A Quasi-Noncommutative Deformation of Classical Fluid Dynamics

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Abstract: This work proposes a deformation of the nonlinear convective term in classical fluid dynamics inspired by the Moyal product, interpreted as a geometric correction rather than a quantum effect. The deformation introduces a controlled nonlocal coupling between velocity gradients through an antisymmetric tensor θ^{ij}, interpreted as an effective geometry induced by the flow itself. Three physically motivated models for θ^{ij} are analyzed: vorticity-driven, constant background anisotropy and strain-rate coupling. Their effects are evaluated on a canonical vortex model, showing that the deformation enhances sensitivity to shear, curvature, and inter-scale interactions beyond standard Navier–Stokes dynamics.
Keywords: Navier-Stokes equation; Moyal product; Fluid dynamics; Vortex dynamics; Noncommutative algebra.

1. Introduction

Classical turbulence arises from nonlinear coupling across scales governed by the convective term (1):

(1) $$(v \cdot \nabla)v$$

This term is local but encodes strong multi-scale interactions. This work proposes a deformation inspired by the Moyal product (2):

(2) $$f \star g = fg + \epsilon \theta^{ij} \partial_i f \, \partial_j g + O(\epsilon^2)$$

without introducing Planck's constant. Instead, ϵ is treated as a phenomenological parameter controlling nonlocality.

For the scope of this research, regarding Navier-Stokes it was considered the references [1], [2] and [3]. Regarding Moyal product and noncommutative applications, references [4], [5], [6] and [7]. Plus [8], [9] and [10] about classic fluids and deformation mathematics.

2. Deformed Navier–Stokes Equation

Consider the convective term (3):

(3) $$v^j \partial_j v^i \to v^j \star \partial_j v^i$$

which yields at first order (4):

(4) $$\partial_t v^i + v^j \partial_j v^i + \epsilon\theta^{ab}(\partial_a v^j)(\partial_b \partial_j v^i) = -\partial_i p + \nu\Delta v^i$$

2.1 Definition of Variables and Physical Roles

Now rigorously define each quantity in the deformed equation and clarify its effect on the model.

1) Velocity Field $\boldsymbol{v}^i(\boldsymbol{x}, \boldsymbol{t})$

(i) Type: Vector field.

(ii) Meaning: Fluid velocity component along direction i.

(iii) Role: Primary dynamical variable and encodes flow structure, vorticity, and transport.

2) Convective Term $\boldsymbol{v}^j \boldsymbol{\partial}_j \boldsymbol{v}^i$

(i) Classical role: Nonlinear advection of momentum and responsible for energy cascade in turbulence.

(ii) Limitation: Strictly local interaction and does not explicitly encode higher-order spatial coupling.

3) Deformation Parameter $\boldsymbol{\epsilon}$

(i) Type: Scalar coefficient (dimension-dependent).

(ii) Interpretation: Strength of nonlocal correction and controls departure from classical Navier–Stokes.

(iii) Physical meaning: Can represent unresolved small-scale effects and is analogous to subgrid modeling intensity in LES.

(iv) Impact:

$\epsilon \to 0$: recovers classical fluid dynamics.

Increasing ϵ: enhances multiscale coupling.

4) Deformation Tensor $\boldsymbol{\theta^{ab}}$

(i) Type: antisymmetric tensor ($\theta^{ab} = -\theta^{ba}$).

(ii) Role: Defines preferred directions of interaction and encodes anisotropy or flow-induced geometry.

(iii) Interpretation: Generates coupling between spatial derivatives and determines how gradients interact nonlocally.

(iv) Impact: Aligns interactions along vortex structures or strain directions and can amplify or suppress instabilities depending on structure.

5) Gradient Term ($\boldsymbol{\partial_a v^j}$)

(i) Meaning: Local shear or velocity gradient.

(ii) Impact: Regions with strong gradients contribute strongly to deformation.

6) Curvature Term ($\boldsymbol{\partial_b \partial_j v^i}$)

(i) Meaning: Spatial curvature of velocity field and sensitive to vortex cores and turning of flow lines.

(ii) Impact:

Enhances corrections near: vortex boundaries, shear layers and regions of rapid spatial change.

7) Pressure Gradient $\boldsymbol{-\partial_i p}$

(i) Remains unchanged.

(ii) Still enforces incompressibility (when coupled with $\nabla \cdot v = 0$).

8) Viscous Term $\boldsymbol{\nu \Delta v^i}$

(i) Role: dissipative regularization.

(ii) Competes with deformation: viscosity smooths gradients and deformation enhances gradient interactions.

2.2 Effect on the Governing Dynamics

The added term (5):

$$\epsilon\theta^{ab}(\partial_a v^j)(\partial_b \partial_j v^i) \tag{5}$$

introduces:

(i) nonlinear gradient coupling (beyond quadratic);

(ii) anisotropic interaction channels;

(iii) enhanced sensitivity to vortex structures.

This transforms Navier–Stokes into a higher-order effective theory, where dynamics depend not only on velocity but on its spatial organization.

3. Interpretation: Effective Geometry of the Flow

The deformation introduces a geometry-like structure encoded by θ^{ij}, making the dynamics dependent on the internal organization of the flow.

3.1 Mathematical Structure of the Deformation Term

The correction term (6),

$$\theta^{ab}(\partial_a v^j)(\partial_b \partial_j v^i) \tag{6}$$

introduces: sensitivity to velocity gradients, coupling between shear and curvature and directional interaction via $\theta^{ab.}$

This allows interpreting θ^{ij} as a geometry induced by the flow itself, analogous to how curvature arises in general relativity.

3.2 Geometric Interpretation of Variables

θ^{ab}: Generator of Effective Geometry

(i) Defines an antisymmetric bilinear form;

(ii)Analogous to a Poisson structure;

(iii) Selects oriented planes in space.

θ^{ab}: Physical meaning

(i) introduces rotational bias;

(ii) couples orthogonal directions;

(iii) encodes internal "twisting" of the flow.

$\boldsymbol{\partial_a v^j}$: Tangent Structure

(i) Represents local deformation of flow lines;

(ii) Interpreted as a "tangent map" of the velocity field.

$\boldsymbol{\partial_b \partial_j v^i}$: Curvature Structure

(i) Measures how velocity gradients change;

(ii) Encodes bending or distortion of flow trajectories.

Combined Effect: equation 6 acts as a direction-dependent coupling between deformation and curvature, modulated by an antisymmetric geometry.

3.3 Physical Interpretation: Flow-Induced Geometry

The key conceptual result: the fluid defines its own effective geometry through θ^{ij}. This implies that regions of strong vorticity generates stronger geometric deformation, regions of strain alter directional coupling and flow becomes self-modulating.

3.4 Impact on Vortex Dynamics

The deformation leads to:

1. Enhanced Vortex Coherence: geometry reinforces rotational structure, stabilizes vortex cores.
2. Modified Interaction Between Vortices: interaction becomes direction-dependent and scale-coupled.
3. Nonlocal Energy Transfer: gradient coupling allows faster cascade and feedback across scales.
4. Anisotropic Transport: determined by θ^{ij}, can produce preferred flow directions.

About limiting behavior, the $\epsilon \to 0$ recovers the classical Navier–Stokes recovered. Small gradients lead to negligible deformation, and strong gradients lead to deformation dominates.

4. Three Physical Models for θ^{ij}

4.1 Vorticity-Induced Geometry

(7) $$\theta^{ij} = \alpha\, \epsilon^{ijk} \omega_k$$

(i) dynamically coupled to the flow;
(ii) amplifies effects near vortex cores;
(iii) introduces feedback: vortex → geometry → dynamics.

4.2 Constant Background Anisotropy

(8) $$\theta^{ij} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

(i) fixed geometry;

(ii) introduces uniform rotational bias;

(iii) useful baseline model.

4.3 Strain-Rate Coupling

(9)
$$S_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$$

Then (10):

(10)
$$\theta^{ij} = \beta(S_{ik}\epsilon^{kjl} - S_{jk}\epsilon^{kil})$$

(i) sensitive to deformation and stretching;

(ii) captures vortex elongation and instability.

5. Comparison with Classical Model

Table 1 - Comparison between classical and deformed Navier-Stokes.

Feature	Classical NS	Deformed Model
Locality	strict	weakly nonlocal
Vortex structure	implicit	explicitly enhanced
Multiscale coupling	emergent	explicit
Geometry	fixed	flow-dependent

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5.1 Advantages, Applicability, and Challenges of the Deformed Model

5.1.1 Principal Advantages

The proposed deformation introduces several structural improvements over the classical Navier–Stokes formulation:

(a) Explicit Multiscale Coupling

The deformation term (11):

(11) $$\epsilon\theta^{ab}(\partial_a v^j)(\partial_b \partial_j v^i)$$

directly couples first- and second-order derivatives of the velocity field, enabling explicit interaction between gradients and curvature, enhanced sensitivity to scale transitions and improved representation of energy cascade mechanisms.

Unlike classical models, where multiscale behavior emerges implicitly, this formulation encodes it at the level of the governing equations.

(b) Flow-Dependent Effective Geometry

When θ^{ij} depends on flow quantities (e.g., vorticity), the model naturally introduces a self-consistent geometric structure and dynamic feedback between flow and governing operators.

This leads to a self-adaptive nonlinear operator, where the flow reorganizes its own dynamics through internal structure.

(c) Improved Vortex Sensitivity

Because the deformation term depends on spatial gradients and curvature, it is naturally amplified in vortex cores, shear layers and boundary regions. This results in enhanced detection of coherent structures and better representation of vortex interaction and stability.

(d) Controlled Departure from Classical Behavior

The parameter ϵ provides a continuous interpolation between classical Navier–Stokes ($\epsilon = 0$) and deformed, geometry-aware dynamics ($\epsilon > 0$). This allows systematic model calibration and compatibility with existing numerical frameworks.

5.1.2 Applications and Relevant Scenarios

The model is particularly well-suited for scenarios where nonlinear spatial structure plays a dominant role.

(a) Vortex-Dominated Flows

Examples: vortex rings, wake flows (e.g., behind cylinders) and rotating fluids.

In these cases, the deformation enhances coherence tracking and improves modeling of vortex merging and splitting.

(b) Transitional and Shear Flows

In flows near instability thresholds: small perturbations grow via nonlinear coupling and strain and curvature interactions become critical. The strain-based θ^{ij} model is particularly effective here.

(c) Large Eddy Simulation (LES) and Subgrid Modeling

The deformation term acts as a physically interpretable subgrid correction: incorporates unresolved gradient effects and reduces reliance on empirical closure models.

(d) Anisotropic or Rotating Systems

For flows with preferred directions like rotating frames, stratified fluids and geophysical flows.

A fixed or structured θ^{ij} can encode directional bias and anisotropic transport mechanisms.

(e) Reduced-Order Modeling

Because the deformation captures higher-order interactions compactly, it may be used to enrich low-dimensional models and improve stability and accuracy of simplified representations.

5.1.3 Principal Challenges

Despite its advantages, the model introduces several nontrivial challenges:

1) Physical Calibration of $\boldsymbol{\theta^{ij}}$

No unique prescription exists for θ^{ij}. Different choices lead to different physical interpretations. Key issue: identifying a physically consistent and experimentally grounded form of θ^{ij}.

2) Parameter Selection ($\boldsymbol{\epsilon}$)

Must balance: stability, physical realism and numerical tractability. Too large: may introduce artificial instabilities. Too small: negligible effect relative to classical terms.

3) Increased Mathematical Complexity

The added term introduces: higher-order derivatives and nonlinear tensor contractions.

Consequences: more difficult analytical treatment, altered stability conditions

4) Numerical Implementation

Challenges include: discretization of mixed derivative terms, maintaining stability in high-gradient regions and compatibility with existing solvers.

5) Energy Consistency and Conservation Laws

The deformation modifies nonlinear interactions, raising questions about energy conservation, dissipation balance and spectral transfer behavior. A rigorous energy analysis is required to ensure physical consistency.

Table 2 - Summary of practical trade-offs.

Aspect	Benefit	Challenge
Multiscale coupling	Explicit representation	Higher complexity
Geometry	Physically rich structure	Requires interpretation
Vortex modeling	Enhanced accuracy	Parameter sensitivity
Numerical modeling	New capabilities	Implementation cost

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6. Analytical Test Case: Deformation-Induced Stabilization in a Simplified Vortex Model

6.1 Objective and Motivation

To provide a tangible validation of the proposed deformation framework, we analyze a simplified and controlled scenario based on a classical vortex configuration. The objective is to isolate the effect of the deformation term, compare directly with classical diffusion-driven behavior and demonstrate how the modified dynamics alter vortex evolution.

6.2 Simplified Governing Equation

Consider the two-dimensional incompressible vorticity equation in reduced form (12).

(12) $$\partial_t \omega + v \cdot \nabla \omega = \nu \Delta \omega$$

To isolate the deformation effect, we focus on a locally diffusive regime and introduce the modified equation (13).

(13) $$\partial_t \omega = \nu \Delta \omega - \epsilon \theta^{ab} (\partial_a v^j)(\partial_b \partial_j \omega)$$

6.3 Choice of Deformation Tensor

Adopt the physically motivated choice (14),

(14) $$\theta^{ab} = \alpha \epsilon^{ab} \omega$$

where:
(i) ϵ^{ab} is the 2D Levi-Civita symbol;
(ii) ω is the local vorticity;
(iii) α is a coupling coefficient.

This choice implies the deformation strength is locally proportional to the vortex intensity, introducing a feedback mechanism.

6.4 Initial Condition: Gaussian Vortex Profile

Consider a Gaussian approximation of a Lamb–Oseen vortex (15),

(15) $$\omega(r,t) = \frac{\Gamma}{4\pi\sigma^2(t)} \exp\left(-\frac{r^2}{4\sigma^2(t)} \right)$$

where:
(i) $r = \sqrt{(x^2 + y^2)}$;
(ii) $\sigma(t)$ represents vortex spreading;
(iii) ν is the viscosity.

6.5 Estimation of Governing Terms

To obtain a tractable expression, it is necessary to estimate the relevant terms.

(a) Diffusion term

(16)
$$\nu\Delta\omega \sim \nu\frac{\omega}{\sigma^2}$$

This represents classical viscous decay.

(b) Velocity gradients

For a vortex (17):

(17)
$$\partial v \sim \frac{v}{r}$$

(c) Second derivatives of vorticity

(18)
$$\partial^2\omega \sim \frac{\omega}{\sigma^2}$$

6.6 Deformation Term Approximation

Substituting into the deformation term (19):

(19) $$\mathcal{D} = \epsilon\theta^{ab}(\partial_a v^j)(\partial_b \partial_j \omega)$$

It was obtained (20) and (21):

(20) $$\mathcal{D} \sim \epsilon\alpha\,\omega\left(\frac{v}{r}\right)\left(\frac{\omega}{\sigma^2}\right)$$

(21) $$= \epsilon\alpha\frac{v}{r}\frac{\omega^2}{\sigma^2}$$

6.7 Reduced Evolution Equation

Combining terms, it was obtained (22):

(22) $$\partial_t\omega = \frac{1}{\sigma^2}\left[\nu\omega - \epsilon\alpha\frac{v}{r}\omega^2\right]$$

6.8 Interpretation as a Nonlinear Growth–Decay Equation

This equation can be rewritten as (23),

(23) $$\frac{d\omega}{dt} = A\omega - B\omega^2$$

where (24) and (25):

(24) $$A = \nu/\sigma^2$$

(25) $$B = \epsilon\alpha\frac{v}{r}/\sigma^2$$

The solution form corresponds to a logistic-type equation (26):

(26) $$\omega(t) = \frac{A}{B}\frac{1}{1 + Ce^{-At}}$$

6.9 Comparison with Classical Model

Classical behavior (27):

(27) $$\partial_t\omega \sim \nu\Delta\omega \Rightarrow \omega \rightarrow 0$$

Where vorticity decays monotonically and vortex dissipates completely. And about the deformed model behavior (28):

(28) $$\omega(t) \rightarrow \frac{A}{B}$$

Where vorticity approaches a finite steady value and vortex does not fully dissipate.

6.10 Physical Interpretation and Implication for Vortex Dynamics

The deformation introduces a term proportional toω^2, which has the following implications:

(a) Nonlinear Self-Reinforcement: stronger vorticity leads to stronger correction and creates resistance to diffusion.

(b) Competition with Viscosity: viscosity promotes decay and deformation counteracts it.

(c) Emergence of a Stable Regime: system evolves toward a balance and leads to sustained vortex intensity.

This simplified model suggests:

(a) Persistence of Coherent Structures: vortex cores remain longer-lived and sharper profiles maintained.

(b) Suppression of Pure Diffusion: diffusion no longer dominates asymptotically and modifies classical spreading laws.

(c) Intrinsic Stabilization Mechanism: no external forcing required and stabilization arises purely from deformation.

7. Conclusion

The Navier–Stokes equations retain a specific point of complexity: vorticity. This paper applies the deformation approach, typically used in semiclassical quantum problems, to a classical fluid dynamics scenario.

Inspired by Moyal quantization, this deformation framework provides a systematic correction to classical fluid dynamics. It offers a novel way to encode vortex structures into geometry, establishing a controllable framework that bridges local and nonlocal dynamics.

As demonstrated by the example in section 6, deformation introduces a nonlinear mechanism that fundamentally alters vortex evolution. This transforms purely diffusive decay into a competitive regime where vorticity is partially preserved.

Consequently, this vorticity-driven model is highly promising, given that vorticity acts as a generator of effective geometry within the fluid flow. In other words, deformation induces an effective self-stabilization of the vorticity.

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