Black-Hole Entropy and Unitary-Complete Evaporation in the Closure-Flat Completed-Cycle Framework

Daegene Song
*Department of Management Information Systems,
Chungbuk National University, Cheongju 28644, Korea*

We formulate black-hole thermodynamics as a positive observer readout of the closure-flat completed-cycle framework. The counted objects are admissible black-hole completed cycles, rather than bare interior microstates, bare forward histories, or independent endpoint data. On the stationary branch, exact Hawking–Unruh purification, committed-sector weights, reversible rank-one coherence, and quarter-normalized detector blocks reproduce the Gibbons–Hawking–Wald area law as an effective completed-cycle capacity. The quarter normalization is not appended after an independent microscopic count: in the completed-cycle framework the classical horizon readout is part of the primitive closure cycle, so the reciprocal quantum context is counted only after the horizon text and its local Einstein response have restricted the admissible sectors. On the quasi-stationary branch, the same readout data, together with the stated support-minimal branch selection, yield a hidden-reservoir transport law, a public release channel, a Page-compatible fine-grained radiation entropy, and temporal gluing to a rank-one terminal endpoint. Quantum-extremal islands appear as semiclassical readouts of hidden-completion branches: the area term measures completion-boundary capacity, while the bulk term measures unresolved completion entropy.

I. INTRODUCTION

Black-hole thermodynamics ties together horizon area, thermal exterior physics, and quantum information in a way that remains conceptually incomplete in the usual semiclassical formulation [1–6]. The area law itself is structurally robust and, in stationary settings, is encoded in the Gibbons–Hawking and Wald formulas [7–9]. At the same time, the information-loss problem appears when one treats the observer-accessible exterior channel as if it were the full quantum system throughout evaporation. The present paper takes a different starting point. It treats the counted black-hole sector not as a set of bare forward histories or bulk interior microstates, but as a restricted quantum sector of admissible completed cycles carrying context choice, committed record, and closure through the singular regime.

The completed-cycle formulation used here is the black-hole positive readout of the closure-flat local-to-global ledger framework. In the companion quantum-gravity construction, physical completed-cycle data are obtained by admissible readout from a closure-flat comparison class. We denote this imported quantum-gravity input schematically by

$$[\eta] \in \mathcal{K}_{\mathrm{c}}, \qquad \mathcal{K}_{\mathrm{c}} = \left\{ \eta \in \mathfrak{C}^{G} : \nabla_{\mathrm{c}} \eta = 0 \right\} / \simeq_{\mathrm{c}} .$$

The construction of ∇_{c}, the closure quotient, and the local-to-global kernel is not repeated here [10]. The present paper uses only the black-hole specialization of that structure: the admissible closure quotient, the closure-stable coherent kernel, the geometric readout, the universal transport normalization, and the singularity-seam continuation needed for entropy counting, quasi-stationary evaporation, and endpoint closure [11–16].

The main claims are twofold. First, on the faithful stationary branch, the admissible committed completed-cycle count reproduces the black-hole area law. Second, on the quasi-stationary evaporation branch, the same architecture yields a hidden-sector transport law, a Page-compatible fine-grained radiation entropy, and a temporal-gluing closure of the late pure endpoint. These claims are conditional sectoral consequences. The imported closure-flat datum supplies the common quotient, ledger-balance, reversible-coherent-kernel, projective-transport, and seam-continuity data only after black-hole readout. The area law and the evaporation profile then require the faithful stationary branch and the support-minimal quasi-stationary branch specified below. Thus the conditions used below are not independent black-hole axioms, but the explicit branch profile is not claimed to follow from the bare closure-flatness condition alone; it follows after the stated black-hole readout and branch selections.

For the entropy argument below, the important point is that the quarter-normalized capacity is imposed at the level of the admissible horizon apparatus. The detailed justification is given in Sec. III; here we only record that the count is performed over text-constrained completed-cycle sectors, not over a substrate-independent Hilbert-space degeneracy.

A further consistency check comes from the quantum-extremal-island description of the Page curve. In that description, the fine-grained entropy of radiation is computed by extremizing and minimizing a generalized entropy consisting of an area term and a bulk entropy term [17–19]. The completed-cycle black-hole sector contains the same structure intrinsically: the area term is the capacity of a hidden-completion boundary, while the bulk entropy term is the entropy of the unresolved completion included with the radiation. Within the completed-cycle reduction, the island entropy prescription is therefore not an independent additional postulate, but the semiclassical geometric readout of hidden-completion purification.

II. COMPLETED-CYCLE BLACK-HOLE SECTOR AND OPERATIONAL CLOSURE

The black-hole sector is formulated on admissible closure classes of completed cycles rather than on unreduced microscopic representatives or smooth interior geometries. The closure-flat local-to-global framework supplies the admissible closure quotient, the closure-stable coherent kernel, exact transport closure on the reversible branch, and the observer-readable geometric branch [10]. In the black-hole specialization, the counted objects are therefore admissible completed cycles carrying a context choice, a committed exterior record, and finite degeneration-stable closure data sufficient to identify the same physical object across singular or degenerate regimes. In the Hodge-theoretic formulation adopted here, those closure data are encoded by admissible limiting mixed Hodge data [20–23]. Let

$$\Xi_{\rm cyc}^{\rm LMHS}(\mathcal{M}) \tag{1}$$

denote the space of admissible black-hole completed cycles at fixed exterior macrodata $\mathcal{M}$, and let

$$\Xi_{\rm BH}^{\rm phys}(\mathcal{M}) := \Xi_{\rm cyc}^{\rm LMHS}(\mathcal{M})/{\sim_{\rm cl}} \tag{2}$$

be the corresponding black-hole physical quotient by admissible closure identifications. The committed sector is obtained by a map

$$\mathrm{Com} : \Xi_{\rm BH}^{\rm phys}(\mathcal{M}) \to \Xi_{\rm com}^{\rm LMHS}(\mathcal{M}), \tag{3}$$

and the basic closure requirement is that the committed record stabilizes the same internal cycle class through the admissible limiting data. The role of the LMHS data here is not to provide a smooth continuation of the singular geometry, but to supply the finite degeneration-stable data needed for the black-hole specialization of the completed-cycle closure quotient.

At the framework level, the black-hole construction is a sectoral readout of a closure-flat comparison class. The physical black-hole readout is

$$\Pi_{\rm BH}(\eta) = \big(X_{\rm BH}(u), \langle K_{\rm BH}(u)\rangle, N_{\rm BH}(u), g_{\rm BH}(u), s_{\rm BH}(u)\big),$$

while the structural data needed to state the black-hole reduction are collected as

$$\mathfrak{G}_{\rm BH}(\eta) := \Pi_{\rm BH}^{\rm str}(\eta) = \big(\Xi_{\rm BH}^{\rm phys}, \pi_{\rm BH}, \mathcal{K}_{\rm BH}^{\rm coh}, \mathcal{U}_{\rm BH}, \mathbb{E}_{\rm BH}, \kappa_{\rm tr}^{\rm BH}\big).$$

Thus $\Pi_{\rm BH}^{\rm str}$ denotes the structural black-hole readout. The entries $\Xi_{\rm BH}^{\rm phys}$, $\pi_{\rm BH}$, $\mathcal{K}_{\rm BH}^{\rm coh}$, $\mathcal{U}_{\rm BH}$, $\mathbb{E}_{\rm BH}$, and $\kappa_{\rm tr}^{\rm BH}$ denote, respectively, the admissible closure quotient, committed-sector pushforward, coherent-kernel readout, geometric readout, sectoral branch law, and black-hole transport normalization. When needed below, the terminal continuation is supplied by the seam-stable LMHS continuation already included in the admissible closure quotient.

Thus the labels BH1–BH5 name the black-hole realizations of the quotient, committed-weight, reversible-coherence, off-anchor-transport, and observer-reduction components just displayed. The condition BH1 is the admissible-quotient readout: all physical states, public records, and thermodynamic weights are defined on admissible closure classes in $\Xi_{\rm BH}^{\rm phys}(\mathcal{M})$, not on unreduced microscopic representatives. The condition BH2 is the committed-sector positive readout: on the faithful stationary branch, equilibrium weights are committed pushforwards of one coherent completion, and faithful committed sectors satisfy reversible micro-indifference. The condition BH3 is the reversible ledger–coherence anchor:

$$X = \frac{A}{4G_{\rm ren}}, \qquad \Delta_{\rm ker}^{\rm rev} := 1 - \mathrm{Tr}\big[(\rho_{\rm coh}^{\rm obs})^2\big] = 0. \tag{4}$$

Here $\rho_{\rm coh}^{\rm obs}$ denotes the normalized observer-side coherent-kernel representative on the reversible support. It is not the exterior KMS density matrix obtained after tracing the completion factor. Thus the reversible condition $\Delta_{\rm ker}^{\rm rev} = 0$ is compatible with the thermal diagonal horizon reduction derived in Sec. III; the latter is the public reduced readout of a rank-one completed implementation.

Thus X is already an apparatus-normalized completed-cycle capacity. The entropy derivation in Sec. III explains how the horizon text and its local Einstein response select the admissible quantum completion sectors.

The same condition BH3 also includes exact projective transport closure of the reversible coherent line. Hence the reversible support carries a rank-one coherent completion unique as a projective reciprocal completion/choice paired object, with only the classwise $\mathrm{U}(1)_{\rm pair}$ gauge left after exact transport closure. The condition BH4 is the off-anchor transport readout of the covariant completed-cycle dynamics:

$$\dot{X} = -\frac{\mathrm{d}}{\mathrm{d}u}\langle K\rangle_{\rm fix} + \dot{N}_c + \dot{\Phi}_{\rm dr} + a_u. \tag{5}$$

The condition BH5 is the positive observer-reduction readout after the preferred modular/Killing context has been selected. Seam-continuity and transport-normalization are not additional black-hole labels; they enter as the terminal and branch-normalization refinements needed for the evaporation branch.

With this division of labor, the entropy theorem and the evaporation theorem have the same logical origin but different technical status. The entropy theorem is the stationary black-hole readout of the admissible quotient, committed-sector weights, reversible rank-one coherence, and quarter-normalized detector anchor. The evaporation theorem is the quasi-stationary black-hole readout of the same quotient and coherent anchor, together with the off-anchor transport, observer-reduction, terminal seam-continuity, and support-minimal branch-selection components. Thus both results are sectoral consequences of one closure-flat completed-cycle datum, while the evaporation theorem additionally uses the support-minimal quasi-stationary branch selection.

III. EXACT THERMAL REDUCTION AND THE STATIONARY HORIZON SECTOR

The Unruh setting already displays the exact completed-cycle pattern underlying unitary-complete thermality in the black-hole sector [24–26]. It therefore provides the simplest exact preliminary model for the stationary observer reduction used below.

Let $a > 0$ be the proper acceleration and fix one Rindler frequency $\omega > 0$. For that mode pair, the Minkowski vacuum may be written as

$$\begin{aligned} |0_{\mathrm{M}}\rangle_\omega = \left(1 - e^{-2\pi\omega/a}\right)^{1/2} \sum_{n=0}^{\infty} e^{-\pi\omega n/a} e^{\mathrm{i}\vartheta_{\omega,n}} \\ \times |n_{\omega,R}\rangle \otimes |n_{\omega,L}\rangle. \end{aligned} \tag{6}$$

This state is pure. Tracing over the left wedge gives

$$\begin{aligned} \rho_{R,\omega} &= \mathrm{Tr}_L\, |0_{\mathrm{M}}\rangle_\omega \langle 0_{\mathrm{M}}|_\omega \\ &= \left(1 - e^{-2\pi\omega/a}\right) \sum_{n=0}^{\infty} e^{-2\pi\omega n/a} |n_{\omega,R}\rangle\langle n_{\omega,R}|, \end{aligned} \tag{7}$$

hence

$$\rho_{R,\omega} = \frac{e^{-\beta_U H_{R,\omega}}}{Z_{R,\omega}}, \qquad \beta_U = \frac{2\pi}{a}, \qquad T_U = \frac{a}{2\pi}. \tag{8}$$

For the full field, the corresponding state is the tensor product over mode pairs, but the single-mode form above is all that is needed for the present exact preliminary model. There is therefore no contradiction between exact thermality and global purity: thermality belongs to the observer reduction, while purity belongs to the completed global implementation.

The stationary black-hole horizon sector is the patchwise modular copy of this same structure. Let K_H be the preferred modular generator selected by the stationary horizon context and let

$$K_H|n\rangle_K = k_n|n\rangle_K. \tag{9}$$

Then the exterior KMS state is

$$\rho_{\mathrm{ext}} = \frac{e^{-K_H}}{Z_H} = \sum_n \frac{e^{-k_n}}{Z_H} |n\rangle_K\langle n|, \qquad Z_H := \sum_n e^{-k_n}. \tag{10}$$

A unitary-complete purification of (10) is

$$\begin{aligned} |\Psi_H\rangle = \sum_n \frac{e^{-k_n/2}}{\sqrt{Z_H}} e^{\mathrm{i}\vartheta_n} |n\rangle_K \otimes |\widetilde{n}\rangle_{\mathrm{comp}}, \\ \langle \widetilde{m}|\widetilde{n}\rangle_{\mathrm{comp}} = \delta_{mn}. \end{aligned} \tag{11}$$

Tracing over the completion sector recovers the exterior KMS state exactly. Thus the stationary Hawking sector is already unitary-complete before one addresses the full evaporation history.

The same formula also makes the reversible coherent-kernel statement explicit. On the preferred modular basis the stationary horizon coherent kernel is

$$\begin{aligned} \mathcal{K}_H(n,m) &= \frac{e^{-(k_n+k_m)/2}}{Z_H} e^{\mathrm{i}(\vartheta_n - \vartheta_m)} \\ &= \Psi_H(n)\overline{\Psi_H(m)}, \end{aligned} \tag{12}$$

which is manifestly rank one. If reversible refinement-loop holonomy is excluded, the remaining classwise phase is only the $\mathrm{U}(1)_{\mathrm{pair}}$ gauge of the reciprocal completion/choice pairing. The stationary Hawking sector is therefore not merely compatible with reversible coherent completion; it is an explicit realization of the projective paired coherent object.

IV. EQUILIBRIUM COUNT AND THE AREA LAW

Let $\Xi^{\mathrm{BH}}_{\mathrm{eq}}(\mathcal{M}) \subset \Xi^{\mathrm{phys}}_{\mathrm{BH}}(\mathcal{M})$ denote the restricted equilibrium sector consisting of those admissible closure classes that satisfy faithful commitment, reversible no-loss coherence, and unitary-complete implementation on the stationary branch. Define the committed image of this equilibrium sector by

$$\Xi^{\mathrm{LMHS}}_{\mathrm{com,eq}}(\mathcal{M}) := \mathrm{Com}\big(\Xi^{\mathrm{BH}}_{\mathrm{eq}}(\mathcal{M})\big), \tag{13}$$

and the fiber over $\chi \in \Xi^{\mathrm{LMHS}}_{\mathrm{com,eq}}(\mathcal{M})$ by

$$\mathcal{F}^{\mathrm{LMHS}}_{\chi} := \mathrm{Com}^{-1}(\chi) \cap \Xi^{\mathrm{BH}}_{\mathrm{eq}}(\mathcal{M}). \tag{14}$$

The role of BH2 is to remove the ambiguity between thermodynamic measure and coherent completion. Let

$$\mathcal{K}_{\mathrm{glob}}(\xi,\xi') = \Psi(\xi)\overline{\Psi(\xi')} \tag{15}$$

be the coherent kernel on the reversible equilibrium support, where Ψ is a coherent representative of a projective reciprocal completion/choice paired class. The committed equilibrium weights are not independent thermodynamic insertions. They are the diagonal pushforward of the same coherent completion,

$$p^{\mathrm{eq}}_{\mathrm{BH}}(\chi) := \sum_{\xi \in \mathcal{F}^{\mathrm{LMHS}}_{\chi}} |\Psi(\xi)|^2, \qquad \sum_{\chi \in \Xi^{\mathrm{LMHS}}_{\mathrm{com,eq}}(\mathcal{M})} p^{\mathrm{eq}}_{\mathrm{BH}}(\chi) = 1. \tag{16}$$

The equilibrium black-hole entropy is therefore

$$S_{\mathrm{BH}}(\mathcal{M}) := -\sum_{\chi \in \Xi^{\mathrm{LMHS}}_{\mathrm{com,eq}}(\mathcal{M})} p^{\mathrm{eq}}_{\mathrm{BH}}(\chi) \ln p^{\mathrm{eq}}_{\mathrm{BH}}(\chi). \tag{17}$$

All logarithms in entropy and effective-dimension expressions are natural logarithms.

On faithful equilibrium sectors, reversible micro-indifference implies that no committed sector carries privileged public weight:

$$p^{\mathrm{eq}}_{\mathrm{BH}}(\chi) = p^{\mathrm{eq}}_{\mathrm{BH}}(\chi') \qquad \text{for all } \chi, \chi' \in \Xi^{\mathrm{LMHS}}_{\mathrm{com,eq}}(\mathcal{M}). \tag{18}$$

Hence the committed equilibrium distribution is forced to be uniform,

$$p^{\rm eq}_{\rm BH}(\chi) = \frac{1}{|\Xi^{\rm LMHS}_{\rm com,eq}(\mathcal{M})|}, \tag{19}$$

and therefore

$$S_{\rm BH}(\mathcal{M}) = \ln |\Xi^{\rm LMHS}_{\rm com,eq}(\mathcal{M})|. \tag{20}$$

The essential point is that projective fiber multiplicity does not survive after admissible quotienting. On the reversible support, each committed equilibrium sector carries one physical projective reciprocal completion/choice paired class rather than a family of inequivalent hidden completions or phase representatives. This closes not only the measure problem but also the quantum overcounting problem: the entropy is not the logarithm of arbitrary phase choices, nor of a basis of the reduced exterior Hilbert space, nor of an independently inserted positive measure. It is the logarithm of the number of admissible committed equilibrium sectors.

To connect the count to the area law one needs the quarter-normalized faithful detector anchor. Write

$$X_\alpha := \frac{A}{\alpha G_{\rm ren}}. \tag{21}$$

On a reversible faithful local stroke the completed-cycle balance is

$$\delta X_\alpha = -\delta\langle K\rangle, \qquad \delta N_c = 0. \tag{22}$$

The minus sign in (22) is the completed-cycle orientation convention for an inward horizon-capacity stroke. In comparing with the local Einstein bridge below, the same physical stroke is put in a common orientation; the compatibility defect therefore measures the normalization mismatch between $A/(\alpha G_{\rm ren})$ and $A/(4G_{\rm ren})$, not an additional sign choice.

On the same local stroke class, the covariant phase-space and modular-universality identities read [11–13]

$$\delta H^{\rm grav}_\zeta(\mathcal{D}_\ell) = \frac{\kappa}{2\pi}\,\delta\!\left(\frac{A}{4G_{\rm ren}}\right) + O(\ell^{d+2}), \tag{23}$$

and

$$\delta\langle K_\omega\rangle = \frac{2\pi}{\kappa}\int_{B_\ell} \delta\langle T_{ab}\rangle\,\zeta^a u^b\,{\rm d}V + O(\ell^{d+2}). \tag{24}$$

If the reversible completed-cycle anchor and the reversible Einstein bridge are required to refer to the same faithful stroke class, then the normalization is not optional. Comparing the α-normalized reversible completed-cycle relation with the Einstein-normalized bridge yields the compatibility defect

$$\Sigma_\alpha := \left(\frac{1}{4} - \frac{1}{\alpha}\right)\frac{A}{G_{\rm ren}}, \qquad \delta\Sigma_\alpha = \left(\frac{1}{4} - \frac{1}{\alpha}\right)\frac{\delta A}{G_{\rm ren}}. \tag{25}$$

Therefore, if the counted equilibrium sector contains a nontrivial family of admissible first-law strokes with $\delta A \neq 0$, compatibility on that same faithful stroke class forces

$$\alpha = 4. \tag{26}$$

Therefore

$$X := \frac{A}{4G_{\rm ren}} \tag{27}$$

is not a free coding normalization. It is the unique faithful reversible detector capacity compatible with the local Einstein bridge.

This admits an apparatus interpretation. The entropy is still a count of quantum completed-cycle sectors; what the classical horizon apparatus supplies is not an additional degeneracy, but the public readout text on which those sectors are physically implemented. In the ledger framework the classical readout is one component of the primitive closure cycle rather than a posterior record added after the quantum count. Hence the reciprocal quantum degrees of freedom associated with that readout cannot be counted independently of the constraints imposed on the readout itself. For a stationary black-hole horizon, these constraints are the area element, null support, seam compatibility, gauge reduction, and the local Einstein response. They restrict the admissible quantum completion sectors before the degeneracy is counted. Equation (26) then fixes the smallest Einstein-compatible reversible detector refinement. Thus the quarter factor enters as an admissibility scale for the quantum sectors, not as a classical multiplier appended after an unconstrained microscopic count.

The detector-theoretic reading is then immediate [27]. On the stationary large-area branch, let

$$N^{\rm eff}_H := \frac{A_H}{4G_{\rm ren}} \tag{28}$$

denote the effective number of quarter-normalized faithful detector slots. When a literal finite partition is required, choose an integer refinement

$$\Sigma = \bigcup_{i=1}^{N^{(r)}_H} D^{(r)}_i, \qquad N^{(r)}_H = N^{\rm eff}_H + o(N^{\rm eff}_H),$$

so that the rounding ambiguity is subextensive and does not affect the leading area law. Faithful one-nat saturation means that each refinement slot $D^{(r)}_i$ carries one elementary public logical slot on the exact equilibrium branch together with the hidden coherent completion required for reversible implementation. This is an effective logarithmic multiplicity statement, not the assertion that each block contains literally e discrete microstates. The faithful thermal optimum selected by the reversible anchor is therefore encoded by

$$s^{\rm eq}_i = 1, \qquad \ln d^{\rm eff}_i = 1. \tag{29}$$

If projectively flat gluing contributes only subextensive corrections to the logarithmic committed multiplicity, then

$$\begin{aligned}\log \dim_{\mathrm{eff}} \Xi^{\mathrm{LMHS}}_{\mathrm{com,eq}}(\mathcal{M}) &= \sum_{i=1}^{N_H^{(r)}} \ln d_i^{\mathrm{eff}} + o(N_H^{\mathrm{eff}}) \\ &= N_H^{\mathrm{eff}} + o(N_H^{\mathrm{eff}}).\end{aligned} \tag{30}$$

Hence

$$\log \dim_{\mathrm{eff}} \Xi^{\mathrm{LMHS}}_{\mathrm{com,eq}}(\mathcal{M}) = \frac{A_H}{4G_{\mathrm{ren}}} + o\left(\frac{A_H}{4G_{\mathrm{ren}}}\right). \tag{31}$$

In the exact effective-capacity saturation limit, the black-hole entropy becomes

$$S_{\mathrm{BH}} = \frac{A_H}{4G_{\mathrm{ren}}}. \tag{32}$$

This is the first main claim of the paper. The black-hole entropy is the thermodynamic image of an admissible quantum completed-cycle count on the faithful stationary equilibrium branch. The qualification "admissible" is essential: the degrees of freedom being counted are quantum completion sectors, but their admissible support is selected by the primitive horizon readout and its Einstein-compatible detector slots. The area law is therefore reconstructed as a text-constrained quantum count on the stationary Einstein branch, not as a free quantum-statistical count with a classical prefactor attached afterward.

V. QUASI-STATIONARY EVAPORATION AND HIDDEN-SECTOR TRANSPORT

The second main claim concerns evaporation. The relevant variables are the remaining horizon support $X_H(u)$, the cumulative public Hawking record $R_{\mathrm{pub}}(u)$, and the hidden coherent/completion reservoir $C_{\mathrm{hid}}(u)$. The completed-cycle capacity split, which is the black-hole form of the total-completion closure used in the dynamical reduction, is

$$X_0 = X_H(u) + C_{\mathrm{hid}}(u) + R_{\mathrm{pub}}(u), \tag{33}$$

where $X_0 := X_H(0)$.

A priori, the hidden sector could be described by a hidden fraction $\lambda(u)$ and an independent release channel $J_{\mathrm{rel}}(u)$ through

$$\dot{C}_{\mathrm{hid}}(u) = \lambda(u)\big[-\dot{X}_H(u)\big] - \dot{J}_{\mathrm{rel}}(u). \tag{34}$$

At this stage the hidden channel appears underdetermined. The point of the present closure-flat framework is precisely that this freedom is removed.

The reversible anchor BH3 fixes the no-loss coherent branch through $\Delta^{\mathrm{rev}}_{\mathrm{ker}} = 0$; it does not by itself determine the off-anchor observer-reduction defect function. On the open quasi-stationary branch, we therefore use the branch-local observer-side purity defect as the natural coarse measure of departure from the rank-one anchor,

$$D^{\mathrm{BH}}_{\mathrm{ker}}(u) := 1 - \mathrm{Tr}\big[(\rho^{\mathrm{obs}}_{\mathrm{rad,qs}}(u))^2\big] \geq 0. \tag{35}$$

The black-hole observer-reduction condition BH5 uses its support-normalized branch representative,

$$\widehat{\Delta}^{\mathrm{BH}}_{\mathrm{ker}}(u) := \mathcal{N}_{\mathrm{BH}}(u)\, D^{\mathrm{BH}}_{\mathrm{ker}}(u), \qquad 0 \leq \widehat{\Delta}^{\mathrm{BH}}_{\mathrm{ker}}(u) \leq 1, \tag{36}$$

where the normalization is fixed by the support-minimal branch convention rather than by an independent Hilbert-space purity scale. The hidden fraction is then identified with this support-normalized observer-reduction defect,

$$\lambda(u) = \widehat{\Delta}^{\mathrm{BH}}_{\mathrm{ker}}(u). \tag{37}$$

This is a branch-local closure law for the quasi-stationary observer reduction, not a statement about the final temporally glued endpoint state discussed in Sec. VIII. On the quasi-stationary Schwarzschild branch, once the preferred KMS data have been reduced to coarse geometry, the remaining macroscopic support is

$$x(u) := \frac{X_H(u)}{X_0}. \tag{38}$$

The support-minimal realization adopted on that branch is

$$\lambda(u) = 1 - x(u), \qquad 0 \leq u < u_{\mathrm{match}} < u_f, \tag{39}$$

where u_{match} denotes the onset of the terminal matching layer. In that final layer the quasi-stationary coarse reduction is no longer assumed to hold, and the endpoint rank-one state is imposed instead by the temporal-gluing condition of Sec. VIII. In particular, the pure endpoint state is not obtained by identifying the limit $u \to u_f^-$ of the quasi-stationary defect variable with the final observer-reduced radiation state.

The hidden reservoir law is the quasi-stationary projected reduction of the horizon-level off-anchor balance (5). After imposing the preferred modular/Killing context, the support-minimal defect law, and the total-completion closure, the black-hole transport compresses to

$$\begin{aligned}\dot{C}_{\mathrm{hid}}(u) + 3H_H(u)C_{\mathrm{hid}}(u) &= \lambda(u)\big[-\dot{X}_H(u)\big], \\ H_H(u) &:= -\frac{1}{2}\frac{\dot{X}_H(u)}{X_H(u)}.\end{aligned} \tag{40}$$

Comparing (40) with (34) gives

$$\dot{J}_{\mathrm{rel}}(u) = 3H_H(u)C_{\mathrm{hid}}(u), \tag{41}$$

so the release channel is not independently postulated. Once λ is fixed, J_{rel} is fixed as well.

To solve the system, treat $C_{\rm hid}$ as a function of X_H. Since

$$\dot{C}_{\rm hid} = \frac{\mathrm{d}C_{\rm hid}}{\mathrm{d}X_H}\dot{X}_H, \tag{42}$$

Eq. (40) becomes

$$\frac{\mathrm{d}C_{\rm hid}}{\mathrm{d}X_H} - \frac{3}{2X_H}C_{\rm hid} = -\left(1-\frac{X_H}{X_0}\right). \tag{43}$$

The integrating factor is $X_H^{-3/2}$. Imposing $C_{\rm hid}(X_0)=0$ yields

$$C_{\rm hid}(X_H) = X_H^{3/2}\int_{X_H}^{X_0}\frac{1-Y/X_0}{Y^{3/2}}\,\mathrm{d}Y, \tag{44}$$

and therefore

$$C_{\rm hid}(x) = 2X_0\, x(1-\sqrt{x})^2. \tag{45}$$

Using (33), the cumulative public record becomes

$$R_{\rm pub}(x) = X_0\big(1-3x+4x^{3/2}-2x^2\big), \tag{46}$$

and integrating (41) gives

$$J_{\rm rel}(x) = X_0\left(\frac{1}{2}-3x+4x^{3/2}-\frac{3}{2}x^2\right), \tag{47}$$

with integration constant fixed by $J_{\rm rel}(1)=0$. The formal continuation of the open-branch solution to $x=0$ gives

$$C_{\rm hid}(0)=0, \qquad R_{\rm pub}(0)=X_0, \qquad J_{\rm rel}(0)=\frac{X_0}{2}. \tag{48}$$

Here $J_{\rm rel}$ should not be identified with the cumulative public record $R_{\rm pub}$. It is the integrated hidden-relaxation contribution appearing in the reservoir equation (34). Therefore the endpoint values $J_{\rm rel}(0)=X_0/2$ and $R_{\rm pub}(0)=X_0$ are not in tension: $R_{\rm pub}$ is fixed by total-completion closure, whereas $J_{\rm rel}$ measures only the portion of the flow passing through the hidden-relaxation channel. These values specify the matched terminal capacity data selected by the quasi-stationary branch formula. In the present paper, however, the actual final observer-reduced state is fixed by the temporal-gluing condition of Sec. VIII, rather than by identifying the endpoint state with the quasi-stationary branch limit itself. This is the first main dynamical closure result: once the support law and the hidden-reservoir transport law are imposed, $C_{\rm hid}$ and $J_{\rm rel}$ are fixed on the open quasi-stationary branch. The separate retarded-time profile is fixed below by the black-hole projection of the transport normalization.

VI. FINE-GRAINED RADIATION ENTROPY AND THE PAGE CURVE

For a pure completed state on an effective split between the public radiation sector and its purifier, purity gives equality of the two reduced entropies and the general bound

$$S^{\rm fine}_{\rm rad} \le \min\{\log\dim_{\rm eff}\mathcal{H}_{\rm rad}, \log\dim_{\rm eff}\mathcal{H}_{\rm pur}\}. \tag{49}$$

In the faithful Page-saturated branch used here, the effective logarithmic capacities are

$$\begin{aligned}\log\dim_{\rm eff}\mathcal{H}_{\rm rad} &= R_{\rm pub}(x),\\ \log\dim_{\rm eff}\mathcal{H}_{\rm pur} &= X_H(x)+C_{\rm hid}(x).\end{aligned} \tag{50}$$

Thus the completed-cycle reduction selects

$$S^{\rm fine}_{\rm rad}(x) = \min\Big\{R_{\rm pub}(x),\, X_H(x)+C_{\rm hid}(x)\Big\}. \tag{51}$$

Thus the Page formula is not a consequence of global purity alone; it is the faithful saturation of the pure completed-cycle split. The purifier side is not merely the surviving black hole. It is the sum of the remaining horizon support and the still-hidden coherent reservoir.

The Page point is determined by

$$R_{\rm pub}(x) = X_H(x)+C_{\rm hid}(x). \tag{52}$$

Using $X_H(x)=X_0x$ together with (45) and (46), the Page condition becomes

$$\begin{aligned}1-3x+4x^{3/2}-2x^2 &= x+2x(1-\sqrt{x})^2,\\ 1-6x+8x^{3/2}-4x^2 &= 0.\end{aligned} \tag{53}$$

With $y=\sqrt{x}$, this is equivalently

$$4y^4-8y^3+6y^2-1=0. \tag{54}$$

The unique root in $0<y<1$ gives

$$\begin{aligned}y_{\rm Page}\approx 0.62452, \qquad x_{\rm Page} &= y_{\rm Page}^2 \approx 0.39002,\\ \frac{M_{\rm Page}}{M_0} &= \sqrt{x_{\rm Page}} \approx 0.62452.\end{aligned} \tag{55}$$

Thus the stated closure conditions give an explicit Page point for the support-minimal quasi-stationary branch.

To express the solution in retarded time, use the quasi-stationary state-to-geometry map on the Schwarzschild branch,

$$M(u) = \sqrt{\frac{X_H(u)}{4\pi G_{\rm ren}}}, \qquad A_H(u) = 16\pi G_{\rm ren}^2 M(u)^2. \tag{56}$$

In the closure-flat completed-cycle framework, the mass-loss law is not inserted as an external semiclassical anchor. The universal off-anchor transport-readout normalization is a readout coefficient associated with the admissible physical projection of the closure-flat datum:

$$\kappa_{\rm tr} > 0. \tag{57}$$

Its black-hole projection is defined by

$$\kappa^{\rm BH}_{\rm tr} := \Pi^{\rm tr}_{\rm BH}(\kappa_{\rm tr}) > 0, \tag{58}$$

where $\Pi^{\rm tr}_{\rm BH}$ denotes the transport-normalization component of the black-hole structural readout. With

$$T_H(M) = \frac{1}{8\pi G_{\rm ren} M},$$

the thermal public-release law is therefore

$$P_{\rm pub}(M) = \kappa^{\rm BH}_{\rm tr}\, A_H(M)\, T_H(M)^4. \tag{59}$$

Since the public-release channel is the observer-accessible asymptotic energy-loss channel on the quasi-stationary branch,

$$\dot M(u) = -\frac{\alpha_H}{G_{\rm ren}^2 M(u)^2}, \qquad \alpha_H := \frac{\kappa^{\rm BH}_{\rm tr}}{256\pi^3}. \tag{60}$$

Integrating gives

$$M(u)^3 = M_0^3 - \frac{3\alpha_H}{G_{\rm ren}^2} u, \tag{61}$$

and hence

$$x(u) = \left(1 - \frac{u}{u_{\rm evap}}\right)^{2/3}, \qquad u_f := u_{\rm evap} = \frac{G_{\rm ren}^2 M_0^3}{3\alpha_H}. \tag{62}$$

Therefore, within the quasi-stationary regime before the terminal matching layer,

$$0 \le u < u_{\rm match} < u_{\rm evap}, \qquad \lambda(u) = 1 - \left(1 - \frac{u}{u_{\rm evap}}\right)^{2/3}. \tag{63}$$

In this form the retarded-time evaporation law is fixed by the black-hole observer reduction of the common off-anchor transport-readout normalization rather than by an independent branch-local coefficient.

$$C_{\rm hid}(u) = 2X_0 \left(1 - \frac{u}{u_{\rm evap}}\right)^{2/3} \left[1 - \left(1 - \frac{u}{u_{\rm evap}}\right)^{1/3}\right]^2, \tag{64}$$

$$\begin{aligned} R_{\rm pub}(u) = X_0 \Bigg[1 - 3\left(1 - \frac{u}{u_{\rm evap}}\right)^{2/3} \\ + 4\left(1 - \frac{u}{u_{\rm evap}}\right) - 2\left(1 - \frac{u}{u_{\rm evap}}\right)^{4/3} \Bigg]. \end{aligned} \tag{65}$$

Combining these with (51) gives the explicit quasi-stationary-branch profile of the fine-grained radiation entropy, up to the onset of the terminal matching layer.

VII. QUANTUM EXTREMAL ISLANDS AS HIDDEN-COMPLETION READOUTS

The completed-cycle Page formula (51) can be compared directly with the quantum-extremal-island prescription. The comparison is structural rather than only analogical. In the completed-cycle formulation, the ingredients of the island formula are already present before passing to a semiclassical geometric readout.

Let R denote the observer-accessible Hawking radiation algebra at a quasi-stationary retarded time u. The completed evaporation state is pure at the level of the full completed object,

$$\rho_{\rm glob}(u) = |\Psi_{\rm glob}(u)\rangle\langle\Psi_{\rm glob}(u)|, \tag{66}$$

where the states at different quasi-stationary times are related by the no-loss temporal gluing map of Sec. VIII. The radiation observer sees the reduced state

$$\rho_R(u) = \mathrm{Tr}_{\rm comp}\, \rho_{\rm glob}(u). \tag{67}$$

Thus the radiation entropy is an observer-reduced entropy, not the entropy of the completed global state.

To make the hidden-completion branch explicit, let $\mathfrak{I}_R^{\rm adm}$ denote the set of completed-cycle-admissible hidden-completion sectors compatible with the radiation algebra R. An element $I \in \mathfrak{I}_R^{\rm adm}$ is not an additional independent semiclassical region at the fundamental level. It is the hidden-completion sector whose semiclassical geometric readout is an island.

On the semiclassical code subspace associated with R, the hidden-completion sectors define a center-block decomposition

$$\mathcal{H}_{\rm code}(R) \simeq \bigoplus_{I \in \mathfrak{I}_R^{\rm adm}} \left(\mathcal{H}^{\rm unres}_{R\cup I} \otimes \mathcal{H}^{\rm unres}_{\overline{R\cup I}} \otimes \mathcal{H}^{\rm edge}_{\partial I}\right). \tag{68}$$

Here $\mathcal{H}^{\rm edge}_{\partial I}$ is the finite edge sector carried by the completion boundary. Its logarithmic dimension is fixed by the completed-cycle capacity law,

$$\log\dim \mathcal{H}^{\rm edge}_{\partial I} = X(\partial I). \tag{69}$$

The geometric area term appears only after the semiclassical readout of this capacity.

Let Π_I denote the projector onto the I-block. Before faithful branch selection, a center-respecting state has the block form

$$\begin{aligned} \rho_{\rm code} &= \bigoplus_{I \in \mathfrak{I}_R^{\rm adm}} p_I \rho_I, \qquad p_I = \mathrm{Tr}(\Pi_I \rho_{\rm code}), \\ \rho_I &= p_I^{-1} \Pi_I \rho_{\rm code} \Pi_I, \qquad (p_I > 0). \end{aligned} \tag{70}$$

Here each ρ_I is normalized on its block. Hence $p_I \ge 0$ and $\sum_I p_I = 1$, and the entropy contains the classical center term

$$S(\rho_{\rm code}) = -\sum_I p_I \log p_I + \sum_I p_I S(\rho_I). \tag{71}$$

The calculation below is therefore a fixed committed-center calculation: faithful branch selection has already selected one stable block I_*, so that $p_{I_*} = 1$, all other p_I vanish, and the classical center Shannon term is absent.

Within such a fixed stable block I, the entropy of the observer radiation algebra is represented by a fixed-center algebraic density operator

$$\rho_{\mathrm{alg}}^{(I)}(R)=\frac{\mathbf{1}_{\partial I}}{\dim\mathcal{H}_{\partial I}^{\mathrm{edge}}}\otimes\rho_{R\cup I}^{\mathrm{unres}}\,. \tag{72}$$

Here $\rho_{R\cup I}^{\mathrm{unres}}$ is the unresolved completion state assigned, in the code-subspace representation, to the radiation together with the hidden-completion sector. Thus $\rho_{\mathrm{alg}}^{(I)}(R)$ should not be read as an ordinary density matrix on a bare Hilbert space $\mathcal{H}_R$, nor as an additional global state. It is the fixed-center algebraic representative whose entropy computes the observer entropy in that branch. This is the completed-cycle analogue of a fixed-area or fixed-center sector in the island calculation.

Using additivity of von Neumann entropy on tensor products,

$$\begin{aligned} S\Big(\rho_{\mathrm{alg}}^{(I)}(R)\Big)&=\log\dim\mathcal{H}_{\partial I}^{\mathrm{edge}}+S(\rho_{R\cup I}^{\mathrm{unres}})\\ &=X(\partial I)+S_{\mathrm{comp}}^{\mathrm{cyc}}(R\cup I),\end{aligned} \tag{73}$$

where

$$S_{\mathrm{comp}}^{\mathrm{cyc}}(R\cup I):=S(\rho_{R\cup I}^{\mathrm{unres}})\,. \tag{74}$$

Thus the completed-cycle generalized entropy is

$$S_{\mathrm{gen}}^{\mathrm{cyc}}(R,I)=X(\partial I)+S_{\mathrm{comp}}^{\mathrm{cyc}}(R\cup I). \tag{75}$$

The admissible completion boundary is selected by no-loss boundary stability. A small deformation of ∂I changes both the edge capacity and the unresolved completion entropy. A stable completed-cycle split between public radiation and hidden completion therefore satisfies

$$\delta_{\partial I}S_{\mathrm{gen}}^{\mathrm{cyc}}(R,I)=\delta_{\partial I}\left[X(\partial I)+S_{\mathrm{comp}}^{\mathrm{cyc}}(R\cup I)\right]=0. \tag{76}$$

Faithful committed branch selection then chooses the stable branch of minimal observer entropy:

$$\begin{aligned} S_R^{\mathrm{cyc}}&=\min_{\substack{I\in\mathfrak{I}_R^{\mathrm{adm}}\\ \delta_{\partial I}S_{\mathrm{gen}}^{\mathrm{cyc}}(R,I)=0}}S_{\mathrm{gen}}^{\mathrm{cyc}}(R,I)\\ &=\min_{I\in\mathfrak{I}_R^{\mathrm{adm}}}\operatorname*{ext}_{I}\left[X(\partial I)+S_{\mathrm{comp}}^{\mathrm{cyc}}(R\cup I)\right].\end{aligned} \tag{77}$$

Here ext_I means extremization with respect to continuous deformations of the completion boundary inside a fixed admissible branch, while the outer minimum compares the resulting stationary admissible branches. Thus the notation is the completed-cycle counterpart of the usual "extremize first, then minimize" prescription in the quantum-extremal-surface formula.

Under the semiclassical geometric readout of a stable completion block,

$$\begin{aligned} I&\longmapsto I_{\mathrm{geom}},\\ X(\partial I)&\longmapsto\frac{A(\partial I_{\mathrm{geom}})}{4G_{\mathrm{ren}}},\\ S_{\mathrm{comp}}^{\mathrm{cyc}}(R\cup I)&\longmapsto S_{\mathrm{bulk}}(R\cup I_{\mathrm{geom}}).\end{aligned} \tag{78}$$

Eq. (77) becomes

$$S(R)=\min_{I_{\mathrm{geom}}}\operatorname*{ext}_{I_{\mathrm{geom}}}\left[\frac{A(\partial I_{\mathrm{geom}})}{4G_{\mathrm{ren}}}+S_{\mathrm{bulk}}(R\cup I_{\mathrm{geom}})\right]. \tag{79}$$

This is the quantum-extremal-island entropy prescription [17–19]. Hence the island prescription is reproduced, at the level of the generalized-entropy functional and its branch competition, as the semiclassical geometric readout of completed-cycle hidden purification. This is not a separate first-principles derivation of the gravitational replica path integral. Rather, within the present framework, the replica-wormhole saddle is interpreted as the semiclassical path-integral presentation of the same hidden-completion branch.

The support-minimal Schwarzschild branch studied above is the two-branch specialization of (77). The no-island branch is the public Hawking branch,

$$S_{\text{no island}}^{\mathrm{cyc}}(x)=R_{\mathrm{pub}}(x), \tag{80}$$

while the island branch is the completed purifier branch,

$$S_{\mathrm{island}}^{\mathrm{cyc}}(x)=X_H(x)+C_{\mathrm{hid}}(x). \tag{81}$$

Therefore

$$\begin{aligned} S_{\mathrm{rad}}^{\mathrm{fine}}(x)&=\min\{S_{\text{no island}}^{\mathrm{cyc}}(x),\,S_{\mathrm{island}}^{\mathrm{cyc}}(x)\}\\ &=\min\{R_{\mathrm{pub}}(x),\,X_H(x)+C_{\mathrm{hid}}(x)\}.\end{aligned} \tag{82}$$

This is exactly Eq. (51). The island transition is therefore the completed-cycle branch transition at

$$R_{\mathrm{pub}}(x)=X_H(x)+C_{\mathrm{hid}}(x). \tag{83}$$

The island/QES construction does not modify the completed-cycle black-hole dynamics. It identifies the semiclassical geometric language in which the hidden-completion branch is read.

VIII. TEMPORAL GLUING AND UNITARY-COMPLETE EVAPORATION

The time-continuous formulas above describe the coarse thermodynamic history. One still wants a precise statement of what "unitary-complete evaporation" means at the level of stepwise completed-cycle transport. Let the evaporation process be coarse-grained into stages $n=0,1,\ldots,N$, with global completed-cycle states

$$|\Psi_n\rangle\in\mathcal{H}_n^{\mathrm{rad}}\otimes\mathcal{H}_n^{\mathrm{comp}}. \tag{84}$$

Unitary-complete temporal gluing means that, for every step, there exists a no-loss isometry on the completed state space,

$$\begin{aligned} V_{n+1,n}:\ \mathcal{H}_n^{\mathrm{rad}}\otimes\mathcal{H}_n^{\mathrm{comp}}&\longrightarrow\mathcal{H}_{n+1}^{\mathrm{rad}}\otimes\mathcal{H}_{n+1}^{\mathrm{comp}},\\ V_{n+1,n}^{\dagger}V_{n+1,n}&=I\end{aligned} \tag{85}$$

on the admissible support. There are also temporal-gluing representative phases $e^{\mathrm{i}\gamma_n^{\mathrm{tg}}}$ such that

$$|\Psi_{n+1}\rangle = e^{\mathrm{i}\gamma_n^{\mathrm{tg}}} V_{n+1,n}|\Psi_n\rangle. \tag{86}$$

The map $V_{n+1,n}$ is not required to factorize as $U^{\mathrm{rad}}_{n+1,n} \otimes U^{\mathrm{comp}}_{n+1,n}$. A factorized local isometry would preserve the Schmidt spectrum across the radiation–completion split and would therefore be too restrictive for an evaporation branch with a changing Page profile. The no-loss condition applies to the completed global state; the radiation and completion sectors are then obtained by the corresponding observer reductions. Define the temporal-gluing defect by

$$\varepsilon_n^{\mathrm{tg}} := 1 - \left|\langle \Psi_{n+1}, V_{n+1,n}\Psi_n\rangle\right|^2. \tag{87}$$

The exact unitary-complete anchor is

$$\varepsilon_n^{\mathrm{tg}} = 0 \qquad \text{for every admissible quasi-stationary step.} \tag{88}$$

The preferred horizon context is preserved by the completed transport at the observer-reduction level. Equivalently, on the admissible quasi-stationary support,

$$V^{\dagger}_{n+1,n}\big(K_{H,n+1} \otimes I_{\mathrm{comp},n+1}\big)V_{n+1,n} = K_{H,n} \otimes I_{\mathrm{comp},n} + O(\delta_n). \tag{89}$$

This condition preserves the modular/Killing context without forcing the global evaporation map to factorize across the radiation–completion split. At each step the observer-accessible reduced state may be written in Gram form

$$\rho_n^{\mathrm{obs}} = \sum_{r,r'} \sqrt{p_n(r)p_n(r')}\, e^{\mathrm{i}(\phi_{n,r}-\phi_{n,r'})}\Gamma^{(n)}_{rr'}|r\rangle\langle r'|. \tag{90}$$

When the preferred modular/Killing context is fixed, one may refine the labels so that

$$K_{H,n}|r,\alpha;n\rangle = \beta_n\omega_{n,r}|r,\alpha;n\rangle, \tag{91}$$

with diagonal observer weights

$$p_n(r,\alpha) = \frac{e^{-\beta_n\omega_{n,r}}}{Z_n}, \qquad Z_n := \sum_{r,\alpha} e^{-\beta_n\omega_{n,r}}. \tag{92}$$

For nonterminal quasi-stationary steps, local detailed balance is not an independent additional postulate. It is a spectral consequence of the preferred horizon context:

$$\frac{p_n(r',\alpha')}{p_n(r,\alpha)} = e^{-\beta_n(\omega_{n,r'}-\omega_{n,r})}. \tag{93}$$

In this sense the Hawking observer reduction is thermal throughout the quasi-stationary regime before the terminal matching layer, while the full temporally glued completed history remains pure. The final rank-one radiation state belongs to the endpoint gluing condition and is not obtained by extending the intermediate KMS reduction beyond its regime of validity.

The endpoint condition of full unitary completion is that the final observer-reduced radiation kernel becomes rank one. In the notation of (90), this means that the completion Gram factor carries no residual endpoint obstruction,

$$\Gamma^{(f)}_{mn} = 1. \tag{94}$$

Equivalently, if

$$c_m := \sqrt{p_m^{(f)}}\, e^{\mathrm{i}\phi_m}, \tag{95}$$

then the final observer-reduced radiation state takes the outer-product form

$$\rho^{\mathrm{obs}}_{\mathrm{rad},\,mn}(u_f) = c_m\overline{c_n}. \tag{96}$$

Thus the endpoint state is rank one. This condition removes a residual singular obstruction in the completion Gram sector; it does not fix an absolute representative phase. The remaining classwise $\mathrm{U}(1)_{\mathrm{pair}}$ is the reciprocal completion/choice pairing gauge of the final rank-one representative. Thermal observer reduction and late pure radiation are therefore compatible: thermality belongs to intermediate reduced observer sectors, while rank-one completion belongs to the full temporally glued completed-cycle history.

This temporal-gluing statement is the black-hole readout of the no-loss transport, rank-one coherence, and seam-continuity components of the closure-flat readout structure in the companion quantum-gravity construction [10]. It is not a microscopic one-sided evaporation Hamiltonian. Rather, it gives a precise mathematical meaning to unitary-complete evaporation: there exists a global pure completed-cycle representative at every quasi-stationary step, these representatives glue by no-loss isometries, the observer-accessible Hawking sector is thermal at each nonterminal quasi-stationary step, and the endpoint radiation state is rank one after the terminal matching layer. In the language of Sec. VII, the same no-loss completion is what allows the island branch to be interpreted as a hidden-completion purifier rather than as an additional independent semiclassical saddle.

IX. RIGIDITY OF THE SUPPORT-MINIMAL QUASI-STATIONARY EVAPORATION BRANCH

The quasi-stationary Schwarzschild evaporation branch is formulated as the black-hole positive readout of a closure-flat datum $\eta \in \mathcal{K}_{\mathrm{c}}$. The closure-flat construction of Ref. [10] is used here only through its admissible closure quotient, closure-stable coherent kernel, exact reversible transport, geometric readout, and transport-readout normalization. After restriction to the preferred modular/Killing context, the required black-hole sectoral data are precisely the readout components introduced in

Sec. II. The purpose of the present section is therefore not to introduce a new black-hole dynamical postulate, but to identify the support-minimal quasi-stationary branch selected by those readout data.

Along any admissible quasi-stationary history $u \mapsto [\xi(u)] \in \Xi^{\rm phys}_{\rm BH}$, the observer-reduction condition supplies the purified completed state and no-loss temporal transport described in Sec. VIII. LMHS-stable singular closure preserves the coherent line projectively, not an absolute phase. The corresponding phase is therefore only a representative of the classwise $\mathrm{U}(1)_{\rm pair}$ gauge and does not define a new physical evaporation branch. Thus temporal gluing is not an additional sector-specific rule at the level of the closure-flat completed-cycle datum; observer-side mixedness and its Page-profile evolution arise only after the black-hole observer reduction.

The earlier quasi-stationary analysis has already fixed the support-minimal branch data:

$$\lambda(x) = 1 - x, \qquad C_{\rm hid}(x) = 2X_0\, x(1-\sqrt{x})^2, \\ R_{\rm pub}(x) = X_0\big(1 - 3x + 4x^{3/2} - 2x^2\big) \tag{97}$$

together with the Page profile (51) and the retarded-time law (60)–(62). The purpose of the present section is not to rederive those formulas, but to record their rigidity status. Once the closure-flat completed-cycle datum, the black-hole transport-normalization readout $\kappa^{\rm BH}_{\rm tr}$, and the support-minimal observer-reduction branch have been selected, these quantities are no longer independent black-hole constitutive inputs. They are fixed features of the quasi-stationary black-hole readout.

The endpoint closes as part of the same temporal-gluing and LMHS-stable closure condition. The formal terminal capacity data selected by the support-minimal quasi-stationary branch are

$$X_H \to 0, \qquad C_{\rm hid} \to 0, \qquad R_{\rm pub} \to X_0. \tag{98}$$

As in Sec. VIII, the actual final state is not obtained by treating the quasi-stationary coarse description as valid all the way through the terminal matching layer. It is fixed by temporal gluing together with LMHS-stable singular closure. Since that closure preserves the coherent line projectively, there is no singular loss of phase coherence at the level of the closure-flat completed-cycle datum. After the hidden completion reservoir is exhausted, the complementary factor carries no nontrivial residual support. Therefore the final state factorizes as

$$|\Psi(u_{\rm evap})\rangle = |\psi_f\rangle_{\rm rad} \otimes |0_f\rangle_{\rm comp} \tag{99}$$

up to the physically irrelevant projective representative gauge. Equivalently,

$$\rho^{\rm obs}_{\rm rad}(u_{\rm evap}) = |\psi_f\rangle\langle\psi_f|, \tag{100}$$

so the late radiation state is rank one. Thus the absence of an endpoint phase obstruction is not an independent black-hole-specific postulate. It follows from the joint statement that singular closure preserves the projective paired coherent object and that the hidden completion reservoir has been exhausted, leaving no nontrivial complementary support in the completion sector.

These relations determine the closed quasi-stationary evaporation branch within the closure-flat completed-cycle framework. The stationary Hawking sector is thermal because the preferred modular/Killing observer reduction is diagonal, while the global history remains pure because every quasi-stationary observer sector descends from the same closure-flat completed-cycle datum. The hidden fraction is the support-normalized branch-local observer-reduction defect of that reduction, the release channel is fixed by the common transport law, the fine-grained radiation entropy is the Page minimum of the public and purifier sides, and the endpoint state is rank one because LMHS-stable singular closure preserves the coherent line while the hidden complementary support is exhausted. In this conditional completed-cycle sense, the quasi-stationary evaporation branch is obtained as one reduction of the common architecture rather than from independent black-hole-specific constitutive assumptions.

The preceding derivations already fixed the three branch ingredients that might otherwise appear independent. The support-minimal defect representative gives

$$\lambda(u) = \widehat{\Delta}^{\rm BH}_{\rm ker}(u), \qquad \lambda(x) = 1 - x, \tag{101}$$

where the second equality is the affine representative with no additional functional shape data on the neutral Schwarzschild branch. This is not uniqueness among all monotone functions with the same endpoint values; it is the rigidity statement for the support-minimal exact-reduction sector.

The geometric readout is fixed by the same quarter-normalized anchor that gives the stationary entropy count:

$$X_H = \frac{A_H}{4G_{\rm ren}} = 4\pi G_{\rm ren} M^2, \qquad A_H = 16\pi G^2_{\rm ren} M^2. \tag{102}$$

Likewise, the leading public release law is fixed by the black-hole projection of the common off-anchor transport-readout normalization,

$$P_{\rm pub}(M) = \kappa^{\rm BH}_{\rm tr} A_H(M) T_H(M)^4, \\ \dot{M} = -\frac{\alpha_H}{G^2_{\rm ren} M^2}, \qquad \alpha_H = \frac{\kappa^{\rm BH}_{\rm tr}}{256\pi^3}. \tag{103}$$

Thus the support law, Schwarzschild state-to-geometry map, and leading public release normalization are not independent black-hole-specific insertions. They are the support-minimal black-hole reduction of the closure-flat completed-cycle framework. The Page-compatible entropy profile, the island branch, and the late rank-one endpoint then follow from the hidden-reservoir law, total-completion closure, faithful Page-saturated observer reduction, global purity, and temporal gluing.

The logical status should nevertheless remain explicit. The result is not a perturbatively exact one-sided quantum-gravity scattering construction and does not constitute a conventional bulk microscopic Hamiltonian derivation. It is a completed-cycle observer-reduction derivation of the support-minimal quasi-stationary evaporation branch. In this sense, and only in this sense, the branch is microscopic inside the completed-cycle framework.

X. CONCLUSION

The paper identifies a concrete statistical object for black-hole entropy and a concrete completed-cycle description of quasi-stationary evaporation. The entropy-carrying objects are admissible black-hole completed cycles rather than bulk interior microstates, bare outcome labels, or bare forward histories. On the stationary branch, exact Unruh/Hawking purification, reversible rank-one coherent completion, faithful quarter-normalized detector blocks, and committed-sector push-forward weights reproduce the black-hole area law. The quarter-normalized detector block is the faithful horizon capacity measure selected by the common completed-cycle implementation of the public horizon readout and its reciprocal quantum completion. On the quasi-stationary branch, the same closure-flat framework yields a hidden-reservoir law, a public release channel, a Page-compatible fine-grained radiation entropy, a horizon state-to-geometry map, and a temporal-gluing closure of the late pure endpoint. When the hidden-completion branch is expressed through a semiclassical completion boundary, the same entropy formula reproduces the quantum-extremal-island prescription.

Thus the paper establishes a conditional sectoral derivation: after the black-hole readout of a closure-flat class and the stated faithful stationary and support-minimal quasi-stationary branch selections, the area law, Page-compatible hidden-reservoir profile, island readout, and endpoint closure follow within one completed-cycle structure. This gives the selected black-hole branch a single closed completed-cycle interpretation rather than a collection of independent sectoral assumptions.

[1] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333–2346 (1973).

[2] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199–220 (1975).

[3] S. W. Hawking, Breakdown of predictability in gravitational collapse, Phys. Rev. D **14**, 2460–2473 (1976).

[4] D. N. Page, Information in black hole radiation, Phys. Rev. Lett. **71**, 3743–3746 (1993).

[5] S. D. Mathur, The information paradox: A pedagogical introduction, Class. Quantum Grav. **26**, 224001 (2009).

[6] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, Black holes: Complementarity or firewalls?, JHEP **02**, 062 (2013).

[7] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D **15**, 2752–2756 (1977).

[8] R. M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D **48**, R3427–R3431 (1993).

[9] V. Iyer and R. M. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy, Phys. Rev. D **50**, 846–864 (1994).

[10] D. Song, Closure-flatness and local-to-global quantum gravity in the ledger framework (2026). Zenodo. `https://zenodo.org/records/20083498`

[11] T. Jacobson, Thermodynamics of spacetime: The Einstein equation of state, Phys. Rev. Lett. **75**, 1260–1263 (1995).

[12] D. D. Blanco, H. Casini, L.-Y. Hung, and R. C. Myers, Relative entropy and holography, JHEP **08**, 060 (2013).

[13] T. Jacobson, Entanglement equilibrium and the Einstein equation, Phys. Rev. Lett. **116**, 201101 (2016).

[14] D. Song, The Born rule as a closure-flat completed-cycle readout in ledger quantum theory (2026). Zenodo. `https://zenodo.org/records/20132761`

[15] D. Song, Closure-flat completed-cycle dynamics and the four-channel transport law (2026). Zenodo. `https://zenodo.org/records/20132795`

[16] D. Song, Closure-flat cosmological readouts, LMHS-stable degeneration, and a unified dark-sector reduction (2026). Zenodo. `https://zenodo.org/records/20150828`

[17] N. Engelhardt and A. C. Wall, Quantum extremal surfaces: Holographic entanglement entropy beyond the classical regime, JHEP **01**, 073 (2015).

[18] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole, JHEP **12**, 063 (2019).

[19] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, The entropy of Hawking radiation, Rev. Mod. Phys. **93**, 035002 (2021).

[20] W. Schmid, Variation of Hodge structure: The singularities of the period mapping, Invent. Math. **22**, 211–319 (1973).

[21] J. Steenbrink and S. Zucker, Variation of mixed Hodge structure. I, Invent. Math. **80**, 489–542 (1985).

[22] M. Kashiwara, A study of variation of mixed Hodge structure, Publ. RIMS Kyoto Univ. **22**, 991–1024 (1986).

[23] E. Cattani, A. Kaplan, and W. Schmid, Degeneration of Hodge structures, Ann. Math. **123**, 457–535 (1986).

[24] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. D **14**, 870–892 (1976).

[25] R. Haag, *Local Quantum Physics*, 2nd ed. (Springer, Berlin, 1996).

[26] H. Araki, Relative entropy of states of von Neumann algebras, Publ. RIMS Kyoto Univ. **11**, 809–833 (1976).

[27] D. Song, Black-hole entropy and the information paradox from a null-boundary ledger architecture, (2026). Zenodo. `https://zenodo.org/records/19028487`