. The map is bilinear because both m_σ and P_r are bilinear/linear. Its image lies in $\mathcal{A}_{\le r}$ by definition of P_r, so it is closed as a modified product.

If $m_\sigma(\mathcal{A}_{\le r}, \mathcal{A}_{\le r}) \subseteq \mathcal{A}_{\le r}$, then P_r acts as the identity on all exact products and $\star_{\sigma,\le r} = m_\sigma$ on $\mathcal{A}_{\le r}$. Conversely, if exact products leave $\mathcal{A}_{\le r}$, then $\star_{\sigma,\le r}$ discards their higher-grade components and is no longer the exact frozen product.

For $r = 1$, $m_\sigma(i, j) = ij \notin \mathcal{A}_{\le 1}$. For $r = 2$, $m_\sigma(ij, k) = ijk \notin \mathcal{A}_{\le 2}$. Hence exact closure fails in both cases. □

7.1 Truncated projections

Recall the grade decomposition

$$\mathcal{A} = \mathcal{A}^{(0)} \oplus \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \mathcal{A}^{(3)},$$

where

$$\mathcal{A}^{(0)} = \operatorname{span}_{\mathbb{R}}\{1\}, \quad \mathcal{A}^{(1)} = \operatorname{span}_{\mathbb{R}}\{i, j, k\},$$
$$\mathcal{A}^{(2)} = \operatorname{span}_{\mathbb{R}}\{ij, ik, jk\}, \quad \mathcal{A}^{(3)} = \operatorname{span}_{\mathbb{R}}\{ijk\}.$$

For $0 \le r \le 3$, set

$$\mathcal{A}_{\le r} = \bigoplus_{q=0}^{r} \mathcal{A}^{(q)}.$$

The truncated projection is

$$P_r := \Pi_{\le r} : \mathcal{A} \to \mathcal{A}_{\le r},$$

and the complementary discarded-grade projection is

$$Q_r := I - P_r : \mathcal{A} \to \bigoplus_{q=r+1}^{3} \mathcal{A}^{(q)}.$$

Thus every $X \in \mathcal{A}$ decomposes uniquely as

$$X = P_r X + Q_r X.$$

The operator P_r retains grades at most r, while Q_r records precisely the grades removed by the truncation.

Remark 7.3 (Projection is signature-independent)**.** The maps P_r and Q_r depend only on the fixed vector-space decomposition of $\mathcal{A}$. They do not depend on the frozen signature σ. By contrast, the exact product m_σ depends on σ. Hence a projected product has two distinct ingredients: a signature-dependent multiplication and a signature-independent truncation.

7.2 The projected product

Definition 7.4 (Projected truncated product)**.** Fix $\sigma \in \Sigma$ and $0 \le r \le 3$. For

$$U, W \in \mathcal{A}_{\le r},$$

define

$$U \star_{\sigma, \le r} W := P_r(U \cdot_\sigma W).$$

Equivalently,

$$U \star_{\sigma, \le r} W = \Pi_{\le r}\big(m_\sigma(U, W)\big).$$

When $r = 1$, we write

$$\star_\sigma := \star_{\sigma, \le 1}.$$

The definition must be read literally: one first uses the exact frozen product in $\mathcal{A}$, and only afterward projects to $\mathcal{A}_{\le r}$. Thus $\star_{\sigma,\le r}$ is not an independent Clifford product on $\mathcal{A}_{\le r}$; it is the truncation of the full product.

Proposition 7.5 (Closed bilinear operation)**.** *For every $\sigma \in \Sigma$ and $0 \le r \le 3$, the operation*

$$\star_{\sigma, \le r} : \mathcal{A}_{\le r} \times \mathcal{A}_{\le r} \to \mathcal{A}_{\le r}$$

is a well-defined bilinear operation. Moreover, 1 is a two-sided unit for $\star_{\sigma,\le r}$.

Proof. The exact product satisfies

$$U \cdot_\sigma W \in \mathcal{A}$$

for all $U, W \in \mathcal{A}$, and the projection P_r maps $\mathcal{A}$ into $\mathcal{A}_{\leq r}$. Therefore

$$P_r(U \cdot_\sigma W) \in \mathcal{A}_{\leq r}.$$

Bilinearity follows from bilinearity of m_σ and linearity of P_r. If $U \in \mathcal{A}_{\leq r}$, then

$$1 \star_{\sigma,\leq r} U = P_r(1 \cdot_\sigma U) = P_r(U) = U,$$

and similarly

$$U \star_{\sigma,\leq r} 1 = P_r(U \cdot_\sigma 1) = P_r(U) = U.$$

Thus 1 is a two-sided unit. □

Remark 7.6 (Closure by projection is not subalgebra closure)**.** Proposition 7.5 does not say that $\mathcal{A}_{\leq r}$ is closed under the exact product m_σ. It says that $\mathcal{A}_{\leq r}$ is closed under the modified operation obtained by composing exact multiplication with projection. This distinction is essential. For $r = 1$, the exact product of i and j is ij, whereas the projected product is zero:

$$i \cdot_\sigma j = ij, \qquad i \star_{\sigma,\leq 1} j = P_1(ij) = 0.$$

7.3 Truncation defect

The difference between exact multiplication and projected multiplication is measured by the discarded component.

Definition 7.7 (Binary truncation defect)**.** For $U, W \in \mathcal{A}_{\leq r}$, define

$$D_{\sigma,\leq r}(U, W) := Q_r(U \cdot_\sigma W).$$

Then

$$U \cdot_\sigma W = U \star_{\sigma,\leq r} W + D_{\sigma,\leq r}(U, W).$$

The defect $D_{\sigma,\leq r}$ records exactly the grades that must be discarded in order to remain inside $\mathcal{A}_{\leq r}$. If

$$D_{\sigma,\leq r}(U, W) = 0,$$

then the projected product agrees with the exact product for that pair:

$$U \star_{\sigma,\leq r} W = U \cdot_\sigma W.$$

If the defect is nonzero, the projected product is an effective shadow of the exact product.

Lemma 7.8 (Exact agreement criterion)**.** *For $U, W \in \mathcal{A}_{\leq r}$, the following are equivalent:*

(i) $U \cdot_\sigma W \in \mathcal{A}_{\leq r}$*;*

(ii) $D_{\sigma,\leq r}(U, W) = 0$*;*

(iii) $U \star_{\sigma,\leq r} W = U \cdot_\sigma W$*.*

Proof. The decomposition

$$U \cdot_\sigma W = P_r(U \cdot_\sigma W) + Q_r(U \cdot_\sigma W)$$

is direct. Hence the exact product lies in $\mathcal{A}_{\leq r}$ if and only if its Q_r-component vanishes. This is exactly the condition that the projection does not alter the product. □

7.4 The four truncation levels

There are only four possible truncation levels in the three-generator carrier.

Level $r = 0$. The sector is

$$\mathcal{A}_{\leq 0} = \mathbb{R} \cdot 1.$$

The projected product is ordinary scalar multiplication:

$$(a1) \star_{\sigma,\leq 0} (b1) = ab1.$$

It is associative because no higher-grade component is generated inside the scalar sector.

Level $r = 1$. The sector is

$$\mathcal{A}_{\leq 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}.$$

This is the main scalar-vector sector of the paper. It is not closed under exact multiplication, because products of distinct generators generate bivectors. The product $\star_{\sigma,\leq 1}$ restores closure only by discarding the bivector part.

Level $r = 2$. The sector is

$$\mathcal{A}_{\leq 2} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk\}.$$

This sector retains bivectors but discards the trivector. It is therefore larger than the scalar-vector sector but still not the full algebraic carrier. Products can still leave $\mathcal{A}_{\leq 2}$, for instance

$$i \cdot_\sigma jk = ijk \notin \mathcal{A}_{\leq 2}.$$

Thus $\star_{\sigma,\leq 2}$ is still a projected effective product.

Level $r = 3$. The sector is the full carrier:

$$\mathcal{A}_{\leq 3} = \mathcal{A}.$$

Here

$$P_3 = I, \qquad Q_3 = 0,$$

and therefore

$$U \star_{\sigma,\leq 3} W = U \cdot_\sigma W.$$

Thus the projected product at level 3 is exactly the frozen Clifford product.

7.5 Level-one projected product

The most important projected product for the motivating expression

$$a + bi + cj + dk$$

is the level-one product

$$\star_\sigma = \star_{\sigma,\leq 1}.$$

Let

$$U = a + bi + cj + dk, \qquad W = \alpha + \beta i + \gamma j + \delta k,$$

with scalar coefficients. Section 3.4 gives the exact product as

$$U \cdot_\sigma W = S_\sigma(U, W) + V_\sigma(U, W) + B_\sigma(U, W),$$

where S_σ is scalar, V_σ is vectorial, and B_σ is bivectorial. Since P_1 discards the bivector part, one obtains

$$\begin{aligned} U \star_\sigma W = a\alpha + \varepsilon_i b\beta + \varepsilon_j c\gamma + \varepsilon_k d\delta \\ + (a\beta + b\alpha)i + (a\gamma + c\alpha)j + (a\delta + d\alpha)k. \end{aligned} \tag{138}$$

The discarded component is

$$D_{\sigma,\leq 1}(U, W) = (b\gamma - c\beta)ij + (b\delta - d\beta)ik + (c\delta - d\gamma)jk. \tag{139}$$

Therefore the projected product is exact if and only if the bivector defect in (139) vanishes.

Corollary 7.9 (Scalar-vector exactness condition)**.** *For scalar-vector elements $U, W \in \mathcal{A}_{\leq 1}$, the level-one projected product agrees with the exact product if and only if*

$$b\gamma - c\beta = 0, \qquad b\delta - d\beta = 0, \qquad c\delta - d\gamma = 0.$$

Equivalently, the vector parts of U and W have zero exterior bivector component.

Proof. By Lemma 7.8, exact agreement is equivalent to vanishing of $D_{\sigma,\leq 1}(U, W)$. Formula (139) gives the three scalar conditions. □

7.6 Coefficient-valued projected products in Regime B

The projections above are blade projections. They act on the $\mathcal{A}$-factor, not on the coefficient factor. In Regime B, where V is an associative real algebra, the ordinary ungraded tensor product algebra $V \otimes \mathcal{A}$ carries the product

$$(v \otimes A) \cdot_\sigma (w \otimes B) := (v \cdot_V w) \otimes (A \cdot_\sigma B).$$

Define

$$P_r^V := I_V \otimes P_r, \qquad Q_r^V := I_V \otimes Q_r.$$

For $U, W \in V \otimes \mathcal{A}_{\leq r}$, define

$$U \star^V_{\sigma,\leq r} W := P_r^V (U \cdot_\sigma W).$$

Theorem 7.10 (Coefficient-valued projected product in Regime B)**.** *Let V be an associative real algebra. For every frozen signature $\sigma \in \Sigma$ and every $0 \leq r \leq 3$, the operation*

$$\star^V_{\sigma,\leq r} : (V \otimes \mathcal{A}_{\leq r}) \times (V \otimes \mathcal{A}_{\leq r}) \to V \otimes \mathcal{A}_{\leq r}$$

is a well-defined bilinear closed operation. Moreover, for all $U, W, Z \in V \otimes \mathcal{A}_{\leq r}$, define

$$\mathrm{Assoc}^V_{\sigma,\leq r}(U, W, Z) := (U \star^V_{\sigma,\leq r} W) \star^V_{\sigma,\leq r} Z - U \star^V_{\sigma,\leq r} (W \star^V_{\sigma,\leq r} Z).$$

Then

$$\boxed{\mathrm{Assoc}^V_{\sigma,\leq r}(U, W, Z) = P_r^V \left(-Q_r^V (U \cdot_\sigma W) \cdot_\sigma Z + U \cdot_\sigma Q_r^V (W \cdot_\sigma Z) \right).}$$

All products inside the right-hand side are exact Regime-B tensor product multiplications before projection.

Proof. Bilinearity follows from bilinearity of $\cdot_V$, bilinearity of m_σ, and linearity of P_r^V. Closure follows because P_r^V maps $V \otimes \mathcal{A}$ into $V \otimes \mathcal{A}_{\le r}$.

For the associator identity, use

$$P_r^V(X) = X - Q_r^V(X).$$

Then

$$\begin{aligned}(U \star_{\sigma,\le r}^V W) \star_{\sigma,\le r}^V Z &= P_r^V \left(P_r^V(U \cdot_\sigma W) \cdot_\sigma Z\right) \\ &= P_r^V \left((U \cdot_\sigma W) \cdot_\sigma Z - Q_r^V(U \cdot_\sigma W) \cdot_\sigma Z\right),\end{aligned}$$

and

$$\begin{aligned}U \star_{\sigma,\le r}^V (W \star_{\sigma,\le r}^V Z) &= P_r^V \left(U \cdot_\sigma P_r^V(W \cdot_\sigma Z)\right) \\ &= P_r^V \left(U \cdot_\sigma (W \cdot_\sigma Z) - U \cdot_\sigma Q_r^V(W \cdot_\sigma Z)\right).\end{aligned}$$

Since V is associative and m_σ is associative, the Regime-B tensor product algebra is associative:

$$(U \cdot_\sigma W) \cdot_\sigma Z = U \cdot_\sigma (W \cdot_\sigma Z).$$

The exact terms cancel, leaving the claimed defect formula. □

Corollary 7.11 (Coefficient-valued non-associativity at intermediate grades)**.** *If V is unital, then for every $\sigma \in \Sigma$,*

$$\star_{\sigma,\le 1}^V \quad \textit{and} \quad \star_{\sigma,\le 2}^V$$

are non-associative on their full coefficient-valued truncated domains.

Proof. For $r = 1$, use

$$U = 1_V \otimes i, \qquad W = 1_V \otimes i, \qquad Z = 1_V \otimes j.$$

The left-associated product gives $1_V \otimes \varepsilon_i j$, while the right-associated product gives 0, because the intermediate exact product $i \cdot_\sigma j = ij$ is discarded by P_1. Since $\varepsilon_i = \pm 1$, the associator is nonzero.

For $r = 2$, use the scalar witness

$$U = 1_V \otimes i, \qquad W = 1_V \otimes j, \qquad Z = 1_V \otimes k.$$

The coefficient factor remains 1_V throughout. Therefore the coefficient-valued associator is exactly the scalar associator tensored with 1_V, and it is nonzero for every frozen signature. Hence $\star_{\sigma,\le 1}^V$ and $\star_{\sigma,\le 2}^V$ are non-associative on their full coefficient-valued truncated domains. □

7.7 Coefficient-valued projections in Regime C

In Regime C, where products move to the hierarchy

$$\mathcal{H}_n = V^{\otimes n} \otimes \mathcal{A}, \qquad n \ge 0,$$

the projection at tensor level n is

$$P_{r,n} := I_{V^{\otimes n}} \otimes P_r, \qquad Q_{r,n} := I_{V^{\otimes n}} \otimes Q_r.$$

At level 0, this is just the scalar blade projection on $\mathcal{A} \cong \mathcal{H}_0$. Thus grade projection remains independent of coefficient bookkeeping: it always acts on the blade factor.

Remark 7.12 (No projection can replace coefficient semantics)**.** Projection solves a grade problem, not a coefficient problem. If V is merely a vector, metric, or topological coefficient space, then P_r^V is meaningful as a linear map on $V \otimes \mathcal{A}$, but the product

$$U \cdot_\sigma W$$

may still be undefined. Thus one cannot use grade projection to avoid specifying coefficient multiplication or tensor-hierarchy semantics.

7.8 Projection table

The four projection regimes may be summarized as follows.

Projection	Domain	Product	Associativity	Status
$\Pi_{\le 0}$	$\mathcal{A}_{\le 0}$	scalar product	yes	exact restricted scalar subalgebra
$\Pi_{\le 1}$	$\mathcal{A}_{\le 1}$	scalar-vector projected product	no on full domain	effective
$\Pi_{\le 2}$	$\mathcal{A}_{\le 2}$	scalar-vector-bivector projected product	no on full domain	effective
$\Pi_{\le 3} = I$	$\mathcal{A}$	full frozen Clifford product	yes	exact full carrier

Remark 7.13 (Meaning of exactness at $r = 0$)**.** The label "exact" for $r = 0$ is restricted to the scalar domain $\mathcal{A}_{\le 0}$. It means that $\mathcal{A}_{\le 0} = \mathbb{R}1$ is a genuine subalgebra of every frozen Clifford algebra $(\mathcal{A}, m_\sigma)$, and that $\star_{\sigma,\le 0}$ agrees with m_σ on this subalgebra. It does not mean that the scalar projection $P_0 \circ m_\sigma$ is the exact Clifford product for arbitrary non-scalar inputs in $\mathcal{A}$.

The non-associative entries are proved in Section 8; the failure holds for every frozen signature on the full truncated domains. The present section only constructs the projected operations and identifies the grades they keep or discard.

7.9 Section conclusion

The projected product

$$U \star_{\sigma,\le r} W = P_r(U \cdot_\sigma W)$$

is a closed operation on $\mathcal{A}_{\le r}$. Its closure is obtained by truncation, not by exact subalgebra closure. The discarded term

$$D_{\sigma,\le r}(U, W) = Q_r(U \cdot_\sigma W)$$

records the information lost by projection. Section 8 shows that these discarded terms are precisely what generate associator defects for $r = 1$ and $r = 2$.

8 General Projection Defect and Non-Associativity

Section 7 introduced the projected products

$$U \star_{\sigma,\le r} W = P_r(U \cdot_\sigma W), \qquad P_r = \Pi_{\le r}.$$

Those products are closed on the truncated grade spaces by definition. Closure, however, is not the same as subalgebra closure. The full frozen product remains associative on $\mathcal{A}$, while the projected product may fail to be associative because the intermediate higher-grade terms have been discarded. The purpose of this section is to make that defect explicit.

The guiding principle is:

$$\boxed{\text{projection creates effective closure by deleting grades; the deleted grades generate the associator defect.}}$$

Convention 8.1 (Domain of associativity statements)**.** Every associativity statement for a projected product

$$\star_{\sigma,\leq r}$$

is understood relative to its declared domain

$$\mathcal{A}_{\leq r} \times \mathcal{A}_{\leq r} \to \mathcal{A}_{\leq r}.$$

Thus the statement that $\star_{\sigma,\leq r}$ is non-associative means that there exist

$$U, V, W \in \mathcal{A}_{\leq r}$$

for which the associator is nonzero. It does not imply that every restricted subsystem or every subspace of $\mathcal{A}_{\leq r}$ is non-associative.

8.1 Projected associators

Fix a frozen signature $\sigma \in \Sigma$ and an integer $0 \leq r \leq 3$. Write

$$P_r = \Pi_{\leq r}, \qquad Q_r = I - P_r.$$

For $U, V, W \in \mathcal{A}_{\leq r}$, define the projected associator by

$$\mathrm{Assoc}_{\sigma,\leq r}(U, V, W) := (U \star_{\sigma,\leq r} V) \star_{\sigma,\leq r} W - U \star_{\sigma,\leq r} (V \star_{\sigma,\leq r} W).$$

The subscript σ will sometimes be suppressed when the frozen signature is fixed.

The exact product in $(\mathcal{A}, m_\sigma)$ is associative:

$$(U \cdot_\sigma V) \cdot_\sigma W = U \cdot_\sigma (V \cdot_\sigma W).$$

Thus any nonzero associator for $\star_{\sigma,\leq r}$ cannot come from the frozen Clifford product itself. It must come from the projection step.

8.2 The general projection-defect formula

Theorem 8.2 (General projection-induced associator defect)**.** *Let* $0 \leq r \leq 3$*, let*

$$P_r = \Pi_{\leq r}, \qquad Q_r = I - P_r,$$

and define

$$U \star_{\sigma,\leq r} V := P_r(U \cdot_\sigma V)$$

for $U, V \in \mathcal{A}_{\leq r}$*. Then, for all* $U, V, W \in \mathcal{A}_{\leq r}$*,*

$$\boxed{\mathrm{Assoc}_{\sigma,\leq r}(U, V, W) = P_r\left(-Q_r(U \cdot_\sigma V) \cdot_\sigma W + U \cdot_\sigma Q_r(V \cdot_\sigma W)\right).}$$

Proof. Since
$$P_r(U \cdot_\sigma V) = U \cdot_\sigma V - Q_r(U \cdot_\sigma V),$$
we have
$$\begin{aligned}(U \star_{\sigma,\le r} V) \star_{\sigma,\le r} W &= P_r\left(P_r(U \cdot_\sigma V) \cdot_\sigma W\right)\\ &= P_r\left((U \cdot_\sigma V - Q_r(U \cdot_\sigma V)) \cdot_\sigma W\right).\end{aligned}$$
Similarly,
$$\begin{aligned}U \star_{\sigma,\le r} (V \star_{\sigma,\le r} W) &= P_r\left(U \cdot_\sigma P_r(V \cdot_\sigma W)\right)\\ &= P_r\left(U \cdot_\sigma (V \cdot_\sigma W - Q_r(V \cdot_\sigma W))\right).\end{aligned}$$
Subtracting gives
$$\begin{aligned}\mathrm{Assoc}_{\sigma,\le r}(U,V,W) = P_r\Big((U \cdot_\sigma V) \cdot_\sigma W - Q_r(U \cdot_\sigma V) \cdot_\sigma W\\ - U \cdot_\sigma (V \cdot_\sigma W) + U \cdot_\sigma Q_r(V \cdot_\sigma W)\Big).\end{aligned}$$
By exact associativity in $(\mathcal{A}, m_\sigma)$,
$$(U \cdot_\sigma V) \cdot_\sigma W = U \cdot_\sigma (V \cdot_\sigma W).$$
The exact associativity terms cancel, leaving
$$\mathrm{Assoc}_{\sigma,\le r}(U,V,W) = P_r\left(-Q_r(U \cdot_\sigma V) \cdot_\sigma W + U \cdot_\sigma Q_r(V \cdot_\sigma W)\right).$$
□

Remark 8.3 (What the formula says)**.** The first term
$$-P_r\big(Q_r(U \cdot_\sigma V) \cdot_\sigma W\big)$$
records the effect of the grades discarded after multiplying U and V. The second term
$$P_r\big(U \cdot_\sigma Q_r(V \cdot_\sigma W)\big)$$
records the effect of the grades discarded after multiplying V and W. The projected associator is the imbalance between these two discarded-grade contributions after they are multiplied back into the retained sector.

8.3 Associativity for the scalar and full cases

The two endpoint cases are associative, but for different reasons.

Proposition 8.4 (Endpoint associativity)**.** *For every frozen signature σ, the products*
$$\star_{\sigma,\le 0} \qquad \textit{and} \qquad \star_{\sigma,\le 3}$$
are associative.

Proof. For $r = 0$, the domain of the truncated product is
$$\mathcal{A}_{\le 0} = \mathbb{R} \cdot 1.$$
If $U = \lambda 1$ and $V = \mu 1$, then
$$U \cdot_\sigma V = (\lambda\mu) 1 \in \mathcal{A}_{\le 0},$$

so

$$U \star_{\sigma,\leq 0} V = P_0(U \cdot_\sigma V) = U \cdot_\sigma V.$$

Thus $\star_{\sigma,\leq 0}$ agrees with the exact frozen product on the scalar subalgebra $\mathcal{A}_{\leq 0}$, where it is ordinary scalar multiplication. This endpoint exactness is only a restricted-domain statement: $P_0 \circ m_\sigma$ is not the exact Clifford product on arbitrary non-scalar inputs.

For $r = 3$, one has

$$P_3 = \Pi_{\leq 3} = I, \qquad Q_3 = 0.$$

Hence

$$U \star_{\sigma,\leq 3} V = P_3(U \cdot_\sigma V) = U \cdot_\sigma V,$$

which is exactly the frozen Clifford product m_σ on $\mathcal{A}$. This product is associative by Lemma 2.11. □

8.4 Failure of associativity in the scalar-vector projection

The first genuinely truncated case is $r = 1$. Here the projection keeps the scalar-vector sector and discards bivectors and trivectors.

Theorem 8.5 (Non-associativity of $\Pi_{\leq 1}$ on the full truncated domain)**.** *For every $\sigma \in \Sigma$, the projected product*

$$\star_{\sigma,\leq 1} : \mathcal{A}_{\leq 1} \times \mathcal{A}_{\leq 1} \to \mathcal{A}_{\leq 1}$$

is not associative as a binary operation on the full truncated domain $\mathcal{A}_{\leq 1}$.

Proof. Take

$$U = i, \qquad V = i, \qquad W = j.$$

Since

$$i \star_{\sigma,\leq 1} j = \Pi_{\leq 1}(ij) = 0,$$

we obtain

$$i \star_{\sigma,\leq 1} (i \star_{\sigma,\leq 1} j) = 0.$$

On the other hand,

$$i \star_{\sigma,\leq 1} i = \Pi_{\leq 1}(i^2) = \varepsilon_i,$$

and therefore

$$(i \star_{\sigma,\leq 1} i) \star_{\sigma,\leq 1} j = \varepsilon_i j.$$

Since $\varepsilon_i = \pm 1$, the element $\varepsilon_i j$ is nonzero. Hence

$$(i \star_{\sigma,\leq 1} i) \star_{\sigma,\leq 1} j \neq i \star_{\sigma,\leq 1} (i \star_{\sigma,\leq 1} j).$$

□

Remark 8.6 (The discarded bivector is responsible)**.** The failure above is caused by

$$Q_1(ij) = ij.$$

The intermediate bivector is invisible after the first projection, but it would have interacted nontrivially with neighboring factors inside the full Clifford carrier. Projecting too early changes the result.

8.5 Failure of associativity in the scalar-vector-bivector projection

The case $r = 2$ retains bivectors but discards trivectors. It is therefore less severe than $r = 1$, but it is still not exact.

Theorem 8.7 (Non-associativity of $\Pi_{\leq 2}$ on the full truncated domain)**.** *For every $\sigma \in \Sigma$, the projected product*

$$\star_{\sigma,\leq 2} : \mathcal{A}_{\leq 2} \times \mathcal{A}_{\leq 2} \to \mathcal{A}_{\leq 2}$$

is not associative as a binary operation on the full truncated domain $\mathcal{A}_{\leq 2}$.

Proof. Take

$$U = i, \qquad V = jk, \qquad W = k.$$

Since

$$i \cdot_\sigma jk = ijk,$$

we have

$$i \star_{\sigma,\leq 2} jk = \Pi_{\leq 2}(ijk) = 0.$$

Thus

$$(i \star_{\sigma,\leq 2} jk) \star_{\sigma,\leq 2} k = 0.$$

On the other hand,

$$jk \cdot_\sigma k = jk^2 = \varepsilon_k j,$$

so

$$jk \star_{\sigma,\leq 2} k = \varepsilon_k j.$$

Therefore

$$i \star_{\sigma,\leq 2} (jk \star_{\sigma,\leq 2} k) = i \star_{\sigma,\leq 2} (\varepsilon_k j) = \varepsilon_k ij.$$

Because $\varepsilon_k = \pm 1$, the bivector $\varepsilon_k ij$ is nonzero. Hence

$$(i \star_{\sigma,\leq 2} jk) \star_{\sigma,\leq 2} k \neq i \star_{\sigma,\leq 2} (jk \star_{\sigma,\leq 2} k).$$

$\square$

Remark 8.8 (The discarded trivector is responsible)**.** For $r = 2$, bivectors are retained, but the trivector part is discarded. In the counterexample above,

$$Q_2(i \cdot_\sigma jk) = ijk.$$

The loss of this trivector is precisely what makes the two bracketings disagree.

8.6 Vanishing criterion

The defect formula also identifies when projected associativity survives on restricted sectors. Here it is important to separate two conditions: vanishing of the associator on a collection of triples and closure of the collection under the projected product.

Definition 8.9 (Restricted projected product sector)**.** Let $S \subseteq \mathcal{A}_{\leq r}$. We say that S is a restricted projected product sector for $\star_{\sigma,\leq r}$ if

$$U \star_{\sigma,\leq r} V \in S \qquad \text{for all } U, V \in S.$$

It is an associative restricted projected product sector if, in addition,

$$(U \star_{\sigma,\leq r} V) \star_{\sigma,\leq r} W = U \star_{\sigma,\leq r} (V \star_{\sigma,\leq r} W)$$

for all $U, V, W \in S$.

Corollary 8.10 (Projection-defect vanishing criterion)**.** *Let $0 \le r \le 3$. For $U, V, W \in \mathcal{A}_{\le r}$, one has*

$$\mathrm{Assoc}_{\sigma,\le r}(U, V, W) = 0$$

if and only if

$$P_r(Q_r(U \cdot_\sigma V) \cdot_\sigma W) = P_r(U \cdot_\sigma Q_r(V \cdot_\sigma W)).$$

In particular, if $S \subseteq \mathcal{A}_{\le r}$ satisfies

$$Q_r(U \cdot_\sigma V) = 0, \qquad Q_r(V \cdot_\sigma W) = 0$$

for every $U, V, W \in S$, then the associator vanishes on S^3. If S is also closed under $\star_{\sigma,\le r}$, then S is an associative restricted projected product sector.

Proof. The equivalence is immediate from Theorem 8.2. If the two displayed discarded-grade contributions agree after applying P_r, then the associator vanishes. Conversely, if the associator vanishes, the two contributions must agree. The sufficient condition follows because $Q_r(U \cdot_\sigma V) = 0$ and $Q_r(V \cdot_\sigma W) = 0$ make both sides equal to zero. The final statement adds the separate closure condition from Definition 8.9, which is needed before S can be treated as a product system rather than merely as a set of triples with vanishing associator. □

Remark 8.11 (Vanishing is not the same as closed associativity)**.** The condition $\mathrm{Assoc}_{\sigma,\le r}|_{S^3} = 0$ is weaker than saying that S is an associative product system. The latter also requires closure under the projected product. This distinction matters because a subset may have vanishing associator on its original triples while products of its elements leave the subset.

Remark 8.12 (Restricted associative sectors)**.** The corollary is not meant to say that projected products are always defective on every restricted subset. It says that their associativity is conditional. A projected product may be associative on a special sector that never produces discarded grades and is closed under the projected product, but $\star_{\sigma,\le 1}$ and $\star_{\sigma,\le 2}$ are not associative as general products on $\mathcal{A}_{\le 1}$ and $\mathcal{A}_{\le 2}$, respectively.

8.7 Summary of the classification

Combining the endpoint associativity with the two counterexamples gives the projected-product classification:

$$\boxed{\begin{array}{c} \star_{\sigma,\le 0} \text{ and } \star_{\sigma,\le 3} \text{ are associative for every } \sigma \in \Sigma, \\ \star_{\sigma,\le 1} \text{ and } \star_{\sigma,\le 2} \text{ are non-associative for every } \sigma \in \Sigma \text{ on their full truncated domains.} \end{array}}$$

The distinction is structural. The case $r = 0$ is scalar and has no room for a discarded non-scalar grade inside its own domain. The case $r = 3$ is the exact full Clifford product. The intermediate cases $r = 1$ and $r = 2$ close only by truncation, and truncation creates the associator defect described by Theorem 8.2.

9 Exact, Projected, and Effective Products

The previous sections constructed several operations that all originate from the same frozen Clifford multiplication, but they do not have the same algebraic status. This section records the product taxonomy used in the rest of the paper. Its purpose is mainly defensive: it prevents an exact frozen product, a projected product, an averaged product, and a coefficient-valued product from being treated as interchangeable.

There are three independent distinctions:

(i) whether the signature is frozen or averaged;

(ii) whether the product is full-grade or projected to a truncated sector;

(iii) whether the coefficients are scalar, algebra-valued, or tensor-hierarchy-valued.

Only after these distinctions are fixed does a product notation have a definite meaning.

9.1 Frozen full products

For a fixed signature

$$\sigma = (\varepsilon_i, \varepsilon_j, \varepsilon_k) \in \Sigma,$$

the exact multiplication is

$$m_\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

Equivalently, for $U, W \in \mathcal{A}$, we write

$$U \cdot_\sigma W = m_\sigma(U, W).$$

This product is the full frozen Clifford product on the carrier

$$\mathcal{A} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}.$$

It is associative:

$$(U \cdot_\sigma V) \cdot_\sigma W = U \cdot_\sigma (V \cdot_\sigma W), \qquad U, V, W \in \mathcal{A}. \tag{140}$$

This is the only exact product in the scalar full-grade theory. The word "exact" will always mean exact with respect to a fixed frozen signature unless explicitly stated otherwise. Thus any statement such as "the algebra is associative" refers to $(\mathcal{A}, m_\sigma)$ for fixed σ, not to a projected or averaged operation.

9.2 Projected products

For $0 \le r \le 3$, the truncated projection

$$P_r = \Pi_{\le r} : \mathcal{A} \to \mathcal{A}_{\le r}$$

defines the projected product

$$U \star_{\sigma,\le r} W := P_r(U \cdot_\sigma W), \qquad U, W \in \mathcal{A}_{\le r}. \tag{141}$$

The product $\star_{\sigma,\le r}$ is closed on $\mathcal{A}_{\le r}$ by definition. It agrees with the exact frozen product on its declared domain only in the endpoint cases $r = 0$ and $r = 3$; for $r = 1, 2$, it is obtained by multiplying exactly in $\mathcal{A}$ and then discarding grades above r.

The classification established in Section 8 is

$$\star_{\sigma,\le 0} \text{ and } \star_{\sigma,\le 3} \text{ are associative,} \tag{142}$$

whereas

$$\star_{\sigma,\le 1} \text{ and } \star_{\sigma,\le 2} \text{ are non-associative for every frozen signature on their full truncated domains.} \tag{143}$$

The endpoint cases are associative for different reasons. For $r = 0$, one is multiplying scalars in $\mathbb{R} \cdot 1$. For $r = 3$, the projection is the identity and the product is the full frozen Clifford product:

$$\star_{\sigma,\le 3} = m_\sigma.$$

The intermediate products $\star_{\sigma,\le 1}$ and $\star_{\sigma,\le 2}$ are closed effective products, not subalgebra products.

Remark 9.1 (Closure versus subalgebra closure)**.** The phrase "closed by projection" does not mean "closed as a subalgebra of $\mathcal{A}$." The sector $\mathcal{A}_{\le r}$ is closed under $\star_{\sigma,\le r}$ because the output is forced back into $\mathcal{A}_{\le r}$. It is a subalgebra under the exact product only if

$$m_\sigma(\mathcal{A}_{\le r}, \mathcal{A}_{\le r}) \subseteq \mathcal{A}_{\le r}.$$

For $r = 1$ and $r = 2$, this exact subalgebra condition fails in general.

9.3 Averaged products

VPSCF1 also introduced averaged multiplication tensors [1]. If π_x is a local one-point law on Σ, the averaged product has the schematic form

$$\bar{m}_x(U, W) := \sum_{\tau\in\Sigma} \pi_x(\tau) m_\tau(U, W), \qquad U, W \in \mathcal{A}. \tag{144}$$

Convention 9.2 (Pointwise interpretation of averaged products)**.** Whenever the averaged product $\bar{m}_x$ is discussed algebraically, the base point x is fixed. Thus associativity, non-associativity, or associator-defect statements for $\bar{m}_x$ are pointwise statements about the bilinear operation

$$\bar{m}_x : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

at that fixed point. For fields, the corresponding pointwise effective product is

$$(\bar{m}(U, W))(x) := \bar{m}_x(U(x), W(x)).$$

This is different from nonlocal, transported, or independently sampled products, which require additional structure and are not part of VPSCF2.

This operation is useful for effective or statistical descriptions. It is a frozen Clifford product exactly at Dirac laws, as the next proposition records. The associativity of every frozen product m_τ does not by itself settle the associativity of the averaged product $\bar{m}_x$; the answer is law-dependent and controlled by finite mixed-signature and moment-factorization identities. The averaged operation depends on the law π_x, not on a single pathwise signature.

VPSCF2 does not develop probability-kernel dynamics or a full stochastic evolution theory. However, the exact/effective taxonomy is made self-contained below by recording the mixed-signature associator expansion, a finite associativity criterion, a moment-factorization classification, and a concrete non-associative averaged law.

Proposition 9.3 (Averaged products are frozen exactly for Dirac laws)**.** *Let $\pi \in \mathcal{P}(\Sigma)$, and define*

$$\bar{m}_\pi = \sum_{\sigma\in\Sigma} \pi(\sigma) m_\sigma.$$

Then there exists $\tau \in \Sigma$ such that

$$\bar{m}_\pi = m_\tau$$

as bilinear maps $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ if and only if

$$\pi = \delta_\tau.$$

Consequently, a non-Dirac averaged product is never an exact frozen Clifford product, even when it happens to be associative.

Proof. If $\pi = \delta_\tau$, then the definition gives $\bar{m}_\pi = m_\tau$.

Conversely, suppose $\bar{m}_\pi = m_\tau$. Evaluate both products on repeated homogeneous grade-one generators. For $a \in \{i, j, k\}$,

$$\bar{m}_\pi(a, a) = \left(\sum_{\sigma \in \Sigma} \pi(\sigma) \varepsilon_a(\sigma) \right) 1 = \mathbb{E}_\pi[\varepsilon_a] 1.$$

On the other hand,

$$m_\tau(a, a) = \varepsilon_a(\tau) 1.$$

Therefore

$$\mathbb{E}_\pi[\varepsilon_a] = \varepsilon_a(\tau) \in \{\pm 1\}$$

for $a = i, j, k$. Since each ε_a is a $\{\pm 1\}$-valued random variable, expectation $+1$ forces $\varepsilon_a = +1$ almost surely, and expectation -1 forces $\varepsilon_a = -1$ almost surely. Hence all three coordinate signs are almost surely equal to the corresponding signs of τ. Thus π is concentrated at τ, namely $\pi = \delta_\tau$. □

Corollary 9.4 (Pointwise Dirac criterion for local averaged products)**.** *Let $x \mapsto \pi_x$ be a local one-point signature kernel and define*

$$\bar{m}_x = \sum_{\sigma \in \Sigma} \pi_x(\sigma) m_\sigma.$$

For a fixed point x, the averaged product $\bar{m}_x$ is equal to a frozen product $m_{\tau(x)}$ for some $\tau(x) \in \Sigma$ if and only if

$$\pi_x = \delta_{\tau(x)}.$$

Thus local averaging is genuinely effective exactly at those points where the local law is non-Dirac.

Definition 9.5 (Mixed-signature associator)**.** For frozen signatures $\rho, \tau \in \Sigma$, define

$$\mathrm{Assoc}^{\mathrm{mix}}_{\rho,\tau}(U, V, W) := m_\rho(m_\tau(U, V), W) - m_\rho(U, m_\tau(V, W)).$$

This measures the defect caused by using m_τ for the inner multiplication and m_ρ for the outer multiplication.

Definition 9.6 (Mixed-signature averaged compatibility condition)**.** Let $\pi \in \mathcal{P}(\Sigma)$, and define

$$\bar{m}_\pi = \sum_{\sigma \in \Sigma} \pi(\sigma) m_\sigma.$$

We say that π satisfies the *mixed-signature averaged compatibility condition* if

$$\sum_{\rho,\tau \in \Sigma} \pi(\rho)\pi(\tau)\, \mathrm{Assoc}^{\mathrm{mix}}_{\rho,\tau}(U, V, W) = 0$$

for all $U, V, W \in \mathcal{A}$. Equivalently, since the diagonal terms $\rho = \tau$ vanish,

$$\sum_{\substack{\rho,\tau \in \Sigma \\ \rho \neq \tau}} \pi(\rho)\pi(\tau)\, \mathrm{Assoc}^{\mathrm{mix}}_{\rho,\tau}(U, V, W) = 0$$

for all $U, V, W \in \mathcal{A}$.

Proposition 9.7 (Averaged associator expansion)**.** *Fix x and write*

$$\bar{m}_x = \sum_{\tau \in \Sigma} \pi_x(\tau) m_\tau.$$

Then

$$\mathrm{Assoc}_{\bar{m}_x}(U, V, W) = \sum_{\rho,\tau \in \Sigma} \pi_x(\rho)\pi_x(\tau)\, \mathrm{Assoc}^{\mathrm{mix}}_{\rho,\tau}(U, V, W).$$

The diagonal terms $\rho = \tau$ vanish because each frozen product m_τ is associative. Hence

$$\mathrm{Assoc}_{\bar{m}_x}(U, V, W) = \sum_{\substack{\rho,\tau \in \Sigma \\ \rho \neq \tau}} \pi_x(\rho)\pi_x(\tau)\, \mathrm{Assoc}^{\mathrm{mix}}_{\rho,\tau}(U, V, W).$$

This expansion motivates the mixed-signature averaged compatibility condition formalized above.

Proof. By bilinearity,

$$\bar{m}_x(\bar{m}_x(U, V), W) = \sum_{\rho,\tau \in \Sigma} \pi_x(\rho)\pi_x(\tau) m_\rho(m_\tau(U, V), W),$$

and

$$\bar{m}_x(U, \bar{m}_x(V, W)) = \sum_{\rho,\tau \in \Sigma} \pi_x(\rho)\pi_x(\tau) m_\rho(U, m_\tau(V, W)).$$

Subtracting gives the claimed formula. If $\rho = \tau$, the corresponding mixed associator is the ordinary associator of m_τ, hence zero. □

Proposition 9.8 (Mixed compatibility is averaged associativity)**.** *For a probability law $\pi \in \mathcal{P}(\Sigma)$, the averaged product*

$$\bar{m}_\pi = \sum_{\sigma \in \Sigma} \pi(\sigma) m_\sigma$$

is associative on $\mathcal{A}$ if and only if π satisfies the mixed-signature averaged compatibility condition of Definition 9.6.

Proof. Apply Proposition 9.7 with $\pi_x = \pi$. The averaged product is associative precisely when

$$\mathrm{Assoc}_{\bar{m}_\pi}(U, V, W) = 0$$

for all $U, V, W \in \mathcal{A}$, and the expansion expresses this associator as exactly the weighted sum appearing in Definition 9.6. □

Remark 9.9 (Three equivalent compatibility languages)**.** For averaged products in the finite rank-three carrier, the same associativity question may be expressed in three equivalent languages:

mixed-signature compatibility,
finite blade cocycle identities,
signature moment factorization.

The mixed-signature formulation is closest to the pathwise/frozen multiplication tensors. The finite blade criterion is the direct algebraic test. The moment-factorization criterion is the probabilistic form of the same condition.

Theorem 9.10 (Finite associativity criterion for averaged products)**.** *Fix a probability law $\pi \in \mathcal{P}(\Sigma)$, and define*

$$\bar{m} = \sum_{\sigma \in \Sigma} \pi(\sigma) m_\sigma.$$

For $S, T \subseteq I$, write

$$\bar{m}(e_S, e_T) = \bar{\lambda}(S, T) e_{S \triangle T}.$$

Then $\bar{m}$ is associative on $\mathcal{A}$ if and only if

$$\bar{\lambda}(S, T)\bar{\lambda}(S \triangle T, R) = \bar{\lambda}(T, R)\bar{\lambda}(S, T \triangle R)$$

for all subsets $S, T, R \subseteq I$.

Proof. Since the blades e_S form a basis of $\mathcal{A}$, bilinearity and trilinearity reduce associativity to triples of basis blades. For such a triple,

$$\begin{aligned} \bar{m}(\bar{m}(e_S, e_T), e_R) &= \bar{\lambda}(S, T)\bar{\lambda}(S \triangle T, R) e_{S \triangle T \triangle R}, \\ \bar{m}(e_S, \bar{m}(e_T, e_R)) &= \bar{\lambda}(T, R)\bar{\lambda}(S, T \triangle R) e_{S \triangle T \triangle R}. \end{aligned}$$

The two sides agree for all basis triples if and only if the displayed scalar identities hold for all S, T, R. This is equivalent to associativity on all of $\mathcal{A}$. □

Theorem 9.11 (Moment-factorization classification of averaged associativity)**.** *Let $\pi \in \mathcal{P}(\Sigma)$, and define signature moments by*

$$M(A) := \sum_{\sigma \in \Sigma} \pi(\sigma) \prod_{a \in A} \varepsilon_a(\sigma), \qquad A \subseteq I.$$

Let

$$\mu_i = M(\{i\}), \quad \mu_j = M(\{j\}), \quad \mu_k = M(\{k\}),$$

and define μ_{ij}, μ_{ik}, μ_{jk}, and μ_{ijk} analogously. Then the averaged product

$$\bar{m} = \sum_{\sigma \in \Sigma} \pi(\sigma) m_\sigma$$

is associative on $\mathcal{A}$ if and only if

$$\mu_{ij} = \mu_i \mu_j, \qquad \mu_{ik} = \mu_i \mu_k, \qquad \mu_{jk} = \mu_j \mu_k,$$

and

$$\mu_{ijk} = \mu_i \mu_j \mu_k.$$

Equivalently, the three coordinate signs $\varepsilon_i, \varepsilon_j, \varepsilon_k$ are mutually independent under π.

Proof. For basis blades,

$$\bar{\lambda}(S, T) = (-1)^{N(S,T)} M(S \cap T).$$

The frozen anticommutation sign $(-1)^{N(S,T)}$ already satisfies the cocycle identity from Proposition 2.9. Therefore Theorem 9.10 reduces associativity of $\bar{m}$ to the moment identities

$$M(S \cap T) M((S \triangle T) \cap R) = M(T \cap R) M(S \cap (T \triangle R))$$

for all $S, T, R \subseteq I$. Choosing suitable triples gives

$$M(\{a, b\}) = M(\{a\}) M(\{b\})$$

for every distinct $a, b \in I$, and

$$M(I) = M(\{i\})M(\{j\})M(\{k\}).$$

Conversely, if these displayed factorization identities hold, then every moment $M(A)$ factors as $\prod_{a\in A} M(\{a\})$. Substitution into the preceding moment criterion makes both sides equal.

For $\{\pm 1\}^3$-valued signs, the joint law is determined by its Walsh–Fourier moments:

$$\pi(\eta_i, \eta_j, \eta_k) = \frac{1}{8} \sum_{A\subseteq I} \left(\prod_{a\in A} \eta_a \right) M(A).$$

Thus full moment factorization is equivalent to factorization of the joint law into the three one-coordinate laws, namely mutual independence. $\square$

Corollary 9.12 (Mixed obstruction criterion)**.** *If there exist $U, V, W \in \mathcal{A}$ such that*

$$\sum_{\substack{\rho,\tau\in\Sigma \\ \rho\neq\tau}} \pi_x(\rho)\pi_x(\tau) \operatorname{Assoc}^{\mathrm{mix}}_{\rho,\tau}(U, V, W) \neq 0,$$

then $\bar{m}_x$ is non-associative.

Example 9.13 (Moment-factorization obstruction)**.** Let π be a probability law on Σ, and define

$$\mu_i = \sum_{\tau\in\Sigma} \pi(\tau)\varepsilon_i(\tau), \qquad \mu_j = \sum_{\tau\in\Sigma} \pi(\tau)\varepsilon_j(\tau), \qquad \mu_{ij} = \sum_{\tau\in\Sigma} \pi(\tau)\varepsilon_i(\tau)\varepsilon_j(\tau).$$

For $\bar{m} = \sum_\tau \pi(\tau) m_\tau$, one has

$$\bar{m}(\bar{m}(ij, i), j) = -\mu_i\mu_j, \qquad \bar{m}(ij, \bar{m}(i, j)) = -\mu_{ij}.$$

Therefore

$$\operatorname{Assoc}_{\bar{m}}(ij, i, j) = \mu_{ij} - \mu_i\mu_j.$$

Hence $\bar{m}$ is non-associative whenever $\mu_{ij} \neq \mu_i\mu_j$.

Example 9.14 (Concrete non-Dirac non-associative law)**.** Let π assign probability 1/2 to $(\varepsilon_i, \varepsilon_j, \varepsilon_k) = (+1, +1, +1)$ and probability 1/2 to $(\varepsilon_i, \varepsilon_j, \varepsilon_k) = (-1, -1, +1)$. Then

$$\mu_i = 0, \qquad \mu_j = 0, \qquad \mu_{ij} = 1.$$

Consequently,

$$\operatorname{Assoc}_{\bar{m}}(ij, i, j) = 1 \neq 0.$$

This non-Dirac averaged multiplication is not associative, even though each frozen multiplication tensor in its support is associative.

Remark 9.15 (Projection and averaging are distinct effective operations)**.** Projection and averaging are different ways of leaving the exact frozen algebra. Projection fixes σ and discards grades. Averaging keeps the full carrier but combines several frozen products using a probability law. At the level of binary multiplication tensors, the fixed projection P_r commutes with finite averaging because it is linear and independent of τ. This linear commutation does not imply that the corresponding projected, averaged, or projected-averaged products share associativity properties, since associativity is an iterated quadratic identity in the product tensor. This paper therefore keeps the constructions formally separate.

Definition 9.16 (Projected averaged product)**.** Fix a law $\pi_x \in \mathcal{P}(\Sigma)$ and let

$$\bar{m}_x = \sum_{\tau \in \Sigma} \pi_x(\tau) m_\tau.$$

For $0 \le r \le 3$, define the projected averaged product

$$U \bar{\star}_{x,\le r} V := P_r(\bar{m}_x(U,V)), \qquad U, V \in \mathcal{A}_{\le r}.$$

This is a closed binary operation on $\mathcal{A}_{\le r}$, but it is neither a frozen Clifford product nor automatically associative.

Lemma 9.17 (Projected associator for an arbitrary product)**.** *Let $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be any bilinear product, not necessarily associative. Define*

$$U \star_{\mu,\le r} V := P_r(\mu(U,V)), \qquad U, V \in \mathcal{A}_{\le r}.$$

Then

$$\begin{aligned} \mathrm{Assoc}_{\star_{\mu,\le r}}(U,V,W) &= P_r\Big(\mu(P_r\mu(U,V),W) - \mu(U, P_r\mu(V,W))\Big) \\ &= P_r\Big(\mathrm{Assoc}_\mu(U,V,W) - \mu(Q_r\mu(U,V),W) + \mu(U, Q_r\mu(V,W))\Big), \end{aligned}$$

where

$$\mathrm{Assoc}_\mu(U,V,W) := \mu(\mu(U,V),W) - \mu(U,\mu(V,W)).$$

Proof. Use $P_r = I - Q_r$. Thus

$$P_r\mu(U,V) = \mu(U,V) - Q_r\mu(U,V),$$

and similarly for $\mu(V,W)$. Substitute these identities into the projected associator and expand by bilinearity. □

Proposition 9.18 (Associator of projected averaged products)**.** *For*

$$\bar{\star}_{x,\le r} = P_r \circ \bar{m}_x,$$

one has

$$\mathrm{Assoc}_{\bar{\star}_{x,\le r}}(U,V,W) = P_r\Big(\mathrm{Assoc}_{\bar{m}_x}(U,V,W) - \bar{m}_x(Q_r\bar{m}_x(U,V),W) + \bar{m}_x(U, Q_r\bar{m}_x(V,W))\Big).$$

Thus projected averaged products have two simultaneous obstruction sources: the averaged associator $\mathrm{Assoc}_{\bar{m}_x}$ *and the projection discard terms* $Q_r\bar{m}_x$*.*

Proof. Apply Lemma 9.17 with $\mu = \bar{m}_x$. □

Theorem 9.19 (Finite associativity criterion for projected averaged products)**.** *Fix $0 \le r \le 3$, a law $\pi \in \mathcal{P}(\Sigma)$, and*

$$\bar{m} = \sum_{\sigma \in \Sigma} \pi(\sigma) m_\sigma.$$

Write

$$\bar{m}(e_S, e_T) = \bar{\lambda}(S,T) e_{S \triangle T}.$$

The projected averaged product

$$e_S \bar{\star}_{\le r} e_T := P_r \bar{m}(e_S, e_T)$$

is associative on $\mathcal{A}_{\le r}$ if and only if, for every

$$S, T, R \subseteq I, \qquad |S|, |T|, |R| \le r,$$

with

$$|S \triangle T \triangle R| \le r,$$

one has

$$\mathbf{1}_{\{|S\triangle T|\le r\}} \bar{\lambda}(S,T)\bar{\lambda}(S\triangle T, R) = \mathbf{1}_{\{|T\triangle R|\le r\}} \bar{\lambda}(T,R)\bar{\lambda}(S, T\triangle R).$$

Proof. It is enough to check triples of basis blades in $\mathcal{A}_{\le r}$. For such blades,

$$e_S \bar{\star}_{\le r} e_T = \begin{cases} \bar{\lambda}(S,T) e_{S\triangle T}, & |S\triangle T| \le r, \\ 0, & |S\triangle T| > r. \end{cases}$$

Therefore

$$(e_S \bar{\star}_{\le r} e_T) \bar{\star}_{\le r} e_R = \mathbf{1}_{\{|S\triangle T|\le r\}} \mathbf{1}_{\{|S\triangle T\triangle R|\le r\}} \bar{\lambda}(S,T)\bar{\lambda}(S\triangle T, R) e_{S\triangle T\triangle R},$$

while

$$e_S \bar{\star}_{\le r} (e_T \bar{\star}_{\le r} e_R) = \mathbf{1}_{\{|T\triangle R|\le r\}} \mathbf{1}_{\{|S\triangle T\triangle R|\le r\}} \bar{\lambda}(T,R)\bar{\lambda}(S, T\triangle R) e_{S\triangle T\triangle R}.$$

If $|S\triangle T\triangle R| > r$, both sides vanish after the final projection. If $|S\triangle T\triangle R| \le r$, equality is exactly the displayed scalar identity. Since the projected associator is trilinear, the basis criterion is necessary and sufficient. □

Corollary 9.20 (Projection-level status of projected averaged products)**.** *For projected averaged products on $\mathcal{A}_{\le r}$, the following finite status table holds.*

r	*Product*	*Associativity status*
0	*scalar projected averaged product*	*associative on $\mathcal{A}_{\le 0}$*
1	*scalar-vector projected averaged product*	*finite criterion; generally non-associative*
2	*scalar-vector-bivector projected averaged product*	*finite criterion; generally non-associative*
3	*full averaged product $\bar{m}$*	*associative iff Theorem 9.11 holds*

Corollary 9.21 (Two-source obstruction for projected averaged products)**.** *The projected averaged product $\bar{\star}_{x,\le r}$ is associative on $\mathcal{A}_{\le r}$ only if the averaged associator and the projected discard terms cancel after applying P_r. In particular, associativity of all frozen products m_τ does not imply associativity of $\bar{\star}_{x,\le r}$.*

9.4 Pathwise convention for varying signatures

If a signature field is modeled as a map

$$\sigma : X \times \Omega \to \Sigma,$$

then exact multiplication is interpreted pointwise and pathwise:

$$(U \cdot_\sigma W)(x,\omega) := m_{\sigma(x,\omega)}(U(x,\omega), W(x,\omega)). \tag{145}$$

Thus for each fixed (x,ω), the algebraic product is the frozen product associated with the realized signature $\sigma(x,\omega)$. This convention is inherited from VPSCF1 [1] and is used throughout VPSCF2.

Convention 9.22 (Pathwise products are algebraic unless regularity is declared)**.** In VPSCF2, the expression

$$(U \cdot_\sigma W)(x, \omega) = U(x, \omega) \cdot_{\sigma(x,\omega)} W(x, \omega)$$

is a pointwise algebraic definition. Measurability, integrability, continuity, smoothness, or Sobolev regularity of the resulting field is not automatic unless the corresponding hypotheses are declared. In particular, regularity in the base variable x may fail when $\sigma(\cdot, \omega)$ jumps or is otherwise irregular. Nonlocal, transported, independently sampled, or kernel-coupled products are different structures and are not part of the algebraic closure theory of VPSCF2.

Convention 9.23 (Measurable structures)**.** Throughout the measurability statements in this paper, the finite signature set

$$\Sigma = \{\pm 1\}^3$$

is equipped with the discrete σ-algebra 2^Σ. The carrier

$$\mathcal{A} = \operatorname{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}$$

is identified with $\mathbb{R}^8$ through the ordered blade coordinates and equipped with the corresponding Euclidean Borel σ-algebra $\mathcal{B}(\mathcal{A})$. The finite-dimensional space $\operatorname{Bil}(\mathcal{A} \times \mathcal{A}, \mathcal{A})$ is likewise equipped with its Euclidean Borel structure after choosing the ordered blade basis.

Proposition 9.24 (Measurability of pathwise frozen products)**.** *Let $(X, \mathcal{B}_X)$ and $(\Omega, \mathcal{F})$ be measurable spaces, and equip $X \times \Omega$ with $\mathcal{B}_X \otimes \mathcal{F}$. Assume*

$$\sigma : X \times \Omega \to \Sigma$$

is measurable with respect to 2^Σ, and

$$U, W : X \times \Omega \to \mathcal{A}$$

are measurable with respect to $\mathcal{B}(\mathcal{A})$. Then the pathwise product

$$(U \cdot_\sigma W)(x, \omega) := U(x, \omega) \cdot_{\sigma(x,\omega)} W(x, \omega)$$

is $\mathcal{B}(\mathcal{A})$-measurable as a map $X \times \Omega \to \mathcal{A}$.

Proof. Since Σ is finite, the sets

$$E_\tau = \{(x, \omega) : \sigma(x, \omega) = \tau\}$$

are measurable and form a finite partition. On E_τ, the product is

$$m_\tau(U(x, \omega), W(x, \omega)).$$

Each $m_\tau : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is bilinear on a finite-dimensional Euclidean space, hence continuous and Borel measurable. Therefore each piece is measurable, and the finite pasted map is measurable. □

Proposition 9.25 (Averaged tensor measurability)**.** *Let $(X, \mathcal{B}_X)$ be a measurable space and let*

$$\pi : X \to \mathcal{P}(\Sigma)$$

be a measurable one-point kernel, meaning that every coordinate function $x \mapsto \pi_x(\tau)$ is measurable. Then

$$x \longmapsto \bar{m}_x := \sum_{\tau \in \Sigma} \pi_x(\tau) m_\tau$$

is a measurable map from X into $\operatorname{Bil}(\mathcal{A} \times \mathcal{A}, \mathcal{A})$*. If $U, W : X \to \mathcal{A}$ are $\mathcal{B}(\mathcal{A})$-measurable, then*

$$x \longmapsto \bar{m}_x(U(x), W(x))$$

is $\mathcal{B}(\mathcal{A})$-measurable.

Proof. The space $\mathrm{Bil}(\mathcal{A}\times\mathcal{A},\mathcal{A})$ is finite-dimensional after choosing the ordered blade basis. Each m_τ is a fixed tensor, and the coefficients $x\mapsto\pi_x(\tau)$ are measurable. Hence $x\mapsto\bar{m}_x$ is a finite measurable linear combination of fixed tensors. Evaluation

$$(T,U,W)\longmapsto T(U,W)$$

is continuous in finite dimensions, so $x\mapsto\bar{m}_x(U(x),W(x))$ is measurable. $\square$

The projected pathwise product is then

$$(U\star_{\sigma,\le r}W)(x,\omega):=\Pi_{\le r}\big(m_{\sigma(x,\omega)}(U(x,\omega),W(x,\omega))\big), \tag{146}$$

whenever the pointwise values lie in $\mathcal{A}_{\le r}$. This is still a pathwise construction, because the realized signature is fixed before multiplication and projection. Averaging, by contrast, replaces the realized signature with a law over signatures.

9.5 Coefficient-valued versions

The same taxonomy persists for coefficient-valued objects, but only after a coefficient regime has been chosen. Let V be a real vector space.

- In Regime A, where V is only a metric or topological coefficient space, there is no canonical product on $V\otimes\mathcal{A}$. One may form coefficient-valued fields, but not coefficient-valued products.
- In Regime B, where $(V,\cdot_V)$ is an associative algebra, one obtains the ordinary ungraded tensor product algebra

  $$(v\otimes A)\cdot_\sigma(w\otimes B)=(v\cdot_V w)\otimes m_\sigma(A,B).$$

  Projected versions are then obtained by applying $\mathrm{id}_V\otimes\Pi_{\le r}$ to the full product.
- In Regime C, where no internal multiplication on V is chosen, products live in the tensor-algebra hierarchy

  $$\mathcal{H}_\bullet=T(V)\otimes\mathcal{A},\qquad \mathcal{H}_n=V^{\otimes n}\otimes\mathcal{A}\ (n\ge 0).$$

  Projected versions are obtained by applying $\mathrm{id}_{V^{\otimes n}}\otimes\Pi_{\le r}$ at the appropriate tensor level.

Thus the projected/exact distinction and the coefficient-regime distinction are orthogonal. Projection answers the question “which grades are kept?” Coefficient semantics answer the question “how are the coefficients combined?”

9.6 Complete product taxonomy

The scalar product taxonomy is summarized in Table 1. The coefficient-valued versions require the additional choices described in Section 6.

The table should be read with three conventions:

(i) $\star_{\sigma,\le 3}=m_\sigma$ is not an additional product; it is the full frozen product expressed in projection notation.

(ii) $\bar{m}_x$ is included as an effective averaged operation inherited from VPSCF1; its associativity is governed by Theorem 9.11.

(iii) $\bar{\star}_{x,\le r}$ denotes the projected averaged product of Definition 9.16; its associator is governed by Proposition 9.18 and its finite basis criterion is Theorem 9.19.

Product	Domain	Codomain	Associative?	Status
$\star_{\sigma,\le 0}$	$\mathcal{A}_{\le 0} \times \mathcal{A}_{\le 0}$	$\mathcal{A}_{\le 0}$	yes	scalar exact sector
$\star_{\sigma,\le 1}$	$\mathcal{A}_{\le 1} \times \mathcal{A}_{\le 1}$	$\mathcal{A}_{\le 1}$	no, for every σ	projected effective product
$\star_{\sigma,\le 2}$	$\mathcal{A}_{\le 2} \times \mathcal{A}_{\le 2}$	$\mathcal{A}_{\le 2}$	no, for every σ	projected effective product
$\star_{\sigma,\le 3} = m_\sigma$	$\mathcal{A} \times \mathcal{A}$	$\mathcal{A}$	yes	exact full Clifford product
$\bar{m}_x$	$\mathcal{A} \times \mathcal{A}$	$\mathcal{A}$	law-dependent; iff moment factorization holds	averaged effective product inherited from VPSCF1
$\bar{\star}_{x,\le r}$	$\mathcal{A}_{\le r} \times \mathcal{A}_{\le r}$	$\mathcal{A}_{\le r}$	finite projection criterion	projected averaged effective product

Table 1: Product taxonomy for VPSCF2. The table separates exact frozen products, projected effective products, and averaged effective products.

9.7 Section conclusion

The product taxonomy established so far is not a linear hierarchy. It is a two-axis separation between signature semantics and grade semantics:

	full-grade output	projected output
frozen signature	m_σ	$P_r \circ m_\sigma$
averaged signature law	$\bar{m}_x = \sum_{\tau\in\Sigma} \pi_x(\tau) m_\tau$	$P_r \circ \bar{m}_x = \sum_{\tau\in\Sigma} \pi_x(\tau)(P_r \circ m_\tau)$.

The equality in the lower-right entry holds at the level of binary multiplication tensors because P_r is linear and independent of the signature. It does not imply that projected averaged products inherit associativity, because associativity is an iterated quadratic identity in the multiplication tensor.

Thus VPSCF2 keeps the following distinctions separate:

frozen versus averaged signature semantics,
full-grade versus projected output,
scalar versus coefficient-valued carrier.

Only m_σ is the exact frozen Clifford product. Projected products are closed effective products. Averaged products are statistical or effective products. Projected averaged products combine both effective operations and require their own associator analysis. The next section applies this taxonomy to the motivating object $a + bi + cj + dk$.

10 Exact Status of $a + bi + cj + dk$

The preceding sections separate three questions that are often collapsed into one notation. The expression

$$a + bi + cj + dk \tag{147}$$

may denote a scalar-vector element, a coefficient-valued scalar-vector element, an element embedded into the full carrier, or an input to a projected effective product. These interpretations are related, but they are not interchangeable. This section records the exact semantic status of (147). Its purpose is partly technical and partly diagnostic: it prevents later sections and later papers from treating a scalar-vector notation as if it already carried every algebraic, analytic, or energetic structure one may eventually want.

10.1 The semantic ladder

There are four layers in the interpretation of (147).

(i) **Scalar-vector layer.** If

$$a, b, c, d \in \mathbb{R},$$

then

$$U = a + bi + cj + dk \in \mathcal{A}_{\leq 1}.$$

This is the scalar-vector sector of the fixed blade carrier.

(ii) **Full-carrier embedding layer.** Since

$$\mathcal{A}_{\leq 1} \subset \mathcal{A},$$

the same element may be regarded as a full Clifford-carrier element with zero bivector and trivector components.

(iii) **Coefficient-valued layer.** If

$$a, b, c, d \in V,$$

then the correct object is

$$U = a \otimes 1 + b \otimes i + c \otimes j + d \otimes k \in V \otimes \mathcal{A}_{\leq 1}. \tag{148}$$

Multiplication is not automatically available. One must choose one of the coefficient regimes of Section 6.

(iv) **Projected effective layer.** If one wants to multiply scalar-vector objects but remain in a truncated grade sector, one must use a projected product

$$U \star_{\sigma,\leq r} W = \Pi_{\leq r}(U \cdot_\sigma W).$$

This operation is effective rather than exact unless $r = 3$, and for $r = 1, 2$ it is non-associative for every frozen signature on the full truncated domain.

The key point is that the same visual expression may sit at different semantic levels. VPSCF2 fixes these levels explicitly.

10.2 Scalar-vector status

In the scalar setting, the expression

$$U = a + bi + cj + dk, \qquad a, b, c, d \in \mathbb{R},$$

is a well-defined element of $\mathcal{A}_{\leq 1}$. It is not, however, an element of a closed scalar-vector algebra under frozen Clifford multiplication.

Proposition 10.1 (Scalar-vector status)**.** *For scalar coefficients, $a+bi+cj+dk$ is an element of $\mathcal{A}_{\leq 1}$ and an embedded element of $\mathcal{A}$. The exact frozen product of two such elements is computed in $\mathcal{A}$, and generally lies in $\mathcal{A}_{\leq 2}$, not in $\mathcal{A}_{\leq 1}$.*

Proof. The inclusion $\mathcal{A}_{\leq 1} \subset \mathcal{A}$ follows from the grade decomposition of Section 2. Section 3 shows that the product of two scalar-vector elements has scalar, vector, and bivector parts. The bivector part is generally nonzero, so the product generally leaves $\mathcal{A}_{\leq 1}$ while remaining in $\mathcal{A}_{\leq 2} \subset \mathcal{A}$. □

Thus the scalar notation is valid, but its multiplicative closure is not the scalar-vector sector. The closed exact carrier is the full blade space $\mathcal{A}$.

10.3 Full-carrier status

A scalar-vector element may always be written as a special full-grade element by assigning zero higher-grade coordinates:

$$a + bi + cj + dk = a + bi + cj + dk + 0\,ij + 0\,ik + 0\,jk + 0\,ijk. \tag{149}$$

This embedding is harmless as a vector-space embedding. It becomes essential for exact multiplication, because the product may create nonzero higher-grade coordinates even if the inputs have none.

Corollary 10.2 (Full carrier as exact multiplicative habitat)**.** *The expression $a + bi + cj + dk$ can be multiplied exactly only by regarding it as an element of the full carrier $\mathcal{A}$. The exact product is the frozen product m_σ, not a product intrinsic to $\mathcal{A}_{\leq 1}$.*

Proof. This is a direct consequence of Proposition 10.1 and the full closure result of Section 4. □

This is the cleanest interpretation of the scalar-vector notation: it is a convenient coordinate expression for a low-grade input inside a larger exact algebra.

10.4 Coefficient-valued status

When the coefficients are not scalars but elements of a vector space V, the expression must be rewritten in tensor notation:

$$a + bi + cj + dk \quad \rightsquigarrow \quad a \otimes 1 + b \otimes i + c \otimes j + d \otimes k.$$

This rewriting is not cosmetic. It records that the blade part and the coefficient part are distinct pieces of data.

Proposition 10.3 (Coefficient semantic dependence)**.** *Let V be a real vector space and let*

$$U = a \otimes 1 + b \otimes i + c \otimes j + d \otimes k \in V \otimes \mathcal{A}_{\leq 1}.$$

Then U is always a well-defined coefficient-valued scalar-vector object. However, the product of two such objects is defined only after one specifies coefficient semantics.

Proof. The tensor expression is well-defined using only the vector-space structure of V. To multiply two such expressions, one must combine coefficient factors such as b with γ, or more generally u_S with v_T. A bare vector space does not specify such a combination. Section 6 lists the three regimes: no canonical product in the pure metric/topological case, ordinary ungraded tensor product multiplication when V is an associative algebra, and tensor-hierarchy bookkeeping when no internal coefficient multiplication is chosen. □

In particular, the scalar product expansion of Section 3 should not be copied into the coefficient-valued setting without first choosing a coefficient regime.

10.5 Projected level-one status

There is a legitimate way to multiply scalar-vector objects and return to the scalar-vector sector: project after multiplying in the full carrier. For scalar inputs,

$$U, W \in \mathcal{A}_{\leq 1},$$

the projected scalar-vector product is

$$U \star_{\sigma,\leq 1} W = \Pi_{\leq 1}(U \cdot_\sigma W). \tag{150}$$

This operation is closed on $\mathcal{A}_{\leq 1}$, but it is not the exact frozen Clifford product. It is obtained by first computing the exact product and then deleting the discarded grades.

Proposition 10.4 (Projected status)**.** *The operation $\star_{\sigma,\leq 1}$ makes $\mathcal{A}_{\leq 1}$ into a closed non-associative projected product space in general. It does not make $\mathcal{A}_{\leq 1}$ into a frozen Clifford algebra.*

Proof. Closure follows from the definition of $\Pi_{\leq 1}$. Non-associativity in general follows from the associator-defect theorem and the explicit counterexample in Section 8. Since the product is obtained by projection after full multiplication, it is not the original frozen product m_σ restricted to a closed subalgebra. □

Thus projected closure should be interpreted as effective truncation, not as exact algebraic closure.

10.6 Correct and incorrect readings

The status of $a + bi + cj + dk$ can now be summarized without ambiguity.

10.7 Semantic firewall for later papers

The classification above is deliberately restrictive. It prevents five common errors:

(1) treating $\mathcal{A}_{\leq 1}$ as if it were a closed Clifford algebra;

Reading	Status
$a + bi + cj + dk \in \mathcal{A}_{\leq 1}$ for scalar coefficients	Correct: it is a scalar-vector element.
$a + bi + cj + dk \in \mathcal{A}$ by embedding	Correct: it is a full-carrier element with zero higher-grade coordinates.
$a + bi + cj + dk$ generates exact products inside $\mathcal{A}$	Correct: exact frozen multiplication is performed in the full carrier.
$a + bi + cj + dk$ belongs to a closed scalar-vector Clifford algebra	Incorrect: $\mathcal{A}_{\leq 1}$ is not closed under m_σ.
$\star_{\sigma,\leq 1}$ is the exact Clifford product	Incorrect: it is a projected effective product.
$\star_{\sigma,\leq 1}$ is associative	Incorrect: projected level-one multiplication is non-associative for every frozen signature.
$a, b, c, d \in V$ can always be multiplied as coefficients	Incorrect: coefficient multiplication requires a chosen regime.
$a + bi + cj + dk$ is already a PDE field or signed-energy object	Incorrect: analytic and signed-form structures belong to later layers.

Table 2: Correct and incorrect readings of the scalar-vector notation.

(2) using projected products as if they were exact frozen products;

(3) multiplying V-coefficients without specifying a product, contraction, or tensor-hierarchy convention;

(4) importing PDE regularity before an analytic field space has been declared;

(5) assigning signed energy before a signed quadratic form has been defined.

This is the semantic firewall supplied by VPSCF2. It does not prevent later papers from defining signed forms, quotient reductions, differential operators, or interface spaces. It only requires that each such construction be attached to a clearly specified carrier, coefficient regime, and product convention.

10.8 Section conclusion

The final status is therefore:

$$\boxed{a + bi + cj + dk \text{ is a scalar-vector element embedded in the full Clifford carrier.}}$$

Exact multiplication takes place in $\mathcal{A}$. Projected multiplication can return to $\mathcal{A}_{\leq 1}$, but only as an effective operation with possible associator defects. If the coefficients lie in a general space V, the expression is only a coefficient-valued carrier until coefficient semantics are specified. This conclusion completes the algebraic clarification needed before the series moves to signed quadratic forms.

11 Interface Contract for VPSCF3

The preceding sections complete the algebraic and coefficient-space tasks assigned to VPSCF2. VPSCF1 [1] supplied the fixed blade carrier, the frozen-signature multiplication tensors, the signature-field state space, and the pathwise-versus-averaged distinction. VPSCF2 has used those ingredients to solve the next structural problem: the scalar-vector expression

$$a + bi + cj + dk$$

is not a closed algebraic universe, and any coefficient-valued version of it requires explicit coefficient semantics. The purpose of this final section is to state precisely what has now been fixed and what may safely be developed in VPSCF3.

Remark 11.1 (No signed-form theory is developed in VPSCF2)**.** This section is an interface contract, not a new signed quadratic-form theory. VPSCF2 does not prove positivity, nondegeneracy, coercivity, conservation, energy estimates, or classification results for signed forms. It only records which additional semantic choices VPSCF3 must declare before such expressions are well typed. None of the closure, coefficient-regime, or projection results of VPSCF2 depends on the optional pairing or involution conventions listed below.

11.1 What VPSCF2 has established

The first established point is the separation between the scalar-vector sector and the full carrier. The sector

$$\mathcal{A}_{\leq 1} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$$

is a linear scalar-vector sector, not a closed Clifford algebra. The full frozen-signature carrier is

$$\mathcal{A} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k, ij, ik, jk, ijk\}.$$

For each frozen signature $\sigma \in \Sigma$, exact multiplication is performed by

$$m_\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

Thus the exact algebraic object is not $(\mathcal{A}_{\leq 1}, m_\sigma)$, which is not defined as a closed algebra, but rather the full frozen algebra $(\mathcal{A}, m_\sigma)$.

The second established point is minimality. Inside the fixed blade carrier, the scalar-vector sector generates all blades under frozen multiplication:

$$\mathrm{BladeAlg}_{m_\sigma}\langle \mathcal{A}_{\leq 1} \rangle = \mathcal{A}.$$

This closure is signature-independent at the level of the generated blade set. The signs $\varepsilon_i, \varepsilon_j, \varepsilon_k$ affect multiplication coefficients and contractions, but they do not change the fact that closure of $1, i, j, k$ produces

$$ij, \qquad ik, \qquad jk, \qquad ijk.$$

This is the full-closure reason why later constructions must not treat the scalar-vector sector as a self-contained exact algebra.

The third established point is coefficient semantics. A coefficient-valued object

$$U = \sum_{S \subseteq I} u_S \otimes e_S \in V \otimes \mathcal{A}$$

is only a coefficient carrier until additional structure is supplied on V. Three regimes have been isolated:

(i) pure metric or topological coefficient spaces, where norms, topologies, or pairings may exist but multiplication is not canonically determined;

(ii) associative coefficient algebras, where an ordinary ungraded tensor product algebra can be formed by

$$(v \otimes A)(w \otimes B) = (v \cdot_V w) \otimes m_\sigma(A, B);$$

(iii) tensor-hierarchy semantics, where product-free coefficients are kept by the tensor-algebra hierarchy

$$\mathcal{H}_\bullet = T(V) \otimes \mathcal{A}, \qquad \mathcal{H}_n = V^{\otimes n} \otimes \mathcal{A} \ (n \geq 0).$$

No later paper should multiply V-coefficients unless it explicitly declares which of these, or which additional coefficient semantics, is being used.

The fourth established point is projection. For

$$P_r = \Pi_{\leq r}, \qquad Q_r = I - P_r,$$

the truncated projected product is

$$U \star_{\sigma,\leq r} W = P_r(U \cdot_\sigma W).$$

The cases $r = 1$ and $r = 2$ are useful effective closures, but they are non-associative for every frozen signature on the full truncated domains. Their associator is governed by the exact formula

$$\mathrm{Assoc}_{\sigma,\leq r}(U, V, W) = P_r\left(-Q_r(U \cdot_\sigma V) \cdot_\sigma W + U \cdot_\sigma Q_r(V \cdot_\sigma W)\right).$$

This identity is the formal obstruction to treating projected truncations as exact Clifford algebras.

11.2 Data that VPSCF3 must declare

The passage from closure theory to signed quadratic-form theory requires more than a carrier and a product. A signed quadratic layer must declare four kinds of data:

(i) a carrier, such as $\mathcal{A}$, $\mathcal{A}_{\leq r}$, $V \otimes \mathcal{A}$, or a projected/effective carrier;

(ii) a coefficient regime, such as scalar coefficients, declared Hilbert or Banach data, Regime-B coefficient multiplication, or the tensor-algebra hierarchy $T(V) \otimes \mathcal{A}$;

(iii) a product semantics, such as exact frozen multiplication m_σ, projected multiplication $\star_{\sigma,\leq r}$, averaged multiplication $\bar{m}_x$, or coefficient-valued projected multiplication $\star^V_{\sigma,\leq r}$;

(iv) a duality, scalar-extraction, involution, or pairing semantics, such as scalar projection, a grade-wise pairing, reversion, grade involution, Clifford conjugation, or a declared coefficient pairing.

Without the fourth datum, expressions such as

$$Q(U) = \langle U, U \rangle, \qquad Q(U) = \mathrm{Sc}(U^\dagger \cdot_\sigma U), \qquad Q(U) = \mathrm{Sc}(U \cdot_\sigma U)$$

are not yet mathematically typed as real-valued quadratic forms. A product on $\mathcal{A}$ generally returns an element of $\mathcal{A}$, not a scalar. The list above is diagnostic rather than constructive: it specifies the type data VPSCF3 must choose, but it does not choose a signed form in VPSCF2.

Definition 11.2 (Standard blade involutions and scalar extraction)**.** Let

$$\mathrm{Sc} : \mathcal{A} \to \mathbb{R}$$

denote projection onto the scalar blade 1. For an ordered blade

$$e_S = e_{a_1} \cdots e_{a_q}, \qquad S = \{a_1 < \cdots < a_q\} \subseteq I,$$

define

$$\alpha(e_S) = (-1)^q e_S,$$

$$\mathrm{rev}(e_S) = (-1)^{q(q-1)/2} e_S,$$

and

$$\overline{e_S} := \alpha(\mathrm{rev}(e_S)) = (-1)^{q(q+1)/2} e_S.$$

Extend these maps linearly to $\mathcal{A}$. Here α is the grade involution, rev is reversion, and $\overline{(\cdot)}$ is Clifford conjugation.

These formulas describe the action on the ordered blade basis. They do not depend on the frozen square signs $\varepsilon_i, \varepsilon_j, \varepsilon_k$; those signs enter the multiplication tensor m_σ, not the basis-level sign pattern of the standard involutions. These are the standard Clifford involution conventions [2, 8].

Proposition 11.3 (Compatibility with frozen multiplication)**.** *For each frozen signature σ, the grade involution is an algebra automorphism of $(\mathcal{A}, m_\sigma)$, while reversion is an anti-automorphism:*

$$\alpha(m_\sigma(U, V)) = m_\sigma(\alpha(U), \alpha(V)),$$

$$\mathrm{rev}(m_\sigma(U, V)) = m_\sigma(\mathrm{rev}(V), \mathrm{rev}(U)).$$

Consequently Clifford conjugation is an anti-automorphism.

Proof. It is enough to check the identities on the generators i, j, k and extend by bilinearity and multiplicativity. The grade involution sends each generator to its negative and therefore preserves the defining relations $e_a^2 = \varepsilon_a$ and $e_a e_b = -e_b e_a$ for $a \neq b$. Reversion fixes each generator and reverses products, so it also preserves the defining square relations while reversing order. The displayed identities follow on the ordered blade basis and hence on all of $\mathcal{A}$. □

Definition 11.4 (Grade-wise and coefficient pairings)**.** A grade-wise scalar pairing is a declared bilinear form

$$\langle \cdot, \cdot \rangle_q : \mathcal{A}^{(q)} \times \mathcal{A}^{(q)} \to \mathbb{R}.$$

In coefficient-valued settings one must also declare a coefficient pairing or bilinear form, for example

$$\langle \cdot, \cdot \rangle_{V,q} : (V \otimes \mathcal{A}^{(q)}) \times (V \otimes \mathcal{A}^{(q)}) \to \mathbb{R}.$$

Such pairings are independent semantic data; they are not determined by the existence of the frozen product m_σ or by the vector-space structure of $V \otimes \mathcal{A}$.

Remark 11.5 (Quadratic forms are not products alone)**.** The square $U \cdot_\sigma U$ is generally an $\mathcal{A}$-valued object. To obtain a real-valued signed quadratic form, one must additionally specify how a scalar is extracted or paired: by Sc, by an involution followed by scalar projection, by a trace-like functional, by a grade-wise pairing, or by a coefficient pairing. This is the extra semantic datum that VPSCF3 must declare before writing energy-like or norm-like quantities.

11.3 What VPSCF3 may now define

The next natural layer is signed quadratic-form theory. Such a layer requires, at minimum, four pieces of structure that VPSCF2 has now separated:

(1) a carrier, such as $\mathcal{A}_{\leq 1}$, $\mathcal{A}_{\leq 2}$, $\mathcal{A}$, $V \otimes \mathcal{A}$, or $T(V) \otimes \mathcal{A}$;

(2) a coefficient regime, such as scalar coefficients, Hilbert coefficients with a declared pairing, an associative coefficient algebra, or the tensor-algebra hierarchy;

(3) a product or projection convention, such as exact frozen multiplication m_σ, projected multiplication $\star_{\sigma,\leq r}$, averaged multiplication $\bar{m}_x$, or coefficient-valued projected multiplication $\star^V_{\sigma,\leq r}$;

(4) a scalar-extraction, involution, grade-wise pairing, trace-like functional, or coefficient pairing that turns the chosen carrier data into a real-valued signed quantity.

Without these choices, a signed form cannot be unambiguously interpreted or even correctly typed. For example, a formal expression measuring the size of

$$U = a + bi + cj + dk$$

requires knowing whether a, b, c, d are scalars, Hilbert-space vectors, algebra elements, fields over a base set, or tensor-hierarchy factors. It also requires knowing whether the intended object is scalar-vector, full-grade, or projected from a higher-grade product.

Thus VPSCF3 may define signed quadratic forms only after declaring the relevant carrier, coefficient regime, product/effective-product semantics, and scalar-extraction or pairing semantics. In the scalar-vector setting, the signs $\varepsilon_i, \varepsilon_j, \varepsilon_k$ provide the first frozen-signature weights, but the rule turning coordinates into a scalar quadratic value must still be stated. In coefficient-valued settings, a pairing or bilinear form on V must also be specified before one can write a meaningful signed quantity. In full-grade settings, the bivector and trivector coordinates must be accounted for by a declared grade-wise pairing, involution-plus-scalar-projection rule, or equivalent scalar-extraction convention rather than silently discarded.

VPSCF2 does not develop those forms. It supplies the algebraic and semantic infrastructure that makes them well-defined.

11.4 Constraints inherited by VPSCF3

VPSCF3 inherits several constraints from the present paper.

First, if VPSCF3 works on $\mathcal{A}_{\leq 1}$, it must remember that this is a linear sector, not an exact algebra. Any multiplication staying inside $\mathcal{A}_{\leq 1}$ must either be absent, projected, or replaced by another declared operation. If products are used, the paper must distinguish between

$$U \cdot_\sigma W \in \mathcal{A}$$

and

$$U \star_{\sigma,\leq 1} W \in \mathcal{A}_{\leq 1}.$$

The former is exact and full-grade; the latter is projected and non-associative on the full scalar-vector domain for every frozen signature.

Second, if VPSCF3 uses coefficient-valued objects, it must not rely on implicit coefficient multiplication. For Hilbert or Banach coefficients, a norm or inner product may support quadratic quantities, but it does not by itself define products of coefficients. If V is an associative algebra, the ordinary ungraded tensor product convention must be stated. If V has no multiplication, tensor-hierarchy semantics must be used or another bilinear operation must be introduced explicitly.

Third, if VPSCF3 uses projected operations, associator defects are not optional technicalities. They are structurally forced by discarded grades. Any signed form or energy functional coupled to $\star_{\sigma,\leq r}$ must state whether it is insensitive to those defects, controls them, or treats them as part of the effective theory.

Fourth, if VPSCF3 averages over signatures, it must preserve the VPSCF1 distinction between exact pathwise frozen multiplication and averaged effective multiplication [1]. Averages over σ produce effective operations; they do not replace the exact frozen product m_σ unless explicitly declared as a separate averaged theory.

Fifth, if VPSCF3 writes a quadratic or energy-like expression, it must declare the scalar-extraction or pairing rule. Expressions such as $\mathrm{Sc}(U \cdot_\sigma U)$, $\mathrm{Sc}(\overline{U} \cdot_\sigma U)$, and $\langle U, U \rangle$ are different constructions unless the relevant involution, scalar projection, trace, grade-wise pairing, or coefficient pairing has been fixed explicitly.

11.5 Recommended starting point for VPSCF3

The cleanest starting point for VPSCF3 is the scalar or Hilbert-coefficient scalar-vector sector with no multiplication assumed inside the sector. One may then define signed quadratic forms on $\mathcal{A}_{\leq 1}$ using the frozen signs and the declared coefficient pairing. This keeps the first signed-form paper close to the motivating expression while avoiding the false claim that $\mathcal{A}_{\leq 1}$ is closed under exact multiplication.

A second, more complete option is to define full-grade signed forms on $\mathcal{A}$. This has the advantage that the carrier is closed under exact multiplication, but it must specify how scalar, vector, bivector, and trivector coordinates contribute to the signed form. Such a formulation is more faithful to full Clifford closure, but it is also heavier and should be introduced only after the scalar-vector form has been made transparent.

A third option is to compare level-one projected forms with full-grade forms. This would allow VPSCF3 to measure the loss caused by projection. The natural quantity to track is not merely the projected value, but also the discarded component

$$Q_r(U \cdot_\sigma V).$$

This would connect signed-form theory directly to the projection-defect mechanism established in Section 8.

For a coherent series, the recommended order is therefore:

$$\boxed{\text{scalar-vector signed form} \longrightarrow \text{full-grade signed form} \longrightarrow \text{projection-loss comparison.}}$$

This sequence keeps VPSCF3 close to the original scalar-vector notation while preparing later papers for quotient reductions, PDE operators, and interface analysis.

11.6 Closing summary

The conclusion of VPSCF2 can be compressed into four rules:

(1) $\mathcal{A}_{\leq 1}$ is useful but not closed under exact frozen Clifford multiplication.

(2) $\mathcal{A}$ is the full exact carrier for frozen-signature algebra.

(3) $V \otimes \mathcal{A}$ requires explicit coefficient semantics before multiplication can be discussed.

(4) Projected products create effective closure only by discarding grades, and the discarded grades generate associator defects.

These rules close the algebraic carrier problem for the second paper of the VPSCF sequence. They do not solve the signed-form problem, the quotient-reduction problem, the differential-operator problem, or the interface problem. Instead, they make those later problems well-posed by fixing the carrier, coefficient, product/projection, and pairing-or-scalar-extraction semantics on which they must depend.

Thus the transition to VPSCF3 is now mathematically clean:

$$\boxed{\begin{array}{c}\text{after full closure, coefficient semantics, and declared scalar-extraction or pairing}\\ \text{semantics, signed quadratic forms can be defined without ambiguity.}\end{array}}$$

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