Stability-Induced Discreteness from the Second Variation of Action

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A common route to discreteness in physics is to postulate a Hilbert-space operator and then solve its eigenvalue problem. Here a different, stability-based route is formulated. The starting point is a variational stability principle: a physically realized stationary state is required not only to satisfy the stationarity condition $\delta S = 0$, but also to be stable with respect to the second variation, $\delta^2 S \geq 0$, understood as a positive stability form. Under standard assumptions on the second variation — symmetry, closedness, lower semiboundedness, and coercivity after a shift — this form defines a self-adjoint stability operator. If the physical boundary conditions make the relevant embedding compact, the stability operator has compact resolvent and therefore a discrete spectrum. In this sense, discreteness is not introduced as an independent quantum postulate; it arises as a spectral consequence of the second variation together with stability and boundary conditions. The result is stated as a theorem and proved using the representation theorem for closed semibounded forms and compactness of the Sobolev embedding. The physical meaning is clarified by distinguishing discrete stability modes from the continuous parameters of the underlying classical system. Periodic classical systems naturally give rise to discrete stability spectra through the second variation and compact boundary conditions.

I. INTRODUCTION

Discreteness appears in many areas of physics. In quantum mechanics it is usually obtained from an operator eigenvalue problem, for example

$$\hat{H}\psi_n = E_n\psi_n. \quad (1)$$

This formulation is extremely successful, but it begins with an already established operator structure. The question addressed here is more elementary: can a discrete spectrum of stationary states or stable perturbations arise before the quantum postulates, from a variational condition of stability?

The guiding idea is that stationarity alone is not sufficient for physical realization. A stationary trajectory or field configuration can be unstable. Therefore a selection rule is needed. The stability principle considered in this paper is

$$\delta S = 0, \qquad \delta^2 S \geq 0. \quad (2)$$

The first condition gives the usual Euler-Lagrange or Hamilton equations [1–3]. The second condition says that the stationary state must be a local minimum of a suitable stability functional. The notation S is used in a broad variational sense: it may denote an action, a generalized action, a thermodynamic potential, an entropy-related functional with a chosen sign convention, or a Lyapunov functional [4, 5]. If S is literally thermodynamic entropy of an isolated system, the usual maximum-entropy convention would give the opposite sign for the entropy itself. Here the sign is chosen so that stable states are minima of the stability functional.

The main result is that the second variation defines a quadratic stability form. Under standard analytic assumptions this form determines a self-adjoint operator [6–8]. When the relevant physical boundary conditions give compactness, the stability operator has a discrete spectrum. Thus the mechanism is

$$\delta S = 0 \Rightarrow \text{stationarity}, \qquad \delta^2 S \geq 0 \Rightarrow \hat{L}_{\rm st} \geq 0, \quad (3)$$

$\hat{L}_{\rm st}$ with compact resolvent $\Rightarrow$ discrete stability spectrum.

This statement is deliberately more modest than claiming that all classical energies become discrete. The theorem gives discreteness of stability modes. The paper is organized as follows. Section II formulates the stability principle and its relation to the second variation. Section III states and proves the theorem. Section IV explains the role of cyclicity and periodic boundary conditions. Section V discusses the physical interpretation and scope of the stability-induced discreteness. The conclusion summarizes the main results.

II. STABILITY PRINCIPLE AND SECOND VARIATION

Let $S[u]$ be a sufficiently smooth functional on a space of admissible configurations. Let u_0 be a stationary point:

$$\delta S[u_0] = 0. \quad (5)$$

For a small admissible variation η, write

$$u = u_0 + \eta. \quad (6)$$

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The Taylor expansion of S has the form

$$S[u_0+\eta] = S[u_0]+\delta S[u_0](\eta)+\frac{1}{2}\delta^2 S[u_0](\eta,\eta)+O(\|\eta\|^3). \quad (7)$$

Because of Eq. (5), the linear term vanishes and local stability is governed by the second variation:

$$S[u_0+\eta] - S[u_0] = \frac{1}{2}\delta^2 S[u_0](\eta,\eta) + O(\|\eta\|^3). \quad (8)$$

The stability condition is

$$\delta^2 S[u_0](\eta,\eta) \geq 0 \quad \text{for all admissible } \eta. \quad (9)$$

Introduce the bilinear form

$$a(\eta,\xi) = \delta^2 S[u_0](\eta,\xi). \quad (10)$$

If this form is closed, symmetric, and lower semibounded, it is represented by a self-adjoint operator [6, 7]. This is the step at which the second variation becomes a spectral object.

The principle can therefore be interpreted as a selection rule:

The role of Eq. (9) is not simply to keep a trajectory bounded. It supplies a positive quadratic form and hence a stability operator.

III. THEOREM: DISCRETENESS FROM THE SECOND VARIATION

We now formulate the central mathematical statement. The result is a standard consequence of the spectral theory of closed semibounded forms [6–8], but its physical interpretation here is that the second variation of the action generates the operator whose spectrum labels stable modes.

Theorem. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Let

$$\mathcal{V} = H_0^1(\Omega) \quad (12)$$

or another Hilbert space encoding the physical boundary conditions, and let

$$H = L^2(\Omega). \quad (13)$$

Let

$$S : \mathcal{V} \to \mathbb{R} \quad (14)$$

be a twice Frechet differentiable functional, and let $u_0 \in \mathcal{V}$ be a stationary point:

$$\delta S[u_0] = 0. \quad (15)$$

Assume that the second variation defines a symmetric, closed, lower-semibounded bilinear form

$$a(\eta,\xi) = \delta^2 S[u_0](\eta,\xi), \qquad \eta,\xi \in \mathcal{V}, \quad (16)$$

and that the stability condition is satisfied:

$$a(\eta,\eta) = \delta^2 S[u_0](\eta,\eta) \geq 0, \qquad \forall \eta \in \mathcal{V}. \quad (17)$$

Assume also ellipticity/coercivity of the shifted form: there exist $\alpha > 0$ and $c > 0$ such that

$$a(\eta,\eta) + \alpha\|\eta\|_L^2 \geq c\|\eta\|_H^2, \qquad \forall \eta \in \mathcal{V}. \quad (18)$$

Then there exists a self-adjoint operator

$$\hat{L}_{\mathrm{st}} \geq 0 \quad (19)$$

in H associated with the form a. Moreover, $\hat{L}_{\mathrm{st}}$ has compact resolvent and its spectrum is discrete:

$$0 \leq \mu_1 \leq \mu_2 \leq \ldots, \qquad \mu_n \to \infty. \quad (20)$$

The eigenfunctions $\{\eta_n\}$ form an orthonormal basis in $L^2(\Omega)$. Thus every admissible stable perturbation can be expanded as

$$\eta = \sum_{n=1}^{\infty} c_n \eta_n, \qquad \hat{L}_{\mathrm{st}}\eta_n = \mu_n \eta_n. \quad (21)$$

Consequently, the stationary states selected by the stability principle are labelled, up to degeneracies and zero modes, by a discrete stability spectrum.

Proof. Since u_0 is stationary, Eq. (7) gives

$$S[u_0+\eta] - S[u_0] = \frac{1}{2}a(\eta,\eta) + O(\|\eta\|^3). \quad (22)$$

The stability condition gives $a(\eta,\eta) \geq 0$, so a is a positive stability form.

By assumption, $a(\cdot,\cdot)$ is symmetric, closed, and lower semibounded. The Friedrichs-Kato representation theorem therefore gives a unique self-adjoint operator $\hat{L}_{\mathrm{st}}$ such that

$$a(\eta,\xi) = \langle \hat{L}_{\mathrm{st}}\eta, \xi\rangle_{L^2}, \qquad \eta \in \mathcal{D}(\hat{L}_{\mathrm{st}}), \quad \xi \in \mathcal{V}. \quad (23)$$

Since $a(\eta,\eta) \geq 0$, the associated operator satisfies $\hat{L}_{\mathrm{st}} \geq 0$.

The shifted coercivity condition, Eq. (18), implies existence and continuity of the weak solution of

$$(\hat{L}_{\rm st} + \alpha I)u = f. \quad (24)$$

Equivalently,

$$(\hat{L}_{\rm st} + \alpha I)^{-1} : L^2(\Omega) \to H_0^1(\Omega) \quad (25)$$

is bounded. Since Ω is bounded, the Rellich-Kondrachov theorem gives the compact embedding [9]

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega). \quad (26)$$

Therefore the composition

$$L^2(\Omega) \xrightarrow{(\hat{L}_{\rm st}+\alpha I)^{-1}} H_0^1(\Omega) \hookrightarrow L^2(\Omega) \quad (27)$$

is a compact operator in $L^2(\Omega)$. Hence the resolvent is compact.

A self-adjoint operator with compact resolvent has a purely point spectrum with finite multiplicities and no finite accumulation point except infinity [7]. Therefore Eq. (20) holds, and the eigenfunctions form an orthonormal basis in $L^2(\Omega)$. The expansion (21) follows. The theorem is proved.

The theorem identifies the precise mathematical place where discreteness appears. It does not follow from non-negativity of $\delta^2 S$ alone. It follows from the combination of a variational principle, positivity of the stability form, self-adjoint representation, and compactness generated by physical boundary conditions.

IV. PERIODICITY AND CYCLIC BOUNDARY CONDITIONS

A particularly important case occurs when the stationary state is periodic:

$$u_0(t + T) = u_0(t). \quad (28)$$

Then admissible perturbations should preserve the periodic boundary condition:

$$\eta(t + T) = \eta(t), \qquad \dot{\eta}(t + T) = \dot{\eta}(t). \quad (29)$$

The interval $[0, T]$ with identified endpoints is compact. This compactness leads to a discrete spectrum of the corresponding stability operator.

The simplest example is the positive operator

$$\hat{L}_{\rm st} = -\frac{d^2}{dt^2} + \Omega^2 \quad (30)$$

on periodic functions. The eigenvalue problem is

$$\left(-\frac{d^2}{dt^2} + \Omega^2\right)\eta_n(t) = \mu_n \eta_n(t), \qquad \eta_n(t+T) = \eta_n(t). \quad (31)$$

The eigenfunctions are Fourier modes,

$$\eta_n(t) = e^{i2\pi nt/T}, \qquad n \in \mathbb{Z}, \quad (32)$$

and the eigenvalues are

$$\mu_n = \left(\frac{2\pi n}{T}\right)^2 + \Omega^2. \quad (33)$$

Thus periodicity supplies a discrete label n, while the second variation supplies the stability operator whose eigenvalues determine the stable modes.

This distinction is important. Fourier decomposition alone gives a discrete representation of periodic functions, but the coefficients remain continuous. The stability principle adds the physical content: the modes are not merely mathematical harmonics; they are eigenmodes of the operator generated by the second variation.

V. DISCUSSION

The stability-induced mechanism proposed here has two logically distinct layers.

First, stationarity:

$$\delta S = 0 \quad (55)$$

gives the classical equations of motion [1–3]. In celestial mechanics this reproduces Keplerian motion in the two-body approximation.

Second, variational stability:

$$\delta^2 S \geq 0 \quad (56)$$

turns the second variation into a positive quadratic form. Under the assumptions of the theorem this form defines a self-adjoint stability operator. Boundary conditions and compactness yield a discrete spectrum of stable modes.

VI. CONCLUSION

A stability-based route to discreteness has been formulated. The central statement is that the second variation of a stability functional defines a quadratic form. If the form is positive, closed, and semibounded, and if the boundary conditions provide compactness, the associated self-adjoint stability operator has compact resolvent and a discrete spectrum. Thus discreteness arises as a consequence of variational stability and physical boundary conditions.

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DATA AVAILABILITY

No numerical data were generated or analyzed in this work.

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