Appendices to the Work
“Three-Level Emergent Architecture of Social Time: From Timeless Information to Collective Temporality”

Dmitry Timofeev

Independent Researcher, Perm, Russia
dimik129@gmail.com

Abstract

This document provides the mathematical and methodological appendices to the theoretical framework of the three-level emergent architecture of social time. The appendices formalise the key concepts of the theory: contextuality (Level 1), measurement as ontological collapse (Level 2), and coherence as emergent social temporality (Level 3). Appendix A presents a comprehensive strategy for empirical validation, combining measures of macro-level coherence (factorial, entropic, and network-based) with formal tests of non-classical logic (CHSH inequalities and order effects). Appendix B develops the mathematical apparatus, including Sorkin’s anhomomorphic logic for potentiality, the CHSH framework for proving contextuality, and multiple metrics for quantifying collective synchronisation. Appendix C establishes the axiomatic foundations of the meta-language, defining operators, axioms, and inference rules that govern the dynamics of measurement and emergence. Appendix D provides rigorous operator-algebraic proofs demonstrating that order effects necessarily follow from non-commutativity of measurements, and that CHSH violations formally exclude classical hidden variable theories. Together, these appendices constitute a complete formal foundation for operationalising and empirically testing the non-classical ontology of social reality.

Contents

A Strategy for Empirical Validation of the Concept

Introduction

The aim of this Appendix is to operationalise the abstract concepts of **coherence** (Level 3) and **contextuality** (Level 1/2) for empirical verification, utilising an apparatus based on non-classical logic. The proposed strategy consists of two complementary components: verification of macro-structure emergence (coherence) and formal proof of non-classical logic (contextuality).

A.1 Validation of Macro-Level Emergence: Measures of Coherence

Social time (Level 3) is understood as the **dynamics of coherence**—the synchronisation of phases across multiple subjective collapses. For its measurement, a multimodal approach is proposed, combining three independent types of metrics.

A.1.1 Factor-based coherence

This approach derives directly from **Q-methodology** and the principle of **synchronised collapse**.

- **Concept:** Coherence is defined as the proportion of total variation explained by a **single coordinated factor** (the first principal component). This factor represents a **mode of synchronised collapse** or collective meaning.
- **Metric:** The ratio of the largest eigenvalue (λ_1) to the sum of all eigenvalues (total variation) is used:

$$\text{Coherence}_F = \frac{\lambda_1}{\sum_k \lambda_k} \tag{1}$$

- **Interpretation:** A high value of Coherence_F indicates that the majority of subjective measurements (opinions, assessments) are aligned along a single axis, evidencing strong **collective synchronisation**.

A.1.2 Entropic Coherence

This approach validates the thesis that the emergence of ordered **social time** is associated with the **reduction of uncertainty** in the potential space.

- **Concept:** Coherence is measured as the degree of entropy reduction in the system during the transition from **potentiality** (high entropy of Level 1) to an **actual ordered state** (low entropy of Level 3).
- **Metric:** Normalised entropy measures or Mutual Information between subjects are used to assess the reduction of uncertainty.
- **Interpretation:** The reduction of system entropy following an act of collective measurement (for example, after a major political event) constitutes an empirical manifestation of the **fixation of collective meaning**.

A.1.3 Network Coherence

This method regards coherence as a **topological property** of the interaction network.

- **Concept:** Coherence is the stability and structural unity of the network arising from correlations between subjects.
- **Metric:** Analysis of the adjacency matrix based on subject correlations. Weights w_{ij} are calculated as Pearson or Spearman correlations between response vectors of subjects i and j. Edges are included in the graph when $|w_{ij}| > \tau$ (recommended threshold $\tau = 0.3$). Metrics such as **modularity** (for detecting stable meaning clusters) or **giant component density** are employed.
- **Interpretation:** Low modularity and high density provide evidence of strong **phase alignment** and robustness of collective meaning.

A.2 Formal Validation of Non-Classical Logic (Level 1/2)

The key proof of the non-classical status of the social system is verification of violations of fundamental laws of classical probability.

A.2.1 Violation of Bell-Type Inequalities (CHSH Checks)

This is a **mandatory test** for proving **ontological contextuality** (the fundamental law of Level 1).

- **Hypothesis:** If correlations in social data are caused by **contextuality** (rather than merely hidden variables or error), they must violate the limits established by classical logic.
- **Test:** The **Clauser–Horne–Shimony–Holt (CHSH)** inequality or its psychological analogues (for example, the **Dzhafarov–Kujala** model [3]) are used.
- **Criterion:** If the observed quantity S (correlation measure) exceeds the classical limit ($S > 2$), this **formally proves** that the data cannot be described using a classical local hidden variable theory, thereby confirming the **non-classical status of social reality**.

A.2.2 Test for Non-Commutativity (Order Effect)

This test validates **Level 2 (the act of collapse)**, confirming that measurement is **ontologically active** (Measurement Creates Reality).

- **Hypothesis:** The order in which two questions (A and B) are posed should influence the joint probability, which is impossible in classical logic ($P(A \text{ then } B) = P(B \text{ then } A)$).
- **Test:** Comparison of probabilities $P(A \text{ then } B)$ and $P(B \text{ then } A)$ in survey data. Observation of a significant difference ($AB \neq BA$) confirms that **measurement constitutes subjective state** rather than passively extracting it [9, 2].

A.3 Methodological Considerations

A.3.1 Dynamic Analysis (Sliding Windows)

Since **social time** is a **dynamic process** (Level 3) rather than a static aggregate, all coherence measures must be analysed in **sliding temporal windows**. This enables tracking of the **dynamics of emergence and dissolution** of collective meanings, corresponding to the concept of **phase transitions** in the quantum model.

A.3.2 Permutation Tests

For establishing the **statistical significance** of detected coherence (variants 1.1–1.3), the use of **permutation tests** is necessary. These tests generate a null distribution of coherence from randomly permuted data, allowing **genuine synchronisation** to be distinguished from random coincidence or noise.

A.3.3 Priority of Combined Measures

The greatest confidence in results is achieved through the **combination of independent measures**. If several independent approaches (for example, high factor-based coherence + low entropy + CHSH violation) simultaneously indicate **strong, non-classical synchronisation** during a particular time period, this serves as powerful confirmation of the **emergent architecture**.

B Formal Mathematical Description

Introduction

This Appendix contains the mathematical apparatus necessary for formalising the key concepts of the **three-level architecture** and their empirical validation.

B.1 Formalisation of the Ontological Foundation (Level 1)

B.1.1 Anhomomorphic Non-Additivity

The notion of anhomomorphic logic originates in Sorkin's programme of Quantum Measure Theory (QMT) [6], where non-classical behaviour is formulated through measures on spaces of histories rather than through classical additive probabilities. In the present framework, however, we do not employ the full formalism of quantum measure theory. The reference to anhomomorphic logic is therefore conceptual rather than technical.

Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space, where Ω denotes the space of possible configurations, $\mathcal{F}$ the event algebra, and μ a valuation measure over potentialities.

Classical Additivity (Classical Sociology). For two disjoint events A and B:

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{2}$$

Contextual Non-Additivity (Level 1). To represent contextual interactions between potential social configurations, we introduce a phenomenological non-additive measure:

$$\mu(A \cup B) = \mu(A) + \mu(B) + \mathcal{I}(A, B) \tag{3}$$

where $\mathcal{I}(A, B)$ is a contextual interaction term. When $\mathcal{I}(A, B) = 0$, the measure reduces to classical additivity. When $\mathcal{I}(A, B) \neq 0$, the joint valuation of two possibilities differs from the simple sum of their separate valuations.

Importantly, this expression is not the formal definition of Sorkin's Quantum Measure Theory. Rather, it serves as an operational representation of non-classical contextuality suitable for modelling social potentialities. In sociological terms, $\mathcal{I}(A, B)$ captures the emergence of new meaning configurations when previously distinct possibilities are considered jointly, producing amplification, suppression, or transformation effects that cannot be represented within a strictly additive framework [8].

B.2 Formalisation of Contextuality and Measurement (Level 2)

Contextuality is the fundamental law of social subjectivity (Level 2), which is proved through violation of classical logical inequalities.

B.2.1 The Clauser–Horne–Shimony–Holt (CHSH) Inequality

The CHSH inequality is used for formal proof that correlations in a social system cannot be explained by a **classical local hidden variable theory** (i.e., classical logic).

Let A_1, A_2 be two possible measurements (for example, question 1, question 2), and B_1, B_2 be two possible contexts (for example, observer N_1, observer N_2). Each measurement yields a result ± 1.

Contextuality-by-Default Interpretation. For empirical social applications, the CHSH construction is implemented following the Contextuality-by-Default framework of Dzhafarov and Kujala [3]. The variables A_1, A_2, B_1, and B_2 do not represent simultaneously measurable properties of the same respondent.

Instead, four distinct experimental contexts are defined:

$$c_1 = (A_1, B_1), \qquad c_2 = (A_1, B_2), \qquad c_3 = (A_2, B_1), \qquad c_4 = (A_2, B_2).$$

Each respondent (or respondent pair) participates in only one context. The sample is randomly partitioned into four sub-samples, and each expectation value is estimated independently within its corresponding context.

Consequently, the variables are treated as context-indexed random variables rather than jointly existing attributes. The CHSH statistic therefore tests contextuality in the sense of Contextuality-by-Default and should not be interpreted as a test of physical non-locality.

The correlation measure between measurement A_i and context B_j is defined as the expected value:

$$E(A_i, B_j) = \frac{N_{++} + N_{--} - N_{+-} - N_{-+}}{N_{\text{Total}}} \tag{4}$$

The CHSH operator S is defined as:

$$S = E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2) \tag{5}$$

Classical Limit (Local Realism). If the system obeys classical logic (additivity, commutativity), then:

$$|S| \leq 2 \tag{6}$$

Non-Classical Limit (Tsirelson's Bound). If the system obeys quantum logic, then the operator S can exceed the classical limit:

$$|S| \leq 2\sqrt{2} \approx 2.828 \tag{7}$$

Validation Criterion. If the empirically measured $|S| > 2$, or if the corresponding Contextuality-by-Default criterion indicates contextuality, the observed correlations cannot be represented by a single classical jointly distributed hidden-variable model. This provides evidence for contextual dependence of social measurements and supports the non-classical interpretation developed in the present framework.

B.3 Formalisation of Coherence (Level 3)

Coherence is the metric of **emergent social time**, reflecting the degree of synchronisation of multiple subjective collapses.

B.3.1 Factorial Coherence

This indicator is based on principal component analysis (PCA) and Q-methodology.

1. **Data matrix:** Matrix $X \in \mathbb{R}^{N \times m}$, where N is the number of subjects/respondents, m is the number of statements/questions.

2. **Covariance matrix:** Matrix $R = \frac{1}{N-1} X^T X$.

3. **Eigenvalues:** Decomposition of R into eigenvalues λ_k and eigenvectors v_k:

$$R v_k = \lambda_k v_k \tag{8}$$

4. **Coherence metric:** Factorial coherence is defined as the proportion of variation explained by the first principal component (λ_1), which represents a unified, collective mode of synchronisation:

$$\text{Coherence}_F = \frac{\lambda_1}{\sum_{k=1}^{m} \lambda_k} \tag{9}$$

   where $\sum_{k=1}^{m} \lambda_k = \text{Trace}(R)$ is the total variation in the system.

B.3.2 Entropic Coherence

Shannon entropy (H) is used for quantitative assessment of the **potentiality** (uncertainty) of the system.

Individual Entropy. The measure of uncertainty in subject X's response (potentiality):

$$H(X) = -\sum_i P(x_i) \log_2 P(x_i) \tag{10}$$

Joint System Entropy. In the context of collective meaning, coherence is inversely proportional to the **joint entropy** $H(X_1, X_2, \ldots, X_N)$.

Coherence as Entropy Reduction. Coherence in the system (Level 3) reflects the degree of reduction of potentiality (Level 1). High coherence corresponds to a large reduction in joint entropy compared to the sum of individual entropies:

$$H_{\text{System}} \ll \sum_i H(X_i) \tag{11}$$

The normalised metric of entropic coherence is defined as:

$$\text{Coherence}_I = 1 - \frac{H_{\text{System}}}{\sum_i H(X_i)} \tag{12}$$

Where $\text{Coherence}_I \in [0, 1]$: a value of 0 corresponds to complete independence of elements ($H_{\text{System}} = \sum_i H(X_i)$), a value of 1 corresponds to complete coherence ($H_{\text{System}} = 0$). This signifies that the system as a whole is more ordered than if its elements acted independently.

C Axiomatic Block of the Meta-Language

C.1 Notation

- Ω_t — space of potential configurations (space of possibilities) in window t.
- $C_i(t) \in \Omega_t$ — collapse of agent i in window t (vector of length m).
- $X(t) \in \mathbb{R}^{N \times m}$ — data matrix in window t, where N is the number of subjects/agents, m is the number of features/statements. Matrix rows: $X(t) = \{C_i(t)\}_{i=1..N}$.
- $\mathcal{S}(\Omega_t)$ — space of distributions over Ω_t.
- $\mathrm{CoherenceOp} : 2^{\Omega_t} \to [0, 1]$ — aggregating coherence operator.

C.2 Operators

- $M_i^B : \mathcal{S}(\Omega_t) \to \mathrm{Outcomes}_i$ — measurement operator of agent i in measurement basis B (context).
- $\mathrm{Coll}_i^B := M_i^B(\cdot) \circ \mathrm{sample}$ — collapse operator: mapping of potential distribution to concrete result $C_i(t)$ under basis B.
- $\mathrm{CoherenceOp}(X(t)) := \Phi\big(C_A(X(t)), C_I(X(t)), C_N(X(t))\big)$ — composition of three component measures:
  1. C_A — factorial proportion (linear coherence, leading eigenvalue proportion);
  2. C_I — information measure (total correlation);
  3. C_N — network coherence (proportion of intra-community weight).

C.3 Axioms

Axiom A1 (Contextuality). There exists $B \neq B'$ such that $\mathrm{Coll}_i^B \neq \mathrm{Coll}_i^{B'}$ for the same initial $\mathcal{S}(\Omega_t)$. The measurement result depends on the measurement basis.

Axiom A2 (Non-Commutativity). There exist i, j, B_1, B_2 such that $M_i^{B_1} \circ M_j^{B_2} \neq M_j^{B_2} \circ M_i^{B_1}$. The order of measurement acts is essential.

Axiom A3 (Emergence). There exists a threshold $\theta \in (0, 1]$ and function G such that when $\mathrm{CoherenceOp}(X(t)) \geq \theta$, a stable macro-structure $S^* = G(X(t))$ emerges (habitus, institution) with high probability of repetition in subsequent windows.

Axiom A4 (Irreversibility). Following realisation of Coll_i^B, restoration of the initial superposition $\mathcal{S}(\Omega_t)$ is impossible without a new act M.

Axiom A5 (Selectivity). For a fixed basis B, there exists a subspace $\Omega_t^B \subset \Omega_t$ of non-zero measure accessible to Coll^B.

C.4 Inference Rules

Rule R1. If $\mathrm{CoherenceOp}(X(t)) \geq \theta_{\text{high}}$, the inference is permitted: "the system is in a synchronisation phase; collective configuration is stabilising".

Rule R2. If the following are observed: (i) significant order effects, (ii) interference deviations from the classical sum of probabilities, (iii) significant $C_I > 0$, then the local hidden variable model is inadequate.

Rule R3. A theoretical conclusion is considered reliable when at least two independent components C_A, C_I, C_N coincide and their exceeding of thresholds is statistically significant (permutation/bootstrap).

C.5 Working Specification of the Aggregated Operator

Weights $\alpha, \beta, \gamma \geq 0$ are specified, with $\alpha + \beta + \gamma = 1$:

$$\text{Coherence}(t) = \alpha\, C_A(t) + \beta\, C_I(t) + \gamma\, C_N(t), \tag{13}$$

where

$$C_A(t) = \frac{\lambda_1}{\sum_k \lambda_k} \quad \text{(SVD/PCA on } X(t)\text{)}, \tag{14}$$

$$C_I(t) = \frac{\mathrm{TC}(X(t))}{\sum_i H(X_i)} \quad \text{(total correlation, normalised)}, \tag{15}$$

$$C_N(t) = \frac{\sum_{i,j \in G^*} w_{ij}}{\sum_{i,j} w_{ij}} \quad \text{(proportion of intra-community weight for leading cluster } G^*\text{)}. \tag{16}$$

Relationship to Factor Entropy (Timofeev, 2025, *Solving*). The component $C_A(t) = \lambda_1 / \sum_k \lambda_k$ is monotonically related to the inter-factor entropy $H_{\text{factors}}(t) = -\sum_i p_i \log_2 p_i$ developed in the companion paper on the meso-macro transition: $H_{\text{factors}} \to 0 \Leftrightarrow C_A \to 1$, and both are derivable from the same eigenvalue vector $\{\lambda_i\}$. The cross-factor discriminability index $D(F_k)$ complements $C_I(t)$ by measuring structural differentiation at the statement level rather than inter-subject total correlation. The network component $C_N(t)$ has no direct analogue in the entropy protocol and represents an additional dimension of synchronisation to be integrated in future empirical implementations. Together, the two frameworks constitute a unified operationalisation of Level III coherence at different levels of granularity.

Centring Convention and Its Interpretation. The factor-based coherence component $C_A(t) = \lambda_1 / \sum_k \lambda_k$ is computed on a **person-centred** data matrix $X(t)$, in accordance with Q-methodology's standard procedure: each subject's response vector is centred relative to that subject's own mean, not relative to the item mean across subjects.

This choice is theoretically motivated. Centring by item (the standard PCA convention) would allow the first component to be dominated by the marginal popularity of responses — a property of the stimulus set rather than of the subjects' meaning configurations. In the present framework, λ_1 is intended to capture **inter-subject synchronisation of collapse patterns**; person-centring removes the baseline popularity effect and ensures this interpretation holds.

On the person-centred matrix $X(t) \in \mathbb{R}^{n \times p}$ (subjects $\times$ items), SVD and PCA are equivalent: the singular value decomposition $X = U \Sigma V^\top$ yields $\lambda_k = \sigma_k^2$, so that $C_A(t) = \sigma_1^2 / \sum_k \sigma_k^2$. The notation "SVD/PCA" in the formula above therefore refers to this unified operation on the person-centred matrix; no ambiguity arises once the centring convention is fixed.

Weight Selection and Sensitivity Analysis. The weighting scheme is a methodological parameter rather than an ontological constant. In the absence of compelling theoretical grounds for privileging one dimension of coherence over another, equal weights are adopted as the baseline specification.

Nevertheless, empirical applications should evaluate the robustness of results under alternative weight configurations.

A sensitivity analysis may be conducted by varying (α, β, γ) across the admissible simplex

$$\alpha + \beta + \gamma = 1, \qquad \alpha, \beta, \gamma \geq 0,$$

and examining the stability of substantive conclusions. Only results robust across a broad range of admissible weights should be interpreted as theoretically significant.

For empirical implementations, the weights may alternatively be estimated from data using principal component analysis (PCA) or related latent-dimension extraction techniques.

Total Correlation is defined as:

$$\text{TC}(X(t)) = \sum_{i=1}^{N} H(X_i) - H(X_1, X_2, \ldots, X_N) \tag{17}$$

where $H(X_i)$ is the entropy of the individual variable, $H(X_1, \ldots, X_N)$ is the joint entropy of the system.

Note: For continuous data, entropy is estimated using the k-nearest neighbours (k-NN) method or through discretisation (binning). For discrete/categorical data, the classical Shannon formula with frequencies is used.

Recommended thresholds: $\theta_{\text{low}} = 0.5$, $\theta_{\text{high}} = 0.75$.

Threshold Calibration. The thresholds

$$\theta_{low} = 0.50, \qquad \theta_{high} = 0.75$$

should be understood as heuristic operational benchmarks rather than universal constants of the theory.

The lower threshold corresponds to the midpoint of the normalized coherence scale and marks the transition from weakly coherent to moderately coherent configurations. The higher threshold identifies configurations exhibiting substantial alignment across the factor, network, and dynamic dimensions simultaneously.

The precise numerical values are therefore methodological choices introduced for operational classification. Empirical applications should evaluate the robustness of substantive conclusions under alternative threshold specifications.

Future implementations may calibrate the thresholds using Bayesian decision criteria, information-theoretic optimization procedures, or simulation studies based on synthetic and empirical datasets.

Accordingly, the theoretical framework depends primarily on the existence of coherence regimes rather than on any particular numerical threshold value.

Survey Weights, Missing Data, and Systematic Bias. When the coherence operator is applied to data from large-scale survey organisations (e.g. VCIOM, Levada Centre), a fundamental pre-processing decision arises: whether to apply or suppress the provider's post-stratification weights.

Survey weights are constructed to align the achieved sample with known population margins (typically sex, age, education, and settlement type). Applying these weights to the coherence computation means that Coherence(t) characterises the **artificially reconstructed population** rather than the observed sample. This is appropriate when the research question concerns population-level synchronisation, but it introduces a layer of modelling assumptions that are external to the present framework.

Suppressing weights and analysing raw data preserves the empirical structure of the sample but may conflate genuine contextual non-additivity with demographic composition effects. The recommended baseline is therefore to **compute coherence on unweighted data** and treat weighted replication as a robustness check.

Two further issues require explicit treatment in any empirical implementation. First, **missing data**: item non-response in Q-sorts or survey batteries should not be imputed using mean substitution, as this artificially inflates $C_A(t)$ by reducing inter-subject variance. Multiple imputation or full-information maximum likelihood are the preferred strategies. Second, **systematic bias**: acquiescence bias (the tendency to agree regardless of content) and social desirability bias (especially salient in politically sensitive surveys) both inflate apparent coherence by driving responses toward a single pole. Ipsative scoring — already implicit in the Q-sort forced-choice procedure — partially controls for acquiescence; for Likert-based instruments, explicit bias correction (e.g. via the MTMM framework) should be applied prior to coherence estimation.

These considerations do not undermine the theoretical framework; they specify the conditions under which empirical estimates of Coherence(t) are interpretable as measures of genuine collective synchronisation rather than artefacts of data collection design.

C.6 Consistency Tests

Test T1 (Order Effects). Randomisation of question order $\rightarrow$ statistical verification of $M_i^B \neq M_i^{B'}$.

Test T2 (Interference). Respondents are randomly allocated to three groups:

- Group 1: assesses probability of event A
- Group 2: assesses probability of event B
- Group 3: assesses probability of disjunction (A or B)

Calculation of interference term: verification of violation of the classical formula $P(A \text{ or } B) \neq P(A) + P(B) - P(A \text{ and } B)$. Significant deviation indicates quantum-like interference.

Test T3 (CHSH-Style). For binary responses (or after binarisation by median/mean), form quartet (A, A', B, B') and estimate S; $S > 2$ indicates strong non-classicality. If data are not initially binary, a binarisation strategy is applied: coding above/below median as $+1/-1$.

Test T4 (Coherence Dynamics). Sliding windows, detection of sharp transitions in Coherence(t); bootstrap / permutations for significance.

D Operator Formalism of Order Effects and Contextuality

Introduction

This Appendix provides a rigorous mathematical proof of the key claims of this work: **(A)** order effects necessarily follow from the non-commutativity of measurements, and **(B)** violations of CHSH inequalities formally prove the impossibility of a classical model with local hidden variables.

D.1 General Formalism

We consider the state space (in idealisation): ρ is the state of the agent/system (in terms of the density operator; ρ is a positive operator with $\operatorname{Tr}\rho = 1$). This constitutes a formal model of **potentials** (Level I).

Measurements. These are defined by families of projectors (or, more generally, by POMs/POVMs), but for simplicity, we use projectors. For a binary measurement A, the outcomes $a \in \{+1, -1\}$ correspond to the projector P_a^A (where $P_+^A + P_-^A = I$). Similarly for B – the projectors P_b^B.

Context. This denotes a specific basis/set of projectors (i.e., the choice of operator/observable). Different contexts $\mathcal{B}$ and $\mathcal{B}'$ correspond to different sets of projectors for B (and similarly for A).

Sequential Measurement. Measuring A then B is modelled using the standard quantum-like formula for conditional / post-selection probability.

D.2 The Order Effect — Formal Proof

Theorem 1 (Order Effect)**.** *If the projectors P_a^A and P_b^B do not commute, then, in general,*

$$P(A = a \textit{ then } B = b) \neq P(B = b \textit{ then } A = a). \tag{18}$$

Proof. 1. The probability of obtaining the outcome a upon measuring A in state ρ (prior probability) is:

$$P(A = a) = \operatorname{Tr}\big(P_a^A\, \rho\big). \tag{19}$$

2. The sequential (subjectively actualising) probability ”first $A = a$, then $B = b$”, derived from the standard formula (projection and normalisation), is:

$$P(A = a \to B = b) = P(A = a)\, P(B = b \mid A = a) = \operatorname{Tr}(P_a^A \rho) \cdot \frac{\operatorname{Tr}\big(P_b^B P_a^A \rho P_a^A\big)}{\operatorname{Tr}(P_a^A \rho)}. \tag{20}$$

Simplifying, we obtain:

$$P(A = a \to B = b) = \operatorname{Tr}\big(P_b^B P_a^A \rho P_a^A\big). \tag{21}$$

3. Similarly, the reverse sequence yields:

$$P(B = b \to A = a) = \operatorname{Tr}\big(P_a^A P_b^B \rho P_b^B\big). \tag{22}$$

4. If the projectors commute: $[P_a^A, P_b^B] = 0$, then $P_b^B P_a^A = P_a^A P_b^B$, and thus

$$P(A = a \to B = b) = \operatorname{Tr}\big(P_b^B P_a^A \rho\big) = \operatorname{Tr}\big(P_a^A P_b^B \rho\big) = P(B = b \to A = a). \tag{23}$$

(Here we have used the cyclic property of the trace and the idempotency of projectors $P_a^2 = P_a$).

5. Conversely, if the projectors do not commute, then in the general case

$$\operatorname{Tr}\big(P_b^B P_a^A \rho P_a^A\big) \neq \operatorname{Tr}\big(P_a^A P_b^B \rho P_b^B\big), \tag{24}$$

which implies that the order of measurements yields different probabilities. The difference is essentially due to the fact that the first act of measurement changes the state (collapse into basis A), thereby altering the distribution for the second measurement B.

□

Interpretation. This is the formal expression of the idea that **"measurement constitutes reality"** (Level 2). Non-commutativity of operators/projectors $\Rightarrow$ order effect (non-equivalence of sequential probabilities). This is the mathematical, operational explanation for the observed order-effect in surveys ($A \to B \neq B \to A$), when modelling the measurement as a state-changing act.

Connection to Quantum Models of Cognition. An analogous operator formalism of order effects has been systematically developed within the quantum approach to cognitive processes [1, 2, 5, 4]. Our approach extends this formalisation to the social level, demonstrating that the same mathematical structure describes not only individual cognition but also collective sense-making processes.

Link to Scales. Different scales (Likert 1-5 vs 1-7) define **different families of projectors**. These families may not commute with the projectors of other questions, which **guarantees** the order effect. Thus, **the construction of context through scales** is mathematically equivalent to the introduction of non-commuting operators.

D.3 CHSH and the Classical Limit

Aim. To demonstrate how, in the operator formulation of contextuality, correlations are possible that exceed the classical limit $|S| \leq 2$. We emphasise that the formalism is a tool; exceeding the classical limit signals the impossibility of explaining the correlations by a joint (additive) probability distribution.

D.3.1 The CHSH Operator

We consider two "observers/contexts" for side A: A_1, A_2, and two for side B: B_1, B_2. All four are binary measurements with spectrum ± 1. In operator notation, we define the operator

$$\hat{S} = \hat{A}_1 \otimes \hat{B}_1 + \hat{A}_1 \otimes \hat{B}_2 + \hat{A}_2 \otimes \hat{B}_1 - \hat{A}_2 \otimes \hat{B}_2, \tag{25}$$

and its expectation value in state ρ (the state of the joint system or joint context) is equal to

$$S = \mathrm{Tr}(\rho\, \hat{S}) = E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2), \tag{26}$$

where $E(A_i, B_j) = \mathrm{Tr}(\rho\, \hat{A}_i \otimes \hat{B}_j)$.

D.3.2 The Classical Limit

Theorem 2 (The Bell-CHSH Inequality)**.** *A classical justification (the existence of a joint distribution for random variables A_1, A_2, B_1, B_2 with deterministic values ± 1 for each "realisation") yields the inequality*

$$|S| \leq 2. \tag{27}$$

Proof Sketch. This is derived algebraically for any ± 1 values: for any numbers $a_1, a_2, b_1, b_2 \in \{\pm 1\}$, it holds that

$$|a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2| \leq 2. \tag{28}$$

The same limit is preserved when taking the expectation over a distribution. The key assumption of the classical derivation is the existence of a single (context-independent) hidden state λ that simultaneously determines the values of all A_i, B_j. This is an additive (homomorphic) logic. □

D.3.3 Operator/Non-Classical Possibility of Exceeding the Limit

If the correlations are defined by an operator model where $\hat{A}_i, \hat{B}_j$ may not commute (and/or no single joint Hilbert operator exists that represents all as compatible), the same algebraic derivation does not apply.

Theorem 3 (Tsirelson's Bound)**.** *For operators in a Hilbert space, there exist operators (e.g., Pauli matrices) and a state ρ such that*

$$|S| \leq 2\sqrt{2} \quad \textit{(Tsirelson bound)}, \tag{29}$$

and the value $S = 2\sqrt{2}$ is achieved for appropriately chosen $\hat{A}_i, \hat{B}_j$ and an entangled state.

This demonstrates that the operator (non-classical) model allows values $2 < |S| \leq 2\sqrt{2}$; that is, the classical limit of 2 can be violated.

Important Note. The bound $2\sqrt{2}$ is derived for a **Hilbert space** (physical quantum systems). For more general anhomorphic systems (in particular, cognitive/semantic ones), this bound is **not mandatory**. Values $S > 2\sqrt{2}$ are possible, which suggests a different mathematical structure, distinct from quantum mechanics, yet still non-classical.

D.3.4 Demonstration Example

Let us take two-level (dichotomous) operators in $\mathbb{C}^2$ – the Pauli matrices σ_x, σ_z. If we choose

$$\hat{A}_1 = \sigma_z, \quad \hat{A}_2 = \sigma_x, \tag{30}$$

$$\hat{B}_1 = \tfrac{\sigma_z + \sigma_x}{\sqrt{2}}, \quad \hat{B}_2 = \tfrac{\sigma_z - \sigma_x}{\sqrt{2}}, \tag{31}$$

and the state ρ is maximally correlated (EPR analogue), the calculation yields $S = 2\sqrt{2}$. This is a standard construction in quantum information; in our context-analogue, it merely shows that, given the appropriate operator structure and an ”entangled” (in terms of connection/context) state, correlations can surpass the classical limit.

Social Operationalisation of the Demonstration Example. The preceding construction serves as a formal existence proof: given the appropriate operator structure, classical limits can be violated. The social analogue requires specifying each component explicitly.

The role of $\hat{A}_1 = \sigma_z$ and $\hat{A}_2 = \sigma_x$ (non-commuting measurement bases for agent A) is played by two **incompatible Q-sort frames**: for instance, a sorting task structured around *economic security* versus one structured around *political identity*. These frames define different projective decompositions of the subject's meaning space — they do not commute because activating one frame changes the salience structure available to the other (order effects, Section A.1.2).

The role of $\hat{B}_1$ and $\hat{B}_2$ (rotated bases for agent B) is played by the **same pair of frames administered to a second subject** whose meaning configuration is correlated with the first — for instance, two members of the same deliberative group or co-workers sharing an institutional context.

The role of the **EPR-analogue state** ρ is played by **prior collective synchronisation**: two subjects who have undergone a shared ritual, media event, or institutional procedure exhibit correlated collapse probabilities that cannot be decomposed into independent marginals. This is precisely the condition $\rho \neq \rho_A \otimes \rho_B$ — entanglement in the formal sense, social synchronisation in the empirical sense.

Under these conditions, the CHSH quantity S is computed from the four cross-correlations between the two subjects' responses across the two frame pairs. A value $S > 2$ (with appropriate statistical controls) would constitute empirical evidence that the joint response distribution cannot be explained by a context-independent hidden variable — i.e., that the subjects' meaning states are **contextually constituted**, not merely correlated.

D.3.5 Interpretation for Social Data

- **Contextuality:** Different measurement contexts (different B's as sets of rules/observers) mean that a single joint distribution of all responses (no ”context-independent” λ) cannot be assumed. Thus, the classical deduction $|S| \leq 2$ is not mandatory.

- **Non-commutativity / Order:** Sequential measurements/surveys change the state (see section above) and thereby violate the possibility of constructing a single joint distribution; this enhances the chance of observing CHSH violations.

- **Empirical Criterion:** If $S > 2$ is observed (with statistical significance) in the data (after correct binning/discretisation), this is evidence that the classical model with local hidden variables (single context) is unable to explain the correlations; consequently, a non-classical (contextual/operator) formalisation is required.

- **Scales as Projectors:** In classical questionnaires, different scales (Likert 1-5, 1-7, binary) define different families of projectors $\{P_1, P_2, \ldots, P_k\}$ with $\sum P_i = I$. These families may not commute with each other or with the projectors of other questions. Thus, the **practice of using different scales** is mathematically equivalent to the **construction of non-commuting measurement bases**, which makes the existence of a single context-independent λ impossible. This is a formal proof of the classical approach's self-refutation: the practice uses contextuality (via scales), while the theory denies it (by assuming a single λ).

D.4 Conclusion

If measurements depend on the context (different operators/bases) and/or the corresponding operators/projectors do not commute, then:

1. the sequential probabilities $P(A \to B)$ and $P(B \to A)$ are generally different (order effect), and

2. the absence of a single context-independent joint probability removes the premise for the classical deduction $|S| \leq 2$; in the operator model, values $2 < |S|$ are possible (in Hilbert space $\leq 2\sqrt{2}$, in more general non-classical measures — even larger), which formally supports a contextual/non-additive logic.

Practical Implication. Classical objections such as "the order effect is explained by priming" or "differences in VCIOM/Levada are explained by fear" prove to be **mathematically redundant**: the effects necessarily follow from the structure of non-commuting measurements. Ad hoc explanations are unnecessary – it is sufficient to acknowledge that **measurement constitutes the state**, rather than passively discovering it. Furthermore, the very practice of classical surveys (using different scales) **mathematically guarantees** contextuality, which renders the classical theory (assuming a single λ) **incompatible** with classical practice.

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