The Fiber Variety of Continuous Endpoint-Compression Words
From One Collision to the Whole Curve

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Fifth paper of the Generative Mechanism Atlas program

Abstract

The fourth paper of the Generative Mechanism Atlas program opened the discrete endpoint pair $M_1(q) = 1/(1+q)$, $M_2(q) = 2/(1+q)$ into the continuous family $M_a(q) = a/(1+q)$, $a > 0$, and proved that freeness is lost: the induced map lands in the three-dimensional group $\mathrm{PGL}_2(\mathbb{R})$, so a length-four word already lies in a one-parameter fiber, exhibited there through a single explicit collision family carrying a square root. That paper established *that* a fiber exists. This paper determines *what* the fiber is. We show that the length-four fiber over a generic projective image is a *rational curve*: writing the image through the three scale-invariant ratios $(u, v, w) = (B/A, C/A, D/A)$ of the word matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, the fiber is parametrized by a single free coordinate $t = c$, with the remaining coefficients a, b, d explicit rational functions of t. The earlier collision family is one chart of this curve; in the rational coordinate the square root disappears, and the positive-coefficient locus is revealed to be the open ray $t \in (t_{\mathrm{pole}}, \infty)$ with $t_{\mathrm{pole}} = (w - uv)/\big(v(u-1)\big)$, strictly larger than the sub-arc displayed before. A positive fiber exists precisely when $w > uv$. We then record the general-length picture: a length-three head saturates the three-dimensional image and the remaining generators act by invertible right multiplication, so the projective image map has rank three at every positive point and the fiber has dimension $\max(0, L-3)$, and a length-four prefix fiber embeds explicitly as a one-parameter subfamily of every longer fiber. The discrete free monoid is thus not merely broken by continuation; its replacement is an explicit, rationally parametrized fiber geometry whose positive locus is a half-line.

1 Introduction

1.1 From the existence of a fiber to its description

The Generative Mechanism Atlas program reads a constant as the canonical trace of a generative map and studies the maps rather than the numbers [1]. Its discrete calculus [2] concerns endpoint-compression words built from the two maps $M_1(q) = 1/(1+q)$ and $M_2(q) = 2/(1+q)$ on the endpoint-ratio coordinate $q \in (0, \infty)$, and established two facts about the type-preserving core: it is a free monoid on $\{M_1, M_2\}$, and a determinant invariant places it properly inside $\mathrm{PGL}_2(\mathbb{Q})$. The third paper [3] turned to the fibers of the trace map on this discrete word space and determined them exactly: the symmetry trace identifies precisely those words differing by a terminal block of 2's, while the dyadic trace is injective. The fourth paper [4] opened the integer pair into the continuous family

$$M_a(q) = \frac{a}{1+q}, \qquad a > 0, \tag{1}$$

and showed that continuation dissolves both rigid features of the discrete theory. The determinant obstruction collapses to a sign, and freeness fails: the induced map lies in the three-dimensional group $\mathrm{PGL}_2(\mathbb{R})$, so a length-four word has more continuous coefficients than the image can record. That paper made non-injectivity concrete through a single explicit one-parameter collision family at length four, parametrized by a variable λ and carrying the square root $\sqrt{9\lambda^2 - 10\lambda + 1}$.

The earlier result was an existence statement: a fiber is present at length four, witnessed by one family. The natural next question, and the one the present paper answers, is structural. What is the fiber, as a variety? Over which images does a positive fiber exist, and what is its exact positive locus? How does the picture scale with word length? This mirrors the step GMA3 took on the discrete side, where the trace fibers were not merely shown to be nontrivial but identified completely. Here we identify the continuous map-level fiber completely at length four and record the general-length dimension.

1.2 Results

Write a length-four word matrix as $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and encode its projective class by the three scale-invariant ratios $(u, v, w) = (B/A, C/A, D/A)$, which are the natural coordinates on the image once $A \neq 0$ (always the case for positive words, since $A = a(c+1) > 0$). The main results are the following.

(i) **The length-four fiber is a rational curve** (Theorem 1). Over a generic image (u, v, w) the fiber is parametrized by the single coordinate $t = c$, with

$$a(t) = \frac{(u-1)\,t}{\delta(t)}, \qquad b(t) = \frac{(w-uv)(t+1)}{\delta(t)}, \qquad c = t, \qquad d(t) = (u-1)(t+1), \quad (2)$$

where $\delta(t) = v(u-1)\,t + (uv - w)$. The induced map is constant along the curve.

(ii) **The square root was a chart artifact** (Section 3). The GMA4 collision family is exactly this curve over the image $(u, v, w) = (\frac{3}{2}, \frac{3}{2}, \frac{5}{2})$; in the rational coordinate t the square root vanishes, and $t = 1$ recovers $(1, 1, 1, 1)$.

(iii) **The positive locus is a half-line** (Theorem 2). A positive fiber exists if and only if $w > uv$, and then the positive-coefficient locus is the open ray $t \in (t_{\mathrm{pole}}, \infty)$ with

$$t_{\mathrm{pole}} = \frac{w - uv}{v(u-1)}. \quad (3)$$

For the GMA4 image this gives $t_{\mathrm{pole}} = 1/3$, whereas GMA4 exhibited only $t \geq 1$, a proper sub-arc.

(iv) **General length** (Proposition 1). A length-three head already saturates the three-dimensional image and the remaining generators act by an invertible right multiplication, so the projective image map has rank exactly three at every positive point; its fiber has dimension $\max(0, L-3)$, and a length-four prefix fiber embeds as an explicit one-parameter subfamily of every longer fiber.

The discrete free monoid of [2, 3] and the continuous fiber of [4] are the two regimes of one construction; the present paper supplies the missing geometric object on the continuous side, an explicit rational fiber whose positive part is a ray. The closest external comparison remains Nathanson's fixed-pair freeness result for nonnegative matrices [5]; the contribution here is complementary, describing what the fiber *is* once the finite alphabet is replaced by the continuous coefficient space.

1.3 Conventions

Throughout, a word matrix is the ordered product of generator matrices $M_a = \left(\begin{smallmatrix} 0 & a \\ 1 & 1 \end{smallmatrix}\right)$, and "the induced map" means the element of $\mathrm{PGL}_2(\mathbb{R})$ it represents, i.e. the matrix up to a nonzero scalar. Unless explicitly stated otherwise, positivity refers to positive generator coefficients. We coordinatize the projective image in the A-*chart*, normalizing by the top-left entry A and using the ratios $(u, v, w) = (B/A, C/A, D/A)$; this is legitimate for every positive word because $A = a(c + 1) > 0$, so A never vanishes on the positive locus. (GMA4 displayed its figure in the D-chart $(A/D, B/D, C/D)$; the two charts carry the same projective information, and the A-chart is the convenient one here since $A > 0$ is immediate from positivity.) The verification script accompanying this paper reproduces the symbolic identities, the GMA4 recovery, the positive-locus scan, the fiber dimensions, and the figures.

2 The length-four fiber is a rational curve

The length-four word matrix is

$$M_a M_b M_c M_d = \begin{pmatrix} a(c+1) & a(c+d+1) \\ b+c+1 & c+(b+1)(d+1) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4}$$

a direct computation. For any positive word, $A = a(c + 1) > 0$, so the projective class is determined by the three ratios $(u, v, w) = (B/A, C/A, D/A)$.

The reason the fiber becomes rational is the order of elimination. The GMA4 parameter solved a collision relation through a quadratic coordinate, which introduced a square root. Here we instead fix the image ratios and keep c itself as the free coordinate; the remaining unknowns d, b, a are then eliminated linearly in succession. The square root is therefore not part of the fiber geometry, but a consequence of the earlier coordinate choice.

Lemma 1. *For every positive length-four word, $u = B/A > 1$.*

Proof. From (4), $u = B/A = (c + d + 1)/(c + 1) = 1 + d/(c + 1)$, and $d/(c + 1) > 0$ because $c, d > 0$. □

Remark 1 (Regular image region)**.** *In Theorem 1, "generic" means the regular part of the real A-chart image where $u > 1$, $v \neq 0$, and $w \neq uv$, so that the normalized matrix $\left(\begin{smallmatrix} 1 & u \\ v & w \end{smallmatrix}\right)$ is nonsingular and the elimination denominator has a genuine t-term. Here $v \neq 0$ is a regularity condition for the real chart computation; on the positive locus it is automatically strengthened to $v > 0$. Thus Theorem 1 describes the regular real chart fiber, whereas Corollary 1 identifies the positive image region as $u > 1$, $v > 0$, $w > uv$. The boundary hypersurfaces $u = 1$, $v = 0$, and $w = uv$ are not part of the main positive fiber geometry: $u = 1$ and $v = 0$ leave the regular image regime used below, while $w = uv$ is the determinant-zero boundary of the normalized image matrix and is not a point of* $\mathrm{PGL}_2(\mathbb{R})$.

Theorem 1. *This theorem concerns the real A-chart fiber, not yet the positive-coefficient part. Work in the A-chart, valid wherever $A = a(c + 1) \neq 0$. Fix an image (u, v, w) with $u > 1$, $v \neq 0$, and $w \neq uv$; equivalently in this range, $v(u - 1) \neq 0$ and the normalized image matrix is invertible. The portion of the real fiber of the length-four image map over (u, v, w) lying in this A-chart, that is the set of real tuples (a, b, c, d) whose word matrix (4) has projective ratios (u, v, w) and $A \neq 0$, is the rational curve parametrized by the single coordinate $t = c$ through (2), with $\delta(t) = v(u - 1)t + (uv - w)$, for*

$$\delta(t) \neq 0, \qquad t \neq 0, \qquad t \neq -1. \tag{5}$$

The additional condition $w \neq uv$ is exactly the nonzero-determinant condition for the image class in $\mathrm{PGL}_2(\mathbb{R})$, since after normalizing by A one has $\det\left(\begin{smallmatrix} 1 & u \\ v & w \end{smallmatrix}\right) = w - uv$. The A-chart condition is exactly $A \neq 0$; the excluded values are precisely the pole of the rational parametrization or the points where $A = a(t)(t+1)$ leaves that chart: at $t = 0$ one has $a(t) = 0$, while at $t = -1$ one has $c + 1 = 0$. In both cases $A = 0$, so these are not points of the A-chart fiber. The induced Möbius map is constant along the curve. In particular, for $L = 4$ the image map $(a, b, c, d) \mapsto \widehat{W} \in \mathrm{PGL}_2(\mathbb{R})$ has one-dimensional fibers on this chart. The positive part of this fiber, when nonempty, is determined in Theorem 2; there the stronger natural condition is $v > 0$.

Proof. The three ratio conditions $B = uA$, $C = vA$, $D = wA$ are, using (4),

$$a(c + d + 1) = u\, a(c + 1), \qquad b + c + 1 = v\, a(c + 1), \qquad c + (b + 1)(d + 1) = w\, a(c + 1). \tag{6}$$

In the first equation a cancels (in the A-chart $A = a(c + 1) \neq 0$, so $a \neq 0$), giving $c + d + 1 = u(c + 1)$, that is

$$d = (u - 1)(c + 1), \tag{7}$$

so d depends only on c and the image. Treat $c = t$ as the free coordinate. The second equation of (6) solves for b linearly,

$$b = v\, a(t + 1) - (t + 1), \tag{8}$$

using $c + 1 = t + 1$. Substituting (7) and (8) into the third equation of (6) and simplifying yields a linear equation for a whose solution is

$$a(t) = \frac{(u - 1)\, t}{v(u - 1)\, t + (uv - w)} = \frac{(u - 1)t}{\delta(t)}. \tag{9}$$

Back-substituting (9) into (8) and simplifying gives

$$b(t) = \frac{(w - uv)(t + 1)}{\delta(t)}. \tag{10}$$

Equations (7), (9), (10) are (2). Conversely, for every t satisfying (5), substituting these expressions into (4) and forming the ratios gives $B/A - u = C/A - v = D/A - w = 0$ identically in t (verified symbolically in the accompanying script), so every point of the curve has the prescribed image, and the image is constant along it. Equivalently, the three equations $B = uA$, $C = vA$, $D = wA$ are eliminated successively in d, b, and a, leaving $t = c$ as the only free variable; hence the displayed rational curve exhausts the A-chart fiber. Since t is the single free coordinate while a, b, d are determined, the chart fiber is one-dimensional. □

Remark 2 (The curve is genuinely rational)**.** *The coefficients in (2) are rational functions of t with a common linear denominator $\delta(t)$. No algebraic extension is needed to describe the fiber: on the A-chart it is a rational curve, isomorphic to a line with the pole of δ and the two chart-boundary values $t = 0, -1$ removed. This is the structural fact underlying the disappearance of the square root in Section 3.*

3 The earlier collision family revisited

GMA4 [4] exhibited the length-four collision family at the image of $(1, 1, 1, 1)$, namely $M_1M_1M_1M_1 = \left(\begin{smallmatrix} 2 & 3 \\ 3 & 5 \end{smallmatrix}\right)$, whose ratios are $(u, v, w) = (3/2, 3/2, 5/2)$. Specializing (2) to these values gives the rational fiber

$$a(t) = \frac{2t}{3t - 1}, \qquad b(t) = \frac{t + 1}{3t - 1}, \qquad c = t, \qquad d(t) = \frac{t + 1}{2}, \tag{11}$$

where the common denominator $\delta(t) = v(u-1)t + (uv - w) = \frac{1}{4}(3t-1)$ has been cleared against the numerators (the constant factor $\frac{1}{4}$ cancels). At $t = 1$ this returns $(a, b, c, d) = (1, 1, 1, 1)$.

The GMA4 family was written in a variable λ with $c = \frac{3}{2}\lambda + \frac{1}{2}\sqrt{9\lambda^2 - 10\lambda + 1} - \frac{1}{2}$ and corresponding square-root expressions for a, b, d. Setting $t = c(\lambda)$ identifies that family with (11): substitution into the rational curve simplifies symbolically to the GMA4 expressions for a, b, d. Thus the two parametrizations trace the same curve, and the rational coordinate t removes the square root. The square root was therefore an artifact of the chart, not a feature of the fiber. The accompanying script checks these symbolic identities and also retains numeric spot checks as a branch guard.

Reading the curve in the rational coordinate also clarifies the positive locus, which the next section treats in general; for this image it is $t > 1/3$, while GMA4 displayed only the sub-arc $\lambda \geq 1$, i.e. $t \geq 1$.

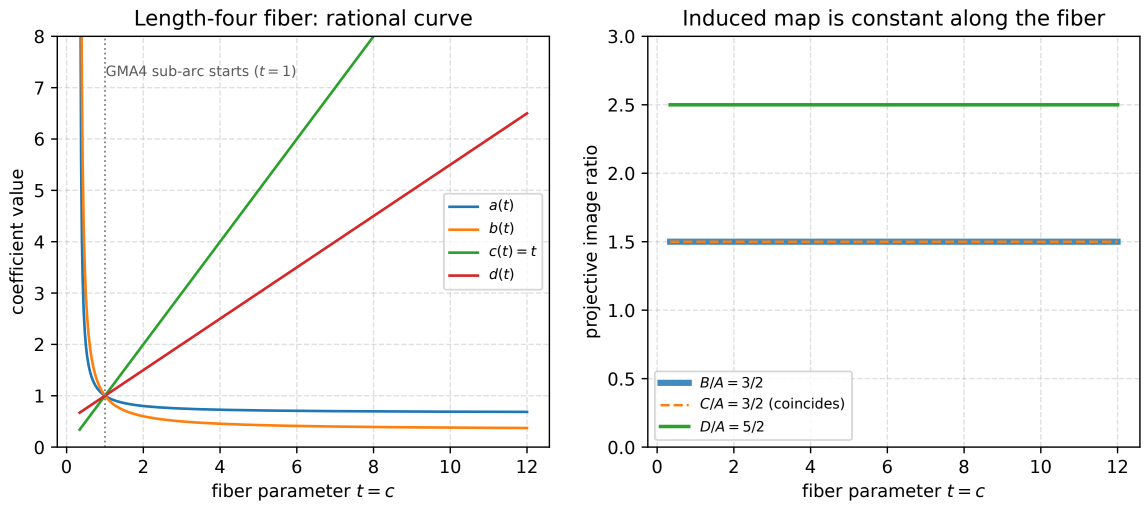

Figure 1: The length-four fiber over the image $(u, v, w) = (3/2, 3/2, 5/2)$ as a rational curve in $t = c$. Left: the coefficients $a(t), b(t), c(t) = t, d(t)$ from (11), passing through $(1, 1, 1, 1)$ at $t = 1$ (dotted line). Right: the projective image ratios $B/A, C/A, D/A$ are constant (3/2, 3/2, 5/2) along the whole curve. Here B/A and C/A coincide because the chosen image is that of the symmetric word $(1, 1, 1, 1)$, which satisfies $u = v$; since the fiber preserves the image, the coincidence persists along the entire curve. This is a special feature of this symmetric image; for a generic image $u \neq v$ and the three ratios separate.

4 The positive locus is a half-line

The rational form makes the positive-coefficient question elementary, because each coefficient is a ratio of explicit linear factors in t. From this point we restrict to the positive-coefficient part of the real fiber; accordingly the natural image-side assumptions are $u > 1$ from Lemma 1 and $v > 0$.

Theorem 2. *Fix an image (u, v, w) with $u > 1$ and $v > 0$. A positive-coefficient fiber point exists if and only if $w > uv$. In that case the positive locus on the curve (2) is exactly the open ray*

$$t \in (t_{\text{pole}}, \infty), \qquad t_{\text{pole}} = \frac{w - uv}{v(u-1)} > 0, \tag{12}$$

and on this ray all four coefficients a, b, c, d are positive.

Proof. The denominator $\delta(t) = v(u-1)t + (uv - w)$ is affine in t with positive slope $v(u-1) > 0$, so it is increasing and vanishes exactly at $t = t_{\text{pole}} = (w - uv)/(v(u-1))$; thus $\delta(t) > 0$ iff $t > t_{\text{pole}}$. We examine the four coefficients on $t > 0$.

$c = t > 0$ on the positive ray. From (7), $d = (u-1)(t+1) > 0$ for all $t > 0$, since $u > 1$. From (9), $a = (u-1)t/\delta(t)$ has positive numerator for $t > 0$, so $a > 0$ iff $\delta(t) > 0$, i.e. $t > t_{\text{pole}}$. From (10), $b = (w - uv)(t+1)/\delta(t)$; the factor $t + 1 > 0$, so on the region $\delta(t) > 0$ the sign of b equals the sign of $w - uv$.

Hence on $t > t_{\text{pole}}$ we have $a, c, d > 0$ automatically, and $b > 0$ iff $w > uv$. If $w > uv$, then $t_{\text{pole}} = (w - uv)/(v(u-1)) > 0$ and every $t > t_{\text{pole}}$ gives all four coefficients positive, so the positive locus is the stated ray. If $w \le uv$, then $b \le 0$ wherever $\delta(t) > 0$ (and $a \le 0$ where $\delta(t) < 0$), so no positive point exists. This proves both the existence criterion and the description of the locus. $\square$

Corollary 1 (Positive image region at length four)**.** *In the A-chart ratio coordinates, the projective images of positive length-four words are exactly the triples*

$$u > 1, \qquad v > 0, \qquad w > uv.$$

Necessity is immediate for $u > 1$ from Lemma 1 and for $v > 0$ from $C/A > 0$ on positive words; Theorem 2 gives the remaining necessary condition $w > uv$. Conversely, for any triple satisfying these three inequalities, take for instance $t = t_{\text{pole}} + 1$. Then $t > t_{\text{pole}}$, and the formulas (2) give $a(t), b(t), c(t), d(t) > 0$ while realizing the prescribed ratios (u, v, w). Thus the positive image region is the open semialgebraic region above, and each of its fibers contains the full ray $t \in (t_{\text{pole}}, \infty)$.

Corollary 2. *For the GMA4 image $(3/2, 3/2, 5/2)$ one has $w - uv = 5/2 - 9/4 = 1/4 > 0$ and $t_{\text{pole}} = \frac{1/4}{(3/2)(1/2)} = 1/3$, so the positive fiber is $t > 1/3$. The arc $t \ge 1$ displayed in [4] is the proper sub-arc reached by the real branch of the square-root parametrization.*

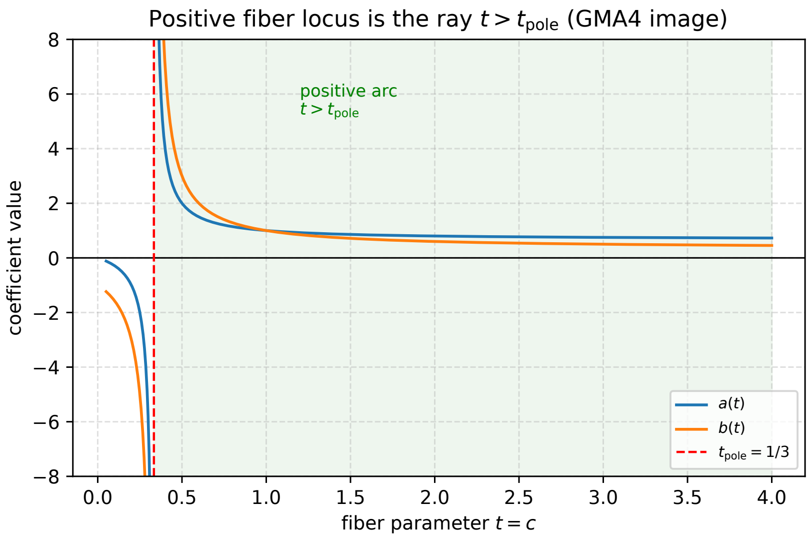

Figure 2: Positive locus for the GMA4 image. The coefficients $a(t)$ and $b(t)$ from (11) change sign across the pole $t_{\text{pole}} = 1/3$ of δ; on the shaded ray $t > 1/3$ all four coefficients are positive ($c, d > 0$ are not shown, being positive throughout $t > 0$). The positive fiber is the half-line $t \in (1/3, \infty)$.

Remark 3 (Degeneration at the pole)**.** *In the positive case $w > uv$ one has $\delta(t) = v(u-1)(t - t_{\text{pole}})$. Hence, as $t \to t_{\text{pole}}^{+}$,*

$$a(t) \sim \frac{t_{\text{pole}}}{v\,(t - t_{\text{pole}})}, \qquad b(t) \sim \frac{(w-uv)(t_{\text{pole}}+1)}{v(u-1)(t-t_{\text{pole}})}.$$

Thus a and b have simple poles and tend to $+\infty$. The word matrix leaves every bounded region of the affine coefficient space; the projective image nonetheless stays fixed, because the image is constant along the whole curve. The pole is therefore the boundary of the positive chart of the fiber, not a discontinuity of the induced map.

5 General length

The length-four analysis fixes the local picture; the dimension of the fiber for general length follows from a rank count on the projective image map, and the explicit length-four curve seeds longer fibers directly.

Lemma 2. *The length-three projective image map $g_3 : (a_1, a_2, a_3) \mapsto (B/A, C/A, D/A) \in \mathbb{R}^3$, defined on positive coefficients, is a local diffeomorphism: its Jacobian determinant is $a_3/a_1^3 \neq 0$ throughout $(0, \infty)^3$.*

Proof. From $M_{a_1}M_{a_2}M_{a_3} = \left(\begin{smallmatrix} a_1 & a_1(a_3+1) \\ a_2+1 & a_2+a_3+1 \end{smallmatrix}\right)$ the ratios are $g_3 = (a_3+1,\ (a_2+1)/a_1,\ (a_2+a_3+1)/a_1)$. A direct computation gives $\det Dg_3 = a_3/a_1^3$, which is nonzero for all positive a_i. □

Remark 4 (Explicit inverse for the length-three chart)**.** *The local diffeomorphism is also visible from the inverse formula. For $(u, v, w) = g_3(a_1, a_2, a_3)$ with $w \neq v$,*

$$a_3 = u - 1, \qquad a_1 = \frac{u-1}{w-v}, \qquad a_2 = v\frac{u-1}{w-v} - 1.$$

Indeed $w - v = a_3/a_1 = (u-1)/a_1$. Thus the length-three head already gives genuine local coordinates on the projective image; in the positive region this inverse lands in $(0, \infty)^3$ exactly when $u > 1$, $w > v$, and $v(u-1)/(w-v) > 1$.

Proposition 1. *For every positive length-L word the projective image map $\Phi_L : (a_1, \ldots, a_L) \mapsto \widehat{W} \in \mathrm{PGL}_2(\mathbb{R})$ has rank $\min(L, 3)$. In particular, for $L \geq 3$ it is a submersion onto the three-dimensional target, and its fiber through any positive point is a regular submanifold of dimension $L - 3$; equivalently the fiber dimension is $\max(0, L-3)$ for all L. Moreover, the length-four positive fiber of Theorem 2 embeds as an explicit one-parameter subfamily of the length-L fiber for every $L \geq 4$.*

Proof. The mechanism is: a length-three head is already a local diffeomorphism, and any fixed positive tail acts by invertible right multiplication. Ranks are computed in any local coordinate chart on the target $\mathrm{PGL}_2(\mathbb{R})$; on the positive locus the A-chart is available throughout. If $T \in GL_2(\mathbb{R})$ is fixed and invertible, the map $[P] \mapsto [PT]$ is well-defined on projective classes because $P \sim sP$ implies $PT \sim s(PT)$, and it is a diffeomorphism with inverse $[P] \mapsto [PT^{-1}]$. Hence right multiplication by an invertible tail is a local coordinate change on this three-dimensional manifold, and the following coordinate rank computations are intrinsic. The target $\mathrm{PGL}_2(\mathbb{R})$ is three-dimensional and the domain L-dimensional, so $\operatorname{rank} \Phi_L \leq \min(L, 3)$. For $L = 1, 2$ the rank is checked directly. At length one the image depends nontrivially on the single coefficient a_1. At length two, $M_{a_1}M_{a_2} = \left(\begin{smallmatrix} a_1 & a_1 \\ 1 & a_2+1 \end{smallmatrix}\right)$, and in the A-chart the two nonconstant ratios $(C/A, D/A) = (1/a_1, (a_2+1)/a_1)$ have Jacobian determinant $-1/a_1^3 \neq 0$. For $L = 3$, Lemma 2 shows Φ_3 is a local diffeomorphism. Thus the rank is exactly L for $L \leq 3$, and the fiber is a point.

For $L \geq 4$ we show the rank is exactly 3 at every positive point by an analytic argument, with no numerics. Split the coefficients as a length-three head (a_1, a_2, a_3) and a tail $(a_4, \ldots, a_L)$, and freeze the tail at any positive values, writing $T = M_{a_4} \cdots M_{a_L}$. Each generator has $\det M_a = -a \neq 0$, so T is invertible and represents a fixed element of $\mathrm{PGL}_2(\mathbb{R})$. On this slice,

$$\Phi_L(a_1, a_2, a_3; \text{fixed tail}) = R_T(\Phi_3(a_1, a_2, a_3)),$$

where $R_T : \mathrm{PGL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$, $P \mapsto PT$, is right multiplication by the fixed invertible class T. Right multiplication is a group diffeomorphism of $\mathrm{PGL}_2(\mathbb{R})$, hence has invertible derivative everywhere, and Φ_3 is a local diffeomorphism by Lemma 2. The composition $R_T \circ \Phi_3$ therefore has rank 3, so the derivative of Φ_L restricted to the head directions already has rank 3. Combined with the ceiling $\operatorname{rank} \Phi_L \leq 3$, this gives $\operatorname{rank} \Phi_L = 3$ at every positive point. Thus for $L \geq 3$ the map is a submersion onto its three-dimensional image chart. By the constant-rank theorem the fiber through any positive point is a regular submanifold of dimension $L - 3$.

For the embedding, take the length-four fiber (2) over a fixed image and append a common positive tail word $T = M_{e_1} \cdots M_{e_{L-4}}$ with the e_j fixed. Along the prefix fiber the length-four matrix is a fixed projective class P_4; right multiplication by the fixed matrix T sends P_4 to the fixed class P_4T, since projective equality is preserved by right multiplication $(P = sP' \Rightarrow PT = s(P'T))$. Thus the whole one-parameter prefix family has a single length-L image, giving an explicit one-parameter subfamily inside the $(L-3)$-dimensional length-L fiber. The remaining $L - 4$ fiber directions mix the tail coefficients with the prefix and are not needed for the embedding. □

Remark 5 (Reading the dimension count)**.** *The rank-three ceiling is the same dimension count that GMA4 used to anticipate non-injectivity, but here it is an analytic theorem rather than a heuristic: the rank is exactly three at every positive point because a length-three head already saturates the three-dimensional target (Lemma 2) and the tail acts by an invertible right multiplication. The fiber dimension* $\max(0, L-3)$ *is the corank of this derivative, and at length four it is realized by the rational curve of Theorem 1. The accompanying script confirms the rank for lengths one through seven, but the count no longer rests on it. The discrete and continuous theories part company exactly at* $L = 4$*, where over a finite alphabet the words remain a separated finite set while over* $(0, \infty)$ *they fill positive-dimensional fibers.*

6 Data and code availability

The symbolic fiber identities, the GMA4 recovery and square-root reparametrization, the positive-locus scan, exact rational rank checks at representative length-four and length-five points, the generic fiber dimensions for lengths one through seven, and the two figures are reproduced by the accompanying verification script `gma5_verify_v01.py`. The script was tested with Python 3.13.5, SymPy 1.14.0, NumPy 2.3.5, and Matplotlib 3.10.8. The reproduction command is

```
python3 gma5_verify_v01.py.
```

A matching dependency file, `requirements.txt`, is included in the package.

7 Conclusion and outlook

The fourth paper showed that continuation of the discrete endpoint pair destroys freeness and produces a length-four fiber, witnessed by one collision family. The present paper identifies that fiber: over a generic projective image it is a rational curve, parametrized by the single coordinate $t = c$ through (2), along which the induced map is constant. The square root in the earlier family was a chart artifact; in the rational coordinate it disappears, and the positive-coefficient locus is the open ray $t > t_{\text{pole}}$, with a positive fiber existing precisely when $w > uv$. At general length the fiber has dimension $\max(0, L - 3)$, and the length-four curve embeds explicitly into every longer fiber through a common tail.

Several questions remain. First, the global geometry of the positive locus as the image (u, v, w) varies: the pole t_{pole} and the existence criterion $w > uv$ cut out a region in image space whose boundary deserves a direct description. Second, the structure of the length-L fiber beyond the embedded prefix subfamily: whether the full $\max(0, L - 3)$-dimensional fiber is rational, irreducible, or connected on its positive part is open. Third, the relation between the continuous rational fiber found here and the discrete trace fibers of [3], which identify words differing by a terminal block of 2's; both are fiber phenomena of the same generator, and a unified account across the discrete and continuous regimes is the natural synthesis once the continuous side is, as here, fully described at length four. Finally, the boundary-value reading of [4] suggests an interpretation of the fiber coordinate t as a gauge freedom in the underlying boundary-data pipeline, which would give the rational curve an operational meaning.

References

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